Elliptic Schlesinger system and Painlevé VI,

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`Dedicated to the centenary of the publication`
`of the Painleve VI equation in the Comptes Rendus`
`de l'Academie des Sciences de Paris`
`by Richard Fuchs in 1905.`

Abstract

We construct an elliptic generalization of the Schlesinger system (ESS) with positions of marked points on an elliptic curve and its modular parameter as independent variables (the parameters in the moduli space of the complex structure). ESS is a non-autonomous Hamiltonian system with pair-wise commuting Hamiltonians. The system is bihamiltonian with respect to the linear and the quadratic Poisson brackets. The latter are the multi-color generalization of the Sklyanin-Feigin-Odeskii classical algebras. We give the Lax form of the ESS. The Lax matrix defines a connection of a flat bundle of degree one over the elliptic curve with first order poles at the marked points. The ESS is the monodromy independence condition on the complex structure for the linear systems related to the flat bundle. The case of four points for a special initial data is reduced to the Painlevé VI equation in the form of the Zhukovsky-Volterra gyrostat, proposed in our previous paper.

Contents

1 Introduction

The Schlesinger system was introduced in [1] is a system of first order differential equations for n matrices $\mathbf{S}^j$ $(j = 1, \ldots, n)$, depending on n points $x_k \in \mathbb{CP}^1$

$$\partial_k \mathbf{S}^j = \frac{[\mathbf{S}^k, \mathbf{S}^j]}{x_k - x_j}\,,\ (k \neq j)\,,\quad \partial_k = \partial_{x_k}\,, \tag{1.1}$$

$$\partial_k \mathbf{S}^k = -\sum_{j\neq k} \frac{[\mathbf{S}^k, \mathbf{S}^j]}{x_k - x_j}\,. \tag{1.2}$$

This system has the Hamiltonian form with respect to the linear (Lie-Poisson) brackets on $\mathrm{sl}(N, \mathbb{C})$. The Hamiltonian

$$H_k = \sum_{j\neq k} \frac{\langle \mathbf{S}^k \mathbf{S}^j \rangle}{x_k - x_j} \quad (\langle\ \rangle = \mathrm{tr})$$

defines the evolution with respect to the time x_k. There exists the tau-function $\exp \mathcal{F}$, related to the Hamiltonians [2]

$$\partial_k \ln \exp \mathcal{F} = H_k\,.$$

The Schlesinger equations are the monodromy preserving conditions for the linear system on $\mathbb{CP}^1$

$$\left(\partial_z + \sum_j \frac{\mathbf{S}^j}{z - x_j} \right) \Psi = 0\,.$$

For two by two matrices and four marked points the Schlesinger system is equivalent to the Painlevé VI equation [3]. In this case the position of three points can be fixed as $(0, 1, \infty)$ while x_4 play the role of an independent variable. Due to $\mathrm{SL}(2, \mathbb{C})$ gauge symmetry we leave with second order differential equation for the matrix element $(1, 2)$ of $\mathbf{S}^4$ (see, for example, [4]).

Here we replace $\mathbb{CP}^1$ by an elliptic curve and define a similar system (the elliptic Schlesinger system (ESS)). In this case, in addition to the coordinates of the marked points a new independent variable appears inevitably. It is the modular parameter of the curve, and thereby we have an additional new Hamiltonian. The similar systems in their integrable versions were considered earlier in [5, 6, 7]

We reproduce the main properties of the Schlesinger system. Moreover, we rewrite the ESS in terms of quadratic Poisson brackets. They are a multi-color generalization of the Sklyanin-Feigin-Odesski classical algebras [8, 9]. In conclusion, for the four point case and the matrices of order two we derive the Painlevé VI equation in the form of the Zhukovsky-Volterra gyrostat, proposed in our previous paper [10]. It was established there that the non-autonomous $\mathrm{SL}(2, \mathbb{C})$ Zhukovsky-Volterra gyrostat is equivalent to the elliptic form of the Painlevé VI equation [11] proposed by P.Painlevé one year later after Fuchs (see, also, [12]). The corresponding isomonodromy problem on an elliptic curve is discovered only recently [13]. This paper is a continuation of [10], though it can be read independently.

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2 Elliptic Schlesinger system

2.1 Definition

Let $\Sigma_\tau = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ be an elliptic curve, with the modular parameter τ, ($\Im m\tau > 0$) and

$$D_n = (x_1, \ldots, x_n)\,,\ x_j \neq x_k\,,\ x_k \in \Sigma_\tau$$

be the divisor of non-coincident points with the condition

$$\sum x_j \in (\mathbb{Z}+\tau\mathbb{Z})\,. \tag{2.1}$$

Consider the space $\mathcal{P}^{(1)}_{n,N}$ of n copies of the Lie coalgebra $\mathfrak{g}^* \sim \mathrm{sl}(N,\mathbb{C})^*$, related to the points of the divisor.

$$\mathcal{P}^{(1)}_{n,N} = \oplus_{j=1}^n \mathfrak{g}^*_j\,,\quad \mathfrak{g}^*_j = \{\mathbf{S}^j = \sum_{\alpha\in\tilde{\mathbb{Z}}^{(2)}_N} S^j_\alpha t^\alpha\}\,, \tag{2.2}$$

where t^α is the basis (B.7).

Introduce three operators that act from $\mathcal{P}^{(1)}_{n,N}$ to the dual space $\oplus_{j=1}^n \mathfrak{g}_j$

$$\mathbf{I}_{kj}\ :\ \mathfrak{g}^*_k \to \mathfrak{g}_j\,,\quad S^k_\gamma \mapsto (I_{kj})_\gamma S^k_\gamma\,,\ (I_{kj})_\gamma = \varphi_\gamma(x_j - x_k)\,, \tag{2.3}$$

$$\mathbf{J}_{jj}\ :\ \mathfrak{g}^*_j \to \mathfrak{g}_j\,,\quad S^j_\gamma \mapsto J_\gamma S^j_\gamma\,,\ J_\gamma = E_2(\breve{\gamma})\,, \tag{2.4}$$

$$\mathbf{J}_{kj}\ :\ \mathfrak{g}^*_k \to \mathfrak{g}_j\,,\quad S^k_\gamma \mapsto (J_{kj})_\gamma S^k_\gamma\,,\ (J_{kj})_\gamma = f_\gamma(x_j - x_k) \tag{2.5}$$

where $\varphi_\gamma(x)$, $E_2(\breve{\gamma})$ and $f_\gamma(x)$ are defined by (B.10) - (B.14).

The positions of the marked points $x_j \in D_n$, satisfying (2.1), and the modular parameter τ are local coordinates in an open cell in the moduli space $\mathcal{M}_{1,n}$ of elliptic curves with n marked points and play the role of times.

Definition 2.1 *The elliptic Schlesinger system (ESS) is the consistent dynamical system on $\mathcal{P}^{(1)}_{n,N}$ with independent variables from $\mathcal{M}_{1,n}$*

$$\partial_j \mathbf{S}^k = [\mathbf{I}_{kj}(\mathbf{S}^j), \mathbf{S}^k]\,,\ (k\neq j)\,,\quad \partial_k = \partial_{x_k}\,, \tag{2.6}$$

$$\partial_k \mathbf{S}^k = -\sum_{j\neq k} [\mathbf{I}_{jk}(\mathbf{S}^j), \mathbf{S}^k]\,, \tag{2.7}$$

$$\partial_\tau \mathbf{S}^j = \sum_{k\neq j} \frac{1}{2\pi\imath}[\mathbf{S}^j, \mathbf{J}_{kj}(\mathbf{S}^k)] + \frac{1}{4\pi\imath}[\mathbf{S}^j, \mathbf{J}_{jj}(\mathbf{S}^j)]\,, \tag{2.8}$$

where the commutators are understand as the coadjoint action of $\mathfrak{g}_j$ on $\mathfrak{g}^*_j$.

The consistency of the system will be proved below.

In the basis t^α ($\alpha \in \tilde{\mathbb{Z}}^{(2)}_N$) (B.7) the ESS takes the form

$$\partial_k S^j_\alpha = \sum_{\gamma\in\tilde{\mathbb{Z}}^{(2)}_N)} \mathbf{C}(\gamma,\alpha) S^k_\gamma S^j_{\alpha-\gamma} \varphi_\gamma(x_j - x_k)\,,\quad (k\neq j)\,, \tag{2.9}$$

$$\partial_k S^k_\alpha = \sum_{\gamma\in\tilde{\mathbb{Z}}^{(2)}_N)} \mathbf{C}(\gamma,\alpha) \sum_{j\neq k} S^j_{\alpha-\gamma} S^k_\gamma \varphi_{\alpha-\gamma}(x_k - x_j)\,, \tag{2.10}$$

$$\partial_\tau S^k = \frac{1}{2\pi\imath} \sum_{\gamma\in\tilde{\mathbb{Z}}^{(2)}_N)} \mathbf{C}(\alpha,\gamma) \left(\sum_{k\neq j} S^k_{\alpha-\gamma} S^j_\gamma f_\gamma(x_k - x_j) + S^k_\gamma S^k_{-\gamma} E_2(\breve{\gamma}) \right)\,. \tag{2.11}$$

Remark 2.1 *Equations (2.9), (2.10) are consistent with the restriction on positions of the marked points (2.1) i.e.* $\sum_{j=1}^{n}\partial_j \mathbf{S}^k = 0$.

Remark 2.2 *In the rational limit (2.9) and (2.10) pass to the standard Schlesinger system (1.1), (1.2) (see (A.9)).*

As in the rational case the ESS has some fundamental properties

- The space $\mathcal{P}^{(1)}_{n,N}$ is Poisson with respect to the linear Lie-Poisson brackets on $\mathfrak{g}^*$

$$\{S^j_\alpha, S^k_\beta\}_1 = \delta^{jk}\mathbf{C}(\alpha,\beta)S_{\alpha+\beta} \tag{2.12}$$

  ESS is a non-autonomous Hamiltonian system with respect to the linear brackets on $\mathcal{P}^{(1)}_{n,N}$

$$\partial_k \mathbf{S}^j = \{H_k, \mathbf{S}^j,\}_1\,,\quad \partial_k = \partial_{x_k}\,,\ (1,\dots,n)\,, \tag{2.13}$$

$$\partial_\tau \mathbf{S}^j = \{H_0, \mathbf{S}^j\}_1\,, \tag{2.14}$$

  where

$$H_k = -\sum_{j\neq k}\langle \mathbf{I}_{kj}(\mathbf{S}^k)\mathbf{S}^j)\rangle = -\sum_{j\neq k}\sum_{\gamma\in\tilde{\mathbb{Z}}^{(2)}_N} S^k_\gamma S^j_{-\gamma}\varphi_\gamma(x_j - x_k)\,, \tag{2.15}$$

$$H_\tau = H_0 = -\frac{1}{2\pi\imath}\left(\sum_{k\neq j}\langle \mathbf{S}^j \mathbf{J}_{kj}(\mathbf{S}^k)\rangle + \sum_j \langle \mathbf{S}^j \mathbf{J}_{jj}(\mathbf{S}^j)\rangle\right) \tag{2.16}$$

$$= -\frac{1}{2\pi\imath}\left(\sum_{k\neq j}\sum_{\gamma\in\tilde{\mathbb{Z}}^{(2)}_N} S^j_\gamma S^k_{-\gamma} f_\gamma(x_k - x_j) + \sum_j \sum_{\gamma\in\tilde{\mathbb{Z}}^{(2)}_N} S^j_\gamma S^j_{-\gamma}E_2(\check{\gamma})\right)\,.$$

  The brackets (2.12) are degenerate. The symplectic leaves are n copies of coadjoint orbits $\mathcal{O}_j$ $(j = 1,\dots,n)$ of $\mathrm{SL}(N,\mathbb{C})$. Let all orbits be generic, and $c^\mu(j)$ be corresponding Casimir functions of order μ $(\mu = 2,\dots,N)$. The phase space of ESS is

$$\mathcal{R}^{(1)}_{n,N} \sim \mathcal{P}^{(1)}_{n,N}/\{c^\mu(j) = c^\mu(j)_0\} \sim \prod \mathcal{O}_j \tag{2.17}$$

$$\dim \mathcal{R}^{(1)}_{n,N} = nN(N-1) \tag{2.18}$$

  The ESS can be considered as a system of interacting non-autonomous $\mathrm{SL}(N,\mathbb{C})$ Euler-Arnold tops, where operators (2.3), (2.4), (2.5) play the role of the inverse inertia tensors.

- The Hamiltonians satisfy the generalized Whitham equations [14]

$$\partial_j H_k - \partial_k H_j = 0\,,\quad (j,k = 0,\dots,n)\,. \tag{2.19}$$

  In other words, the flows commute and the equations (2.6), (2.7) and (2.8) are consistent. These conditions provide the existence of the tau-function $\exp\mathcal{F}$

$$H_j = \partial_j\mathcal{F}\,,\quad H_0 = \partial_\tau\mathcal{F}\,.$$

- ESS is the monodromy preserving condition for flat rank N and degree one bundles over Σ_τ with respect to deformations of its moduli.

While the first two statements can be checked directly the last one should be considered separately. In next subsection we prove all of them by the symplectic reduction from trivial, though infinite Hamiltonian system.

2.2 Derivation of ESS

Here we derive the ESS starting with a bundle over the elliptic curve Σ_τ. Deformations of the complex structure of Σ_τ allows us to introduce the times and the Hamiltonians. The ESS arises on the symplectic quotient of the space of vector bundles with respect to the action of the $\mathrm{SL}(N,\mathbb{C})$ gauge group.

2.2.1 Vector bundles of degree one over elliptic curves

Let E_N be a degree one and rank N bundle over the elliptic curve $\Sigma_{\tau_0} \sim \mathbb{C}/(\mathbb{Z}+\tau_0\mathbb{Z})$ and $Conn(E_N)=\{\mathcal{A}\}$ be the space of its C^∞ connections. It is a symplectic space with the form

$$\omega^0 = \frac{1}{2}\int_\Sigma \langle \delta\mathcal{A}\wedge\delta\mathcal{A}\rangle\,.$$

Let $(z,\bar z)$ be the complex coordinates on Σ_{τ_0}

$$z = x+\tau_0 y\,,\ \ \bar z = x+\bar\tau_0 y\,,\quad (0<x\,,y\leq 1)\,.$$

For generic degree one bundles the transition matrices corresponding to the two basic cycles can be chosen as

$$\begin{array}{c}\mathcal{A}(z+1,\bar z+1) = Q\mathcal{A}(z,\bar z)Q^{-1}\,,\\ \mathcal{A}(z+\tau_0,\bar z+\bar\tau_0) = \tilde\Lambda\mathcal{A}(z,\bar z)\tilde\Lambda^{-1}+\frac{2\pi\imath}{N}dz\,,\end{array} \tag{2.20}$$

where $\tilde\Lambda(z,\tau) = -\mathbf{e}_N(-z-\frac{\tau_0}{2})\Lambda$ and Q,Λ (B.1), (B.2). It means that there are no moduli parameters for degree one bundles.

The complex structure on Σ_τ allows us to introduce the complex structure on $Conn(E_N)$. Let

$$d' = \partial + A\,,\quad d''=\bar\partial+\bar A\,,\quad (\partial=\partial_z\,,\,\bar\partial=\partial_{\bar z})$$

be the corresponding components of the connection $\mathcal{A}$.

In addition, we fix a quasi-parabolic structure at n marked points. It means that A has simple poles at the marked points and

$$ResA|_{z=x_j^0} = \mathbf{S}^j = g^{-1}\mathbf{S}_0^j g\in\mathcal{O}_j\subset\mathfrak{g}_j^*$$

while $\bar A$ is regular. The symplectic form acquires the additional Kirillov-Kostant terms

$$\omega^0 = \int_\Sigma\langle\delta A\wedge\delta\bar A\rangle - \sum_{j=1}^n\langle\mathbf{S}_0^j g_j^{-1}\delta g_j g_j^{-1}\wedge\delta g_j\rangle\,,\ g_j\in\mathrm{SL}(N,\mathbb{C})\,. \tag{2.21}$$

We denote the set $Conn(E_N)$ with the quasi-parabolic structure at the marked points as $\tilde{\mathcal{R}}^{(1)}_{N,\tau,n}(\mathbf{S}_0^j)$.

In fact, we will work with the larger space

$$\tilde{\mathcal{P}}^{(1)}_{n,N} = \{Conn(E_N)\,;\oplus_{j=1}^n\mathfrak{g}_j^*\} = \{(A,\bar A)\,,\mathbf{S}^j\,,(j=1,\ldots,n)\}$$

equipped with the Poisson brackets

$$\{A_\alpha,\bar A_\beta\} = \delta_{\alpha,-\beta}\,, \tag{2.22}$$

$$\{S_\alpha^j,S_\beta^k\} = \delta_{jk}\mathbf{C}(\alpha,\beta)S_{\alpha+\beta}\,. \tag{2.23}$$

By fixing the values of the Casimir functions to come down to $\tilde{\mathcal{R}}^1_{N,\tau,n}(\mathbf{S}_0^j)$.

2.2.2 Introducing Hamiltonians by deformation of complex structure

Deform the complex structure as

$$\left\{\begin{array}{l} w = z - \epsilon(z,\bar z)\,, \\ \bar w = \bar z\,; \end{array}\right. \quad dw = (1-\partial\epsilon)dz - \bar\partial\epsilon d\bar z\,. \tag{2.24}$$

The Beltrami differential

$$\mu = \frac{\bar\partial\epsilon(z,\bar z)}{1-\partial\epsilon(z,\bar z)}\left(\frac{\partial}{\partial z}\otimes d\bar z\right)\,, \quad (\bar\partial = \partial_{\bar z})$$

defines the new holomorphic structure - the deformed antiholomorphic operator annihilates dw, while the antiholomorphic structure is kept unchanged

$$\partial_{\bar w} = \bar\partial + \mu\partial, \quad \partial_w = \partial.$$

In addition, assume that μ vanishes at the marked points $\mu(z,\bar z)|_{x_j^0} = 0$.

We specify the dependence of μ on the positions of the marked points in the following way. Let $\mathcal{U}_j' \supset \mathcal{U}_j$ be two vicinities of the marked point x_a such that $\mathcal{U}_j' \cap \mathcal{U}_k' = \emptyset$ for $j \neq k$. Let $\chi_j(z,\bar z)$ be a smooth function

$$\chi_j(z,\bar z) = \left\{\begin{array}{ll} 1, & z \in \mathcal{U}_j \\ 0, & z \in \Sigma_g \setminus \mathcal{U}_j'. \end{array}\right.$$

Introduce times related to the positions of the marked points $t_j = x_j - x_j^0$. Then

$$\mu_j = t_j\mu_j^0 = t_j\bar\partial\chi_j(z,\bar z)\,, \quad t_j = x_j - x_j^0\,. \tag{2.25}$$

The dependence of the modular parameter takes the form

$$\mu_\tau = t_\tau\mu_0^0 = \frac{t_\tau}{\tau_0-\bar\tau_0}\bar\partial(\bar z - z)(1-\sum_{j=1}^n\chi_j(z,\bar z))\,, \quad t_\tau = \tau - \tau_0\,. \tag{2.26}$$

The functions μ_j^0 $(j = 0,\dots,n)$ can be considered as a basis in a big cell $\mathcal{M}_{1,n}^0$ of the moduli space $\mathcal{M}_{1,n}$. The introduced above times play the role of coordinates in this basis

$$\mu = t_\tau\mu_\tau^0 + \sum_{j=1}^n t_j\mu_j^0\,. \tag{2.27}$$

We deform ω^0 by means of the Beltrami differentials in a such way that it acquires nontrivial Hamiltonians. Let us go to a new pair of the connection components

$$(A,\bar A) \to (A, \bar A' = \bar A - \mu A)$$

It changes the form of ω^0 (2.21) as

$$\omega = \omega_0 - \frac{1}{2}\int_{\Sigma_\tau}\delta\langle A^2\rangle\delta\mu\,. \tag{2.28}$$

Expanding μ in the basis (2.27) we obtain

$$\omega = \omega^0 - \sum_{j=0}^n \delta\tilde H_j\delta t_j\,, \quad t_0 = t_\tau\,, \tag{2.29}$$

where

$$\tilde{H}_j = \frac{1}{2}\int_{\Sigma_\tau} \langle A^2\rangle \bar{\partial}\chi_j(z,\bar{z})\,, \quad (j=1,\ldots,n) \tag{2.30}$$

$$\tilde{H}_0 = \frac{1}{2}\int_{\Sigma_\tau} \langle A^2\rangle \bar{\partial}(\bar{z}-z)(1-\sum_{j=1}^{n}\chi_j(z,\bar{z}))\,. \tag{2.31}$$

The form ω is defined on $\mathcal{R}^1_N(\Sigma_\tau \backslash D_n)\times\mathcal{M}^0_{1,n}$. The brackets (2.22), (2.23) and the Hamiltonians $\tilde{H}_j$ lead to the equations of motion

$$1.\,\partial_j\bar{A} = A\mu^0_j\,, \quad 2.\,\partial_j A = 0\,, \quad 3.\,\partial_j g_k = 0\,, \quad (\partial_j = \partial_{t_j})\,. \tag{2.32}$$

Evidently, these flows pairwise commute. Moreover, we have from (2.22), (2.30), and (2.31)

$$\{\tilde{H}_j, \tilde{H}_k\} = 0 \tag{2.33}$$

Remark 2.3 *It easy to see that for general non-autonomous multi-time Hamiltonian systems, as, for example, ESS, the commutativity of flows amounts to the quasi-classical flatness*

$$\partial_j H_k - \partial_k H_j + \{H_k, H_j\} = 0\,.$$

If, moreover, (2.33) holds, then these conditions provide the existence of the tau-function $\partial_i \exp\mathcal{F} = H_i$. *In particular, the tau-function exists for the flows (2.32).*

2.2.3 ESS as symplectic quotient

Let $\mathcal{G} = \{f(w,\bar{w})\}$ be the group of smooth maps of Σ_τ to $\mathrm{SL}(N,\mathbb{C})$ with the quasi-periodicity

$$f(w+1,\bar{w}+1) = Q^{-1}f(w,\bar{w})Q\,, \quad f(w+\tau,\bar{w}+\bar{\tau}) = \tilde{\Lambda}^{-1}(w)f(w,\bar{w})\tilde{\Lambda}(w)\,. \tag{2.34}$$

Define its action on the fields as

$$A \to f^{-1}\partial_w f + f^{-1}Af\,, \quad \bar{A} \to f^{-1}\partial_{\bar{w}} f + f^{-1}\bar{A}f\,, \tag{2.35}$$

$$g_j \to g_j f_j\,, \quad f_j = f(z,\bar{z})|_{z=x_j}\,.$$

The form ω is invariant with respect to this action. Therefore we can pass to the symplectic quotient

$$\mathcal{R}^{(1)}_{N,\tau,n}(\mathbf{S}^j_0) = \tilde{\mathcal{R}}^{(1)}_{N,\tau,n}(\mathbf{S}^j_0)//\mathcal{G}\,.$$

Proposition 2.1

- *The symplectic quotient is the product of the coadjoint orbits*

$$\mathcal{R}^{(1)}_{N,\tau,n}(\mathbf{S}^j_0) \sim \times^n_{j=1}\mathcal{O}_j$$

- *The ESS is a result of the symplectic reduction of the system (2.32). Its Hamiltonians (2.15), (2.16) are reduction of (2.30), (2.31) to* $\mathcal{R}^{(1)}_{N,\tau,n}(\mathbf{S}^j_0)$.
- *There exists the tau-function* $\exp\mathcal{F}$ *for the ESS*

$$\partial_j \exp\mathcal{F} = H_j\,.$$

Proof.
The symplectic quotient is characterized by the conditions:
i. the moment constraints

$$F(A,\bar{A})=\sum_{j=1}^{n}\mathbf{S}_j\delta(w-x_j,\bar{w}-\bar{x}_j)-N\delta(w,\bar{w})t^0\,,\quad \mathbf{S}^j=g_j^{-1}\mathbf{S}_0^jg_j\,, \tag{2.36}$$

where $F(A,\bar{A})=\bar{\partial}A+\partial(\mu A)+[\bar{A},A]$. Note that the last term in the r.h.s. of (2.36) comes from (2.34) and (2.35).
ii. the gauge fixing

$$A_{\bar{w}}=0\,. \tag{2.37}$$

It means that any $A_{\bar{w}}$ can be represented as the pure gauge $A_{\bar{w}}=f^{-1}[A_{\bar{w}}]\partial_{\bar{w}}f[A_{\bar{w}}]$. As a result $\mathcal{R}^{(1)}_{N,\tau,n}(\mathbf{S}_0^j)$ is described by the Lax matrix

$$L=-\partial_w ff^{-1}+fAf^{-1}\,,\quad f=f[A_{\bar{w}}]\,.$$

The Lax matrix is a solution of the equation

$$\partial_{\bar{w}}L=\sum_{j=1}^{n}\mathbf{S}^j\delta(w-x_j,\bar{w}-\bar{x}_j)-N\delta(w,\bar{w})t^0$$

with the quasi-periodicity (2.20). From (A.13) and (B.15) we get

$$L(w)=-\frac{1}{N}E_1(w)T_0+\sum_{j=1}^{n}\sum_{\gamma\in\tilde{\mathbb{Z}}_N^{(2)}}S_\gamma^j\varphi_\gamma(w-x_j)T_\gamma\,. \tag{2.38}$$

Here, for convenience we have used the basis T_γ instead of t^γ. We stay only with finite degrees of freedom described by the ESS variables $\mathbf{S}^j$. Thereby, the symplectic quotient $\mathcal{R}^{(1)}_{N,\tau,n}(\mathbf{S}_0^j)$ coincides with the phase space of the ESS (2.17).

The following Lemma complete the essential part of the proof.

Lemma 2.1 • *The equations of motion (2.32) on the reduced space $\mathcal{R}^{(1)}_{N,\tau,n}(\mathbf{S}_0^j)$ take the Lax form*

$$\partial_kL-\partial_wM^k+[M^k,L]=0\,,\quad (k=0,\dots,n), \tag{2.39}$$

where

$$M^k=-\sum_{\gamma\in\tilde{\mathbb{Z}}_N^{(2)}}S_\gamma^k\varphi_\gamma(w-x_k)T_\gamma\,,\quad (k\neq 0)\,, \tag{2.40}$$

$$M^0=-\frac{1}{N}\partial_\tau\ln\vartheta(w|\tau)T_0+\frac{1}{2\pi\imath}\sum_{l=1}^{n}\sum_{\gamma\in\tilde{\mathbb{Z}}_N^{(2)}}S_\gamma^lf_\gamma(w-x_l)T_\gamma\,. \tag{2.41}$$

• *(2.39) coincides with the ESS (2.9), (2.10), (2.11).*

Proof.
Substituting in the equation of motion for A (2.32 (2))

$$A=f^{-1}\partial f+f^{-1}Lf$$

and defining $M^k = -\partial_k f f^{-1}$ we come to (2.39). It follows from (2.32 (1)) that M^k satisfies the equation $\partial_{\bar{w}} M^k = -L\mu_k^0$ with the same quasi-periodicity as L for $j \neq 0$. To define M^j we have used (B.15) and (B.16). The Lax equation with M^j $(j \neq 0)$ leads directly to (2.9). The Lax equation with M^0 follows from the heat equation (A.11) and the Calogero equation (A.18). □

After the reduction the Poisson space $\tilde{\mathcal{P}}^{(1)}_{n,N}$ passes to $\mathcal{P}^{(1)}_{n,N}$ with the brackets (2.23). It follows from (2.30), (2.31) that the Hamiltonians H_j on $\mathcal{P}^{(1)}_{n,N}$ can be read off from the expansion of $\mathrm{tr}(L^2)$ on the basis of the elliptic functions

$$\frac{1}{2}\mathrm{tr}(L(w))^2 = \sum_{j=1}^{n} \left(H_{2,j} E_2(w - x_j) + H_{1,j} E_1(w - x_j) \right) + H_0' \,,$$

where $H_0 = -\frac{1}{2\pi\imath}(H_0' - \frac{4\eta_1}{N})$ and $\sum_j H_{1,j} = 0$. Here $H_{2,j} = \frac{1}{2}\sum_\gamma S_\gamma^j S_{-\gamma}^j$ are the quadratic Casimir functions corresponding to the orbits $\mathcal{O}_j$. It can be find that $H_{1,j}$ coincide with (2.15), and H_0 with (2.16). The Hamiltonians commute since their pre-images commute on $\tilde{\mathcal{P}}^{(1)}_{n,N}$. Therefore, we have proved the consistency of ESS and the existence of the tau-function. □

2.2.4 Isomonodromy problem

Let $\Psi \in \Gamma$ be a section of a degree one vector bundle over Σ_τ. Consider the linear system

$$\left\{ \begin{array}{l} (\partial_w + A)\Psi = 0 \,, \\ (\partial_{\bar{w}} + \bar{A})\Psi = 0 \,, \\ \partial_k \Psi = 0 \,, \quad (k = 0, \ldots, n) \,. \end{array} \right. \tag{2.42}$$

The compatibility conditions of the first two equations is the flatness condition of the bundle. The equations of motion (2.32) are the compatibility conditions of the last equations with the two first equations. Let γ be a closed path on Σ_τ, Ψ_γ is the corresponding transformed solution and Θ_γ is the monodromy matrix

$$\Psi_\gamma = \Psi \Theta_\gamma \,.$$

Then the last equations implies the independence of Θ_γ on the moduli times t_k. Therefore, the equations of motion are the monodromy preserving conditions.

Let f be the gauge transformations $\Psi \to f\Psi$ that "kills" $A_{\bar{w}}$. Then (2.42) takes the form

$$\left\{ \begin{array}{l} (\partial_w + L)\Psi = 0 \,, \\ \partial_{\bar{w}} \Psi = 0 \,, \\ (\partial_k + M^k)\Psi = 0 \,, \quad (k = 0, \ldots, n) \,, \end{array} \right. \tag{2.43}$$

where L (2.38) and M^k (2.40),(2.41). The compatibility conditions of the last equations with the first one is the ESS in the Lax form (2.39). They are the monodromy preserving conditions for the linear system of the first two equations.

3 Bihamiltonian structure of ESS

3.1 Quadratic Poisson algebra

Consider a complex space of dimension nN^2. We organize it in the following way. Attribute to the marked points of the divisor D_n n copies of the $\mathrm{GL}(N, \mathbb{C})$-valued elements

$$x_j \to S_0^j T_0 + \mathbf{S}^j = \sum_{a \in \mathbb{Z}_N^{(2)}} S_a^j T_a \,.$$

Add to this set a variable $S_0\in\mathbb{C}$ and define

$$\mathcal{P}^{(2)}_{n,N}=\{S_0\,,\,(S_0^j\,,\mathbf{S}^j\,,j=1,\ldots,n)\,|\,\sum_{j=1}^{n}S_0^j=0\}\,.$$

Proposition 3.1 *The space $\mathcal{P}^{(2)}_{n,N}$ is Poisson with respect to the quadratic brackets*

$$\{S_0,S_0^j\}_2=\{S_0^j,S_0^k\}_2=\{S_\alpha^j,S_\alpha^k\}_2=0\,, \tag{3.1}$$

$$\{S_0,S_\alpha^k\}_2=\sum_{\gamma\neq\alpha}\mathbf{C}(\alpha,\gamma)\left(S^k_{\alpha-\gamma}S^k_\gamma E_2(\breve{\gamma})-\sum_{j\neq k}S^j_{-\gamma}S^k_{\alpha+\gamma}f_\gamma(x_k-x_j)\right)\,, \tag{3.2}$$

$$\begin{aligned}\{S_\alpha^k,S_\beta^k\}_2=\;&\mathbf{C}(\alpha,\beta)S_0S^k_{\alpha+\beta}+\sum_{\gamma\neq\alpha,-\beta}\mathbf{C}(\gamma,\alpha-\beta)S^k_{\alpha-\gamma}S^k_{\beta+\gamma}\mathbf{f}_{\alpha,\beta,\gamma}\\&+\mathbf{C}(\alpha,\beta)S_0^kS^k_{\alpha+\beta}(E_1(\breve{\alpha}+\breve{\beta})-E_1(\breve{\alpha})-E_1(\breve{\beta}))\\&-\mathbf{C}(\alpha,\beta)\sum_{j\neq k}[S_0^kS^j_{\alpha+\beta}\varphi_{\alpha+\beta}(x_k-x_j)-S_0^jS^k_{\alpha+\beta}E_1(x_k-x_j)]\}\\&-2\sum_{j\neq k}\mathbf{C}(\gamma,\alpha-\beta)S^k_{\alpha-\gamma}S^k_{\beta+\gamma}\varphi_{\beta+\gamma}(x_k-x_j)\}\,,\end{aligned} \tag{3.3}$$

where $\mathbf{f}_{\alpha,\beta,\gamma}$ is defined by (B.14). For $j\neq k$

$$\begin{aligned}\{S_\alpha^j,S_\beta^k\}_2=&\sum_{\gamma\neq\alpha,-\beta}\mathbf{C}(\gamma,\alpha-\beta)S^j_{\alpha-\gamma}S^k_{\beta+\gamma}\varphi_\gamma(x_j-x_k)\\&-\mathbf{C}(\alpha,\beta)\left(S_0^jS^k_{\alpha+\beta}\varphi_\alpha(x_j-x_k)-S_0^kS^j_{\alpha+\beta}\varphi_{-\beta}(x_k-x_j)\right)\,,\end{aligned} \tag{3.4}$$

and

$$\{S_0^j,S_\beta^k\}_2=\left\{\begin{array}{lc}2\sum_\gamma\mathbf{C}(\gamma,-\beta)S^j_{-\gamma}S^k_{\beta+\gamma}\varphi_\gamma(x_k-x_j)\,, & j\neq k\,,\\ -2\sum_{m\neq k}\sum_\gamma\mathbf{C}(\gamma,-\beta)S^k_{-\gamma}S^m_{\beta+\gamma}\varphi_{\beta+\gamma}(x_k-x_m)\,, & j=k\,.\end{array}\right. \tag{3.5}$$

The brackets are extracted from the classical exchange algebra

$$\{L_1^{group}(z),L_2^{group}(w)\}_2=[r(z-w),L_1^{group}(z)\otimes L_2^{group}(w)]\,,$$

where r is the classical Belavin-Drinfeld r-matrix $r(z)=\sum_\gamma\varphi_\gamma(z)T_\gamma\otimes T_{-\gamma}$ [17], and L^{group} is the modified Lax operator

$$L^{group}=\left(S_0+\sum_{j=1}^{n}S_0^jE_1(z-x_j)\right)T_0+\tilde{L}_j\,,\quad\tilde{L}_j=\sum_\alpha S^j_\alpha\varphi_\alpha(z-x_j)T_\alpha\,.$$

The Jacobi identity for $\mathcal{P}^{(2)}_{n,N}$ follows from the classical Yang-Baxter equation for $r(z)$. The Poisson algebra $\mathcal{P}^{(2)}_{n,N}$ defines the structure of the Poisson-Lie group on the product of G_j attached to the marked points x_j. The proof of Lemma will be given in a separate publication.

Remark 3.1 *For $n=1$ we come to the classical Feigin-Odesski-Sklyanin algebras [8, 9]*

$$\{S_0,S_\alpha\}_2=\sum_{\gamma\neq\alpha}\mathbf{C}(\alpha,\gamma)S_{\alpha-\gamma}S_\gamma E_2(\breve{\gamma})\,, \tag{3.6}$$

$$\{S_\alpha,S_\beta\}_2=S_0S_{\alpha+\beta}\mathbf{C}(\alpha,\beta)+\sum_{\gamma\neq\alpha,-\beta}\mathbf{C}(\gamma,\alpha-\beta)S_{\alpha-\gamma}S_{\beta+\gamma}\mathbf{f}(\breve{\alpha},\breve{\beta},\breve{\gamma})\,, \tag{3.7}$$

3.2 Bihamiltonian structure

The quadratic brackets on $\mathcal{P}^{(2)}_{n,N}$ are degenerate. The function $\det L(z)$ is the generating function for the Casimir functions $C^{\mu}(j)$ [1] (see [16]). Since it is a double periodic function it can be expanded in the basis of elliptic functions (A.6)

$$\det L(z) = C^0 + \sum_j^n C^1(j)E_1(z-x_j) + C^2(j)E_2(z-x_j) + \ldots + C^N(j)E_N(z-x_j)\,. \tag{3.8}$$

In particular, for the second order matrices $N = 2$

$$C^0 = S_0^2 + 4\eta_1 \sum_{j=1}^n (S_0^j)^2 + \sum_\gamma \left(\sum_{j=1}^n E_2(\check{\gamma}) S_\gamma^j S_\gamma^j + 2\sum_{k\neq j} S_\gamma^j S_{-\gamma}^k f_\gamma(x_k - x_j) \right)\,, \tag{3.9}$$

$$C^1(j) = S_0 S_j + \sum_{k\neq j} S_0^j S_0^k E_1(x_j - x_k) + \sum_{k\neq j}\sum_\gamma S_\gamma^j S_\gamma^k \phi_\gamma(x_j - x_k)\,, \tag{3.10}$$

$$C^2(j) = (S_0^j)^2 - \sum_\gamma (S_\gamma^j)^2\,. \tag{3.11}$$

Due to the condition

$$\sum_{j=1}^n C^1(j) = 0\,, \tag{3.12}$$

the number of the independent Casimir functions is Nn. The generic symplectic leaf

$$\mathcal{R}^2_{n,N} \sim \mathcal{P}^{(2)}_{n,N}/\{(C^{\mu}(j) = C^{\mu}(j)_{(0)})\,,\ \mu = 1,\ldots,N\,, j = 1,\ldots,N\}\,.$$

has dimension

$$\dim(\mathcal{R}^2_{n,N}) = nN(N-1)\,. \tag{3.13}$$

It coincides with the dimension of the ESS phase space $\mathcal{R}^{(1)}_{N,\tau,n}(\mathbf{S}_0^j)$ defined in terms of the linear brackets.

We can extend the linear Poisson manifold $\mathcal{P}^{(1)}_{n,N}$ (2.2) by adding the variables $S_0\,, S_0^j$. In terms of the linear brackets they are the Casimir functions and therefore preserve the phase space $\mathcal{R}^{(1)}_{N,\tau,n}(\mathbf{S}_0^j)$ (2.17).

The form of brackets (3.2), (3.5) and the Casimir functions (3.9), (3.10) suggests the following statement:

Proposition 3.2 *In terms of the quadratic brackets the ESS takes the form*

$$\partial_k S_\alpha^j = \frac{1}{2}\{S_0^k, S_\alpha^j\}_2\,, \quad (j,k = 1,\ldots,n)\,,$$

$$\partial_\tau S_\alpha^j = \frac{1}{2}\{S_0, S_\alpha^j\}_2\,.$$

We have more for the second order matrices. The Casimir functions of the quadratic brackets serve as Hamiltonians in the representations ESS by the linear brackets

$$\partial_k S_\alpha^j = \{C^1(k), S_\alpha^j\}_1\,, \quad (j,k = 1,\ldots,n)\,,$$

$$\partial_\tau S_\alpha^j = \frac{1}{2\pi\imath}\{C_0, S_\alpha^j\}_1\,.$$

[1] To distinguish them from the Casimir functions of the linear algebra we denote them by capital letters.

Therefore, for $N=2$ the trajectories of the ESS lie on the intersection of the symplectic leaves of $\mathcal{P}^{(2)}_{n,2}$ and $\mathcal{P}^{(1)}_{n,N}$. This phenomena is a manifestation of the compatibility of the linear and the quadratic Poisson brackets. The existence of compatible Poisson structures implies the bihamiltonian structure of integrable hierarchies related to these brackets [18]. We don't touch this point here.

4 Reduction to the PVI

Consider the rank two case ($N=2$) with four marked points $n=4$. We slightly change here our notations and enumerate the marked points as x_j, $j=0,1,2,3$. Replace the basis T_α with the Pauli matrices

$$T_{(1,0)} \to \sigma_3\,,\ \ T_{(0,1)} \to \sigma_1\,,\ \ T_{(1,1)} \to \sigma_2\,,$$

and the basis index $\alpha = 1,2,3$. As an initial data we put the marked points on $z=0$ and the half-periods of Σ_τ

$$x_0 = 0\,,\ x_1 = \frac{\tau}{2} = \omega_2\,,\ x_2 = \frac{1+\tau}{2} = \omega_1+\omega_2\,,\ x_3 = \frac{1}{2} = \omega_1\,,$$

and assume that

$$S^j_\alpha = \delta^j_\alpha \tilde{\nu}_\alpha\,,\ (j=1\,,2\,,3)\,, \tag{4.1}$$

while $S^0_\alpha = S_\alpha$ are arbitrary. Since for $N=2$ $\check{\gamma} \sim -\check{\gamma}$ it is not difficult to see that the Hamiltonians H_j $(j=1,2,3)$ (2.15) vanish for this configuration, while (2.16) assume the form

$$H_\tau = \sum_{\gamma=1,2,3} (S_\gamma)^2 E_2(\check{\gamma}) + S_\gamma \nu'_\gamma\,,\ \ \nu'_\alpha = -\tilde{\nu}_\alpha \mathbf{e}(-\omega_\alpha \partial_\tau \omega_\alpha) \left(\frac{\vartheta'(0)}{\vartheta(\omega_\alpha)}\right)^2 .$$

Therefore, the initial data (4.1) stay unchanged and we leave with the two-dimensional phase space $\mathcal{R}^{(1)} \subset \mathcal{R}^1_{4,2}$. It is described by $\mathbf{S} = (S_1\,,S_2\,,S_3)$ with the linear $\mathrm{sl}(2,\mathbb{C})$ brackets and the Casimir function

$$c^2 = \sum_{\gamma=1,2,3} S^2_\gamma\,. \tag{4.2}$$

The equations of motion on $\mathcal{R}^{(1)}$ take the form of the non-autonomous Zhukovsky-Volterra gyrostat [10].

$$\partial_\tau S_\alpha = 2\imath \epsilon_{\alpha\beta\gamma} \left(S_\beta S_\gamma E_2(\check{\gamma}) + \nu'_\beta S_\gamma\right)\,. \tag{4.3}$$

Here $\vec{S} = (S_1\,,S_2\,,S_3)$ is the momentum vector, $\vec{J} = (E_2(\omega_2)\,,E_2(\omega_1+\omega_2)\,,E_2(\omega_1))$ is the inverse inertia vector, and $\vec{\nu}' = (\nu'_1\,,\nu'_2\,,\nu'_3)$ is the gyrostat momentum. This equation has the bihamiltonian structure based on the generalized Sklyanin algebra [10].

It was proved in [10] that there exists a transformation that allows to pass from the elliptic form of the Painlevé VI [11] to the non-autonomous Zhukovsky-Volterra gyrostat (4.3).

The Lax matrices can be read off from their representations for the ESS (2.38), (2.41)

$$L = -\frac{1}{2}\partial_w \ln \vartheta(w;\tau)\sigma_0 + \sum_\alpha (S_\alpha \varphi_\alpha(w) + \nu_\alpha \varphi_\alpha(w-\omega_\alpha))\sigma_\alpha\,.$$

$$M = -\frac{1}{2}\partial_\tau \ln \vartheta(w;\tau)\sigma_0 + \sum_\alpha -S_\alpha \frac{\varphi_1(w)\varphi_2(w)\varphi_3(w)}{\varphi_\alpha(w)}\sigma_\alpha + E_1(w)L'\,.$$

where $L' = \sum_\alpha (S_\alpha \varphi_\alpha(w) + \nu_\alpha \varphi_\alpha(w-\omega_\alpha))\sigma_\alpha$. The former matrix define the linear problem for (4.3) in the form (2.43).

5 Appendix

5.1 Appendix A. Elliptic functions.

We assume that $q = \exp 2\pi i\tau$, where τ is the modular parameter of the elliptic curve E_τ.

The basic element is the theta function:

$$\vartheta(z|\tau) = q^{\frac{1}{8}} \sum_{n\in\mathbf{Z}} (-1)^n \mathbf{e}(\frac{1}{2}n(n+1)\tau + nz) = \quad (\mathbf{e} = \exp 2\pi\imath) \tag{A.1}$$

The Eisenstein functions

$$E_1(z|\tau) = \partial_z \log\vartheta(z|\tau), \quad E_1(z|\tau) \sim \frac{1}{z} - 2\eta_1 z, \tag{A.2}$$

where

$$\eta_1(\tau) = \frac{24}{2\pi i}\frac{\eta'(\tau)}{\eta(\tau)}, \quad \eta(\tau) = q^{\frac{1}{24}} \prod_{n>0}(1-q^n). \tag{A.3}$$

is the Dedekind function.

$$E_2(z|\tau) = -\partial_z E_1(z|\tau) = \partial_z^2 \log\vartheta(z|\tau), \quad E_2(z|\tau) \sim \frac{1}{z^2} + 2\eta_1 . \tag{A.4}$$

Relation to the Weierstrass functions

$$\zeta(z,\tau) = E_1(z,\tau) + 2\eta_1(\tau)z, \quad \wp(z,\tau) = E_2(z,\tau) - 2\eta_1(\tau). \tag{A.5}$$

The highest Eisenstein functions

$$E_j(z) = \frac{(-1)^j}{(j-1)!}\partial^{(j-2)}E_2(z), \quad (j>2). \tag{A.6}$$

The next important function is

$$\phi(u,z) = \frac{\vartheta(u+z)\vartheta'(0)}{\vartheta(u)\vartheta(z)}. \tag{A.7}$$

$$\phi(u,z) = \phi(z,u), \quad \phi(-u,-z) = -\phi(u,z). \tag{A.8}$$

It has a pole at $z = 0$ and

$$\phi(u,z) = \frac{1}{z} + E_1(u) + \frac{z}{2}(E_1^2(u) - \wp(u)) + \dots . \tag{A.9}$$

$$\partial_u\phi(u,z) = \phi(u,z)(E_1(u+z) - E_1(u))|_{z\to 0} = -E_2(u). \tag{A.10}$$

Heat equation

$$\partial_\tau\phi(u,w) - \frac{1}{2\pi i}\partial_u\partial_w\phi(u,w) = 0. \tag{A.11}$$

Quasi-periodicity

$$\vartheta(z+1) = -\vartheta(z), \quad \vartheta(z+\tau) = -q^{-\frac{1}{2}}e^{-2\pi iz}\vartheta(z), \tag{A.12}$$

$$E_1(z+1) = E_1(z), \quad E_1(z+\tau) = E_1(z) - 2\pi i, \tag{A.13}$$

$$E_2(z+1)=E_2(z)\,,\quad E_2(z+\tau)=E_2(z)\,, \tag{A.14}$$

$$\phi(u,z+1)=\phi(u,z)\,,\quad \phi(u,z+\tau)=e^{-2\pi\imath u}\phi(u,z)\,. \tag{A.15}$$

$$\partial_u\phi(u,z+1)=\partial_u\phi(u,z)\,,\quad \partial_u\phi(u,z+\tau)=e^{-2\pi\imath u}\partial_u\phi(u,z)-2\pi\imath\phi(u,z)\,. \tag{A.16}$$

The Fay three-section formula:

$$\phi(u_1,z_1)\phi(u_2,z_2)-\phi(u_1+u_2,z_1)\phi(u_2,z_2-z_1)-\phi(u_1+u_2,z_2)\phi(u_1,z_1-z_2)=0\,. \tag{A.17}$$

Particular cases of this formula are the functional equations

$$\phi(u,z)\partial_v\phi(v,z)-\phi(v,z)\partial_u\phi(u,z)=(E_2(v)-E_2(u))\phi(u+v,z)\,, \tag{A.18}$$

$$\phi(u,z_1)\phi(-u,z_2)=\phi(u,z_2-z_1)(E_1(z_1)-E_1(z_2))-\partial_u\phi(u,z_2-z_1)\,. \tag{A.19}$$

$$\phi(u,z)\phi(-u,z)=E_2(z)-E_2(u)\,. \tag{A.20}$$

5.2 Appendix B. Lie algebra $\mathrm{sl}(N,\mathbb{C})$ and elliptic functions

Introduce the notation

$$\mathbf{e}_N(z)=\exp(\frac{2\pi i}{N}z)$$

and two matrices

$$Q=\mathrm{diag}(\mathbf{e}_N(1),\dots,\mathbf{e}_N(m),\dots,1) \tag{B.1}$$

$$\Lambda=\delta_{j,j+1}\,,\quad (j=1,\dots,N\,,\ mod\,N)\,. \tag{B.2}$$

Let

$$\mathbb{Z}_N^{(2)}=(\mathbb{Z}/N\mathbb{Z}\oplus\mathbb{Z}/N\mathbb{Z})\,,\quad \tilde{\mathbb{Z}}_N^{(2)})=\mathbb{Z}_N^{(2)}\setminus(0,0) \tag{B.3}$$

be the two-dimensional lattice of order N^2 and N^2-1 correspondingly. The matrices $Q^{a_1}\Lambda^{a_2}$, $a=(a_1,a_2)\in\mathbb{Z}_N^{(2)}$ generate a basis in the group $\mathrm{GL}(N,\mathbb{C})$, while $Q^{\alpha_1}\Lambda^{\alpha_2}$, $\alpha=(\alpha_1,\alpha_2)\in\tilde{\mathbb{Z}}_N^{(2)}$ generate a basis in the Lie algebra $\mathrm{sl}(N,\mathbb{C})$. More exactly, we introduce the following basis in $\mathrm{GL}(N,\mathbb{C})$. Consider the projective representation of $\mathbb{Z}_N^{(2)}$ in $\mathrm{GL}(N,\mathbb{C})$

$$a\to T_a=\frac{N}{2\pi i}\mathbf{e}_N(\frac{a_1a_2}{2})Q^{a_1}\Lambda^{a_2}\,, \tag{B.4}$$

$$T_aT_b=\frac{N}{2\pi i}\mathbf{e}_N(-\frac{a\times b}{2})T_{a+b}\,,\quad (a\times b=a_1b_2-a_2b_1) \tag{B.5}$$

Here $\frac{N}{2\pi i}\mathbf{e}_N(-\frac{a\times b}{2})$ is a non-trivial two-cocycle in $H^2(\mathbb{Z}_N^{(2)},\mathbb{Z}_{2N})$. The matrices T_α, $\alpha\in\tilde{\mathbb{Z}}_N^{(2)}$ generate a basis in $\mathrm{sl}(N,\mathbb{C})$. It follows from (B.5) that

$$[T_\alpha,T_\beta]=\mathbf{C}(\alpha,\beta)T_{\alpha+\beta}\,, \tag{B.6}$$

where $\mathbf{C}(\alpha,\beta)=\frac{N}{\pi}\sin\frac{\pi}{N}(\alpha\times\beta)$ are the structure constants of $\mathrm{sl}(N,\mathbb{C})$.

The Lie coalgebra $\mathfrak{g}^*=\mathrm{sl}(N,\mathbb{C})$ has the dual basis

$$\mathfrak{g}^*=\{\mathbf{S}=\sum_{\tilde{\mathbb{Z}}_N^{(2)}}S_\gamma t^\gamma\}\,,\quad t^\gamma=\frac{2\pi\imath}{N^2}T_{-\gamma}\,,\quad \langle T_\alpha t^\beta\rangle=\delta_\alpha^{-\beta}\,. \tag{B.7}$$

It follows from (B.6) that $\mathfrak{g}^*$ is a Poisson space with the linear brackets

$$\{S_\alpha,S_\beta\}=\mathbf{C}(\alpha,\beta)S_{\alpha+\beta}\,. \tag{B.8}$$

The coadjoint action in these basises takes the form

$$\mathrm{ad}^*_{T_\alpha} t^\beta = \mathbf{C}(\alpha,\beta) t^{\alpha+\beta}\,. \tag{B.9}$$

Let $\breve{\gamma} = \frac{\gamma_1+\gamma_2\tau}{N}$. Then introduce the following constants on $\tilde{\mathbb{Z}}^{(2)}$:

$$\vartheta(\breve{\gamma}) = \vartheta(\frac{\gamma_1+\gamma_2\tau}{N})\,, \quad E_1(\breve{\gamma}) = E_1(\frac{\gamma_1+\gamma_2\tau}{N})\,, \quad E_2(\breve{\gamma}) = E_2(\frac{\gamma_1+\gamma_2\tau}{N})\,, \tag{B.10}$$

$$\phi_\gamma(z) = \phi(\breve{\gamma}, z)\,, \tag{B.11}$$

$$\varphi_\gamma(z) = \mathbf{e}_N(\gamma_2 z)\phi_\gamma(z)\,, \tag{B.12}$$

$$f_\gamma(z) = \mathbf{e}_N(\gamma_2 z)\partial_u \phi(u,z)|_{u=\breve{\gamma}} = \varphi_\gamma(z)(E_1(\breve{\gamma}+z) - E_1(\breve{\gamma}))\,. \tag{B.13}$$

$$\mathbf{f}_{\alpha,\beta,\gamma} = E_1(\breve{\gamma}) - E_1(\breve{\alpha} - \breve{\beta} - \breve{\gamma}) + E_1(\breve{\alpha} - \breve{\gamma}) - E_1(\breve{\beta} - \breve{\gamma})\,. \tag{B.14}$$

It follows from (A.7) that

$$\varphi_\gamma(z+1) = \mathbf{e}_N(\gamma_2)\varphi_\gamma(z)\,, \quad \varphi_\gamma(z+\tau) = \mathbf{e}_N(-\gamma_1)\varphi_\gamma(z)\,. \tag{B.15}$$

$$f_\gamma(z+1) = \mathbf{e}_N(\gamma_2) f_\gamma(z)\,, \quad f_\gamma(z+\tau) = \mathbf{e}_N(-\gamma_1) f_\gamma(z) - 2\pi\imath\varphi_\gamma(z)\,. \tag{B.16}$$

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