On the cohomology of categories, universal Toda brackets and homotopy pairs

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ON THE COHOMOLOGY OF CATEGORIES, UNIVERSAL TODA BRACKETS AND HOMOTOPY PAIRS

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Let $\underline{\underline{Top}}^*/\simeq$ be the homotopy category of pointed spaces and let $\underline{\underline{C}}$ be a subcategory consisting of either suspensions ΣX or loop spaces ΩY. The 'universal Toda bracket' [3] of the category $\underline{\underline{C}}$ is a cohomology class

$$\langle\underline{\underline{C}}\rangle_\Sigma \in H^3(\underline{\underline{C}}, D_\Sigma), \quad \text{resp.} \quad \langle\underline{\underline{C}}\rangle_\Omega \in H^3(\underline{\underline{C}}, D_\Omega)$$

in the third cohomology of $\underline{\underline{C}}$ where the coefficients D_Σ, resp. D_Ω, are also determined by the category $\underline{\underline{C}}$. Very little is known on these important characteristic classes of homotopy theory.

Associated to the category $\underline{\underline{Pair}}(\underline{\underline{C}})$ of pairs in $\underline{\underline{C}}$ Hardie [13] studies the 'category of homotopy pairs' for $\underline{\underline{C}}$ which as we show represents a cohomology class in the second cohomology of $\underline{\underline{Pair}}(\underline{\underline{C}})$. In fact, we obtain a natural transformation

$$\lambda : H^{n+1}(\underline{\underline{C}}, D) \to H^n(\underline{\underline{Pair}}(\underline{\underline{C}}), D^\sharp)$$

which for $n = 2$ carries the universal Toda bracket of $\underline{\underline{C}}$ to the cohomology class represented by the category of homotopy pairs for $\underline{\underline{C}}$. Hence the elements $\lambda\langle\underline{\underline{C}}\rangle_\Sigma$, resp. $\lambda\langle\underline{\underline{C}}\rangle_\Omega$, determine the category of homotopy pairs for $\underline{\underline{C}}$ up to equivalence. We also show in § 4 that the elements $\lambda\langle\underline{\underline{C}}\rangle_\Sigma$, resp. $\lambda\langle\underline{\underline{C}}\rangle_\Omega$, characterize algebraically homotopy categories of cofibers $C(f)$, $f \in \underline{\underline{C}}$, resp. fibers $P(f)$, $f \in \underline{\underline{C}}$. The coefficients D_Σ and D_Ω are described explicitly (§ 3) and examples of these coefficients are computed (§ 4). As special cases we study the homotopy categories of CW-complexes with cells only in dimension m and n; dually we characterize the homotopy category of two stage Postnikov systems X with non trivial homotopy groups only in dimension m and n (§ 5). Computations of the universal Toda bracket for one point unions of spheres (§ 6), of homotopy pairs between Hopf maps (§ 7), and homotopy pairs between Pontrjagin maps (§ 8) are given.

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§ 1 The cohomology of categories and linear extensions

For the convenience of the reader and in order to fix notation we recall basic definitions concerning the cohomology of categories, linear extensions and linear track extensions of categories.

(1.1) Definition. Let $\underline{\underline{C}}$ be a category. The category of factorizations in $\underline{\underline{C}}$, denoted by $F\underline{\underline{C}}$, is given as follows. Objects are morphisms $f, g, \ldots$ in $\underline{\underline{C}}$ and morphisms $f \to g$ are pairs (α, β) for which

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & A' \\ \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle g} \\ B & \xleftarrow{\beta} & B' \end{array}$$

commutes in $\underline{\underline{C}}$. Here $\alpha f \beta$ is a factorization of g. Composition is defined by $(\alpha', \beta')(\alpha, \beta) = (\alpha'\alpha, \beta\beta')$. We clearly have $(\alpha, \beta) = (\alpha, 1)(1, \beta) = (1, \beta)(\alpha, 1)$. A natural system (of abelian groups) on $\underline{\underline{C}}$ is a functor $D : F\underline{\underline{C}} \to \underline{\underline{Ab}}$. The functor D carries the object f to $D_f = D(f)$ and carries the morphism $(\alpha, \beta) : f \to g$ to the induced homomorphism

$$D(\alpha, \beta) = \alpha_* \beta^* : D_f \to D_{\alpha f \beta} = D_g$$

Here we set $D(\alpha, 1) = \alpha_*$, $D(1, \beta) = \beta^*$.

We have a canonical forgetful functor $\pi : F\underline{\underline{C}} \to \underline{\underline{C}}^{op} \times \underline{\underline{C}}$ so that each bifunctor $D : \underline{\underline{C}}^{op} \times \underline{\underline{C}} \to \underline{\underline{Ab}}$ yields a natural system $D\pi$, as well denoted by D. Such a bifunctor is also called a $\underline{\underline{C}}$- bimodule.

(1.2) Definition. Let D be a natural system on $\underline{\underline{C}}$. We say that

$$D \overset{+}{\rightarrowtail} \underline{\underline{E}} \overset{p}{\twoheadrightarrow} \underline{\underline{C}}$$

is a linear extensions of the category $\underline{\underline{C}}$ by D if (a), (b) and (c) hold.

(a) $\underline{\underline{E}}$ and $\underline{\underline{C}}$ have the same objects and p is a full functor which is the identity on objects.

(b) For each $f : A \to B$ in $\underline{\underline{C}}$ the abelian group D_f acts transitively and effectively on the subset $p^{-1}(f)$ of morphisms in $\underline{\underline{C}}$. We write $f_0 + \alpha$ for the action of $\alpha \in D_f$ on $f_0 \in p^{-1}(f)$.

(c) The action satisfies the linear distributivity law:

$$(f_0 + \alpha)(g_0 + \beta) = f_0 g_0 + f_* \beta + g^* \alpha.$$

Two linear extensions $\underline{\underline{E}}$ and $\underline{\underline{E}}'$ are equivalent if there is an isomorphism of categories $\epsilon : \underline{\underline{E}} \cong \underline{\underline{E}}'$ with $p'\epsilon = p$ and with $\epsilon(f_0 + \alpha) = \epsilon(f_0) + \alpha$ for $f_0 \in Mor(\underline{\underline{E}})$, $\alpha \in D_{pf_0}$. The extension $\underline{\underline{E}}$ is split if there is a functor $s : \underline{\underline{C}} \to \underline{\underline{E}}$ with $ps = 1$.

Let $\underline{\underline{C}}$ be a small category and let $M(\underline{\underline{C}}, D)$ be the set of equivalence classes of linear extensions of $\underline{\underline{C}}$ by $\underline{\underline{D}}$. Then there is a canonical bijection

(1.3) $$\psi : M(\underline{\underline{C}}, D) \cong H^2(\underline{\underline{C}}, D)$$

which maps the split extension to the zero element; see [2] and IV § 6 in [4]. Here $H^n(\underline{\underline{C}}, D)$ denotes the cohomology of $\underline{\underline{C}}$ with coefficients in D defined below. We obtain a representing cocycle Δ_t of the cohomology class $\{\underline{\underline{E}}\} = \psi \underline{\underline{E}} \in H^2(\underline{\underline{C}}, D)$ as follows. Let t be a "splitting" function for p which associates with each morphism $f : A \to B$ in $\underline{\underline{C}}$ a morphism $f_0 = t(f)$ in $\underline{\underline{E}}$ with $pf_0 = f$. Then t yields a cocycle Δ_t by the formula

(1.4) $$t(gf) = t(g)t(f) + \Delta_t(g, f)$$

with $\Delta_t(g, f) \in D(gf)$. The cohomology class $\{\underline{\underline{E}}\} = \{\Delta_t\}$ is trivial if and only if $\underline{\underline{E}}$ is a split extension. We call

$$D \overset{+}{\rightarrowtail} \underline{\underline{E}} \overset{p'}{\longrightarrow} \underline{\underline{C}}'$$

a <u>weak</u> linear extension if there is a linear extension as in (1.2) together with an equivalence of categories $e : \underline{\underline{C}} \overset{\sim}{\longrightarrow} \underline{\underline{C}}'$ such that $p' = ep$.

Next we define the cohomology of a category $\underline{\underline{C}}$ with coefficients in a natural system D on $\underline{\underline{C}}$. In order to get cohomology groups which are actually sets we have to assume that $\underline{\underline{C}}$ is a small category; by change of universe it would also be possible to define the cohomology in case $\underline{\underline{C}}$ is not small.

(1.5) Definition. Let $\underline{\underline{C}}$ be a category and let $N_n(\underline{\underline{C}})$ be the set of sequences $(\lambda_1, \dots, \lambda_n)$ of n composable morphisms in $\underline{\underline{C}}$ (which are the n-simplices of the <u>nerve</u> of $\underline{\underline{C}}$). For $n = 0$ let $N_0(C) = Ob(\underline{\underline{C}})$ be the set of objects in $\underline{\underline{C}}$. The $n-th$ cochain group $F^n = F^n(\underline{\underline{C}}, D)$ is the abelian group of all functions

$$c : N_n(\underline{\underline{C}}) \to \bigcup_{g \in Mor(\underline{\underline{C}})} D_g \tag{1}$$

with $c(\lambda_1, \dots, \lambda_n) \in D_{\lambda_1 \circ \dots \circ \lambda_n}$. Addition in F^n is given by adding pointwise in the abelian groups D_g. The coboundary $\delta : F^{n+1} \to F^n$ is defined by the formula

$$\begin{aligned}(\delta c)(\lambda_1, \dots, \lambda_n) = {} & (\lambda_1)_* c(\lambda_2, \dots, \lambda_n) \\ & + \sum_{i=1}^{n-1} (-1)^i c(\lambda_1, \dots, \lambda_i \lambda_{i+1}, \dots, \lambda_n) \\ & + (-1)^n \lambda_n^* c(\lambda_1, \dots, \lambda_{n-1})\end{aligned} \tag{2}$$

For $n = 1$ we have $(\delta c)(\lambda) = \lambda_* c(A) - \lambda^* c(B)$ for $\lambda : A \to B \in N_1(\underline{\underline{C}})$. One can check that $\delta c \in F^n$ for $c \in F^{n-1}$ and that $\delta\delta = 0$. Hence the <u>cohomology groups</u> [2]

$$H^n(\underline{\underline{C}}, D) = H^n(F^*(\underline{\underline{C}}, D), \delta) \tag{3}$$

are defined, $n \geq 0$. A functor $\phi : \underline{\underline{C}}' \to \underline{\underline{C}}$ induces the homomorphism

$$\phi^* : H^n(\underline{\underline{C}}, D) \to H^n(\underline{\underline{C}}', \phi^* D) \tag{4}$$

where $\phi^* D$ is the natural system given by $(\phi^* D)_f = D_{\phi(f)}$. On cochains the map ϕ^* is given by the formula

$$(\phi^* f)(\lambda_1', \ldots, \lambda_n') = f(\phi\lambda_1', \ldots, \phi\lambda_n')$$

where $(\lambda_1', \ldots, \lambda_n') \in N_n(\underline{\underline{C}}')$. If ϕ is an equivalence of categories then ϕ^* is an isomorphism. A natural transformation $\tau : D \to D'$ between natural systems induces a homomorphism

$$\tau_* : H^n(\underline{\underline{C}}, D) \to H^n(\underline{\underline{C}}, D') \tag{5}$$

by $(\tau_* f)(\lambda_1, \ldots, \lambda_n) = \tau_\lambda f(\lambda_1, \ldots, \lambda_n)$ where $\tau_\lambda : D_\lambda \to D'_\lambda$ with $\lambda = \lambda_1 \circ \ldots \circ \lambda_n$ is given by the transformation τ. Now let

$$D'' \overset{\iota}{\rightarrowtail} D \overset{\tau}{\twoheadrightarrow} D'$$

be a short exact sequence of natural systems on $\underline{\underline{C}}$. Then we obtain as usual the natural long exact sequence

(1.6) $$\longrightarrow H^n(\underline{\underline{C}}, D') \xrightarrow{\iota_*} H^n(\underline{\underline{C}}, D) \xrightarrow{\tau_*} H^n(\underline{\underline{C}}, D'') \xrightarrow{\beta} H^{n+1}(\underline{\underline{C}}, D') \longrightarrow$$

where β is the Bockstein homomorphism. For a cocycle c'' representing a class $\{c''\}$ in $H^n(\underline{\underline{C}}, D'')$ we obtain $\beta\{c''\}$ by choosing a cochain c as in (1.5) (1) with $\tau c = c''$. This is possible since τ is surjective. Then $\iota^{-1}\delta c$ is a cocycle which represents $\beta\{c''\}$.

(1.7) Remark. The cohomology (1.5) generalizes the cohomology of a group. In fact, let G be a group and let $\underline{\underline{G}}$ be the corresponding category with a single object and with morphisms given by the elements in G. A G-module D yields a natural system $\bar{D} : F\underline{\underline{G}} \to \underline{\underline{Ab}}$ by $\bar{D}g = D$ for $g \in G$. The induced maps are given by $f^*(x) = x^f$ and $h_*(y) = y$, $f, h \in G$. Then the classical definition of the cohomology $H^n(G, D)$ coincides with the definition of

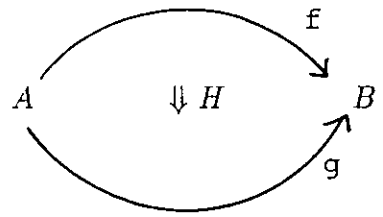
$$H^n(\underline{\underline{G}}, \bar{D}) = H^n(G, D)$$

given by (1.5). Further results and applications of the cohomology of categories can be found in [2], [3], [4], [5], [11], [23], [24].

Recall that $\underline{\underline{K}}(A, B)$ denotes the set of all morphisms $A \to B$ in a category $\underline{\underline{K}}$. Assume for all objects A, B in $\underline{\underline{K}}$ we have an equivalence relation $\simeq$ on $\underline{\underline{K}}(A, B)$. Then $\simeq$ is said to be a natural equivalence relation on $\underline{\underline{K}}$ if $f \simeq g$ and $x \simeq y$ implies $xf \simeq yg$ for $f, g \in Mor(A, B)$ and $x, y \in Mor(B, C)$. In this case we obtain the quotient category $\underline{\underline{K}}/\simeq$ which has the same objects as $\underline{\underline{K}}$ and for which the morphism $A \to B$ are the equivalence classes $\{f\}$ in $\underline{\underline{K}}(A,B)/\simeq$. We often denote the equivalence class $\{f\}$ as well by f.

(1.8) Definition. A track category $T\underline{\underline{K}}$ or $T \rightrightarrows \underline{\underline{K}}$ is a category $\underline{\underline{K}}$ with the following additional structure T of tracks.

(i) For $f, g \in \underline{\underline{K}}(A,B)$ a set $T(f,g)$ is given. We write $f \simeq g$ if $T(f,g)$ is non empty and we write $H : f \simeq g$ if $H \in T(f,g)$. We call H a track from f to g and we indicate H also by the diagram



(ii) An element $0 = 0_f \in T(f,f)$ and functions

$$+ : T(f,g) \times T(g,h) \to T(f,h),$$
$$- : T(f,g) \to T(g,f)$$

are given. We call 0 the trivial track and $+$ is the addition of tracks H and G denoted by $H + G : f \simeq h$. The function $-$ maps $H : f \simeq g$ to the negative $-H : g \simeq f$ of H.

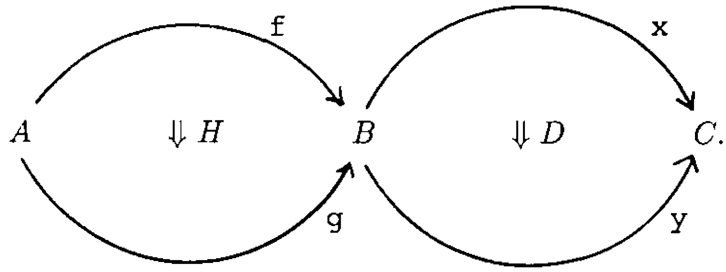
(iii) Induced functions

$$b_* : T(f,g) \to T(bf, bg),$$
$$a^* : T(f,g) \to T(fa, ga)$$

are given for $b : B \to B'$ and $a : A' \to A$ where $A, A', B, B' \in Ob(\underline{\underline{K}})$.

(iv) This structure satisfies the following conditions

$$H + (G + F) = (H + G) + F,$$
$$H + 0 = 0 + H = H,$$
$$H + (-H) = 0,\ (-H) + H = 0,$$
$$a^*(H + G) = (a^*H) + (a^*G),$$
$$a^*(-H) = -a^*(H),$$
$$b_*(H + G) = (b_*H) + (b_*G),$$
$$b_*(-H) = -(b_*H),$$
$$(a'a)^* = a^*(a')^*,\ 1^* = \text{identity}$$
$$(b'b)_* = (b')_*b_*,\ 1_* = \text{identity},$$
$$b_*a^* = a^*b_*, \text{ and}$$
$$f^*D + y_*H = x_*H + g^*D \quad \text{for}$$



One readily checks that the relation $\simeq$ in (i) is a natural equivalence relation on $\underline{\underline{K}}$. We call $\underline{\underline{K}}/\simeq$ the homotopy category of the track category and we write

$$(1.9) \qquad T\underline{\underline{K}} = (T \rightrightarrows \underline{\underline{K}} \overset{p}{\twoheadrightarrow} \underline{\underline{C}})$$

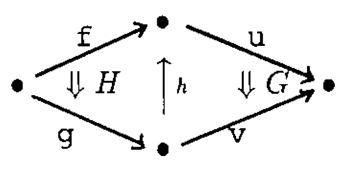
if p is a full functor which is the identity on objects and which satisfies $p(f) = p(g)$ iff $f \simeq g$. Hence p induces the isomorphism $\underline{\underline{K}}/\simeq \cong \underline{\underline{C}}$. A track category $T\underline{\underline{K}}$ as above has the following additional properties. Recall that a groupoid is a small category whose morphisms are invertible. We define a groupoid $\underline{\underline{T}}(A, B)$ for objects $A, B \in \underline{\underline{K}}$ as follows. The set of objects in $\underline{\underline{T}}(A, B)$ is the set $\underline{\underline{K}}(A, B)$ and the elements $H \in T(f, g)$ are the morphisms

$$(1.10) \qquad H : g \to f \quad \text{in} \quad \underline{\underline{T}}(A, B).$$

The composition law is given by the operation $+$ of tracks above and the identical morphism of f is 0_f. We obtain a bifunctor

$$* : \underline{\underline{T}}(B, C) \times \underline{\underline{T}}(A, B) \to \underline{\underline{T}}(A, C)$$
$$D * H = f^*D + y_*H = x_*H + g^*D$$

by the final equation in (iv). This bifunctor satisfies $0_y * H = y_*H$ and $D * 0_f = f^*D$. Moreover the *-operation is associative. This shows that, up to the convention in (1.10), a track category is the same as a groupoid enriched category or equivalently a category based on the monoidal category of groupoids. The operations on tracks can be combined to give the more general operation of pasting. For example



is meant to indicate the track $u_*H + f^*G : uf \simeq uhg \simeq vg$. We define a functor $t : T\underline{\underline{K}} \to T'\underline{\underline{K}}'$ between track categories by a functor $t : \underline{\underline{K}} \to \underline{\underline{K}}'$ and by functions

$$(1.11) \qquad t = t_{f,g} : T(f, g) \to T'(tf, tg)$$

which are compatible with the structure (ii) and (iii) in (3.1), that is $t(0) = 0$, $t(H + G) = (tH) + (tG)$, $t(-H) = -(tH)$, $t(b_*H) = (tb)_*(tH)$, $t(a^*H) = (ta)^*(tH)$.

Clearly a functor $t : T\underline{\underline{K}} \to T'\underline{\underline{K}}'$ induces a functor $t : \underline{\underline{K}}/\simeq \to \underline{\underline{K}}'/\simeq$ between homotopy categories.

(1.12) Definition. Let $\underline{\underline{C}}$ be a category and let D be a natural system on $\underline{\underline{C}}$. A <u>linear track extension</u> $T\underline{\underline{K}}$ of $\underline{\underline{C}}$ by D, denoted by

$$D \xrightarrow{+} T \rightrightarrows \underline{\underline{K}} \xrightarrow[p]{} \underline{\underline{C}}, \tag{1}$$

is defined by a track category, a functor p and an action of D on T as follows. The functor p is the identity on objects and is full, moreover p satisfies

$$p(f) = p(g) \Longleftrightarrow f \simeq g \tag{2}$$

so that p induces an isomorphism $\underline{\underline{K}}/\simeq \cong \underline{\underline{C}}$. The <u>action of</u> D <u>on</u> T is given by isomorphisms of groups

$$\sigma = \sigma_f : D_{pf} \cong T(f,f),\ f \in Mor\,\underline{\underline{K}}, \tag{3}$$

such that (4) and (5) hold:

$$\sigma_f(\alpha) + H = H + \sigma_h(\alpha) \qquad \text{for} \qquad H \in T(f,h) \tag{4}$$

We also write $H + \alpha = H + \sigma_k(\alpha)$.

$$\left.\begin{aligned} g^*\sigma_f(\alpha) &= \sigma_{fg}(g^*\alpha),\ \alpha \in D_{pf}, \\ f_*\sigma_g(\beta) &= \sigma_{fg}(f_*\beta),\ \beta \in D_{pg}. \end{aligned}\right\} \tag{5}$$

We now consider maps between linear track extensions. Let $T\underline{\underline{K}}$ and $T'\underline{\underline{K}}'$ be both linear track extensions of $\underline{\underline{C}}$ by D. A D-<u>equivariant map over</u> $\underline{\underline{C}}$, $t : T\underline{\underline{K}} \to T'\underline{\underline{K}}'$, is a functor t as in (1.11) which satisfies

$$pt = p \qquad \text{and} \qquad t_{f,f}\sigma_f = \sigma_{tf} \tag{6}$$

for $f \in Mor\,\underline{\underline{K}}$. Hence all linear track extensions of $\underline{\underline{C}}$ by D and D-equivariant maps over $\underline{\underline{C}}$ form a category which we denote by $\underline{\underline{Track}}\,(\underline{\underline{C}}, D)$. Two objects in this category are <u>equivalent</u> and we write $T\underline{\underline{K}} \sim T'\underline{\underline{K}}'$, if there exist maps

$$T\underline{\underline{K}} \longleftarrow T''\underline{\underline{K}}'' \longrightarrow T'\underline{\underline{K}}'$$

in $\underline{\underline{Track}}\,(\underline{\underline{C}}, D)$. The set of equivalence classes

$$\pi_0\underline{\underline{Track}}(\underline{\underline{C}}, D) = Ob(\underline{\underline{Track}}(\underline{\underline{C}}, D))/\sim \tag{7}$$

is the set of <u>connected components</u> of the category $\underline{\underline{Track}}(\underline{\underline{C}}, D)$. We define the <u>trivial track extension</u> in $\underline{\underline{Track}}(\underline{\underline{C}}, D)$ by $\underline{\underline{K}} = \underline{\underline{C}}$, $p = 1_{\underline{\underline{C}}}$, and

$$T(f,g) = \begin{cases} D(f) & \text{if} \quad f = g \\ \emptyset & \text{otherwise,} \end{cases} \tag{8}$$

with $\sigma_f = 1_{D(f)}$ and with zero tracks given by zero elements in $D(f)$, $f \in Mor(\underline{\underline{C}})$.

The next result is a fundamental property of linear track extensions which is of similar nature as the classification of linear extensions in (1.3); it is proved in [3].

(1.13) Theorem. *There is a canonical bijection*

$$\psi : \pi_0 \underline{\underline{Track}}(\underline{\underline{C}}, D) \cong H^3(\underline{\underline{C}}, D)$$

which carries the trivial track extension to the zero element of the cohomology group $H^3(\underline{\underline{C}}.D)$ *defined in (1.5).*

(1.14) Definition. We define the bijection ψ in (1.12) as follows. Let $T\underline{\underline{K}}$ be a linear track extension of $\underline{\underline{C}}$ by D. We choose functions

$$\left.\begin{array}{l} t : Mor\,\underline{\underline{C}} \to Mor\,\underline{\underline{K}} \\ H : N_2\underline{\underline{C}} \to \bigcup\limits_{f,g \in Mor(\underline{\underline{K}})} T(f,g) \end{array}\right\} \tag{1}$$

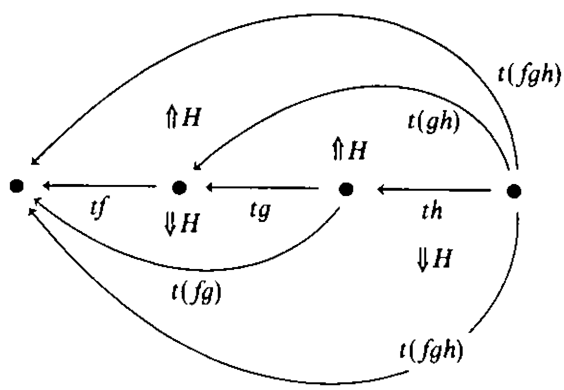
with $pt = 1$ and

$$H(f,g) \in T(tf \circ tg,\, t(fg)). \tag{2}$$

Using such choices of t and H we obtain the cochain

$$c(t,H) : N_3(\underline{\underline{C}}) \to \bigcup_{f \in Mor\,\underline{\underline{C}}} D(f) \tag{3}$$

by the element $c(t,H)(f,g,h) \in D(fgh)$. This element is obtained by the "operation of pasting" in the following diagram



(4)

that is

$$c(t,H)(f,g,h) = \sigma^{-1}_{t(fgh)}(\Delta) \qquad \text{with}$$
$$\Delta = -H(f,gh) - (tf)_* H(g,h) + (th)^* H(f,g) + H(fg,h). \tag{5}$$

One can check that $c(t,H)$ is a cocycle which represents the <u>characteristic cohomology class</u>

$$\psi\{t\underline{\underline{K}}\} = \{c(t,H)\} \in H^3(\underline{\underline{C}}, D). \tag{6}$$

This cohomology class depends only on the equivalence class $\{T\underline{\underline{K}}\}$ of $T\underline{\underline{K}}$ in $\pi_0 \underline{\underline{Track}}(\underline{\underline{C}}, D)$ so that ψ in (1.13) is well defined.

§ 2 The category of homotopy pairs

The category of homotopy pairs was introduced by Hardie [13] and later studied in a series of papers [13, ..., 22]. We here show that the definition of homotopy pairs yields a natural transformation of cohomology groups

$$H^3(\underline{\underline{C}}, D) \xrightarrow{\lambda} H^2(\underline{\underline{Pair}}(\underline{\underline{C}}), D^\sharp) \tag{2.1}$$

Here $\underline{\underline{Pair}}(\underline{\underline{C}})$ is the category of pairs in $\underline{\underline{C}}$; the objects are the morphisms in $\underline{\underline{C}}$ and morphisms $f \to g$ in $\underline{\underline{Pair}}(\underline{\underline{C}})$ are commutative diagrams

$$\begin{array}{ccc} A & \xrightarrow{\xi} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow[\eta]{} & Y \end{array}$$

in $\underline{\underline{C}}$. Given a natural system D on $\underline{\underline{C}}$ we obtain the natural system $D^\sharp$ on $\underline{\underline{Pair}}(\underline{\underline{C}})$ by the quotient group

$$D^\sharp(\xi, \eta) = D(g\xi)/g_* D(\xi) + f^* D(\eta) \tag{2.2}$$

Induced maps for $D^\sharp$ are in the obvious way given by the induced maps for D.

(2.3) Definition. A track category

$$T\underline{\underline{K}} = (T \rightrightarrows \underline{\underline{K}} \xrightarrow{p} \underline{\underline{C}})$$

yields the <u>category $\underline{\underline{Hopair}}(T\underline{\underline{K}})$ of homotopy pairs</u> as follows: Objects are the morphisms in $\underline{\underline{C}}$ and a morphism $\{\xi, \eta, H\} : f \to g$ is an equivalence class of triple (ξ, η, H) defined as follows. We fix for each morphism $f : A \to B$ in $\underline{\underline{C}}$ a morphism $\tilde{f} : A \to B$ in $\underline{\underline{K}}$ representing f, that is $\tilde{f}$ induces $f = p(\tilde{f})$ via the functor p. A triple (ξ, η, H) is given by a diagram in $T\underline{\underline{K}}$, $H \in T(\eta\tilde{f}, \tilde{g}\xi)$

$$\begin{array}{ccc} A & \xrightarrow{\xi} & X \\ {\scriptstyle \tilde{f}}\downarrow & \overset{H}{\Longrightarrow} & \downarrow{\scriptstyle \tilde{g}} \\ B & \xrightarrow{\eta} & Y \end{array}$$

and the equivalence relation for such triple is defined by

$$(\xi, \eta, H) \sim (\xi', \eta', \tilde{f}^* H_0 + H + \tilde{g}_* H_1)$$

with $H_0 \in T(\eta', \eta)$, $H_1 \in T(\xi, \xi')$, that is

$$
\begin{array}{ccc}
A & \xrightarrow{\xi} & X \\
\bar{f}\downarrow & \overset{H}{\Longrightarrow} & \downarrow\bar{g} \\
B & \xrightarrow{\eta} & Y
\end{array}
\quad \sim \quad
\begin{array}{ccc}
A & \xrightarrow{\xi'} & X \\
\| & \overset{H_1}{\Longrightarrow} & \| \\
A & \xrightarrow{\xi} & X \\
\bar{f}\downarrow & \overset{H}{\Longrightarrow} & \downarrow\bar{g} \\
B & \xrightarrow{\eta} & Y \\
\| & \overset{H_2}{\Longrightarrow} & \| \\
B & \xrightarrow{\eta'} & Y
\end{array}
$$

Let $\{\xi, \eta, H\}$ be the equivalence class of (ξ, η, H). Composition is clearly given by

$$\{\xi, \eta, H\}\{\xi', \eta', H'\} = (\xi\xi', \eta\eta', \eta_* H' + (\xi')^* H)$$

There is a well defined full functor

$$p : \underline{\underline{Hopair}}(T\underline{\underline{K}}) \to \underline{\underline{Pair}}(\underline{\underline{C}}) \tag{1}$$

which is the identity on objects and which carries $\{\xi, \eta, H\}$ to the pair of homotopy classes $(\{\xi\}, \{\eta\})$. Now suppose that $T\underline{\underline{K}}$ is part of a linear track extension

$$D \xrightarrow{+} T \rightrightarrows \underline{\underline{K}} \to \underline{\underline{C}}$$

Then this track extension yields a linear extension of categories

$$D^\sharp \xrightarrow{+} \underline{\underline{Hopair}}(T\underline{\underline{K}}) \xrightarrow{p} \underline{\underline{Pair}}(\underline{\underline{C}}) \tag{2}$$

Here the action of $\{\alpha\} \in D^\sharp(\xi, \eta)$ on $\{\xi, \eta, H\}$ is defined by

$$\{\xi, \eta, H\} + \{\alpha\} = \{\xi, \eta, H + \alpha\},\ \alpha \in D(g\xi).$$

One readily checks that (2) is a well defined linear extension. The following result relies on the bijections (1.3) and (1.13).

(2.4) Proposition. *There is a well defined binatural homomorphism λ in (2.1) which carries the class $\psi\{T\underline{\underline{K}}\}$ represented by a linear track extension $T\underline{\underline{K}}$, to the class $\psi\{\underline{\underline{Hopair}}(T\underline{\underline{K}})\}$ represented by the linear extension (2.3) (2).*

Proof. The map λ is well defined since a morphism $t : T\underline{\underline{K}} \to T'\underline{\underline{K}}'$ in $\underline{\underline{Track}}(\underline{\underline{C}}, D)$ induces an equivalence of linear extensions

$$t_* : \underline{\underline{Hopair}}(T\underline{\underline{K}}) \to \underline{\underline{Hopair}}(T'\underline{\underline{K}}')$$

defined as follows. Let $\approx{f} \in \underline{\underline{K}}'$ be the choice of $\approx{f}$ with $p\approx{f} = f$ corresponding to $\tilde{f}$ in $\underline{\underline{K}}$; see (2.3). Then we choose a track $H_f \in T'(\approx{f}, t\tilde{f})$ and we define

$$t_*\{\xi, \eta, H\} = \{t\xi, t\eta, (t\eta)_* H_f + t(H) - (t\xi)^* H_g\}$$

q.e.d.

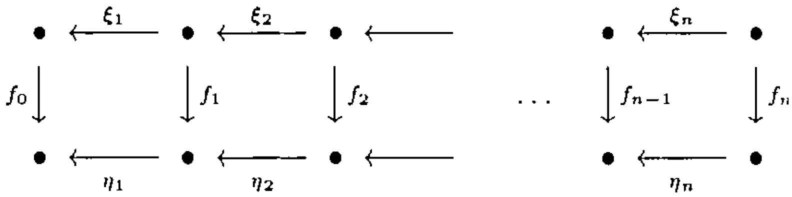
(2.5) Theorem. *There is a binatural homomorphism*

$$\lambda : H^{n+1}(\underline{\underline{C}}, D) \to H^n(\underline{\underline{Pair}}(\underline{\underline{C}}), D^\sharp)$$

which for $n = 2$ coincides with the map λ in (2.4). For a cocycle c with $\{c\} \in H^{n+1}(\underline{\underline{C}}, D)$ we define a cocycle λc with $\lambda\{c\} = \{\lambda c\} \in H^n(\underline{\underline{Pair}}(\underline{\underline{C}}), D^\sharp)$ by the formula

$$(\lambda c)((\xi_1, \eta_1), \dots, (\xi_n, \eta_n)) = \sum_{i=0}^{n} (-1)^i c(\eta_1, \dots, \eta_i, f_i, \xi_{i+1}, \dots, \xi_n)$$

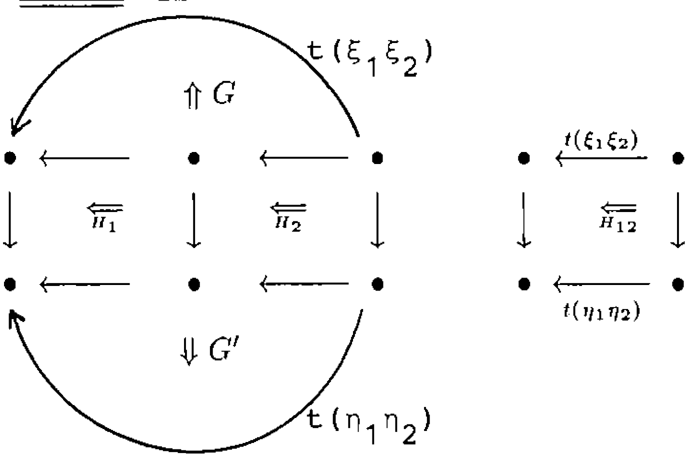
Here



is an n-simplex in $\underline{\underline{Pair}}(\underline{\underline{C}})$.

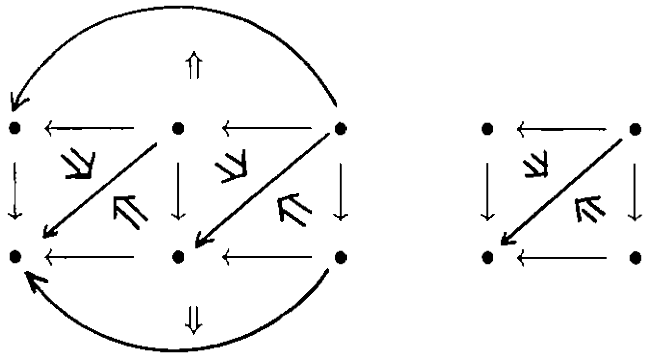
The formula for λc in the theorem is similar to the formula of the homotopy h in (1.15) of [2].

Proof of (2.5). We compare λ in (2.5) with λ in (2.4). For this we first describe the cocycle $\Delta \in \{\underline{\underline{Hopair}}(T\underline{\underline{K}})\}$ in (1.4) by addition of tracks



$$\Delta((\xi_1, \eta_1), (\xi_2, \eta_2)) = -G - H_1 - H_2 + G' + H_{12}$$

On the right hand side we omit the obvious induced maps. The cocycle Δ can also be described by subdividing the squares as follows.



Here the diagonals are $t(f_0\xi_1) = t(\eta_1 f_1)$, $t(f_1\xi_2) = t(\eta_2 f_2)$, $t(f_0\xi_1\xi_2) = t(\eta_1\eta_2 f_1)$. The subdivision corresponds to the cocycle λc with c given by $\{T\underline{\underline{K}}\}$ as in (1.14) (4). Hence we get $\Delta = \lambda c$. A somewhat tedious but straightforward calculation shows that λ in (2.5) is well defined by the formula for λc. For this the definition of $D^\sharp$ in (2.1) is crucial.

q.e.d.

§ 3 The universal Toda bracket

Our standard example of a track category is the category $\underline{\underline{Top}}^*$ of pointed topological spaces. Let I be the unit interval and let $IX = I \times X / I \times \{*\}$ be the reduced cylinder of $X \in \underline{\underline{Top}}^*$. We have the maps

$$X \vee X \xrightarrow{(i_0, i_1)} IX \xrightarrow{p} X$$

where $X \vee X$ is the one point union. Here we set $i_t(x) = (t, x)$ and $p(t, x) = x$, $t \in I$, $x \in X$. For maps $f, g : X \to Y \in \underline{\underline{Top}}^*$ let

$$T(f, g) = [IX, Y]^{(f,g)}$$

be the set of homotopy classes relative $X \vee X$ of map $H : IX \to Y$ with $H(i_0, i_1) = (f, g)$. An element $H \in T(f, g)$ is termed a <u>track</u> $H : f \simeq g$. This defines the track category

$$T \rightrightarrows \underline{\underline{Top}}^* \to \underline{\underline{Top}}^* / \simeq$$

which yields the following linear track extensions.

(3.1) Theorem. *(A) Let $\underline{\underline{K}}$ be a full category of $\underline{\underline{Top}}^*$ such that the objects of $\underline{\underline{K}}$ are suspensions. Then there is a natural system D_Σ on $\underline{\underline{K}}/\simeq$ together with a linear track extension*

$$D_\Sigma \xrightarrow{+} T \rightrightarrows \underline{\underline{K}} \to \underline{\underline{K}}/\simeq$$

(B) Let $\underline{\underline{K}}'$ be a full subcategory of $\underline{\underline{Top}}_*$ such that the objects of $\underline{\underline{K}}'$ are loop spaces. Then there is a natural system D_Ω on $\underline{\underline{K}}'/\simeq$ together with a linear track extension

$$D_\Omega \xrightarrow{+} T \rightrightarrows \underline{\underline{K}}' \to \underline{\underline{K}}'/\simeq$$

The corresponding cohomology classes from (1.13)

$$\langle \underline{\underline{K}} \rangle_\Sigma = \psi\{T\underline{\underline{K}}\} \in H^3(\underline{\underline{K}}, D_\Sigma)$$
$$\langle \underline{\underline{K}}' \rangle_\Omega = \psi\{T\underline{\underline{K}}'\} \in H^3(\underline{\underline{K}}', D_\Omega)$$

are called the <u>universal Toda brackets</u> [3] for $\underline{\underline{K}}$ and $\underline{\underline{K}}'$ respectively. All classical triple Toda brackets $\langle f, g, h \rangle$ in $\underline{\underline{K}}$ are determined by $\langle \underline{\underline{K}} \rangle_\Sigma$, that is $\langle f, g, h \rangle = (f, g, h)^* \langle \underline{\underline{K}} \rangle_\Sigma$; compare 3.3 in [3]. Recently the universal Toda bracket $\langle K \rangle_\Sigma$ plays a role in the work of Smirnov on homotopy groups of spheres [26]. For the definition of the natural systems D_Σ and D_Ω in (3.1) we need the partial suspension E and the partial loop operation L. For $X, Y \in \underline{\underline{Top}}^*$ let $X \vee Y$ be the coproduct (i.e. one point union) with inclusions $i_1 : X \to X \vee Y$ and $i_2 : Y \to X \vee Y$. Moreover let $X \times Y$ be the product with projections $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$. A zero map $0 : A \to B$ is given by $A \to * \to B$. We say that maps

$$
(3.2) \qquad \begin{cases} f : A \to X \vee Y \\ g : X \times Y \to B \end{cases}
$$

are <u>trivial</u> on Y if the compositions $(0,1)f : A \to X \vee Y \to Y$ and $g(0,1) : Y \to X \times Y \to B$ are homotopic to the zero map. Let $[X,Y]$ be the set of homotopy classes in $\underline{\underline{Top}}^*/\simeq$ and let $[A, X \vee Y]_2$ and $[X \times Y, B]_2$ be the sets of homotopy classes in $\underline{\underline{Top}}^*/\simeq$ which are trivial on Y. If A is a suspension $A = \Sigma A'$ and if B is a loop space $B = \Omega B'$ there are natural homomorphisms of groups

$$
(3.3) \qquad \begin{cases} E : [A, X \vee Y]_2 \to [\Sigma A, (\Sigma X) \vee Y]_2 \\ L : [X \times Y, B]_2 \to [(\Omega X) \times Y, B]_2 \end{cases}
$$

which are termed the <u>partial suspension</u> and the <u>partial loop operation</u> respectively. For the definition of E, resp. L, we use the functors Σ_*, resp. Ω_*, on $\underline{\underline{Top}}^*$ which are given by

$$
\begin{array}{ll}
\Sigma_* X = S^1 \times X / S^1 \times * & \text{(quotient space)} \\
\Omega_* X = (X^{S^1}, 0) & \text{(function space)}
\end{array}
$$

Here $S^1 = I/\partial I$ is the 1-sphere. We have canonical natural maps $j : \Sigma_* X \to X$, $j : X \to \Omega_* X$, $\pi : \Sigma_* X \to \Sigma X$, and $\pi : \Omega X \to \Omega_* X$. Now $E\xi$, resp. $L\eta$, are the unique maps in $\underline{\underline{Top}}^*/\simeq$, trivial on Y, for which the following diagrams commute in $\underline{\underline{Top}}^*/\simeq$

$$
\begin{array}{ccc}
\Sigma_* A & \xrightarrow{\Sigma_* \xi} & \Sigma_*(X \vee Y) \\
 & & \| \\
\pi \big\downarrow & & \Sigma_* X \vee \Sigma_* Y \\
 & & \big\downarrow \pi \vee j \\
\Sigma A & \xrightarrow{E\xi} & (\Sigma X) \vee Y
\end{array}
\qquad\qquad
\begin{array}{ccc}
\Omega_* B & \xleftarrow{\Omega_* \eta} & \Omega_*(X \times Y) \\
 & & \| \\
\big\uparrow \pi & & \Omega_* X \times \Omega_* Y \\
 & & \big\uparrow \pi \times j \\
\Omega B & \xleftarrow{L\eta} & (\Omega X) \times Y
\end{array}
$$

Compare [1] and [4], in particular (II.11.12) in [4], for a list of properties of E and L. The definition of L is dual to the definition of E.

Let $f : A' \to A$ be a map between suspensions and let $g : B \to B'$ be a map between loop spaces. Then we can use addition of maps to define the difference elements

$$
(3.4) \qquad \begin{cases} \nabla f = -i_2 f + (i_2 + i_1) f \in [A', A \vee A]_2 \\ \nabla g = -g p_2 + g(p_2 + p_1) \in [B \times B, B']_2 \end{cases}
$$

and hence we get

$$(3.5)\qquad \begin{cases} E \bigtriangledown f \in [\Sigma A', (\Sigma A) \vee A]_2 \\ L \bigtriangledown g \in [(\Omega B) \times B, \, \Omega B']_2 \end{cases}$$

Here $E \bigtriangledown f$ satisfies the formula ($\dim A' < \infty$)

$$E \bigtriangledown f = i_1 \Sigma f + \sum_{n \geq 2} w_n(\Sigma \lambda_n f)$$

where Σf is the suspension and $\lambda_n f$ is the James-Hopf invariant. Moreover w_n is the iterated Whitehead product with $w_1 = i_1$ and $w_n = [w_{n-1}, i_2]$ for $n \geq 2$. Compare 3.3.13 in [1]. Using these constructions we are ready to define D_Σ and D_Ω in (3.1) as follows.

(3.6) Definition. (A) Let $\underline{\underline{K}}/ \simeq \subset \underline{\underline{Top}}^*/ \simeq$ be a full subcategory consisting of suspensions. Then we define a natural system D_Σ on $\underline{\underline{K}}/ \simeq$ by the abelian group

$$D_\Sigma(h) = [\Sigma A, B] \qquad \text{for} \qquad h : A \to B \in \underline{\underline{K}}/ \simeq$$

Induced maps for $f : A' \to A$ and $g : B \to B'$ are given by

$$\begin{aligned} &f^* : D_\Sigma(h) = [\Sigma A, B] \to D_\Sigma(hf) = [\Sigma A', B] \\ &f^*(\alpha) = (\alpha, h)(E \bigtriangledown f) : \Sigma A' \to \Sigma A \vee A \to B \\ &g_* : D_\Sigma(h) = [\Sigma A, B] \to D_\Sigma(gh) = [\Sigma A, B'] \\ &g_*(\alpha) = g\alpha : \Sigma A \to B \to B' \end{aligned}$$

(B) Let $\underline{\underline{K}}'/ \simeq \subset \underline{\underline{Top}}^*/ \simeq$ be a full subcategory consisting of loop spaces. Then we define a natural system D_Ω on $\underline{\underline{K}}'/ \simeq$ by the abelian group

$$D_\Omega(h) = [A, \Omega B] \qquad \text{for} \qquad h : A \to B \in \underline{\underline{K}}'/ \simeq$$

Induced maps for $f : A' \to A$ and $g : B \to B'$ are given by

$$\begin{aligned} &f^* : D_\Omega(h) = [A, \Omega B] \to D_\Omega(hf) = [A', \Omega B] \\ &f^*(\alpha) = \alpha f : A' \to A \to \Omega B \\ &g_* : D_\Omega(h) = [A, \Omega B] \to D_\Omega(gh) = [A, \Omega B'] \\ &g_*(\alpha) = (L \bigtriangledown g)(\alpha, h) : A \to \Omega B \times B \to \Omega B' \end{aligned}$$

The properties of E and L show that D_Σ and D_Ω are well defined natural systems.

Remark. The functors D_Σ and D_Ω with $D_\Sigma(h) = [\Sigma A, B]$ and $D_\Omega(h) = [A, \Omega B]$ look like bimodules. The induced maps above however show that D_Σ, D_Ω are actually natural systems of abelian groups. Accordingly in § 3 of [2] and in (I.3.13) of [5] one has to replace the word 'bimodule' by 'natural system'.

Proof of (3.1). We only prove part (A). The proof of (B) is the dual version. If A is a suspension we obtain the map $i_2+i_1 : A \to A\vee A$ which yields the composition

$$\Sigma_*A \xrightarrow{\Sigma_*(i_2+i_1)} \Sigma_*(A\vee A) = \Sigma_*A\vee\Sigma_*A \xrightarrow{\pi\vee p} \Sigma A\vee A$$

which is homotopic to a map $s : \Sigma_*A \to \Sigma A\vee A$ under A. Since s is actually a homotopy equivalence under A we see that for $h : A\to B \in \underline{\underline{K}}$ we get the isomorphism of groups

$$\sigma_h : D(h) = [\Sigma A, B] = [\Sigma A\vee A, B]^h \xrightarrow{s^*} [\Sigma_*A, B]^h = T(h,h)$$

which defines the action of $D(h)$ on $T(h,h)$. Compare also (II.10.18) in Baues [4]. The following diagram commutes in $\underline{\underline{Top}}^*/\simeq$

$$\begin{array}{ccc} A\vee A & \xleftarrow{i_2+i_1} & A \\ \downarrow{\scriptstyle (\nabla f, i_2 f)} & & \downarrow{\scriptstyle f} \\ A'\vee A' & \xleftarrow{i_2+i_1} & A' \end{array}$$

Therefore also the next diagram commutes in $\underline{\underline{Top}}^*/\simeq$.

$$\begin{array}{ccccc} (\Sigma A)\vee A & \xleftarrow{\pi\vee p} & \Sigma_*(A\vee A) & \xleftarrow{\Sigma_*(i_2+i_1)} & \Sigma_*A \\ \downarrow{\scriptstyle (E\xi, i_2 f)} & & \downarrow{\scriptstyle \Sigma_*(\nabla f, i_2 f)} & & \downarrow{\scriptstyle \Sigma_* f} \\ (\Sigma A')\vee A' & \xleftarrow{\pi\vee p} & \Sigma_*(A'\vee A') & \xleftarrow[\Sigma_*(i_2+i_1)]{} & \Sigma_*A' \end{array}$$

This shows that $f^*\sigma_h = \sigma_{fh}f^*$.

q.e.d.

(3.7) Remark. We can replace $\underline{\underline{Top}}^*$ in (3.1) by any cofibration category $\underline{\underline{C}}$, [4]. That is, if $\underline{\underline{K}}$ is a full subcategory of $\underline{\underline{C}}_{cf}$ consisting of suspensions in $\underline{\underline{C}}$ then D_Σ is defined in the same way on $\underline{\underline{K}}/\simeq$ and one obtains a linear track extension as in (3.1) (A) for $\underline{\underline{K}}$. This way one also obtains the dual track extension in any fibration category with (3.1) (B) as a special case. A different approach for the computation of D_Σ, D_Ω can be deduced from [25]; see 2.3 in [22].

§ 4 Homotopy categories for principal maps

We consider the fiber functor P and the cofiber functor C

$$\underline{\underline{Top}}^*/\simeq \xleftarrow{P} \underline{\underline{Hopair}}(\underline{\underline{Top}}^*) \xrightarrow{C} \underline{\underline{Top}}^*/\simeq \tag{4.2}$$

defined as follows; compare [17]. Given an object $h : A \to B$ in $\underline{\underline{Hopair}}(\underline{\underline{Top}}^*)$ represented by $\tilde{h} : A \to B$ in $\underline{\underline{Top}}^*$ we obtain the cofiber (or mapping cone)

$$C(h) = CA \cup_{\tilde{h}} B$$

and the fiber (or mapping path space)

$$P(h) = PB \times_{\tilde{h}} A$$

Here $CA = IA/i_1A$ is the cone on A and $PB = \{\sigma \in B^I, \sigma(1) = *\}$ in the contractible path space. Each morphism $\{\xi, \eta, H\} : h \to g$ in $\underline{\underline{Hopair}}(\underline{\underline{Top}}^*)$ induces well defined homotopy classes, termed principal maps in [4]:

$$C\{\xi, \eta, H\} : C(h) \to C(g),$$
$$P\{\xi, \eta, H\} : P(h) \to P(g)$$

They are represented by the well known maps associated to the triple (ξ, η, H); compare for example (V. § 2) in [4]. This completes the definition of the functors P and C in (4.1).

For a class $\mathcal{X}$ of morphisms in $\underline{\underline{Top}}^*/\simeq$ let $\underline{\underline{Hopair}}(\mathcal{X})$ be the full subcategory of $\underline{\underline{Hopair}}(\underline{\underline{Top}}^*)$ consisting of objects which are elements in $\mathcal{X}$. We write $\mathcal{X} = \mathcal{X}_\Sigma$ if all elements of $\mathcal{X}$ are maps between suspensions and we write $\mathcal{X} = \mathcal{X}_\Omega$ if all elements of $\mathcal{X}$ are maps between loop spaces. Moreover let $\underline{\underline{Pair}}(\mathcal{X})$ be the full subcategory of $\underline{\underline{Pair}}(\underline{\underline{Top}}^*)$ consisting of objects which are elements in $\mathcal{X}$. By theorem (3.1) and (2.3) we have linear extensions of categories

$$D^\sharp_\Sigma \xrightarrow{+} \underline{\underline{Hopair}}(\mathcal{X}_\Sigma) \xrightarrow{p} \underline{\underline{Pair}}(\mathcal{X}_\Sigma)$$
$$D^\sharp_\Omega \xrightarrow{+} \underline{\underline{Hopair}}(\mathcal{X}_\Omega) \xrightarrow{p} \underline{\underline{Pair}}(\mathcal{X}_\Omega) \tag{4.2}$$

Using λ in (2.1) we get the following result which shows that the extensions (4.2) are determined up to equivalence by universal Toda brackets $\langle \underline{\underline{K}} \rangle_\Sigma$, resp. $\langle \underline{\underline{K}}' \rangle_\Omega$.

(4.3) Theorem. *(A) Let* $\underline{\underline{K}}$ *be a full homotopy category of suspensions with* $\mathcal{X}_\Sigma \subset \underline{\underline{K}}$*. Then the class*

$$\psi\{\underline{\underline{Hopair}}(\mathcal{X}_\Sigma)\} \in H^2(\underline{\underline{Pair}}(\mathcal{X}_\Sigma), D^\sharp_\Sigma)$$

is a restriction of $\lambda\langle\underline{\underline{K}}\rangle_\Sigma$.
(B) Let $\underline{\underline{K}}'$ *be a full homotopy category of loop spaces with* $\mathcal{X}_\Omega \subset \underline{\underline{K}}'$. *Then the class*

$$\psi\{\underline{\underline{Hopair}}(\mathcal{X}_\Omega)\} \in H^2(\underline{\underline{Pair}}(\mathcal{X}_\Omega),\, D^\sharp_\Omega)$$

is a restriction of $\lambda\langle\underline{\underline{K}}'\rangle_\Omega$.

The fiber functor P and the cofiber functor C in (4.1) are compatible with the linear extensions (4.2) in the following sense. Let

$$\text{(4.4)} \qquad \underline{\underline{C}}(\mathcal{X}_\Sigma),\ \underline{\underline{P}}(\mathcal{X}_\Omega) \subset \underline{\underline{Top}}^*/\simeq$$

be the full homotopy categories consisting of $C(f)$, $f \in \mathcal{X}_\Sigma$ and $P(g)$, $g \in \mathcal{X}_\Omega$ respectively.

(4.5) <u>Theorem</u>. *Let* $2 \leq b < a$ *and* $\mathcal{X}_\Sigma$ *be a class of maps* $h : A \to B$ *between suspensions* $A = \Sigma A'$, $B = \Sigma B'$ *of CW-complexes* A', B' *such that* A *is* $(a-1)$ *-connected,* B *is* $(b-1)$ *-connected,* $dim(A) \leq a+b-2$, $dim(B) \leq a-1$. *Then there exists a commutative diagram of linear extensions*

$$\begin{array}{ccccc}
D^\sharp_\Sigma & \xrightarrow[+]{} & \underline{\underline{Hopair}}(\mathcal{X}_\Sigma) & \longrightarrow & \underline{\underline{Pair}}(\mathcal{X}_\Sigma) \\
{\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle C} & & \| \\
\Gamma_\Sigma & \xrightarrow[+]{} & \underline{\underline{C}}(\mathcal{X}_\Sigma) & \longrightarrow & \underline{\underline{Pair}}(\mathcal{X}_\Sigma)
\end{array}$$

where the functor C*, given by (4.1), is full and* τ *is the surjective natural transformation in (4.6) below. In fact* C *and* τ *are isomorphisms if for all* $h : A \to B \in \mathcal{X}_\Sigma$ *we have* $dim(A) < a+b-2$.

Let $h : A \to B$, $g : X \to Y \in \mathcal{X}_\Sigma$ with $\mathcal{X}_\Sigma$ as in (4.5) and let

$$(\xi, \eta) : h \to g \in \underline{\underline{Pair}}(\mathcal{X}_\Sigma)$$

We define the natural system $\Gamma_\Sigma(\xi,\eta) = cokernel(q(g,1)_*)$ and the natural quotient map $\tau : D^\sharp_\Sigma(\xi,\eta) \twoheadrightarrow \Gamma_\Sigma(\xi,\eta)$ by the cokernel of

$$\text{(4.6)} \qquad [\Sigma A, X \vee Y]_2 \xrightarrow{(g,1)_*} [\Sigma A, Y] \xrightarrow{q} D^\sharp_\Sigma(\xi,\eta)$$

where q is the quotient map. Compare the notation in (3.2).

(4.7) <u>Corollary</u>. *Let* $\mathcal{X}_\Sigma$ *be a class of maps as in (4.5). Then the homotopy catgeory* $\underline{\underline{C}}(\mathcal{X}_\Sigma)$ *is determined by the universal Toda bracket* $\langle\underline{\underline{K}}\rangle_\Sigma$ *where* $\underline{\underline{K}}$ *is a category as in (4.3) (A) with* $\mathcal{X}_\Sigma \subset \underline{\underline{K}}$. *That is, the class*

$$\psi\{\underline{\underline{C}}(\mathcal{X}_\Sigma)\} \in H^2(\underline{\underline{Pair}}(\mathcal{X}_\Sigma), \Gamma_\Sigma)$$

is the image under τ_* of the restriction of $\lambda\langle\underline{\underline{K}}\rangle_\Sigma$ to $\underline{\underline{Pair}}(\mathcal{X}_\Sigma)$.

Proof of (4.5). The assumptions on $\mathcal{X}_\Sigma$ show that $\underline{\underline{C}}(\mathcal{X}_\Sigma) = PRIN(\mathcal{X}_\Sigma)$ where the right hand side is defined in (V. §. 3) of [4]. Hence the result follows from (V.7.17), (V.7.18) in [4].

q.e.d.

Next we consider the category $\underline{\underline{P}}(\mathcal{X}_\Omega)$ which is dual to $\underline{\underline{C}}(\mathcal{X}_\Sigma)$ in (4.4). We write $hodim(X) \leq n$ if $\pi_i(X) = 0$ for $i > n$.

(4.8) Theorem. *Let $1 \leq a < b$ and let $\mathcal{X}_\Omega$ be a class of maps $h : A \to B$ between loop spaces $A = \Omega A'$, $B = \Omega B'$ of CW-complexes A', B' such that A is $(a-1)$ -connected, B is $(b-1)$ -connected, $hodim(A) < b-1$, $hodim(B) < a+b$. Then there exists a commutative diagram of linear extensions*

$$\begin{array}{ccccc} D^\sharp_\Omega & \xrightarrow{+} & \underline{\underline{Hopair}}(\mathcal{X}_\Omega) & \longrightarrow & \underline{\underline{Pair}}(\mathcal{X}_\Omega) \\ \| & & \downarrow P & & \| \\ D^\sharp_\Omega & \xrightarrow{+} & \underline{\underline{P}}(\mathcal{X}_\Omega) & \longrightarrow & \underline{\underline{Pair}}(\mathcal{X}_\Omega) \end{array}$$

where the functor P, given by (4.1), is an isomorphism of categories.

(4.9) Corollary. *Let $\mathcal{X}_\Omega$ be a class of maps as in (4.8). Then the homotopy category $\underline{\underline{P}}(\mathcal{X}_\Omega)$ is determined by the universal Toda bracket $\langle\underline{\underline{K}}'\rangle_\Omega$ where $\underline{\underline{K}}'$ is a category as in (4.3) (B) with $X_\Omega \subset \underline{\underline{K}}'$. That is the class*

$$\psi\{\underline{\underline{P}}(\mathcal{X}_\Omega)\} \in H^2(\underline{\underline{Pair}}(\mathcal{X}_\Omega), D^\sharp_\Omega)$$

is the restriction of $\lambda\langle\underline{\underline{K}}'\rangle_\Omega$ to $\underline{\underline{Pair}}(\mathcal{X}_\Omega)$.

Proof of (4.8). The assumptions on $\mathcal{X}_\Omega$ imply that $\underline{\underline{P}}(\mathcal{X}_\Omega) = PRIN(\mathcal{X}_\Omega)$ where the right hand side is defined in (V. § 6) of [4]. Hence the result follows from (V.10.19) in [4].

q.e.d.

§ 5 Two stage Postnikov towers and two stage CW-complexes

A bifunctor $D : \underline{\underline{C}}^{op} \times \underline{\underline{C}} \to \underline{\underline{Ab}}$ yields the Grothendieck-construction $\underline{\underline{Gro}}(D)$ which is the following category. Objects are triple (A, B, h) where A, B are objects in $\underline{\underline{C}}$ and $h \in D(A, B)$. A morphism $(\xi, \eta) : (A, B, h) \to (X, Y, g)$ is a pair of morphisms $\xi : A \to X$, $\eta : B \to Y$ in $\underline{\underline{C}}$ satisfying $\xi^* g = \eta_* h$. We shall use the following bifunctors. Let

$$H^n_{(m)} : \underline{\underline{Ab}}^{op} \times \underline{\underline{Ab}} \to \underline{\underline{Ab}} \tag{5.1}$$

be the Eilenberg-Mac Lane functor given by cohomology group

$$H^n_{(m)}(A, B) = H^n(K(A, m), B) = [K(A, m), K(B, n)]$$

where $K(A, m)$ denotes the Eilenberg-Mac Lane space of A in degree m. The algebraic properties of the bifunctor $H^n_{(m)}$ are fairly well understood; compare [12] and [10]. In the next result we describe the full homotopy

$$\underline{\underline{types}}(m, n) \subset \underline{\underline{Top}}^* / \simeq \tag{5.2}$$

consisting of CW-spaces X with $\pi_i(X) = 0$ for $i \notin \{m, n\}$, $1 < m < n$. Using the Postnikov decomposition each such space in the fiber $X = P(k)$ of a map

$$k : K(A, m) \to K(B, n + 1)$$

where $A = \pi_m(X)$, $B = \pi_n X$. Here $k = k_X \in H^{n+1}_m(A, B)$ is called the k-invariant of X. The fiber $P(k)$ is also called a two-stage Postnikov tower.

(5.3) Theorem. *Let $1 < m < n$ and let $\mathcal{X}_\Omega$ be the class of maps $k : K(A, m) \to K(B, n + 1)$ with $A, B \in \underline{\underline{Ab}}$. Then there are equivalent linear extensions of categories*

$$\begin{array}{ccccc}
\bar{H}^n_{(m)} & \xrightarrow{+} & \underline{\underline{types}}(m, n) & \xrightarrow{k} & \underline{\underline{Gro}}(H^{n+1}_{(m)}) \\
\| & & \| & & \| \\
D^\sharp_\Omega & \xrightarrow{+} & \underline{\underline{P}}(\mathcal{X}_\Omega) & \longrightarrow & \underline{\underline{Pair}}(\mathcal{X}_\Omega) \\
\| & & \| & & \| \\
D^\sharp_\Omega & \xrightarrow{+} & \underline{\underline{Hopair}}(\mathcal{X}_\Omega) & \longrightarrow & \underline{\underline{Pair}}(\mathcal{X}_\Omega)
\end{array}$$

Here the functor k carries X to $(\pi_m X, \pi_n X, k_X)$. The natural system $\bar{H}^n_{(m)}$ is defined by

$$\bar{H}^n_{(m)}(\xi, \eta) = H^n_{(m)}(A, Y)$$

which is actually a bimodule with induced maps determined by the bifunctor $H^n_{(m)}$.

Proof. We can apply theorem (4.8). We get $D_\Omega = \bar{H}^n_{(m)}$ since $\Omega K(Y, n+1) = K(Y, n)$ and $[K(A,n), \Omega K(X,n)] = 0$ and $[K(B,m), \Omega K(Y,m)] = 0$.

q.e.d.

Next we consider the full homotopy category

$$(5.4) \qquad \underline{\underline{CW}}(m,n) \subset \underline{\underline{Top}}^* / \simeq$$

consisting of CW-complexes X with cells only in dimension m and n, $1 < m < n-1$. Then $X = C(b)$ is the cofiber of a map

$$b = b_X : M(A, n-1) \to M(B, m)$$

where $A = H_n(X)$, $B = H_m(X)$ are free abelian groups. Here $M(B,m)$ is the Moore space of B given by a one point union of m-spheres. We call the cofiber $X = C(h)$ a two-stage CW-complex. Let $\underline{\underline{ab}} \subset \underline{\underline{Ab}}$ be the category of free abelian groups and let

$$(5.5) \qquad \Gamma^k_m : \underline{\underline{ab}} \to \underline{\underline{Ab}}$$

be the functor given by the homotopy group $\Gamma^k_m(A) = \pi_{m+k} M(A, m)$. Using homotopy groups of spheres $\pi_{m+k}(S^m)$ and primary homotopy operations it is possible to compute the functors Γ^k_m explicitely by the Hilton-Milnor theorem. For example we have

$$(5.6) \qquad \begin{cases} \Gamma^k_m(A) = A \otimes \pi_{m+k}(S^m) & \text{for} \quad k < m-1 \\ \Gamma^k_m(A) = A \otimes \pi_{m+k}\{S^m\} & \text{for} \quad k < 2m-2 \end{cases}$$

For $k < m-1$ we use the tensor product of abelian groups; while for $k < 2m-2$ we use the following quadratic tensor product of A and the quadratic $\mathbb{Z}$ -module $\pi_{m+k}\{S^m\}$ in (5.8).

(5.7) Definition [6]. A quadratic $\mathbb{Z}$ -module

$$M = (M_e \xrightarrow{H} M_{ee} \xrightarrow{P} M_e) \tag{1}$$

is a pair of abelian groups M_e, M_{ee} together with homomorphisms H, P which satisfy

$$PHP = 2P \quad \text{for} \quad HPH = 2H. \tag{2}$$

Then $T = HP - 1$ is an involution on M_{ee}, i.e. $TT = 1$. A morphism $f : M \to N$ between quadratic $\mathbb{Z}$ -modules is a pair of homomorphisms $f = (f_e, f_{ee})$ which commute with H and P respectively, $f_e P = p f_{ee}$, $f_{ee} H = H f_e$. Let $\underline{\underline{QM}}(\mathbb{Z})$ be the category of quadratic $\mathbb{Z}$ -modules which is an abelian category. We identify an abelian group Π with the quadratic $\mathbb{Z}$ -module $\Pi = (\Pi \to 0 \to \Pi)$, this yields the inclusion $\underline{\underline{Ab}} \subset \underline{\underline{QM}}(\mathbb{Z})$. We define the quadratic tensor product

$$\otimes_{\mathbf{Z}} : \underline{\underline{Ab}} \times \underline{\underline{QM}}(\mathbb{Z}) \to \underline{\underline{Ab}} \tag{3}$$

which generalizes the classical tensor product of abelian groups. Here $A \otimes_{\mathbf{Z}} M$ is the abelian group generated by the symbols $a \otimes m$, $[a,b] \otimes m$ with $a, b \in A$, $m \in M_e$, $n \in M_{ee}$. The relations are

$$\begin{cases} (a+b) \otimes m = a \otimes m + b \otimes m + [a,b] \otimes H(m), \\ \quad [a,a] \otimes n = a \otimes P(n), \end{cases} \tag{4}$$

where $a \otimes m$ is linear in m and $[a,b] \otimes n$ is linear in each variable a, b and n. One has the natural homomorphism

$$A \otimes_Z M \xrightarrow{H} A \otimes A \otimes M_{ee} \xrightarrow{P} A \otimes_{\mathbf{Z}} M \tag{5}$$

with

$$\begin{aligned} H(a \otimes m) &= a \otimes a \otimes H(m), \\ H([a,b] \otimes n) &= a \otimes b \otimes n + b \otimes a \otimes T(n), \\ P(a \otimes b \otimes n) &= [a,b] \otimes n, \end{aligned}$$

where $T = HP - 1$ is the involution.

Homotopy groups of spheres yield for $k < 2m - 2$ the quadratic $\mathbb{Z}$ -module

$$(5.8) \qquad \pi_{m+k}\{S^m\} = (\pi_{m+k}(S^m) \xrightarrow{H} \pi_{m+k}(S^{2m-1}) \xrightarrow{P} \pi_{m+k}(S^m))$$

where $H = \gamma_2$ is the Hopf invariant and where P is induced by the Whitehead product square $[i_n, i_n]$, that is $P(\alpha) = [i_n, i_n] \circ \alpha$. In (5.8) we get the involution $T = HP - 1 = (-1)^n$. For $k < 19$ the quadratic $\mathbb{Z}$ -modules $\pi_{m+k}\{S^m\}$ are computed in Toda's book [27]. For example

$$\pi_3\{S^2\} = \{\mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{2} \mathbb{Z}\}$$

and $\Gamma : \underline{\underline{Ab}} \to \underline{\underline{Ab}}$ defined by

$$(5.9) \qquad \Gamma(A) = A \otimes \pi_3\{S^2\}$$

is J.H.C. Whitehead's quadratic functor [28].

(5.10) ***Definition.*** Let $2 < m < n-1 < 3m-3$ and consider a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\xi} & X \\ f\downarrow & & \downarrow g \\ B \otimes \pi_{n-1}\{S^m\} & \xrightarrow{\eta\otimes 1} & Y \otimes \pi_{n-1}\{S^m\} \end{array}$$

where $\xi : A \to X$, $\eta : B \to Y$ are homomorphisms between free abelian groups. Then $(\xi, \eta) : f \to g$ is a morphism in the Grothendieck construction $\underline{\underline{Gro}}(D)$ of the bifunctor $D : \underline{\underline{ab}}^{op} \times \underline{\underline{ab}} \to \underline{\underline{Ab}}$ which carries A, B to $D(A,B) = Hom(A, B \otimes \pi_{n-1}\{S^m\})$. We define a natural system $D^{\sharp}_{\Sigma} = \Gamma_{\Sigma}$ on $\underline{\underline{Gro}}(D)$ by the quotient

(1) $\Gamma_{\Sigma}(\xi,\eta) = Hom(A, Y \otimes \pi_n\{S^m\})/g_* Hom(A, X \otimes \mathbb{Z}/2) + f^* Hom(B, Y \otimes \mathbb{Z}/2)$

Induced maps for Γ_{Σ} are given by the bifunctor $(A,Y) \longmapsto Hom(A, Y \otimes \pi_n\{S^m\})$. Here the homomorphisms

(2) $$g_* : Hom(A, X \otimes \mathbb{Z}/2) \to Hom(A, Y \otimes \pi_n\{S^m\})$$

(3) $$f^* : Hom(B, Y \otimes \mathbb{Z}/2) \to Hom(A, Y \otimes \pi_n\{S^m\})$$

are defined as follows. The suspension Σ induces a map

(4) $$\Sigma : \pi_{n-1}\{S^m\} \to \pi_n\{S^{m+1}\}$$

between quadratic $\mathbb{Z}$ -modules where $\Sigma_e = \Sigma$ is the suspension and where $\Sigma_{ee} = 0$ is trivial. The image $\Sigma\pi_{n-1}\{S^m\} = \Sigma\pi_{n-1}(S^m)$ is an abelian group. Moreover the Hopf-maps $\eta_m \in \pi_{m+1}S^m$, $\eta_{n-1} \in \pi_n S^{n-1}$ induce maps between quadratic $\mathbb{Z}$ -modules

(5) $$\pi_{n-1}\{S^m\} \otimes \mathbb{Z}/2 \xrightarrow{\eta^{\bullet}_{n-1}} \pi_n\{S^m\} \xleftarrow{(\eta_m)_*} \mathbb{Z}/2 \otimes \Sigma\pi_{n-1}(S^m)$$

Now g_* above carries $\alpha : A \to X \otimes \mathbb{Z}/2$ to the composition

(6) $$A \xrightarrow{\alpha} X \otimes \mathbb{Z}/2 \xrightarrow{g\otimes\mathbb{Z}/2} Y \otimes \pi_{n-1}\{S^m\} \otimes \mathbb{Z}/2 \xrightarrow{Y\otimes\eta^{\bullet}_{n-1}} Y \otimes \pi_n\{S^m\}$$

Moreover f^* above carries $\beta : B \to Y \otimes \mathbb{Z}/2$ to the sum of the following two homomorphisms

(7) $$A \xrightarrow{f} B \otimes \pi_{n-1}\{S^m\} \xrightarrow{\beta\otimes\Sigma} Y \otimes \mathbb{Z}/2 \otimes \Sigma\pi_{n-1}(S^m) \xrightarrow{Y\otimes(\eta_m)_*} Y \otimes \pi_n\{S^m\}$$

(8) $$A \xrightarrow{f} B \otimes \pi_{n-1}\{S^m\} \xrightarrow{H} B \otimes B \otimes \pi_{n-1}(S^{2m-1}) \xrightarrow{\eta\otimes\beta\otimes 1}$$
$$Y \otimes Y \otimes \mathbb{Z}/2 \otimes \pi_{n-1}S^{2m-1} \xrightarrow{\eta^{\bullet}_{2m-1}} Y \otimes Y \otimes \pi_n S^{2m-1} \xrightarrow{P} Y \otimes \pi_n\{S^m\}$$

This completes the algebraic definition of the natural system Γ_Σ which is used in the next theorem.

The functor Γ_m^k in (5.5) yields the bifunctor

$$D = Hom(-, \Gamma_m^k) : \underline{\underline{ab}}^{op} \times \underline{\underline{ab}} \to \underline{\underline{Ab}} \tag{5.11}$$

which carries (A, B) to the group $Hom(A, \Gamma_m^k B) = [M(A, m+k), M(B, m)]$. By use of (5.6) we see that the bifunctor D in (5.10) is a special case of (5.11).

(5.12) Theorem. *Let $2 \le m < n-1$ and let $\mathcal{X}_\Sigma$ be the class of maps $b : M(A, n-1) \to M(B, m)$ with $A, B \in \underline{\underline{ab}}$. Then there is a commutative diagram of linear extensions of categories, $k = n - m$,*

$$\begin{array}{ccccc}
Hom(-, \Gamma_n^k)/I & \xrightarrow{+} & \underline{\underline{CW}}(m, n) & \xrightarrow{b} & \underline{\underline{Gro}}(Hom(-, \Gamma_n^{k-1})) \\
\parallel & & \parallel & & \parallel \\
\Gamma_\Sigma & \xrightarrow{+} & \underline{\underline{C}}(\mathcal{X}_\Sigma) & \longrightarrow & \underline{\underline{Pair}}(\mathcal{X}_\Sigma) \\
\uparrow{\scriptstyle \tau} & & \uparrow{\scriptstyle C} & & \parallel \\
D_\Sigma^\sharp & \longrightarrow & \underline{\underline{Hopair}}(\mathcal{X}_\Sigma) & \longrightarrow & \underline{\underline{Pair}}(\mathcal{X}_\Sigma)
\end{array}$$

Here the functor b carries X to $(H_m X, H_n X, b_X)$. The natural system Γ_Σ is a quotient, $\Gamma_\Sigma = Hom(-, \Gamma_n^k)/I$, by the definition in (4.6). Moreover C and τ are isomorphisms for $m \ge 3$ and for $3 \le m < n-1 < 3m-3$ the natural system $\Gamma_\Sigma = D_\Sigma^\sharp$ is defined in (5.10).

Proof of (5.12). The theorem is a special case of theorem (4.5). The explicit computation of $\Gamma_\Sigma = D_\Sigma^\sharp$ for $3 \le m < n-1 < 3m-3$, given in (5.10), is obtained by the following arguments. Consider the natural system D_Σ in (3.6). Then it is clear that g_* in (3.6) corresponds to g_* in (5.10). We have to show that also f^* in (3.6) coincides with f^* in (5.10). For this we know that $E \nabla f = i_1 \Sigma f + [i_1, i_2](\Sigma \gamma_2 f)$, so that

$$\begin{aligned} f^*(\beta) = (\beta, \eta) E \nabla f &= \beta \Sigma f + [\beta, \eta] \Sigma \gamma_2 f \\ &= \beta \Sigma f + [1, 1](\beta \sharp \eta)(\Sigma \gamma_2 f) \end{aligned}$$

Here 1 is the identity of $M(Y, m)$ and $\beta \sharp \eta = \Sigma(\beta' \wedge \eta')$ is given by the smash product $\beta' \wedge \eta'$ with $\Sigma \beta' = \beta$, $\Sigma \eta' = \eta$. Now Σ in (5.10) (4) induces the homomorphism

$$\Sigma : \pi_{n-1} M(A, m) = A \otimes \pi_{n-1}\{S^m\} \xrightarrow{1 \otimes \Sigma} A \otimes \pi_n \{S^{m+1}\} = \pi_n M(A, m+1)$$

which is the suspension on $\pi_{n-1} M(A, m)$. Therefore $\beta(\Sigma f)$ corresponds to (5.10) (8). Moreover $[1, 1](\beta \sharp \eta) \Sigma \gamma_2 f$ corresponds to (5.10) (9) since for $\Sigma R = M(Z, m)$, $Z \in \underline{\underline{ab}}$, we have the commutative diagram, $t < 2m - 2$,

$$\begin{array}{ccccc} Z \otimes \pi_t\{S^m\} & \xrightarrow{H} & Z \otimes Z \otimes \pi_t(S^{2m-1}) & \xrightarrow{P} & Z \otimes \pi_t\{S^m\} \\ \| & & \| & & \| \\ \pi_t(\Sigma R) & \xrightarrow{\gamma_2} & \pi_t(\Sigma R \wedge R) & \xrightarrow{[1,1]_*} & \pi_t(\Sigma R) \end{array}$$

This shows that H in (5.7) (5) corresponds to the James-Hopf invariant γ_2 and P in (5.7) (5) is induced by the generalized Whitehead product $[1,1] : \Sigma R \wedge R \to \Sigma R$.

q.e.d.

§ 6 The Toda bracket of one point unions of 2-spheres

Let Γ be the quadratic functor of J.H.C. Whitehead [28] in (5.9) with

$$\Gamma(A) = \pi_3 M(A,2) = H_4 K(A,2) \tag{6.1}$$

for $A \in \underline{\underline{Ab}}$. We have the natural exact sequence in $\underline{\underline{Ab}}$

$$\Gamma(A) \xrightarrow{H} A \otimes A \xrightarrow{p} \Lambda^2(A) \to 0 \tag{6.2}$$

which is short exact if A is free abelian. Here $\Lambda^2(A) = A \otimes A / \{a \otimes a \sim 0, a \in A\}$ is the exterior square of A. For a free abelian group A let G_A be a free group with abelianization $(G_A)^{ab} = A$ and let $E_A = G_A/\Gamma_3(G_A)$ where $\Gamma_3(G_A)$ is the subgroup of triple commutators in G_A. Then one has the central extension of groups

$$\Lambda^2(A) \overset{\omega}{\rightarrowtail} E_A \overset{p}{\twoheadrightarrow} A \tag{6.3}$$

where p is the abelianization and ω is the commutator map with $\omega(\{a\} \wedge \{b\}) = a^{-1}b^{-1}ab$ for $a, b \in E_A$ and $\{a\} = p(a)$. Let $\underline{\underline{nil}}$ be the full subcategory of groups E_A, $A \in \underline{\underline{ab}}$, where $\underline{\underline{ab}}$ is the category of free abelian groups. Then one has the linear extension of categories

$$Hom(-,\Lambda^2) \xrightarrow{+} \underline{\underline{nil}} \xrightarrow{p} \underline{\underline{ab}} \tag{6.4}$$

where p carries E_A to A and where $Hom(-,\Lambda^2)$ is a bifunctor on $\underline{\underline{ab}}$. The action $+$ on $\xi : E_A \to E_B$ is defined by

$$\xi + \alpha = \xi + \omega\alpha p \quad \text{for} \quad \alpha \in Hom(A, \Lambda^2 B).$$

Hence we obtain the canonical class

$$\{\underline{\underline{nil}}\} \in H^2(\underline{\underline{ab}}, Hom(-,\Lambda^2)) \tag{6.5}$$

which is non trivial. The exact sequence (6.2) induces the short exact sequence of $\underline{\underline{ab}}$ -bimodules

$$0 \to Hom(-,\Gamma) \to Hom(-,\otimes^2) \to Hom(-,\Lambda^2) \to 0$$

and hence the associated Bockstein homomorphism

$$\beta : H^2(\underline{\underline{ab}}, Hom(-,\Lambda^2)) \to H^3(\underline{\underline{ab}}, Hom(-,\Gamma))$$

Now let $\underline{\underline{S}}(2)$ be the full homotopy category of Moore spaces $M(A,2)$ where A is free abelian. Hence $\underline{\underline{S}}(2) \cong \underline{\underline{ab}}$. The natural system D_Σ on $\underline{\underline{S}}(2)$ is via (6.1) given by the $\underline{\underline{ab}}$ -bimodule $D_\Sigma = Hom(-,\Gamma)$ so that the universal Toda bracket $\langle \underline{\underline{S}}(2) \rangle_\Sigma$ is an element

$$\langle \underline{\underline{S}}(2) \rangle_\Sigma \in H^3(\underline{\underline{ab}}, Hom(-,\Gamma))$$

This element has the following algebraic description in terms of $\{\underline{\underline{nil}}\}$ above, [3].

(6.6) Theorem.

$$\langle \underline{\underline{S}}(2)\rangle_\Sigma = \beta\{\underline{\underline{nil}}\}$$

We use the category $\underline{\underline{nil}}$ also for the example of § 7 below given by Hopf-maps.

§ 7 Homotopy pairs for Hopf maps

Let A be a free abelian group. A generalized Hopf map for A is a map

$$(7.1) \qquad \eta_A : M(\Gamma(A), 3) \to M(A, 2)$$

which induces the identity of $\Gamma(A)$,

$$1 : \Gamma(A) = \pi_3 M(\Gamma(A), 3) \xrightarrow{(\eta_A)_*} \pi_3 M(A, 2) = \Gamma(A).$$

Such Hopf maps exist and are well defined up to homotopy. For $A = \mathbb{Z}$ the map $\eta_{\mathbb{Z}}$ is the classical Hopf map. Let $\mathcal{X}_{Hopf}$ be the class of all Hopf maps η_A, $A \in \underline{\underline{ab}}$.

The cofiber $C(\eta_A)$ of a Hopf map is the 5-skeleton of $K(A, 2)$. This implies that there are isomorphisms of categories

$$(7.2) \qquad \underline{\underline{C}}(\mathcal{X}_{Hopf}) = \underline{\underline{Pair}}(\mathcal{X}_{Hopf}) = \underline{\underline{ab}}$$

The natural system Γ_Σ for $\mathcal{X}_{Hopf}$ in (5.7) is trivial; the natural system $D^\sharp_\Sigma$, however, is non-trivial. In fact $D^\sharp_\Sigma$ on $\underline{\underline{ab}}$ coincides with the following natural system L.

(7.3) Definition. We define a natural system L on the category $\underline{\underline{ab}}$. Let $x, y, z \in A \in \underline{\underline{ab}}$ and

$$(1) \qquad [[x, y], z] = (x \otimes y + y \otimes x) \otimes z - z \otimes (x \otimes y + y \otimes x)$$

in $\otimes^3 A = A \otimes A \otimes A$. Let $L(A, 1)_3$ be the subgroup of $\otimes^3 A$ generated by all $[[x, y], z]$. A homomorphism ξ induces

$$\xi_\sharp : Hom(A, B \otimes B) \to Hom(\Gamma(A), L(B, 1)_3)$$

where $\xi_\sharp(\alpha)$ is the composition

$$\xi_\sharp(\alpha) : \Gamma(A) \xrightarrow{H} A \otimes A \xrightarrow{\alpha \otimes \xi} \otimes^3 B \xrightarrow{q} L(B, 1)_3.$$

Here q carries $x \otimes y \otimes z$ to $-[[z, x], y]$. The natural system L associates with $\xi : A \to B \in \underline{\underline{ab}}$ the group $L(\xi) = \text{cokernel}\,(\xi_\sharp H_*)$ where

(2) $\qquad \xi_\sharp H_* : Hom(A,\Gamma(B)) \to Hom(A, B\otimes B) \to Hom(\Gamma(A), L(B,1)_3)$

Induced maps for $L(\xi)$ are obtained from the bifunctor $(A,B) \longmapsto Hom(A, L(B,1)_3)$. Let $L' = Hom(-,\Lambda^2)$ be the natural system on $\underline{\underline{ab}}$ given by the bimodule $L'(\xi) = Hom(A,\Lambda^2 B)$. Then there is a canonical natural transformation $\tau : L' \to L$ of natural systems on $\underline{\underline{ab}}$,

$$(3) \qquad \tau : L'(\xi) = Hom(A,\Lambda^2 B) \to L(\xi),$$

which carries $\beta \in L'(\xi)$ to $\tau(\beta) = \{\xi_\sharp(\bar\beta)\}$. Here $\bar\beta : A \to \otimes^2 B$ is a homomorphism which projects to $\beta : A \to \Lambda^2 B$ by p in (6.2).

(7.4) Theorem. *The category of homotopy pairs between Hopf maps, $\underline{\underline{Hopair}}(\mathcal{X}_{Hopf})$, is characterized algebraically by the fact that there is a commutative diagram of linear extensions*

$$\begin{array}{ccccc} Hom(-,\Lambda^2) & \xrightarrow{+} & \underline{\underline{nil}} & \longrightarrow & \underline{\underline{ab}} \\ \tau\downarrow & & \downarrow & & \| \\ L & \xrightarrow{+} & \underline{\underline{Hopair}}(\mathcal{X}_{Hopf}) & \longrightarrow & \underline{\underline{ab}} \end{array}$$

Here τ is the natural transformation in (7.2).

Equivalently the theorem can be expressed by the equation

$$(7.5) \qquad \tau_*\{\underline{\underline{nil}}\} = \{\underline{\underline{Hopair}}(\mathcal{X}_{Hopf})\} \in H^2(\underline{\underline{ab}}, L)$$

Theorem (7.4) requires a highly sophisticated proof; compare [8].

§ 8 Homotopy pairs for Pontrjagin maps

Let A be an abelian group. A Pontrjagin map for A is a map

$$(8.1) \qquad \tau_A : K(A,2) \to K(\Gamma(A),4)$$

which induces the identity of $\Gamma(A)$,

$$1 : \Gamma(A) = H_4K(A,2) \xrightarrow{(\tau_A)_*} H_4K(\Gamma(A),4) = \Gamma(A)$$

Such Pontrjagin maps exist and are well defined up to homotopy. The map τ_A induces the Pontrjagin square which is the cohomology operation [28]

$$H^2(X,A) = [X, K(A,2)] \xrightarrow{(\tau_A)_*} [X, K(\Gamma(A),4)] = H^4(X,\Gamma(A))$$

Let $\mathcal{X}_{Pontrjagin}$ be the class of all Pontrjagin maps τ_A, $A \in \underline{\underline{Ab}}$. The fiber $P(\tau_A)$ of a Pontrjagin map is the 3-type of the Moore space $M(A,2)$. Let

$$Ext(-,\Gamma) : \underline{\underline{Ab}}^{op} \times \underline{\underline{Ab}} \to \underline{\underline{Ab}}$$

be the bimodule which carries (A,B) to the group $Ext(A,\Gamma(B))$.

(8.2) Theorem. *The category of homotopy pairs between Pontrjagin maps, $\underline{\underline{Hopair}}(\mathcal{X}_{Pontrjagin})$, is part of the following diagram of non-split linear extension:*

$$\begin{array}{ccccc}
Ext(-,\Gamma) & \xrightarrow{+} & \underline{\underline{Hopair}}(\mathcal{X}_{Pontrjagin}) & \longrightarrow & \underline{\underline{Ab}} \\
\| & & \| & & \| \\
Ext(-,\Gamma) & \xrightarrow{+} & \underline{\underline{P}}(\mathcal{X}_{Pontrjagin}) & \longrightarrow & \underline{\underline{Pair}}(\mathcal{X}_{Pontrjagin}) \\
\| & & \| & & \| \\
Ext(-,\Gamma) & \xrightarrow{+} & \underline{\underline{M}}^2 & \longrightarrow & \underline{\underline{Ab}}
\end{array}$$

Here $\underline{\underline{M}}^2$ is the full homotopy category of Moore spaces $M(A,2)$, $A \in \underline{\underline{Ab}}$.

Proof. The result is essentially a special case of (5.3) since

$$H^3_{(2)}(A,\Gamma(B)) = H^3(K(A,2),\Gamma(B)) = Ext(A,\Gamma(B))$$

The functor $\underline{\underline{M}}^2 \to \underline{\underline{P}}(\mathcal{X}_{Pontrjagin})$ carries the Moore space $M(A,2)$ to its 3-type. The linear extension for $\underline{\underline{M}}^2$ is the bottom row is also described in (V.3a.2) of [4] where we show that the extension is non-split.

q.e.d.

We study the linear extension (8.2) in more detail in [9].

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