Blowup equivalence for smooth compact manifolds

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ABSTRACT. In this paper I study the equivalence relation of smooth compact manifolds (manifolds with boundary) generated by blowups along proper smooth submanifolds. I prove that two smooth connected manifolds are blowup equivalent if and only if they have the same number of boundary components. This result extends the result of [3] on the blowup equivalence of smooth closed manifolds.

The main result of the paper is the following theorem.

Theorem 1. *If M and N are two smooth connected compact manifolds and the number of components of ∂M and ∂N is the same then M and N are blowup equivalent.*

Similarly to the case of smooth closed manifolds [3] this allows to turn the set $\mathrm{Diff}_k(n)$ of diffeomorphism types of the smooth compact connected n-dimensional manifolds with k connected components into a metric space. The integer-valued distance in $\mathrm{Diff}_k(n)$ is defined as the minimal length of a blowup equivalence sequence connecting the manifolds.

1. DEFINITIONS

Let M be a smooth compact manifold and let $L \subset M$ be a smooth compact submanifold of M such that

$$\partial L = L \cap \partial M.$$

Such a submanifold is called *proper* if L is transverse to ∂M. The *normal bundle* of

$$\nu_M(L) : T \to L$$

is defined as the quotient $j^* \tau(M)/\tau(L)$ of the vector bundle induced by the inclusion $j : L \hookrightarrow M$ from the tangent bundle of M by the tangent bundle of L. The *tubular neighborhood theorem* states that there exists an embedding $\mathrm{em} : T \to M$ sending the zero section of $\nu_M(L)$ to L and such that the inverse image of ∂M coincides with the total space of $\nu_M(L)|_{\partial L}$.

$$\begin{array}{ccc} T & \xrightarrow{\nu_M(L)} & L \\ {\scriptstyle \mathrm{em}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ M & \hookleftarrow & L \end{array}$$

Any neighborhood of L representable as the image of such an embedding of the normal bundle is called the *tubular neighborhood* of L. For convenience we identify the tubular neighborhood of a submanifold and the total space of its normal bundle, so we view T as a submanifold of M of codimension 0 and L as a submanifold of T (given by the zero section).

The *projectivization* of $\nu_M(L)$ is the bundle

$$\operatorname{proj}\nu_M(L) : \tilde{L} \to L$$

with the total space $\tilde{L}$ consisting of all 1-dimensional subspaces of the fibers of $\nu_M(L)$ and with the map $\operatorname{proj}\nu_M(L)$ sending each of these 1-dimensional subspaces to the point in L where this subspace projects under $\nu_M(L)$. The *tautological* bundle

$$\tilde{\nu} : \tilde{T} \to \tilde{L},$$

is the bundle formed by the fibers consisting of the elements of the 1-dimensional subspaces which are themselves the points of projection of these fibers (over each fiber of $\tilde{L} \to L$ the map is the Hopf map $\mathbb{R}P_0^{k+1} \to \mathbb{R}P^k$).

We note that each point of $T - L$ is contained in the unique 1-dimensional subspace of the unique fiber of $\nu_M(L)$. Therefore,

$$\tilde{T} - \tilde{L} = T - L$$

and, in particular,

$$\partial\tilde{T} = \partial T.$$

There is a well-defined smooth map

$$\beta : \tilde{T} \to T,$$

such that $\beta|_{\tilde{T}-\tilde{L}} : \tilde{T} - \tilde{L} \to T - L$ is identity and $\beta|_{\tilde{L}}$ is the projection map $\operatorname{proj}\nu_M(L) : \tilde{L} \to L$.

Definition. *The result of blowup* of M along L is the manifold

$$\tilde{M} = B(M, L) = (M - L) \cup_{T-L} \tilde{T}.$$

The blowup of M along L is the map

$$\beta(M, L) : \tilde{M} \to M$$

defined by $\beta(M, L)|_T = \beta$ and $\beta(M, L)|_{M-L} = \mathrm{id}$. The submanifold $L \subset M$ is called *the center of the blowup.* The submanifold $\tilde{L} \subset \tilde{M}$ is called *the exceptional divisor of the blowup*

Assertion 1.1. *The exceptional divisor $\tilde{L}$ is not $\mathbb{Z}_2$-homologous to zero in $\tilde{M}$.*

Proof. The fiber F of $\tilde{L} \to L$ is diffeomorphic to $\mathbb{R}P^q$. The $\mathbb{Z}_2$-intersection number in $\tilde{M}$ of $\tilde{L}$ and $\mathbb{R}P^1 \subset F$ is 1. $\square$

Definition. *The proper transform* $\tilde{V}$ of a subset $V \subset M$ under the blowup $\beta(M, L)$ is the closure in $\tilde{M}$ of the image of $V - L$ in $\tilde{M}$ under the natural identification $M - L = \tilde{M} - \tilde{L}$.

Assertion 1.2. *If $V \subset M$ is a smooth proper submanifold transversal to L then the proper transform $\tilde{V}$ under $\beta(M, L)$ is diffeomorphic to $B(V, V \cap L)$.*

The assertion is a consequence of the coincidence of the normal bundle of $V \cap L$ in V and the restriction to $V \cap L$ of the normal bundle of L in M.

Definition. Smooth compact manifolds M and N are called *blowup equivalent* if there exists a sequence of smooth compact manifolds

$$M = M_0, M_1, \ldots, M_n = N$$

such that either M_{j-1} is the result of blowup of M_j or M_j is the result of blowup of M_{j-1}, $j \in \{1, \ldots, n\}$.

This defines the blowup equivalence as the equivalence generated by the operation of blowup.

Assertion 1.3. *If M and N are blowup equivalent then $M \times R$ and $N \times R$ are blowup equivalent for any smooth compact manifold R.*

Proof. To get the blowup equivalence between $M \times R$ and $N \times R$ we multiply by R every blowup in the blowup sequence connecting M and N. □

Corollary 1.4. *If M is blowup equivalent to N and M' is blowup equivalent to N' then $M \times M'$ is blowup equivalent to $N \times N'$.*

If M and N are of the same dimension then by the same argument $M \# N$ is blowup equivalent to $N \# N'$.

Definition. Manifold R is called *the result of multiblowup* of M if there exists a sequence of blowups $R = M_r \to M_{r-1} \to \cdots \to M_0 = M$. The composition of blowups is called *the multiblowup. The center of multiblowup* $R \to M$ is the union of images in M of the centers of all blowups in the sequence. The multiblowup is called *disjoint from a subset* $S \subset M$ if the intersection of its center and S is empty.

Proposition 1.5. *If M and N are blowup equivalent then there exists a manifold R which is the result of a multiblowup of M and, in the same time, the result of a multiblowup of N.*

This proposition follows from Lemma 1.6 by induction.

Definition. Pairs (M, A) and (N, B) are called *blowup equivalent* if there exists a manifold R which is the result of a multiblowup of both M and N and the proper transforms of A and B in R coincide. Pair (M, A) is called blowup equivalent to (N, B) *over a subset* $U \subset M$ if the center of multiblowup $R \to M$ is contained in U. Manifolds M and N are called blowup equivalent *away from a subset* $S \subset M$ if $R \to M$ is disjoint from S.

Let $M = B(P, L_M)$ and $N = B(P, L_N)$ be the results of two blowups of a smooth manifold P along submanifolds $L_M, L_N \subset P$ transverse to each other.

Lemma 1.6. *There exists a manifold R which is the result of a blowup of M and, in the same time, is the result of a blowup of N.*

$$\begin{array}{ccc} R & \xrightarrow{\beta(N,K_N)} & N \\ {\scriptstyle \beta(M,K_M)}\downarrow & & \downarrow{\scriptstyle \beta(P,L_N)} \\ M & \xrightarrow{\beta(P,L_M)} & P \end{array}$$

Proof. Assertion 1.2 implies that the proper transform $K_M \subset M$ of L_N under $\beta(P, L_M)$ and the proper transform $K_N \subset N$ of L_M under $\beta(P, L_M)$ are smooth manifolds. We define R as $B(M, K_M)$, then R is also diffeomorphic to $B(N, K_N)$, since the pullbacks of the tubular (and respecting L_M and L_N) neighborhood of the submanifold $L_M \cap L_N \subset P$ under $\beta(M, K_M)\beta(P, L_M)$ and $\beta(N, K_N)\beta(P, L_N)$ are both equal to the bundle over $L_M \cap L_N$ associated to $\nu_{L_N}(L_M \cap L_N) \oplus \nu_{L_M}(L_M \cap L_N)$ with the fiber $\mathbb{R}P^p_0 \times \mathbb{R}P^q_0$, where $p = \dim P - \dim L_N$, $q = \dim P - \dim L_M$. $\square$

2. Example of blowup equivalence

Proposition 2.1. *$S^p \times D^q$ and D^{p+q} are blowup equivalent, if $p > 0$ and $q > 1$.*

Proof. Subsequent applications of Lemma 2.2 and Assertion 1.3 imply that D^{p+q} is blowup equivalent to $S^1 \times D^{p+q-1}$, ... and, finally, to $S^1 \times \cdots \times S^1 \times D^2$. By Corollary 3.3 of [3], S^p is blowup equivalent to $S^1 \times \cdots \times S^1$. Thus, $S^p \times D^q$ is blowup equivalent to $S^1 \times \cdots \times S^1 \times D^2$ and, therefore, to D^{p+q}. $\square$

Lemma 2.2. *$S^1 \times D^n$ is blowup equivalent to D^{n+1} if $n > 1$.*

Proof. Note that

$$B(D^{n+1}, D^{n-1}) = \mathbb{R}P^1 \tilde{\times} D^n = S^1 \tilde{\times} D^n,$$

since $D^{n+1} = \bigcup_{x \in \mathbb{R}P^1} D^n_x$, where $D^n_x \subset D^{n+1}$ is the n-dimensional disk containing D^{n-1} and corresponding to the direction x in the orthogonal complement of D^{n-1} in D^{n+1}.

Thus, it suffices to show that $S^1 \tilde{\times} D^n$ and $S^1 \times D^n$ are blowup equivalent. If $n > 2$ then each of these bundles admits a fiberwise linearly independent pair of sections. The orthogonal complements of these sections produce the codimension 2 embeddings

$$S^1 \tilde{\times} D^{n-2} \hookrightarrow S^1 \tilde{\times} D^n \quad \text{and} \quad S^1 \times D^{n-2} \hookrightarrow S^1 \times D^n.$$

The blowing up along the images of these embeddings produces two bundles over $S^1 \times S^1$ with fiber D^{n-1}. Both of them are D^{n-1}-bundles associated to non-trivial principal $\mathbb{Z}_2$-bundles over $S^1 \times S^1$ (the nontrivial element of $\mathbb{Z}_2$ acts on D_2 by a linear involution reversing the orientation of D^{n-1}). Their total spaces are diffeomorphic, since a nontrivial double covering of $S^1 \times S^1$ is unique up to diffeomorphism.

For $n = 2$ we construct a blowup equivalence between $S^1 \tilde{\times} D^2$ and $S^1 \times D^2$ separately. We blow up each bundle along the zero section. The result of the first (resp. second) blowup is a D^1-bundle over $S^1 \times S^1$ (resp. $S^1 \tilde{\times} S^1$). Blowing up of each of these two bundles along a point in the interior produces a nontrivial D^1-bundle over $\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$, but all the non-trivial D^1-bundles over $\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$ are diffeomorphic. $\square$

3. Key lemmata

The following lemma is useful when we need to perform a blowing up or down in ∂M (recall that in M we may blowup only along proper submanifolds).

Lemma 3.1. *If $V \subset \partial M$ is a smooth compact submanifold such that $\nu_{\partial M}(V)|_{\partial V}$ admits a non-vanishing section then there exists a multiblowup $\mu : \tilde{M} \to M$ disjoint from ∂V and a proper embedding*

$$\tilde{V} \times [0,1] \hookrightarrow \tilde{M}$$

($\tilde{V} \times [0,1]$ is a manifold with boundary $\tilde{V} \times \{0\} \cup \tilde{V} \times \{1\} \cup \partial\tilde{V} \times [0,1]$) such that $\tilde{V} = \tilde{V} \times \{0\} \subset \partial\tilde{M}$ is the proper transform of V

Proof. We construct the embedding of $\tilde{V} \times [0,1] \hookrightarrow \partial\tilde{M}$ by finding a non-vanishing section of $\nu_{\tilde{M}}(\tilde{V})$ and then we push the interior of $\tilde{V} \times [0,1]$ into the interior of $\tilde{M}$ to make the embedding proper.

Let η be a section of $\nu_{\partial M}(V)$ non-vanishing on ∂V and transverse to V so that $Z_1 = V \cap \eta(V)$ is a smooth closed submanifold of V. If $Z_1 = \emptyset$ then η does not vanish on V and the proof is finished. Inductively, if $Z_j \neq \emptyset$ then we define the smooth submanifold $Z_{j+1} \subset Z_j$ as the zero set of a section of $\nu_{\partial M}(V)|_{Z_j}$ transverse to Z_j. Note that $\dim Z_j - \dim Z_{j+1} = \dim \partial M - \dim V > 0$ and, therefore, $Z_{k+1} = \emptyset$ for some k so $\nu_{\partial M} V|_{Z_k}$ admits a non-vanishing section. This gives us an embedding $Z_k \times [0,1] \subset \partial M$ such that $Z_k \times [0,1] \cap V = Z_k \times \{0\} = Z_k$. If we push $Z_k \times (0,1)$ inside the interior of M we get a proper submanifold $L \subset M$ diffeomorphic to $Z_k \times [0,1]$. Let $\bar{M} = B(M,L)$ and let

$$\bar{V} \supset \bar{Z}_1 \supset \cdots \supset \bar{Z}_{k-1}$$

be the proper transforms of

$$V \supset Z_1 \supset \cdots \supset Z_{k-1}.$$

$\bar{Z}_{j+1} \subset \bar{Z}_j$ is a smooth submanifold representable as the zero set of a generic section of $\nu_{\partial\bar{M}}(\bar{V})|_{Z_j}$, but $\nu_{\partial\bar{M}}\bar{V}|_{\bar{Z}_{k-1}}$ admits a non-vanishing section. Inductively, we obtain a multiblowup $\tilde{M} \to \bar{M} \to M$ and a non-vanishing section of $\nu_{\tilde{M}}(\tilde{V})$ for the proper transform $\tilde{V}$ of V. $\square$

The next lemma and addendum are the versions for manifolds with boundary of Lemma 6.5 and Addendum 6.6 of [3] and the proof of them is similar.

Lemma 3.2. *Let $\tilde{W} \subset M$ be a smooth compact proper submanifold. If $\tilde{W} = B(W,K)$ and $\nu_M(\tilde{W})$ is trivial then there exists a manifold M' containing W as a submanifold such that $\nu_{M'}(W)$ is trivial and $(M,\tilde{W})$ is blowup equivalent to (M',W) over $\tilde{K}$.*

Let $\tilde{K} \subset \tilde{W}$ be the exceptional divisor of $\beta(W,K)$.

Addendum 3.3. *If $S \subset W - \tilde{K}$ is a subset closed in W and a trivialization τ of $\nu_M(\tilde{W})|_S$ extends to a trivialization of $\nu_M(\tilde{W})$ then τ extends to a trivialization of $\nu_{M'}(W)$.*

Proof of Lemma 3.2 *and Addendum* 3.3. Let $\tilde{M} = B(M, \tilde{K})$. The proper transform of $\tilde{W}$ under $\beta(M, \tilde{W})$ is diffeomorphic to $\tilde{W}$, we denote it by the same symbol $\tilde{W} \subset \tilde{M}$. By adjunction formula, the normal bundle $\nu_{\tilde{M}}(\tilde{K})$ is isomorphic to the tensor product of the (one-dimensional) bundle $\nu_{\tilde{M}}(\tilde{K})$ and the trivial bundle ϵ^{q+1}, $q+1 = \dim M - \dim \tilde{K}$,

$$\nu_{\tilde{M}}(\tilde{K}) = \nu_{\tilde{W}}(\tilde{K}) \otimes \epsilon^{q+1}.$$

Therefore, the projectivization E of $\nu_{\tilde{M}}(\tilde{K})$ is diffeomorphic to $\tilde{K} \times \mathbb{R}P^q$ and the projectivization of $\nu_{\tilde{W}}(\tilde{K}) \subset \nu_{\tilde{M}}(\tilde{K})$ corresponds to the embedding

$$F = \tilde{K} \times \{x\} \subset \tilde{K} \times \mathbb{R}P^q = E$$

for a point $x \in \mathbb{R}P^q$.

Let $\bar{M} = B(\tilde{M}, \tilde{K})$. The proper transform $\bar{W}$ of $\tilde{W}$ under $\beta(\tilde{M}, \tilde{K})$ is diffeomorphic to $\tilde{W}$ and $\nu_{\bar{W}}(F) \approx \nu_{\tilde{W}}(\tilde{K})$. The latter implies that the tubular neighborhood of E in $\bar{M}$ is diffeomorphic to $B(T, K \times \mathbb{R}P^q)$, where T is the total space of $\nu_{W \times \mathbb{R}P^q}(K \times \mathbb{R}P^q)$. Therefore,

$$\bar{M} = B(M', K \times \mathbb{R}P^q)$$

for a manifold $M' \supset W$ and

$$W \cap (K \times \mathbb{R}P^q) = K \times \{x\}.$$

Note that $\nu_{M'}(W)$ is trivial, since $\nu_{\bar{M}}(\bar{W})$ is trivial (the latter is isomorphic to the tensor product of the trivial bundle $\nu_M(\tilde{W})$ and the square of one-dimensional bundle over $\tilde{W}$ dual to $\tilde{K}$) and a trivialization of $\nu_M(\tilde{W})$ produces a trivialization of $\nu_{M'}(W)$. $\square$

We need the next lemma to drop the assumption of triviality of $\nu_M(\tilde{W})$ in Lemma 3.2.

Lemma 3.4. *Suppose that $W \subset M$ is a proper submanifold, $S \subset W$ is a closed subset and the sections $\epsilon_1, \ldots, \epsilon_q$ give a trivialization of the bundle $\nu_M(W)|_S$. Then there exists a multiblowup $\mu : \tilde{M} \to M$ disjoint from S such that the trivialization $(\epsilon_1, \ldots, \epsilon_q)$ extends to a trivialization of $\nu_{\tilde{M}} \tilde{W}$ for the proper transform $\tilde{W} \subset \tilde{M}$ of W.*

Proof (cf. the proof of Proposition 6.4 of [3]*).* Let ξ_1' be a section of $\nu_M(W)$ extending ϵ_1 and transversal to the zero section. The zero set of ξ_1' is a proper smooth submanifold $Z_1 \subset W$. Let $M_1 = B(M, Z_1)$ and let $W_1 \subset M_1$ be the proper transform of W under $\beta(M, Z_1)$. Then ϵ_1 extends to a non-vanishing section ξ_1 of $\nu_{M_1}(W_1)$.

Inductively, we suppose that $\mu_k : M_k \to M$ is a multiblowup such that $\epsilon_1, \ldots, \epsilon_k$ extend to linearly independent sections $\xi_1, \ldots, \xi_k$ of $\nu_{M_k}(W_k)$, where W_k is the proper transform of W under μ_k, $k < q$. Let ξ_{k+1}' be a section of $\nu_{M_k}(W_k)$ extending ϵ_{k+1} transverse to the zero section and orthogonal to the sections $\xi_1, \ldots, \xi_k$. The zero set of ξ_{k+1}' is a proper smooth submanifold $Z_{k+1} \subset W_k \subset M_k$. The projectivization of the subbundle of $\nu_{M_k}(Z_{k+1})$ orthogonal to $\xi_1, \ldots, \xi_k$ produces a smooth submanifold

$$F_k \subset \bar{M}_k = B(M_k, Z_{k+1}).$$

Let $M_{k+1} = B(\tilde{M}_k, F_k)$ and let $W_{k+1} \subset M_{k+1}$ be the proper transform of W_k. Sections $\epsilon_1, \dots, \epsilon_{k+1}$ of $\nu_{M_{k+1}}(W_{k+1})|_S = \nu_M(W)|_S$ extend to non-vanishing sections $\xi_1, \dots, \xi_{k+1}$ of $\nu_{M_{k+1}}(W_{k+1})$. □

Corollary 3.5. *Let $\tilde{W} \subset M$ be a smooth compact proper submanifold. If $\tilde{W} = B(W, K)$ then there exists a manifold M' containing W as a submanifold such that $\nu_{M'}(W)$ is trivial and $(M, \tilde{W})$ is blowup equivalent to (M', W). If we fix a trivialization τ of $\nu_M(\tilde{W})|_S$ for a subset $S \subset \tilde{W} - \tilde{K}$ closed in $\tilde{W}$ then the blowup equivalence can be made disjoint from S and so that τ extends to a trivialization of $\nu_{M'}(W)$.*

Proof. Lemma 3.4 gives the multiblowup sequence

$$\mu : M_k \to \dots M_1 \to M$$

and the sequence of proper transforms

$$\beta(W, K) \circ \mu|_{W_k} : W_k \to \dots W_1 \to \tilde{W} \to W$$

such that $\nu_{M_k}(W_k)$ is trivial. The application of Lemma 3.2 to $M_k \supset W_k \to W_{k-1}$ gives a pair (M'_{k-1}, W'_{k-1}) blowup equivalent to $(M, \tilde{W})$ and such that W'_{k-1} is diffeomorphic to W_{k-1} and $\nu_{M'_{k-1}}(W'_{k-1})$ is trivial. Inductively, we get the pair (M', W) blowup equivalent to $(M, \tilde{W})$. □

4. Proof of Theorem 1

It suffices to prove that any smooth n-dimensional compact manifold M is blowup equivalent to the n-dimensional sphere punctured k times, where k is the number of components of ∂M. This follows from Proposition 4.1 and Theorem 1 of [3]. Indeed, the manifold N produced by Proposition 4.1 after gluing each component of the boundary with D^n is a smooth closed manifold and by Theorem 1 of [3] it is blowup equivalent to S^n. This produces an equivalence between N and the punctured sphere (after isotoping the centers of the blowup equivalence away from the attached D^n).

Proposition 4.1. *Any smooth compact manifold M is blowup equivalent to a manifold N with the boundary ∂N diffeomorphic to the disjoint union of k copies of S^{n-1}.*

Lemma 4.2. *Any smooth compact manifold M is blowup equivalent to a manifold N such that every component of ∂N is null-cobordant.*

Proof. Suppose that a component A of ∂M is not cobordant to zero. Then there exists another component B of ∂M not cobordant to zero (since ∂M is cobordant to zero). Let γ be a path connecting A and B. The closed tubular neighborhood of γ is diffeomorphic to $D^{n-1} \times [0,1]$. By Theorem 1 of [3] S^{n-1} and A are blowup equivalent and, therefore, $D^{n-1} \times [0,1]$ and $A_0 \times [0,1]$ are blowup equivalent, $A_0 = A - D^{n-1}$. The application of the latter blowup equivalence to the neighborhood $D^{n-1} \times [0,1] \subset M$ of γ gives the manifold M'. The union of all the components of $\partial M'$ but two is diffeomorphic to $\partial M - (A \cup B)$ and the two components are diffeomorphic to $A\#A$ (which is null-cobordant) and $A\#B$. Repeating this procedure we make all the components of ∂M null-cobordant. □

Proof of Proposition 4.1. Lemma 4.2 allows to assume that every component of ∂M is cobordant to zero. This implies that the existence of a sequence of surgeries between every component of ∂M and S^{n-1}. The surgery on ∂M is defined by a sphere $S^{p-1} = P \subset \partial M$ and a trivialization τ of $\nu_{\partial M}(P)$. The surgery on ∂M is the boundary of the following surgery on M

$$N = M \cup_{T(P)} H, \quad H \approx D^p \times D^q, \ n = p + q,$$

where $T(P) \approx S^{p-1} \times D^q \subset D^p \times D^q$ is the tubular neighborhood of P and the diffeomorphism between $T(P)$ and $S^{p-1} \times D^q$ is given by τ.

To prove the proposition it suffices to show that M and N are blowup equivalent, We prove it by induction on dimension of M. The proof splits into three cases:

(1) P is $\mathbb{Z}_2$-homologous to zero in ∂M,
(2) P is not $\mathbb{Z}_2$-homologous to zero in M,
(3) P is $\mathbb{Z}_2$-homologous to zero in M but not $\mathbb{Z}_2$-homologous to zero in ∂M.

Proposition 4.4 of [3] allows us to assume in the first case that the surgery on P is not odd, i.e. the parallel copy of P defined by τ is $\mathbb{Z}_2$-homologous to zero in $\partial M - P$. Proposition 4.1 follows from Proposition 5.2, Proposition 6.2 and Proposition 7.3. □

5. If the surgery sphere is $\mathbb{Z}_2$-homologous to zero in ∂M

Lemma 5.1. *If $[P] = 0 \in H_{p-1}(\partial M; \mathbb{Z}_2)$ then there exists a multiblowup $\tilde{M} \to M$ disjoint from P such that $P = \partial\tilde{V} \subset \partial\tilde{M}$ for some smooth submanifold $\tilde{V} \subset \partial\tilde{M}$.*

Proof. Proposition 5.3 of [3] implies the existence of a multiblowup

$$\mu : C' - P \to C - P$$

for the component C of ∂M containing P and a smooth submanifold $V \subset C'$ such that $P = \partial M$.

We need to produce such V by a multiblowup of M. We do this by induction on the length of μ. Let $\beta(C, L) : \hat{C} \to C$, $L \subset C$, be the first blowup of μ. We apply Lemma 3.1 to get the proper embedding

$$\bar{L} \times [0,1] \hookrightarrow \bar{M}$$

into the result of a multiblowup $(\bar{M}, \bar{L}) \to (M, L)$. Lemma 1.6 implies that the proper transform $\bar{C} \subset \bar{M}$ of C is the result of a multiblowup of $\hat{C}$ disjoint from P.

Induction gives a multiblowup

$$(\tilde{M}, \tilde{C}) \to (M, C)$$

disjoint from P and such that $\tilde{C}$ is the result of a multiblowup μ' of C'. To finish the proof we isotop $V \subset C'$ into a position transverse to μ' and define $\tilde{V} \subset \tilde{C}$ as the proper transform of V. □

Proposition 5.2. *If $[P] = 0 \in H_{p-1}(\partial M; \mathbb{Z}_2)$ then then the result N of surgery of M along P is blowup equivalent to M.*

Proof. By Lemma 5.1 we may assume that $P = \partial V \subset \partial M$ for a smooth submanifold $V \subset \partial M$. By Lemma 3.1 we may assume (possibly after a multiblowup) that the embedding $V = V \times \{0\} \hookrightarrow \partial M$ extends to a proper embedding

$$V \times [0,1] \hookrightarrow M.$$

By Theorem 1 of [3] V is blowup equivalent to D^p away from P. We apply induction on the length l of blowup sequence between V and D^q.

Let U be the manifold next to V in the blowup sequence between V and D^p. If

$$U = B(V, L), \ L \subset V - P,$$

then we define

$$M' = B(M, L \times [0,1]).$$

If

$$V = B(U, L), \ L \subset U - P,$$

then we take

$$W = U \times [0,1], \ K = L \times [0,1], \ \tilde{W} = V \times [0,1] = B(W, K)$$

and use Corollary 3.5 to get M'. In both cases

$$P = \partial U \subset M',$$

where M' is blowup equivalent to M and U can be connected to D^p with a blowup sequence of length $l-1$. In the second case Corollary 3.5 implies that τ extends to U. The last operation in the blowup equivalence sequence connecting V and D^q is always a blowdown (D^q is not the result of a blowup by Assertion 1.1).

The result N' of attaching a handle to M' along P equipped with τ is blowup equivalent to N (since the blowup between U and V is disjoint from P). Therefore, to finish the proof we need only to show that if $V \approx D^p$ and τ extends to V then M is blowup equivalent to N. But in this case N is the boundary connected sum of M and $S^p \times D^q$, so M and N are blowup equivalent by Proposition 2.1. □

6. If the surgery sphere is not homologous to zero in M

Lemma 6.1. *If $[P] \neq 0 \in H_*(M;\mathbb{Z}_2)$ then there exists a multiblowup $\tilde{M} \to M$ disjoint from P and a proper submanifold $W \subset \tilde{M}$ such that P intersects W transversely at one point and ∂W is connected.*

Proof. Consider the closed manifold

$$D = M \cup_{\partial M} M.$$

The surgery sphere P is not $\mathbb{Z}_2$-homologous to zero in D since P is not $\mathbb{Z}_2$-homologous to zero in M. Lemma 5.1 of [3] produces a multiblowup $\mu : \tilde{D} \to D$ disjoint from P and a smooth submanifold $\tilde{V} \subset \tilde{D}$ intersecting P transversely at one point.

We may assume that μ transversal to $\partial M \subset D$ (changing $M \subset D$ by a small isotopy). The restriction $\mu|_{\tilde{M}} : \tilde{M} \to M$ is then a multiblowup by Assertion 1.2. The smooth submanifold $W = \tilde{V} \cap \tilde{M} \subset \tilde{M}$ is proper and intersects P transversely at one point. To make ∂W connected we add small tubes to W in $\tilde{M}$. □

Proposition 6.2. *If $[P] \neq 0 \in H_*(M;\mathbb{Z}_2)$ then then the result N of surgery of M along P is blowup equivalent to M.*

Proof. By Lemma 6.1 we may assume that P intersects a smooth proper submanifold $W \subset M$ transversely at one point and ∂W is connected. By the assumption of induction ($\dim W < \dim M$) W is blowup equivalent to D^{q+1}. We may assume that the blowup sequence is disjoint from the point $W \cap P$. Similar to the proof of Proposition 5.2 we use induction on the length l of blowup sequence between V and D^{q+1}.

Let U be the manifold next to W in the blowup sequence between W and D^{q+1}. If $U = B(W,L)$, $L \subset W$, then we put $M' = B(M,L)$. If $W = B(U,L)$, $L \subset U$, then we use Corollary 3.5 to get M'. In both cases

$$P = \partial U \subset M',$$

where M' is blowup equivalent to M and U can be connected to D^p with a blowup sequence of length $l-1$. The result N' of attaching a handle to M' along P equipped with τ is blowup equivalent to N.

Induction reduces the proof to finding a blowup equivalence between M and N in the case when $W \approx D^{q+1}$. The regular neighborhood T of $P \cup W$ is diffeomorphic to $P \times D^{q+1} \approx S^{p-1} \times D^{q+1}$ and $\partial T - \partial M$ is diffeomorphic to D^{p+q-1}. Therefore, M decomposes as the boundary connected sum $T\#(M-T)$ and it suffices to prove that the result of surgery of T along P is blowup equivalent to T. But the result of surgery of $S^{p-1} \times D^{q+1}$ along the core sphere S^{p-1} is diffeomorphic to D^{p+q} which is blowup equivalent to $S^{p-1} \times D^{q+1}$ by Proposition 2.1. □

7. If the surgery sphere iz $\mathbb{Z}_2$-homologous to zero in M but not $\mathbb{Z}_2$-homologous to zero in ∂M

The next lemma generalizes Lemma 5.1 of [3] to the case of manifolds with boundary.

Lemma 7.1. *For any homology class $\beta \in H_*(N;\mathbb{Z}_2)$ in a compact manifold N and any proper smooth submanifold $Q \subset N$ such that $\beta.[Q] = 1 \in \mathbb{Z}_2$ there exist a multiblowup $\mu : \tilde{N} \to N$ disjoint from Q and a closed smooth submanifold $\tilde{W} \subset \tilde{N}$ intersecting Q transversely at one point and such that*

$$\mu_*([\tilde{W}]) = \beta \in H_*(N;\mathbb{Z}_2).$$

Proof (cf. the proof of Lemma 5.1 of [3]*).* Using the Thom's Théorème III.2 [5] we represent β as the image of a smooth closed manifold B under a smooth map $f : B \to N$. The Nash theorem [4] allows us to assume that B and N are nonsingular real algebraic varieties. Using the Akbulut-King normalization Theorem 2.8.3 of [1] we approximate f by a rational map $F : Z \to N$, $F_*(Z) = \beta \in H_*(N;\mathbb{Z}_2)$. The algebraic closure $\bar{Z}$ is a $(\dim B)$-dimensional algebraic subset of N which has at least one nonsingular point. We isotop Q to general position with respect to $\bar{Z}$, then the points of $Q \cap \bar{Z}$ are nonsingular. The Hironaka resolution theorem [2] produces a blowup $\tilde{N} \to N$ resolving $\bar{Z}$. The proper transform $\tilde{Z}$ of Z is a smooth closed submanifold intersecting Q transversely in an odd number of points. To get $\tilde{W}$ we connect all the points of $Q \cap \tilde{Z}$ but one with tubes about Q. □

Corollary 7.2. *If $0 \neq P \in \ker(H_*(\partial M;\mathbb{Z}_2) \to H_*(M;\mathbb{Z}_2))$ then there exists a multiblowup $\tilde{M} \to M$ disjoint from P such that $P = \partial W$ for a smooth proper submanifold $W \subset \tilde{M}$.*

Proof. We apply Lemma 7.1 to the cocore Q of the surgery of M along Q

$$Q = \{0\} \times D^q \subset D^p \times D^q = H \subset N = M \cup_{T(P)} H.$$

Note that a homology class β such that $\beta.[Q] = 1$ exists, since P is homologous to zero in M. Changing $\tilde{W} \subset \tilde{N}$ with an isotopy we may assume that $\tilde{W} \cap H = D^p \times \{0\}$. We define W as the closure of $\tilde{W} - H$. □

Proposition 7.3. *If $0 \neq P \in \ker(H_*(\partial M;\mathbb{Z}_2) \to H_*(M;\mathbb{Z}_2))$ then the result N of surgery of M along P is blowup equivalent to M.*

Proof. By Corollary 7.2 we may assume that $P = \partial W \subset \partial M$ for a smooth proper submanifold $W \subset M$. By Lemma 5.1 of [3] and Lemma 3.1 we may assume that there exists a smooth closed manifold $V \subset \partial M$ intersecting P transversely at one point (cf. the proof of Lemma 5.1). By the assumption of induction ($\dim W < \dim M$) W is blowup equivalent to D^p. By Theorem 1 of [3] V is blowup equivalent to S^q and we may assume that the blowup equivalence is disjoint from the point $P \cap V$.

We make V into S^q similarly to the proof of Proposition 5.2. By Lemma 3.1 we may assume that the embedding $V = V \times \{0\} \hookrightarrow \partial M$ extends to a proper embedding $V \times [0,1] \hookrightarrow M$. We use induction on the length of blowup sequence between V and S^q. Let Y be the manifold next to V in the blowup sequence between V and D^p. If $Y = B(V, L)$, $L \subset V - P$, then we put $M' = B(M, L \times [0,1])$. If $V = B(Y, L)$, $L \subset Y$, then we use Corollary 3.5 to get M'.

We make W into D^p similarly to the proof of Proposition 6.2. We use induction on the length of blowup sequence between W and D^p. Let U be the manifold next to W in the blowup sequence between W and D^p. If $U = B(W, L)$, $L \subset W - P$, then we put $M' = B(M, L)$. If $W = B(U, L)$, $L \subset U - P$, then we use Corollary 3.5 to get M'. In the second case Corollary 3.5 assures that τ extends to U so if $U \approx D^p$ than τ extends to U by Assertion 1.1.

To finish the proof we need only to show that if $W \approx D^p$, τ extends to W and there exists a smooth submanifold $V \approx S^q$ of ∂M intersecting P transversely at one point then M is blowup equivalent to N. But in this case M decomposes as the boundary connected sum $T\#(M - T)$, where $T \approx D^p \times S^q$ is the regular neighborhood of $V \cup W$ in M (cf. the proof of Proposition 6.2). In the same way N decomposes as $T'\#(M - T)$, where $T' \approx S^p \times S^q - \operatorname{int} D^{p+q}$ is the result of the surgery of $T \approx D^p \times S^q$ along $P \approx S^{p-1} \times x$, $x \in S^q$, equipped with the standard 0-framing τ. Proposition 2.1 and Corollary 3.3 of [3] imply that T and T' are blowup equivalent and, therefore, M and N are blowup equivalent. □

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