SYMPLECTIC STRUCTURES ON MODULI SPACES OF SHEAVES VIA THE ATIYAH CLASS

A. KUZNETSOV AND D. MARKUSHEVICH

ABSTRACT. It is proven that the composition of the Yoneda coupling with the semiregularity map is a closed 2-form on moduli spaces of sheaves. Two examples are given when this 2-form is symplectic. Both of them are moduli spaces of torsion sheaves on the cubic 4-fold Y. The first example is the Fano scheme of lines in Y. Beauville and Donagi showed that it is symplectic but did not construct an explicit symplectic form on it. We prove that our construction provides a symplectic form. The other example is the moduli space of torsion sheaves which are supported on the hyperplane sections $H \cap Y$ of Y and are cokernels of the Pfaffian representations of $H \cap Y$.

INTRODUCTION

The existence of symplectic or Poisson structures on various moduli spaces in differential and algebraic geometry is rather a general phenomenon. Starting from the seminal work of Atiyah–Bott [AB], such structures appeared mostly via the method of symplectic reduction, which was further developed in order to produce Kähler and hyper-Kähler structures [KN]. Nowadays, many results on holomorphic Poisson structures have been obtained by techniques of algebraic geometry for moduli spaces of sheaves [Muk-1], [Bot-1], [O'G], [K] or related objects, including Higgs pairs [Hi], [Bot-2], [BR], parabolic bundles [Bot-3], regular or meromorphic connections [IIS].

Mukai [Muk-1] proved that any moduli space of simple sheaves on a K3 or abelian surface has a nondegenerate holomorphic 2-form. Its closedness was proved later for vector bundles in [Muk-2], [Bot-1], [O'G] and for sheaves in [HL]. Mukai's result was extended to moduli spaces of vector bundles over Poisson surfaces [Bot-1] and over surfaces of general type [Tyu]. A higher-dimensional generalization was obtained by Kobayashi [K], who proved that moduli spaces of simple vector bundles on a hyper-Kähler manifold are holomorphically symplectic. In all these situations, the nondegenerate 2-form on the moduli space of sheaves is induced by that on the base space of the sheaves (or on its open part for Poisson surfaces and surfaces of general type).

Beauville–Donagi [BD] discovered that the variety $F(Y)$ of lines in a smooth cubic 4-fold $Y \subset \mathbb{P}^5$ is holomorphically symplectic. Their proof is indirect, they identified $F(Y)$ for a special Y with the length-2 punctual Hilbert scheme of a K3 surface, which is known to be an irreducible symplectic manifold. This means that it is compact, simply connected, hyper-Kähler and has a unique holomorphic symplectic structure. Then the assertion for any smooth 4-dimensional cubic follows by a deformation argument: any Kähler deformation of an irreducible symplectic manifold is also irreducible symplectic. This approach does not provide a recipe for *constructing* a symplectic form on $F(Y)$.

One can interprete $F(Y)$ as the moduli space parameterizing the structure sheaves of all the lines on Y. The authors of [MT2] found another holomorphically symplectic moduli space

1991 *Mathematics Subject Classification.* 14J60, 14J45, 14F05.

A.K. was partially supported by RFFI grant 05-01-01034, Russian Presidential grant for young scientists No. MK-6122.2006.1, CRDF Award No. RUM1-2661-MO-05, the Russian Science Support Foundation, and gratefully acknowledges the support of the Pierre Deligne grant based on his 2004 Balzan prize in mathematics.

of sheaves $P(Y)$ on the cubic 4-fold Y. It parameterizes the torsion sheaves which are rank-2 vector bundles on the hyperplane sections of Y with Chern numbers $c_1 = 0$, $c_2 = 2$ (see Theorem 7.1 for other equivalent characterizations of these sheaves). These examples both differ from the previous ones by the fact that the symplectic structures on them are nomore induced by a symplectic structure on Y itself: Y has no holomorphic forms at all.

One of the objectives of this paper is to find a general reason for these moduli spaces to be symplectic. We propose a way to construct closed holomorphic 2-forms on all moduli spaces of sheaves on an arbitrary smoth complex projective variety. We show that the symplectic structures on $F(Y)$ and $P(Y)$ are obtained in this way.

Our construction of the 2-form involves the following steps. The tangent space to the moduli space at a point $[\mathcal{F}]$ representing a stable (or just simple) sheaf $\mathcal{F}$ is canonically isomorphic to $\mathrm{Ext}^1(\mathcal{F},\mathcal{F})$, so we have to associate a complex number to two elements of $\mathrm{Ext}^1(\mathcal{F},\mathcal{F})$. The first step is the Yoneda coupling

$$\mathrm{Ext}^1(\mathcal{F},\mathcal{F}) \times \mathrm{Ext}^1(\mathcal{F},\mathcal{F}) \longrightarrow \mathrm{Ext}^2(\mathcal{F},\mathcal{F}).$$

When the base space of the sheaves is a symplectic surface S with a symplectic form $\omega^{2,0} \in H^0(S,\Omega^2_S)$, Mukai composes the Yoneda coupling with the map

$$\mathrm{Ext}^2(\mathcal{F},\mathcal{F}) \xrightarrow{\mathrm{Tr}} H^2(S,\mathcal{O}_S) \xrightarrow{\cup\,\omega^{2,0}} H^2(S,\Omega^2_S) = \mathbb{C},$$

and this ends the construction in the surface case. Over a n-dimensional base Y, we insert an intermediate step: compose the Yoneda coupling with an exterior power of the Atiyah class $\mathrm{At}(\mathcal{F}) \in \mathrm{Ext}^1(\mathcal{F},\mathcal{F}\otimes\Omega^1_Y)$:

$$\mathrm{Ext}^2(\mathcal{F},\mathcal{F}) \ni \xi \mapsto \mathrm{At}(\mathcal{F})^{\wedge q}\circ\xi, \quad \mathrm{Ext}^q(\mathcal{F},\mathcal{F}\otimes\Omega^q_Y)\times\mathrm{Ext}^2(\mathcal{F},\mathcal{F}) \xrightarrow{\circ} \mathrm{Ext}^{q+2}(\mathcal{F},\mathcal{F}\otimes\Omega^q_Y). \quad (1)$$

The exponent q should be chosen in such a way that $h^{q,q+2}(Y) \neq 0$. Then we pick up an element $\omega = \omega^{n-q,n-q-2} \in H^{n-q-2}(\Omega^{n-q}_Y)$, and, to end up in $\mathbb{C}$, compose with the map

$$\mathrm{Ext}^{q+2}(\mathcal{F},\mathcal{F}\otimes\Omega^q_Y) \xrightarrow{\mathrm{Tr}} H^{q+2}(\Omega^q_Y) \xrightarrow{\cup\,\omega^{n-q,n-q-2}} H^n(\Omega^n_Y) = \mathbb{C}, \quad (2)$$

which provides the 2-form α_ω on the moduli space. The idea to couple $\mathrm{Ext}^2(\mathcal{F},\mathcal{F})$ with powers of $\mathrm{At}(\mathcal{F})$ was used by Buchweitz–Flenner [BuF1], [BuF2] to define an analog of the Bloch semiregularity map for deformations of sheaves. Thus our approach combines the ideas of Mukai and Buchweitz–Flenner. We also give a simpler shortcut formula for calculation of the 2-form in the case when the moduli space under consideration is (the partial compactification of) the relative Picard of some Hilbert scheme of equidimensional l. c. i. subschemes of Y. This formula implies that the 2-form lifts from the Hilbert scheme of Y.

Unlike Mukai's case, in which the nondegeneracy of the 2-form immediately follows from the Serre duality, our forms may be degenerate, and it is not so easy to prove that they are nondegenerate or even nonzero in particular examples. For a cubic 4-fold, $h^{1,3} = 1$, so that our construction provides a unique, up to a constant factor, 2-form α on every moduli space of sheaves on Y. We prove the following general sufficient condition: α is nondegenerate at $[\mathcal{F}]$ if

$$H^i(\mathcal{F}) = H^i(\mathcal{F}(-1)) = H^i(\mathcal{F}(-2)) = 0 \quad \text{for all} \quad i \in \mathbb{Z}. \quad (3)$$

This condition is verified for all sheaves in $P(Y)$, but not for the structure sheaves $\mathcal{O}_\ell$ of lines $\ell \subset Y$. However, we manage to apply this criterion to $F(Y)$ upon replacing $\mathcal{O}_\ell$ by the second syzygy sheaf of $\mathcal{O}_\ell(1)$, see the last part of Section 4 and Section 5 for more details.

The adequate techniques for the proof of the nondegeneracy criterion are those of derived category. The real reason of its validity is that the full triangulated subcategory $\mathcal{C}_Y \subset \mathcal{D}^b(Y)$ of those $\mathcal{F}$ satisfying (3) is a kind of deformation of the derived category of a K3 surface (see [Ku2]), and the moduli of sheaves in it behave like moduli of sheaves on a K3 surface. The

K3-type Serre duality $\operatorname{Ext}^i(\mathcal{F},\mathcal{G}) \cong \operatorname{Ext}^{2-i}(\mathcal{G},\mathcal{F})^\vee$ on $\mathcal{C}_Y$ is defined via the composition with the so called linkage class $\epsilon_\mathcal{G} \in \operatorname{Ext}^2(\mathcal{G},\mathcal{G}\otimes\mathcal{N}^\vee_{Y/\mathbb{P}^5})$:

$$\operatorname{Ext}^i(\mathcal{F},\mathcal{G}) \times \operatorname{Ext}^{2-i}(\mathcal{G},\mathcal{F}) \xrightarrow{\circ} \operatorname{Ext}^2(\mathcal{G},\mathcal{G}) \xrightarrow{\epsilon_\mathcal{G}\circ\cdot} \operatorname{Ext}^4(\mathcal{G},\mathcal{G}\otimes\Omega^4_Y) \xrightarrow{\operatorname{Tr}} H^4(\Omega^4_Y) = \mathbb{C},$$

where we use an isomorphism $\mathcal{N}^\vee_{Y/\mathbb{P}^5} \cong \Omega^4_Y$. This implies the nondegeneracy of the forms α by virtue of Theorem 3.2, which affirms that $\epsilon_\mathcal{G}$ factors through $\operatorname{At}(\mathcal{G})$.

Another outcome of our construction is a simple explicit formula for the Beauville–Donagi symplectic form on $F(Y)$. For a line $\ell \subset Y$, the tangent space to $F(Y)$ at the point $\{\ell\}$ is $H^0(\mathcal{N}_{\ell/Y})$. Formulas (27), (28) associate the skew-symmetric product $\alpha(v_1, v_2)$ to two sections v_1, v_2 of $\mathcal{N}_{\ell/Y}$, given in coordinates. A different approach to an explicit calculation of the Beauville–Donagi form, using the embedding $F(Y) \subset G(2,6)$, can be found in [IMan], Theorem 1.

Obviously, our construction of 2-forms can be literally extended to p-forms, just by taking in its first step the p-linear Yoneda product on $\operatorname{Ext}^1(\mathcal{F},\mathcal{F})$ in place of the bilinear one. We deliberately chose to limit ourselves to the 2-forms, firstly, in view of the particular importance of this case, including symplectic structures, and secondly, because we do not know nice examples with $p \geq 3$. Neither do we dwell on the case $p = 1$, in which our map $\omega \mapsto \alpha_\omega$ is nothing but the adjoint of the infinitesimal Abel-Jacobi map. This is easily seen from [Gri], Theorem 2.5 for a Hilbert scheme, the formula of Griffiths being the exact ($p = 1$)-analog of our formula (25). It is also worth noting that Welters in [W], 2.8, in order to calculate the infinitesimal Abel-Jacobi map for curves on a 3-fold X, embeds X into a 4-fold W and obtains a description, equivalent to our recipe of taking product with the linkage class of the embedding $X \subset W$.

In Section 1, we gather reminders on the tools needed in the sequel: trace map, functors Li^*, $i^!$ and duality for a closed embedding $i : Z \hookrightarrow Y$, evaluation (or integral transform) and the Atiyah class. In Section 2, we describe the construction of the 2-form α and prove that it is closed in adapting to our case the proof of the closedness of Mukai's form in [HL]. In Section 3, we define the linkage class $\epsilon_\mathcal{F}$ of a sheaf $\mathcal{F}$ supported on a locally complete intersection subscheme Y in a variety M and show that $\epsilon_\mathcal{F}$ factors through $\operatorname{At}(\mathcal{F})$. In Section 4, we show that on a cubic 4-fold, the product with $\epsilon_\mathcal{G}$ induces an isomorphism from $\operatorname{Ext}^\bullet(\mathcal{F},\mathcal{G})$ onto $\operatorname{Ext}^{\bullet+2}(\mathcal{F},\mathcal{G}(-3))$ whenever $\mathcal{F},\mathcal{G} \in \mathcal{C}_Y$, which implies the nondegeneracy of the 2-form α on any moduli space parameterizing sheaves from $\mathcal{C}_Y$. Section 5 represents the family $F(Y)$ of lines on a cubic 3-fold as a connected component of a moduli space of sheaves from $\mathcal{C}_Y$, hence the nondegeneracy of the 2-form α on it is a consequence of the results of the previous section. Section 6 provides a simplified formula for the calculation of the 2-form α on a Hilbert scheme and explicit formulas in coordinates for the case of lines in a cubic 4-fold. The concluding Section 7 describes the 10-dimensional moduli space $P(Y)$.

Acknowledgements. The second author thanks Hubert Flenner for discussions and acknowledges the hospitality of the Max-Planck-Institut für Mathematik in Bonn, where was done a part of the work on the present paper.

1. Preliminaries

1.1. **Notations and conventions.** Throughout the paper we use the field of complex numbers $\mathbb{C}$ as the base field. Certainly, all results remain true for any algebraically closed field of zero characteristic. On the other hand, the Hodge theory is used in the proof of closedness of the forms, so this probably fails in positive characteristic. By an algebraic variety we mean an integral separated scheme of finite type over the base field.

Given an algebraic variety Y we denote by $\mathrm{Coh}(Y)$ the abelian category of coherent sheaves on Y, and by $\mathcal{D}^b(\mathrm{Coh}(Y))$ its bounded derived category. It is defined (see [Ve]) as the localization of the homotopy category of bounded complexes of coherent sheaves with respect to the class of quasiisomorphisms of complexes. There are also some unbounded versions of the derived category: the bounded above derived category $\mathcal{D}^-(\mathrm{Coh}(Y))$, the bounded below derived category $\mathcal{D}^+(\mathrm{Coh}(Y))$, and the unbounded derived category $\mathcal{D}(\mathrm{Coh}(Y))$. The derived category is triangulated, it comes equipped with the *shift functor* $F \mapsto F[1]$ (induced by the shift of grading on complexes) and with a class of *distinguished* (or *exact*) *triangles*, sequences of the form $F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow F_1[1]$ (generalizing short exact sequences of complexes), satisfying a number of axioms.

All the standard functors on categories of coherent sheaves give rise to their derived functors between the derived categories (see, e.g. [Ha]). Derived functors are compatible with triangulated structures, i.e. commute with the shift functor and take exact triangles to exact triangles. In particular, the tensor product $\otimes : \mathrm{Coh}(Y) \times \mathrm{Coh}(Y) \to \mathrm{Coh}(Y)$ gives rise to the derived tensor product $\overset{\mathrm{L}}{\otimes} : \mathcal{D}^-(\mathrm{Coh}(Y)) \times \mathcal{D}^-(\mathrm{Coh}(Y)) \to \mathcal{D}^-(\mathrm{Coh}(Y))$, the functor of local homomorphisms $\mathcal{H}om : \mathrm{Coh}(Y)^\circ \times \mathrm{Coh}(Y) \to \mathrm{Coh}(Y)$ gives rise to the derived local-Hom functor $\mathcal{RH}om : \mathcal{D}^-(\mathrm{Coh}(Y))^\circ \times \mathcal{D}^+(\mathrm{Coh}(Y)) \to \mathcal{D}^+(\mathrm{Coh}(Y))$, and the functor of global homomorphisms $\mathrm{Hom} : \mathrm{Coh}(Y)^\circ \times \mathrm{Coh}(Y) \to \mathrm{Vect}$ gives rise to the derived global Hom-functor $\mathrm{RHom} : \mathcal{D}^-(\mathrm{Coh}(Y))^\circ \times \mathcal{D}^+(\mathrm{Coh}(Y)) \to \mathcal{D}^+(\mathrm{Vect})$, where Vect stands for the category of vector spaces. Similarly, given a proper map $f : X \to Y$ of algebraic varieties, the pushforward functor $f_* : \mathrm{Coh}(X) \to \mathrm{Coh}(Y)$ gives rise to the derived pushforward $Rf_* : \mathcal{D}^b(\mathrm{Coh}(X)) \to \mathcal{D}^b(\mathrm{Coh}(Y))$ and the pullback functor $f^* : \mathrm{Coh}(Y) \to \mathrm{Coh}(X)$ gives rise to the derived pullback $Lf^* : \mathcal{D}^-(\mathrm{Coh}(Y)) \to \mathcal{D}^-(\mathrm{Coh}(X))$. In particular, when $f : X \to \mathrm{Spec}\,\mathbb{C}$ is the projection to the point, the pushforward functor f_* is nothing but the functor of global sections $\Gamma(X, -)$ and its derived functor is denoted by $\mathrm{R}\Gamma(X, -)$.

Given an object $F \in \mathcal{D}(\mathrm{Coh}(Y))$ we can consider its k-th cohomology sheaf $\mathcal{H}^k(F)$. The cohomology sheaves of the derived functors applied to sheaves are the classical derived functors, e.g. $\mathcal{H}^{-k}(F \overset{\mathrm{L}}{\otimes} G) = \mathrm{Tor}_k(F, G)$, $\mathcal{H}^k(\mathcal{RH}om\,(F, G)) = \mathcal{E}xt^{\,k}(F, G)$, $\mathcal{H}^k(\mathrm{RHom}(F, G)) = \mathrm{Ext}^k(F, G)$, $\mathcal{H}^k(Rf_*(F)) = R^k f_*(F)$, $\mathcal{H}^{-k}(Lf^*(G)) = L_k f^*(G)$, and $\mathcal{H}^k(\mathrm{R}\Gamma(X, F)) = H^k(X, F)$. This gives a useful interpretation of Ext's. We have

$$\mathrm{Ext}^k(F, G) = \mathcal{H}^k(\mathrm{RHom}(F, G)) = \mathcal{H}^0(\mathrm{RHom}(F, G)[k]) = \mathcal{H}^0(\mathrm{RHom}(F, G[k])) = \mathrm{Hom}(F, G[k]),$$

so we can consider an element of the space $\mathrm{Ext}^k(F, G)$ as a morphism $F \to G[k]$. In this interpretation the Yoneda multiplication of Ext's corresponds to the composition of Hom's.

The standard isomorphisms between functors give rise to isomorphisms between derived functors. E.g., $\mathrm{Hom}(F, G) \cong \Gamma(X, \mathcal{H}om\,(F, G))$ gives $\mathrm{RHom}(F, G) \cong \mathrm{R}\Gamma(X, \mathcal{RH}om\,(F, G))$. Considering the cohomology sheaves, these isomorphisms give rise to spectral sequences, such as the local-to-global spectral sequence

$$E_2^{p,q} = H^p(X,\ \mathcal{E}xt^{\,q}(\mathcal{F}, \mathcal{G})) \Longrightarrow \mathrm{Ext}^{p+q}(\mathcal{F}, \mathcal{G}). \tag{4}$$

Since in general the derived functors may take an object of the bounded derived category to an unbounded object, the conditions ensuring boundedness are very useful. We will mention two of them. An object $F \in \mathcal{D}^b(\mathrm{Coh}(Y))$ is called a *perfect complex* if it is locally quasiisomorphic to a finite complex of locally free sheaves of finite rank. If F is a perfect complex then the functors $\mathcal{RH}om\,(F, -)$ and $\mathrm{RHom}(F, -)$ preserve boundedness. Moreover, the derived pullback of a perfect complex is a perfect complex. Perfect complexes form a triangulated subcategory of $\mathcal{D}^b(\mathrm{Coh}(Y))$ called the category of perfect complexes.

A map $f : X \to Y$ is said to have finite Tor-dimension if the structure sheaf $\mathcal{O}_X$ has finite Tor-dimension over $\mathcal{O}_Y$, i.e. for any point $x \in X$ the local ring $\mathcal{O}_{X,x}$ admits a finite flat resolution over $\mathcal{O}_{Y,f(x)}$. If $f : X \to Y$ has finite Tor-dimension then the derived pullback functor preserves boundedness and the derived pushforward functor takes perfect complexes to perfect complexes.

1.2. **Traces.** Let Y be an algebraic variety and E a vector bundle on Y. For every coherent sheaf $\mathcal{F}$ on Y which is a perfect complex consider the composition

$$\mathcal{RH}om\,(\mathcal{F}, \mathcal{F} \otimes E) \cong \mathcal{F}^\vee \overset{\mathrm{L}}{\otimes} \mathcal{F} \otimes E \to E,$$

where $\mathcal{F}^\vee = \mathcal{RH}om\,(\mathcal{F}, \mathcal{O}_Y)$ is the derived dual of $\mathcal{F}$. The first map above is the canonical isomorphism (it uses perfectness of $\mathcal{F}$) and the second is the "contraction" map. Taking the k-th cohomology, we obtain a natural map

$$\mathrm{Tr} : \mathrm{Ext}^k(\mathcal{F}, \mathcal{F} \otimes E) \to H^k(Y, E),$$

the *trace map* (see [Ill]).

The most important property of the trace is *additivity*: if $\mathcal{F}_1 \xrightarrow{\phi_1} \mathcal{F}_2 \xrightarrow{\phi_1} \mathcal{F}_3 \xrightarrow{\phi_3} \mathcal{F}_1[1]$ is a distinguished triangle and a collection $\mu_i \in \mathrm{Ext}^k(\mathcal{F}_i, \mathcal{F}_i \otimes E)$ is compatible with the triangle (i.e. the diagram

$$\begin{array}{ccccccc}
\mathcal{F}_1 & \xrightarrow{\phi_1} & \mathcal{F}_2 & \xrightarrow{\phi_2} & \mathcal{F}_3 & \xrightarrow{\phi_3} & \mathcal{F}_1[1] \\
\downarrow{\scriptstyle \mu_1} & & \downarrow{\scriptstyle \mu_2} & & \downarrow{\scriptstyle \mu_3} & & \downarrow{\scriptstyle -\mu_1} \\
\mathcal{F}_1 \otimes E[k] & \xrightarrow{\phi_1 \otimes 1_E} & \mathcal{F}_2 \otimes E[k] & \xrightarrow{\phi_2 \otimes 1_E} & \mathcal{F}_3 \otimes E[k] & \xrightarrow{\phi_3 \otimes 1_E} & \mathcal{F}_1 \otimes E[k+1]
\end{array}$$

is commutative), then

$$\mathrm{Tr}(\mu_1) - \mathrm{Tr}(\mu_2) + \mathrm{Tr}(\mu_3) = 0$$

in $H^k(Y, E)$.

Another important property is *multiplicativity*: if $\mu \in \mathrm{Ext}^k(\mathcal{F}, \mathcal{F} \otimes E)$ and $\varphi \in \mathrm{Ext}^l(E, E')$ then

$$\varphi \circ \mathrm{Tr}(\mu) = \mathrm{Tr}((\mathrm{id}_{\mathcal{F}} \otimes \varphi) \circ \mu)$$

in $H^{k+l}(Y, E')$.

1.3. **Sheaves on a subvariety.** Let $i : Z \hookrightarrow Y$ be a closed embedding. If $Z \subset Y$ is a locally complete intersection, we denote by $\mathcal{N}_{Z/Y}$ the normal bundle of Z in Y. Now let us compute the cohomology sheaves $L_k i^* \mathcal{F}$ of the derived pullback functor for $\mathcal{F} = i_* F$, where F is a coherent sheaf on Z.

Lemma 1.3.1. *If $Z \subset Y$ is a locally complete intersection of codimension m then we have $L_k i^* i_* F \cong F \otimes \Lambda^k \mathcal{N}^\vee_{Z/Y}$ for $0 \le k \le m$, and $L_{>m} i^* i_* F = 0$.*

Proof. Since i is a closed embedding, it suffices to check that

$$i_* L_k i^* i_* F \cong i_*(F \otimes \Lambda^k \mathcal{N}^\vee_{Z/Y}), \quad \text{for } 0 \le k \le m, \qquad \text{and} \qquad i_* L_{>m} i^* i_* F = 0.$$

By the projection formula we have $i_* L_k i^* i_* F \cong \mathrm{Tor}_k(i_* F, i_* \mathcal{O}_Z)$. Since Z is a locally complete intersection, it can be represented locally as the zero locus of a regular section of a rank m vector bundle $\mathcal{E}$ on Y. Therefore, locally we have the Koszul resolution

$$0 \to \Lambda^m \mathcal{E}^\vee \to \Lambda^{m-1} \mathcal{E}^\vee \to \cdots \to \Lambda^2 \mathcal{E}^\vee \to \mathcal{E}^\vee \to \mathcal{O}_Y \to i_* \mathcal{O}_Z \to 0.$$

Using it to compute Tor-s we see that $\mathrm{Tor}_k(i_* \mathcal{F}, i_* \mathcal{O}_Y) \cong i_* \mathcal{F} \otimes \Lambda^k \mathcal{E}^\vee \cong i_*(\mathcal{F} \otimes \Lambda^k \mathcal{E}^\vee_{|Z})$ and it remains to note that $\mathcal{N}^\vee_{Z/Y} \cong \mathcal{E}^\vee_{|Z}$. ☐

Now let us compute $\mathcal{E}xt^k(\mathcal{F}, \mathcal{G})$ for $\mathcal{F} = i_* F$, $\mathcal{G} = i_* G$, where F and G are coherent sheaves on Z.

Lemma 1.3.2. *Assume that $Z \subset Y$ is a locally complete intersection of codimension m. Let F and G be coherent sheaves on Z and assume that F is locally free. Then we have*

(i) $\mathcal{E}xt^k(i_*F, i_*G) \cong i_*(\wedge^k \mathcal{N}_{Z/Y} \otimes F^\vee \otimes G)$ *if* $0 \le k \le m$, *and* $\mathcal{E}xt^k(i_*F, i_*G) = 0$ *otherwise.*

(ii) Multiplication $\mathcal{E}xt^l(i_*G, i_*H) \otimes \mathcal{E}xt^k(i_*F, i_*G) \to \mathcal{E}xt^{k+l}(i_*F, i_*H)$ *corresponds under isomorphisms (i) to the map* $\wedge^l \mathcal{N}_{Z/Y} \otimes G^\vee \otimes H \otimes \wedge^k \mathcal{N}_{Z/Y} \otimes F^\vee \otimes G \to \wedge^{k+l} \mathcal{N}_{Z/Y} \otimes F^\vee \otimes H$ *given by the wedge product* $\wedge^l \mathcal{N}_{Z/Y} \otimes \wedge^k \mathcal{N}_{Z/Y} \to \wedge^{k+l} \mathcal{N}_{Z/Y}$ *and the contraction* $G^\vee \otimes G \to \mathcal{O}_Z$.

Proof. For (i) we note that $R\mathcal{H}om\,(i_*F, i_*G) \cong i_* R\mathcal{H}om\,(Li^* i_* F, G)$ by the standard adjunction between the pushforward and the pullback (see [Ha]). On the other hand, by Lemma 1.3.1 the complex $Li^* i_* F$ has cohomology sheaves $F \otimes \wedge^k \mathcal{N}^\vee_{Z/Y}$, which are locally free. Therefore, $R\mathcal{H}om\,(Li^* i_* F, G)$ has cohomology $(F \otimes \wedge^k \mathcal{N}^\vee_{Z/Y})^\vee \otimes G \cong \wedge^k \mathcal{N}_{Z/Y} \otimes F^\vee \otimes G$.

Assertion (ii) is local, so we may assume that Z is the zero locus of a regular section of a rank m vector bundle $\mathcal{E}$ on Y, and that F, G and H are the restrictions from Y to Z of sheaves $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$. Then the tensor product of $\mathcal{F}$ and of the Koszul resolution of $i_*\mathcal{O}_Z$ is a resolution of $i_*F \cong i_* i^* \mathcal{F} \cong \mathcal{F} \otimes i_* \mathcal{O}_Z$, similarly for i_*G, and we use these resolutions to compute $R\mathcal{H}om$-s. It is clear that the multiplication $R\mathcal{H}om\,(i_*F, i_*G) \otimes R\mathcal{H}om\,(i_*G, i_*H) \to R\mathcal{H}om\,(i_*F, i_*H)$ is induced by the wedge product $\wedge^k \mathcal{E} \otimes \wedge^l \mathcal{E} \to \wedge^{k+l} \mathcal{E}$ and by the contraction $\mathcal{G}^\vee \otimes \mathcal{G} \to \mathcal{O}_Y$. Restricting to the cohomology we deduce the claim. ∎

The local-to-global spectral sequence (4) allows us to compute $\operatorname{Ext}^k(\mathcal{F}, \mathcal{G})$ for the sheaves $\mathcal{F} = i_*F$, $\mathcal{G} = i_*G$. For the computation of the Yoneda multiplication on Ext-s the following lemma is very useful.

Lemma 1.3.3. *The maps*

$$H^{p_1}(V, \mathcal{E}xt^{q_1}(\mathcal{G}, \mathcal{H})) \otimes H^{p_2}(V, \mathcal{E}xt^{q_2}(\mathcal{F}, \mathcal{G})) \to H^{p_1+p_2}(V, \mathcal{E}xt^{q_1+q_2}(\mathcal{F}, \mathcal{H}))$$

induced by the composition on local $\mathcal{E}xt$'s and by the cup-product on the cohomology commute with the differentials of the spectral sequence and differ from the maps induced by the Yoneda multiplication $\operatorname{Ext}^{k_1}(\mathcal{G}, \mathcal{H}) \otimes \operatorname{Ext}^{k_2}(\mathcal{F}, \mathcal{G}) \to \operatorname{Ext}^{k_1+k_2}(\mathcal{F}, \mathcal{H})$ *by the sign* $(-1)^{p_1 q_2}$.

Proof. Follows from the fact that the isomorphism of functors $\mathrm{R}\Gamma \circ \mathcal{RH}om \cong \mathrm{RHom}$ is compatible with multiplication upon an appropriate change of signs in the double complex on the l. h. s. ∎

1.4. **Traces and duality.** Let $f : Z \to Y$ be a projective morphism of finite Tor-dimension. It is well known that the (derived) pushforward functor $Rf_* : \mathcal{D}^b(\operatorname{Coh}(Z)) \to \mathcal{D}^b(\operatorname{Coh}(Y))$ has a right adjoint functor, called the *twisted pullback functor* $f^! : \mathcal{D}^b(\operatorname{Coh}(Y)) \to \mathcal{D}^b \operatorname{Coh}(Z))$ (see [Ha]). The twisted pullback functor differs from the usual (derived) pullback functor by the relative canonical class (and a shift), explicitly

$$f^! \mathcal{F} \cong Lf^* \mathcal{F} \otimes \omega_{Z/Y}[\dim Z - \dim Y].$$

Since $f^!$ is right adjoint to Rf_* we have a canonical *duality* isomorphism

$$\operatorname{Hom}(\mathcal{G}, f^! \mathcal{F}) \cong \operatorname{Hom}(Rf_* \mathcal{G}, \mathcal{F}) \tag{5}$$

for all $\mathcal{F} \in \mathcal{D}^b(\operatorname{Coh}(Y))$, $\mathcal{G} \in \mathcal{D}^b(\operatorname{Coh}(Z))$. In particular, taking $\mathcal{G} = f^! \mathcal{F}$ we denote by can the map $Rf_* f^! \mathcal{F} \to \mathcal{F}$ corresponding to the identity map $f^! \mathcal{F} \to f^! \mathcal{F}$ under the duality isomorphism.

Lemma 1.4.1. *Take any morphism $\varphi : \mathcal{G} \to f^! \mathcal{F}$. Then the morphism $Rf_* \mathcal{G} \to \mathcal{F}$ associated to φ by the duality isomorphism coincides with the composition*

$$Rf_* \mathcal{G} \xrightarrow{Rf_* \varphi} Rf_* f^! \mathcal{F} \xrightarrow{\mathsf{can}} \mathcal{F}. \tag{6}$$

In particular, every map $Rf_*\mathcal{G} \to \mathcal{F}$ has a factorization (6) *for some $\varphi : \mathcal{G} \to f^!\mathcal{F}$.*

Proof. This is a standard consequence of adjointness. □

We will use the twisted pullback functor in case when f is a closed embedding of a locally complete intersection subscheme. So, assume that $Z \subset Y$ is a locally complete intersection of codimension m, and $i : Z \to Y$ is the embedding. Then $\omega_{Z/Y} = \wedge^m \mathcal{N}_{Z/Y}$ and we have

$$i^!\mathcal{F} \cong Li^*\mathcal{F} \otimes \wedge^m \mathcal{N}_{Z/Y}[-m]. \tag{7}$$

In particular, if $\mathcal{F}$ is a vector bundle on Y, then $i^!\mathcal{F}$ is a vector bundle on Z shifted by $-m$.

Proposition 1.4.2. *Let Y be an algebraic variety, $Z \subset Y$ a locally complete intersection subscheme, $i : Z \to Y$ the embedding, E a vector bundle on Y, and F a coherent sheaf on Z which is a perfect complex. Then the trace map* $\mathrm{Tr} : \mathrm{Ext}^k(i_*F, i_*F \otimes E) \to H^k(Y, E)$ *factors through $H^k(Y, i_*i^!E) \xrightarrow{\mathsf{can}} H^k(Y, E)$.*

Proof. By definition, the trace map is induced by the contraction map $(i_*F)^\vee \overset{\mathrm{L}}{\otimes} i_*F \otimes E \to E$ and by the projection formula we have $(i_*F)^\vee \overset{\mathrm{L}}{\otimes} i_*F \otimes E \cong i_*(Li^*((i_*F)^\vee \otimes E) \overset{\mathrm{L}}{\otimes} F)$, hence the contraction map factors through $i_*i^!E \xrightarrow{\mathsf{can}} E$ by Lemma 1.4.1. By functoriality of the cohomology, the trace map factors as well. □

In the case when $k = m$ is the codimension of Z in Y, and F is locally free, the factorization of the trace map can be described rather explicitly. In this case we can consider the following composition of maps

$$\begin{aligned}\mathrm{Ext}^m(i_*F, i_*F \otimes E) &\to H^0(Y, \mathcal{E}xt^m(i_*F, i_*F \otimes E)) \cong \\ &\cong H^0(Z, F^\vee \otimes F \otimes \wedge^m \mathcal{N}_{Z/Y} \otimes E_{|Z}) \to H^0(Z, \wedge^m \mathcal{N}_{Z/Y} \otimes E_{|Z}) \cong \\ &\cong H^m(Z, i^!E) \cong H^m(Y, i_*i^!E) \xrightarrow{\mathsf{can}} H^m(Y, E).\end{aligned} \tag{8}$$

The first map here is the canonical projection, the second is the isomorphism of Lemma 1.3.2 (i), the third is the trace map on Z, the fourth is the isomorphism (7), the fifth is evident, and the last one is the canonical map.

Proposition 1.4.3. *Let Y be an algebraic variety, $Z \subset Y$ a locally complete intersection subscheme of codimension m, $i : Z \to Y$ the embedding, E a vector bundle on Y, and F a vector bundle on Z. Then the composition of the maps in* (8) *coincides with the trace map* $\mathrm{Tr} : \mathrm{Ext}^m(i_*F, i_*F \otimes E) \to H^m(Y, E)$.

Proof. Indeed, by Lemma 1.3.2 (i) the complex $(i_*F)^\vee \overset{\mathrm{L}}{\otimes} i_*F \otimes E$ has nontrivial cohomology only in degrees from 0 to m, while $i_*i^!E$ is concentrated in degree m by (7). Therefore, the contraction map $(i_*F)^\vee \overset{\mathrm{L}}{\otimes} i_*F \otimes E \to i_*i^!E$ factors through the m-th cohomology sheaf of $(i_*F)^\vee \overset{\mathrm{L}}{\otimes} i_*F \otimes E$, i.e. through $\mathcal{E}xt^m(i_*F, i_*F \otimes E)$. The remaining part is evident. □

Finally, we will need the following description of the map $H^m(Y, i_*i^!E) \xrightarrow{\mathsf{can}} H^m(Y, E)$.

Lemma 1.4.4. *Let Y and Z be smooth varieties, $Z \subset Y$, $m = \mathrm{codim}_Y Z$. The map $H^m(Y, i_*i^!E) \xrightarrow{\mathsf{can}} H^m(Y, E)$ is dual to the map*

$$\begin{aligned}H^m(Y, E)^\vee &\cong H^{n-m}(Y, E^\vee \otimes \omega_Y) \to H^{n-m}(Z, E^\vee_{|Z} \otimes \omega_{Y|Z}) \cong \\ &\cong H^0(Z, E_{|Z} \otimes \omega^{-1}_{Y|Z} \otimes \omega_Z)^\vee \cong H^0(Z, E_{|Z} \otimes \wedge^m \mathcal{N}_{Z/Y})^\vee \cong H^m(Z, i^!E)^\vee \cong H^m(Y, i_*i^!E)^\vee\end{aligned}$$

where the first and the third isomorphisms are given by the Serre duality on Y and Z respectively, the second map is the restriction from Y to Z, the fourth isomorphism is the adjunction formula for ω_Z, the fifth is (7)*, and the last one is obvious.*

Proof. The standard relation between the right adjoint and the left adjoint functors. When the Serre duality takes place, the right adjoint functor (and its canonical map) is obtained from the left adjoint functor by conjugation with the Serre functors. ☐

1.5. **Evaluation.** Let K be an object of the derived category $\mathcal{D}^-(\mathrm{Coh}(X \times Y))$. Let $\mathrm{pr}_1, \mathrm{pr}_2 : X \times Y \to X, Y$ be the projections. For every $\mathcal{F} \in \mathcal{D}^-(\mathrm{Coh}(X))$ we consider the object

$$\Phi_K(\mathcal{F}) = R\,\mathrm{pr}_{2*}(\mathrm{pr}_1^* \mathcal{F} \overset{\mathrm{L}}{\otimes} K) \in \mathcal{D}^-(\mathrm{Coh}(Y)).$$

The object $\Phi_K(\mathcal{F})$ will be called the *evaluation* of K on $\mathcal{F}$. It is clear that evaluation is functorial, both in $\mathcal{F}$ and in K. Functoriality in $\mathcal{F}$ means that Φ_K is a functor from the derived category $\mathcal{D}^-(\mathrm{Coh}(X))$ to the derived category $\mathcal{D}^-(\mathrm{Coh}(Y))$ (in other terminology such functors are referred to as *integral* or *kernel* functors and the objects K are referred to as *kernels*). Functoriality in K means that to every morphism of kernels $\phi : K_1 \to K_2$ in $\mathcal{D}^-(\mathrm{Coh}(X \times Y))$ corresponds a morphism $\Phi_{K_1}(\mathcal{F}) \to \Phi_{K_2}(\mathcal{F})$ in $\mathcal{D}^-(\mathrm{Coh}(Y))$ which we call *evaluation* of ϕ on $\mathcal{F}$.

1.6. **Atiyah classes.** The Atiyah class was introduced in [At] for the case of vector bundles and in [Ill] for any complex of coherent sheaves $\mathcal{F}$. Let Y be an algebraic variety. Let $\Delta : Y \to Y \times Y$ denote the diagonal embedding. Let $\Delta(Y)^{(2)} \subset Y \times Y$ denote the second infinitesimal neighborhood of the diagonal $\Delta(Y) \subset Y \times Y$. In other words, if $\mathcal{I}_\Delta$ is the sheaf of ideals of the diagonal $\Delta(Y) \subset Y \times Y$, then $\Delta(Y)^{(2)}$ is the closed subscheme of $Y \times Y$ defined by the sheaf of ideals $\mathcal{I}_\Delta^2$. Note that $\mathcal{I}_\Delta/\mathcal{I}_\Delta^2 \cong \mathcal{N}^\vee_{\Delta(Y)/Y\times Y} \cong \Omega_Y$, hence we have the following exact sequence

$$0 \to \Delta_*\Omega_Y \to \mathcal{O}_{\Delta(Y)^{(2)}} \to \Delta_*\mathcal{O}_Y \to 0. \tag{9}$$

The corresponding class $\widetilde{\mathrm{At}} \in \mathrm{Ext}^1(\Delta_*\mathcal{O}_Y, \Delta_*\Omega_Y)$ is called the *universal Atiyah class* of Y.

Evaluation produces from the universal Atiyah class the usual Atiyah classes of sheaves on Y. Indeed, it is clear that $\Phi_{\Delta_*\mathcal{O}_Y}(\mathcal{F}) \cong \mathcal{F}$, $\Phi_{\Delta_*\Omega_Y}(\mathcal{F}) \cong \mathcal{F} \otimes \Omega_Y$, so evaluation of $\widetilde{\mathrm{At}}$ on $\mathcal{F}$ gives a class $\mathrm{At}_\mathcal{F} \in \mathrm{Ext}^1(\mathcal{F}, \mathcal{F} \otimes \Omega_Y)$ which is just the Atiyah class of $\mathcal{F}$. See [HL], 10.1.5 for a representation of $\mathrm{At}_\mathcal{F}$ by a Čech cocycle, or [ALJ] for an approach via simplicial spaces.

2. Closed 2-forms on moduli spaces of sheaves

Let Y be a smooth complex projective variety of dimension n. Consider a moduli space $\mathfrak{M}$ of stable sheaves on Y as defined in Sect. 1 of [Si] (see also [LP]) or, more generally, any pre-scheme representing an open part of the moduli functor $\mathbf{Spl}_Y$ of simple sheaves [AK]. All the considerations of this section have also their analytic counterpart in the case when Y is a compact Kähler manifold and $\mathfrak{M}$ is the analytic moduli space (maybe, non-Hausdorff) of simple vector bundles on Y, see [K]. However, we will need for later applications the case when $\mathfrak{M}$ is a component of the moduli space of torsion sheaves, so we cannot restrict ourselves to vector bundles.

For any sheaf $\mathcal{F}$ whose isomorphism class $[\mathcal{F}] \in \mathfrak{M}$, the vector space $\mathrm{Ext}^1(\mathcal{F}, \mathcal{F})$ is naturally identified with the tangent space $T_{[\mathcal{F}]}\mathfrak{M}$, and $\mathrm{Ext}^2(\mathcal{F}, \mathcal{F})$ is the obstruction space (see [Muk-1]). The Yoneda pairing

$$\mathrm{Ext}^i(\mathcal{F}, \mathcal{F}) \times \mathrm{Ext}^j(\mathcal{F}, \mathcal{F}) \xrightarrow{\text{Yoneda}} \mathrm{Ext}^{i+j}(\mathcal{F}, \mathcal{F}),\ (a, b) \mapsto a \circ b$$

for $i = j = 1$ provides the bilinear map

$$\Lambda : \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \times \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \longrightarrow \mathrm{Ext}^2(\mathcal{F}, \mathcal{F}). \tag{10}$$

The obstruction map $\mathrm{ob}: \mathrm{Ext}^1(\mathcal{F},\mathcal{F}) \longrightarrow \mathrm{Ext}^2(\mathcal{F},\mathcal{F})$ is expressed in terms of the Yoneda pairing by $a \mapsto a \circ a$. It has the following sense: an element $a \in \mathrm{Ext}^1(\mathcal{F},\mathcal{F})$ defines the isomorphism class of a flat deformation $\mathcal{F}^{(1)}$ of $\mathcal{F}$ over $Y \times \mathrm{Spec}\, \mathbb{C}[t]/(t^2)$, and $\mathrm{ob}(a) = 0$ if and only if $\mathcal{F}^{(1)}$ can be extended further to a flat deformation $\mathcal{F}^{(2)}$ over $Y \times \mathrm{Spec}\, \mathbb{C}[t]/(t^3)$. In particular, if $[\mathcal{F}]$ is a smooth point of $\mathfrak{M}$, then all the obstructions vanish: $\mathrm{ob}(a) = 0\ \forall\ a \in \mathrm{Ext}^1(\mathcal{F},\mathcal{F})$. Thus we have the following statement:

Lemma 2.1. *Let $\mathfrak{M}^{\mathrm{sm}}$ denote the smooth locus of $\mathfrak{M}$, and let $[\mathcal{F}]$ be a point of $\mathfrak{M}^{\mathrm{sm}}$. Then the bilinear map Λ defined by* (10) *is skew symmetric: $\Lambda(a,b) = -\Lambda(b,a)$ for all $a,b \in \mathrm{Ext}^1(\mathcal{F},\mathcal{F})$.*

Buchweitz and Flenner [BuF1], [BuF2] define the map

$$\sigma = \sum_{q\geq 0} \sigma_q : \mathrm{Ext}^2(\mathcal{F},\mathcal{F}) \longrightarrow \bigoplus_{q \geq 0} H^{q+2}(X, \Omega_X^q),\ \ \sigma : c \mapsto \mathrm{Tr}\,(\exp(-\mathrm{At}(\mathcal{F})) \circ c), \tag{11}$$

which coincides with Bloch's semiregularity map in the case when $\mathcal{F} = \mathcal{O}_Z$ for a subscheme Z. It is proved in loc. cit. that for any simple coherent sheaf $\mathcal{F}$, the map σ plays the same role for the moduli space of sheaves as Bloch's semiregularity map for subschemes: $\mathfrak{M}$ is smooth in $[\mathcal{F}]$ if σ is injective. This is the reason to call it semiregularity map for sheaves.

Composing Λ with $\sigma : \mathrm{Ext}^2(\mathcal{F},\mathcal{F}) \longrightarrow \bigoplus H^{q+2}(Y, \Omega_Y^q)$ for $\mathcal{F} \in \mathfrak{M}^{\mathrm{sm}}$, we obtain a family of skew-symmetric bilinear forms on $\mathrm{Ext}^1(\mathcal{F},\mathcal{F}) = T_{[\mathcal{F}]}\mathfrak{M}$, each one of which corresponds to some element of the dual space $(\bigoplus H^{q+2}(Y, \Omega_Y^q))^\vee = \bigoplus H^{n-q-2}(Y, \Omega_Y^{n-q})$. These forms fit into exterior 2-forms on $\mathfrak{M}^{\mathrm{sm}}$ and are linear combinations of the forms α_ω with $\omega \in H^{n-q-2}(Y, \Omega_Y^{n-q})$, defined as follows:

$$\alpha_\omega(v_1, v_2) = \mathrm{Tr}(\mathrm{At}^q_{\mathcal{F}} \circ v_1 \circ v_2) \cup \omega \cap [Y], \tag{12}$$

where $v_1, v_2 \in T_{[\mathcal{F}]}\mathfrak{M}$ and $[Y]$ is the fundamental class of Y.

Theorem 2.2. *Let Y be a smooth complex projective variety of dimension n, $\mathfrak{M}$ a moduli space of stable or simple sheaves on Y, $\omega \in H^{n-q-2}(Y, \Omega_Y^{n-q})$. Then formula* (12) *defines a closed 2-form $\alpha_\omega \in H^0(\mathfrak{M}^{\mathrm{sm}}, \Omega^2)$.*

Proof. This statement generalizes the known result for the case when Y is a surface (then σ is just the trace map with values in $H^2(Y, \mathcal{O}_Y)$). Different proofs in the surface case for moduli of vector bundles can be found in [Muk-2], [O'G], [Bot-1]. A proof for moduli of sheaves on a surface was given in [HL]. Our proof, given below, is obtained by a slight modification of the argument from [HL].

It suffices to prove that given a smooth affine variety S, for any S-flat sheaf $\mathcal{F}$ on $S \times Y$ defining a classifying morphism $\tau : S \longrightarrow \mathfrak{M}^{\mathrm{sm}}$, $s \mapsto [\mathcal{F}_s]$, the pullback $\tau^*(\alpha_\omega) \in H^0(S, \Omega_S^2)$ is closed. The latter 2-form is the following map:

$$\begin{aligned} T_sS \times T_sS &\xrightarrow{KS\times KS} \mathrm{Ext}^1(\mathcal{F}_s,\mathcal{F}_s) \times \mathrm{Ext}^1(\mathcal{F}_s,\mathcal{F}_s) \xrightarrow{\text{Yoneda}} \mathrm{Ext}^2(\mathcal{F}_s,\mathcal{F}_s) \\ &\xrightarrow{\mathrm{At}(\mathcal{F}_s)^q} \mathrm{Ext}^{q+2}(\mathcal{F}_s, \mathcal{F}_s \otimes \Omega_Y^q) \xrightarrow{\mathrm{Tr}} H^{q+2}(Y, \Omega_Y^q) \xrightarrow{\cup\omega} H^n(Y, \Omega_Y^n) \simeq \mathbb{C}\ , \end{aligned} \tag{13}$$

where KS stands for the Kodaira–Spencer map.

The Kodaira–Spencer map has the following description in terms of the Atiyah class $\mathrm{At}_{S\times Y}(\mathcal{F})$. Denote by $A(\mathcal{F})$ the image of $\mathrm{At}_{S\times Y}(\mathcal{F})$ in $H^0(S, \mathcal{E}xt^{\,1}_{\mathrm{pr}_1}(\mathcal{F}, \mathcal{F}\otimes\Omega^1_{S\times Y}))$, where pr_i is the projection of $S \times Y$ to the i-th factor ($i = 1, 2$), and $\mathcal{E}xt^{\,i}_{\mathrm{pr}_1}$ stands for the i-th derived functor of $\mathrm{pr}_{1*} \circ \mathcal{H}om : \mathcal{D}^b(\mathrm{Coh}(S \times Y))^\circ \times \mathcal{D}^b(\mathrm{Coh}(S \times Y)) \to \mathcal{D}^b(\mathrm{Coh}(S))$. Note that we have a relative analog of the local-to-global spectral sequence $H^p(S, \mathcal{E}xt^{\,q}_{\mathrm{pr}_1}(\mathcal{F},\mathcal{G})) \Longrightarrow \mathrm{Ext}^{p+q}(\mathcal{F},\mathcal{G})$. Since S is affine this spectral sequence degenerates in the second term, so that we have an isomorphism $\mathrm{Ext}^i(\mathcal{F},\mathcal{G}) \cong H^0(S, \mathcal{E}xt^{\,i}_{\mathrm{pr}_1}(\mathcal{F},\mathcal{G}))$.

Write $A(\mathcal{F}) = A'(\mathcal{F}) + A''(\mathcal{F})$ according to the direct sum decomposition $\Omega^1_{S\times Y} = \mathrm{pr}_1^*\Omega^1_S \oplus \mathrm{pr}_2^*\Omega^1_Y$. Then KS is the composition

$$\mathcal{T}S \xrightarrow{1\otimes A'(\mathcal{F})} \mathcal{T}S \otimes \mathcal{E}xt^{\,1}_{\mathrm{pr}_1}(\mathcal{F}, \mathcal{F}\otimes \mathrm{pr}_1^*\Omega^1_S) \longrightarrow \\ \longrightarrow \mathcal{E}xt^{\,1}_{\mathrm{pr}_1}(\mathcal{F}, \mathcal{F}\otimes \mathrm{pr}_1^*(\Omega^{1\vee}_S \otimes \Omega^1_S)) \longrightarrow \mathcal{E}xt^{\,1}_{\mathrm{pr}_1}(\mathcal{F},\mathcal{F}). \quad (14)$$

Combining (13) and (14), we see that

$$\tau^*(\alpha_\omega) = (\omega\cup\gamma)\cap[Y], \quad \gamma = \mathrm{Tr}(A''(\mathcal{F})^q \circ A'(\mathcal{F})^2) \in H^0(S,\Omega^2_S)\otimes H^{q+2}(Y,\Omega^q_Y).$$

Consider also the class $\tilde{\gamma} = \mathrm{Tr}(\mathrm{At}_{S\times Y}(\mathcal{F})^{q+2}) \in H^{q+2}(S\times Y, \Omega^{q+2}_{S\times Y})$. According to [HL], Sect. 10.1.6, it is $d_{S\times Y}$-closed, where $d_{S\times Y} = d_S\otimes 1 + 1\otimes d_Y$ is the natural differential on $H^p(S\times Y, \Omega^q_{S\times Y})$ induced by the De Rham differential on $\Omega^\bullet_{S\times Y}$. Hence

$$d_{S\times Y}(\omega\cup\tilde{\gamma}) = d_{S\times Y}(\omega)\cup\tilde{\gamma} + (-1)^{n-q}\omega\cup d_{S\times Y}(\tilde{\gamma}) = 0.$$

Recall that the groups $H^p(S\times Y,\Omega^q_{S\times Y})$ have a Künneth decomposition

$$H^p(S\times Y,\Omega^q_{S\times Y}) = \bigoplus_{i,j} H^i(S,\Omega^j_S)\otimes H^{p-i}(Y,\Omega^{q-j}_Y).$$

Since S is affine we have $H^i(S,\Omega^j_S) = 0$ for $i>0$, therefore $\omega\cup\tilde{\gamma} \in H^n(S\times Y,\Omega^{n+2}_{S\times Y})$ is the sum of the Künneth components $f_j \in H^0(S,\Omega^j_S)\otimes H^n(Y,\Omega^{n+2-j}_Y)$, $j\geq 2$, and we have $f_2 = \binom{q+2}{2}\omega\cup\gamma \in H^0(S,\Omega^2_S)\otimes H^n(Y,\Omega^n_Y)$. As Y is projective, d_Y vanishes on $H^{p-i}(Y,\Omega^{q-j}_Y)$, hence the closedness of $\omega\cup\tilde{\gamma}$ implies that of f_j for any j. In particular, $\omega\cup\gamma$ is closed. Let us represent $\omega\cup\gamma$ in the form $\tau^*(\alpha_\omega)\otimes\eta$ with $\tau^*(\alpha_\omega)\in H^0(S,\Omega^2_S)$, where η is a generator of $H^n(Y,\Omega^n_Y)$, dual to $[Y]$. Then

$$0 = d_{S\times Y}(\omega\cup\gamma) = d_{S\times Y}(\tau^*(\alpha_\omega)\otimes\eta) = d_S(\tau^*(\alpha_\omega))\otimes\eta,$$

which implies that $d_S(\tau^*(\alpha_\omega)) = 0$. □

Thus we have constructed closed 2-forms α_ω on $\mathfrak{M}^{\mathrm{sm}}$. In general, these forms may be degenerate, but, as we will see, they are symplectic in some examples.

3. The linkage class

Let M be an algebraic variety and $Y\subset M$, a locally complete intersection subvariety of codimension m. Denote by $i: Y\to M$ the embedding. Let $\mathcal{F}$ be a coherent sheaf on Y, then $i_*\mathcal{F}$ is a coherent sheaf on M supported on Y. As we have shown in Lemma 1.3.1, the derived pullback $Li^*(i_*\mathcal{F})$ considered as an object of the derived category $\mathcal{D}^b(\mathrm{Coh}(Y))$ is a complex with $(m+1)$ nontrivial cohomology, $\mathcal{F}$ at degree 0, $\mathcal{F}\otimes\mathcal{N}^\vee_{Y/M}$ at degree -1, and so on. Consider the canonical filtration of this object. Its associated graded factors are the shifted cohomology sheaves, explicitly $\mathcal{F}\otimes\wedge^k\mathcal{N}^\vee_{Y/M}[k]$. So the object $Li^*i_*\mathcal{F}$ provides us with extension classes $\epsilon^k_{\mathcal{F}} \in \mathrm{Ext}^1(\mathcal{F}\otimes\wedge^k\mathcal{N}^\vee_{Y/M}[k], \mathcal{F}\otimes\wedge^{k+1}\mathcal{N}^\vee_{Y/M}[k+1]) \cong \mathrm{Ext}^2(\mathcal{F}\otimes\wedge^k\mathcal{N}^\vee_{Y/M}, \mathcal{F}\otimes\wedge^{k+1}\mathcal{N}^\vee_{Y/M})$. The most important of them, $\epsilon_{\mathcal{F}} := \epsilon^0_{\mathcal{F}} \in \mathrm{Ext}^2(\mathcal{F},\mathcal{F}\otimes\mathcal{N}^\vee_{Y/M})$ will be called *the linkage class* of $\mathcal{F}$.

If $Y\subset M$ is a divisor, the linkage class $\epsilon_{\mathcal{F}} \in \mathrm{Ext}^2(\mathcal{F},\mathcal{F}\otimes\mathcal{N}^\vee_{Y/M}) \cong \mathrm{Ext}^2(\mathcal{F},\mathcal{F}\otimes\mathcal{O}_Y(-Y))$ completely determines the derived pullback $Li^*i_*\mathcal{F}$, namely there is a distinguished triangle

$$Li^*i_*\mathcal{F} \longrightarrow \mathcal{F} \xrightarrow{\epsilon_{\mathcal{F}}} \mathcal{F}\otimes\mathcal{O}_Y(-Y)[2]. \quad (15)$$

In other words, $Li^*i_*\mathcal{F}$, up to a shift, is a cone of $\epsilon_{\mathcal{F}}$.

Proposition 3.1. *The linkage class $\epsilon_{\mathcal{F}} \in \mathrm{Ext}^2(\mathcal{F}, \mathcal{F} \otimes \mathcal{N}^\vee_{Y/M})$ is defined for any object $\mathcal{F}$ of the derived category $\mathcal{D}^b(\mathrm{Coh}(Y))$ and is functorial, i. e. for any morphism $\varphi : \mathcal{F} \to \mathcal{G}$ in $\mathcal{D}^b(\mathrm{Coh}(Y))$ we have a commutative diagram*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\epsilon_{\mathcal{F}}} & \mathcal{F} \otimes \mathcal{N}^\vee_{Y/M}[2] \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi\otimes 1} \\ \mathcal{G} & \xrightarrow{\epsilon_{\mathcal{G}}} & \mathcal{G} \otimes \mathcal{N}^\vee_{Y/M}[2] \end{array}$$

Proof. Let $\Delta : Y \to Y \times Y$ be the diagonal embedding, and denote $\tilde{\imath} = (1 \times i) : Y \times Y \to Y \times M$. Then by Lemma 1.3.1, for any coherent sheaf F on $Y \times Y$ the cohomology sheaves of the derived pullback $L\tilde{\imath}^*\tilde{\imath}_*F$ are isomorphic to $F \otimes \mathrm{pr}_2^* \wedge^k \mathcal{N}^\vee_{Y/M}$. Therefore, we have an extension class $\tilde{\epsilon}_F \in \mathrm{Ext}^2(F, F \otimes \mathrm{pr}_2^* \mathcal{N}^\vee_{Y/M})$. Take $F = \Delta_*\mathcal{O}_Y$ and consider $\tilde{\epsilon} := \tilde{\epsilon}_{\Delta_*\mathcal{O}_Y}$ as a morphism $\Delta_*\mathcal{O}_Y \to \Delta_*\mathcal{O}_Y \otimes \mathrm{pr}_2^*\mathcal{N}^\vee_{Y/M}[2] \cong \Delta_*\mathcal{N}^\vee_{Y/M}[2]$. Now for any object $\mathcal{F} \in \mathcal{D}^b(\mathrm{Coh}(Y))$ the evaluation of $\tilde{\epsilon}_{\Delta_*\mathcal{O}_Y}$ gives a morphism

$$\mathcal{F} \cong \mathrm{pr}_{2*}(\mathrm{pr}_1^* \mathcal{F} \otimes \Delta_*\mathcal{O}_Y) \xrightarrow{\tilde{\epsilon}} \mathrm{pr}_{2*}(\mathrm{pr}_1^* \mathcal{F} \otimes \Delta_*\mathcal{N}^\vee_{Y/M}[2]) \cong \mathcal{F} \otimes \mathcal{N}^\vee_{Y/M}[2]. \qquad (16)$$

It is clear that if $\mathcal{F}$ is a coherent sheaf, then this morphism considered as an element of $\mathrm{Ext}^2(\mathcal{F}, \mathcal{F} \otimes \mathcal{N}^\vee_{Y/M})$ coincides with the linkage class $\epsilon_{\mathcal{F}}$ defined above, so (16) can be considered as an extension of the definition of the linkage class to the whole derived category. Moreover, (16) also shows that $\epsilon_{\mathcal{F}}$ is the evaluation of the universal class $\tilde{\epsilon}$ and hence is functorial. □

Actually, the linkage class can be expressed in terms of the Atiyah class. Consider the adjunction exact sequence

$$0 \longrightarrow \mathcal{N}^\vee_{Y/M} \xrightarrow{\kappa} \Omega_{M|Y} \xrightarrow{\rho} \Omega_Y \longrightarrow 0 \qquad (17)$$

and denote by $\kappa = \kappa_{Y/M} : \mathcal{N}^\vee_{Y/M} \to \Omega_{M|Y}$, $\rho = \rho_{Y/M} : \Omega_{M|Y} \to \Omega_Y$ the maps in (17), and by $\nu = \nu_{Y/M} \in \mathrm{Ext}^1(\Omega_Y, \mathcal{N}^\vee_{Y/M})$ the extension class of (17).

Theorem 3.2. *Let $i : Y \to M$ be a locally complete intersection.*

(i) For any $\mathcal{F} \in \mathcal{D}^b(\mathrm{Coh}(Y))$, the linkage class $\epsilon_{\mathcal{F}} \in \mathrm{Ext}^2(\mathcal{F}, \mathcal{F} \otimes \mathcal{N}^\vee_{Y/M})$ is the product of the Atiyah class $\mathrm{At}_{\mathcal{F}} \in \mathrm{Ext}^1(\mathcal{F}, \mathcal{F} \otimes \Omega_Y)$ with $\nu_{Y/M}$. In other words $\epsilon_{\mathcal{F}} = (1_{\mathcal{F}} \otimes \nu_{Y/M}) \circ \mathrm{At}_{\mathcal{F}}$.

(ii) For any $\mathcal{G} \in D^b(\mathrm{Coh}(M))$ we have

$$\mathrm{At}_{Li^*\mathcal{G}} \cong \rho_*((\mathrm{At}_{\mathcal{G}})_{|Y}),$$

where $\rho_* : \mathrm{Ext}^1(Li^*\mathcal{G}, Li^*\mathcal{G} \otimes \Omega_{M|Y}) \to \mathrm{Ext}^1(Li^*\mathcal{G}, Li^*\mathcal{G} \otimes \Omega_Y)$ is the pushout via $\rho : \Omega_{M|Y} \to \Omega_Y$.

(iii) For any $\mathcal{F} \in D^b(\mathrm{Coh}(Y))$ the image of the Atiyah class $\mathrm{At}_{i_*\mathcal{F}} \in \mathrm{Ext}^1(i_*\mathcal{F}, i_*\mathcal{F} \otimes \Omega_M)$ in $H^0(M, \mathcal{E}xt^1(i_*\mathcal{F}, i_*\mathcal{F} \otimes \Omega_M)) = H^0(M, i_*(\mathcal{F}^\vee \otimes \mathcal{F} \otimes \mathcal{N}_{Y/M} \otimes \Omega_{M|Y})) = \mathrm{Hom}(\mathcal{F} \otimes \mathcal{N}^\vee_{Y/M}, \mathcal{F} \otimes \Omega_{M|Y})$ equals $1_{\mathcal{F}} \otimes \kappa$.

Proof. Let $\Delta_Y : Y \to Y \times Y$ and $\Delta_M : M \to M \times M$ denote the diagonal embeddings, and let $\Gamma : Y \to M \times Y$ be the graph of $i : Y \to M$. Then we have a commutative diagram

$$\begin{array}{ccccc} Y & = & Y & \xrightarrow{i} & M \\ \downarrow{\scriptstyle\Delta_Y} & & \downarrow{\scriptstyle\Gamma} & & \downarrow{\scriptstyle\Delta_M} \\ Y \times Y & \xrightarrow{i\times 1} & M \times Y & \xrightarrow{1\times i} & M \times M \end{array}$$

Note that both squares of the diagram are cartesian. Let $\Delta_M(M)^{(2)}$, $\Gamma(Y)^{(2)}$ and $\Delta_Y(Y)^{(2)}$ denote the second infinitesimal neighborhoods of $\Delta_M(M)$ in $M \times M$, $\Gamma(Y)$ in $M \times Y$ and $\Delta_Y(Y)$ in $Y \times Y$ respectively. On $M \times M$ we have the short exact sequence

$$0 \to \Delta_{M*}\Omega_M \to \mathcal{O}_{\Delta_M(M)^{(2)}} \to \Delta_{M*}\mathcal{O}_M \to 0$$

representing the universal Atiyah class on M. Consider its pullback to $M \times Y$. Since $M \times Y$ intersects $\Delta_M(M)$ transversely, the higher derived inverse images of $1 \times i$ are zero, and we have an isomorphism $(1 \times i)^*\Delta_{M*}\mathcal{G} \cong \Gamma_* i^*\mathcal{G}$ for any coherent sheaf $\mathcal{G}$ on M. Moreover, it is clear that $(1 \times i)^*(\mathcal{O}_{\Delta_M(M)^{(2)}}) \cong \mathcal{O}_{\Gamma(Y)^{(2)}}$, hence we obtain the exact sequence

$$0 \to \Gamma_*\Omega_{M|Y} \to \mathcal{O}_{\Gamma(Y)^{(2)}} \to \Gamma_*\mathcal{O}_Y \to 0. \tag{18}$$

By definition, this sequence represents the restriction to Y of the universal Atiyah class of M. On the other hand, it is clear that the map $\mathcal{O}_{\Gamma(Y)^{(2)}} \to \Gamma_*\mathcal{O}_Y$ factors through $(i \times 1)_*\mathcal{O}_{\Delta_Y(Y)^{(2)}}$, hence we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma_*\Omega_{M|Y} & \longrightarrow & \mathcal{O}_{\Gamma(Y)^{(2)}} & \longrightarrow & \Gamma_*\mathcal{O}_Y & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\rho} & & \downarrow & & \| & & \\
0 & \longrightarrow & (i \times 1)_*\Delta_*\Omega_Y & \longrightarrow & (i \times 1)_*\mathcal{O}_{\Delta_Y(Y)^{(2)}} & \longrightarrow & (i \times 1)_*\Delta_*\mathcal{O}_Y & \longrightarrow & 0
\end{array}$$

Since the bottom line represents the universal Atiyah class of Y, we deduce part (ii).

Now consider the pullback of (18) to $Y \times Y$. This time the intersection of $Y \times Y$ with $\Gamma(Y)$ is not transversal. Actually, we have $\Gamma_*\mathcal{F} \cong (1 \times i)_*\Delta_{Y_*}\mathcal{F}$ for any coherent sheaf $\mathcal{F}$ on Y, hence by Lemma 1.3.1 we have $(1 \times i)^*\Gamma_*\mathcal{F} \cong \Delta_{Y*}\mathcal{F}$, $L_1(1 \times i)^*\Gamma_*\mathcal{F} \cong \Delta_{Y*}(\mathcal{F} \otimes \mathcal{N}^\vee_{Y/M})$, and so on. Moreover, it is clear that $(i \times 1)^*(\mathcal{O}_{\Gamma(Y)^{(2)}}) \cong \mathcal{O}_{\Delta_Y(Y)^{(2)}}$, hence we obtain the following long exact sequence

$$\cdots \to L_1(1 \times i)^*\mathcal{O}_{\Gamma(Y)^{(2)}} \to \Delta_{Y*}\mathcal{N}^\vee_{Y/M} \to \Delta_{Y*}\Omega_{M|Y} \to \mathcal{O}_{\Delta_Y(Y)^{(2)}} \to \Delta_{Y*}\mathcal{O}_Y \to 0. \tag{19}$$

The map $\Delta_{Y*}\mathcal{N}^\vee_{Y/M} \to \Delta_{Y*}\Omega_{M|Y}$ in this sequence is the image of the universal Atiyah class of M, restricted to Y, under the natural map

$$\begin{aligned}
&\mathrm{Ext}^1(\Gamma_*\mathcal{O}_Y, \Gamma_*\Omega_{M|Y}) \cong \mathrm{Ext}^1((i \times 1)_*\Delta_*\mathcal{O}_Y, (i \times 1)_*\Delta_*\Omega_{M|Y}) \cong \\
&\quad \cong \mathrm{Ext}^1(L(i \times 1)^*(i \times 1)_*\Delta_*\mathcal{O}_Y, \Delta_*\Omega_{M|Y}) \to \mathrm{Hom}(L_1(i \times 1)^*(i \times 1)_*\Delta_*\mathcal{O}_Y, \Delta_*\Omega_{M|Y}) \to \\
&\qquad \to \mathrm{Hom}((\mathcal{N}^\vee_{Y/M} \boxtimes \mathcal{O}_Y) \otimes \Delta_*\mathcal{O}_Y, \Delta_*\Omega_{M|Y}) \cong \mathrm{Hom}(\Delta_*\mathcal{N}^\vee_{Y/M}, \Delta_*\Omega_{M|Y}),
\end{aligned}$$

so for part (iii) it suffices to check that this map is κ. Comparing (19) with the sequence

$$0 \to \Delta_{Y*}\Omega_Y \to \mathcal{O}_{\Delta_Y(Y)^{(2)}} \to \Delta_{Y*}\mathcal{O}_Y \to 0,$$

we see that the map $\Delta_{Y*}\Omega_{M|Y} \to \mathcal{O}_{\Delta_Y(Y)^{(2)}}$ in (19) factors through $\Delta_{Y*}\Omega_Y$. It is clear that the arising map $\Omega_{M|Y} \to \Omega_Y$ is the restriction of differential forms, hence its kernel is isomorphic to $\mathcal{N}^\vee_{Y/M}$. Thus we see that the map $L_1(1 \times i)^*\mathcal{O}_{\Gamma(Y)^{(2)}} \to \Delta_{Y*}\mathcal{N}^\vee_{Y/M}$ in (19) must be zero and the last 4 terms of (19) form an exact sequence

$$0 \longrightarrow \Delta_{Y*}\mathcal{N}^\vee_{Y/M} \xrightarrow{\kappa} \Delta_{Y*}\Omega_{M|Y} \longrightarrow \mathcal{O}_{\Delta_Y(Y)^{(2)}} \longrightarrow \Delta_{Y*}\mathcal{O}_Y \longrightarrow 0. \tag{20}$$

So, part (iii) follows. Finally, the Yoneda class of the extension (20) by definition equals $\tilde{\epsilon} \in \mathrm{Ext}^2(\Delta_{Y*}\mathcal{O}_Y, \Delta_{Y_*}\mathcal{N}^\vee_{Y/M})$. On the other hand, as we have seen above this class factors as $\tilde{\epsilon} = \Delta_{Y*}(\nu_{Y/M}) \circ \widetilde{\mathrm{At}}$, where $\widetilde{\mathrm{At}} \in \mathrm{Ext}^1(\Delta_{Y*}\mathcal{O}_Y, \Delta_{Y*}\Omega_Y)$ is the universal Atiyah class. Evaluating this equality on any $\mathcal{F} \in \mathcal{D}^b(\mathrm{Coh}(Y))$ we deduce part (i) of the theorem. $\square$

4. Application to cubic fourfold

From now on we take $M = \mathbb{P}^5$ and $Y \subset \mathbb{P}^5$, a smooth cubic fourfold. Let $i : Y \to \mathbb{P}^5$ denote the embedding. Note that $\Omega_Y^4 \cong \mathcal{O}_Y(-3) \cong \mathcal{N}^\vee_{Y/\mathbb{P}^5}$. This coincidence will be important below. Consider the full triangulated subcategory $\mathcal{C}_Y \subset \mathcal{D}^b(\mathrm{Coh}(Y))$ defined by

$$\begin{aligned} \mathcal{C}_Y &= \{\mathcal{F} \in \mathcal{D}^b(\mathrm{Coh}(Y)) \mid \mathrm{Ext}^\bullet(\mathcal{O}_Y, \mathcal{F}) = \mathrm{Ext}^\bullet(\mathcal{O}_Y(1), \mathcal{F}) = \mathrm{Ext}^\bullet(\mathcal{O}_Y(2), \mathcal{F}) = 0\} \\ &= \{\mathcal{F} \in \mathcal{D}^b(\mathrm{Coh}(Y)) \mid H^\bullet(Y, \mathcal{F}) = H^\bullet(Y, \mathcal{F}(-1)) = H^\bullet(Y, \mathcal{F}(-2)) = 0\}. \end{aligned} \tag{21}$$

Proposition 4.1. *Assume that $\mathcal{F}, \mathcal{G} \in \mathcal{C}_Y$. Then the multiplication by the linkage class $\epsilon_\mathcal{G} \in \mathrm{Ext}^2(\mathcal{G}, \mathcal{G}(-3))$ induces an isomorphism $\mathrm{Ext}^p(\mathcal{F}, \mathcal{G}) \cong \mathrm{Ext}^{p+2}(\mathcal{F}, \mathcal{G}(-3))$ for all p.*

Proof. Consider the Beilinson spectral sequence for $i_*\mathcal{G}$ (see [Bei, OSS])

$$E_1^{-p,q} = H^q(\mathbb{P}^5, i_*\mathcal{G}(-p)) \otimes \Omega^p_{\mathbb{P}^5}(p) \Longrightarrow i_*\mathcal{G} \qquad (p = 0, 1, \ldots, 5).$$

Note that $H^\bullet(\mathbb{P}^5, i_*\mathcal{G}(-p)) = H^\bullet(Y, \mathcal{G}(-p))$, hence $E_1^{0,q} = E_1^{-1,q} = E_1^{-2,q} = 0$ for all q since $\mathcal{G} \in \mathcal{C}_Y$. It follows that the derived pullback $Li^*i_*\mathcal{G}$ is contained in the triangulated subcategory of $\mathcal{D}^b(\mathrm{Coh}(Y))$ generated by $i^*\Omega^3_{\mathbb{P}^5}(3)$, $i^*\Omega^4_{\mathbb{P}^5}(4)$, and $i^*\Omega^5_{\mathbb{P}^5}(5)$. On the other hand, the standard resolutions

$$0 \to \mathcal{O}_{\mathbb{P}^5}(-3) \to \mathcal{O}_{\mathbb{P}^5}(-2)^{\oplus 6} \to \mathcal{O}_{\mathbb{P}^5}(-1)^{\oplus 15} \to \Omega^3_{\mathbb{P}^5}(3) \to 0,$$
$$0 \to \mathcal{O}_{\mathbb{P}^5}(-2) \to \mathcal{O}_{\mathbb{P}^5}(-1)^{\oplus 6} \to \Omega^4_{\mathbb{P}^5}(4) \to 0,$$

and an isomorphism $\mathcal{O}_{\mathbb{P}^5}(-1) \cong \Omega^5_{\mathbb{P}^5}(5)$ show that this subcategory coincides with the subcategory of $\mathcal{D}^b(\mathrm{Coh}(Y))$ generated by $\mathcal{O}_Y(-3)$, $\mathcal{O}_Y(-2)$ and $\mathcal{O}_Y(-1)$. But note that the Serre duality gives

$$\begin{aligned} \mathrm{Ext}^p(\mathcal{F}, \mathcal{O}_Y(-3)) &\cong \mathrm{Ext}^{4-p}(\mathcal{O}_Y, \mathcal{F})^\vee = 0, \\ \mathrm{Ext}^p(\mathcal{F}, \mathcal{O}_Y(-2)) &\cong \mathrm{Ext}^{4-p}(\mathcal{O}_Y(1), \mathcal{F})^\vee = 0, \\ \mathrm{Ext}^p(\mathcal{F}, \mathcal{O}_Y(-1)) &\cong \mathrm{Ext}^{4-p}(\mathcal{O}_Y(2), \mathcal{F})^\vee = 0, \end{aligned}$$

which implies that $\mathrm{Ext}^\bullet(\mathcal{F}, Li^*i_*\mathcal{G}) = 0$. Applying the functor $\mathrm{Hom}(\mathcal{F}, -)$ to the distinguished triangle (15) for $\mathcal{G}$, we deduce the proposition. □

Remark 4.2. Combining the isomorphism $\mathrm{Ext}^p(\mathcal{F}, \mathcal{G}) \cong \mathrm{Ext}^{p+2}(\mathcal{F}, \mathcal{G}(-3))$ of the proposition with the Serre duality $\mathrm{Ext}^{p+2}(\mathcal{F}, \mathcal{G}(-3)) \cong \mathrm{Ext}^{2-p}(\mathcal{G}, \mathcal{F})^\vee$, we obtain a duality

$$\mathrm{Ext}^p(\mathcal{F}, \mathcal{G}) \cong \mathrm{Ext}^{2-p}(\mathcal{G}, \mathcal{F})^\vee \qquad \text{for any} \quad \mathcal{F}, \mathcal{G} \in \mathcal{C}_Y. \tag{22}$$

This duality, in fact, is the Serre duality for the triangulated category $\mathcal{C}_Y$. In other words, the Serre functor (see [BK]) of $\mathcal{C}_Y$ equals to the shift by 2 functor. This fact was proved earlier in [Ku1] by the same argument.

Using the isomorphism $\mathrm{Ext}^1(\Omega_Y, \mathcal{N}^\vee_{Y/\mathbb{P}^5}) \cong H^1(Y, \mathcal{T}_Y \otimes \mathcal{O}_Y(-3)) \cong H^1(Y, \Omega^3_Y)$, we consider the extension class $\nu_{Y/\mathbb{P}^5} \in \mathrm{Ext}^1(\Omega_Y, \mathcal{N}^\vee_{Y/\mathbb{P}^5})$ of the adjunction sequence (17) as an element of $H^1(Y, \Omega^3_Y)$.

Theorem 4.3. *Let $\mathfrak{M}$ be a moduli space of stable sheaves on a cubic 4-fold Y such that for every sheaf $\mathcal{F}$ with $[\mathcal{F}] \in \mathfrak{M}$ we have $H^\bullet(Y, \mathcal{F}) = H^\bullet(Y, \mathcal{F}(-1)) = H^\bullet(Y, \mathcal{F}(-2)) = 0$. Then the closed 2-form $\alpha_\nu \in H^0(\mathfrak{M}^{\mathrm{sm}}, \Omega^2)$ corresponding to the class $\nu = \nu_{Y/\mathbb{P}^5} \in H^1(Y, \Omega^3_Y)$ is nondegenerate.*

Proof. Let $[\mathcal{F}] \in \mathfrak{M}^{\mathrm{sm}}$. Recall that for any $a, b \in \mathrm{Ext}^1(\mathcal{F}, \mathcal{F})$ the form $\alpha_\nu(a, b)$ is defined as $\nu \circ \mathrm{Tr}(\mathrm{At}_\mathcal{F} \circ a \circ b)$. By multiplicativity of the trace, this is equal to $\mathrm{Tr}((1_\mathcal{F} \otimes \nu) \circ \mathrm{At}_\mathcal{F} \circ a \circ b)$. In other words, we apply the Yoneda multiplication map

$$\mathrm{Ext}^1(\mathcal{F} \otimes \Omega_Y, \mathcal{F} \otimes \mathcal{N}^*_{Y/\mathbb{P}^5}) \otimes \mathrm{Ext}^1(\mathcal{F}, \mathcal{F} \otimes \Omega_Y) \otimes \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \otimes \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \to \mathrm{Ext}^4(\mathcal{F}, \mathcal{F} \otimes \mathcal{N}^*_{Y/\mathbb{P}^5})$$

to $(1_{\mathcal{F}} \otimes \nu) \otimes \mathrm{At}_{\mathcal{F}} \otimes a \otimes b$ and then the trace map

$$\mathrm{Tr} : \mathrm{Ext}^4(\mathcal{F}, \mathcal{F} \otimes \mathcal{N}^{\vee}_{Y/\mathbb{P}^5}) \cong \mathrm{Ext}^4(\mathcal{F}, \mathcal{F}(-3)) \to H^4(Y, \mathcal{O}_Y(-3)) = \mathbb{C}.$$

Since the Yoneda multiplication is associative we have

$$\mathrm{Tr}((1_{\mathcal{F}} \otimes \nu) \circ \mathrm{At}_{\mathcal{F}} \circ a \circ b) = \mathrm{Tr}((\epsilon_{\mathcal{F}} \circ a) \circ b)$$

by Theorem 3.2 (i). It remains to note that the map $\mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \to \mathrm{Ext}^3(\mathcal{F}, \mathcal{F}(-3))$, $a \mapsto \epsilon_{\mathcal{F}} \circ a$ is an isomorphism by Proposition 4.1, and that the Serre duality pairing $\mathrm{Ext}^3(\mathcal{F}, \mathcal{F}(-3)) \otimes \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \xrightarrow{\circ} \mathrm{Ext}^4(\mathcal{F}, \mathcal{F}(-3)) \xrightarrow{\mathrm{Tr}} H^4(Y, \mathcal{O}_Y(-3)) = \mathbb{C}$ is nondegenerate. □

There are two well-known examples of symplectic moduli spaces of sheaves on a cubic fourfold Y. The first one [BD] is the Hilbert scheme of lines on Y, which we will denote by $F(Y)$. It was shown in [BD] that $F(Y)$ is an irreducible symplectic variety of dimension 4. The second one [MT2] is (an open subset of) the moduli space of torsion sheaves of the form $i_*\mathcal{E}$, where $\mathcal{E}$ is a vector bundle of rank 2 with $c_1 = 0$ and $c_2 = 2[\ell]$ on a smooth hyperplane section Y' of Y, and $i : Y' \to Y$ is the embedding. This variety $P(Y)$ is 10-dimensional and the map $P(Y) \to \check{\mathbb{P}}^5$ taking a sheaf $i_*\mathcal{E}$ to its support hyperplane section $Y' \subset Y$, considered as a point of the dual projective space $\check{\mathbb{P}}^5$ was shown in [MT2] to be a Lagrangian fibration.

We will show that our results provide constructions of symplectic forms on both these moduli spaces. The 10-dimensional moduli space $P(Y)$ can be dealt with in a straightforward way. We will explain in Section 7 that all sheaves $i_*\mathcal{E}$ belonging to this moduli space are contained in the subcategory $\mathcal{C}_Y$ of $\mathcal{D}^b(\mathrm{Coh}(Y))$, hence Theorem 4.3 applies and gives a symplectic form on $P(Y)$. The details of the construction can be found in Section 7.

The case of the variety $F(Y)$ of lines on Y is slightly more complicated. For technical reasons it is more convenient to consider $F(Y)$ as the moduli space of twisted ideal sheaves $\mathcal{I}_\ell(1)$, where $\ell \subset Y$ is a line. Certainly, the sheaves $\mathcal{I}_\ell(1)$ are not contained in $\mathcal{C}_Y$ (nor any other twist of $\mathcal{O}_\ell$ or of $\mathcal{I}_\ell$). However, it turns out that a simple endofunctor $\mathbb{L} : \mathcal{D}^b(\mathrm{Coh}(Y)) \to \mathcal{D}^b(\mathrm{Coh}(Y))$ (the left mutation in $\mathcal{O}_Y$) takes $\mathcal{I}_\ell(1)$ to the subcategory $\mathcal{C}_Y$ for every line ℓ. Explicitly, $\mathbb{L}(\mathcal{I}_\ell(1))$ is just the "second syzygy sheaf" of ℓ twisted by 1, that is the kernel of the natural map $\mathcal{O}_Y^{\oplus 4} = H^0(Y, \mathcal{I}_\ell(1)) \otimes \mathcal{O}_Y \to \mathcal{I}_\ell(1)$. In other words, $\mathcal{F}_\ell := \mathbb{L}(\mathcal{I}_\ell(1))$ is the reflexive sheaf on Y defined by the following exact sequence

$$0 \to \mathcal{F}_\ell \to \mathcal{O}_Y^{\oplus 4} \to \mathcal{O}_Y(1) \to \mathcal{O}_\ell(1) \to 0. \tag{23}$$

It is easy to see that for each ℓ the sheaf $\mathcal{F}_\ell$ is stable and is contained in the subcategory $\mathcal{C}_Y$ of $\mathcal{D}^b(\mathrm{Coh}(Y))$. Therefore, Theorem 4.3 gives a symplectic form on the module space $F'(Y)$ of stable sheaves containing sheaves $\mathcal{F}_\ell$. We will show in Section 5 that the map $\mathbb{L} : F(Y) \to F'(Y)$, $[\ell] \mapsto [\mathcal{F}_\ell]$ is an open embedding (so $F(Y)$, being projective, is identified with a connected component of $F'(Y)$) hence the symplectic form on $F'(Y)$ restricts to a symplectic form on $F(Y)$. Moreover, we will show that this form coincides with the form α_ν defined in Section 2.

Remark 4.4. Another example of a symplectic variety associated to a cubic fourfold Y was constructed recently by Iliev and Manivel [IMan]. Unfortunately, we do not know whether it is possible to realize it as a moduli space of sheaves on Y. It would be interesting to find such a realization. Then Theorem 4.3 would give a construction of a symplectic form on this moduli space.

5. The variety of lines

Let $Y \subset \mathbb{P}^5$ be a smooth cubic 4-fold. Let $F(Y)$ be the Hilbert scheme of lines on Y. We consider $F(Y)$ as the moduli space of sheaves $\mathcal{I}_\ell(1)$ where $\mathcal{I}_\ell \subset \mathcal{O}_Y$ is the ideal of a line ℓ.

Consider the functor $\mathbb{L} : \mathcal{D}^b(\mathrm{Coh}(Y)) \to \mathcal{D}^b(\mathrm{Coh}(Y))$ defined as follows

$$\mathbb{L}(F) = \mathsf{Cone}\{\, H^\bullet(Y, F) \otimes \mathcal{O}_Y \xrightarrow{\mathsf{ev}} F \,\},$$

Here ev stands for the evaluation homomorphism and Cone stands for the cone of a morphism in the derived category. The functor $\mathbb{L}$ actually is the left mutation through the exceptional line bundle $\mathcal{O}_Y$, see [Bon, GR].

Lemma 5.1. *Let $\ell \subset Y$ be a line. Then $\mathbb{L}(\mathcal{I}_\ell(1))[-1]$ is isomorphic in $\mathcal{D}^b(\mathrm{Coh}(Y))$ to a reflexive sheaf $\mathcal{F}_\ell$ of rank 3 on Y, which fits into the exact sequence (23). Moreover, $\mathcal{F}_\ell \in \mathcal{C}_Y$.*

Proof. We have $H^\bullet(Y, \mathcal{I}_\ell(1)) = \mathbb{C}^4$, hence $\mathbb{L}(\mathcal{I}_\ell(1)) = \mathsf{Cone}\{\, \mathcal{O}_Y^{\oplus 4} \xrightarrow{\mathsf{ev}} \mathcal{I}_\ell(1) \,\}$. Note that the space of global sections of $\mathcal{O}_Y^{\oplus 4}$ here is spanned by linear functions on $\mathbb{P}^5$ vanishing on ℓ, and the map $\mathcal{O}_Y^{\oplus 4} \to \mathcal{I}_\ell(1)$ is induced by considering these functions as sections of $\mathcal{I}_\ell(1)$. Since the sheaf of ideals of a line is generated by these linear functions, the evaluation homomorphism is surjective, hence $\mathbb{L}(\mathcal{I}_\ell(1)) = \mathcal{F}_\ell[1]$, where $\mathcal{F}_\ell = \mathsf{Ker}\{\, \mathcal{O}_Y^{\oplus 4} \xrightarrow{\mathsf{ev}} \mathcal{I}_\ell(1) \,\}$. Combining the sequence

$$0 \to \mathcal{F}_\ell \to \mathcal{O}_Y^{\oplus 4} \to \mathcal{I}_\ell(1) \to 0 \tag{24}$$

with the sequence $0 \to \mathcal{I}_\ell \to \mathcal{O}_Y \to \mathcal{O}_\ell \to 0$ twisted by $\mathcal{O}_Y(1)$, we deduce (23).

Moreover, using (23) to compute $H^\bullet(Y, \mathcal{F}(-q))$ for $q = 0, 1, 2$, we conclude that $\mathcal{F}_\ell \in \mathcal{C}_Y$. □

Proposition 5.2. *For any line $\ell \subset Y$, the sheaf $\mathcal{F}_\ell$ is stable. Moreover, for $\ell \neq \ell'$ we have $\mathcal{F}_\ell \not\cong \mathcal{F}_{\ell'}$.*

Proof. The sheaf $\mathcal{F}_\ell$ is reflexive of rank 3 with $c_1(\mathcal{F}_\ell) = -1$, hence for stability it suffices to check that $H^0(Y, \mathcal{F}_\ell) = H^0(Y, \mathcal{F}_\ell^\vee(-1)) = 0$. But $H^0(Y, \mathcal{F}_\ell) = 0$ since $\mathcal{F}_\ell \in \mathcal{C}_Y$, and by Serre duality $H^0(Y, \mathcal{F}_\ell^\vee(-1)) = H^4(Y, \mathcal{F}_\ell(-2)) = 0$ since $\mathcal{F}_\ell \in \mathcal{C}_Y$.

Further, note that (23) implies that $\mathcal{E}xt^1(\mathcal{F}_\ell, \mathcal{O}_Y) \cong \mathcal{E}xt^3(\mathcal{O}_\ell(1), \mathcal{O}_Y) \cong \mathcal{O}_\ell$, whereof it follows that $\mathcal{F}_\ell \not\cong \mathcal{F}_{\ell'}$ for $\ell \neq \ell'$. □

Corollary 5.3. *For any line $\ell \subset Y$, we have* $\dim \mathrm{Hom}(\mathcal{F}_\ell, \mathcal{F}_\ell) = \dim \mathrm{Ext}^2(\mathcal{F}_\ell, \mathcal{F}_\ell) = 1$, $\dim \mathrm{Ext}^1(\mathcal{F}_\ell, \mathcal{F}_\ell) = 4$.

Proof. The equality $\dim \mathrm{Hom}(\mathcal{F}_\ell, \mathcal{F}_\ell) = 1$ follows from the stability of $\mathcal{F}_\ell$, and $\dim \mathrm{Ext}^2(\mathcal{F}_\ell, \mathcal{F}_\ell) = 1$ follows from (22). It also follows from (22) that $\mathrm{Ext}^p(\mathcal{F}_\ell, \mathcal{F}_\ell) = 0$ for $p > 2$. Therefore, $\dim \mathrm{Ext}^1(\mathcal{F}_\ell, \mathcal{F}_\ell)$ can be computed by Riemann–Roch. □

Proposition 5.4. *The map $\mathbb{L} : \mathrm{Ext}^1(\mathcal{I}_\ell(1), \mathcal{I}_\ell(1)) \to \mathrm{Ext}^1(\mathcal{F}_\ell, \mathcal{F}_\ell)$ induced by the functor $\mathbb{L}$ is an isomorphism.*

Proof. Applying the functor $\mathrm{Hom}(-, \mathcal{F}_\ell)$ to (24) and taking into account that $\mathrm{Hom}(\mathcal{O}_Y, \mathcal{F}_\ell) = 0$, we conclude that

$$\mathrm{Ext}^p(\mathcal{F}_\ell, \mathcal{F}_\ell) \cong \mathrm{Ext}^{p+1}(\mathcal{I}_\ell(1), \mathcal{F}_\ell)$$

for all p. Further, note that by Serre duality we have

$$\mathrm{Ext}^p(\mathcal{I}_\ell(1), \mathcal{O}_Y) \cong \mathrm{Ext}^{4-p}(\mathcal{O}_Y, \mathcal{I}_\ell(-2))^\vee \cong H^{3-p}(\ell, \mathcal{O}_\ell(-2))^\vee = \begin{cases} \mathbb{C}, & \text{if } p = 2 \\ 0, & \text{otherwise} \end{cases}$$

Therefore, applying the functor $\mathrm{Hom}(\mathcal{I}_\ell(1), -)$ to (24), we obtain the exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}(\mathcal{I}_\ell(1), \mathcal{F}_\ell) &\to 0 \to \mathrm{Hom}(\mathcal{I}_\ell(1), \mathcal{I}_\ell(1)) \to \\ \to \mathrm{Ext}^1(\mathcal{I}_\ell(1), \mathcal{F}_\ell) &\to 0 \to \mathrm{Ext}^1(\mathcal{I}_\ell(1), \mathcal{I}_\ell(1)) \to \\ \to \mathrm{Ext}^2(\mathcal{I}_\ell(1), \mathcal{F}_\ell) &\to \mathbb{C}^4 \to \mathrm{Ext}^2(\mathcal{I}_\ell(1), \mathcal{I}_\ell(1)) \to \\ \to \mathrm{Ext}^3(\mathcal{I}_\ell(1), \mathcal{F}_\ell) &\to 0 \to \mathrm{Ext}^3(\mathcal{I}_\ell(1), \mathcal{I}_\ell(1)) \to \\ \to \mathrm{Ext}^4(\mathcal{I}_\ell(1), \mathcal{F}_\ell) &\to 0 \to \mathrm{Ext}^4(\mathcal{I}_\ell(1), \mathcal{I}_\ell(1)) \to 0. \end{aligned}$$

The composition of the map $\mathrm{Ext}^1(\mathcal{I}_\ell(1),\mathcal{I}_\ell(1)) \to \mathrm{Ext}^2(\mathcal{I}_\ell(1),\mathcal{F}_\ell)$ with the isomorphism $\mathrm{Ext}^2(\mathcal{I}_\ell(1),\mathcal{F}_\ell) \cong \mathrm{Ext}^1(\mathcal{F}_\ell,\mathcal{F}_\ell)$ clearly coincides with the map induced by the functor $\mathbb{L}$, hence $\mathbb{L} : \mathrm{Ext}^1(\mathcal{I}_\ell(1),\mathcal{I}_\ell(1)) \to \mathrm{Ext}^1(\mathcal{F}_\ell,\mathcal{F}_\ell)$ is injective. But $\dim \mathrm{Ext}^1(\mathcal{I}_\ell(1),\mathcal{I}_\ell(1)) = 4$, since this is the tangent space to the smooth 4-dimensional moduli space $F(Y)$, and $\dim \mathrm{Ext}^1(\mathcal{F}_\ell,\mathcal{F}_\ell) = 4$ by Corollary 5.3. Hence the map $\mathbb{L}$ is an isomorphism. $\square$

Let $F'(Y)$ denote the moduli space of stable sheaves on Y containing sheaves $\mathcal{F}_\ell$. Consider the map $F(Y) \to F'(Y)$ defined by the functor $\mathbb{L}$, $\ell \mapsto \mathbb{L}(\mathcal{I}_\ell(1))[-1] = \mathcal{F}_\ell$.

Proposition 5.5. *The map $\mathbb{L} : F(Y) \to F'(Y)$, $\ell \mapsto \mathcal{F}_\ell$ is an isomorphism of $F(Y)$ with a connected component of $F'(Y)$.*

Proof. We already know that $\mathbb{L}$ induces an isomorphism on tangent spaces, hence it is étale. On the other hand, if $\ell \neq \ell'$ then $\mathcal{F}_\ell \not\cong \mathcal{F}_{\ell'}$. Hence $\mathbb{L}$ is injective. Thus $\mathbb{L}$ has to be an open embedding. Since $F(Y)$ is a projective variety, its image is closed. Therefore, $\mathbb{L}$ is an isomorphism onto a connected component. $\square$

It is an interesting question, whether $F'(Y) = F(Y)$, or not.

Theorem 5.6. *The map $\mathbb{L} : F(Y) \to F'(Y)$, $\ell \mapsto \mathcal{F}_\ell$ agrees up to a sign with the forms α_ν. In particular, the form α_ν on $F(Y)$ is symplectic.*

Proof. Let us denote the form α_ν on $F(Y)$ by α, and the form α_ν on $F'(Y)$ by α'. Take any line ℓ on Y and any $a, b \in \mathrm{Ext}^1(\mathcal{I}_\ell(1),\mathcal{I}_\ell(1))$. As it was shown in the proof of Theorem 4.3 the value $\alpha(a,b)$ is the trace of the following composition of morphisms $\mathcal{I}_\ell(1) \xrightarrow{b} \mathcal{I}_\ell(1)[1] \xrightarrow{a} \mathcal{I}_\ell(1)[2] \xrightarrow{\epsilon_{\mathcal{I}_\ell(1)}} \mathcal{I}_\ell(-2)[4]$ in the derived category $\mathcal{D}^b(\mathrm{Coh}(Y))$, and $\alpha'(\mathbb{L}(a),\mathbb{L}(b))$ is the trace of the composition $\mathcal{F}_\ell \xrightarrow{\mathbb{L}(b)} \mathcal{F}_\ell[1] \xrightarrow{\mathbb{L}(a)} \mathcal{F}_\ell[2] \xrightarrow{\epsilon_{\mathcal{F}_\ell}} \mathcal{F}_\ell(-3)[4]$. By functoriality of the Yoneda multiplication and of the linkage class we have the following commutative diagram in $\mathcal{D}^b(\mathrm{Coh}(Y))$

$$\begin{array}{ccccccc}
\mathcal{F}_\ell & \longrightarrow & \mathcal{O}_Y^{\oplus 4} & \longrightarrow & \mathcal{I}_\ell(1) & \longrightarrow & \mathcal{F}_\ell[1] \\
\downarrow{\scriptstyle \mathbb{L}(b)} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -\mathbb{L}(b)} \\
\mathcal{F}_\ell[1] & \longrightarrow & \mathcal{O}_Y^{\oplus 4}[1] & \longrightarrow & \mathcal{I}_\ell(1)[1] & \longrightarrow & \mathcal{F}_\ell[2] \\
\downarrow{\scriptstyle \mathbb{L}(a)} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle -\mathbb{L}(a)} \\
\mathcal{F}_\ell[2] & \longrightarrow & \mathcal{O}_Y^{\oplus 4}[2] & \longrightarrow & \mathcal{I}_\ell(1)[2] & \longrightarrow & \mathcal{F}_\ell[3] \\
\downarrow{\scriptstyle \epsilon_{\mathcal{F}_\ell}} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \epsilon_{\mathcal{I}_\ell(1)}} & & \downarrow{\scriptstyle -\epsilon_{\mathcal{F}_\ell}} \\
\mathcal{F}_\ell(-3)[4] & \longrightarrow & \mathcal{O}_Y(-3)^{\oplus 4}[4] & \longrightarrow & \mathcal{I}_\ell(-2)[1] & \longrightarrow & \mathcal{F}_\ell(-3)[5]
\end{array}$$

Since the trace is additive and the trace of the second column is 0, we conclude that $\alpha'(\mathbb{L}(a),\mathbb{L}(b)) = -\alpha(a,b)$. $\square$

Remark 5.7. The same argument shows that the 2-form on $F(Y)$ considered as the moduli space of ideal sheaves $\mathcal{I}_\ell$ agrees up to a sign with the 2-form on $F(Y)$ considered as the moduli space of structure sheaves $\mathcal{O}_\ell$.

In the next section we will compute the form α on $F(Y)$ explicitly in coordinates.

6. Closed forms on Hilbert schemes

Let Y be a projective variety. Fix an ample line bundle $\mathcal{O}(1)$ and a Hilbert polynomial h of some reduced equidimensional proper closed subscheme $Z_0 \subset Y$. Consider the moduli space $\mathcal{P}$

of stable sheaves on Y with Hilbert polynomial h. Then it has an open subscheme $\mathcal{P}_0$ parameterizing torsion free sheaves of rank 1 on subschemes $Z \subset Y$ with the same Hilbert polynomial as Z_0. We obtain a morphism $p : \mathcal{P}_0 \to \mathrm{Hilb}^h(Y)$ to the Hilbert scheme parameterizing the subschemes $Z \subset Y$ with Hilbert polynomial h. The fiber of $p : \mathcal{P}_0 \to \mathrm{Hilb}^h(Y)$ over a point $[Z] \in \mathrm{Hilb}^h(Y)$, if nonempty, is a partial compactification of the generalized Picard scheme $\mathrm{Pic}^0(Z)$.

In this section we will give another interpretation of the forms α_ω constructed in (12) in the special case when the moduli space is $\mathcal{P}_0$. Actually we will show that for some ω the form α_ω is the pullback of a 2-form on $\mathrm{Hilb}^h(Y)$, at least over the open subset of $\mathcal{P}_0$ consisting of line bundles on locally complete intersection subschemes $Z \subset Y$. As an application we will give an explicit formula for the symplectic form on the Hilbert scheme of lines on a cubic fourfold.

Let $n = \dim Y$ and $m = n - \deg h$ be the codimension in Y of subschemes parameterized by $\mathrm{Hilb}^h(Y)$. The tangent space to $\mathrm{Hilb}^h(Y)$ at a point $[Z] \in \mathrm{Hilb}^h(Y)$ is canonically isomorphic to $H^0(Z, \mathcal{N}_{Z/Y})$. Let $\mathrm{Hilb}^h_{lci}(Y) \subset \mathrm{Hilb}^h(Y)$ denote the open subset of equidimensional locally complete intersection subschemes. Recall that for any $[Z] \in \mathrm{Hilb}^h_{lci}(Y)$ we have the adjunction exact triple

$$0 \to \mathcal{N}^\vee_{Z/Y} \xrightarrow{\kappa} \Omega^1_{Y|Z} \to \Omega_Z \to 0.$$

Denote by $\kappa \in H^0(Z, \mathcal{N}_{Z/Y} \otimes \Omega^1_{Y|Z}) = \mathrm{Hom}(\mathcal{N}^\vee_{Z/Y}, \Omega^1_{Y|Z})$ the element corresponding to the first morphism in this triple. Take any $\omega \in H^{n-m}(Y, \Omega_Y^{n-m+2})$ and denote by $\omega_{|Z} \in H^{n-m}(Z, \Omega^{n-m+2}_{Y|Z})$ its restriction to Z. We define a 2-form β_ω on $\mathrm{Hilb}^h_{lci}(Y)$ as follows. For any $s_1, s_2 \in H^0(Z, \mathcal{N}_{Z/Y})$ we set

$$\beta_\omega(s_1, s_2) = (\kappa^{\wedge(m-2)} \wedge s_1 \wedge s_2 \wedge \omega_{|Z}) \cap [Z], \tag{25}$$

where $\kappa^{\wedge(m-2)}$ is the $(m-2)$-fold wedge product of κ in $H^0(Z, \wedge^{m-2}\mathcal{N}_{Z/Y} \otimes \Omega^{m-2}_{Y|Z})$, so that $\kappa^{\wedge(m-2)} \wedge s_1 \wedge s_2 \wedge \omega_{|Z} \in H^{n-m}(Z, \wedge^m \mathcal{N}_{Z/Y} \otimes \Omega^n_{Y|Z}) \cong H^{n-m}(Z, \Omega^m_Z)$.

Let $\mathcal{P}^\circ \subset p^{-1}(\mathrm{Hilb}^h_{lci}(Y))$ denote the open subset of $\mathcal{P}_0$ consisting of line bundles on subschemes $Z \subset Y$.

Theorem 6.1. *We have $\alpha_\omega = p^*\beta_\omega$ on $\mathcal{P}^\circ$.*

Proof. Take any $\mathcal{F} \in \mathcal{P}^\circ$ and put $[Z] = p([\mathcal{F}])$. By definition of p this means that $\mathcal{F} \cong i_*F$ for some line bundle F on Z, where $i : Z \to Y$ is the embedding and Z is a locally complete intersection in Y. Take any $v_1, v_2 \in \mathrm{Ext}^1(\mathcal{F}, \mathcal{F})$.

By the definition of α, we should compute the product $\mathrm{At}_Y^{m-2}(\mathcal{F}) \circ v_1 \circ v_2$ in $\mathrm{Ext}^m(\mathcal{F}, \mathcal{F} \otimes \Omega_Y^{m-2})$, then take its trace in $H^m(Y, \Omega_Y^{m-2})$ and finally couple it with $\omega \in H^{n-m}(Y, \Omega_Y^{n-m+2})$. By Proposition 1.4.3 the trace factors through

$$H^0(Y, \mathcal{E}xt^{\,m}(\mathcal{F}, \mathcal{F} \otimes \Omega_Y^{m-2})) \cong H^0(Z, F^\vee \otimes F \otimes \wedge^m \mathcal{N}_{Z/Y} \otimes \Omega^{m-2}_{Y|Z}) = H^0(Z, \wedge^m \mathcal{N}_{Z/Y} \otimes \Omega^{m-2}_{Y|Z}),$$

hence it suffices to compute the image of $\mathrm{At}_Y^{m-2}(\mathcal{F}) \circ v_1 \circ v_2$ in $H^0(Z, \wedge^m \mathcal{N}_{Z/Y} \otimes \Omega^{m-2}_{Y|Z})$ and then apply the canonical map $\mathsf{can} : H^0(Z, \wedge^m \mathcal{N}_{Z/Y} \otimes \Omega^{m-2}_{Y|Z}) \to H^m(Y, \Omega_Y^{m-2})$. By Lemma 1.3.3 this image coincides with the product of the $(m-2)$-th wedge power of the image of $\mathrm{At}_Y(\mathcal{F})$ in $H^0(Y, \mathcal{E}xt^{\,1}(\mathcal{F}, \mathcal{F} \otimes \Omega^1_Y)) = H^0(Z, \mathcal{N}_{Z/Y} \otimes \Omega^1_{Y|Z})$ and of the images of v_1, v_2 in $H^0(Y, \mathcal{E}xt^{\,1}(\mathcal{F}, \mathcal{F})) = H^0(Z, \mathcal{N}_{Z/Y})$. By Theorem 3.2 (*iii*) the image of $\mathrm{At}_Y(\mathcal{F})$ in $H^0(Z, \mathcal{N}_{Z/Y} \otimes \Omega^1_{Y|Z})$ equals κ. The images of v_i are equal to $p_*(v_i)$. So, by Lemma 1.3.2 (*ii*) we have

$$\alpha_\omega(v_1, v_2) = \mathsf{can}(\kappa^{\wedge(m-2)} \wedge p_*(v_1) \wedge p_*(v_2)) \cup \omega \cap [Y].$$

Further, we note that the cap-products $\cap[Y]$ and $\cap[Z]$ are nothing but the Serre duality on Y and Z respectively. Therefore, by Lemma 1.4.4 we have

$$\mathsf{can}(-)\cup\omega\cap[Y] = (-)\cup\omega_{|Z}\cap[Z]$$

which completes the proof. □

Now we apply Theorem 6.1 in the following case. We take for Y a smooth cubic hypersurface in $\mathbb{P}^5$ (a cubic 4-fold), and $h(n) = n+1$.

Lemma 6.2. *If $\mathcal{F}$ is a semistable sheaf on Y with Hilbert polynomial $h_{\mathcal{F}}(n) = n+1$ then $\mathcal{F} = \mathcal{O}_\ell$, the structure sheaf of a line $\ell \subset Y$.*

Proof. By Riemann–Roch we have $\mathsf{ch}(\mathcal{F}) = [\ell] - \frac{1}{2}[\mathsf{pt}]$, hence $\mathcal{F}$ is a rank 1 sheaf on a line. By semistability $\mathcal{F}$ has no 0-dimensional torsion, hence $\mathcal{F} \cong \mathcal{O}_\ell(k)$ for some $k \in \mathbb{Z}$. Finally, computing $h_{\mathcal{O}_\ell(k)}(n) = n + (k+1)$, we conclude that $k = 0$. □

We conclude that $\mathcal{P} = \mathrm{Hilb}^h(Y) = F(Y)$ is the Hilbert scheme of lines and the projection map $p : \mathcal{P} \to \mathrm{Hilb}^h(Y)$ is the identity. Thus, Theorem 6.1 gives a way to compute the form α_ω. We are going to do this explicitly.

Recall that $\Omega^4_Y \cong \mathcal{O}_Y(-3) \cong \mathcal{N}^\vee_{Y/\mathbb{P}^5}$. So, we take the form $\omega \in H^1(Y, \Omega^3_Y)$ corresponding to the extension $\nu_{Y/\mathbb{P}^5} \in \mathrm{Ext}^1(\Omega_Y, \mathcal{N}^\vee_{Y/\mathbb{P}^5})$ under the isomorphisms

$$\mathrm{Ext}^1(\Omega_Y, \mathcal{N}^\vee_{Y/\mathbb{P}^5}) = H^1(Y, \mathcal{T}_Y \otimes \mathcal{O}_Y(-3)) \cong H^1(Y, \mathcal{T}_Y \otimes \Omega^4_Y) \cong H^1(Y, \Omega^3_Y).$$

According to (25) we have to compute $\kappa \wedge \omega_{|\ell}$.

Lemma 6.3. *The wedge product*

$$\kappa \wedge \omega_{|\ell} \in H^1(\ell, \mathcal{N}_{\ell/Y} \otimes \Omega^4_{Y|Z}) \cong H^1(\ell, \mathcal{N}_{\ell/Y}(-3)) \cong H^1(\ell, \mathcal{N}_{\ell/Y} \otimes \mathcal{N}^\vee_{Y/\mathbb{P}^5|\ell})$$

is given by the extension class of the normal bundles exact sequence

$$0 \longrightarrow \mathcal{N}_{\ell/Y} \longrightarrow \mathcal{N}_{\ell/\mathbb{P}^5} \longrightarrow \mathcal{N}_{Y/\mathbb{P}^5}|_\ell \longrightarrow 0. \tag{26}$$

Proof. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{N}^\vee_{Y/\mathbb{P}^5}|_\ell & = & \mathcal{N}^\vee_{Y/\mathbb{P}^5}|_\ell & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{N}^\vee_{\ell/\mathbb{P}^5} & \longrightarrow & \Omega_{\mathbb{P}^5|\ell} & \longrightarrow & \Omega_\ell & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{N}^\vee_{\ell/Y} & \xrightarrow{\kappa} & \Omega_{Y|\ell} & \longrightarrow & \Omega_\ell & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Note that the form $\omega_{|\ell}$ by definition is given by the restriction of the extension $\nu_{Y/\mathbb{P}^5}$ to ℓ which coincides with the middle column of the diagram. Therefore, $\kappa \wedge \omega_{|\ell}$ equals the extension class of the left column. It remains to note that the extension classes of dual exact sequences coincide. □

So, to compute $\alpha_\omega = \beta_\omega$, it remains to identify the element $\sigma \in H^1(\ell, \mathcal{N}_{\ell/Y}(-3))$ corresponding to the extension class of (26).

Recall that according to [CG], there are two possibilities for the normal bundle $\mathcal{N}_{\ell/Y}$ of a line ℓ in Y. We have either $\mathcal{N}_{\ell/Y} = \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1)$, or $\mathcal{N}_{\ell/Y} = \mathcal{O}(-1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(1)$. The lines

ℓ with the normal bundle of the first type fill an open subset of $F(Y)$ and the lines with the normal bundle of the second type fill a subset of codimension 2 in $F(Y)$.

First of all consider the case of $\mathcal{N}_{\ell/Y} = \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1)$. Denote by e_1, e_2, e_3 rational sections of twists of $\mathcal{N}_{\ell/Y}$ such that $\mathcal{N}_{\ell/Y} = \mathcal{O}e_1 \oplus \mathcal{O}e_2 \oplus \mathcal{O}(1)e_3$. Then the sequence (26) twisted by $\mathcal{O}(-3)$ takes form

$$0 \to \mathcal{O}_\ell(-3)e_1 \oplus \mathcal{O}_\ell(-3)e_2 \oplus \mathcal{O}_\ell(-2)e_3 \to \mathcal{O}_\ell(-2)^{\oplus 4} \to \mathcal{O}_\ell \to 0.$$

We see that σ is a generator of the kernel of the map

$$H^1(\ell, \mathcal{O}_\ell(-3)e_1 \oplus \mathcal{O}_\ell(-3)e_2 \oplus \mathcal{O}_\ell(-2)e_3) \to H^1(\ell, \mathcal{O}_\ell(-2)^{\oplus 4}).$$

This implies that the component σ_3 of σ in the third summand $\mathcal{O}_\ell(-2)$ of $\mathcal{N}_{\ell/Y}(-3)$ is zero, while the components σ_1 and σ_2 in the first two summands are linearly independent elements of $H^1(\ell, \mathcal{O}_\ell(-3))$. Using the Serre duality $H^1(\ell, \mathcal{O}_\ell(-3)) \cong H^0(\ell, \mathcal{O}(1))^\vee$, let us endow ℓ with homogeneous coordinates t_0, t_1, dual to σ_1, σ_2. The sections $v_i \in H^0(\ell, \mathcal{N}_\ell)$ can be written as

$$v_i = a_i e_1 + b_i e_2 + (c_i t_0 + d_i t_1)e_3.$$

Then it is clear that

$$\alpha_\omega(v_1, v_2) = \sigma \wedge v_1 \wedge v_2 = b_1c_2 - b_2c_1 - a_1d_2 + a_2d_1. \tag{27}$$

The case of $\mathcal{N}_{\ell/Y} = \mathcal{O}(-1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(1)$ is considered similarly. Denote by e_1, e_2, e_3 rational sections of twists of $\mathcal{N}_{\ell/Y}$ such that $\mathcal{N}_{\ell/Y} = \mathcal{O}(-1)e_1 \oplus \mathcal{O}(1)e_2 \oplus \mathcal{O}(1)e_3$. Then the sequence (26) twisted by $\mathcal{O}(-3)$ takes form

$$0 \to \mathcal{O}_\ell(-4)e_1 \oplus \mathcal{O}_\ell(-2)e_2 \oplus \mathcal{O}_\ell(-2)e_3 \to \mathcal{O}_\ell(-2)^{\oplus 4} \to \mathcal{O}_\ell \to 0.$$

Then σ is a generator of the kernel of the map

$$H^1(\ell, \mathcal{O}_\ell(-4)e_1 \oplus \mathcal{O}_\ell(-2)e_2 \oplus \mathcal{O}_\ell(-2)e_3) \to H^1(\ell, \mathcal{O}_\ell(-2)^{\oplus 4}).$$

This implies that the components σ_2 and σ_3 of σ are zero, while the component σ_1 is a nondegenerate element of $H^1(\ell, \mathcal{O}_\ell(-4)) \cong S^2H^1(\ell, \mathcal{O}_\ell(-3))$. Using the Serre duality $H^1(\ell, \mathcal{O}_\ell(-3)) \cong H^0(\ell, \mathcal{O}(1))^\vee$, let us endow ℓ with homogeneous coordinates t_0, t_1, which are isotropic for σ_1. The sections $v_i \in H^0(\ell, \mathcal{N}_\ell)$ can be written as

$$v_i = (a_i t_0 + b_i t_1)e_2 + (c_i t_0 + d_i t_1)e_3.$$

Then it is clear that

$$\alpha_\omega(v_1, v_2) = \sigma \wedge v_1 \wedge v_2 = b_1c_2 - b_2c_1 + a_1d_2 - a_2d_1. \tag{28}$$

7. A 10-dimensional example

Let X be a smooth 3-dimensional cubic hypersurface in $\mathbb{P}^4$. By a normal elliptic quintic in X, we mean a curve $C \subset \mathbb{P}^4$, contained in X and projectively equivalent to the image of an elliptic curve E by the linear system $|5o|$, where $o \in E$ is a point of E. Equivalently, C is a smooth connected curve in X of degree 5 and of genus 1 such that its linear span $\langle C \rangle$ is $\mathbb{P}^4$, see [Hu].

To each C as above, we associate the vector bundle $\mathcal{E} = \mathcal{E}_C$ obtained from C by Serre's construction:

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\ s\ } \mathcal{E}(1) \xrightarrow{\ t\ } \mathcal{I}_C(2) \longrightarrow 0\ , \tag{29}$$

where $\mathcal{I}_C = \mathcal{I}_{C,X}$ is the ideal sheaf of C in X, and s is a section of $\mathcal{E}(1)$ which has C is its zero locus. Since the class of C modulo algebraic equivalence is $5[\ell]$, where ℓ is a line in X and $[\ell]$ as its class in $H^4(X, \mathbb{Z})$, (29) implies that $c_1(\mathcal{E}) = 0, c_2(\mathcal{E}) = 2[\ell]$. Further, $\det \mathcal{E}$ is trivial, and hence $\mathcal{E}$ is self-dual as soon as it is a vector bundle (that is, $\mathcal{E}^\vee \simeq \mathcal{E}$). See [MT2, Sect. 2] for further details on this construction. As follows from [IMar], [B1], [Dr] (see also [B2], where the relevant results of the other three papers are summarized) or [Ku1], the vector bundles $\mathcal{E}$ of this type have several other equivalent characterizations:

Theorem 7.1. *Let $\mathcal{E}$ be a rank-2 vector bundle on X. Then the following properties are equivalent:*

(i) $\mathcal{E}$ is stable with Chern classes $c_1 = 0$, $c_2 = 2[\ell]$.

(ii) $\mathcal{E}$ is isomorphic to a vector bundle obtained by Serre's construction (29) *from a normal elliptic quintic $C \subset X$.*

(iii) $\mathcal{E}$ has Chern classes $c_1 = 0$, $c_2 = 2[\ell]$ and the intermediate cohomology of the twists of $\mathcal{E}$ vanishes:

$$H^i(X, \mathcal{E}(j)) = 0 \text{ for } i = 1, 2 \text{ and for all } j \in \mathbb{Z}.$$

(iv) There exists a Pfaffian representation of X, that is a skew-symmetric 6 by 6 matrix M of linear forms on $\mathbb{P}^4$ such that the equation of X is $\mathrm{Pf}(M) = 0$, and $\mathcal{E} \simeq \mathcal{K}(1)$, where $\mathcal{K}$ is the kernel bundle of M: it is defined as a rank 2 subbundle of the trivial rank-6 bundle $\mathcal{O}_X^{\oplus 6}$ over X whose fiber K_x over $x \in X$ is the kernel of the rank-4 linear map $M(x) : \mathbb{C}^6 \longrightarrow \mathbb{C}^6$. Equivalently, $\mathcal{E}(1) \simeq \mathcal{C}$, where $\mathcal{C}$ is the cokernel bundle of M.

(v) There exists a skew-symmetric 6 by 6 matrix M of linear forms on $\mathbb{P}^4$ such that $\mathcal{E}(1)$ considered as a sheaf on $\mathbb{P}^4$ can be included in the following exact sequence:

$$0 \longrightarrow \mathcal{O}(-1)^{\oplus 6} \xrightarrow{M} \mathcal{O}^{\oplus 6} \longrightarrow \mathcal{E}(1) \longrightarrow 0.$$

The vector bundles as in the above theorem possess the following property:

Lemma 7.2. *Let $\mathcal{E}$ be a vector bundle on X satisfying any of the equivalent conditions (i)–(v) of Theorem* 7.1. *Then $H^\bullet(X, \mathcal{E}) = H^\bullet(X, \mathcal{E}(-1)) = H^\bullet(X, \mathcal{E}(-2)) = 0$.*

Proof. Follows immediately from Theorem 7.1 (v). ☐

Let $M_X = M_X(2; 0, 2)$ be the moduli space of stable rank-2 vector bundles with Chern classes $c_1(\mathcal{E}) = 0, c_2(\mathcal{E}) = 2[\ell]$. There is a natural map $\phi_X : M \longrightarrow J(X)$ to the intermediate Jacobian $J(X)$ of X, well-defined modulo a constant translation in $J(X)$. It can be described as follows. According to [Mur], the Chow group $A_1(0)_X$ of algebraic 1-cycles of degree 0 on a smooth cubic threefold X modulo rational equivalence is canonically isomorphic to the intermediate Jacobian $J(X)$. Taking any 1-cycle Z_0 of degree d on X, we obtain also an identification $A_1(d)_X \longrightarrow A_1(0)_X = J(X)$ for the set $A_1(d)_X$ of rational equivalence classes of degree-d cycles, $(Z) \in A_1(d)_X \mapsto (Z - Z_0) \in J(X)$, where (Z) denotes the class of Z modulo rational equivalence. This is nothing but the Abel–Jacobi map on the algebraic cycles of degree d.

Grothendieck defined in [Gro] the Chern classes with values in the Chow groups of algebraic cycles modulo rational equivalence. Let us denote these Grothendieck–Chern classes by $\mathfrak{c}_i(\mathcal{E})$. Then the wanted map ϕ_X can be defined by $[\mathcal{E}] \in M_X \mapsto \mathfrak{c}_2(\mathcal{E}) - (Z_0)$ for some fixed reference 1-cycle Z_0 of degree 2. One can choose, for example, $Z_0 = 2\ell$. Remark that if $\mathcal{E} = \mathcal{E}_C$ is obtained by Serre's construction from a normal elliptic quintic C, then $\mathfrak{c}_2(\mathcal{E}) = (C) - h^2$, where h^2 is the class of plane cubic curve, a linear section $\mathbb{P}^2 \cap X$.

It follows from the results of [MT1], [IMar] and [Dr] that ϕ_X is an open immersion, thus $M_X(2; 0, 2)$ is isomorphic to an open subset of $J(X)$ (see also [B2] and [Ku1]).

Now we are passing to dimension 4. Let $Y \subset \mathbb{P}^5$ be a nonsingular cubic fourfold. Denote by $P(Y)$ the moduli space of sheaves on Y of the form $i_*\mathcal{E}$, where $[\mathcal{E}] \in M_X(2; 0, 2)$, X is a nonsingular hyperplane section of Y, and $i : X \to Y$ is the embedding. There is a natural map $\pi : P(Y) \longrightarrow \check{\mathbb{P}}^5$ whose image is the complement of $Y^\vee$, the projectively dual variety of Y. According to [MT2], $P(Y)$ is a nonsingular 10-dimensional variety.

Let ν be the generator of $H^1(Y, \Omega_Y^3) \simeq \mathbb{C}$ defined in Theorem 4.3, and α_ν the associated 2-form on $P(Y)$. We have already proved its closedness in Section 2. Now we will see its nondegeneracy.

Theorem 7.3. *The 2-form α_ν on the moduli space $P(Y)$ is nondegenerate.*

Proof. By Lemma 7.2 (i) we have $\mathcal{E} \in \mathcal{C}_Y$ for every $[\mathcal{E}] \in P(Y)$. So, Theorem 4.3 applies. □

Remark 7.4. The paper [MT2] provides a construction of a nondegenerate 2-form on $P(Y)$, but does not prove its closedness. It is just the Yoneda pairing Λ, as defined by (10), and one can treat it as a global 2-form on $P(Y)$ because the 1-dimensional vector spaces $\mathrm{Ext}^2(i_*\mathcal{E}, i_*\mathcal{E})$ fit into a trivial line bundle on $P(Y)$ as $\mathcal{E}$ runs over $P(Y)$. It is also proved in loc. cit. that π is a Lagrangian fibration for Λ. As α_ν factors through Λ, the same holds for α_ν.

Remark 7.5. The second Chern class mappings $\mathcal{E} \mapsto \mathfrak{c}_2(\mathcal{E}) \in A_1(2)_X$ over the smooth hyperplane sections X of Y identify $P(Y)$ with an open subset of the family $\mathcal{A}$ of varieties $A_1(2)_X$. The latter family is an algebraic torsor under the relative intermediate Jacobian $\mathcal{J}$ of the family of smooth hyperplane sections of Y. By [DM], 8.5.2 , $\mathcal{J}$ has a natural symplectic structure $\alpha_{\mathcal{J}}$ such that the map $\mathcal{J} \longrightarrow \check{\mathbb{P}}^5$ is a Lagrangian fibration; let us say for short that $\alpha_{\mathcal{J}}$ is a Lagrangian structure on $\mathcal{J}/\check{\mathbb{P}}^5$. It is easy to see that a Lagrangian structure on a family of abelian varieties induces a Lagrangian structure on any its algebraic torsor. Let us denote the thus induced Lagrangian structure on $\mathcal{A}/\check{\mathbb{P}}^5$ by $\alpha_{\mathcal{A}}$. Then it is plausible that α_ν coincides with the restriction of $\alpha_{\mathcal{A}}$ up to a constant factor. A way to prove this might be to find a partial compactification $\overline{P(Y)}$ of $P(Y)$, such that $h^0(\Omega^2_{\overline{P(Y)}}) = 1$ and both α_ν, $\alpha_{\mathcal{A}}$ extend to $\overline{P(Y)}$.

References

[AB] Atiyah, M. F., Bott, R.: *The Yang-Mills equations over Riemann surfaces*, Philos. Trans. Roy. Soc. London Ser. **A 308** (1983), 523–615.

[AK] Altman, A. B., Kleiman, S. L.: *Compactifying the Picard scheme,* Adv. in Math. **35** (1980), 50–112.

[ALJ] Angéniol, B., Lejeune-Jalabert, M.: *Calcul différentiel et classes caractéristiques en géométrie algébrique*, Travaux en Cours, 38, Hermann, Paris, 1989.

[At] Atiyah, M. F.: *Complex analytic connections in fiber bundles,* Trans. Amer. Math. Soc. **85** (1957), 181–207.

[B1] Beauville, A.: *Determinantal hypersurfaces,* Michigan Math. J. **48** (2000), 39–64.

[B2] Beauville, A.: *Vector bundles on the cubic threefold*, Bertram, Aaron (ed.) et al., Symposium in honor of C. H. Clemens, University of Utah, March 2000, Contemp. Math. **312**, 71–86 (2002).

[BD] Beauville, A., Donagi, R.: *La variété des droites d'une hypersurface cubique de dimension 4,* C. R. Acad. Sci. Paris Ser. I Math. **301**, no. 14, 703–706 (1985).

[Bei] Beilinson, A.: *Coherent sheaves on P^n and problems in linear algebra* (Russian) Funktsional. Anal. i Prilozhen. **12** (1978), no. 3, 68–69.

[Bon] Bondal, A.: *Representations of associative algebras and coherent sheaves*, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 1, 25–44; translation in Math. USSR-Izv. **34** (1990), no. 1, 23–42.

[BK] Bondal, A., Kapranov, M.: *Representable functors, Serre functors, and reconstructions*, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 6, 1183–1205, 1337; translation in Math. USSR-Izv. **35** (1990), no. 3, 519–541.

[Bot-1] Bottacin, F.: *Poisson structures on moduli spaces of sheaves over Poisson surfaces*, Invent. Math. **121** (1995), 421–436.

[Bot-2] Bottacin, F.: *Symplectic geometry on moduli spaces of stable pairs*, Ann. Sci. E'cole Norm. Sup. (4) **28** (1995), 391–433.

[Bot-3] Bottacin, F.: *Poisson structures on moduli spaces of parabolic bundles on surfaces*, Manuscripta Math. **103** (2000), 31–46.

[BR] Biswas, I., Ramanan, S.: *An infinitesimal study of the moduli of Hitchin pairs*, J. London Math. Soc. (2) **49** (1994), 219231.

[BuF1] Buchweitz, R.-O., Flenner, H.: *The Atiyah-Chern character yields the semiregularity map as well as the infinitesimal Abel-Jacobi map*, The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), 33–46, CRM Proc. Lecture Notes, 24, Amer. Math. Soc., Providence, 2000.

[BuF2] Buchweitz, R.-O., Flenner, H.: *A Semiregularity Map for Modules and Applications to Deformations*, Compositio Math. **137** (2003), 135–210.

[CG] Clemens, C. H., Griffiths, P. A.: *The intermediate Jacobian of the cubic threefold,* Annals of Math. **95**(1972), 281–356.

[DM] Donagi, R., Markman, E.: *Spectral covers, algebraically completely integrable Hamiltonian systems, and moduli of bundles,* In: Francaviglia, M. (ed.) et al., Integrable systems and quantum groups, CIME Lectures, Italy, June 14-22, 1993. Lect. Notes Math., 1620, Springer-Verlag, Berlin, 1996, 1–119.

[Dr] Druel, S.: *Espace des modules des faisceaux semi-stables de rang 2 et de classes de Chern* $c_1 = 0$, $c_2 = 2$ *et* $c_3 = 0$ *sur une hypersurface cubique lisse de* $\mathbb{P}^4$, Internat. Math. Res. Notices **2000**, No. 19, 985–1004.

[Gri] Griffiths, P. A.: *Periods of integrals on algebraic manifolds, II, Local study of the period mapping*, Amer. J. Math. **90** (1968), 805–865.

[GR] Gorodentsev, A., Rudakov, A.: *Exceptional vector bundles on projective spaces*, Duke Math. J. **54** (1987), no. 1, 115–130.

[Gro] Grothendieck, A.: *La théorie des classes de Chern*, Bull. Soc. Math. France **86** (1958), 137–154.

[Ha] Hartshorne, R.: *Residues and duality,* A seminar on the work of A. Grothendieck, Harvard, 1963/64, Lecture Notes in Math., **20** Springer, Berlin–N. Y., 1966.

[Hi] Hitchin, N.: *Stable bundles and integrable systems,* Duke Math. J. **54** (1987), 91–114.

[Hu] Hulek, K.: *Projective geometry of elliptic curves,* Astérisque **137** (1986).

[HL] Huybrechts, D., Lehn, M.: *The geometry of moduli spaces of sheaves,* Aspects of Mathematics, E31, Braunschweig, Vieweg, 1997.

[IIS] Inaba, M., Iwasaki, K., Saito, M.-H.: *Moduli of Stable Parabolic Connections, Riemann-Hilbert correspondence and Geometry of Painleve' equation of type VI*, I: math.AG/0309342, II: math.AG/0605025.

[IMan] Iliev, A., Manivel, L.: *Cubic hypersurfaces and integrable systems*, math.AG/0605260.

[IMar] Iliev, A., Markushevich, D.: *The Abel–Jacobi map for a cubic threefold and periods of Fano threefolds of degree 14*, Doc. Math. **5** (2000), 23–47.

[Ill] Illusie, L.: *Complexe cotangent et déformations*, I, II, Lecture Notes in Math., 239, 283, Springer Verlag, Berlin-Heidelberg- New York, 1971, 1972.

[K] Kobayashi, S.: *Simple vector bundles over symplectic Kaehler manifolds*, Proc. Japan Acad., Ser. A **62** (1986), 21–24.

[KN] Kronheimer, P. B., and Nakajima, H.: *Yang-Mills instantons on ALE gravitational instantons*, Math. Ann. **288:2** (1990), 263–307.

[Ku1] Kuznetsov, A. G.: *Derived category of a cubic threefold and the variety* V_{14}, Proc. Steklov Inst. Math. **246** (2004), 171–194.

[Ku2] Kuznetsov, A. G.: *Homological projective duality for Grassmannians of lines*, math.AG/0610957.

[LP] Le Potier, J.: *Faisceaux semi-stables et systèmes cohérents*, Vector bundles in algebraic geometry (Durham, 1993), Ed. N. Hitchin et al., 179–239, London Math. Soc. Lecture Note Ser., 208, Cambridge Univ. Press, Cambridge, 1995.

[MT1] Markushevich, D., Tikhomirov, A. S.: *The Abel–Jacobi map of a moduli component of vector bundles on the cubic threefold*, J. Algebraic Geom. **10** (2001), 37–62.

[MT2] Markushevich, D., Tikhomirov, A. S.: *Symplectic structure on a moduli space of sheaves on the cubic fourfold*, Izv. Ross. Akad. Nauk Ser. Mat. **67** (2003), 131–158.

[Muk-1] Mukai, S.: *Symplectic structure of the moduli of sheaves on an abelian or* $K3$ *surface,* Invent. Math. **77** (1984), 101–116.

[Muk-2] Mukai, S.: *Moduli of vector bundles on* $K3$ *surfaces and symplectic manifolds*, Sugaku Expositions **1** (1988), 139–174.

[Mur] Murre, J. P.: *Some results on cubic threefolds*, In: Classification of algebraic varieties and compact complex manifolds, 140–160, Lecture Notes in Math., Vol. 412, Springer, Berlin, 1974.

[O'G] O'Grady, K. G.: *Algebro-geometric analogues of Donaldson's polynomials*, Invent. Math. **107** (1992), 351–395.

[OSS] Okonek, C., Schneider, M., Spindler, H.: *Vector bundles on complex projective spaces* Progress in Mathematics, 3. Birkhaüser, Boston, Mass., 1980. vii+389 pp.

[Si] Simpson, C. T.: *Moduli of representations of the fundamental group of a smooth projective variety I,* Publ. Math. I.H.E.S. **79** (1994), 47–129.

[Tyu] Tyurin, A.N.: *Symplectic structures on the varieties of moduli of vector bundles on algebraic surfaces with* $p_g > 0$, Math. USSR, Izv. **33** (1989), 139–177 = Izv. Akad. Nauk SSSR, Ser. Mat. **52** (1988), 813–852.

[Ve] Verdier, J.-L.: *Des catégories dérivées des catégories abéliennes*, Astérisque No. 239 (1996), xii+253 pp.

[W] Welters, G.E.: *Abel–Jacobi isogenies for certain types of Fano threefolds,* Mathematical Centre Tracts **141**, Amsterdam, 1981.

A. K.: Algebra Section, Steklov Mathematical Institute, 8 Gubkin str., Moscow 119991 Russia

E-mail address: akuznet@mi.ras.ru

D. M.: Mathématiques - bât. M2, Université Lille 1, F-59655 Villeneuve d'Ascq Cedex, France

E-mail address: markushe@math.univ-lille1.fr