Remark on abstract definition for finite reflection groups

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1. Introduction

A reflection in unitary space $(V, \langle\ ,\ \rangle)$ is a linear transformation with the property that all but one of its characteristic values are equal to 1. For $\alpha \in V^*$ and $\zeta \in \mathbf{C}$, $|\zeta| = 1$, define $w(\alpha, \zeta)$ to be the unique reflection with

$$w(\alpha, \zeta)z = z - (1 - \zeta)\alpha(z)/\langle\alpha, \alpha\rangle$$

for any $z \in V$.

A finite group generated by unitary reflections (or a reflection group) can be decomposed as a product of irreducible reflection groups. Shephard and Todd [5] have classified the irreducible reflection groups and gave their presentations. The aim of this paper is to introduce canonical (but highly overdetermined) presentation for all finite reflection groups, except the imprimitive groups $G(m, p, n)$, which are not the Coxeter groups (also see the final remark).

2. Main result

Henceforth Γ will be an irreducible reflection group in $V = \mathbf{C}^l$ and $W(\Gamma) = \{w(\alpha, \zeta)\}$ be the set of all reflections of the group Γ. Note that the set $W(\Gamma)$ is invariant under the conjugation by the elements of Γ and that we have

(i) $$w(\alpha, \zeta)\ w(\alpha, \mu) = w(\alpha, \zeta\mu)$$

(ii) $$w(\alpha, \zeta)w(\beta, \mu)w^{-1}(\alpha, \zeta) = w(w(\alpha, \zeta)\beta, \mu)$$

Theorem. *Let Γ be a finite primitive reflection group or an imprimitive Coxeter group. Then the generators $w(\alpha, \zeta) \in W(\Gamma)$ and defining relators (i), (ii) make the presentation of Γ.*

Remark 1. R. Steinberg proved the Theorem for all Coxeter groups (see the Appendix in his book "Lectures on Chevalley groups")

3. Preparatory results 1. The group $\widetilde{\Gamma}$

Let $\widetilde{\Gamma}$ be an abstract group with the generators $W\left(\widetilde{\Gamma}\right) = \{\widetilde{w}(\alpha, \zeta)\}$, where $w(\alpha, \zeta)$ ranges over the all reflections of Γ, and defining relators

(i) $$\widetilde{w}(\alpha, \zeta)\widetilde{w}(\alpha, \mu) = \widetilde{w}(\alpha, \zeta\mu)$$

(ii) $$\widetilde{w}(\alpha,\zeta)\widetilde{w}(\beta,\mu)\widetilde{w}^{-1}(\alpha,\zeta) = \widetilde{w}(w(\alpha,\zeta)\beta,\mu)$$

Lemma 1. *The natural epimorphism* $\varphi : \widetilde{\Gamma} \to \Gamma$, $\varphi(\widetilde{w}(\alpha,\zeta)) = w(\alpha,\zeta)$, *is a central extension of the group* Γ.

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Suppose, that $\varphi(\widetilde{g}) = 1$, $\widetilde{g} \in \widetilde{\Gamma}$. Then we have $\widetilde{g}\ \widetilde{w}(\beta,\mu)\widetilde{g}^{-1} = \widetilde{w}(g(\beta),\mu) = \widetilde{w}(\beta,\mu)$ for any generator $\widetilde{w}(\beta,\mu)$ of the group $\widetilde{\Gamma}$. It means that $\widetilde{g} \in Z\left(\widetilde{\Gamma}\right)$.

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4. Preparatory results 2. The graph construction

The following graph construction was called to my attention by Victor Kulikov. We define the graph $Gr\left(\widetilde{\Gamma}\right)$ to be a 1–dimensional CW-complex with the set $W\left(\widetilde{\Gamma}\right)$ as the vertex set and the set of edges (1-cells), corresponding to the set of (ii)-type relators. This means that we join two vertices by an edge iff the corresponding elements of $W\left(\widetilde{\Gamma}\right)$ satisfiy (ii). It is clear that the same construction defines the graph $Gr(\Gamma)$ and that $Gr\left(\widetilde{\Gamma}\right) \simeq Gr(\Gamma)$ (isomorphism of graphs).

We call the generator $\widetilde{w}(\alpha,\zeta)$ primitive if it is not a power of any other generator. It is clear that the graph $Gr\left(\widetilde{\Gamma}\right)$ has a finite number of connected components. We call the component Gr^0 primitive if any generator $\widetilde{w}(\alpha,\zeta) \in Gr^0$ is primitive. Then the following lemma is easily seen to hold.

Lemma 2. *Let* $m_1,\cdots,m_k$ *be the orders of the elements of all the primitive components of* $Gr\left(\widetilde{\Gamma}\right)$. *Then*

$$\widetilde{\Gamma}^{ab} = \Gamma^{ab} = \bigoplus_{i=1}^{k} \mathbb{Z}/m_i\mathbb{Z}.$$

Remark 2. Every path in the graph $Gr(\Gamma)$ gives us a relation in the group Γ which is derivable from defining relators (ii).

5. Preparatory results 3. The good presentation

We say that the group Γ has a good presentation if it is generated by reflections $w_1,\cdots,w_N$ with defining relators of the following types:

a. $w_i^{r_i} = 1$, $i = 1,\cdots,N$
b. $w_i w_j w_i^{-1} = w_k$ (for some i,j,k)

c. $w_p \rightleftarrows R(w_1, \cdots, w_N)$ (a generator w_p commutes with some element $R(w_1, \cdots, w_N)$ of Γ)

d. $\underbrace{w_p w_q w_p \cdots}_{n_{pq}} = \underbrace{w_q w_p w_q \ldots}_{n_{pq}}$ for some p, q.

Lemma 3. *The relators of the types (a), (b), (c), (d) are derivable from relators (i), (ii).*

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It is clear that we must check only the types (c) and (d). But in the first case we can write (c) as $R\ w_p R^{-1} = w_p$ and in the second - rewrite (d) as

$$\underbrace{\cdots w_p w_q w_p w_q^{-1} w_p^{-1} \cdots}_{n_{pq}} = \underbrace{\cdots w_q^{-1} w_p^{-1} w_q w_p w_q \cdots} \quad (n_{pq} \text{ is odd})$$

$$\underbrace{\cdots w_p^{-1} w_q^{-1} w_p w_q w_p \cdots}_{n_{pq}} = \underbrace{\cdots w_p w_q w_p w_q^{-1} w_p^{-1} \cdots}_{n_{pq}+1} \quad (n_{pq} \text{ is even})$$

In both cases we see that the relators (c) and (d) give us the path in the graph $Gr(\Gamma)$ and therefore are derivable from (i) and (ii) by Remark 2.

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6. Preparatory results 4. The Schur multiplier

Let $M(\Gamma)$ be the Schur multiplier of the group Γ. We will need the following result.

Lemma 4 ([4]). *Let H be any group, A a central subgroup of finite index and let $G = H/A$. Then the group $[H, H] \cap A$ is a homomorphic image of $M(\Gamma)$.*

Now we present the partial proof of the Theorem.

Proposition 1. *If the group $M(\Gamma)$ is trivial, then $\widetilde{\Gamma} = \Gamma$.*

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To prove, let us apply the result of Lemma 4 to the central extension $1 \to A \to \widetilde{\Gamma} \xrightarrow{\varphi} \Gamma \to 1$ (Lemma 1). Then we have $A \cap \left[\widetilde{\Gamma}, \widetilde{\Gamma}\right] = \{1\}$. But by Lemma 2 we should have $A \subset \left[\widetilde{\Gamma}, \widetilde{\Gamma}\right]$. Therefore $A = \text{Ker}\, \varphi = \{1\}$.

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7. The proof of the Theorem

Case 1: Coxeter groups.

The Coxeter group is well known to have the good presentation ([1]), hence the Theorem follows by Lemma 3.

Case 2: Shephard-Todd primitive groups.

The result of the paper [3] shows that the group $M(\Gamma) = \{1\}$ for all the groups Γ but the groups NN 5, 7, 9, 11, 13, 15, 24, 27, 29, 31. Therefore according the Proposition 1 we have only to check this exceptional groups.

Case 3: Exceptional groups.

The groups NN 5, 9, 24, 29, 31 have the good presentations (see [1], [6], [7]).

The groups NN 7, 11 have the presentation ([2])

generators: reflections w_1, w_2, w_3
relators: $w_1^{n_1} = w_2^{n_2} = w_3^{n_3} = 1,\ w_1w_2w_3 = w_3w_1w_2 = w_2w_3w_1$

But the relator $w_1w_2w_3 = w_3w_1w_2$ we can present in the form $w_1^{-1}w_3w_1 = w_2w_3w_2^{-1}$, where it obviously comes as the consequence of (ii).

The group N 13 is generated by three reflections w_1, w_2, w_3 and has the presentation ([2]):

$$\Gamma = \langle w_1, w_2, w_3 | w_1^2 = w_2^2 = w_3^2 = 1,\ \underbrace{w_1w_2\cdots}_{6} = \underbrace{w_2w_1\cdots}_{6},$$
$$\underbrace{w_1w_3\cdots}_{8} = \underbrace{w_3w_1\cdots}_{8},\ w_1w_3w_1w_3 = \underbrace{w_2w_3w_2w_3\cdots}_{8} \rangle$$

Now we can write the last relator as $w_1w_3w_1^{-1} = w_2w_3w_2w_3w_2^{-1}w_3^{-1}w_2^{-1}$ and then argue as above.

The group N 15 is generated by three reflections w_1, w_2, w_3 with defining relators ([2]): $w_1^2 = w_2^2 = w_3^2 = 1$,

$$\underbrace{w_2w_3\cdots}_{6} = \underbrace{w_3w_2\cdots}$$
$$w_2w_1w_3 = w_1w_3w_2$$
$$w_3w_1w_3w_1w_2 = w_1w_2w_3w_1w_3.$$

It is easy to see the relevant form of two last relators to be $w_1^{-1}w_2w_1 = w_3w_2w_3^{-1}$ and $w_3w_1w_3w_1^{-1}w_3^{-1} = w_1w_2w_3w_2^{-1}w_1^{-1}$ respectively.

We have worked out the last case and hence completes the proof of the Theorem.

Final remark. According to the presentation given in [7], the Theorem still holds for the imprimitive group $G(m, 1, n)$.

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