NEW COMPONENT OF THE MODULI SPACE M(2;0,3) OF STABLE VECTOR BUNDLES ON THE DOUBLE SPACE P^3 OF INDEX TWO

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Introduction

In 1998 D.Markushevich and the author found a first relation between the moduli spaces of stable vector bundles on Fano threefolds and their intermediate Jacobians (see [MT-1]). They investigated the case of a cubic hypersurfuce, and their results were developed in further works [MT-2],[IM],[D],[B]. In this paper we are interested in the case of the quartic double solid X, the next after cubic in the series of Fano varieties of index 2 with nontrivial intermediate Jacobian. We consider stable rank-2 vector bundles on X. Let W be a smooth quartic surface in $\mathbf{P}^3$ and $\pi : X \to \mathbf{P}^3$ the double covering ramified in W. (It is clear that X is defined by the embedding $W \hookrightarrow \mathbf{P}^3$ uniquely up to an isomorphism.) We study the moduli scheme $M(2; 0, 3)$ of rank-2 stable vector bundles with Chern classes $c_1 = 0, c_2 = 3$ on X. We show that there is a distinguished component M in $M_X(2; 0, 3)$, of the prescribed dimension 9, obtained via the Serre construction from the family H of smooth elliptic quintics on X. It is proved that the Stein factorization of the Abel-Jacobi map of H into the intermediate Jacobian $J(X)$ of X gives a quasi-finite (probably birational) morphism of (an open part of) M to (a translate of) the Poincare theta divisor Θ of $J(X)$. As a byproduct we obtain a new parametrisation of Θ via Abel-Jacobi image of the family of elliptic quintics on X (the known parametrisation of Θ via sextics of genus 3 was found in [T-1]).

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1. Construction of the theta divisor of the double space X via the family of elliptic quintics on X

1.1. **Notations.** We adopt the standard notation for the degree $\deg(C)$ of a curve C in X as the degree $\deg(C) := \deg(\mathcal{O}_X(1)|C)$, where $\mathcal{O}_X(1)$ is the positive generator of $Pic(X)$.

Let

- $H^* := \{C \in \mathrm{Hilb}(X) |\ C$ is a smooth irreducible elliptic quintic curve in $X\}$,
- H closure of H^* in $\mathrm{Hilb}(X)$.

By a standard parameter counting one proves the following

Lemma 1.2. *H is nonempty and equidimensional of dimension 10.*

For any irreducible component H_0 of H a general curve $C \in H_0$ lies in a unique $K3$-surface $S(C)$ of the linear series $|\mathcal{O}_X(2)|$ and satisfies the condition that $\pi : C \to \pi(C)$ is an isomorphism.

We also can formulate here

Conjecture 1.3. *H is irreducible.*

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Now let H_0 be any fixed irreducible component of H and denote

- $H_{6,4}(\mathbf{P}^3) := \{C \in \mathrm{Hilb}(\mathbf{P}^3) |\ C$ is a smooth complete intersection curve of a cubic and a quadric surfaces in $\mathbf{P}^3\}$,
- $H_{12,19} := \{C \in \mathrm{Hilb}(X) |\ C = \pi^{-1}(C_0), C_0 \in H_{6,4}(\mathbf{P}^3)\}$,
- $\mathcal{H} := \{C \in \mathrm{Hilb}(X) |\ ={}_1 \cup C_2,$ where $C_1 \in H_0,\ C_2 \in H_{12,19},\ \pi : C_1 \to \pi(C_1)$ is an isomorphism, and C_1 and C_2 intersect transversally at 10 points$\}$.

Remark 1.4. Clearly, $H_{12,19} \simeq H_{6,4}$ is a rational irreducible variety, and any curve $C \in H_{12,19}$ has degree 12 and arithmetic genus 19. Respectively, any curve $C \in \mathcal{H}$ has degree 17 and arithmetic genus 28.

We have a natural embedding $\mathcal{H} \hookrightarrow H_0 \times H_{12,19} : C = {}_1 \cup C_2 \mapsto (C_1, C_2)$, and lemma 1.2 together with an easy dimension count implies

$$\overline{pr_1(\mathcal{H})} = \overline{H_0}. \tag{1}$$

Convention: for any subvariety $\mathcal{B}$ of the Hilbert scheme of curves on X we denote by $\Phi_{\mathcal{B}} : \mathcal{B} \to J(X)$ the Abel-Jacobi map (defined uniquely up to a choice of the reference point in $\mathcal{B}$).

The remark 1.4 above shows immediately that

$$\Phi_{H_{12,19}}(H_{12,19}) = \{\mathrm{point}\}. \tag{2}$$

Let $\mathcal{F}$ be the Fano surface of X, i.e. the base of the family of lines on X. (This is a smooth irreducible surface, see, e.g., [T-2] or [We].)

Lemma 1.5. *(i) For any curve $C \in \mathcal{H}$, $h^0(\mathcal{J}_{C,X}(4)) = 4$.*

(ii) For a general point $(l, C_1) \in \mathcal{F} \times H_0$ there exists a curve $C = C_1 \cup C_2 \in \mathcal{H}$ such that $C \cap l = \emptyset$ and there exists a surface $S \in |\mathcal{O}_X(4)|$ containing the curve $C \cup l$.

Proof. (i) We have $C = C_1 \cup C_2$, where $Z := C_1 \cap C_2 = \{x_1, ..., x_{10}\}$, $C_1 \in H_0$, $\pi : C_1 \to C_{10} := \pi(C_0)$ is an isomorphism, and $C_2 = \pi^{-1}(C_{20})$, where $C_{20} \in H_{6,4}$. This means, in particular, that C_{20} lies in a uniquely defined quadric, say, Q. Let $Z_0 := \pi(Z)$, $\deg(Z_0) = \deg(Z) = 10$. Since by construction $C_{10} \cap Q \supset Z_0$ (as a scheme-theoretic inclusion) and $\deg(C_{10} \cap Q) = 10 = \deg(Z_0)$, it follows that

$$Z_0 = C_{10} \cap Q. \tag{3}$$

Using (3) and the fact that $\mathcal{O}_Q(C_{20}) = \mathcal{O}_Q(3)$ one derives the exact triple:

$$0 \to \mathcal{J}_{Q \cup C_{10}, \mathbf{P}^3}(4) \to \mathcal{J}_{C_{10} \cup C_{20}, \mathbf{P}^3}(4) \to \mathcal{O}_Q(1) \to 0. \tag{4}$$

Next, take any smooth cubic surface $S \subset \mathbf{P}^3$ through the smooth quintic curve C_{10}. We have the exact triple $0 \to \mathcal{J}_{Q \cup S, \mathbf{P}^3} \to \mathcal{J}_{Q \cup C_{20}, \mathbf{P}^3} \to \mathcal{J}_{(S \cap Q) \cup C_{10}, S} \to 0$. Here $\mathcal{J}_{(S \cap Q) \cup C_{10}, S} \simeq \mathcal{O}_S(-2)(-C_{10})$ since S is smooth, hence twisting the above triple by $\mathcal{O}_{\mathbf{P}^3}(4)$ we obtain:

$$0 \to \mathcal{O}_{\mathbf{P}^3}(-1) \to \mathcal{J}_{Q \cup C_{20}, \mathbf{P}^3}(4) \to \mathcal{O}_S(2)(-C_{10}) \to 0. \tag{5}$$

One quickly checks that the natural map $H^0(\mathcal{O}_S(2)) \to H^0(\mathcal{O}_{C_{10}}(2))$ is surjective, so that $h^1(\mathcal{O}_S(2)(-C_{10})) = 0$. Besides, $h^1(\mathcal{O}_{\mathbf{P}^3}(-1) = 0$. Hence (5) implies $h^1(\mathcal{J}_{Q \cup C_{10}, \mathbf{P}^3}(4)) = 0$. This together with (4) and the equality $h^1(\mathcal{O}_Q(1)) = 0$ gives

$$h^1(\mathcal{J}_{C_{10} \cup C_{20}, \mathbf{P}^3}(4)) = 0. \tag{6}$$

Hence the exact triple $0 \to \mathcal{J}_{C_{10}\cup C_{20},\mathbf{P}^3}(4) \to \mathcal{J}_{C_{20},\mathbf{P}^3}(4) \to \mathcal{J}_{Z_0,C_{10}}(4) \to 0$ implies the surjective map

$$H^0(\mathcal{J}_{C_{20},\mathbf{P}^3}(4)) \twoheadrightarrow H^0(\mathcal{J}_{Z_0,C_{10}}(4)). \tag{7}$$

Now since $\pi : C_1 \to C_{10}$ is an izomorphism, we have $\pi_*\mathcal{J}_{Z,C_1}(4) \simeq \mathcal{J}_{Z_0,C_{10}}(4)$, this giving the exact triple

$$0 \to \mathcal{J}_{Z_0,C_{10}}(4) \to \pi_*\mathcal{O}_C(4) \to \pi_*\mathcal{O}_{C_2}(4) \to 0. \tag{8}$$

Whence one quickly obtains:

$$h^0(\pi_*\mathcal{O}_C(4)) = \chi(\mathcal{O}_C(4)) = 40. \tag{9}$$

Next, consider the composition of natural (restriction) maps: $g : \pi_*\mathcal{O}_X(4) \twoheadrightarrow \pi_*\mathcal{O}_C(4) \twoheadrightarrow \pi_*\mathcal{O}_{C_2}(4)$. Here $\ker(g) = \pi_*\mathcal{J}_{C_2,X}(4)$, so passing to the natural (restriction) map of sections

$$f : H^0(\pi_*\mathcal{O}_X(4)) \to H^0(\pi_*\mathcal{O}_C(4)), \tag{10}$$

we obtain:

$$\mathrm{coker}(f) = \mathrm{coker}(H^0(\pi_*\mathcal{J}_{C_2,X}(4)) \stackrel{res}{\to} H^0(\mathcal{J}_{Z_0,C_{10}}(4))). \tag{11}$$

Next, by the projection formula $\pi_*\mathcal{J}_{C_2,X}(4) = \pi_*\pi^*\mathcal{J}_{C_{20},\mathbf{P}^3}(4) = \pi_*\pi^*\mathcal{J}_{C_{20},\mathbf{P}^3}(4) \oplus \pi_*\pi^*\mathcal{J}_{C_{20},\mathbf{P}^3}(2)$, and the composition $H^0(\pi_*\pi^*\mathcal{J}_{C_{20},\mathbf{P}^3}(4)) \hookrightarrow H^0(\pi_*\mathcal{J}_{C_2,X}(4)) \stackrel{res}{\to} H^0(\mathcal{J}_{Z_0,C_{10}}(4))$ coincides with the surjective map (6), so (11) implies $\mathrm{coker}(f) = 0$. This together with (9) and the equality $h^0(\mathcal{O}_X(4)) = h^0(\mathcal{O}_{\mathbf{P}^3}(4)) \oplus h^0(\mathcal{O}_{\mathbf{P}^3}(2)) = 45$ gives:

$$H^0(\mathcal{J}_{C,X}(4)) = H^0(\pi_*\mathcal{J}_{C,X}(4)) = \ker(f) = k^5. \tag{12}$$

(ii) This is a standard dimension count. For simplicity one could look for the surface S with the prescribed properties, e.g., in the linear subseries $\pi^*|\mathcal{O}_{\mathbf{P}^3}(4)|$ of $|\mathcal{O}_X(4)|$. Lemma is proved.

□

Now we invoke the following important technical statement due to G.Welters [We, Prop. 6.17].

Proposition 1.6. *Let T be a smooth irreducible (not necessarily complete) variety parametrizing a family $\{C_t | t \in T\}$ of reduced curves of degree $d = k(k+1) - 3$ on X with the following properties:*

(i) The Abel-Jacobi map $\Phi_T : T \to J(X)$ is dominating;

(ii) for all $t \in T$ the linear system of surfaces $|\mathcal{J}_{C_t,X}(k)|$ has dimension k and the surfaces of this linear system don't contain all lines of X;

(iii) the incidence divisor between T and the family of lines $\mathcal{F}$ (the Fano surface) of X is reduced.

Then for any line $l \in \mathcal{F}$ the subvariety of T defined as $T_l = \{t \in T | C_t \cap l = \emptyset\}$ is mapped into (a translate of) the theta divisor Θ of $J(X)$.

Now take the closure $\tilde{T}$ of $\mathcal{H}$ in $\mathrm{Hilb}(X)$ with a reduced structure, i.e. as an irreducible variety, and let $T = \tilde{T} \setminus \mathrm{Sing}(\tilde{T})$ be its smooth open part. In view of the statement (i) of lemma 1.5 we can apply the above proposition to T with $k = 4,\ d = 17$. (Here we possibly remove from T a proper subset so that the condition (iii) on the incidence divisor is satisfied.) We thus obtain $\Phi_T(T) \subset \Theta + \mathrm{const}$, hence also

$$\Phi_{\tilde{T}}(\tilde{T}) \subset \Theta + \mathrm{const}. \tag{13}$$

As a corollary we have

$$\Phi_{\mathcal{H}}(\mathcal{H}) \subset \Theta + \text{const.} \tag{14}$$

Now remark that, since by construction $\mathcal{H} \subset H_0 \times H_{12,19}$, hence for any point $C = (C_1, C_2) \in \mathcal{H}$ we have $\Phi_{\mathcal{H}}(C) = \Phi_{H_0}(C_1) + \Phi_{H_{12,19}}(C_2)$. Thus,

$$\Phi_{\mathcal{H}}(\mathcal{H}) \subset \Phi_{H_0}(H_0) + \Phi_{H_{12,19}}(H_{12,19}). \tag{15}$$

Now (2) together with (14) and (15) gives the inclusion:

$$\Phi_{H_0}(H_0) \subset \Theta + \text{const.} \tag{16}$$

Next, the description of the differential of the composition $H_0 \stackrel{\Phi_{H_0}}{\to} \Phi_{H_0}(H_0) \stackrel{\gamma}{\to} \mathbf{P}^9 = P(T_{J(X)}(0))$, (where γ is the Gauss map) given in [We], we quickly obtain in view of lemma 1.2 that $\Phi_{H_0}(H_0)$ is a divisor in $J(X)$. Hence, since Θ is irreducible, (16) implies the following

Theorem 1.7. $\Phi_{H_0}(H_0) = \Theta + \text{const.}$

2. H.Clemens' approach to the description of the Abel-Jacobi map of the family of elliptic quintics on X

Let $\{F = 0\}$ be the equation of a smooth branch quartic W of the double covering $\pi : X \to \mathbf{P}^3$ and let Q be a genericaly chosen quadric in $\mathbf{P}^3$ with equation $\{G = 0\}$, so that $D = W \cap Q$ is a smooth curve. We consider a family $\pi_t : X_t \to \mathbf{P}^3$ of double solids defined by the branch quartics of the pencil $W_t = \{F + tG^2 = 0\}$. Thus W_t degenerate at $t = 0$ to $2Q$, and the double solids X_t degenerate to the reducible threefold X_0 which is the union of two copies, say $\mathbf{P}'$ and $\mathbf{P}''$, of $\mathbf{P}^3$ intersecting along the quadric Q. Moreover, according to H.Clemens [C-1],[C-2] the corresponding family of intermediate Jacobians X_t has Neron model $\mathbf{J}$ with fibre over $t = 0$ given by $J(X_0)$, where $J(X_0)$ is made up of two copies of a $\mathbf{C}^*$-extension J_0 of the Jacobian $J(D)$. The extension data that characterizes J_0 is given by $(L \cdot D) - (M \cdot D)$, where L and M are the two rulings on Q. Now one makes $\mathbf{J}$ proper over the t-line $\mathbf{A}^1$ by pasting in two copies of $J(D)$ in the fiber over $t = 0$. Denote by $\bar{\mathbf{J}}$ this compacified Neron model.

Following the arguments of H.Clemens [C-1],[C-2], we consider the family H_t of smooth elliptic quintics on X_t. Every smooth elliptic quintic that lies in X_0 must lie in one of the two copies $\mathbf{P}'$ and $\mathbf{P}''$ of $\mathbf{P}^3$. Since curves that are not locally complete intersection of X_0 with a surface (inside the total space of the family $\{X_t\}$) cannot deform, the image in $\mathbf{P}^3$ of any curve from Z_0 must meet D in 10 points. Next, any continuous choice of linear equivalence class $\{C_t\}$ in the "carrier" $K3$-surface $S(C_t)$ (see notations of lemma 1.2) gives a (possibly multivalued) normal function $\nu(t)$ near $t = 0$ with values in $J(X_t)$.

We are going to reprove theorem 1.7 and at the same time obtain some additional piece of information about the fibers of the Abel-Jacobi map Φ_{H_t}. For this, we follow the divisor $\Phi_{H_t}(H_t)$ as t approaches zero, and we show that there is a divizor $\boldsymbol{\mathcal{D}}$ in $\bar{\mathbf{J}}$ such that

(i) $\boldsymbol{\mathcal{D}}$ gives $\Phi_{H_t}(H_t)$ for $t \neq 0$,

(ii) $\boldsymbol{\mathcal{D}}$ cuts the two components of the compactification of $J(X_0)$ in divisors Σ' and Σ'' respectively, such that exactly one of the two, say Σ', is reduced and projects birationally onto $J(D)$, while the other projects into a proper subvariety of $J(D)$. (Here inside the fiber $J(X_t)$ we have a real-dimensional two-torus, which represents the homology class given by the Pontryagin product of of the vanishing cycle and a transverse one-cycle and degenerates to a curve made up of the closure of the union of a $\mathbf{C}^*$-fiber from each component of $J(X_0)$. The condition (ii) ensures that the intersection number of $\boldsymbol{\mathcal{D}}$ with this torus is exactly *one*.)

Now copying the argument of H.Clemens from [C-2, pp.96-97] (applied to the families H_t of elliptic quintics on X_t's instead of the families of sextics of genus 3 treated therein) we see that, to check (i)–(ii), we need to show the following for the family $H_t = \{C_t\}$ of elliptic quintics:

(iii) Given a generic divisor Δ of degree 10 on the curve D, there is a unique elliptic quintic C_0 that meets D at the points of Δ.

We are going to check the condition (iii). For this we need to do some preparation. First introduce some more notation. Let $\mathrm{Hilb}^n\mathbf{P}^3$ be the Hilbert scheme of 0-dimensional subschemes of length n of $\mathbf{P}^3$, W_n its irreducible open subscheme consisting of reduced subschemes (i.e. of n-ples of distinct points in $\mathbf{P}^3$) and $\bar{W}_n$ its closure in $\mathrm{Hilb}^n\mathbf{P}^3$. Consider the closed subset $\mathbf{B} = \{\Delta \in W_{10} | \text{there exists a quadric in } \mathbf{P}^3 \text{ containing } \Delta\}$ of W_{10}. Clearly,

(a) the closure $\bar{\mathbf{B}}$ of $\mathbf{B}$ in $\bar{W}_{10}$ is an irreducible divisor in $\bar{W}_{10}$, and

(b) $\bar{\mathbf{B}} \not\subset \mathrm{Sing}\,\bar{W}_{10}$.

Next let $\mathcal{Y}$ be the open subset of $\mathrm{Hilb}\mathbf{P}^3$ consisting of smooth irreducible elliptic quintics in $\mathbf{P}^3$. This is an irreducible 20-dimensional variety (see, e.g., [H-H, Prop. 6.1]). Let $\bar{\mathcal{Y}}$ be the closure of $\mathcal{Y}$ in $\mathrm{Hilb}\mathbf{P}^3$. The following result is apparently well known, but we give here its proof because of the lack of an appropriate reference.

Lemma 2.1. *Through a general set of points $\Delta \in W_{10}$ there pass a finite number of quintics from $\bar{\mathcal{Y}}$.*

Proof. Take a reducible elliptic quintic curve $C = C_0 \cup l$ consisting of a line l and s a smooth elliptic quartic C_0:

$$C_0 = Q_1 \cap Q_2, \tag{17}$$

where Q_1, Q_2 are smooth quadrics in $\mathbf{P}^3$ and where l and C_0 intersect transversally at a point, say, x_0. Clearly, $C \in \bar{\mathcal{Y}}$; besides, by (32)

$$N_{C_0/\mathbf{P}^3} \simeq \mathcal{O}_{C_0}(2) \oplus \mathcal{O}_{C_0}(2), \quad N_{l/\mathbf{P}^3} \simeq \mathcal{O}_l(1) \oplus \mathcal{O}_l(1). \tag{18}$$

As it is known (see, e.g., [H-H, Prop.3.1]), the restricted normal bundle $N_{C/\mathbf{P}^3}|C_0$ (respectively, $N_{C/\mathbf{P}^3}|l$) is an elementary transform of $N_{C_0/\mathbf{P}^3}$ (respectively, of $N_{l/\mathbf{P}^3}$) by means of the skyscraper sheaf $k(x_0)$, so that (18) imply:

$$N_{C/\mathbf{P}^3}|C_0 \simeq \mathcal{O}_{C_0}(2)(x_0) \oplus \mathcal{O}_{C_0}(2), \quad N_{C/\mathbf{P}^3}|l \simeq \mathcal{O}_l(1) \oplus \mathcal{O}_l(2). \tag{19}$$

Now the result of gluing of these locally free sheaves on components of C into the sheaf $N_{C/\mathbf{P}^3}$ leads evidently to one of the possibilities:

(I) $N_{C/\mathbf{P}^3} \simeq \mathcal{E}_1 \oplus \mathcal{E}_2$, where $\mathcal{E}_1|C_0 = \mathcal{O}_{C_0}(2)(x_0)$, $\mathcal{E}_1|l \simeq \mathcal{O}_l(1)$, $\mathcal{E}_2|l \simeq \mathcal{O}_{C_0}(2)$, $\mathcal{E}_2|l \simeq \mathcal{O}_l(2)$, or

(II) $N_{C/\mathbf{P}^3} \simeq \mathcal{E}_1 \oplus \mathcal{E}_2$, where $\mathcal{E}_1|C_0 = \mathcal{O}_{C_0}(2)(x_0)$, $\mathcal{E}_1|l \simeq \mathcal{O}_l(2)$, $\mathcal{E}_2|l \simeq \mathcal{O}_{C_0}(2)$, $\mathcal{E}_2|l \simeq \mathcal{O}_l(1)$.

Let Q be the quadric of the pencil spanned by Q_1 and Q_2 that touches the line l at the point x_0, and let $\mathbf{P}^2_{x_0}$ be the projective plane in $\mathbf{P}^3$ touching Q at x_0. Take a simple effective divisor D' of degree 8 on C_0 (i.e. a set of 8 distinct points on C_0), respectively, a pair D'' of distinct points on l such that

$$\mathcal{O}_{C_0}(D') \neq \mathcal{O}_{C_0}(2), \quad x_0 \notin D'', \tag{20}$$

and consider $\Delta = D' \cup D'' \in W_{10}$ as a divisor on C. We show now that the case (II) above is impossible. In fact, this case would mean that there exists a section $0 \neq s \in H^0(\mathcal{E}_1(-D)) \subset H^0(N_{C/\mathbf{P}^3}(-D))$ such that the corresponding infinitesimal deformation C_s of the curve C would consist of two components C'_s and C''_s corresponding to the restrictions of the section s onto C_0 and l respectively, where by construction $C'_s \subset Q$, $C''_s \subset \mathbf{P}^2_{x_0}$, and in view of (20) $x_0 \notin C'_s \cup C''_s$. Since clearly $l \not\subset Q$ (for, otherwise, C wouldn't belong to $\bar{\mathcal{Y}}$), it follows that $C'_s \cap C''_s = \emptyset$, contrary to the fact that $C_s' \cup C_s''$ connected as a deformation of C. Thus the case (II) is excluded. Now in view of (20) an easy computation shows that in case (I) $h^0(\mathcal{E}_1(-D)) = h^0(\mathcal{E}_2(-D)) = 0$, hence also $h^0(N_{C/\mathbf{P}^3}(-D)) = 0$. This precisely means that through Δ there passes a finite set of curves from $\bar{\mathcal{Y}}$ (including the curve C itself). This property is open by an easy dimension count, hence the same is true for a general point $\Delta \in W_{10}$. $\square$

Since $\bar{\mathcal{Y}}$ is irreducible, from this lemma and the above properties (a) and (b) of the variety $\mathbf{B}$ follows

Corollary 2.2. *Through a general set of points $\Delta \in \mathbf{B}$ there pass a finite number of quintics from $\bar{\mathcal{Y}}$, and these quintics are smooth.*

Now consider in $W_5 \times W_5$ the open subset $\mathcal{U} = \{(\mathcal{P}, \mathcal{P}') \in W_5 \times W_5 | \mathcal{P} \cap \mathcal{P}' = \emptyset\}$ and let $\phi : \mathcal{U} \to W_{10}$ be the natural map $(\mathcal{P}, \mathcal{P}') \mapsto \mathcal{P} \cup \mathcal{P}'$. Denote $\mathcal{B} = \phi^{-1}(\mathbf{B})$. Since $\mathbf{B}$ is irreducible, $\mathcal{B}$ is irreducible as well. Let $\mathcal{B}^*$ be the open subset of $\mathcal{B}$ consisting of points $z = (\mathcal{P}, \mathcal{P}')$ such that:

(1) $\Delta = \mathcal{P} \cup \mathcal{P}'$ lies on a unique smooth quadric Q and the set of curves from $\bar{\mathcal{Y}}$ passing through Δ is finite and consists of smooth elliptic quintics;

(2) take any such elliptic quintic C through Δ and consider the linear series $|\mathcal{J}_{C,\mathbf{P}^3}(3)| \simeq \mathbf{P}^4$ and its restriction $res_Q(\mathbf{P}^4)$ onto Q. (By construction, $res_Q(\mathbf{P}^4) \subset |\mathcal{J}_{\Delta,Q}(3,3)| \simeq \mathbf{P}^5$; next, for each point $x \in \mathcal{P}$ consider the linear subseries $\mathbf{P}^2_x = \{X \in res_Q(\mathbf{P}^4) | x \in \operatorname{Sing} X\}$, take the Grassmanian $G = Gr(1, res_Q(\mathbf{P}^4))$ and let $M_x = \{\mathbf{P}^1 \in G | \mathbf{P}^1 \cap \mathbf{P}^2_x \neq \emptyset\}$; here clearly $M_z := \bigcap_{x \in \mathcal{P}} M_x \neq \emptyset$; now for any pencil of curves $\mathbf{P}^1 \in M_z$ on Q let $B(\mathbf{P}^1)$ be its base-point set (by construction this set contains Δ as a subscheme); the open condition that we impose on z is:

$$length_x(B(\mathbf{P}^1)) = 2, \quad x \in \mathcal{P}, \tag{21}$$

for a general point $\mathbf{P}^1$ of any irreducible component of M_z. (Here by the length of a 0-dimensional subscheme of Q at a given point x we denote the length of its subscheme with a support at x.)

One can quickly find a point $z \in \mathcal{U}$ satisfying the condition (2) above; respectively, the condition (1) above is also satisfied by the general point of $\mathcal{U}$ because of the corollary 2.2. Hence $\mathcal{B}^*$ is dense open in $\mathcal{B}$.

Now consider the set $\mathcal{T}$ of smooth irreducible rational quartic curves in $\mathbf{P}^3$. This is an irreducible 16-dimensional open subset of Hilb$\mathbf{P}^3$ (see, e.g. [H-H, Prop. 6.1]). Let $\bar{\mathcal{T}}$ be its closure in Hilb$\mathbf{P}^3$. Fix a point $z = (\mathcal{P}, \mathcal{P}') \in \mathcal{B}^*$, where $\Delta = \mathcal{P} \cup \mathcal{P}'$ lies on a smooth quadric Q, and let $\mathcal{C}_\Delta = \{C_1, ..., C_N\}$ be the set of all quintics from $\bar{\mathcal{Y}}$ passing through Δ (these quintics all being smooth by the definition of $\mathcal{B}^*$). Next, consider the set $\mathcal{T}_\mathcal{P} = \{C \in \mathcal{T} | C \supset \mathcal{P}\}$ and its open subset $\mathcal{T}^*_\mathcal{P} = \{C \in \mathcal{T}_\mathcal{P} | \operatorname{mult}_x(C \cap Q) = 1 \text{ for each } x \in \mathcal{P}\}$, respectively, its closure $\bar{\mathcal{T}}_\mathcal{P}$ in $\bar{\mathcal{T}}$. Since each smooth rational quartic $C \in \mathcal{T}$ lies on a unique quadric which is smooth, it follows immediately that $\bar{\mathcal{T}}_\mathcal{P}$ is an irreducible 6-dimensional variety and $\mathcal{T}^*_\mathcal{P}$ is dense open in $\mathcal{T}_\mathcal{P}$.

Consider the two linear series $\mathbf{P}^5_\Delta = |\mathcal{J}_{\Delta,Q}(3,3)|$, $\mathbf{P}^9_\Delta = |\mathcal{J}_{\Delta,\mathbf{P}^3}(3)|$ and the corresponding Grassmanians $Gr = Gr(1, \mathbf{P}^5_\Delta)$, $Gr' = Gr(1, \mathbf{P}^9_\Delta)$. The natural rational restriction map $res_Q : \mathbf{P}^9_\Delta \dashrightarrow \mathbf{P}^5_\Delta : S \mapsto S \cap Q$ induces the corresponding restriction map $res_Q : Gr' \dashrightarrow Gr$ and thus the rational composition map

$$f: \ \mathcal{T}_\mathcal{P} \xrightarrow{\psi} Gr' \overset{res_Q}{\dashrightarrow} Gr : C \mapsto \mathbf{P}^1(C) := |\mathcal{J}_{C \cup \mathcal{P}',\mathbf{P}^3}(3)| \overset{res_Q}{\mapsto} \mathbf{P}^1(Q). \tag{22}$$

(here ψ is clearly an embedding). Next, for each $x \in \mathcal{P}$ we consider the linear subseries $\mathbf{P}^3_x = \{C \in \mathbf{P}^5_\Delta | x \in \operatorname{Sing} C\}$, and let $L_x = \{\mathbf{P}^1 \in Gr | \mathbf{P}^1 \cap \mathbf{P}^3_x \neq \emptyset\}$ be the corresponding hyperplane section of Gr. Denote $\mathcal{C}_{\mathcal{P}} = f(\mathcal{T}) \cap (\bigcap_{x\in\mathcal{P}} L_x)$, $\mathcal{C}^*_{\mathcal{P}} = f(\mathcal{T}^*) \cap (\bigcap_{x\in\mathcal{P}} L_x)$ and let $\mathcal{X}_{\mathcal{P}}$ be the closure in Gr' of the set $f^{-1}(\mathcal{C}_{\mathcal{P}})$. Since $res_Q : Gr' \dashrightarrow Gr$ is a linear rational projection, $\mathcal{X}_{\mathcal{P}}$ is an intersection of 5 hyperplane sections of Gr', hence it is connected. Besides, since ψ is an embedding, we can identify $\mathcal{X}_{\mathcal{P}}$ with its preimage in $\bar{\mathcal{T}}_{\mathcal{P}}$ under ψ. Consider the map $g : \mathcal{X}_{\mathcal{P}} \to \bar{\mathcal{Y}} : C \mapsto \mathbf{P}^1(C) \mapsto F \cap F' \smallsetminus C$ for any two cubics $F, F' \in \mathbf{P}^1(C)$. By construction, for a general point C of any irreducible component of $\mathcal{X}_{\mathcal{P}}$ we have $g(C) \in \mathcal{C}_\Delta$; hence, since $\mathcal{X}_{\mathcal{P}}$ is connected, $g(\mathcal{X}_{\mathcal{P}}) \subset \mathcal{C}_\Delta$, i.e. it is a finite set. Again, since $\mathcal{X}_{\mathcal{P}}$ is connected, $g(\mathcal{X}_{\mathcal{P}})$ is a point:

$$g(\mathcal{X}_{\mathcal{P}}) = \{C'\}, \quad C' \in \mathcal{C}_\Delta \subset \mathcal{Y}. \tag{23}$$

Now show that, moreover,

$$g(\mathcal{X}_{\mathcal{P}}) = \mathcal{C}_\Delta. \tag{24}$$

In fact, take any smooth elliptic quintic $C' \in \mathcal{C}_\Delta$ and, in notations of (21) pick a general point $\mathbf{P}^1$ of some irreducible component of M_z satisfying (21), where z is our point $(\mathcal{P}, \mathcal{P}')$ fixed above. Let $C = F \cap F' \smallsetminus C'$. By construction, $C' = g(C)$, and we are done. Now (23) and (24) imply that $\mathcal{C}_\Delta = \{C'\}$. We thus have proved

Proposition 2.3. *For a general quadric Q in $\mathbf{P}^3$ and a general 10-ple of points $\Delta \subset Q$ there is a unique smooth elliptic quintic curve passing through Δ.*

Clearly one can take a general quartic surface W through Δ, so that $\Delta \subset D$ for $D = W \cap Q$; thus the proposition 2.3 is just the condition (iii) we were looking for.

Moreover, from proposition 2.3 we obtain certain information about the Abel-Jacobi map Φ_{H_t}. Namely, by construction, for $t = 0$,

a) the preimage $\Phi^{-1}_{H_0}(\Sigma')$ of the component Σ' of $\Phi_{H_0}(H_0)$, where Σ' is birational to $J(D)$, coincides with the closure $\bar{H}'$ in $\bar{\mathcal{Y}}$ of the set $H' = \{C \in \mathcal{Y} | C \text{ intersects } D \text{ in 10 points}\}$ (clearly, H' is irreducible, as $\bar{\mathcal{Y}}$ and D are irreducible);

b) the composition map $\Phi_0 : H' \xrightarrow{\Phi_{H_0}} \Sigma' \xrightarrow{bir} J(D)$ coincides with the map

$$C \mapsto cl(C \cap Q - \Delta_0), \tag{25}$$

where Δ_0 is some fixed divisor of degree 10 on D. Now, by proposition 2.3, for a general 10-ple of points Δ on C through a general divisor of the linear series $|\Delta|$ there passes a unique elliptic quintic from H', and $|\Delta| \simeq \mathbf{P}^1$. Thus it follows quickly by dimension reasons (since H' is irreducible) that through *any* divisor of the linear series $|\Delta|$ there passes a unique quintic from H'. By (25), this precisely means that, for a general point $y = cl(\Delta - \Delta_0) \in J(D)$, the fiber $\Phi^{-1}(y)$ equals $|\Delta| \simeq \mathbf{P}^1$. Thus the fiber of the Abel-Jacobi map Φ_{H_0} over a general point of Σ' equals $\mathbf{P}^1$. One can expect that the similar property of the Abel-Jacobi map is true also for $t \neq 0$. In other words, we may formulate the following

Conjecture 2.4. *In conditions and notations of theorem 1.7 the general fiber of the Abel-Jacobi map $\Phi_{H_0} : H_0 \to \Theta + \mathrm{const}$ is isomorphic to $\mathbf{P}^1$.*

3. The component of the moduli space $M(2;0,3)$ of rank-2 stable vector bundles on X associated to the family of elliptic quintics on X

Consider the irreducible component H_0 of $\mathrm{Hilb}(X)$ from theorem 1.7 and, using notations of subsection 1.1, let

- $H^*_0 := \{C \in H^* \cap H_0 | \ h^0(\mathcal{J}_{C,X}(2)) = 1 \text{ and } \pi : C \to \pi(C) \text{ is an isomorphism}\}$.

According to lemma 1.2, H_0^* is a dense open subset of H_0. Now take any curve $C \in H_0^*$ and consider the so called Serre construction defining the $\mathcal{O}_X$-sheaf $\mathcal{E}_C$ as an extension of $\mathcal{O}_X$-sheaves

$$0 \to \mathcal{O}_X(-1) \to \mathcal{E}_C \to \mathcal{J}_{C,X}(1) \to 0. \tag{26}$$

Any such extension is determined uniquely up to an isomorphism by a 1-dimensional subspace of the group $\mathrm{Ext}^1(\mathcal{J}_{C,X}(1), \mathcal{O}_X(-1)) \simeq H^0(\mathcal{E}xt^1(\mathcal{J}_{C,X}(1), \mathcal{O}_X(-1))) \simeq H^0(\mathcal{E}xt^2(\mathcal{O}_C(1), \mathcal{O}_X(-1))) \simeq H^0(\mathcal{E}xt^2(\mathcal{O}_C, \omega_X)) \simeq H^0(\omega_C) \simeq \mathbf{C}$ (here we use the well known equalities $h^i\mathcal{O}_X(-2) = 0, \;\; i < 3$, and the equality $\omega_C \simeq \mathcal{O}_C$ for the smooth elliptic curve C). Thus $\mathcal{E}_C$ is defined uniquely up to an isomorphism by the curve C. Besides, since for the elliptic curve C a nonzero section of ω_C vanishes nowhere on C, it follows that $\mathcal{E}_C$ is locally free (this is the well known property of the Serre construction). Now from (26) one quickly computes the Chern classes of the vector bundle $\mathcal{E}_C$:

$$c_1(\mathcal{E}_C) = 0, \qquad c_2(\mathcal{E}_C) = 3 \tag{27}$$

(here we understand, as usual, the second Chern class c_2 of a given $\mathcal{O}_X$-shef as an element of the group B_1 of classes of algebraic equivalence of 1-dimensional cycles on X, where $B_1 = \mathbb{Z} \cdot l \simeq \mathbb{Z}$ and l is the class of a line on X). Next, since $\pi : C \to \pi(C)$ is an isomorphism onto a quintic $\pi(C)$ not lying in a plane and, as it is easy to see, $H^0(\mathcal{O}_X(1)) = \pi^* H^0(\mathcal{O}_{\mathbf{P}^3}(1))$, it follows quickly that $H^0(\mathcal{J}_{C,X}(1)) \simeq H^0(\mathcal{J}_{\pi(C),\mathbf{P}^3}(1)) = 0$. Hence the triple (26) implies that $h^0(\mathcal{E}_C) = 0$. This clearly means that $\mathcal{E}_C$ is stable in the sense of Gieseker (cf. [MT-1, Prop. 2.6]). Thus, denoting

- $M(2; 0, 3)$ the Gieseker-Maruyama moduli scheme of stable rank-2 vector bundles on X with Chern classes $c_1 = 0, \;\; c_2 = 3$,

we see from (27) that

$$[\mathcal{E}_C] \in M(2; 0, 3) \tag{28}$$

(here we use the standard notation $[\mathcal{E}]$ for the isomorphism class of a given $\mathcal{O}_X$-sheaf $\mathcal{E}$). Remark that the equality $h^0(\mathcal{J}_{C,X}(2)) = 1$ immediately implies $h^i(\mathcal{J}_{C,X}(2)) = 0, \;\; i > 0$; hence by (26) also

$$h^i(\mathcal{E}_C) = 0, \;\; i > 0. \tag{29}$$

Now let $\tilde{M} = \{[\mathcal{E}] \in M(2; 0, 3) | \; h^i(\mathcal{E}) = 0, \;\; i > 0\}$. By semicontinuity, this is clearly an open subset of $M(2; 0, 3)$ containing a point $\mathcal{E}_C$ in view of (29). Let M be an irreducible component of $\tilde{M}$ containing this point $\mathcal{E}_C$. We then have a natural map

$$f : H_0^* \to M : C' \mapsto \mathcal{E}_{C'}. \tag{30}$$

It is a standard matter to show that f is a dominating morphism (cf. [MT-1, lemmas 5.2 and 5.3]), with the fibre $f^{-1}(f(C'))$ open in $P(H^0(\mathcal{E}_{C'}(1)) \simeq \mathbf{P}^1$:

$$\overline{f^{-1}(f(C'))} = \mathbf{P}^1, \quad C' \in H_0^*. \tag{31}$$

(Here the closure is taken in H_0; besides, we keep in mind that, by Riemann-Roch, $h^0(\mathcal{E}_{C'}(1)) = 2$, since by the property of M, $h^i(\mathcal{E}_{C'}(1)) = 0, \;\; i > 0$.) Remark that the equality $h^0(\mathcal{J}_{C',X}(2)) = 1$ means that the linear series $|\mathcal{J}_{C',X}(2))|$ consists of a unique K3-surface $S(C')$, and by the Serre construction, the fibre $\overline{f^{-1}(f(C'))} = \mathbf{P}^1$ is just the pencil of elliptic curves on $S(C')$, these curves being the zero loci of sections from $H^0(\mathcal{E}_{C'}(1))$:

$$\overline{f^{-1}(f(C'))} = |\mathcal{O}_{S(C')}(C')| = \mathbf{P}^1. \tag{32}$$

Now consider the Stein decomposition of the Abel-Jacobi map $\Phi = \Phi_{H_0}|H_0^* : H_0^* \to \Theta +$ const. Using the above description (32) of fibres of f and the well-known interpretation of the differential of the Abel-Jacobi map Φ in terms of the branch quartic W of the double covering

$\pi : X \to \mathbf{P}^3$ (see, e.g., [T-1, corollary 1] or [We, Prop.2.13]), we can apply the argument similar to that used in the proof of Theorem 5.6 of [MT-1] and obtain the following

Theorem 3.1. *There exists a quasi-finite dominating morphism of (an open part of) the component M of the moduli space $M(2;0,3)$ to (a translate of) the theta divisor:*

$$g : M \to \Theta + \mathrm{const} \tag{33}$$

which gives the Stein decomposition $\Phi = g \circ f$ of the Abel-Jacobi map Φ of H_0^*:

$$\begin{tikzcd} H_0^* \arrow[rr, "\Phi"] \arrow[dr, "f"'] & & \Theta + \mathrm{const} \\ & M \arrow[ur, "g"'] & \end{tikzcd}, \tag{34}$$

where the the fibers of $f : H_0^* \to M$ are open subsets in $\mathbf{P}^1$.

Now theorem 3.1 and conjecture 2.4 give a hint to the following

Conjecture 3.2. *The morphism $g : M \to \Theta + \mathrm{const}$ in (34) is birational.*

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