HOMOLOGY OF GENERALIZED STEINBERG VARIETIES AND WEYL GROUP INVARIANTS

J. MATTHEW DOUGLASS AND GERHARD RÖHRLE

Abstract. Let G be a complex, connected, reductive algebraic group. In this paper we show analogues of the computations by Borho and MacPherson of the invariants and anti-invariants of the cohomology of the Springer fibres of the cone of nilpotent elements, $\mathcal{N}$, of $\mathrm{Lie}(G)$ for the Steinberg variety Z of triples.

Using a general specialization argument we show that for a parabolic subgroup $W_P \times W_Q$ of $W \times W$ the space of $W_P \times W_Q$-invariants and the space of $W_P \times W_Q$-anti-invariants of $H_{4n}(Z)$ are isomorphic to the top Borel-Moore homology groups of certain generalized Steinberg varieties introduced in [5].

The rational group algebra of the Weyl group W of G is isomorphic to the opposite of the top Borel-Moore homology $H_{4n}(Z)$ of Z, where $2n = \dim \mathcal{N}$. Suppose $W_P \times W_Q$ is a parabolic subgroup of $W \times W$. We show that the space of $W_P \times W_Q$-invariants of $H_{4n}(Z)$ is $e_Q \mathbb{Q}W e_P$, where e_P is the idempotent in the group algebra of W_P affording the trivial representation of W_P and e_Q is defined similarly. We also show that the space of $W_P \times W_Q$-anti-invariants of $H_{4n}(Z)$ is $\epsilon_Q \mathbb{Q}W \epsilon_P$, where ϵ_P is the idempotent in the group algebra of W_P affording the sign representation of W_P and ϵ_Q is defined similarly.

1. Introduction

Suppose G is a complex, reductive algebraic group and $\mathcal{B}$ is the variety of Borel subgroups of G. Then $\mathcal{B}$ is a smooth, projective variety. Let T be a maximal torus in G and choose a Borel subgroup, B, of G containing T. Let $W = N_G(T)/T$ be the Weyl group of (G, T). Then W acts on G/T on the right, the natural projection $G/T \to G/B$ has the structure of a vector bundle, and the varieties G/B and $\mathcal{B}$ are isomorphic. Thus, W acts on the singular cohomology with rational coefficients of $\mathcal{B}$ via the isomorphisms $H^\bullet(\mathcal{B}) \cong H^\bullet(G/B) \cong H^\bullet(G/T)$.

Now suppose P is a parabolic subgroup of G containing B and $\mathcal{P}$ is the variety of G-conjugates of P. Then $\mathcal{P}$ is again a smooth, projective variety and it is a classical result that $H^\bullet(\mathcal{P})$ is isomorphic to the space of W_P-invariants in $H^\bullet(\mathcal{B})$ where $W_P = N_P(T)/T$ is the Weyl group of (P, T) (see [9]).

Borho and MacPherson have generalized this result to fixed point subvarieties of $\mathcal{B}$ as follows. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathcal{N}$ the cone of nilpotent elements in $\mathfrak{g}$. There is a *moment map*, $\mu_0 \colon T^*\mathcal{B} \to \mathcal{N}$, where $T^*\mathcal{B}$ is the cotangent bundle of $\mathcal{B}$. For x in $\mathcal{N}$, set $\mathcal{B}_x = \mu_0^{-1}(x)$. The variety $\mathcal{B}_x$ may be identified with the variety of all Borel subgroups of

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G, whose Lie algebra contains x. The varieties $\mathcal{B}_x$ vary from a point, when x is regular, to $\mathcal{B}$, when $x = 0$. The moment map factors as $\mu_0 = \eta_0 \circ \xi_0$ where $\xi_0^{-1}(x)$ may be identified with the variety, $\mathcal{P}_x$, of all subgroups in $\mathcal{P}$ whose Lie algebra contains x. There is also a moment map, $\mu_0^{\mathcal{P}}$ from the cotangent bundle of $\mathcal{P}$ to $\mathcal{N}$, and $(\mu_0^{\mathcal{P}})^{-1}(x)$ may be identified with the variety of all subgroups in $\mathcal{P}$ whose Lie algebras contain x in their nilradical. Set $\mathcal{P}_x^0 = (\mu_0^{\mathcal{P}})^{-1}(x)$.

Springer [17] has defined an action of W on $H^\bullet(\mathcal{B}_x)$ and Borho and MacPherson [3] have shown that if W acts on $H^\bullet(\mathcal{B}_x)$ by the tensor product of Springer's action with the sign representation, then:

(1.1) $H^\bullet(\mathcal{P}_x)$ is isomorphic to the space of W_P-invariants in $H^\bullet(\mathcal{B}_x)$.
(1.2) $H^\bullet(\mathcal{P}_x^0)$ is isomorphic to the subspace of $H^\bullet(\mathcal{B}_x)$ on which W_P acts as the sign representation.

In a different direction, the *Steinberg variety* of G is the fibred product $T^*\mathcal{B} \times_{\mathcal{N}} T^*\mathcal{B}$ which may be identified with the closed subvariety

$$Z = \{ (x, B', B'') \in \mathcal{N} \times \mathcal{B} \times \mathcal{B} \mid x \in \operatorname{Lie}(B') \cap \operatorname{Lie}(B'') \}$$

of $\mathcal{N} \times \mathcal{B} \times \mathcal{B}$. Kazhdan and Lusztig [12] have defined an action of $W \times W$, on $H_\bullet(Z)$, the rational, Borel-Moore homology of Z, and they showed that the representation of $W \times W$ on the top-dimensional homology group of Z, $H_{4n}(Z)$, where $n = \dim \mathcal{B}$, is equivalent to the two-sided regular representation of W.

Tanisaki [19] and, more recently, Chriss and Ginzburg [4] have strengthened the connection between $H_\bullet(Z)$ and W by defining a $\mathbb{Q}$-algebra structure on $H_\bullet(Z)$ so that $H_i(Z) \cdot H_j(Z) \subseteq H_{i+j-4n}(Z)$ and $H_{4n}(Z)^{\mathrm{op}}$ is isomorphic to the group algebra $\mathbb{Q}W$.

In this paper we prove analogs of (1.1) and (1.2) for the Steinberg variety.

Suppose Q is a parabolic subgroup of G containing B (a special case is when $Q = P$), W_Q is the Weyl group of (Q, T), and $\mathcal{Q}$ is the conjugacy class of parabolic subgroups that contains Q. In [5] we defined generalized Steinberg varieties

$$X^{\mathcal{P},\mathcal{Q}} = \{ (x, P', Q') \in \mathcal{N} \times \mathcal{P} \times \mathcal{Q} \mid x \in \operatorname{Lie}(P') \cap \operatorname{Lie}(Q') \}$$

and

$$Y^{\mathcal{P},\mathcal{Q}} = \{(x, P', Q') \in \mathcal{N} \times \mathcal{P} \times \mathcal{Q} \mid x \in \operatorname{Lie}(U_{P'}) \cap \operatorname{Lie}(U_{Q'}) \}$$

where $U_{P'}$ and $U_{Q'}$ are the unipotent radicals of P' and Q' respectively. It was shown in [5] that $X^{\mathcal{P},\mathcal{Q}}$ is purely $2n$-dimensional and $Y^{\mathcal{P},\mathcal{Q}}$ is purely $(2n - f)$-dimensional where $f = \dim P/B + \dim Q/B$.

The first analogs of (1.1) and (1.2) are:

(1.1′) $H_{4n}(X^{\mathcal{P},\mathcal{Q}})$ is isomorphic to the space of $W_P \times W_Q$-invariants in $H_{4n}(Z)$.
(1.2′) $H_{4n-2f}(Y^{\mathcal{P},\mathcal{Q}})$ is isomorphic to the subspace of $H_{4n}(Z)$ on which $W_P \times W_Q$ acts as the sign representation.

We prove both of these statements in this paper.

More generally we have:

(1.1″) $H_\bullet(X^{\mathcal{P},\mathcal{Q}})$ is isomorphic to the space of $W_P \times W_Q$-invariants in $H_\bullet(Z)$.
(1.2″) $H_\bullet(Y^{\mathcal{P},\mathcal{Q}})$ is isomorphic to the subspace of $H_\bullet(Z)$ on which $W_P \times W_Q$ acts as the sign representation.

In §3 we prove a general specialization result, in the spirit of [3], which has (1.1″) as a special case. Obviously (1.1′) follows immediately from (1.1″). It seems likely that (1.2″) is true, but our proof of (1.2′) uses dimension computations from [5] that are not available for $H_i(Y^{\mathcal{P},\mathcal{Q}})$ for $i < 4n - 2f$.

In §4 we prove a general equivariance result in the spirit of [4]. A special case of this result is that there is a $W \times W$-equivariant isomorphism

$$\mathrm{Ext}^{4n-\bullet}_{\mathcal{N}}(R(\mu_0)_!\mathbb{Q}_{T^*\mathcal{B}}, R(\mu_0)_!\mathbb{Q}_{T^*\mathcal{B}}) \xrightarrow{\simeq} H_\bullet(Z)^{\mathrm{op}}.$$

Borho and MacPherson [2] have shown that the $\mathbb{Q}$-algebras $\mathbb{Q}W$ and $\mathrm{End}_{\mathcal{N}}(R(\mu_0)_!\mathbb{Q}_{T^*\mathcal{B}})$ are isomorphic and Chriss and Ginzburg [4, §8.6] have shown that that

$$\mathrm{Ext}^{4n-\bullet}_{\mathcal{N}}(R(\mu_0)_!\mathbb{Q}_{T^*\mathcal{B}}, R(\mu_0)_!\mathbb{Q}_{T^*\mathcal{B}}) \cong H_\bullet(Z)^{\mathrm{op}}.$$

Thus, taking $\bullet = 4n$ we get $W \times W$-equivariant, $\mathbb{Q}$-algebra isomorphisms

$$\mathbb{Q}W \xrightarrow{\simeq} \mathrm{End}_{\mathcal{N}}(R(\mu_0)_!\mathbb{Q}_{T^*\mathcal{B}}) \xrightarrow{\simeq} H_{4n}(Z)^{\mathrm{op}}.$$

where $W \times W$ acts on $\mathbb{Q}W$ by $(w, w') \cdot v = w'vw^{-1}$ for w and w' in W and v in $\mathbb{Q}W$.

Using the isomorphism between $\mathbb{Q}W$ and $H_{4n}(Z)^{\mathrm{op}}$ we may formulate (1.1′) and (1.2′) in terms of the group algebra of W:

(1.1‴) If e_P is the primitive idempotent in $\mathbb{Q}W_P$ corresponding to the trivial representation of W_P and e_Q is defined similarly, then $H_{4n}(X^{\mathcal{P},\mathcal{Q}})$ is isomorphic to the subspace $e_Q\mathbb{Q}We_P$ of $\mathbb{Q}W$.

(1.2‴) If ϵ_P is the primitive idempotent in $\mathbb{Q}W_P$ corresponding to the sign representation of W_P and ϵ_Q is defined similarly, then $H_{4n-2f}(Y^{\mathcal{P},\mathcal{Q}})$ is isomorphic to the subspace $\epsilon_Q\mathbb{Q}W\epsilon_P$ of $\mathbb{Q}W$.

In [5] we defined generalized Steinberg varieties $X^{\mathcal{P},\mathcal{Q}}_{c,d}$. Statements (1.1‴) and (1.2‴) together with computations in some special cases suggest that the Borel-Moore homology of a general $X^{\mathcal{P},\mathcal{Q}}_{c,d}$ is given as follows.

A generalized Steinberg variety, $X^{\mathcal{P},\mathcal{Q}}_{c,d}$, depends on a pair of nilpotent adjoint orbits in $\mathrm{Lie}(P/U_P)$ and $\mathrm{Lie}(Q/U_Q)$ respectively. We will not recall the precise definition here but instead refer the interested reader to [5]. In turn, these nilpotent orbits determine irreducible representations of W_P and W_Q, say ρ_c and ρ_d respectively, corresponding to the trivial representation of the component groups of the orbits via the Springer correspondence as defined in [2]. Let e_c and e_d denote primitive idempotents in $\mathbb{Q}W_P$ and $\mathbb{Q}W_Q$ affording ρ_c and ρ_d respectively. In [5, Corollary 2.6] we have given a sharp upper bound, $\delta^{\mathcal{P},\mathcal{Q}}_{c,d}$, for the dimension of $X^{\mathcal{P},\mathcal{Q}}_{c,d}$. We conjecture that

- $H_{2\delta^{\mathcal{P},\mathcal{Q}}_{c,d}}(X^{\mathcal{P},\mathcal{Q}}_{c,d})$ is isomorphic to $e_d\mathbb{Q}We_c$.

More generally, we conjecture that

- $H_\bullet(X^{\mathcal{P},\mathcal{Q}}_{c,d})$ is isomorphic to $e_dH_\bullet(Z)e_c$ where we consider e_c and e_d in $H_\bullet(Z)$ via the isomorphism $\mathbb{Q}W \cong H_{4n}(Z)^{\mathrm{op}}$.

In much of this paper (§2 – §4 and the Appendix) we are concerned with general sheaf theory. Most of our conclusions about the Borel-Moore homology of generalized Steinberg varieties are straightforward applications of more general results. The main theorems, which are described briefly below, are the specialization results, Theorem 3.1.2 and Corollary 3.5.2, and the equivariance results discussed in §4.1. We hope these general results will have applications outside the realm of generalized Steinberg varieties.

Our computation of the Borel-Moore homology of $X^{\mathcal{P},\mathcal{Q}}$ and $Y^{\mathcal{P},\mathcal{Q}}$ is given in §5. Although the results depend on facts proved in §3 and §4, this section may be read independently of the other sections.

The rest of this paper is organized as follows.

In §2 we fix notation and collect some sheaf-theoretic results that are used in subsequent sections for which we could not find a suitable reference.

In §3 we give an axiomatic approach to a specialization result which allows us to identify a direct image map in Borel-Moore homology with the averaging map for a group action. The basic idea goes back to Lusztig [14] and Borho-MacPherson [3]. A result which is similar in spirit, but which is in a sense dual to our result, and does not apply to Borel-Moore homology, has been used by Spaltenstein in [16]. Statement (1.1″) is a straightforward consequence of the main result in this section, Theorem 3.1.2.

In §4 we continue the axiomatic approach from §3 and prove an equivariance result for two-sided group actions that is key for our application to generalized Steinberg varieties. The crucial result is Theorem 4.4.1 which when applied to the Steinberg variety implies that there is a $W \times W$-equivariant isomorphism between $\operatorname{Ext}^{4n-\bullet}_{\mathcal{N}}(R(\mu_0)_!\mathbb{Q}_{T^*\mathcal{B}}, R(\mu_0)_!\mathbb{Q}_{T^*\mathcal{B}})$ and $H_\bullet(Z)$. This result is similar in spirit to the results in [4, §8.6].

In §5 we specialize the results in the previous sections to the case of generalized Steinberg varieties and prove (1.1″), (1.2′), (1.1‴), and (1.2‴).

In the Appendix, we prove two results about the natural transformation $\xi^* \to \xi^![2l]$ for a morphism $\xi\colon X \to Y$, where $l = \dim Y - \dim X$. These results are needed in the proof of Theorem 4.4.1.

For simplicity, in this paper we have chosen to work with complex algebraic groups and Borel-Moore homology, but our arguments are essentially categorical and make sense in the setting of algebraic groups over arbitrary algebraically closed fields and l-adic cohomology.

2. Preliminaries

2.1. First, we fix some assumptions and notation that will be used throughout the rest of this paper. The reader is urged to skim this section quickly to become familiar with the notation and refer back to the results used in the sequel when necessary. The main references for sheaf-theoretic notation and results used in this paper are the article [1] by Borel (with the collaboration of N. Spaltenstein) and the book [11] by Kashiwara and Shapira.

The topological spaces we consider are complex algebraic varieties endowed with their Euclidean topologies, although many arguments apply as well to pseudomanifolds as defined in [8, §1.1].

The "dimension" of a space always means its dimension as a complex algebraic variety.

If X is a variety, then $D(X)$ denotes the derived category of the category of sheaves of $\mathbb{Q}$-vector spaces on X, $D^b(X)$ denotes the full subcategory of $D(X)$ consisting of complexes with bounded cohomology, and $D^b_c(X)$ denotes the full subcategory of $D^b(X)$ consisting of complexes with constructible cohomology.

For complexes A and B in $D(X)$, $\operatorname{Ext}^j(A,B)$ is defined to be $H^j(R\operatorname{Hom}(A,B))$ and it is shown in [1, §5.17] that $\operatorname{Ext}^j(A,B) = \operatorname{Hom}_{D(X)}(A, B[j])$. Define

$$\operatorname{Ext}^j_X(A,B) = \operatorname{Hom}_{D(X)}(A, B[j]).$$

Since $D^b_c(X)$ is a full subcategory of $D(X)$, if A and B are complexes in $D^b_c(X)$, then $\mathrm{Hom}_{D^b_c(X)}(A,B) = \mathrm{Hom}_{D(X)}(A,B)$. To simplify the notation, we denote both of these spaces by $\mathrm{Hom}_X(A,B)$. Also, we denote the complex $A \overset{L}{\otimes} B$ simply by $A \otimes B$.

The constant sheaf on X, considered as a complex concentrated in degree 0, is denoted by $\mathbb{Q}_X$ and the dualizing complex of X is denoted by $\mathbb{D}_X$.

If A is a complex of sheaves of $\mathbb{Q}$-vector spaces on X, then $A^\vee = R\mathcal{H}om(A, \mathbb{D}_X)$ denotes the Verdier dual of A. There is a canonical isomorphism between $\mathbb{D}_X$ and $\mathbb{Q}_X^\vee$ we denote by dc_X, so

$$\mathrm{dc}_X \colon \mathbb{D}_X \xrightarrow{\simeq} \mathbb{Q}_X^\vee .$$

If $f\colon A \to B$ is a morphism in $D(X)$, and C is a complex in $D(X)$, then f induces natural morphisms in $D(X)$,

$$f^\sharp \colon R\mathcal{H}om(B,C) \longrightarrow R\mathcal{H}om(A,C) \quad \text{and} \quad f_\sharp \colon R\mathcal{H}om(C,A) \longrightarrow R\mathcal{H}om(C,B).$$

In the special case when $C = \mathbb{D}_X$, we have $R\mathcal{H}om(A,C) = A^\vee$ and $R\mathcal{H}om(B,C) = B^\vee$. We usually write $f^\vee$ instead of $f^\sharp$ in this case, so $f^\vee \colon B^\vee \to A^\vee$ is the Verdier dual of f.

Similarly, f induces natural linear transformations

$$f^\sharp \colon \mathrm{Ext}^\bullet_X(B,C) \longrightarrow \mathrm{Ext}^\bullet_X(A,C) \quad \text{and} \quad f_\sharp \colon \mathrm{Ext}^\bullet_X(C,A) \longrightarrow \mathrm{Ext}^\bullet_X(C,B).$$

The j^{th} Borel-Moore homology group of a locally compact, Hausdorff topological space, X, has several equivalent definitions (see [4, §2.6]). In this paper we use the canonical isomorphisms,

$$H^{-j}(X, \mathbb{D}_X) \cong H^{-j}(X, R\mathcal{H}om(\mathbb{Q}_X, \mathbb{D}_X)) \cong \mathrm{Ext}^{-j}_X(\mathbb{Q}_X, \mathbb{D}_X)$$

where $H^{-j}(X, \mathbb{D}_X)$ is the hypercohomology of X with coefficients in $\mathbb{D}_X$ and we *define* the j^{th} Borel-Moore homology group of X by

$$H_j(X) = \mathrm{Ext}^{-j}_X(\mathbb{Q}_X, \mathbb{D}_X).$$

2.2. Now suppose that $\xi\colon X \to Y$ is a morphism of varieties. Then ξ determines natural isomorphisms

$$\phi_\xi \colon R\mathcal{H}om(R\xi_! A, B) \xrightarrow{\simeq} R\xi_* R\mathcal{H}om(A, \xi^! B)$$

and

$$\mathrm{nat}_\xi \colon \xi^! R\mathcal{H}om(B,C) \xrightarrow{\simeq} R\mathcal{H}om(\xi^* B, \xi^! C)$$

for A in $D(X)$ and B and C in $D(Y)$.

There are canonical isomorphisms,

$$\alpha_\xi \colon \xi^* \mathbb{Q}_Y \xrightarrow{\simeq} \mathbb{Q}_X \qquad \text{and} \qquad \beta_\xi \colon \mathbb{D}_X \xrightarrow{\simeq} \xi^! \mathbb{D}_Y$$

in the category of sheaves on X and $D^b_c(X)$ respectively. It is straightforward to check that α_ξ and β_ξ have the following properties:

(2.2.1) The maps $\beta_\xi \colon \mathbb{D}_X \to \xi^! \mathbb{D}_Y$ and $(\beta_\xi)_\sharp \colon R\mathcal{H}om(\xi^* \mathbb{Q}_Y, \mathbb{D}_X) \longrightarrow R\mathcal{H}om(\xi^* \mathbb{Q}_Y, \xi^! \mathbb{D}_Y)$ are related by $(\beta_\xi)_\sharp \circ \alpha_\xi^\vee \circ \mathrm{dc}_X = \mathrm{nat}_\xi \circ \xi^!(\mathrm{dc}_Y) \circ \beta_\xi$ where dc_X and dc_Y are as in §2.1.

(2.2.2) If $\eta\colon Y \to Z$ is another morphism of varieties, then $\alpha_{\eta\xi} = \alpha_\xi \circ \xi^*(\alpha_\eta)$ and $\beta_{\eta\xi} = \xi^!(\beta_\eta) \circ \beta_\xi$.

2.3. Let $\delta\colon X \to X \times X$ be the diagonal embedding and let p and q denote the projections of $X \times X$ on the first and second factors respectively. In [1, Theorem 10.25] it is shown that there is a natural isomorphism,

$$\lambda\colon A^\vee \boxtimes B \xrightarrow{\simeq} R\mathcal{H}om(p^*A, q^!B)$$

in $D^b_c(X \times X)$. It follows that $\mathrm{nat}_\delta \circ \delta^!(\lambda)$ is a natural isomorphism between $\delta^!(A^\vee \boxtimes B)$ and $R\mathcal{H}om(A, B)$.

Proposition 2.3.1. *Suppose A and B are in $D^b_c(X)$, $u\colon A \to A$ is an endomorphism of A, and $v\colon B \to B$ is an endomorphism of B. Then the diagram*

$$\begin{array}{ccc} A^\vee \boxtimes B & \xrightarrow{u^\vee \boxtimes v} & A^\vee \boxtimes B \\ \lambda \downarrow & & \downarrow \lambda \\ R\mathcal{H}om(p^*A, q^!B) & \xrightarrow[(p^*u)^\sharp \circ (q^!v)_\sharp]{} & R\mathcal{H}om(p^*A, q^!B) \end{array}$$

commutes.

Proof. By definition $A^\vee \boxtimes B = p^* R\mathcal{H}om(A, \mathbb{D}_X) \otimes q^*B$ and $u^\vee \boxtimes v = p^*(u^\sharp) \otimes q^*v$.

In the special case when A is the constant sheaf, the isomorphism λ may be identified with a natural isomorphism $\lambda'\colon p^*\mathbb{D}_X \otimes q^*B \to q^!B$ as in [1, ¶10.24]. Then for an arbitrary A, the isomorphism λ is defined as the composition $\lambda'_\sharp \circ h_2 \circ h_1$, where h_1 and h_2 are the natural maps

$$h_1\colon p^*R\mathcal{H}om(A, \mathbb{D}_X) \otimes q^*B \longrightarrow R\mathcal{H}om(p^*A, p^*\mathbb{D}_X) \otimes q^*B$$

and

$$h_2\colon R\mathcal{H}om(p^*A, p^*\mathbb{D}_X) \otimes q^*B \longrightarrow R\mathcal{H}om(p^*A, p^*\mathbb{D}_X \otimes q^*B).$$

It is straightforward to check that

$$h_1 \circ \big(p^*(u^\sharp) \otimes q^*v\big) = \big((p^*u)^\sharp) \otimes q^*v\big) \circ h_1$$

and

$$h_2 \circ \big((p^*u)^\sharp) \otimes q^*v\big) = \big((p^*u)^\sharp \circ (id \otimes q^*v)_\sharp\big) \circ h_2.$$

Moreover, it follows from the naturality of λ' that

$$\lambda'_\sharp \circ \big((p^*u)^\sharp \circ (id \otimes q^*v)_\sharp\big) = \big((p^*u)^\sharp \circ (q^!v)_\sharp\big) \circ \lambda'_\sharp.$$

Therefore $\lambda \circ (u^\vee \boxtimes v) = \big((p^*u)^\sharp \circ (q^!v)_\sharp\big) \circ \lambda$, as desired. □

Corollary 2.3.2. *With the preceding notation, the diagram*

$$\begin{array}{ccc} \delta^!\,(A^\vee \boxtimes B) & \xrightarrow{\delta^!(u^\vee \boxtimes v)} & \delta^!\,(A^\vee \boxtimes B) \\ \mathrm{nat}_\delta \circ \delta^!(\lambda) \downarrow & & \downarrow \mathrm{nat}_\delta \circ \delta^!(\lambda) \\ R\mathcal{H}om(A, B) & \xrightarrow[u^\sharp \circ v_\sharp]{} & R\mathcal{H}om(A, B) \end{array}$$

commutes.

Proof. We have just seen that $\lambda \circ (u^\vee \boxtimes v) = \big((p^*u)^\sharp \circ (q^!v)_\sharp\big) \circ \lambda$, so

$$\delta^!(\lambda) \circ \delta^!(u^\vee \boxtimes v) = \delta^! \left((p^*u)^\sharp \circ (q^!v)_\sharp\right) \circ \delta^!(\lambda).$$

It is straightforward to check that

$$\mathrm{nat}_\delta \circ \delta^! \left((p^*u)^\sharp \circ (q^!v)_\sharp\right) = \left((\delta^* p^* u)^\sharp \circ (\delta^! q^! v)_\sharp\right) \circ \mathrm{nat}_\delta = \left(u^\sharp \circ v_\sharp\right) \circ \mathrm{nat}_\delta$$

so

$$\mathrm{nat}_\delta \circ \delta^!(\lambda) \circ \delta^!(u^\vee \boxtimes v) = \mathrm{nat}_\delta \circ \delta^! \left((p^*u)^\sharp \circ (q^!v)_\sharp\right) \circ \delta^!(\lambda) = \left(u^\sharp \circ v_\sharp\right) \circ \mathrm{nat}_\delta \circ \delta^!(\lambda).$$

This proves the corollary. □

It is shown in [11, §2.6] that for A, B, and C in $D(X)$ there is a natural isomorphism $\mathrm{Hom}_X(C \otimes A, B) \cong \mathrm{Hom}_X(C, R\mathcal{H}om(A, B))$. It follows that there is an isomorphism of graded vector spaces $\mathrm{Ext}^\bullet_X(C \otimes A, B) \cong \mathrm{Ext}^\bullet_X(C, R\mathcal{H}om(A, B))$. Taking $C = \mathbb{Q}_X$ and using the canonical isomorphism $\mathbb{Q}_X \otimes A \cong A$ we get a natural isomorphism of graded vector spaces

$$\mathrm{can}\colon \mathrm{Ext}^\bullet_X(A, B) \xrightarrow{\simeq} \mathrm{Ext}^\bullet_X(\mathbb{Q}_X, R\mathcal{H}om(A, B)).$$

The next proposition follows from the naturality of can.

Proposition 2.3.3. *Suppose A and B are in $D(X)$, $u\colon A \to A$ is an endomorphism of A, and $v\colon B \to B$ is an endomorphism of B. Then the diagram*

$$\begin{array}{ccc} \mathrm{Ext}^\bullet_X(A, B) & \xrightarrow{\mathrm{can}} & \mathrm{Ext}^\bullet_X(\mathbb{Q}_X, R\mathcal{H}om(A, B)) \\ \downarrow{\scriptstyle u^\sharp \circ v_\sharp} & & \downarrow{\scriptstyle (u^\sharp \circ v_\sharp)_\sharp} \\ \mathrm{Ext}^\bullet_X(A, B) & \xrightarrow[\mathrm{can}]{} & \mathrm{Ext}^\bullet_X(\mathbb{Q}_X, R\mathcal{H}om(A, B)) \end{array}$$

commutes.

2.4. As in §2.2, $\xi\colon X \to Y$ is a morphism of varieties. The functors ξ^* and $R\xi_*$ form an adjoint pair. We denote by

$$\Psi_\xi\colon \mathrm{Hom}_X(\xi^* B, A) \xrightarrow{\simeq} \mathrm{Hom}_Y(B, R\xi_* A)$$

the adjunction mapping for A in $D(X)$ and B in $D(Y)$ and by χ^ξ the unit of the adjunction. Although χ^ξ is a natural transformation, $\chi^\xi_B\colon B \to R\xi_*\xi^* B$, in order to simplify the notation we omit the subscript and just write χ^ξ instead of χ^ξ_B. The appropriate subscript is always uniquely determined by the context and so this should cause no confusion.

Similarly, the functors $R\xi_!$ and $\xi^!$ form an adjoint pair. We denote by

$$\Phi_\xi\colon \mathrm{Hom}_Y(R\xi_! A, B) \xrightarrow{\simeq} \mathrm{Hom}_X(A, \xi^! B)$$

the adjunction mapping and by ϵ^ξ the counit of the adjunction.

We need the following identities for morphisms $f\colon R\xi_! A \to B$ and $k\colon B \to B'$ in $D(Y)$ and $g\colon A \to \xi^! B$ and $h\colon A' \to A$ in $D(X)$ (see [13, IV.1]):

(2.4.1) $$\epsilon^\xi = \Phi_\xi^{-1}(id) \qquad \Phi_\xi^{-1}(g) = \epsilon^\xi \circ R\xi_!(g)$$

(2.4.2) $$\Phi_\xi(f \circ R\xi_!(h)) = \Phi_\xi(f) \circ h \qquad \Phi_\xi(k \circ f) = \xi^!(k) \circ \Phi_\xi(f)$$

(2.4.3) $$\Phi_\xi^{-1}(g \circ h) = \Phi_\xi^{-1}(g) \circ R\xi_!(h) \qquad \Phi_\xi^{-1}(\xi^!(k) \circ g) = k \circ \Phi_\xi^{-1}(g)$$

Verdier duality defines contravariant automorphisms of the subcategories $D_c^b(X)$ and $D_c^b(Y)$ of $D(X)$ and $D(Y)$ respectively. In these subcategories we can use standard identities for Verdier duality in [1, §10] to express Φ_ξ and ϵ^ξ in terms of Ψ_ξ and χ^ξ as follows.

Suppose A is in $D_c^b(X)$, B is in $D_c^b(Y)$, and f is in $\mathrm{Hom}_Y(R\xi_! A, B)$. Then $\Psi_\xi^{-1}(f^\vee)^\vee$ is in $\mathrm{Hom}_X(A, \xi^! B)$. Clearly, $f \mapsto \Psi_\xi^{-1}(f^\vee)^\vee$ is natural in A and B and so we may *define* Φ_ξ by $\Phi_\xi(f) = \Psi_\xi^{-1}(f^\vee)^\vee$.

Similarly, taking the Verdier dual of $\chi_B^\xi \colon B \to R\xi_*\xi^* B$ we get $\left(\chi_B^\xi\right)^\vee \colon R\xi_!\xi^! B^\vee \to B^\vee$ and we conclude that $\left(\chi_B^\xi\right)^\vee = \epsilon_{B^\vee}^\xi$.

2.5. Next, consider a cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{\ i\ } & X \\ {\scriptstyle \eta}\big\downarrow & & \big\downarrow{\scriptstyle \xi} \\ Y' & \xrightarrow{\ j\ } & Y \end{array} \tag{2.5.1}$$

where ξ and η are proper morphisms. Then $\Psi_j^{-1}(R\xi_*(\chi^i)) \colon j^* R\xi_* \to R\eta_* i^*$ is a natural equivalence of functors from $D(X)$ to $D(Y)$. Restricting to $D_c^b(X)$ and $D_c^b(Y)$ and taking the Verdier dual we conclude that $\Psi_j^{-1}(R\xi_*(\chi^i))^\vee \colon R\eta_! i^! \to j^! R\xi_!$ is a natural equivalence. It follows from the discussion in §2.4 above that

$$\Psi_j^{-1}(R\xi_*(\chi^i))^\vee = \Phi_j\left((R\xi_*(\chi^i))^\vee\right) = \Phi_j\left(R\xi_!((\chi^i)^\vee)\right) = \Phi_j\left((R\xi_!(\epsilon^i)\right).$$

Define

$$\mathrm{bc}_{\eta,i} \colon j^! \circ R\xi_! \longrightarrow R\eta_! \circ i^! \quad \text{by} \quad \mathrm{bc}_{\eta,i} = \Phi_j\left(R\xi_!(\epsilon^i)\right)^{-1}.$$

Then $\mathrm{bc}_{\eta,i}$ is a natural equivalence and $\mathrm{bc}_{\eta,i}^{-1} = \Phi_j\left(R\xi_!(\epsilon^i)\right)$.

Lemma 2.5.2. *Suppose that in diagram (2.5.1) the maps i and j are open embeddings. Then, for A in $D_c^b(X)$ and B in $D_c^b(Y)$, the diagram*

$$\begin{array}{ccccc} j^! R\xi_! R\mathcal{H}om(A, \xi^! B) & \xrightarrow{\ j^!(\phi_\xi^{-1})\ } & j^! R\mathcal{H}om(R\xi_! A, B) & \xrightarrow{\ \mathrm{nat}_j\ } & R\mathcal{H}om(j^* R\xi_! A, j^! B) \\ {\scriptstyle \mathrm{bc}}\big\downarrow & & & & \big\downarrow{\scriptstyle (\mathrm{bc}^{-1})^\sharp} \\ R\eta_! i^! R\mathcal{H}om(A, \xi^! B) & \xrightarrow[R\eta_!(\mathrm{nat}_i)]{} & R\eta_! R\mathcal{H}om(i^* A, i^! \xi^! B) & \xrightarrow[\phi_\eta^{-1}]{} & R\mathcal{H}om(R\eta_! i^! A, j^! B) \end{array}$$

commutes in $D_c^b(Y')$, where $\mathrm{bc} = \mathrm{bc}_{\eta,i}$.

Proof. Since i and j are open embeddings, we have $i^! = i^*$ and $j^! = j^*$, so the statement of the lemma makes sense and is easily proved for sheaves on X and Y. The result then follows using standard arguments for derived functors. □

2.6. If U is an open, dense subvariety of X, and L is a local system on U, then we denote the intersection complex, as in [1], middle perversity, by $\mathrm{IC}(X, L)$. It is a complex of sheaves in $D_c^b(X)$. It is shown in [8, Theorem 3.5] that IC defines a fully faithful functor from the category of local systems on U to $D_c^b(X)$.

Notice that if we start with a complex, A, on an open, dense subvariety of X with $H^p(A) = 0$ for $p \neq 0$, then we may construct a complex $\mathrm{IC}(X, A)$ as in [1, §2.2] starting with A. The complexes $\mathrm{IC}(X, A)$ and $\mathrm{IC}(X, H^0(A))$ are isomorphic in $D^b_c(X)$.

3. Specialization

3.1. In this section we axiomatize a specialization argument that allows us to compute invariants in Borel-Moore homology. There are various schemes that allow one to use generic information to prove (co)-homological results about special fibres, or more generally closed subvarieties (see [7], [4], [15]). Our approach, which is based on an idea of Lusztig in [14] that was generalized by Borho and MacPherson [3], is to use intersection complexes of local systems on open, dense subvarieties of a variety, N, to obtain information about the Borel-Moore homology groups of a closed subvariety, N_0, of N.

We start with what we call the "basic commutative diagram" of morphisms of complex, algebraic varieties consisting of cartesian squares:

(3.1.1)
$$\begin{array}{ccccc} M_0 & \xrightarrow{\eta_0} & P_0 & \xrightarrow{\xi_0} & N_0 \\ {\scriptstyle j_M}\downarrow & & {\scriptstyle j_P}\downarrow & & {\scriptstyle j_N}\downarrow \\ M & \xrightarrow{\eta} & P & \xrightarrow{\xi} & N \\ {\scriptstyle i_M}\uparrow & & {\scriptstyle i_P}\uparrow & & {\scriptstyle i_N}\uparrow \\ M_r & \xrightarrow{\eta_r} & P_r & \xrightarrow{\xi_r} & N_r \end{array}$$

Define

$$\mu = \xi\eta, \quad \mu_r = \xi_r\eta_r, \quad \text{and} \quad \mu_0 = \xi_0\eta_0.$$

We assume that this basic commutative diagram has the following properties:

- D1 The varieties M, P, and N are purely d-dimensional.
- D2 The varieties M and P are rational homology manifolds.
- D3 The morphisms ξ and μ are surjective, proper morphisms that are *small* (see [8, §6.2]) in the sense that for all $r > 0$,

$$\dim\{\, z \in N \mid \dim \xi^{-1}(z) \geq r \,\} < \dim N - 2r$$

  and

$$\dim\{\, z \in N \mid \dim \mu^{-1}(z) \geq r \,\} < \dim N - 2r.$$

- D4 The morphisms i_M, i_P, and i_N are open embeddings.
- D5 The morphisms j_M, j_P, and j_N are closed embeddings.
- D6 A finite group, Σ, acts on M_r on the right so that $N_r \cong M_r/\Sigma$ and μ_r may be identified with the orbit map.
- D7 There is a subgroup, Σ', of Σ, so that $P_r \cong M_r/\Sigma'$ and η_r may be identified with the orbit map.

Since η and ξ are proper morphisms and the squares in the basic commutative diagram are cartesian, it follows that all the horizontal maps in the basic commutative diagram are proper morphisms and that μ, μ_r, and μ_0 are proper morphisms. Thus, if f is any of the morphisms in the basic commutative diagram except i_M, i_P, or i_N, then $Rf_* = Rf_!$. Since i_M, i_P, and i_N are open embeddings, we have $i_M^* = i_M^!$, $i_P^* = i_P^!$, and $i_N^* = i_N^!$. Finally, since

η_r, ξ_r, and μ_r are finite covering maps, we have $\eta_r^! = \eta_r^*$, $\xi_r^! = \xi_r^*$, $\mu_r^! = \mu_r^*$, $R(\eta_r)_! = (\eta_r)_!$, $R(\xi_r)_! = (\xi_r)_!$, and $R(\mu_r)_! = (\mu_r)_!$.

In this section we prove the following theorem.

Theorem 3.1.2. *The group Σ acts on $H_\bullet(M_0)$ and there is an isomorphism $h' \colon H_\bullet(P_0) \cong H_\bullet(M_0)^{\Sigma'}$ so that if $\mathrm{Av}\colon H_\bullet(M_0) \to H_\bullet(M_0)^{\Sigma'}$ is the averaging map, then the diagram*

$$\begin{array}{ccc} H_\bullet(M_0) & \xrightarrow{(\eta_0)_*} & H_\bullet(P_0) \\ & {\scriptstyle \mathrm{Av}}\searrow \quad \swarrow{\scriptstyle h'}\,{\scriptstyle \simeq} & \\ & H_\bullet(M_0)^{\Sigma'} & \end{array}$$

of graded vector spaces commutes.

The idea of the argument is a standard one and is given in the next three subsections. In §3.2 we prove Proposition 3.2.1, the analog of Theorem 3.1.2 for local systems on M_r, P_r, and N_r. In §3.3 we apply IC and use that ξ and μ are small maps to identify the intersection complexes with higher direct images of constant sheaves. Thus we obtain a sheaf-theoretic version of Theorem 3.1.2 for complexes of sheaves in $D_c^b(N)$. In §3.4 we complete the proof of the theorem by restricting to N_0, applying $\mathrm{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j_N^!(\,\cdot\,))$, and showing that the induced map in Borel-Moore homology is $(\eta_0)_*$. Since we are concerned not only with complexes of sheaves, but also the precise maps between them, most of the work involved is in keeping track of morphisms as we apply the various functors.

Finally, in §3.5 we discuss a two variable version of Theorem 3.1.2. Here M, P, and N are replaced by $M \times M$, $P \times Q$, and $N \times N$ respectively, M_0 and P_0 are replaced by the fibred products $Z = (M \times M) \times_{N \times N} N_0$ and $X = (P \times Q) \times_{N \times N} N_0$ respectively, and j_N is replaced by $\delta j_N \colon N_0 \to N \times N$, where δ is the diagonal map. In the application we are mainly interested in (see (5.1.2)), $M \times M = \widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}}$, Z is the Steinberg variety of G, and X is the generalized Steinberg variety $X^{\mathcal{P},\mathcal{Q}}$.

As we have observed above, all the horizontal maps in the basic commutative diagram are proper, so direct image and direct image with proper support are the same functors for these maps. Direct image with proper support is better adapted to Borel-Moore homology, so the following argument is phrased in terms of direct image with proper support.

3.2. First, μ_r may be identified with the orbit map from M_r to M_r/Σ and so Σ acts as automorphisms on the local system $(\mu_r)_!\mathbb{Q}_{M_r}$ on N_r. Similarly, Σ' acts as automorphisms on the local system $(\xi_r)_!\mathbb{Q}_{P_r}$ on N_r.

Next, local systems on N_r form an abelian category so we may consider the Σ'-invariants of the local system $(\mu_r)_!\mathbb{Q}_{M_r}$. Let

$$\mathrm{Av}\colon (\mu_r)_!\mathbb{Q}_{M_r} \longrightarrow \left((\mu_r)_!\mathbb{Q}_{M_r}\right)^{\Sigma'}$$

denote the projection onto the local system of Σ'-invariants given by averaging over Σ'.

Finally, recall from §2.2 that $\alpha_{\eta_r} \colon \eta_r^*\mathbb{Q}_{P_r} \to \mathbb{Q}_{M_r}$ is the natural isomorphism. Since $\eta_r^* = \eta_r^!$, we may consider α_{η_r} as a map from $\eta_r^!\mathbb{Q}_{P_r}$ to $\mathbb{Q}_{M_r}$ and so we may apply $\Phi_{\eta_r}^{-1}$ to $\alpha_{\eta_r}^{-1}$ and get a map from $(\eta_r)_!\mathbb{Q}_{M_r}$ to $\mathbb{Q}_{P_r}$. Define

$$\gamma_r \colon (\eta_r)_!\mathbb{Q}_{M_r} \longrightarrow \mathbb{Q}_{P_r} \quad \text{by} \quad \gamma_r = \Phi_{\eta_r}^{-1}(\alpha_{\eta_r}^{-1}) = \epsilon^{\eta_r} \circ (\eta_r)_!(\alpha_{\eta_r}^{-1}).$$

The following proposition is easily proved either directly, or by using the correspondence between local systems and representations of fundamental groups.

Proposition 3.2.1. *There is an isomorphism* $h_r\colon (\xi_r)_!\mathbb{Q}_{P_r} \cong ((\mu_r)_!\mathbb{Q}_{M_r})^{\Sigma'}$ *so that the diagram*

$$\begin{array}{ccc} (\mu_r)_!\mathbb{Q}_{M_r} & \xrightarrow{(\xi_r)_!(\gamma_r)} & (\xi_r)_!\mathbb{Q}_{P_r} \\ & {\scriptstyle \mathrm{Av}}\searrow \qquad \swarrow{\scriptstyle h_r}\ \simeq & \\ & ((\mu_r)_!\mathbb{Q}_{M_r})^{\Sigma'} & \end{array}$$

of local systems on N_r *commutes.*

3.3. In this subsection we prove the following proposition, the analog of Proposition 3.2.1 for M, P, and N.

Proposition 3.3.1. *There is a map* $\gamma\colon R\eta_!\mathbb{Q}_M \to \mathbb{Q}_P$ *and an isomorphism* $h\colon R\xi_!\mathbb{Q}_P \to (R\mu_!\mathbb{Q}_M)^{\Sigma'}$ *so that the diagram*

$$\begin{array}{ccc} R\mu_!\mathbb{Q}_M & \xrightarrow{R\xi_!(\gamma)} & R\xi_!\mathbb{Q}_P \\ & {\scriptstyle \mathrm{Av}}\searrow \qquad \swarrow{\scriptstyle h}\ \simeq & \\ & (R\mu_!\mathbb{Q}_M)^{\Sigma'} & \end{array}$$

of complexes in $D^b_c(N)$ *commutes.*

We can apply the functor $\mathrm{IC}(N, \cdot\,)$ to the diagram of local systems in Proposition 3.2.1 and obtain a commutative triangle of complexes in $D^b_c(N)$. Since the functor $\mathrm{IC}(N, \cdot\,)$ takes its values in an abelian category of perverse sheaves on N and is an additive functor by construction, we may consider $\mathrm{IC}(N, \cdot\,)$ as an additive functor between abelian categories. It follows that Σ acts on $\mathrm{IC}(N, (\mu_r)_!\mathbb{Q}_{M_r})$, that

$$\mathrm{IC}(N, ((\mu_r)_!\mathbb{Q}_{M_r})^{\Sigma'}) \cong \mathrm{IC}(N, (\mu_r)_!\mathbb{Q}_{M_r})^{\Sigma'},$$

and that if Av is the averaging map, the diagram

$$\begin{array}{ccc} \mathrm{IC}(N, (\mu_r)_!\mathbb{Q}_{M_r}) & \xrightarrow{\mathrm{IC}(N,(\xi_r)_!(\gamma_r))} & \mathrm{IC}(N, (\xi_r)_!\mathbb{Q}_{P_r}) \\ & {\scriptstyle \mathrm{Av}}\searrow \qquad \swarrow{\scriptstyle \mathrm{IC}(h_r)}\ \simeq & \\ & \mathrm{IC}(N, (\mu_r)_!\mathbb{Q}_{M_r})^{\Sigma'} & \end{array}$$

of complexes in $D^b_c(N)$ commutes.

Since ξ and μ are small maps, it follows from the axioms characterizing intersection complexes (see [1, §4.13]) that $\mathrm{IC}(N, (\mu_r)_!\mathbb{Q}_{M_r})$ and $\mathrm{IC}(N, (\xi_r)_!\mathbb{Q}_{P_r})$ are isomorphic in $D^b_c(N)$ to the direct images $R\mu_!\mathbb{Q}_M$ and $R\xi_!\mathbb{Q}_P$ respectively. Moreover, since the Σ-action on $R\mu_!\mathbb{Q}_M$ comes from transport of structure from $(\mu_r)_!\mathbb{Q}_{M_r}$ it follows that there are isomorphisms, $\overline{\mu}$,

$\overline{\xi}$, and h, so that if $g = \overline{\xi}^{-1} \circ \mathrm{IC}(N, (\xi_r)_!(\gamma_r)) \circ \overline{\mu}$, then the diagram

$$\begin{array}{ccccc} \mathrm{IC}(N, (\mu_r)_!\mathbb{Q}_{M_r}) & & \xrightarrow{\mathrm{IC}(N,(\xi_r)_!(\gamma_r))} & & \mathrm{IC}(N, (\xi_r)_!\mathbb{Q}_{P_r}) \\ \uparrow{\scriptstyle \overline{\mu}} & & & & \uparrow{\scriptstyle \overline{\xi}} \\ R\mu_!\mathbb{Q}_M & & \xrightarrow{g} & & R\xi_!\mathbb{Q}_P \\ & \searrow^{\mathrm{Av}} & & \swarrow^{h} & \\ & & (R\mu_!\mathbb{Q}_M)^{\Sigma'} & & \end{array} \tag{3.3.2}$$

in $D^b_c(N)$ commutes. We can apply the functor $\mathrm{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j^!_N(\,\cdot\,))$ to the bottom triangle in (3.3.2) and obtain a commutative triangle of Ext-groups that are isomorphic to the Borel-Moore homology groups in the statement of Theorem 3.1.2. In order to show that the resulting horizontal map is indeed the direct image map in Borel-Moore homology induced by η_0, we need to choose the isomorphisms $\overline{\mu}$ and $\overline{\xi}$ appropriately and identify the map g in (3.3.2). This is accomplished in the next lemma and the following corollary.

Since P is a purely d-dimensional, rational homology manifold, we have $\mathbb{D}_P \cong \mathbb{Q}_P[2d]$ in $D^b_c(P)$. We denote by ν_P a fixed isomorphism, $\nu_P \colon \mathbb{D}_P \to \mathbb{Q}_P[2d]$ in $D^b_c(P)$.

Now $i^!_M\mathbb{D}_M[-2d]$ and $i^!_M\mathbb{Q}_M$ are in $D^b_c(M_r)$ and $i^!_M(\alpha_\eta) \circ i^!_M(\eta^!(\nu_P) \circ \beta_\eta)$ is an isomorphism between them, so $i^!_M\mathbb{D}_M[-2d]$ is in fact a local system on M_r. Notice that $\eta^!(\nu_P) \colon \eta^!\mathbb{D}_P[-2d] \to \eta^!\mathbb{Q}_P$ and $\alpha_\eta \colon \eta^*\mathbb{Q}_P \to \mathbb{Q}_M$, so the composition $\alpha_\eta \circ \eta^!(\nu_P)$ is not defined. However,

$$i^!_M\eta^! = (\eta i_M)^! = (i_P\eta_r)^! = \eta_r^! i^!_P = \eta_r^* i^*_P = i^*_M\eta^* = i^!_M\eta^*,$$

so the composition $i^!_M(\alpha_\eta) \circ i^!_M(\eta^!(\nu_P))$ is defined.

By [1, Lemma 4.11] there is a unique isomorphism of local systems on M that restricts to $i^!_M(\alpha_\eta) \circ i^!_M(\eta^!(\nu_P) \circ \beta_\eta)$. The statement in [1] assumes that M is a manifold, but the argument applies when M is a variety that is a rational homology manifold. Denote this isomorphism by ν^P_M, so $\nu^P_M \colon \mathbb{D}_M[-2d] \to \mathbb{Q}_M$ and

$$i^!_M(\nu^P_M) = i^!_M(\alpha_\eta) \circ i^!_M(\eta^!(\nu_P) \circ \beta_\eta). \tag{3.3.3}$$

Define $\gamma \colon R\eta_!\mathbb{Q}_M \to \mathbb{Q}_P$ by

$$\gamma = \nu_P \circ \Phi^{-1}_\eta(\beta_\eta \circ (\nu^P_M)^{-1}) = \nu_P \circ \epsilon^\eta \circ R\eta_!(\beta_\eta \circ (\nu^P_M)^{-1}) = \Phi^{-1}_\eta(\eta^!(\nu_P) \circ \beta_\eta \circ (\nu^P_M)^{-1}).$$

Lemma 3.3.4. *The diagram*

$$\begin{array}{ccc} i^!_N R\mu_!\mathbb{Q}_M & \xrightarrow{i^!_N R\xi_!(\gamma)} & i^!_N R\xi_!\mathbb{Q}_P \\ {\scriptstyle (\mu_r)_!(\alpha_{i_M})\circ \mathrm{bc}_{\mu_r,i_M}}\downarrow & & \downarrow{\scriptstyle (\xi_r)_!(\alpha_{i_P})\circ \mathrm{bc}_{\xi_r,i_P}} \\ (\mu_r)_!\mathbb{Q}_{M_r} & \xrightarrow[(\xi_r)_!(\gamma_r)]{} & (\xi_r)_!\mathbb{Q}_{P_r} \end{array}$$

of complexes in $D^b_c(N_r)$ commutes.

Proof. Since $\mathrm{bc}_{\mu_r,i_M} = (\xi_r)_!(\mathrm{bc}_{\eta_r,i_M}) \circ \mathrm{bc}_{\xi_r,i_P}$, we need to show that

$$(\xi_r)_!(\alpha_{i_P}) \circ \mathrm{bc}_{\xi_r,i_P} \circ i^!_N R\xi_!(\gamma) = (\xi_r)_!(\gamma_r) \circ (\mu_r)_!(\alpha_{i_M}) \circ (\xi_r)_!(\mathrm{bc}_{\eta_r,i_M}) \circ \mathrm{bc}_{\xi_r,i_P}.$$

Using the naturality of the base change morphism bc_{ξ_r, i_P} we see that it is enough to show that

$$(\xi_r)_!(\alpha_{i_P}) \circ (\xi_r)_! i_P^!(\gamma) = (\xi_r)_!(\gamma_r) \circ (\mu_r)_!(\alpha_{i_M}) \circ (\xi_r)_!(\mathrm{bc}_{\eta_r, i_M}).$$

Since $\gamma_r = \epsilon^{\eta_r} \circ (\eta_r)_!(\alpha_{\eta_r}^{-1})$ it's enough to show that

$$\alpha_{i_P} \circ i_P^!(\gamma) = \epsilon^{\eta_r} \circ (\eta_r)_!(\alpha_{\eta_r}^{-1} \circ \alpha_{i_M}) \circ \mathrm{bc}_{\eta_r, i_M}.$$

Equivalently, it's enough to show that

$$i_P^!(\gamma) \circ \mathrm{bc}_{\eta_r, i_M}^{-1} = \alpha_{i_P}^{-1} \circ \epsilon^{\eta_r} \circ (\eta_r)_!(\alpha_{\eta_r}^{-1} \circ \alpha_{i_M}).$$

Finally, $\eta i_M = i_P \eta_r$ and so $\Phi_{i_M} \Phi_\eta = \Phi_{\eta_r} \Phi_{i_P}$ and hence $\Phi_{i_P} \Phi_\eta^{-1} = \Phi_{\eta_r}^{-1} \Phi_{i_M}$. Therefore:

$$\begin{aligned}
i_P^!(\gamma) \circ \mathrm{bc}_{\eta_r, i_M}^{-1} &= i_P^! \left(\Phi_\eta^{-1}(\eta^!(\nu_P) \circ \beta_\eta \circ (\nu_M^P)^{-1})\right) \circ \Phi_{i_P}(R\eta_!(\epsilon^{i_M})) \\
&= \Phi_{i_P} \left(\Phi_\eta^{-1}(\eta^!(\nu_P) \circ \beta_\eta \circ (\nu_M^P)^{-1}) \circ R\eta_!(\epsilon^{i_M})\right) && \text{(by 2.4.2)} \\
&= \Phi_{i_P} \Phi_\eta^{-1} \left(\eta^!(\nu_P) \circ \beta_\eta \circ (\nu_M^P)^{-1} \circ \epsilon^{i_M}\right) && \text{(by 2.4.3)} \\
&= \Phi_{\eta_r}^{-1} \Phi_{i_M} \left(\eta^!(\nu_P) \circ \beta_\eta \circ (\nu_M^P)^{-1} \circ \epsilon^{i_M}\right) && (\Phi_{i_P} \Phi_\eta^{-1} = \Phi_{\eta_r}^{-1} \Phi_{i_M}) \\
&= \Phi_{\eta_r}^{-1} \left(i_M^!(\eta^!(\nu_P) \circ \beta_\eta \circ (\nu_M^P)^{-1})\right) && \text{(by 2.4.2)} \\
&= \Phi_{\eta_r}^{-1} \left(i_M^!(\alpha_\eta^{-1})\right) && \text{(by 3.3.3)} \\
&= \Phi_{\eta_r}^{-1} \left(i_M^*(\alpha_\eta^{-1})\right) \\
&= \Phi_{\eta_r}^{-1} \left(\eta_r^!(\alpha_{i_P}^{-1}) \circ \alpha_{\eta_r}^{-1} \circ \alpha_{i_M}\right) && \text{(by 2.2.2)} \\
&= \alpha_{i_P}^{-1} \circ \Phi_{\eta_r}^{-1} \left(\alpha_{\eta_r}^{-1} \circ \alpha_{i_M}\right) && \text{(by 2.4.3)} \\
&= \alpha_{i_P}^{-1} \circ \epsilon^{\eta_r} \circ (\eta_r)_! \left(\alpha_{\eta_r}^{-1} \circ \alpha_{i_M}\right) && \text{(by 2.4.1)}
\end{aligned}$$

This completes the proof of the lemma. □

Corollary 3.3.5. *There are isomorphisms,*

$$\overline{\mu}\colon R\mu_!\mathbb{Q}_M \longrightarrow \mathrm{IC}(N, (\mu_r)_!\mathbb{Q}_{M_r}) \quad \text{and} \quad \overline{\xi}\colon R\xi_!\mathbb{Q}_P \longrightarrow \mathrm{IC}(N, (\xi_r)_!\mathbb{Q}_{P_r}),$$

so that the diagram

$$\begin{array}{ccc}
R\mu_!\mathbb{Q}_M & \xrightarrow{\quad R\xi_!(\gamma) \quad} & R\xi_!\mathbb{Q}_P \\
{\scriptstyle \overline{\mu}} \big\downarrow & & \big\downarrow {\scriptstyle \overline{\xi}} \\
\mathrm{IC}(N, (\mu_r)_!\mathbb{Q}_{M_r}) & \xrightarrow[\mathrm{IC}((\xi_r)_!(\gamma_r))]{} & \mathrm{IC}(N, (\xi_r)_!\mathbb{Q}_{P_r})
\end{array}$$

of complexes in $D_c^b(N)$ commutes.

Proof. We have already observed that since ξ and μ are small maps, the direct images, $R\xi_!\mathbb{Q}_P$ and $R\mu_!\mathbb{Q}_M$ are isomorphic in $D_c^b(N)$ to $\mathrm{IC}(N, \xi_!\mathbb{Q}_{P_r})$ and $\mathrm{IC}(N, \mu_!\mathbb{Q}_{M_r})$ respectively. Thus, $R\xi_!\mathbb{Q}_P$ and $R\mu_!\mathbb{Q}_M$ are in the image of IC. It is shown in [8, Theorem 3.5] that on the image of IC, the composition $\mathrm{IC}(N, \cdot\,) \circ i_N^*$ is naturally equivalent to the identity so there are isomorphisms,

$$\mathrm{ic}_\mu\colon R\mu_!\mathbb{Q}_M \xrightarrow{\simeq} \mathrm{IC}(N, i_N^! R\mu_!\mathbb{Q}_M) \quad \text{and} \quad \mathrm{ic}_\xi\colon R\xi_!\mathbb{Q}_P \xrightarrow{\simeq} \mathrm{IC}(N, i_N^! R(\xi_r)_!\mathbb{Q}_{P_r}),$$

in $D(N)$ with $i_N^*(\mathrm{ic}_\mu) = id$ and $i_N^*(\mathrm{ic}_\xi) = id$. Since IC is fully faithful, it follows that the diagram

$$\begin{array}{ccc} R\mu_!\mathbb{Q}_M & \xrightarrow{R\xi_!(\gamma)} & R\xi_!\mathbb{Q}_P \\ {\scriptstyle \mathrm{ic}_\mu}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{ic}_\xi} \\ \mathrm{IC}(N, i_N^! R\mu_!\mathbb{Q}_M) & \xrightarrow[\mathrm{IC}(i_N^! R\xi_!(\gamma))]{} & \mathrm{IC}(N, i_N^! R(\xi_r)_!\mathbb{Q}_{P_r}) \end{array}$$

commutes.

If we apply IC to the commutative diagram in the lemma we get a commutative diagram:

$$\begin{array}{ccc} \mathrm{IC}(N, i_N^! R\mu_!\mathbb{Q}_M) & \xrightarrow{\mathrm{IC}(i_N^! R\xi_!(\gamma))} & \mathrm{IC}(N, i_N^! R(\xi_r)_!\mathbb{Q}_{P_r}) \\ {\scriptstyle \mathrm{IC}((\mu_r)_!(\alpha_{i_M})\circ \mathrm{bc}_{\mu_r,i_M})}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{IC}((\xi_r)_!(\alpha_{i_P})\circ \mathrm{bc}_{\xi_r,i_P})} \\ \mathrm{IC}(N, (\mu_r)_!\mathbb{Q}_{M_r}) & \xrightarrow[\mathrm{IC}((\xi_r)_!(\gamma_r))]{} & \mathrm{IC}(N, (\xi_r)_!\mathbb{Q}_{P_r}) \end{array}$$

Therefore, if we define $\overline{\mu} = \mathrm{IC}((\mu_r)_!(\alpha_{i_M}) \circ \mathrm{bc}_{\mu_r,i_M}) \circ \mathrm{ic}_\mu$ and $\overline{\xi} = \mathrm{IC}((\xi_r)_!(\alpha_{i_P}) \circ \mathrm{bc}_{\xi_r,i_P}) \circ \mathrm{ic}_\xi$, the corollary follows. □

Since $\overline{\mu}\colon R\mu_!\mathbb{Q}_M \to \mathrm{IC}(N, (\mu_r)_!\mathbb{Q}_{M_r})$ is an isomorphism, it follows that Σ acts on $R\mu_!\mathbb{Q}_M$ by transport of structure and that $\overline{\mu}$ induces an isomorphism between Σ'-invariants, say $\overline{\mu}'\colon (R\mu_!\mathbb{Q}_M)^{\Sigma'} \to \mathrm{IC}(N, (\mu_r)_!\mathbb{Q}_{M_r})^{\Sigma'}$, which commutes with the respective averaging maps.

Now consider the diagram:

$$\begin{array}{ccccccc} R\mu_!\mathbb{Q}_M & \xrightarrow{\overline{\mu}} & \mathrm{IC}(N, (\mu_r)_!\mathbb{Q}_{M_r}) & \xrightarrow{\mathrm{IC}((\xi_r)_!(\gamma_r))} & \mathrm{IC}(N, (\xi_r)_!\mathbb{Q}_{P_r}) & \xleftarrow{\overline{\xi}} & R\xi_!\mathbb{Q}_P \\ {\scriptstyle \mathrm{Av}}\big\downarrow & & & {\scriptstyle \mathrm{Av}}\searrow \quad \swarrow{\scriptstyle \mathrm{IC}(h_r)} & & & \big\downarrow{\scriptstyle h} \\ (R\mu_!\mathbb{Q}_M)^{\Sigma'} & & \xrightarrow[\overline{\mu}']{} & \mathrm{IC}(N, (\mu_r)_!\mathbb{Q}_{M_r})^{\Sigma'} & \xleftarrow[\overline{\mu}']{} & & (R\mu_!\mathbb{Q}_M)^{\Sigma'} \end{array}$$

If h is defined by $h = (\overline{\mu}')^{-1} \circ \mathrm{IC}(h_r) \circ \overline{\xi}$, then the diagram commutes. By Corollary 3.3.5, the composition across the top row is just $R\xi_!(\gamma)$ and so tracing around the outside of the diagram we see that $h \circ R\xi_!(\gamma) = \mathrm{Av}$. This completes the proof of Proposition 3.3.1.

3.4. In this subsection, we complete the proof of Theorem 3.1.2.

Lemma 3.4.1. *There are isomorphisms of graded vector spaces,*

$$J'\colon H_{2d-\bullet}(M_0) \xrightarrow{\simeq} \mathrm{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j_N^! R\mu_!\mathbb{Q}_M)$$

and

$$J_1'\colon H_{2d-\bullet}(P_0) \xrightarrow{\simeq} \mathrm{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j_N^! R\xi_!\mathbb{Q}_P)$$

so that the diagram

$$\begin{array}{ccc} H_{2d-\bullet}(M_0) & \xrightarrow{(\eta_0)_!} & H_{2d-\bullet}(P_0) \\ \downarrow{\scriptstyle J'} & & \downarrow{\scriptstyle J'_1} \\ \mathrm{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j_N^! R\mu_! \mathbb{Q}_M) & \xrightarrow{(j_N^! R\xi_!(\gamma))_\sharp} & \mathrm{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j_N^! R\xi_! \mathbb{Q}_P) \\ & {\scriptstyle \mathrm{Av}}\searrow \quad \swarrow{\scriptstyle (j_N^!(h))_\sharp} & \\ & \mathrm{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j_N^! R\mu_! \mathbb{Q}_M)^{\Sigma'} & \end{array}$$

commutes.

Assuming for a moment that the lemma has been proved, we complete the proof of Theorem 3.1.2 using the argument at the end of §3.3 as follows.

Since J' is an isomorphism, Σ acts on $H_{2d-\bullet}(M_0)$ by transport of structure and J' induces an isomorphism between Σ'-invariants, say $\overline{J}$, which commutes with the respective averaging maps.

Now consider the diagram

$$\begin{array}{ccccccc} H_{2d-\bullet}(M_0) & \xrightarrow{J'} & E_1 & \xrightarrow{(j_N^! R\xi_!(\gamma))_\sharp} & E_2 & \xleftarrow{J'_1} & H_{2d-\bullet}(P_0) \\ \downarrow{\scriptstyle \mathrm{Av}} & & & {\scriptstyle \mathrm{Av}}\searrow \quad \swarrow{\scriptstyle (j_N^!(h))_\sharp} & & & \vdots{\scriptstyle h'} \\ H_{2d-\bullet}(M_0)^{\Sigma'} & & \xrightarrow[\overline{J}]{} & E_3 & \xleftarrow[\overline{J}]{} & & H_{2d-\bullet}(M_0)^{\Sigma'} \end{array}$$

where

$$E_1 = \mathrm{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j_N^! R\mu_! \mathbb{Q}_M), \quad E_2 = \mathrm{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j_N^! R\xi_! \mathbb{Q}_P), \quad \text{and}$$
$$E_3 = \mathrm{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j_N^! R\mu_! \mathbb{Q}_M)^{\Sigma'}.$$

If h' is defined by $h' = (\overline{J})^{-1} \circ (j_N^!(h))_\sharp \circ J'_1$, then the diagram commutes. By Lemma 3.4.1, the composition across the top row is $(\eta_0)_*$ and so tracing around the outside of the diagram we see that $h' \circ (\eta_0)_* = \mathrm{Av}$. This proves Theorem 3.1.2.

It remains to prove Lemma 3.4.1.

First, we apply the functor $\mathrm{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j_N^!(\,\cdot\,))$ to the diagram in Proposition 3.3.1 and obtain a commutative triangle of graded vector spaces. Since the functor $\mathrm{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j_N^!(\,\cdot\,))$ restricted to the abelian category of perverse sheaves in which $\mathrm{IC}(N, \cdot\,)$ takes its values is an additive functor between abelian categories, it follows that Σ acts on $\mathrm{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j_N^! R\mu_! \mathbb{Q}_M)$, that

$$\mathrm{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j_N^!(R\mu_! \mathbb{Q}_M)^{\Sigma'}) \cong \mathrm{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j_N^! R\mu_! \mathbb{Q}_M)^{\Sigma'},$$

and that if Av is the averaging map, the diagram of graded vector spaces

$$\begin{array}{ccc} \mathrm{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j_N^! R\mu_! \mathbb{Q}_M) & \xrightarrow{(j_N^! R\xi_!(\gamma))_\sharp} & \mathrm{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j_N^! R\xi_! \mathbb{Q}_P) \\ & {\scriptstyle \mathrm{Av}}\searrow \quad \swarrow{\scriptstyle (j_N^!(h))_\sharp} & \\ & \mathrm{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j_N^! R\mu_! \mathbb{Q}_M)^{\Sigma'} & \end{array}$$

commutes.

Next, recall that $\gamma = \nu_{_P} \circ \epsilon^\eta \circ R\eta_!(\beta_\eta \circ (\nu^P_{_M})^{-1})$, so using (2.4.1) we get

$$\nu_{_P} \circ \Phi_\eta^{-1}(\beta_\eta) = \gamma \circ R\eta_!(\nu^P_{_M}) \colon R\eta_!\mathbb{D}_M[-2d] \longrightarrow \mathbb{Q}_P.$$

Applying $j^!_{_N} R\xi_!$ we get $j^!_{_N} R\xi_!(\nu_{_P}) \circ j^!_{_N} R\xi_!(\Phi_\eta^{-1}(\beta_\eta)) = j^!_{_N} R\xi_!(\gamma) \circ j^!_{_N} R\mu_!(\nu^P_{_M})$. This shows that the diagram

$$\begin{array}{ccc}
\operatorname{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j^!_{_N} R\mu_!\mathbb{D}_M) & \xrightarrow{(j^!_{_N} R\xi_!(\Phi_\eta^{-1}(\beta_\eta)))_\sharp} & \operatorname{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j^!_{_N} R\xi_!\mathbb{D}_P) \\
{\scriptstyle (j^!_{_N} R\mu_!(\nu^P_{_M}))_\sharp}\downarrow & & \downarrow{\scriptstyle (j^!_{_N} R\mu_!(\nu_P))_\sharp} \\
\operatorname{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j^!_{_N} R\mu_!\mathbb{Q}_M) & \xrightarrow[(j^!_{_N} R\xi_!(\gamma))_\sharp]{} & \operatorname{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j^!_{_N} R\xi_!\mathbb{Q}_P)
\end{array}$$

commutes.

Finally, we show that there are isomorphisms

$$J \colon H_{2d-\bullet}(M_0) \xrightarrow{\simeq} \operatorname{Ext}^\bullet_{N_0}(\mathbb{D}_{N_0}, j^!_{_N} R\mu_!\mathbb{Q}_M)$$

and

$$J_1 \colon H_{2d-\bullet}(P_0) \xrightarrow{\simeq} \operatorname{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j^!_{_N} R\xi_!\mathbb{D}_P)$$

so that the diagram

(3.4.2)
$$\begin{array}{ccc}
\operatorname{Ext}^\bullet_{M_0}(\mathbb{Q}_{M_0}, \mathbb{D}_{M_0}) & \xrightarrow{(\eta_0)_*} & \operatorname{Ext}^\bullet_{P_0}(\mathbb{Q}_{P_0}, \mathbb{D}_{P_0}) \\
{\scriptstyle J}\downarrow & & \downarrow{\scriptstyle J_1} \\
\operatorname{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j^!_{_N} R\mu_!\mathbb{D}_M) & \xrightarrow[(j^!_{_N} R\xi_!(\Phi_\eta^{-1}(\beta_\eta)))_\sharp]{} & \operatorname{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j^!_{_N} R\xi_!\mathbb{D}_P)
\end{array}$$

commutes. Once this has been done, set $J' = (j^!_{_N} R\mu_!(\nu^P_{_M}))_\sharp \circ J$ and $J_1' = (j^!_{_N} R\xi_!(\nu_{_P}))_\sharp \circ J_1$. Then $J_1' \circ (\eta_0)_! = (j^!_{_N} R\xi_!(\gamma))_\sharp \circ J'$ and so the diagram in the statement of Lemma 3.4.1 commutes as claimed.

Recall that since η_0 is a proper map, it induces a map in Borel-Moore homology. If Ψ_{η_0} is the adjunction of the adjoint pair $(\eta_0^*, (R\eta_0)_*)$, then $(\eta_0)_*$ is the composition,

$$\begin{aligned}
H_{-\bullet}(M_0) &= \operatorname{Ext}^\bullet_{M_0}(\mathbb{Q}_{M_0}, \mathbb{D}_{M_0}) \\
&\cong \operatorname{Ext}^\bullet_{M_0}(\eta_0^*\mathbb{Q}_{P_0}, \mathbb{D}_{M_0}) && \text{by } \alpha^\sharp_{\eta_0} \\
&\cong \operatorname{Ext}^\bullet_{P_0}(\mathbb{Q}_{P_0}, R(\eta_0)_*\mathbb{D}_{M_0}) && \text{by } \Psi_{\eta_0} \\
&\cong \operatorname{Ext}^\bullet_{P_0}(\mathbb{Q}_{P_0}, R(\eta_0)_!\eta_0^!\mathbb{D}_{P_0}) && \text{by } (R(\eta_0)_!(\beta_{\eta_0}))_\sharp \\
&\longrightarrow \operatorname{Ext}^\bullet_{P_0}(\mathbb{Q}_{P_0}, \mathbb{D}_{P_0}) && \text{by } (\epsilon^{\eta_0})_\sharp \\
&= H_{-\bullet}(P_0),
\end{aligned}$$

so

$$(\eta_0)_* = (\epsilon^{\eta_0} \circ R(\eta_0)_!(\beta_{\eta_0}))_\sharp \circ \Psi_{\eta_0} \circ \alpha^\sharp_{\eta_0} = \Phi^{-1}_{\eta_0}(\beta_{\eta_0})_\sharp \circ \Psi_{\eta_0} \circ \alpha^\sharp_{\eta_0}.$$

Now consider the diagram

$$\begin{array}{ccccc}
\operatorname{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j^!_N R\mu_!\mathbb{D}_M) & \xrightarrow{(\dagger)} & \operatorname{Ext}^\bullet_{P_0}(\mathbb{Q}_{P_0}, R(\eta_0)_*\mathbb{D}_{M_0}) & \xleftarrow{\Psi_{\eta_0}\circ\alpha^\sharp_{\eta_0}} & \operatorname{Ext}^\bullet_{M_0}(\mathbb{Q}_{M_0}, \mathbb{D}_{M_0}) \\
{\scriptstyle (*)}\downarrow & & {\scriptstyle (**)}\downarrow & \swarrow{\scriptstyle (\eta_0)_*} & \\
\operatorname{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j^!_N R\xi_!\mathbb{D}_P) & \xrightarrow[(\dagger\dagger)]{} & \operatorname{Ext}^\bullet_{P_0}(\mathbb{Q}_{P_0}, \mathbb{D}_{P_0}) & &
\end{array}$$

where $(*) = \big(j^!_N R\xi_!(\Phi^{-1}_\eta(\beta_\eta))\big)_\sharp$, $(**) = \Phi^{-1}_{\eta_0}(\beta_{\eta_0})_\sharp$, and $(\dagger)$ and $(\dagger\dagger)$ are given by the compositions

$$\begin{aligned}
\operatorname{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j^!_N R\mu_!\mathbb{D}_M) &\cong \operatorname{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, R(\xi_0)_! j^!_P R\eta_!\mathbb{D}_M) && \text{by } \big(\mathrm{bc}_{\xi_0, j_P},\big)_\sharp \\
&\cong \operatorname{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, R(\xi_0)_! R(\eta_0)_! j^!_M \mathbb{D}_M) && \text{by } \big(R(\xi_0)_!(\mathrm{bc}_{\eta_0, j_M})\big)_\sharp \\
&\cong \operatorname{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, R(\xi_0)_! R(\eta_0)_! \mathbb{D}_{M_0}) && \text{by } \Big(R(\xi_0)_* R(\eta_0)_*(\beta^{-1}_{j_M})\Big)_\sharp \\
&\cong \operatorname{Ext}^\bullet_{P_0}(\xi_0^*\mathbb{Q}_{N_0}, R(\eta_0)_!\mathbb{D}_{M_0}) && \text{by } \Psi^{-1}_{\xi_0} \\
&\cong \operatorname{Ext}^\bullet_{P_0}(\mathbb{Q}_{P_0}, R(\eta_0)_!\mathbb{D}_{M_0}) && \text{by } \big(\alpha^{-1}_{\xi_0}\big)_\sharp
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j^!_N R\xi_!\mathbb{D}_P) &\cong \operatorname{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, R(\xi_0)_! j^!_P \mathbb{D}_P) && \text{by } \big(\mathrm{bc}_{\xi_0, j_P},\big)_\sharp \\
&\cong \operatorname{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, R(\xi_0)_! \mathbb{D}_{P_0}) && \text{by } \Big(R(\xi_0)_*(\beta^{-1}_{j_P})\Big)_\sharp \\
&\cong \operatorname{Ext}^\bullet_{P_0}(\xi_0^*\mathbb{Q}_{N_0}, \mathbb{D}_{P_0}) && \text{by } \Psi^{-1}_{\xi_0} \\
&\cong \operatorname{Ext}^\bullet_{P_0}(\mathbb{Q}_{P_0}, \mathbb{D}_{P_0}) && \text{by } \big(\alpha^{-1}_{\xi_0}\big)_\sharp
\end{aligned}$$

respectively, so

$$(\dagger) = (\alpha^{-1}_{\xi_0})^\sharp \circ \Psi^{-1}_{\xi_0} \circ \Big(R(\xi_0)_* R(\eta_0)_*(\beta^{-1}_{j_M}) \circ R(\xi_0)_*(\mathrm{bc}_{\eta_0, j_M}) \circ \mathrm{bc}_{\xi_0, j_P}\Big)_\sharp$$

and

$$(\dagger\dagger) = (\alpha^{-1}_{\xi_0})^\sharp \circ \Psi^{-1}_{\xi_0} \circ \Big(R(\xi_0)_*(\beta^{-1}_{j_P}) \circ \mathrm{bc}_{\xi_0, j_P}\Big)_\sharp .$$

Assume for a moment that $(**) \circ (\dagger) = (\dagger\dagger) \circ (*)$ and define

$$J = (\dagger)^{-1} \circ \Psi_{\eta_0} \circ \alpha^\sharp_{\eta_0} \colon \operatorname{Ext}^\bullet_{M_0}(\mathbb{Q}_{M_0}, \mathbb{D}_{M_0}) \longrightarrow \operatorname{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j^!_N R\mu_!\mathbb{D}_M)$$

and

$$J_1 = (\dagger\dagger)^{-1} \colon \operatorname{Ext}^\bullet_{P_0}(\mathbb{Q}_{P_0}, \mathbb{D}_{P_0}) \longrightarrow \operatorname{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j^!_N R\xi_!\mathbb{D}_P).$$

Then J and J_1 are isomorphisms and

$$J_1 \circ (\eta_0)_* = (\dagger\dagger)^{-1} \circ (\eta_0)_* = (j^!_N R\xi_!(\Phi^{-1}_\eta(\beta_\eta)))_\sharp \circ [(\dagger)^{-1} \circ \Psi_{\eta_0} \circ \alpha^\sharp_{\eta_0}] = (j^!_N R\xi_!(\Phi^{-1}_\eta(\beta_\eta)))_\sharp \circ J$$

so diagram (3.4.2) commutes as claimed.

It remains to show that $(**) \circ (\dagger) = (\dagger\dagger) \circ (*)$. Suppose h is in $\operatorname{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j^!_N R\mu_!\mathbb{D}_M)$. Then

$$((**) \circ (\dagger))(h) = \Phi^{-1}_{\eta_0}(\beta_{\eta_0}) \circ \Psi^{-1}_{\xi_0}\Big(R(\mu_0)_!(\beta^{-1}_{j_M}) \circ R(\xi_o)_!(\mathrm{bc}_{\eta_0, j_M}) \circ \mathrm{bc}_{\xi_0, j_P} \circ h\Big) \circ \alpha^{-1}_{\xi_0}$$

$$= \Psi_{\xi_0}^{-1}\Big(R(\xi_0)_!(\Phi_{\eta_0}^{-1}(\beta_{\eta_0}) \circ R(\eta_0)_!(\beta_{j_M}^{-1}) \circ \mathrm{bc}_{\eta_0, j_M}) \circ \mathrm{bc}_{\xi_0, j_P} \circ h\Big) \circ \alpha_{\xi_0}^{-1}.$$

On the other hand, using the naturality of the base change bc_{ξ_0, j_P} we have

$$\begin{aligned} ((\dagger\dagger) \circ (*))(h) &= \Psi_{\xi_0}^{-1}\Big(R(\xi_0)_!(\beta_{j_P}^{-1}) \circ \mathrm{bc}_{\xi_0, j_P} \circ j_N^! R\xi_!(\Phi_\eta^{-1}(\beta_\eta)) \circ h\Big) \circ \alpha_{\xi_0}^{-1} \\ &= \Psi_{\xi_0}^{-1}\Big(R(\xi_0)_!(\beta_{j_P}^{-1}) \circ R(\xi_0)_! j_P^!(\Phi_\eta^{-1}(\beta_\eta)) \circ \mathrm{bc}_{\xi_0, j_P} \circ h\Big) \circ \alpha_{\xi_0}^{-1} \\ &= \Psi_{\xi_0}^{-1}\Big(R(\xi_0)_!(\beta_{j_P}^{-1} \circ j_P^!(\Phi_\eta^{-1}(\beta_\eta))) \circ \mathrm{bc}_{\xi_0, j_P} \circ h\Big) \circ \alpha_{\xi_0}^{-1} \end{aligned}$$

so it is enough to show that

$$\Phi_{\eta_0}^{-1}(\beta_{\eta_0}) \circ R(\eta_0)_!(\beta_{j_M}^{-1}) \circ \mathrm{bc}_{\eta_0, j_M} = \beta_{j_P}^{-1} \circ j_P^!(\Phi_\eta^{-1}(\beta_\eta)).$$

This last equality is easily proved by a computation similar to the computation in the proof of Lemma 3.3.4 using the definition of $\mathrm{bc}_{\eta_0, j_M}$ from §2.5; the identities (2.4.1), (2.4.2), and (2.4.3); the equality $\Phi_{j_P}\Phi_\eta^{-1} = \Phi_{\eta_0}^{-1}\Phi_{j_M}$; and (2.2.2). We omit the details. This completes the proof of Lemma 3.4.1 and Theorem 3.1.2.

3.5. From now on we denote η and ξ by η^P and ξ^P respectively.

In this subsection we consider the case when we have two factorizations of μ, $\mu = \xi^P \circ \eta^P = \xi^Q \circ \eta^Q$, and the spaces M and N in the basic commutative diagram (3.1.1) are replaced by $M \times M$ and $N \times N$ respectively. So, suppose that Q is a purely d-dimensional, rational homology manifold and that in addition to the assumptions already made concerning the basic commutative diagram, the diagram

$$\begin{array}{ccccc} M & \xrightarrow{\eta^Q} & Q & \xrightarrow{\xi^Q} & N \\ \uparrow & & \uparrow & & \uparrow \\ M_r & \xrightarrow{\eta_r^Q} & Q_r & \xrightarrow{\xi_r^Q} & N_r \end{array}$$

satisfies conditions D1, D2, D3, D4, and D7 with P replaced by Q and Σ' replaced by a possibly different subgroup, Σ'', of Σ.

Let $\delta\colon N \to N \times N$ be the diagonal embedding. Then $\delta j_N\colon N_0 \to N \times N$ is a closed embedding. Define X to be the fibred product $(P \times Q) \times_{N \times N} N_0$ and define Z to be the fibred product $(M \times M) \times_{N \times N} N_0$. It follows immediately from the definition that a cartesian product of two small morphisms is again a small morphism. Therefore, modifying the notation as indicated, the diagram

(3.5.1)
$$\begin{array}{ccccc} Z & \xrightarrow{\eta_0} & X & \xrightarrow{\xi_0} & N_0 \\ {\scriptstyle j_Z}\downarrow & & {\scriptstyle j_X}\downarrow & & {\scriptstyle j_N}\downarrow \\ M \times M & \xrightarrow{\eta^P \times \eta^Q} & P \times Q & \xrightarrow{\xi^P \times \xi^Q} & N \times N \\ \uparrow & & \uparrow & & \uparrow \\ M_r \times M_r & \longrightarrow & P_r \times Q_r & \longrightarrow & N_r \times N_r \end{array}$$

satisfies conditions D1 – D7 in §3.1.

We have the following corollary to Theorem 3.1.2.

Corollary 3.5.2. *The group $\Sigma \times \Sigma$ acts on the local system $(\mu_r \times \mu_r)_! \mathbb{Q}_{M_r \times M_r}$. This action induces an action of $\Sigma \times \Sigma$ on $R(\mu \times \mu)_! \mathbb{Q}_{M \times M}$ and hence an action of $\Sigma \times \Sigma$ on $H_\bullet(Z)$ by functoriality and transport of structure via the isomorphism*

$$J' \colon H_\bullet(Z) \longrightarrow \mathrm{Ext}^{4d-\bullet}_{N_0}(\mathbb{Q}_{N_0}, j_N^! \delta^! R(\mu \times \mu)_! \mathbb{Q}_{M \times M}).$$

There is an isomorphism $h' \colon H_\bullet(X) \to H_\bullet(Z)^{\Sigma' \times \Sigma''}$ so that if $\mathrm{Av} \colon H_\bullet(Z) \to H_\bullet(Z)^{\Sigma' \times \Sigma''}$ is the averaging map, then the diagram

$$\begin{array}{ccc} H_\bullet(Z) & \xrightarrow{\ \eta_*\ } & H_\bullet(X) \\ & {\scriptstyle \mathrm{Av}} \searrow \quad \swarrow {\scriptstyle h'}\ {\scriptstyle \simeq} & \\ & H_\bullet(Z)^{\Sigma' \times \Sigma''} & \end{array}$$

of graded vector spaces commutes.

4. Equivariance

4.1. In this section we continue the analysis of diagram (3.5.1) and consider isomorphisms of graded vector spaces from [4, §8.6]

$$H_\bullet(Z) \xrightarrow{J} \mathrm{Ext}^{-\bullet}_{N_0}\big(\mathbb{Q}_{N_0}, j_N^! \delta^! R(\mu \times \mu)_! \mathbb{D}_{M \times M}\big) \xrightarrow{K} \mathrm{Ext}^{4n-\bullet}_{N_0}(R(\mu_0)_! \mathbb{Q}_{M_0}, R(\mu_0)_! \mathbb{Q}_{M_0})$$

where $\dim N_0 = 2n$, J is as in §3.4, and K is defined below. Notice that

$$\mathrm{End}_{N_0}(R(\mu_0)_! \mathbb{Q}_{M_0}) = \mathrm{Ext}^0_{N_0}(R(\mu_0)_! \mathbb{Q}_{M_0}, R(\mu_0)_! \mathbb{Q}_{M_0}) \cong H_{4n}(Z).$$

Recall that $\dim M = \dim N = d$ and define $l = \mathrm{codim}_N N_0 = d - 2n$. From now on, we assume that M_0 and N_0 are purely $2n$-dimensional, rational, homology manifolds. We also assume that the fibred products X and Z in §3.5 are purely $2n$-dimensional varieties.

The graded vector space $\mathrm{Ext}^{4n-\bullet}_{N_0}(R(\mu_0)_! \mathbb{Q}_{M_0}, R(\mu_0)_! \mathbb{Q}_{M_0})$ is a graded $\mathbb{Q}$-algebra and the composition $K \circ J$ can be used to give $H_\bullet(Z)$ a $\mathbb{Q}$-algebra structure with $H_i(Z) \cdot H_j(Z) \subseteq H_{i+j-4n}(Z)$. Since the multiplication in $\mathrm{Ext}^{4n-\bullet}_{N_0}(R(\mu_0)_! \mathbb{Q}_{M_0}, R(\mu_0)_! \mathbb{Q}_{M_0})$ is composition, we have

$$\mathrm{Ext}^{4n-\bullet}_{N_0}(R(\mu_0)_! \mathbb{Q}_{M_0}, R(\mu_0)_! \mathbb{Q}_{M_0}) \cong H_\bullet(Z)^{\mathrm{op}}.$$

We saw in §3.3 that Σ acts on $R\mu_! \mathbb{Q}_M$. This action induces a degree-preserving action of $\Sigma \times \Sigma$ on $\mathrm{Ext}^{4n-\bullet}_{N_0}(R(\mu_0)_! \mathbb{Q}_{M_0}, R(\mu_0)_! \mathbb{Q}_{M_0})$. On the other hand, as in §3.5, $\Sigma \times \Sigma$ acts on $R(\mu \times \mu)_! \mathbb{Q}_{M \times M}$. This action induces a degree-preserving $\Sigma \times \Sigma$-action on $\mathrm{Ext}^{-\bullet}_{N_0}\big(\mathbb{Q}_{N_0}, j_N^! \delta^! R(\mu \times \mu)_! \mathbb{D}_{M \times M}\big)$.

In this section we show that the isomorphisms J and K are $\Sigma \times \Sigma$-equivariant. It then follows that if $\Sigma \times \Sigma$ acts on the group algebra $\mathbb{Q}\Sigma$ in the usual way, then there are $\Sigma \times \Sigma$-equivariant, $\mathbb{Q}$-algebra homomorphisms

$$\mathbb{Q}\Sigma \longrightarrow \mathrm{End}_{N_0}(R(\mu_0)_! \mathbb{Q}_{M_0}) \xrightarrow{\ \simeq\ } H_{4n}(Z)^{\mathrm{op}}. \tag{4.1.1}$$

In §4.2 we describe the $\Sigma \times \Sigma$-action on $\mathrm{Ext}^{4n-\bullet}_{N_0}(R(\mu_0)_! \mathbb{Q}_{M_0}, R(\mu_0)_! \mathbb{Q}_{M_0})$. In §4.3 we describe the $\Sigma \times \Sigma$-action on $\mathrm{Ext}^{-\bullet}_{N_0}\big(\mathbb{Q}_{N_0}, j_N^! \delta^! R(\mu \times \mu)_! \mathbb{D}_{M \times M}\big)$ and observe that J is $\Sigma \times \Sigma$-equivariant. In §4.4 we define the map K, and in §4.5–§4.8 we show that K is $\Sigma \times \Sigma$-equivariant.

4.2. We first consider the $\Sigma\times\Sigma$-action on $\mathrm{Ext}^\bullet_{N_0}(R(\mu_0)_!\mathbb{Q}_{M_0}, R(\mu_0)_!\mathbb{Q}_{M_0})$. Returning to our original basic commutative diagram (3.1.1), Σ acts on the direct image, $(\mu_r)_!\mathbb{Q}_{M_r}$. This action induces a $\mathbb{Q}$-algebra homomorphism

$$L_r\colon \mathbb{Q}\Sigma \longrightarrow \mathrm{End}_{N_r}\left((\mu_r)_!\mathbb{Q}_{M_r}\right).$$

Applying IC and transporting the action via the isomorphism $\overline{\mu}\colon R\mu_!\mathbb{Q}_M \to \mathrm{IC}(N, (\mu_r)_!\mathbb{Q}_{M_r})$ from Corollary 3.3.5 gives rise to a $\mathbb{Q}$-algebra homomorphism

$$L\colon \mathbb{Q}\Sigma \longrightarrow \mathrm{End}_N(R\mu_!\mathbb{Q}_M)$$

with $L(\sigma) = \overline{\mu}^{-1}\circ \mathrm{IC}(L_r(\sigma))\circ\overline{\mu}$.

Since L is a ring homomorphism, we get an action of $\Sigma\times\Sigma$ on $\mathrm{End}_N(R\mu_!\mathbb{Q}_M)$ with

$$(\sigma,\sigma')\cdot f = L(\sigma')\circ f\circ L(\sigma^{-1})$$

for f in $\mathrm{End}_N(R\mu_!\mathbb{Q}_M)$.

Clearly, if $\Sigma\times\Sigma$ acts on $\mathbb{Q}\Sigma$ by $(\sigma,\sigma')\cdot x = \sigma' x\sigma^{-1}$, then L is $\Sigma\times\Sigma$-equivariant.

Let $\mathrm{bc}^*\colon j_{_N}^* R\mu_! \to R(\mu_0)_! j_{_M}^*$ be as in §2.5. Then $R(\mu_0)_!(\alpha_{j_M})\circ\mathrm{bc}^*$ is an isomorphism between $j_{_N}^*R\mu_!\mathbb{Q}_M$ and $R(\mu_0)_!\mathbb{Q}_{M_0}$. We define

$$L_0\colon \mathbb{Q}\Sigma \longrightarrow \mathrm{End}_{N_0}\left(R(\mu_0)_!\mathbb{Q}_{M_0}\right)$$

by

$$L_0(\sigma) = R(\mu_0)_!(\alpha_{j_M})\circ \mathrm{bc}^*\circ j_{_N}^*L(\sigma)\circ(\mathrm{bc}^*)^{-1}\circ R(\mu_0)_!(\alpha_{j_M}^{-1}).$$

Since $\mathrm{Ext}^j_{N_0}(R(\mu_0)_!\mathbb{Q}_{M_0}, R(\mu_0)_!\mathbb{Q}_{M_0}) = \mathrm{Hom}_{N_0}(R(\mu_0)_!\mathbb{Q}_{M_0}, R(\mu_0)_!\mathbb{Q}_{M_0}[j])$ is naturally an $\mathrm{End}_{N_0}(R(\mu_0)_!\mathbb{Q}_{M_0})$-bimodule, we may define an action of $\Sigma\times\Sigma$ on the graded vector space $\mathrm{Ext}^\bullet_{N_0}(R(\mu_0)_!\mathbb{Q}_{M_0}, R(\mu_0)_!\mathbb{Q}_{M_0})$ by

$$(\sigma,\sigma')\cdot g = L_0(\sigma')\circ g\circ L_0(\sigma^{-1})$$

for σ and σ' in Σ and g in $\mathrm{Ext}^\bullet_{N_0}(R(\mu_0)_!\mathbb{Q}_{M_0}, R(\mu_0)_!\mathbb{Q}_{M_0})$.

4.3. Next we consider the $\Sigma\times\Sigma$-action on $\mathrm{Ext}^\bullet_{N_0}\left(\mathbb{Q}_{N_0}, j_{_N}^!\delta^! R(\mu\times\mu)_!\mathbb{D}_{M\times M}\right)$. Since M is a rational homology manifold, so is $M\times M$ and we denote by $\nu_{M\times M}$ a fixed isomorphism, $\nu_{M\times M}\colon \mathbb{D}_{M\times M}\to\mathbb{Q}_{M\times M}[4d]$.

As in §3.4 and §3.5, $\Sigma\times\Sigma$ acts as automorphisms on $R(\mu\times\mu)_!\mathbb{Q}_{M\times M}$ and we transport the group action on $R(\mu\times\mu)_!\mathbb{Q}_{M\times M}$ to an action on $R(\mu\times\mu)_!\mathbb{D}_{M\times M}$ using $R(\mu\times\mu)_!(\nu_{M\times M})$. The group actions induce ring homomorphisms

$$L_2\colon \mathbb{Q}(\Sigma\times\Sigma)\longrightarrow \mathrm{End}_{N\times N}(R(\mu\times\mu)_!\mathbb{Q}_{M\times M})$$

and

$$L_2'\colon \mathbb{Q}(\Sigma\times\Sigma)\longrightarrow \mathrm{End}_{N\times N}(R(\mu\times\mu)_!\mathbb{D}_{M\times M})$$

where L_2 and L_2' are related by

$$L_2'(\sigma,\sigma') = R(\mu\times\mu)_!(\nu_{M\times M}^{-1})\circ L_2(\sigma,\sigma')\circ R(\mu\times\mu)_!(\nu_{M\times M}).$$

Notice that L_2' depends on the choice of the orientation $\nu_{M\times M}$.

Applying $\mathrm{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j_{_N}^!\delta^!(\,\cdot\,))$ to $R(\mu\times\mu)_!\mathbb{D}_{M\times M}$ and using L_2' we get an action of $\Sigma\times\Sigma$ on $\mathrm{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j_{_N}^!\delta^! R(\mu\times\mu)_!\mathbb{D}_{M\times M})$ with

$$(\sigma,\sigma')\cdot f = (j_{_N}^!\delta^! L_2'(\sigma,\sigma'))_\sharp(f) = j_{_N}^!\delta^! L_2'(\sigma,\sigma')\circ f$$

for f in $\mathrm{Ext}^{\bullet}_{N_0}(\mathbb{Q}_{N_0}, j^!_N\delta^! R(\mu\times\mu)_!\mathbb{D}_{M\times M})$.

As in §3.4 and §3.5, the $\Sigma\times\Sigma$-action on $\mathrm{Ext}^{\bullet}_{N_0}(\mathbb{Q}_{N_0}, j^!_N\delta^! R(\mu\times\mu)_!\mathbb{Q}_{M\times M})$ induces an action of $\Sigma\times\Sigma$ on $H_\bullet(Z)$ by transport of structure using the isomorphism

$$J' = (j^!_N\delta^! R(\mu\times\mu)_!(\nu_{M\times M}))_\sharp \circ J\colon H_{-\bullet}(Z) \longrightarrow \mathrm{Ext}^{\bullet}_{N_0}(\mathbb{Q}_{N_0}, j^!_N\delta^! R(\mu\times\mu)_!\mathbb{Q}_{M\times M}).$$

It follows from the definitions that $(j^!_N\delta^! R(\mu\times\mu)_!(\nu_{M\times M}))_\sharp$ is $\Sigma\times\Sigma$-equivariant. This proves the following proposition.

Proposition 4.3.1. *The isomorphism*

$$J\colon H_\bullet(Z) \longrightarrow \mathrm{Ext}^{-\bullet}_{N_0}(\mathbb{Q}_{N_0}, j^!_N\delta^! R(\mu\times\mu)_!\mathbb{D}_{M\times M})$$

is $\Sigma\times\Sigma$-equivariant.

4.4. Define

$$K\colon \mathrm{Ext}^{\bullet}_{N_0}(\mathbb{Q}_{N_0}, j^!_N\delta^! r(\mu\times\mu)_!\mathbb{D}_{M\times M}) \longrightarrow \mathrm{Ext}^{\bullet+4n}_{N_0}(R(\mu_0)_!\mathbb{Q}_{M_0}, R(\mu_0)!\mathbb{Q}_{M_0})$$

to be the composition

$$\begin{aligned}
\mathrm{Ext}^{\bullet}_{N_0}(\mathbb{Q}_{N_0}, j^!_N\delta^! R(\mu\times\mu)_!\mathbb{D}_{M\times M}) &\xrightarrow{j^!_N\delta^!(k')_\sharp} \mathrm{Ext}^{\bullet}_{N_0}\big(\mathbb{Q}_{N_0}, j^!_N\delta^!(R\mu_!\mathbb{D}_M\boxtimes R\mu_!\mathbb{D}_M)\big)\\
&\xrightarrow{j^!_N\delta^!(c^{-1}\boxtimes id)_\sharp} \mathrm{Ext}^{\bullet}_{N_0}\big(\mathbb{Q}_{N_0}, j^!_N\delta^!((R\mu_!\mathbb{Q}_M)^\vee\boxtimes R\mu_!\mathbb{D}_M)\big)\\
&\xrightarrow{j^!_N\delta^!(\lambda)_\sharp} \mathrm{Ext}^{\bullet}_{N_0}\big(\mathbb{Q}_{N_0}, j^!_N\delta^!(R\mathcal{H}om(p^*R\mu_!\mathbb{Q}_M, q^!R\mu_!\mathbb{D}_M)\big)\\
&\xrightarrow{(\mathrm{nat}_{\delta j_N})_\sharp} \mathrm{Ext}^{\bullet}_{N_0}\big(\mathbb{Q}_{N_0}, R\mathcal{H}om(j^*_N R\mu_!\mathbb{Q}_M, j^!_N R\mu_!\mathbb{D}_M)\big)\\
&\xrightarrow{\mathrm{can}^{-1}} \mathrm{Ext}^{\bullet}_{N_0}\big(j^*_N R\mu_!\mathbb{Q}_M, j^!_N R\mu_!\mathbb{D}_M\big)\\
&\xrightarrow{(a^{-1})^\sharp\circ b_\sharp} \mathrm{Ext}^{\bullet+4n}_{N_0}(R(\mu_0)_!\mathbb{Q}_{M_0}, R(\mu_0)_!\mathbb{Q}_{M_0})
\end{aligned}$$

where the notation is as follows:

- $k'\colon R(\mu\times\mu)_!\mathbb{D}_{M\times M}\to R\mu_!\mathbb{D}_M\boxtimes R\mu_!\mathbb{D}_M$ is the Künneth isomorphism (recall that μ is proper).
- $c = R\mu_!(\mathrm{dc}_M^{-1}\circ(\beta_\mu^{-1})_\sharp))\circ\phi_\mu\colon (R\mu_!\mathbb{Q}_M)^\vee\to R\mu_!\mathbb{D}_M$ where dc_M is as in §2.1 and β_μ and ϕ_μ are as in §2.2. Notice that c is an isomorphism in $D^b_c(N)$, so $j^!_N\delta^!(c^{-1}\boxtimes id)_\sharp$ makes sense.
- λ, nat_δ, and can are as in §2.
- $a = R(\mu_0)_!(\alpha_{j_M})\circ\mathrm{bc}^*\colon j^*_N R\mu_!\mathbb{Q}_M\to R(\mu_0)_!\mathbb{Q}_{M_0}$ (see §4.1).
- $b = R(\mu_0)_!(\nu_{M_0}\circ\beta^{-1}_{j_M})\circ\mathrm{bc}^!\colon j^!_N R\mu_!\mathbb{D}_M\to R(\mu_0)_!\mathbb{Q}_{M_0}$ where $\mathrm{bc}^!\colon j^!_N R\mu_!\to R(\mu_0)_! j^!_M$ is as in §2.5, β_{j_M} is as in §2.2, and $\nu_{M_0}\colon \mathbb{D}_{M_0}\to\mathbb{Q}_{M_0}[4n]$ is an isomorphism in $D^b_c(M_0)$ (recall that M_0 is a rational homology manifold).

Since K is a composition of isomorphisms of graded vector spaces, it follows that K is an isomorphism of graded vector spaces that increases the grading by $4n$.

Theorem 4.4.1. *The isomorphism*

$$K\colon \mathrm{Ext}^{\bullet}_{N_0}(\mathbb{Q}_{N_0}, j^!_N\delta^!(\mu\times\mu)_!\mathbb{D}_{M\times M}) \longrightarrow \mathrm{Ext}^{\bullet+4n}_{N_0}(R(\mu_0)_!\mathbb{Q}_{M_0}, R(\mu_0)_!\mathbb{Q}_{M_0})$$

is $\Sigma \times \Sigma$-equivariant.

To prove the theorem we show that $j_N^! \delta^! (k')_\sharp$, $j_N^! \delta^! (c^{-1} \boxtimes id)_\sharp$, $\mathrm{can}^{-1} \circ (\mathrm{nat}_{\delta j_N} \circ j_N^! \delta^!(\lambda))_\sharp$, and $(a^{-1})^\sharp \circ b_\sharp$ are $\Sigma \times \Sigma$-equivariant in §4.5, §4.6, §4.7, and §4.8 respectively.

4.5. In the situation of §3.5 we have two factorizations of μ: $\mu = \xi^P \eta^P = \xi^Q \eta^Q$. Let ν_M^P and ν_M^Q be two isomorphisms, $\mathbb{D}_M \xrightarrow{\simeq} \mathbb{Q}_M[2d]$. Then $\nu_M^P \boxtimes \nu_M^Q \colon \mathbb{D}_M \boxtimes \mathbb{D}_M \to \mathbb{Q}_M \boxtimes \mathbb{Q}_M[4d]$ is an isomorphism in $D_c^b(M \times M)$. The superscripts P and Q do not necessarily have anything to do with P and Q, but are convenient for distinguishing between the factors.

Using the orientations ν_M^P and ν_M^Q we can define $\mathbb{Q}$-algebra homomorphisms

$$L'_P \colon \mathbb{Q}\Sigma \longrightarrow \mathrm{End}_N(R\mu_! \mathbb{D}_M) \qquad \text{and} \qquad L'_Q \colon \mathbb{Q}\Sigma \longrightarrow \mathrm{End}_N(R\mu_! \mathbb{D}_M)$$

by $L'_P(\sigma) = R\mu_!(\nu_M^P)^{-1} \circ L(\sigma) \circ R\mu_!(\nu_M^P)$ and $L'_Q(\sigma) = R\mu_!(\nu_M^Q)^{-1} \circ L(\sigma) \circ R\mu_!(\nu_M^Q)$ respectively.

In the following, we always assume that $\nu_{M\times M}$ is chosen so that

$$\nu_{M\times M} = (k'')^{-1} \circ (\nu_M^P \boxtimes \nu_M^Q) \circ k'$$

where

$$k' \colon R(\mu \times \mu)_! \mathbb{D}_{M\times M} \xrightarrow{\simeq} R\mu_! \mathbb{D}_M \boxtimes R\mu_! \mathbb{D}_M$$

and

$$k'' \colon R(\mu \times \mu)_! \mathbb{Q}_{M\times M} \xrightarrow{\simeq} R\mu_! \mathbb{Q}_M \boxtimes R\mu_! \mathbb{Q}_M$$

are Künneth isomorphisms.

The next lemma follows from the naturality of k'.

Lemma 4.5.1. *For σ and σ' in Σ, the diagram*

$$\begin{array}{ccc} R(\mu \times \mu)_! \mathbb{D}_{M\times M} & \xrightarrow{k'} & R\mu_! \mathbb{D}_M \boxtimes R\mu_! \mathbb{D}_M \\ \Big\downarrow {\scriptstyle L'_2(\sigma,\sigma')} & & \Big\downarrow {\scriptstyle L'_P(\sigma) \boxtimes L'_Q(\sigma')} \\ R(\mu \times \mu)_! \mathbb{D}_{M\times M} & \xrightarrow{k'} & R\mu_! \mathbb{D}_M \boxtimes R\mu_! \mathbb{D}_M \end{array}$$

commutes.

The lemma shows that if $\Sigma \times \Sigma$ acts on $\mathrm{Ext}^\bullet_{N_0}\left(\mathbb{Q}_{N_0}, j_N^! \delta^! (R\mu_! \mathbb{D}_M \boxtimes R\mu_! \mathbb{D}_M)\right)$ by

$$(\sigma, \sigma') \cdot f = (j_N^! \delta^! (L'_P(\sigma) \boxtimes L'_Q(\sigma')) \circ f,$$

then $j_N^! \delta^! (k')_\sharp$ is $\Sigma \times \Sigma$-equivariant.

4.6. In this subsection we show that if $\Sigma \times \Sigma$ acts on $\mathrm{Ext}^\bullet_{N_0}\left(\mathbb{Q}_{N_0}, j_N^! \delta^! ((R\mu_! \mathbb{Q}_M)^\vee \boxtimes R\mu_! \mathbb{D}_M)\right)$ by

$$(\sigma, \sigma') \cdot f = (j_N^! \delta^! (L(\sigma^{-1})^\vee \boxtimes L'_Q(\sigma')) \circ f,$$

then $j_N^! \delta^! (c^{-1} \boxtimes id)_\sharp$ is $\Sigma \times \Sigma$-equivariant. In order to do this, it is enough to show that $c \colon (R\mu_! \mathbb{Q}_M)^\vee \to R\mu_! \mathbb{D}_M$ intertwines $L(\sigma^{-1})^\vee$ and $L'_P(\sigma)$ for σ in Σ.

In the rest of this subsection, we denote ν_M^P and L'_P simply by ν_M and L' respectively.

It is shown in [1, Theorem 9.8] that the Verdier dual of the intersection complex of a local system is, up to a shift, the intersection complex of the dual local system. Also, in the equivalence between local systems and representations of the fundamental group, the dual of a local system corresponds to the contragredient representation and the direct image of local systems corresponds to the induced representation. On the representation theory side, we are considering permutation representations, which are obviously equivalent to their contragredients, so it is natural to expect that for σ in Σ, the Verdier dual of σ, acting on $(R\mu_! \mathbb{Q}_M)^\vee$, may be identified with σ^{-1} acting on $(R\mu_! \mathbb{Q}_M)$. This is indeed the case and the next proposition gives the precise formulation we need.

Proposition 4.6.1. *If* $c = R\mu_!(\mathrm{dc}_M^{-1} \circ (\beta_\mu^{-1})_\sharp)) \circ \phi_\mu$, *then the diagram*

$$\begin{CD} (R\mu_! \mathbb{Q}_M)^\vee @>{L(\sigma^{-1})^\vee}>> (R\mu_! \mathbb{Q}_M)^\vee \\ @V{c}VV @VV{c}V \\ R\mu_! \mathbb{D}_M @>>{L'(\sigma)}> R\mu_! \mathbb{D}_M^\vee \end{CD}$$

of isomorphisms of complexes in $D_c^b(N)$ *commutes for every* σ *in* Σ.

Proof. It follows from [1, Theorem 9.8] that there is a unique isomorphism,

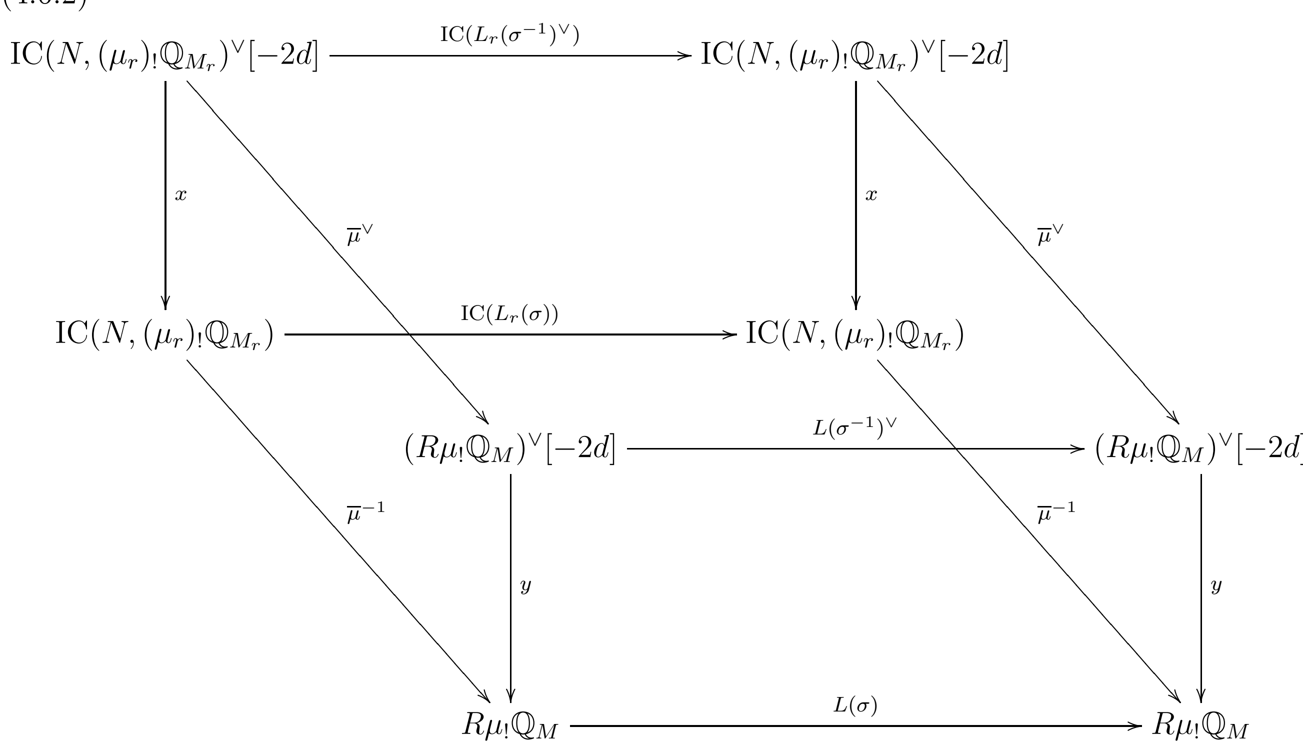
$$\mathrm{vd}\colon \mathrm{IC}(N, (\mu_r)_! \mathbb{Q}_{M_r})^\vee[-2d] \longrightarrow \mathrm{IC}(N, ((\mu_r)_! \mathbb{Q}_{M_r})^\vee[-2d])$$

with the property that $i_{_N}^*(\mathrm{vd}) = (\beta_{i_N}^{-1})_\sharp \circ \mathrm{nat}_{i_N}$.

Define $\nu_{_{M_r}} = \alpha_{i_M} \circ i_{_M}^!(\nu_{_M}) \circ \beta_{i_M}$, so $\nu_{_{M_r}} : \mathbb{D}_{M_r} \to \mathbb{Q}_{M_r}[2d]$ is an isomorphism.

Now consider the "cube"

(4.6.2)



where $x = \mathrm{IC}\left((\mu_r)_!(\nu_{_{M_r}} \circ \mathrm{dc}_{M_r}^{-1} \circ (\beta_{\mu_r}^{-1})_\sharp) \circ \phi_{\mu_r}\right) \circ \mathrm{vd}$, $y = R\mu_!(\nu_{_M}) \circ c$, and ϕ_μ is as in §2.2. It follows from the definitions of y and L' that it is enough to show that the front face commutes. We show that all faces besides the front face commute and so the front face must commute also.

The top and bottom faces of (4.6.2) commute by definition and the left and right faces are equal, so we need to show that the back face and the left face commute.

To show that the left face of (4.6.2) commutes we need to show that the diagram

$$
\begin{array}{ccc}
\mathrm{IC}\left(N, (\mu_r)_!\mathbb{Q}_{M_r}\right)^\vee[-2d] & \xrightarrow{\overline{\mu}^\vee} & \left(R\mu_!\mathbb{Q}_M\right)^\vee[-2d] \\
\Big\downarrow{\scriptstyle \mathrm{vd}} & & \Big\downarrow{\scriptstyle R\mu_!((\beta_\mu^{-1})_\sharp)\circ\phi_\mu} \\
\mathrm{IC}\left(N, ((\mu_r)_!\mathbb{Q}_{M_r})^\vee[-2d]\right) & & \\
\Big\downarrow{\scriptstyle \mathrm{IC}((\mu_r)_!((\beta_{\mu_r}^{-1})_\sharp)\circ\phi_{\mu_r})} & & \\
\mathrm{IC}\left(N, (\mu_r)_!\mathbb{Q}_{M_r}^\vee[-2d]\right) & \overset{z}{\dashrightarrow} & R\mu_!\mathbb{Q}_M^\vee[-2d] \\
\Big\downarrow{\scriptstyle \mathrm{IC}((\mu_r)_!(\mathrm{dc}_{M_r}^{-1}))} & & \Big\downarrow{\scriptstyle R\mu_!(\mathrm{dc}_M^{-1})} \\
\mathrm{IC}\left(N, (\mu_r)_!\mathbb{D}_{M_r}[-2d]\right) & \overset{w}{\dashrightarrow} & R\mu_!\mathbb{D}_M[-2d] \\
\Big\downarrow{\scriptstyle \mathrm{IC}((\mu_r)_!(\nu_{M_r}))} & & \Big\downarrow{\scriptstyle R\mu_!(\nu_M)} \\
\mathrm{IC}\left(N, (\mu_r)_!\mathbb{Q}_{M_r}\right) & \xrightarrow{\overline{\mu}^{-1}} & R\mu_!\mathbb{Q}_M
\end{array}
\tag{4.6.3}
$$

commutes.

For the rest of this proof, set $\mathrm{bc} = \mathrm{bc}_{\mu_r, i_M}$.

As in the proof of Corollary 3.3.5, since we have $R\mu_!\mathbb{Q}_M^\vee[-2d] \cong \mathrm{IC}\left(N, (\mu_r)_!\mathbb{Q}_{M_r}^\vee[-2d]\right)$ and $R\mu_!\mathbb{D}_M[-2d] \cong \mathrm{IC}\left(N, (\mu_r)_!\mathbb{D}_{M_r}[-2d]\right)$, there are isomorphisms,

$$\mathrm{ic}_c\colon R\mu_!\mathbb{Q}_M^\vee[-2d] \longrightarrow \mathrm{IC}\left(N, (\mu_r)_!\mathbb{Q}_{M_r}^\vee[-2d]\right)$$

and

$$\mathrm{ic}_d\colon R\mu_!\mathbb{D}_M[-2d] \longrightarrow \mathrm{IC}\left(N, (\mu_r)_!\mathbb{D}_{M_r}[-2d]\right),$$

in $D(N)$ with $i_N^*(\mathrm{ic}_c) = id$ and $i_N^*(\mathrm{ic}_d) = id$. Define

$$z = \mathrm{ic}_c^{-1} \circ \mathrm{IC}(\mathrm{bc}^{-1} \circ (\mu_r)_!(\mathrm{nat}_{i_M}^{-1} \circ (\beta_{i_M})_\sharp \circ \alpha_{i_M}^\sharp)), \quad w = \mathrm{ic}_d^{-1} \circ \mathrm{IC}(\mathrm{bc}^{-1} \circ (\mu_r)_!(\beta_{i_M}))$$

and recall that $\overline{\mu}^{-1} = \mathrm{ic}_\mu^{-1} \circ \mathrm{IC}(\mathrm{bc}^{-1} \circ (\mu_r)_!(\alpha_{i_M}^{-1}))$.

Since all the complexes in (4.6.3) are in the image of IC, it is enough to show that (4.6.3) commutes after applying i_N^*.

First, it follows from the definition of ν_{M_r} and the naturality of bc that

$$
\begin{aligned}
i_N^*\left(\overline{\mu}^{-1} \circ \mathrm{IC}((\mu_r)_!(\nu_{M_r}))\right) &= \mathrm{bc}^{-1} \circ (\mu_r)_!(\alpha_{i_M}^{-1}) \circ (\mu_r)_!(\nu_{M_r}) \\
&= i_N^* R\mu_!(\nu_M) \circ \mathrm{bc}^{-1} \circ (\mu_r)_!(\beta_{i_M}) \\
&= i_N^*\left(R\mu_!(\nu_M) \circ w\right).
\end{aligned}
$$

Second, it follows from (2.2.1) applied to i_M and the naturality of bc that

$$
\begin{aligned}
i_N^*\left(R\mu_!(\mathrm{dc}_M^{-1}) \circ z\right) &= i_N^* R\mu_!(\mathrm{dc}_M^{-1}) \circ \mathrm{bc}^{-1} \circ (\mu_r)_!(\mathrm{nat}_{i_M}^{-1} \circ (\beta_{i_M})_\sharp \circ \alpha_{i_M}^\sharp) \\
&= \mathrm{bc}^{-1} \circ (\mu_r)_!(\beta_{i_M} \circ \mathrm{dc}_{M_r}^{-1}) \\
&= i_N^*\left(w \circ \mathrm{IC}((\mu_r)_!(\mathrm{dc}_{M_r}^{-1}))\right).
\end{aligned}
$$

Lastly, it follows from the naturality of nat_{i_N}, nat_{i_M}, bc, ϕ_μ, and ϕ_{μ_r}, Lemma 2.5, (2.2.2), and the equality $i_N^*(\mathrm{vd}) = (\beta_{i_N}^{-1})_\sharp \circ \mathrm{nat}_{i_N}$ that

$$\begin{aligned} i_N^* \left(R\mu_!((\beta_\mu^{-1})_\sharp) \circ \phi_\mu \circ \overline{\mu}^\vee\right) &= i_N^* \left(R\mu_!((\beta_\mu^{-1})_\sharp) \circ \phi_\mu\right) \circ \mathrm{nat}_{i_N} \circ^{-1} \left((\mu_r)_!(\alpha_{i_M}) \circ \mathrm{bc}\right)^\sharp \circ \mathrm{nat}_{i_N} \\ &= \mathrm{bc}^{-1} \circ (\mu_r)_!(\mathrm{nat}_{i_M}^{-1} \circ (\beta_{i_M})_\sharp \circ \alpha_{i_M}^\sharp \circ (\beta_{\mu_r}^{-1})_\sharp) \circ \phi_{\mu_r} \circ (\beta_{i_N}^{-1})_\sharp \circ \mathrm{nat}_{i_N} \\ &= i_N^* \left(z \circ \mathrm{IC}((\mu_r)_!((\beta_{\mu_r}^{-1})_\sharp) \circ \phi_{\mu_r}) \circ \mathrm{vd}\right). \end{aligned}$$

Finally, consider the back face of diagram (4.6.2). It follows from the uniqueness of vd that $\mathrm{vd} \circ \mathrm{IC}(L(\sigma^{-1}))^\vee = \mathrm{IC}(L_r(\sigma^{-1})^\vee) \circ \mathrm{vd}$. Thus, to show that the back face commutes, it is enough to show that

$$(\mu_r)_!(\nu_{M_r} \circ \mathrm{dc}_{M_r}^{-1} \circ (\beta_{\mu_r}^{-1})_\sharp) \circ \phi_{\mu_r} \circ L_r(\sigma^{-1})^\vee = L_r(\sigma) \circ (\mu_r)_!(\nu_{M_r} \circ \mathrm{dc}_{M_r}^{-1} \circ (\beta_{\mu_r}^{-1})_\sharp) \circ \phi_{\mu_r}.$$

In other words, we need to show that the diagram of local systems

$$\text{(4.6.4)} \qquad \begin{CD} ((\mu_r)_!\mathbb{Q}_{M_r})^\vee[-2d] @>{L_r(\sigma^{-1})^\vee}>> ((\mu_r)_!\mathbb{Q}_{M_r})^\vee[-2d] \\ @V{(\mu_r)_!((\beta_{\mu_r}^{-1})^\sharp) \circ \phi_{\mu_r}}VV @VV{(\mu_r)_!((\beta_{\mu_r}^{-1})^\sharp) \circ \phi_{\mu_r})}V \\ (\mu_r)_!\mathbb{Q}_{M_r}^\vee[-2d] @. (\mu_r)_!\mathbb{Q}_{M_r}^\vee[-2d] \\ @V{(\mu_r)_!(\mathrm{dc}_{M_r}^{-1})}VV @VV{(\mu_r)_!(\mathrm{dc}_{M_r}^{-1})}V \\ (\mu_r)_!\mathbb{D}_{M_r}[-2d] @. (\mu_r)_!\mathbb{D}_{M_r}[-2d] \\ @V{(\mu_r)_!(\nu_{M_r})}VV @VV{(\mu_r)_!(\nu_{M_r})}V \\ (\mu_r)_!\mathbb{Q}_{M_r} @>{L_r(\sigma)}>> (\mu_r)_!\mathbb{Q}_{M_r} \end{CD}$$

commutes.

Using that $((\mu_r)_!\mathbb{Q}_{M_r})^\vee[-2d]$ is isomorphic to the dual local system, $((\mu_r)_!\mathbb{Q}_{M_r})^*$, and $(\mu_r)_!\mathbb{Q}_{M_r}^\vee[-2d]$ is isomorphic to $(\mu_r)_!\mathbb{Q}_{M_r}^*$, since M_r and N_r are rational homology manifolds, it is straightforward to show that for x in N_r, the diagram obtained from diagram (4.6.4) by taking the stalk at x commutes. It follows that diagram (4.6.4) commutes as desired. □

4.7. In this subsection we show that if $\Sigma \times \Sigma$ acts on $\mathrm{Ext}_{N_0}^\bullet\left(j_N^* R\mu_!\mathbb{Q}_M, j_N^! R\mu_!\mathbb{D}_M\right)$ by

$$(\sigma, \sigma') \cdot f = j_N^! L_Q'(\sigma') \circ f \circ j_N^* L(\sigma^{-1}) = \left(j_N^* L(\sigma^{-1})^\sharp \circ j_N^! L_Q'(\sigma')_\sharp\right)(f),$$

then $\mathrm{can}^{-1} \circ (\mathrm{nat}_{\delta j_N} \circ j_N^! \delta^!(\lambda))_\sharp$ is $\Sigma \times \Sigma$-equivariant.

Suppose σ and σ' are in Σ and f is in $\mathrm{Ext}_{N_0}^\bullet\left(\mathbb{Q}_{N_0}, j_N^! \delta^!((R\mu_!\mathbb{Q}_M)^\vee \boxtimes R\mu_!\mathbb{D}_M)\right)$. Then, setting $u = L(\sigma^{-1})$, $v = L_Q'(\sigma')$ and using $\mathrm{nat}_{\delta j_N} = \mathrm{nat}_{j_N} \circ j_N^!(\mathrm{nat}_\delta)$, Corollary 2.3.2, the naturality of nat_{j_N}, and Proposition 2.3.3 we have:

$$\begin{aligned} \mathrm{can}^{-1} \circ \left(\mathrm{nat}_{\delta j_N} \circ j_N^! \delta^!(\lambda)\right)_\sharp ((\sigma, \sigma') \cdot f) &= \mathrm{can}^{-1}\left(\mathrm{nat}_{\delta j_N} \circ j_N^! \delta^!(\lambda \circ (u^\vee \boxtimes v)) \circ f\right) \\ &= \mathrm{can}^{-1}\left(\mathrm{nat}_{j_N} \circ j_N^!(\mathrm{nat}_\delta \circ \delta^!(\lambda \circ (u^\vee \boxtimes v))) \circ f\right) \\ &= \mathrm{can}^{-1}\left(\mathrm{nat}_{j_N} \circ j_N^!(u^\sharp \circ v_\sharp \circ \mathrm{nat}_\delta \circ \delta^!(\lambda)) \circ f\right) \\ &= \mathrm{can}^{-1}\left(j_N^*(u)^\sharp \circ j_N^!(v)_\sharp \circ \mathrm{nat}_{j_N} \circ j_N^!(\mathrm{nat}_\delta \circ \delta^!(\lambda)) \circ f\right) \\ &= \mathrm{can}^{-1}\left((j_N^*(u)^\sharp \circ j_N^!(v)_\sharp)_\sharp(\mathrm{nat}_{\delta j_N} \circ j_N^! \delta^!(\lambda) \circ f)\right) \end{aligned}$$

$$= j_N^*(u)^\sharp \circ j_N^!(v)_\sharp \circ \mathrm{can}^{-1}(\mathrm{nat}_{\delta j_N} \circ j_N^! \delta^!(\lambda) \circ f)$$
$$= (\sigma, \sigma') \cdot \big(\mathrm{can}^{-1} \circ (\mathrm{nat}_{\delta j_N} \circ j_N^! \delta^!(\lambda))_\sharp(f)\big)\,.$$

4.8. To complete the proof of Theorem 4.4.1 we need to show that

$$\mathrm{Ext}^\bullet_{N_0}\left(j_N^* R\mu_! \mathbb{Q}_M, j_N^! R\mu_! \mathbb{D}_M\right) \xrightarrow{(a^{-1})^\sharp \circ b_\sharp} \mathrm{Ext}^{\bullet+4n}_{N_0}\left(R(\mu_0)_!\mathbb{Q}_{M_0}, R(\mu_0)_!\mathbb{Q}_{M_0}\right)$$

is $\Sigma \times \Sigma$-equivariant where $\Sigma \times \Sigma$ acts on the domain as in §4.7. For this, it is enough to show that

$$b \circ j_N^! L'_Q(\sigma') \circ f \circ j_N^* L(\sigma^{-1}) \circ a^{-1} = L_0(\sigma') \circ b \circ f \circ a^{-1} \circ L_0(\sigma^{-1})$$

for σ and σ' in Σ and f in $\mathrm{Ext}^\bullet_{N_0}\left(j_N^* R\mu_! \mathbb{Q}_M, j_N^! R\mu_! \mathbb{D}_M\right)$.

Recall from §4.2 and §4.4 that $a = R(\mu_0)_!(\alpha_{j_M}) \circ \mathrm{bc}^*$ and

$$L_0(\sigma^{-1}) = R(\mu_0)_!(\alpha_{j_M}) \circ \mathrm{bc}^* \circ j_N^* L(\sigma^{-1}) \circ (R(\mu_0)_!(\alpha_{j_M}) \circ \mathrm{bc}^*)^{-1} = a \circ j_N^* L(\sigma^{-1}) \circ a^{-1}.$$

Therefore, to show that $(a^{-1})^\sharp \circ b_\sharp$ is $\Sigma \times \Sigma$-equivariant, it is enough to show that $b \circ j_N^! L'_Q(\sigma') = L_0(\sigma') \circ b$.

Recall from §4.4 that $b = R(\mu_0)_!(\nu_{M_0} \circ \beta_{j_M}^{-1}) \circ \mathrm{bc}^!$, so we need to show that

$$R(\mu_0)_!(\nu_{M_0} \circ \beta_{j_M}^{-1}) \circ \mathrm{bc}^! \circ j_N^!(R\mu_!(\nu_M^Q)^{-1} \circ L(\sigma) \circ R\mu_!(\nu_M^Q)) = \\ R(\mu_0)_!(\alpha_{j_M}) \circ \mathrm{bc}^* \circ j_N^* L(\sigma) \circ (\mathrm{bc}^*)^{-1} \circ R(\mu_0)_!(\alpha_{j_M}^{-1}) \circ R(\mu_0)_!(\nu_{M_0} \circ \beta_{j_M}^{-1}) \circ \mathrm{bc}^!$$

for σ in Σ.

Setting $\nu_M = \nu_M^Q$, and using the naturality of $\mathrm{bc}^!$, it follows that it is enough to show that

$$j_N^! L(\sigma) \circ (\mathrm{bc}^!)^{-1} \circ R(\mu_0)_!(j_M^!(\nu_M) \circ \beta_{j_M} \circ \nu_{M_0}^{-1} \circ \alpha_{j_M}) \circ \mathrm{bc}^* = \\ (\mathrm{bc}^!)^{-1} \circ R(\mu_0)_!(j_M^!(\nu_M) \circ \beta_{j_M} \circ \nu_{M_0}^{-1} \circ \alpha_{j_M}) \circ \mathrm{bc}^* \circ j_N^* L(\sigma).$$

Set $\tau = (\mathrm{bc}^!)^{-1} \circ R(\mu_0)_!(j_M^!(\nu_M) \circ \beta_{j_M} \circ \nu_{M_0}^{-1} \circ \alpha_{j_M}) \circ \mathrm{bc}^*$. Then τ is an isomorphism in $D_c^b(N_0)$, $\tau\colon j_N^* R\mu_! \mathbb{Q}_M \to j_N^! R\mu_! \mathbb{Q}_M[2l]$, where $l = \mathrm{codim}_N N_0 = \mathrm{codim}_M M_0$, and we need to show that

$$j_N^! L(\sigma) \circ \tau = \tau \circ j_N^* L(\sigma). \tag{4.8.1}$$

We prove the following proposition in the Appendix.

Proposition 4.8.2. *There is a natural transformation, $\rho^{j_N}\colon j_N^* \to j_N^![2l]$, so that $\tau = \rho^{j_N}_{R\mu_!\mathbb{Q}_M}$.*

Given the truth of the proposition, it follows from the naturality of ρ^{j_N} that $j_N^!(g) \circ \tau = \tau \circ j_N^*(g)$ for g in $\mathrm{End}_N(R\mu_!\mathbb{Q}_M)$ and so in particular (4.8.1) holds for σ in Σ. This completes the proof of Theorem 4.4.1.

5. Generalized Steinberg Varieties

5.1. In this section we apply the results of §3 and §4 to generalized Steinberg varieties.

We start with the following incarnation of the basic commutative diagram (3.1.1) as in [3]:

(5.1.1)
$$\begin{array}{ccccc}
\widetilde{\mathcal{N}} & \xrightarrow{\eta_0^{\mathcal{P}}} & \widetilde{\mathcal{N}}^{\mathcal{P}} & \xrightarrow{\xi_0^{\mathcal{P}}} & \mathcal{N} \\
\downarrow{\scriptstyle j_{\widetilde{\mathcal{N}}}} & & \downarrow{\scriptstyle j_{\widetilde{\mathcal{N}}^{\mathcal{P}}}} & & \downarrow{\scriptstyle j_{\mathcal{N}}} \\
\widetilde{\mathfrak{g}} & \xrightarrow{\eta^{\mathcal{P}}} & \widetilde{\mathfrak{g}}^{\mathcal{P}} & \xrightarrow{\xi^{\mathcal{P}}} & \mathfrak{g} \\
\uparrow & & \uparrow & & \uparrow \\
\widetilde{\mathfrak{g}}_{\mathrm{rs}} & \xrightarrow{\eta_{\mathrm{rs}}^{\mathcal{P}}} & \widetilde{\mathfrak{g}}_{\mathrm{rs}}^{\mathcal{P}} & \xrightarrow{\xi_{\mathrm{rs}}^{\mathcal{P}}} & \mathfrak{g}_{\mathrm{rs}}
\end{array}$$

The notation is as follows:

- G is a connected, reductive, complex algebraic group with Lie algebra $\mathfrak{g}$, $\mathcal{N}$ is the cone of nilpotent elements in $\mathfrak{g}$, and $\mathfrak{g}_{\mathrm{rs}}$ is the open subvariety of regular semisimple elements in $\mathfrak{g}$.
- $\mathcal{P}$ is a conjugacy class of parabolic subgroups of G.
- $\widetilde{\mathfrak{g}} = \{\, (x, B) \in \mathfrak{g} \times \mathcal{B} \mid x \in \mathrm{Lie}(B) \,\}$ where $\mathcal{B}$ is the variety of Borel subgroups of G and $\widetilde{\mathfrak{g}}^{\mathcal{P}} = \{\, (x, P) \in \mathfrak{g} \times \mathcal{P} \mid x \in \mathrm{Lie}(P) \,\}$.
- The maps $\eta_?^{\mathcal{P}}$ are defined by $\eta_?^{\mathcal{P}}(x, B) = (x, P)$ where P is the unique subgroup in $\mathcal{P}$ that contains B.
- The maps $\xi_?^{\mathcal{P}}$ are projection on the first factor.
- $\mu = \xi^{\mathcal{P}} \circ \eta^{\mathcal{P}}$ is the projection on the first factor.
- $\widetilde{\mathfrak{g}}_{\mathrm{rs}} = \mu^{-1}(\mathfrak{g}_{\mathrm{rs}}) = \{\, (x, B) \in \mathfrak{g}_{\mathrm{rs}} \times \mathcal{B} \mid x \in \mathrm{Lie}(B) \,\}$ and $\widetilde{\mathfrak{g}}_{\mathrm{rs}}^{\mathcal{P}} = (\xi^{\mathcal{P}})^{-1}(\mathfrak{g}_{\mathrm{rs}}) = \{\, (x, P) \in \mathfrak{g}_{\mathrm{rs}} \times \mathcal{P} \mid x \in \mathrm{Lie}(P) \,\}$.
- $\widetilde{\mathcal{N}} = \mu^{-1}(\mathcal{N}) = \{\, (x, B) \in \mathcal{N} \times \mathcal{B} \mid x \in \mathrm{Lie}(B) \,\}$ and $\widetilde{\mathcal{N}}^{\mathcal{P}} = (\xi^{\mathcal{P}})^{-1}(\mathcal{N}) = \{\, (x, P) \in \mathcal{N} \times \mathcal{P} \mid x \in \mathrm{Lie}(P) \,\}$.

In accordance with the notation above, we assume also that $\dim G = \dim \mathfrak{g} = d$, $\dim \mathcal{N} = 2n$, and $l = d - 2n = \mathrm{codim}_{\mathfrak{g}} \mathcal{N}$.

It is shown in [3] that diagram (5.1.1) has properties D1 – D7 of the basic commutative diagram. For the convenience of the reader, we recall the group action involved in properties D6 and D7.

Fix a maximal torus, T and a Borel subgroup, B, of G with $T \subset B$. Define $\mathfrak{t} = \mathrm{Lie}(T)$ and $\mathfrak{t}_{\mathrm{reg}} = \mathfrak{t} \cap \mathfrak{g}_{\mathrm{rs}}$, so $\mathfrak{t}_{\mathrm{reg}}$ is the set of regular semisimple elements in $\mathfrak{t}$. Let $W = N_G(T)/T$ be the Weyl group of (G, T). Then W acts on $\mathfrak{t}_{\mathrm{reg}} \times G/T$ on the right by $(t, gT) \cdot w = (\mathrm{Ad}(w^{-1})t, gwT)$ for w in W, t in $\widetilde{\mathfrak{g}}_{\mathrm{rs}}$ and g in G. It is well-known and easy to check that the rule $(t, gT) \mapsto (\mathrm{Ad}(g)t, gBg^{-1})$ defines an isomorphism of varieties $\mathfrak{t}_{\mathrm{reg}} \times G/T \cong \widetilde{\mathfrak{g}}_{\mathrm{rs}}$ and we use this isomorphism to transport the W-action from $\mathfrak{t}_{\mathrm{reg}} \times G/T$ to $\widetilde{\mathfrak{g}}_{\mathrm{rs}}$. It is also well-known and easy to prove that the projection on the first factor, from $\widetilde{\mathfrak{g}}_{\mathrm{rs}}$ to $\mathfrak{g}_{\mathrm{rs}}$, is an orbit map for the right W-action on $\widetilde{\mathfrak{g}}_{\mathrm{rs}}$. Thus, diagram (5.1.1) has property D6.

Next, let P be the subgroup in $\mathcal{P}$ with $B \subseteq P$ and set $W_P = N_P(T)/T$, the Weyl group of P, so W_P is a subgroup of W. It is straightforward to check that $\eta^{\mathcal{P}}|_{\widetilde{\mathfrak{g}}_{\mathrm{rs}}}$ is an orbit map for the action of W_P on $\widetilde{\mathfrak{g}}_{\mathrm{rs}}$. Thus, diagram (5.1.1) has property D7.

If $\mathcal{Q}$ is a second conjugacy class of parabolic subgroups of G, then the two variable version of diagram (5.1.1), as in §3.5, is the following:

(5.1.2)
$$\begin{array}{ccccc}
Z & \xrightarrow{\quad\eta\quad} & X & \xrightarrow{\quad\xi\quad} & \mathcal{N} \\
\downarrow{\scriptstyle j_Z} & & \downarrow{\scriptstyle j_X} & & \downarrow{\scriptstyle \delta\circ j_{\mathcal{N}}} \\
\widetilde{\mathfrak{g}}\times\widetilde{\mathfrak{g}} & \xrightarrow{\eta^{\mathcal{P}}\times\eta^{\mathcal{Q}}} & \widetilde{\mathfrak{g}}^{\mathcal{P}}\times\widetilde{\mathfrak{g}}^{\mathcal{Q}} & \xrightarrow{\xi^{\mathcal{P}}\times\xi^{\mathcal{Q}}} & \mathfrak{g}\times\mathfrak{g} \\
\uparrow & & \uparrow & & \uparrow \\
\widetilde{\mathfrak{g}}_{\mathrm{rs}}\times\widetilde{\mathfrak{g}}_{\mathrm{rs}} & \longrightarrow & \widetilde{\mathfrak{g}}^{\mathcal{P}}_{\mathrm{rs}}\times\widetilde{\mathfrak{g}}^{\mathcal{Q}}_{\mathrm{rs}} & \longrightarrow & \mathfrak{g}_{\mathrm{rs}}\times\mathfrak{g}_{\mathrm{rs}}
\end{array}$$

Since $(\widetilde{\mathfrak{g}}\times\widetilde{\mathfrak{g}})\times_{\mathfrak{g}\times\mathfrak{g}}\mathcal{N}=\{\,((x,B'),(x,B''))\mid x\in\operatorname{Lie}(B')\cap\operatorname{Lie}(B'')\,\}$, we may identify Z with the Steinberg variety of G. Then $j_Z\colon Z\to\widetilde{\mathfrak{g}}\times\widetilde{\mathfrak{g}}$ by $j_Z(x,B',B'')=((x,B'),(x,B''))$.

Also, since $(\widetilde{\mathfrak{g}}^{\mathcal{P}}\times\widetilde{\mathfrak{g}}^{\mathcal{Q}})\times_{\mathfrak{g}\times\mathfrak{g}}\mathcal{N}=\{\,((x,P'),(x,Q'))\mid x\in\operatorname{Lie}(P')\cap\operatorname{Lie}(Q')\,\}$, we may identify X with the generalized Steinberg variety $X^{\mathcal{P},\mathcal{Q}}$ from §1. Then $j_X\colon X^{\mathcal{P},\mathcal{Q}}\to\widetilde{\mathfrak{g}}^{\mathcal{P}}\times\widetilde{\mathfrak{g}}^{\mathcal{Q}}$ by $j_X(x,P',Q')=((x,P'),(x,Q'))$.

Applying Theorem 3.5.2 we have our first main result.

Theorem 5.1.3. *If $H_\bullet(Z)$ is given the $W\times W$-action induced from the $W\times W$-action on $(\mu_{\mathrm{rs}}\times\mu_{\mathrm{rs}})_!\mathbb{Q}_{\widetilde{\mathfrak{g}}_{\mathrm{rs}}\times\widetilde{\mathfrak{g}}_{\mathrm{rs}}}$, then there is an isomorphism of vector spaces, $H_\bullet(X^{\mathcal{P},\mathcal{Q}})\cong H_\bullet(Z)^{W_P\times W_Q}$, so that the diagram*

$$\begin{array}{ccc}
H_\bullet(Z) & \xrightarrow{\qquad\eta_*\qquad} & H_\bullet(X^{\mathcal{P},\mathcal{Q}}) \\
& {\scriptstyle \mathrm{Av}}\searrow \qquad \nearrow{\scriptstyle \simeq} & \\
& H_\bullet(Z)^{W_P\times W_Q} &
\end{array}$$

commutes.

5.2. Now we consider the special case of Theorem 5.1.3 when $\bullet=2\dim Z=4n$ as in §4.1. Borho and MacPherson [2] have shown that the $\mathbb{Q}$-algebra homomorphism, $\mathbb{Q}W\to\operatorname{End}_{\mathcal{N}}(R(\mu_0)_!\mathbb{Q}_{\widetilde{\mathcal{N}}})$ from §4.2 is an isomorphism. Therefore, from (4.1.1) we get the result originally proved by Kazhdan and Lusztig [12] and strengthened by Chriss and Ginzburg [4].

Theorem 5.2.1. *If $W\times W$ acts on $\mathbb{Q}W$ by $(w,w')\cdot x=w'xw^{-1}$, then there are $W\times W$-equivariant isomorphisms*

$$\mathbb{Q}W\xrightarrow{\ \simeq\ }\operatorname{End}_{\mathcal{N}}(R(\mu_0)_!\mathbb{Q}_{\widetilde{\mathcal{N}}})\xrightarrow{\ \simeq\ }H_{4n}(Z)^{\mathrm{op}}.$$

Recall that e_P denotes the primitive idempotent in $\mathbb{Q}W_P$ corresponding to the trivial representation of W_P. Since $(\mathbb{Q}W)^{W_P\times W_Q}=e_Q\mathbb{Q}We_P$, the next corollary follows immediately from Theorems 5.1.3 and 5.2.1.

Corollary 5.2.2. *The $W\times W$-equivariant isomorphism $\mathbb{Q}W\xrightarrow{\simeq}H_{4n}(Z)^{\mathrm{op}}$ in Theorem 5.2.1 induces an isomorphism between the subspace $e_Q\mathbb{Q}We_P$ of $\mathbb{Q}W$ and $H_{4n}(X^{\mathcal{P},\mathcal{Q}})$, the top Borel-Moore homology group of the generalized Steinberg variety, $X^{\mathcal{P},\mathcal{Q}}$:*

$$\begin{array}{ccc}
\mathbb{Q}W & \xrightarrow{\ \simeq\ } & H_{4n}(Z)^{\mathrm{op}} \\
\downarrow{\scriptstyle \mathrm{Av}} & & \downarrow{\scriptstyle \eta_*} \\
e_Q\mathbb{Q}We_P & \xrightarrow[\ \simeq\ ]{} & H_{4n}(X^{\mathcal{P},\mathcal{Q}})
\end{array}$$

5.3. In this subsection, we use Corollary 5.2.2 to compute the action of a simple reflection in W on $H_{4n}(Z)$. What we prove is the analog for $H_{4n}(Z)$ of the "easy" part of Hotta's transformations for the action of a simple reflection in the cohomology of a Springer fibre. Our argument is inspired by Hotta's argument in [10].

It is well-known that W indexes the G-orbits on $\mathcal{B}\times\mathcal{B}$ and that if Z_w denotes the preimage of the orbit indexed by w in W under the projection of Z onto $\mathcal{B}\times\mathcal{B}$ given by the projection on the second and third factors, then the dimension of Z_w is $2n$ and the irreducible components of Z are the closures of the Z_w's (see [18]). Thus, if $[\overline{Z_w}]$ denotes the canonical class of $\overline{Z_w}$ in $H_{4n}(Z)$, it follows that $\{\, [\overline{Z_w}] \mid w \in W \,\}$ is a basis of $H_{4n}(Z)$.

Recall that we have fixed a Borel subgroup, B, of G containing T. The choice of B determines a set of Coxeter generators of W and hence a length function and a partial order, the Bruhat order, on W.

For the time being we fix a simple reflection, s, in W and let $\mathcal{P}_s$ denote the conjugacy class of minimal parabolic subgroups of G determined by s. Then $\mathcal{P}_s$ and $\mathcal{B}$ are conjugacy classes of parabolic subgroups of G and we may consider $\eta_*\colon H_{4n}(Z) \to H_{4n}(X^{\mathcal{P}_s\times\mathcal{B}})$.

Let P_s be the subgroup in $\mathcal{P}_s$ that contains B. It is shown in [5, §3] that if w is in W, then $\dim \eta(Z_w) = \dim Z_w$ if and only if w is minimal in its (W_{P_s}, W_B)-double coset. Since $W_{P_s} = \{\, 1, s \,\}$ and $W_B = \{\, 1 \,\}$, it follows that w is minimal in its double coset if and only if $sw > w$ in the Bruhat order. Therefore, if $sw < w$ we have $\eta_*([\overline{Z_w}]) = 0$. It follows that $\dim \ker \eta_* \geq |W|/2$.

On the other hand, by Corollary 5.2.2, we may identify η_* with the averaging map onto the set of $W_{P_s}\times W_B$-invariants in $\mathbb{Q}W$. In this case, the averaging map from $\mathbb{Q}W$ to $(\mathbb{Q}W)^{W_{P_s}\times W_B}$ is $x \mapsto \frac{1}{2}(x + xs)$ and so its kernel is $\{\, x \in \mathbb{Q}W \mid xs = -x \,\}$ and has dimension equal $|W|/2$. Therefore, the kernel of η_* is the subspace $\{\, c \in H_{4n}(Z) \mid s\cdot c = -c \,\}$ and it has dimension equal $|W|/2$. Since $\ker \eta_*$ contains the linearly independent set $\{\, [\overline{Z_w}] \mid sw < w \,\}$, it follows that $\{\, [\overline{Z_w}] \mid sw < w \,\}$ is a basis of $\ker \eta_*$. This proves the following theorem.

Theorem 5.3.1. *If s is a simple reflection in W, then $\{\, [\overline{Z_w}] \mid sw < w \,\}$ is a basis of the subspace $\{\, c \in H_{4n}(Z) \mid s \cdot c = -c \,\}$ of $H_{4n}(Z)$. In particular, if w is in W and s is a simple reflection, then $s \cdot [\overline{Z_w}] = -[\overline{Z_w}]$ if and only if $sw < w$ in the Bruhat order.*

5.4. We now turn to computing the top Borel-Moore homology group of the generalized Steinberg variety $Y^{\mathcal{P},\mathcal{Q}}$. Recall that we have fixed parabolic subgroups, P in $\mathcal{P}$ and Q in $\mathcal{Q}$, with $B \subseteq P \cap Q$. Then

$$Y^{\mathcal{P},\mathcal{Q}} = \{\, (x, P', Q') \in \mathcal{N} \times \mathcal{P} \times \mathcal{Q} \mid x \in \operatorname{Lie}(U_{P'}) \cap \operatorname{Lie}(U_{Q'}) \,\} \subseteq X^{\mathcal{P},\mathcal{Q}}$$

and

$$Z^{\mathcal{P},\mathcal{Q}} = \eta^{-1}(Y^{\mathcal{P},\mathcal{Q}}).$$

Thus, we have a cartesian square

$$\begin{array}{ccc} Z^{\mathcal{P},\mathcal{Q}} & \xrightarrow{\;j\;} & Z \\ {\scriptstyle \overline{\eta}}\downarrow & & \downarrow{\scriptstyle \eta} \\ Y^{\mathcal{P},\mathcal{Q}} & \longrightarrow & X^{\mathcal{P},\mathcal{Q}} \end{array}$$

where the horizontal arrows are inclusions and $\overline{\eta}$ is the restriction of η to $Z^{\mathcal{P},\mathcal{Q}}$.

It follows from the definitions that $\overline{\eta}$ is a fibre bundle with smooth fibres isomorphic to $P/B \times Q/B$.

Define $W^{P,Q}$ to be the set of maximal length (W_P, W_Q)-double coset representatives in W, so $W^{P,Q}$ indexes the G-orbits on $\mathcal{P} \times \mathcal{Q}$.

It was shown in [5, §4] that if Y_w denotes the preimage of the orbit indexed by w in $W^{P,Q}$ under the projection of $Y^{\mathcal{P},\mathcal{Q}}$ onto $\mathcal{P} \times \mathcal{Q}$ given by the projection on the second and third factors, then the dimension of Y_w is $\dim \mathcal{P} + \dim \mathcal{Q}$ and the irreducible components of $Y^{\mathcal{P},\mathcal{Q}}$ are the closures of the Y_w's.

It was also shown in [5, §4] that $\{\, \overline{Z_w} \mid w \in W^{P,Q} \,\}$ is the set of irreducible components of $Z^{\mathcal{P},\mathcal{Q}}$. Clearly $\overline{\eta}(Z_w) \subseteq Y_w$ and so since $\overline{\eta}$ is proper, Z_w and Y_w are irreducible, and the fibres $\overline{\eta}$ all have the same dimension, it follows that $\overline{\eta}(\overline{Z_w}) = \overline{Y_w}$.

Since $\overline{\eta}$ is a fibre bundle with smooth fibres, if $f = \dim P/B + \dim Q/B$, then there is an inverse image map in Borel-Moore homology, $\overline{\eta}^*\colon H_\bullet(Y^{\mathcal{P},\mathcal{Q}}) \to H_{\bullet+2f}(Z^{\mathcal{P},\mathcal{Q}})$ (see [4, 8.3.31]).

It is straightforward to check that if $[\overline{Y_w}]$ denotes the canonical class of $\overline{Y_w}$ in $H_{4n-2f}(Y^{\mathcal{P},\mathcal{Q}})$, then $\overline{\eta}^*([\overline{Y_w}])$ is a multiple of $[\overline{Z_w}]$ (see [6]). Since $\dim H_{4n-2f}(Y^{\mathcal{P},\mathcal{Q}}) = \dim H_{4n}(Z^{\mathcal{P},\mathcal{Q}})$ it follows that $\overline{\eta}^*$ is injective.

Next, $Z^{\mathcal{P},\mathcal{Q}}$ is a closed subvariety of Z, so if j denotes the inclusion, there is a direct image map in Borel-Moore homology, $j_*\colon H_\bullet(Z^{\mathcal{P},\mathcal{Q}}) \to H_\bullet(Z)$. It follows immediately that $j_*([\overline{Z_w}]) = [\overline{Z_w}]$ for w in $W^{P,Q}$ and that j_* is injective.

Combining the results in the last two paragraphs we have proven the next proposition.

Proposition 5.4.1. *The mapping* $\overline{\eta}^*\colon H_{4n-2f}(Y^{\mathcal{P},\mathcal{Q}}) \to H_{4n}(Z^{\mathcal{P},\mathcal{Q}})$ *is an isomorphism of vector spaces and the mapping* $j_*\colon H_{4n}(Z^{\mathcal{P},\mathcal{Q}}) \to H_{4n}(Z)$ *is injective with image equal the span of* $\{\, [\overline{Z_w}] \mid w \in W^{P,Q} \,\}$.

5.5. We identify the image of $H_{4n}(Z^{\mathcal{P},\mathcal{Q}})$ with its image in $H_{4n}(Z)$. Then $H_{4n}(Z^{\mathcal{P},\mathcal{Q}})$ is the span of $\{\, [\overline{Z_w}] \mid w \in W^{P,Q} \,\}$ in $H_{4n}(Z)$ and $H_{4n}(Z^{\mathcal{P},\mathcal{Q}}) \cong H_{4n-2f}(Y^{\mathcal{P},\mathcal{Q}})$. Define $H^{P,Q}$ to be the subspace of c in $H_{4n}(Z)$ with the property that $s \cdot c = -c$ and $c \cdot t = -c$ for all simple reflections, s in W_P and t in W_Q. It follows from Theorem 5.3.1 that $H_{4n}(Z) \subseteq H^{P,Q}$.

Recall that ϵ_P and ϵ_Q denote the primitive idempotents in W_P and W_Q corresponding to the sign representations of W_P and W_Q respectively. Then $\dim \epsilon_Q \mathbb{Q}W \epsilon_P = |W^{P,Q}|$ and $\epsilon_Q \mathbb{Q}W \epsilon_P$ is the set of all x in $\mathbb{Q}W$ with the property that $sx = -x$ and $xt = -x$ for all simple reflections, s in W_Q, and t in W_P. It follows from Theorem 5.2.1 that under the isomorphism $\mathbb{Q}W \xrightarrow{\simeq} H_{4n}(Z)^{\mathrm{op}}$, the subspace $H^{P,Q}$ is the image of $\epsilon_Q \mathbb{Q}W \epsilon_P$. Therefore, $\dim H^{P,Q} = |W^{P,Q}|$ and hence $H^{P,Q} = H_{4n}(Z^{\mathcal{P},\mathcal{Q}})$. This proves the following theorem.

Theorem 5.5.1. *The* $W \times W$*-equivariant isomorphism* $\mathbb{Q}W \xrightarrow{\simeq} H_{4n}(Z)^{\mathrm{op}}$ *in Theorem 5.2.1 induces an isomorphism between the subspace* $\epsilon_Q \mathbb{Q}W \epsilon_P$ *of* $\mathbb{Q}W$ *and* $H_{4n-2f}(Y^{\mathcal{P},\mathcal{Q}})$*, the top Borel-Moore homology group of the generalized Steinberg variety,* $Y^{\mathcal{P},\mathcal{Q}}$*,*

$$\begin{array}{ccc} \epsilon_Q \mathbb{Q}W \epsilon_P & \xrightarrow{\simeq} & H_{4n-2f}(Y^{\mathcal{P},\mathcal{Q}}) \\ \big\downarrow & & \big\downarrow {\scriptstyle j_*\overline{\eta}^*} \\ \mathbb{Q}W & \xrightarrow[\simeq]{} & H_{4n}(Z)^{\mathrm{op}} \end{array}$$

where the left vertical arrow is inclusion.

Appendix A

A.1. In this appendix we change notation slightly from §2.4. For a morphism, $\xi\colon X \to Y$, of complex, algebraic varieties, the units of the adjoint pairs (ξ^*, ξ_*) and $(\xi^!, \xi_!)$ are denoted by η^*_ξ and $\eta^!_\xi$ respectively. Similarly, the counits are denoted by ϵ^*_ξ and $\epsilon^!_\xi$ respectively.

Suppose $\xi\colon X \to Y$ is a morphism between complex, algebraic varieties that are rational homology manifolds. For a fixed choice of isomorphisms $\nu_X\colon \mathbb{D}_X \xrightarrow{\simeq} \mathbb{Q}_X[2\dim X]$ and $\nu_Y\colon \mathbb{D}_Y \xrightarrow{\simeq} \mathbb{Q}_Y[2\dim Y]$ there is a natural transformation $\rho^\xi\colon \xi^* \to \xi^![2l]$ where $l = \dim Y - \dim X$. For a complex A in $D^b_c(Y)$, $\rho^\xi = \rho^\xi_A$ is defined to be the composition

$$\xi^*A \xrightarrow{m_1^{-1}} \mathbb{Q}_X \otimes \xi^*A \xrightarrow{\omega_\xi \otimes id} \xi^!\mathbb{Q}_X \otimes \xi^*A[2l] \xrightarrow{\eta^!_\xi} \xi^! R\xi_!(\xi^!\mathbb{Q}_X \otimes \xi^*A)[2l] \xrightarrow{\xi^!(\mathrm{pr}_\xi^{-1})}$$
$$\xi^!(R\xi_!\xi^!\mathbb{Q}_X \otimes A)[2l] \xrightarrow{\xi^!(\epsilon^!_\xi \otimes id)} \xi^!(\mathbb{Q}_X \otimes A)[2l] \xrightarrow{\xi^!(m_1)} \xi^!A[2l]$$

where the notation is as follows:

- $m_1\colon \mathbb{Q}_X \otimes B \xrightarrow{\simeq} B$ is the natural isomorphism for B in $D(X)$.
- $\omega_\xi = \xi^!(\nu_Y) \circ \beta_\xi \circ \nu_X^{-1}\colon \mathbb{Q}_X \xrightarrow{\simeq} \xi^!\mathbb{Q}_Y[2l]$, so ω_ξ is an isomorphism in $D^b_c(X)$.
- $\eta^!_\xi$ and $\epsilon^!_\xi$ are as above.
- For B in $D^b(X)$ and C in $D^b(Y)$, $\mathrm{pr}_\xi\colon R\xi_!B \otimes C \xrightarrow{\simeq} R\xi_!(B \otimes \xi^*C)$ is the projection isomorphism.

Notice that ρ^ξ is a natural transformation since each map in the definition of ρ^ξ_A is natural in A.

Now consider a cartesian square

(A.1.1)
$$\begin{array}{ccc} M_0 & \xrightarrow{\mu_0} & N_0 \\ {\scriptstyle j_M}\downarrow & & \downarrow{\scriptstyle j_N} \\ M & \xrightarrow[\mu]{} & N \end{array}$$

satisfying the following conditions:

C1 The spaces are all complex, algebraic varieties that are rational homology manifolds.
C2 The maps are all proper morphisms.
C3 j_M and j_N are closed embeddings.
C4 $\dim M_0 = \dim N_0 = 2n$, $\dim M = \dim N = d$, and $l = d - 2n$.

For a cartesian square as in (A.1.1), we have base change isomorphisms

$$\mathrm{bc}^*\colon j_N^* R\mu_! \xrightarrow{\simeq} R(\mu_0)_! j_M^* \qquad \text{and} \qquad \mathrm{bc}^!\colon j_N^! R\mu_! \xrightarrow{\simeq} R(\mu_0)_! j_M^!$$

defined as in §2.5.

We prove the following lemmas in the next two subsections.

Lemma A.1.2. *If X and Y are complex, algebraic varieties that are rational homology manifolds and $\xi\colon X \to Y$ is a proper morphism, then $\rho^\xi_{\mathbb{Q}_Y} = \omega_\xi \circ \alpha_\xi$.*

Lemma A.1.3. *If ν_{M_0} is chosen appropriately, then in the cartesian square (A.1.1) the morphisms $\rho^{j_M}_{\mathbb{Q}_M}$ and $\rho^{j_N}_{R\mu_!\mathbb{Q}_M}$ are related by*

$$\mathrm{bc}^! \circ \rho^{j_N}_{R\mu_!\mathbb{Q}_M} = R(\mu_0)_!(\rho^{j_M}_{\mathbb{Q}_M}) \circ \mathrm{bc}^*.$$

Recall from §4.8 that in the setting of (A.1.1) we have $\tau \colon j_N^* R\mu_!\mathbb{Q}_M \xrightarrow{\simeq} j_N^! R\mu_!\mathbb{Q}_M[2l]$, by

$$\tau = (\mathrm{bc}^!)^{-1} \circ R(\mu_0)_!(j_M^!(\nu_M) \circ \beta_{j_M} \circ \nu_{M_0}^{-1} \circ \alpha_{j_M}) \circ \mathrm{bc}^*$$

where $\nu_M \colon \mathbb{D}_M \xrightarrow{\simeq} \mathbb{Q}_M[2d]$ and $\nu_{M_0} \colon \mathbb{D}_{M_0} \xrightarrow{\simeq} \mathbb{Q}_{M_0}[4n]$ are isomorphisms in $D^b_c(M)$ and $D^b_c(M_0)$ respectively.

Assuming Lemmas A.1.2 and A.1.3 have been proved we have:

$$\begin{aligned} \rho^{j_N}_{R\mu_!\mathbb{Q}_M} &= (\mathrm{bc}^!)^{-1} \circ R(\mu_0)_!(\rho^{j_M}_{\mathbb{Q}_M}) \circ \mathrm{bc}^* \\ &= (\mathrm{bc}^!)^{-1} \circ R(\mu_0)_!(\omega_{j_M} \circ \alpha_{j_M}) \circ \mathrm{bc}^* \\ &= (\mathrm{bc}^!)^{-1} \circ R(\mu_0)_!(j_M^!(\nu_M) \circ \beta_{j_M} \circ \nu_{M_0}^{-1} \circ \alpha_{j_M}) \circ \mathrm{bc}^* \\ &= \tau \end{aligned}$$

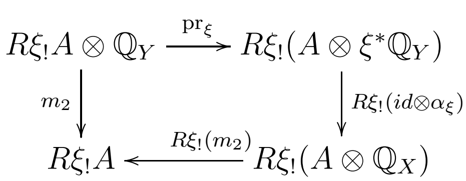
This proves Proposition 4.8.2.

A.2. In this subsection we prove Lemma A.1.2. Before doing so, we need some preliminary results.

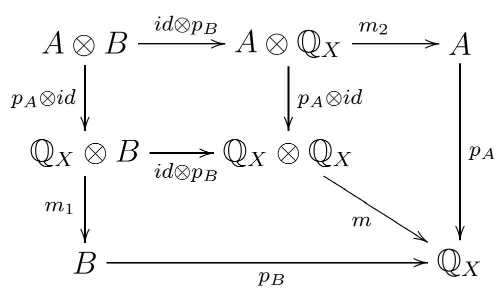
If A is in $D^b_c(X)$ we denote the canonical isomorphisms $\mathbb{Q}_X \otimes A \xrightarrow{\simeq} A$ and $A \otimes \mathbb{Q}_X \xrightarrow{\simeq} A$ by m_1 and m_2 respectively. When $A = \mathbb{Q}_X$ we set $m = m_1 = m_2$.

The proof of the next lemma is a straightforward computation using stalks and is omitted.

Lemma A.2.1. *Suppose A and B are in $D^b_c(X)$ and $p_A \colon A \to \mathbb{Q}_X$ and $p_B \colon B \to \mathbb{Q}_X$ are two morphisms in $D^b_c(X)$. Then the diagrams*



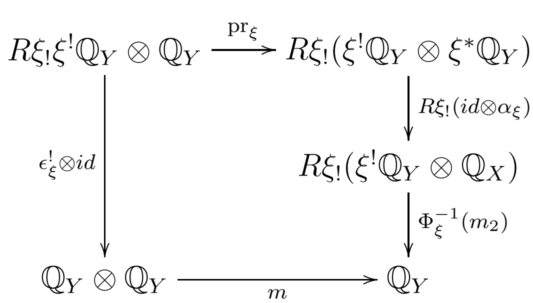
and



commute.

Let $\mathrm{nat}^{\otimes}_{\xi} \colon \xi^*(A \otimes B) \xrightarrow{\simeq} \xi^*A \otimes \xi^*B$ denote the canonical isomorphism in $D^b(X)$.

Lemma A.2.2. *The diagram*



commutes.

Proof. Using the definition of Φ_ξ^{-1} we have

$$\Phi_\xi^{-1}(m_2) \circ R\xi_!(id \otimes \alpha_\xi) \circ \mathrm{pr}_\xi = \epsilon_\xi^! \circ R\xi_!(m_2) \circ R\xi_!(id \otimes \alpha_\xi) \circ \mathrm{pr}_\xi.$$

Also, using the naturality of $\epsilon_\xi^!$ we have $m \circ (\epsilon_\xi^! \otimes id) = \epsilon_\xi^! \circ m_2$, so it is enough to show that

$$m_2 = R\xi_!(m_2) \circ R\xi_!(id \otimes \alpha_\xi) \circ \mathrm{pr}_\xi. \tag{A.2.3}$$

Since ξ is proper, we have $\mathrm{pr}_\xi = \Psi_\xi((\epsilon_\xi^* \otimes id) \circ \mathrm{nat}_\xi^\otimes)$.

The proof of (A.2.3) is a straightforward computation using the formula for pr_ξ and Lemma A.2.2. We omit the details. □

We can now complete the proof of Lemma A.1.2. Recall that

$$\rho_{\mathbb{Q}_Y}^\xi = \xi^! \left(m_1 \circ \epsilon_\xi^! \circ \mathrm{pr}_\xi^{-1}\right) \circ \eta_\xi^! \circ (\omega_\xi \otimes id) \circ m_1^{-1} = \Phi_\xi(m_1 \circ \epsilon_\xi^! \circ \mathrm{pr}_\xi^{-1}) \circ (\omega_\xi \otimes id) \circ m_1^{-1},$$

so to prove the lemma, we need to show that

$$\Phi_\xi(m_1 \circ \epsilon_\xi^! \circ \mathrm{pr}_\xi^{-1}) = \omega_\xi \circ \alpha_\xi \circ m_1 \circ (\omega_\xi^{-1} \otimes id).$$

Taking $A = \xi^!\mathbb{Q}_Y[2l]$, $p_A = \omega_\xi^{-1}$, $B = \xi^*\mathbb{Q}_Y$, and $p_B = \alpha_\xi$ in Lemma A.2.1 we get

$$\alpha_\xi \circ m_1 \circ \circ(\omega_\xi^{-1} \otimes id) = \omega_\xi^{-1} \circ m_2 \circ (id \otimes \alpha_\xi),$$

so it is enough to show that $\Phi_\xi(m_1 \circ \epsilon_\xi^! \circ \mathrm{pr}_\xi^{-1}) = m_2 \circ (id \otimes \alpha_\xi)$. This last equality follows immediately from Lemma A.2.2. This completes the proof of Lemma A.1.2.

A.3. In this subsection we prove Lemma A.1.3.

The proof is accomplished by showing that the diagrams (A.3.1) and (A.3.2) below are commutative. Then juxtaposing these diagrams and tracing around the outside gives the desired result.

It is easy to see that any unlabeled regions of diagrams (A.3.1) and (A.3.2) commute. The commutativity of the labeled regions is shown in the corresponding statements below.

To make the diagrams as clear as possible, we need to simplify the notation. First, for a morphism, $\xi\colon X \to Y$, we denote the derived functors $R\xi_*$ and $R\xi_!$ simply by ξ_* and $\xi_!$ respectively. Second, we denote j_N simply by j. Third, we label the maps in the diagrams using only the core maps or natural transformations involved. For example, we write α_{μ_0} instead of $(\mu_0)_!(\alpha_{\mu_0} \otimes id)$ and bc^* instead of $j^! j_!(id \otimes \mathrm{bc}^*)$.

If $\xi\colon X \to Y$, and A and B are complexes, then

$$\mathrm{pr}^1\colon \xi_! A \otimes B \xrightarrow{\simeq} \xi_!(A \otimes \xi^* B) \quad \text{and} \quad \mathrm{pr}^2\colon A \otimes \xi_! B \xrightarrow{\simeq} \xi_!(\xi^* A \otimes B).$$

With this notation we have $\rho^\xi = \xi^!(m_1 \circ (\epsilon_\xi^! \otimes id) \circ (\mathrm{pr}_\xi^1)^{-1}) \circ \eta_\xi^! \circ (\omega_\xi \otimes id) \circ m_1^{-1}$.

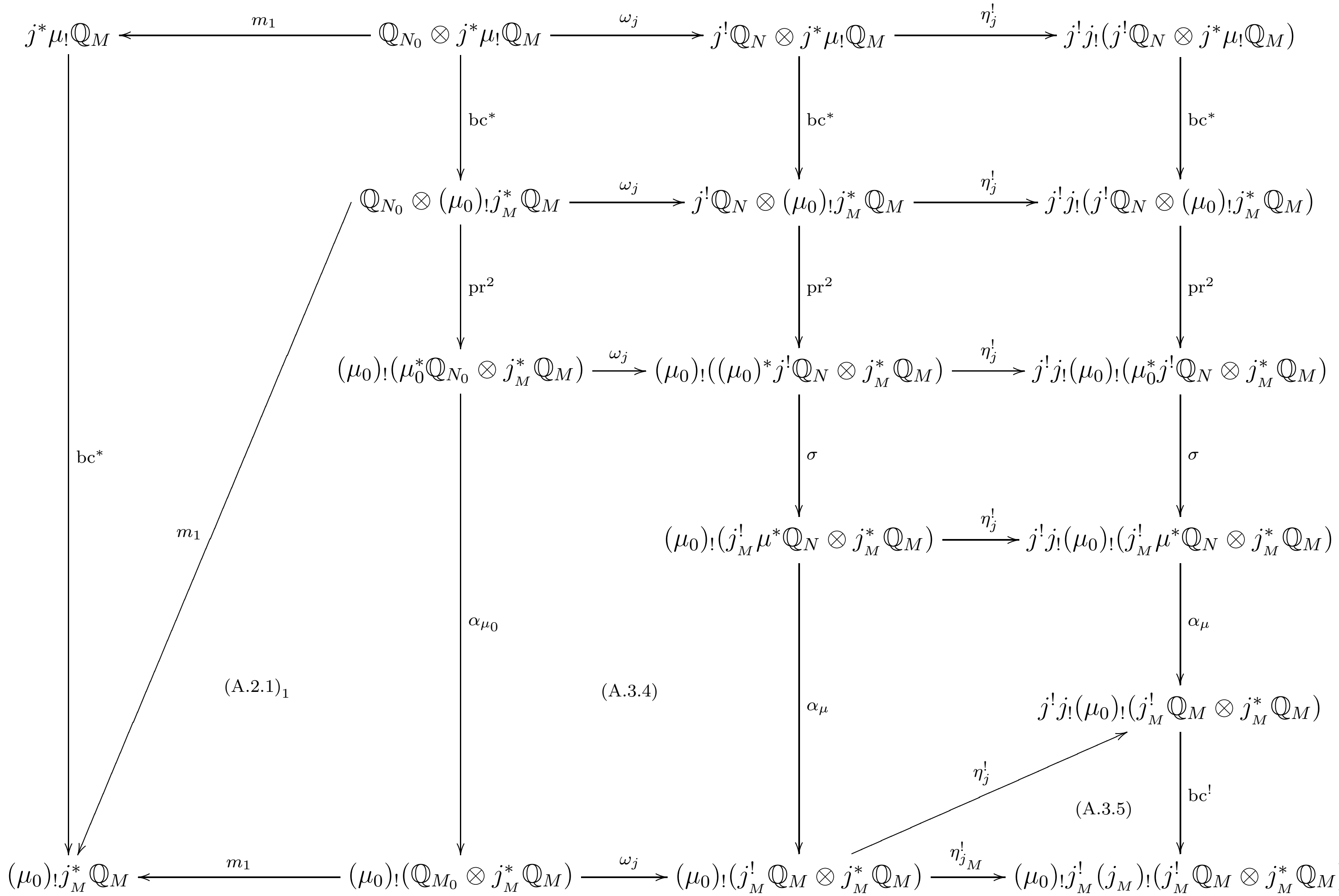
Notice that if ξ is proper, then $\mathrm{pr}_\xi^1 = \Psi_\xi((\epsilon_\xi^* \otimes id) \circ \mathrm{nat}_\xi^\otimes)$ and $\mathrm{pr}_\xi^2 = \Psi_\xi((id \otimes \epsilon_\xi^*) \circ \mathrm{nat}_\xi^\otimes)$.

For a cartesian square as in (A.1.1) we have a base change isomorphism,

$$\widetilde{\mathrm{bc}}\colon \mu^* R(j_N)_! \xrightarrow{\simeq} R(j_M)_! \mu_0^*,$$

defined by $\widetilde{\mathrm{bc}} = \Psi_{j_M}(\mu_0^*(\epsilon_{j_N}^*))$. Define $\sigma\colon \mu_0^* j_N^! \to j_M^! \mu^*$ by $\sigma = \Phi_{j_M}(\mu^*(\epsilon_{j_N}^!) \circ \widetilde{\mathrm{bc}}^{-1})$. Then σ is a natural transformation.

(A.3.1)


$j^*\mu_!\mathbb{Q}_M$
m_1
$\mathbb{Q}_{N_0} \otimes j^*\mu_!\mathbb{Q}_M$
ω_j
$j^!\mathbb{Q}_N \otimes j^*\mu_!\mathbb{Q}_M$
$\eta^!_j$
$j^! j_!(j^!\mathbb{Q}_N \otimes j^*\mu_!\mathbb{Q}_M)$
bc*
bc*
bc*
$\mathbb{Q}_{N_0} \otimes (\mu_0)_! j^*_M \mathbb{Q}_M$
ω_j
$j^!\mathbb{Q}_N \otimes (\mu_0)_! j^*_M \mathbb{Q}_M$
$\eta^!_j$
$j^! j_!(j^!\mathbb{Q}_N \otimes (\mu_0)_! j^*_M \mathbb{Q}_M)$
pr^2
pr^2
pr^2
$(\mu_0)_!(\mu_0^*\mathbb{Q}_{N_0} \otimes j^*_M \mathbb{Q}_M)$
ω_j
$(\mu_0)_!((\mu_0)^* j^!\mathbb{Q}_N \otimes j^*_M \mathbb{Q}_M)$
$\eta^!_j$
$j^! j_!(\mu_0)_!(\mu_0^* j^!\mathbb{Q}_N \otimes j^*_M \mathbb{Q}_M)$
bc*
σ
σ
m_1
$(\mu_0)_!(j^!_M \mu^*\mathbb{Q}_N \otimes j^*_M \mathbb{Q}_M)$
$\eta^!_j$
$j^! j_!(\mu_0)_!(j^!_M \mu^*\mathbb{Q}_N \otimes j^*_M \mathbb{Q}_M)$
α_{μ_0}
α_μ
(A.2.1)$_1$
(A.3.4)
α_μ
$j^! j_!(\mu_0)_!(j^!_M \mathbb{Q}_M \otimes j^*_M \mathbb{Q}_M)$
$\eta^!_j$
bc$^!$
(A.3.5)
$(\mu_0)_! j^*_M \mathbb{Q}_M$
m_1
$(\mu_0)_!(\mathbb{Q}_{M_0} \otimes j^*_M \mathbb{Q}_M)$
ω_j
$(\mu_0)_!(j^!_M \mathbb{Q}_M \otimes j^*_M \mathbb{Q}_M)$
$\eta^!_{j_M}$
$(\mu_0)_! j^!_M (j_M)_!(j^!_M \mathbb{Q}_M \otimes j^*_M \mathbb{Q}_M)$

$$
\begin{array}{ccccccc}
j^!j_!(j^!\mathbb{Q}_N \otimes j^*\mu_!\mathbb{Q}_M) & \xleftarrow{\mathrm{pr}^1} & j^!(j_!j^!\mathbb{Q}_N \otimes \mu_!\mathbb{Q}_M) & \xrightarrow{\epsilon^!_j} & j^!(\mathbb{Q}_N \otimes \mu_!\mathbb{Q}_M) & \xrightarrow{m_1} & j^!\mu_!\mathbb{Q}_M \\
\downarrow{\scriptstyle \mathrm{bc}^*} & (\mathrm{A.3.6}) & \downarrow{\scriptstyle \mathrm{pr}^2} & & \downarrow{\scriptstyle \mathrm{pr}^2} & (\mathrm{A.2.1})_2 & \\
j^!j_!(j^!\mathbb{Q}_N \otimes (\mu_0)_!j^*_M\mathbb{Q}_M) & & j^!\mu_!(\mu^*j_!j^!\mathbb{Q}_N \otimes \mathbb{Q}_M) & \xrightarrow{\epsilon^!_j} & j^!\mu_!(\mu^*\mathbb{Q}_N \otimes \mathbb{Q}_M) & & \\
\downarrow{\scriptstyle \mathrm{pr}^2} & & \downarrow{\scriptstyle \widetilde{\mathrm{bc}}} & (\mathrm{A.3.7}) & & & \\
j^!j_!(\mu_0)_!(\mu_0^*j^!\mathbb{Q}_N \otimes j^*_M\mathbb{Q}_M) & \xleftarrow{\mathrm{pr}^1} & j^!\mu_!((j_M)_!\mu^*j^!\mathbb{Q}_N \otimes \mathbb{Q}_M) & & \| & m_1 & \downarrow{\scriptstyle \mathrm{bc}^!} \\
\downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} & & & & \\
j^!j_!(\mu_0)_!(j^!_M\mu^*\mathbb{Q}_N \otimes j^*_M\mathbb{Q}_M) & \xleftarrow{\mathrm{pr}^1} & j^!\mu_!((j_M)_!j^!_M\mu^*\mathbb{Q}_N \otimes \mathbb{Q}_M) & \xrightarrow{\epsilon^!_{j_M}} & j^!\mu_!(\mu^*\mathbb{Q}_N \otimes \mathbb{Q}_M) & & \\
\downarrow{\scriptstyle \alpha_\mu} & & \downarrow{\scriptstyle \alpha_\mu} & & \downarrow{\scriptstyle \alpha_\mu} & & \\
j^!j_!(\mu_0)_!(j^!_M\mathbb{Q}_M \otimes j^*_M\mathbb{Q}_M) & \xleftarrow{\mathrm{pr}^1} & j^!\mu_!((j_M)_!j^!_M\mathbb{Q}_M \otimes \mathbb{Q}_M) & \xrightarrow{\epsilon^!_{j_M}} & j^!\mu_!(\mathbb{Q}_M \otimes \mathbb{Q}_M) & \nearrow_{m_1} & \\
\downarrow{\scriptstyle \mathrm{bc}^!} & & \downarrow{\scriptstyle \mathrm{bc}^!} & & \downarrow{\scriptstyle \mathrm{bc}^!} & & \\
(\mu_0)_!j^!_M(j_M)_!(j^!_M\mathbb{Q}_M \otimes j^*_M\mathbb{Q}_M) & \xleftarrow{\mathrm{pr}^1} & (\mu_0)_!j^!_M((j_M)_!j^!_M\mathbb{Q}_M \otimes \mathbb{Q}_M) & \xrightarrow{\epsilon^!_{j_M}} & (\mu_0)_!j^!_M(\mathbb{Q}_M \otimes \mathbb{Q}_M) & \xrightarrow{m_1} & (\mu_0)_!j^!_M\mathbb{Q}_M
\end{array}
\tag{A.3.2}
$$

The regions labeled $(\mathrm{A.2.1})_1$ and $(\mathrm{A.2.1})_2$ commute using the analog of the first rectangle in Lemma A.2.1 with m_1 instead of m_2, taking $A = j_M^! \mathbb{Q}_M$ and $A = \mathbb{Q}_M$ respectively.

Lemma A.3.3. *The mapping* $\sigma_{\mathbb{Q}_N}\colon \mu_0^* j_N^! \mathbb{Q}_N \to j_M^! \mu^* \mathbb{Q}_N$ *is an isomorphism in* $D_c^b(M_0)$.

Proof. We have

$$\sigma = \Phi_{j_M}(\mu^*(\epsilon^!_{j_N} \circ \widetilde{\mathrm{bc}}^{-1}) = j_M^! \mu^*(\epsilon^!_{j_N}) \circ j_M^!(\widetilde{\mathrm{bc}}^{-1}) \circ \eta^!_{j_M}.$$

Since j_M is a closed embedding, $\eta^!_{j_M}$ is an isomorphism, so it is enough to show that $j_M^! \mu^*(\epsilon^!_{j_N})\colon j_M^! \mu^* (j_N)_! j_N^! \mathbb{Q}_N \longrightarrow j_M^! \mu^* \mathbb{Q}_N$ is an isomorphism. Since M and N are purely d-dimensional, rational homology manifolds and M_0 and N_0 are purely $2n$-dimensional, rational homology manifolds, it follows that $j_M^! \mu^* (j_N)_! j_N^! \mathbb{Q}_N$ and $j_M^! \mu^* \mathbb{Q}_N$ are both isomorphic to $\mathbb{Q}_{M_0}[-2l]$. It follows that $j_M^! \mu^*(\epsilon^!_{j_N})$ is an isomorphism. □

Since $\sigma_{\mathbb{Q}_N}$ is an isomorphism, the composition

$$\beta_{j_M}^{-1} \circ j_M^!(\nu_M^{-1} \circ \alpha_\mu) \circ \sigma \circ \mu_0^*(\omega_{j_N}^{-1}) \circ \alpha_\mu^{-1}\colon \mathbb{Q}_{M_0} \longrightarrow \mathbb{D}_{M_0}[-4n]$$

is an isomorphism, so we may choose ν_{M_0} so that

$$\nu_{M_0}^{-1} = \beta_{j_M}^{-1} \circ j_M^!(\nu_M^{-1} \circ \alpha_\mu) \circ \sigma_{\mathbb{Q}_N} \circ \mu_0^*(\omega_{j_N}^{-1}) \circ \alpha_\mu^{-1}.$$

It then follows that

$$\omega_{j_M} \circ \alpha_{\mu_0} = j_M^!(\alpha_\mu) \circ \sigma_{\mathbb{Q}_N} \circ \mu_0^*(\omega_{j_N}). \tag{A.3.4}$$

As in §2.5, $(\mathrm{bc}^!)^{-1}\colon (\mu_0)_! j_M^! \to j_N^! \mu_!$ by $(\mathrm{bc}^!)^{-1} = \Phi_{j_N}(\mu_!(\epsilon^!_{j_M}))$. Thus, using (2.4.2) we have

$$\begin{aligned}
(\mathrm{bc}^!)^{-1} \circ (\mu_0)_!(\eta^!_{j_M}) &= \Phi_{j_N}(\mu_!(\epsilon^!_{j_M})) \circ (\mu_0)_!(\eta^!_{j_M}) \\
&= \Phi_{j_N}(\mu_!(\epsilon^!_{j_M}) \circ j_!(\mu_0)_!(\eta^!_{j_M})) \\
&= \Phi_{j_N}(\mu_!(\epsilon^!_{j_M} \circ j_{M\,!}(\eta^!_{j_M}))) \\
&= \Phi_{j_N}(\mu_!(\Phi_{j_M}^{-1}(\eta^!_{j_M}))) \\
&= \Phi_{j_N}(id) \\
&= \eta^!_{j_N}.
\end{aligned}$$

Therefore,

$$\mathrm{bc}^! \circ \eta^!_{j_N} = (\mu_0)_!(\eta^!_{j_M}). \tag{A.3.5}$$

Lemma A.3.6. *The diagram*

$$\begin{array}{ccc}
R(j_N)_! j_N^! A \otimes R\mu_! B & \xrightarrow{\mathrm{pr}^1_{j_N}} & R(j_N)_! \left(j_N^! A \otimes j_N^* R\mu_! B \right) \\
\Big\downarrow {\scriptstyle \mathrm{pr}^2_\mu} & & \Big\downarrow {\scriptstyle R(j_N)_!(id \otimes \mathrm{bc}^*)} \\
R\mu_! \left(\mu^* R(j_N)_! j_N^! A \otimes B \right) & & R(j_N)_! \left(j_N^! A \otimes R(\mu_0)_! j_M^* B \right) \\
\Big\downarrow {\scriptstyle R\mu_!(\widetilde{\mathrm{bc}} \otimes id)} & & \Big\downarrow {\scriptstyle R(j_N)_!(\mathrm{pr}^2_{\mu_0})} \\
R\mu_! \left(R(j_M)_! \mu_0^* j_N^! A \otimes B \right) & \xrightarrow[R\mu_!(\mathrm{pr}^!_{j_M})]{} & R(j_N \mu_0)_! \left(\mu_0^* j_N^! A \otimes j_M^* B \right)
\end{array}$$

commutes for A in $D^b(N)$ and B in $D^b(M)$.

Proof. First, using the formulas for $\mathrm{pr}^1_{j_N}$ and pr^2_μ and the analogs of equation (2.4.2) for Ψ_{j_N} and Ψ_μ we see that it is enough to show that

$$\Psi_\mu \left(\mathrm{pr}^1_{j_M} \circ (\widetilde{\mathrm{bc}} \otimes \epsilon^*_\mu) \circ \mathrm{nat}^\otimes_\mu \right) = \Psi_{j_N} \left(\mathrm{pr}^2_{\mu_0} \circ (\epsilon^*_{j_N} \otimes \mathrm{bc}^*) \circ \mathrm{nat}^\otimes_{j_N} \right).$$

Next, using the formulas for $\mathrm{pr}^1_{j_M}$ and $\mathrm{pr}^2_{\mu_0}$ and the analogs of equation (2.4.2) for Ψ_{j_M} and Ψ_{μ_0} we see that it is enough to show that

$$\Psi_\mu \Psi_{j_M} \left((\epsilon^*_{j_M} \otimes id) \circ \mathrm{nat}^\otimes_{j_M} \circ j_M^*((\widetilde{\mathrm{bc}} \otimes \epsilon^*_\mu) \circ \mathrm{nat}^\otimes_\mu) \right)$$
$$= \Psi_{j_N} \Psi_{\mu_0} \left((id \otimes \epsilon^*_{\mu_0}) \circ \mathrm{nat}^\otimes_{\mu_0} \circ \mu_0^*((\epsilon^*_{j_N} \otimes \mathrm{bc}^*) \circ \mathrm{nat}^\otimes_{j_N}) \right).$$

Now using that $\Psi_f \Psi_g = \Psi_{fg}$ and the naturality of $\mathrm{nat}^\otimes_{j_M}$ and $\mathrm{nat}^\otimes_{\mu_0}$, we see that it is enough to show that

$$(\epsilon^*_{j_M} \otimes id) \circ (j_M^*(\widetilde{\mathrm{bc}}) \otimes j_M^*(\epsilon^*_\mu)) \circ \mathrm{nat}^\otimes_{j_M} \circ j_M^*(\mathrm{nat}^\otimes_\mu)$$
$$= (id \otimes \epsilon^*_{\mu_0}) \circ (\mu_0^*(\epsilon^*_{j_N}) \otimes \mu_0^*(\mathrm{bc}^*)) \circ \mathrm{nat}^\otimes_{\mu_0} \circ \mu_0^*(\mathrm{nat}^\otimes_{j_N}).$$

Since $\mathrm{nat}^\otimes_g \circ g^*(\mathrm{nat}^\otimes_f) = \mathrm{nat}_{fg}$, we only need to show that

$$\epsilon^*_{j_M} \circ j_M^*(\widetilde{\mathrm{bc}}) = \mu_0^*(\epsilon^*_{j_N}) \quad \text{and} \quad j_M^*(\epsilon^*_\mu) = \epsilon^*_{\mu_0} \circ \mu_0^*(\mathrm{bc}^*)$$

which is the same as

$$\Psi^{-1}_{j_M}(\widetilde{\mathrm{bc}}) = \mu_0^*(\epsilon^*_{j_N}) \quad \text{and} \quad j_M^*(\epsilon^*_\mu) = \Psi^{-1}_{\mu_0}(\mathrm{bc}^*).$$

These last two equations follow immediately from the definitions $\widetilde{\mathrm{bc}} = \Psi_{j_M}(\mu_0^*(\epsilon^*_{j_N}))$ and $\mathrm{bc}^* = \Psi_{\mu_0}(j_M^*(\epsilon^*_\mu))$ above. □

Since $\Phi_{j_M}(\xi) = j_M^!(\xi) \circ \eta^!_{j_M}$, using the naturality of $\epsilon^!_{j_M}$ and the fact that $\epsilon^!_{j_M} \circ \eta^!_{j_M} = id$, we have

$$\begin{aligned}
\epsilon^!_{j_M} \circ (j_M)_!(\sigma) \circ \widetilde{\mathrm{bc}} &= \epsilon^!_{j_M} \circ (j_M)_! \left(j_M^!(\mu^*(\epsilon_{j_N}) \circ \widetilde{\mathrm{bc}}^{-1}) \right) \circ \eta^!_{j_M} \circ \widetilde{\mathrm{bc}} \\
&= \mu^*(\epsilon_{j_N}) \circ \widetilde{\mathrm{bc}}^{-1} \circ \epsilon^!_{j_M} \circ \eta^!_{j_M} \circ \widetilde{\mathrm{bc}} \\
&= \mu^*(\epsilon_{j_N}).
\end{aligned} \tag{A.3.7}$$

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Department of Mathematics, University of North Texas, Denton TX, USA 76203
E-mail address: douglass@unt.edu
URL: http://hilbert.math.unt.edu

School of Mathematics, University of Birmingham, Birmingham B15 2TT, UK
E-mail address: ger@for.mat.bham.ac.uk
URL: http://www.mat.bham.ac.uk