The Geometry of some Kähler–Einstein manifolds

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0. Introduction

This paper is an analysis of the geometry of some compact, inhomogeneous Kähler-Einstein manifolds with $c_1(M) > 0$. More precisely, I look at those examples of Koiso and Sakane, which are $\mathbb{C}P^1$-bundles, describe their isometry groups, and add some more explicit examples of cohomogeneity one.

0.1. In chapter 1, I give a description of the metrics of Koiso and Sakane in the terminology of principal bundles and connections. Proofs are only indicated, but I hope that the reader may find this description easy to understand nevertheless. The proof of inhomogeneity depends on the basic curvature calculation at the end of this chapter.

0.2. It has turned out that the computation of (infinitesimal) isometries is easier to survey if you look at Kaluza-Klein metrics first. This is done in chapter 2.

0.3. In chapter 3 I prove that the cohomogeneity formula

$$\operatorname{coh}(\hat{P}) = \operatorname{coh}(M) + 1$$

holds without any further assumptions.

0.4. In chapter 4 I give some (perhaps new) cohomogeneity 1-examples in every even dimension ≥ 4. For completeness I include manifolds of higher cohomogeneity, which are essentially due to Koiso and Sakane.

0.5. There is an appendix with short proofs for well-known facts, e.g. for "Kählerform closed implies Kähler".

Some of the results stated here already appear in my Diplomarbeit written at the University of Bonn under the supervision of Prof. Hermann Karcher. I wrote this paper while being at the Max-Planck-Insititut für Mathematik in Bonn.

1. The metrics of Koiso and Sakane

1.1 Notation. Our notation is an extension of the one in [Bes, Chap. 2]. Let $L \xrightarrow{\pi} M$ be a Hermitian holomorphic line bundle over a Kähler-Einstein manifold with $c_1(M) > 0$. Hence we may assume the Einstein condition on M to be $\check{r} = \check{g}$. We denote the (real-valued) norm function on L by

$$\nu(l) = \sqrt{h(l,l)}\,.$$

The principal $\mathbb{C}^*$-bundle corresponding to L is just

$$P = L \setminus \{\text{zero-section}\},$$

the Chern connection is

$$\theta = d' \log \nu^2,$$

and its curvature (form) is given by

$$-d\theta = \Omega = \pi^* i\rho^\nabla,$$

where ρ^∇ is a form of type $(1,1)$ on M representing the Chern class of L up to the factor $\frac{1}{2\pi}$. We assume that the corresponding symmetric tensor field

$$B(u,v) = \rho^\nabla(u, Jv)$$

has constant eigenvalues in $]-1,1[$ w.r.t. $\check{g}$. To exclude the trivial bundle we assume that $B \neq 0$.

1.2. We are looking for "natural" metrics on P. In the situation at hand it is natural to consider the so-called *Kaluza-Klein metric*

$$g(u,v) = \pi^* \check{g}(u,v) + \langle \theta u, \theta v \rangle_{\mathbb{C}}.$$

The following facts are well-known:

- (a) $P \xrightarrow{\pi} M$ is a Riemannian submersion and the horizontal distributions for the connection and the submersion coincide; especially
- (a') $\mathcal{H} \perp \mathcal{V}$, where $\mathcal{V} = \ker d\pi\,, \mathcal{H} = \ker \theta$.
- (b) The θ-parallel displacement on P is isometrical, hence all fibres are isometric and totally geodesic, cf. [Vil, 3.5].
- (c) $\mathbb{C}^*$ acts isometrically on the right $P \times \mathbb{C}^* \to P$.

1.3. Unfortunately the Kaluza-Klein metric will never be Kähler, and by Myer's theorem no metric with $\check{r} = \check{g}$ on P will be complete. So we are interested in metrics that may be continuated to the compactification

$$\hat{P} = P \times_{\mathbb{C}^*} \hat{\mathbb{C}}.$$

Clearly, the isometry group will then be compact.

Claim. The metrics g on P satisfying

(a') *as above*

(b) *as above*

(c) $S^1 \subset \mathbb{C}^*$ *acts isometrically* $P \times S^1 \to P$

(d) g *is Hermitian*

(e) $d\omega = 0$.

are all given by

$$g = \pi^*\check{g} - U\pi^*B + U'\langle\theta\,,\theta\,\rangle.$$

Here $\check{g}$ is Kähler, $U: P \to \mathbb{R}$ is a function, $U' = d_{\hat{1}}U$ is its derivative , and $U' > 0$. $\hat{1}$ and $\hat{\imath}$ denote the fundamental fields generated by 1 and $i \in \mathbb{C}$, the Lie algebra of $\mathbb{C}^*$.

"Proof". Check the conditions in the order they are listed. When you check (e), compute $d\omega$ on vectors which are basic or fundamental. Then use:

Let X, Y be basic and U, V be fundamental. Then

$$[U,V] = [X,V] = 0$$
$$\theta[X,Y] = \Omega(X,Y), \quad \mathcal{H}[X,Y] \text{ is basic.}$$

□

Furthermore U satisfies

$$d_X U = d_{\hat{\imath}} U = 0 \qquad (\forall X \in \mathcal{H}) \tag{1.3}$$

and therefore depends only on the $\mathbb{R}^+$-factor in the decomposition

$$P = \nu^{-1}(1) \times \mathbb{R}^+$$

given by the norm function. Additional, $U|\nu^{-1}(1) = 0$. For functions with (1.3), differential equations involving $'$ become ODEs when you choose the coordinate $\log\nu$ on the $\mathbb{R}^+$-factor. Hence there is a function F with $F' = U$, and one can show that

$$\omega = \pi^*\check{\omega} + dd^c F.$$

1.4. The Ricci curvature is intimately related to the volume form μ, see [Bes, 2.100 f]. Here

$$\begin{aligned}\mu &= Q(U)\,\pi^*\check{\mu}\ \wedge\ U'(\operatorname{Re}\theta)\wedge(\operatorname{Im}\theta)\\ &= U'Q(U)\,\pi^*\check{\mu}\ \wedge\ \omega_{\mathbb{C}}(\theta\,,\theta\,),\end{aligned} \tag{1.4a}$$

where $Q(s)$ is the polynomial

$$Q(s)=\prod_{i=1}^{m}(1-\lambda_i s)\,,\qquad m=\dim_{\mathbb{C}} M,$$

i.e. Q(s) is just $\left(\det_{\mathbb{R}}(\check{g}-sB)\right)^{1/2}$ w.r.t. $\check{g}$.

The Einstein condition ($r=g$) reduces to

$$\check{r}=\check{g}\quad\text{and}\quad 2U+\left(\log U'Q(u)\right)'=0. \tag{1.4b}$$

This ODE implies

$$U|\nu^{-1}(1)=-1,\qquad U|\nu^{-1}(1)=1$$

$$\int_{-1}^{1} sQ(s)\,ds=0.$$

The ODE (1.4b) may be integrated once to obtain

$$U'=\left(1-U^2\right)\psi(u)\quad\text{with}\quad \psi(s)=\frac{-2\int_{-1}^{s}\tau Q(\tau)\,d\tau}{(1-s^2)\,Q(s)}$$

1.5. This and the function

$$V:=\frac{\nu^2-1}{\nu^2+1}=\frac{1-1/\nu^2}{1+1/\nu^2}$$

as a better coordinate for the $\mathbb{R}^+$-factor helps to show the regularity of g near the "poles" $\nu^{-1}(0)$ and $\nu^{-1}(\infty)$.

Sketch of the proof. The proof can be divided into several steps:

1) V satisfies $V'=1-V^2$.
2) $g_V=\pi^*\check{g}-V\pi^*B+V'(\theta\,,\theta\,)$ does exist on $\hat{P}$ and is regular.
3) Write $U=:u\circ V$. Then $u\colon[-1,1]\to[-1,1]$ is regular.
4) $\omega_U=\omega_V+dd^c(F\circ V)$ for $F(s)=\int_{-1}^{s}\frac{u(t)-t}{1-t^2}dt$ □

g_v induces the standard metric on the fibres $\hat{\mathbb{C}}=S^2$, with V as the standard height function. Using this V and its volume form similar to (1.4a) and performing the product integration, you see that the condition $\int_{-1}^{1} sQ(s)\,ds=0$ is equivalent to the vanishing of Futaki's functional.

1.6. The function U is a moment map for the S^1-action (i.e. $\hat{\imath} = J \operatorname{grad} U$), and hence $\Delta U = 2U$, [Lic, Cor. in sec. 96], [Bes, 11.52]. Indeed, under conditions (a') to (e) in 1.3, (1.4b) is equivalent to

$$\check{r} = \check{g} \quad \text{and} \quad \Delta U = 2U.$$

This equation also helps you to find unit vectors $X \in \mathcal{H}$, $W \in \mathcal{V}$ on $\nu^{-1}(0)$, such that $K(X, W) < 0$. Thus none of these metrics has positive sectional curvature.

By direct computation, the curvature of the fibres is given by

$$K\left(\hat{\imath}, \hat{\imath}\right) = \left(-\frac{1}{2}\frac{U''}{U'}\right)' \Big/ U',$$

which is never constant. Indeed, if it were constant, curvature would be 1 by Gauß-Bonnet. Then you might {multiply by U' and integrate} twice to obtain

$$U^2 + U' = c_1 U + c_2.$$

But the constants may be determined by the values of U and U' on $\nu^{-1}(0)$ and $\nu^{-1}(\infty)$. We obtain

$$U' = 1 - U^2, \qquad \text{i.e.} \quad \psi \equiv 1.$$

Hence $Q \equiv 1$ and $B \equiv 0$. But this was excluded from the very beginning. Hence the fibres are inhomogeneous. Looking at the principal orbits, we see that the height function U must be preserved by isometries (because otherwise we could use the θ-parallel displacement and the S^1-action to construct 2-dimensional orbits in the fibres, but these have cohomogeneity at most one).

2. Isometries of torus bundles

In chapter 3 we will describe the orbits under the connected isometry group $I^0(\hat{P})$ in terms of $I^0(M)$. Remember that the cohomogeneity of a Riemannian manifold may be defined as

$$\operatorname{coh}(M) = \inf_{p \in M} \operatorname{codim} I^0(M) \cdot p,$$

and similarly for other Lie group operations on a differentiable manifold. The Lie algebra of the isometry group $I(M)$ is the Lie algebra $i(M)$ of Killing fields, which are characterized by $L_X g = 0$. Let us first consider the simpler case of a Kaluza-Klein metric on a principal torus bundle.

2.1. Let

$(M, \check{g})$ be a Riemannian manifold with $H^1(M, \mathbb{R}) = 0$,
$P \longrightarrow M$ be a principal torus bundle with fibre T,
$\langle\ ,\ \rangle_{\mathbf{t}}$ be a metric on T (biinvariant, of course),
θ be a principal connection with curvature form
$\Omega = -d\theta = \pi^* \eta$ (such an η does exist),
η be harmonic.

These manifolds occur "in real life", cf. [Wa-Zi]. Consider the Kaluza-Klein metric

$$g = \pi^* \check{g} + \langle \theta\ , \theta \rangle_{\mathbf{t}}.$$

We are interested in the closed subgroup $F \subset I^0(\hat{P})$ of isometries mapping fibres to fibres and its Lie algebra $\mathbf{f}$ of projectable Killing fields.

2.2 Properties. Let X be a projectable Killing field, and $\check{X}$ its projection. Then

$$0 = L_X g = \pi^* L_{\check{X}} \check{g} + \langle L_X \theta\ , \theta \rangle_{\mathbf{t}} + \langle \theta\ , L_X \theta \rangle_{\mathbf{t}}. \tag{2.2}$$

The horizontal part $\mathcal{H}X$ is described by $\check{X}$, and the vertical part $\mathcal{V}X$ may be described by the function $\theta X \colon P \to \mathbf{t}$. Since $\mathcal{V}X$ is Killing along each fibre (see 5.2) and $i(T) \equiv \mathbf{t}$,

(0) θX is constant along each fibre.
(1) Evaluation of (2.2) on 2 horizontal vectors yields $L_{\check{X}} \check{g} = 0$, i.e. $\check{X}$ is Killing.
(2) Evaluation of $Y \in \mathcal{H}$, $V \in \mathcal{V}$ yields $L_X \theta(Y) = 0$, i.e. $L_X \theta$ is vertical.
(3) But $L_X \theta = i_X d\theta + d i_X \theta = -i_X \Omega + d\theta X$ is also horizontal by (0).

Thus X is projectable Killing iff $L_{\check{X}} \check{g} = 0$ and $L_X \theta = 0$.

2.3 Existence. Let $\check{X}$ be a Killing field on M. Find a lift X with

$$L_X\theta = -i_X\Omega + d\,\theta X = 0.$$

By (0), there must be a function $f\colon M \to \mathbf{t}$ with $\theta X = \pi^* f$. So our condition becomes

$$\pi^*\left(i_{\check{X}}\eta\right) = i_X\Omega = d\,\theta X = \pi^* df.$$

Since η is harmonic, $di_{\check{X}}\eta = L_{\check{X}}\eta = 0$, so there is an f with $i_{\check{X}}\eta = df$. Once f is chosen, there is a unique lift X with $\theta X = \pi^* f$. This is the desired Killing field.

2.4 Further properties. $\check{X} = 0$ implies $f = \text{const}$, hence $\ker\pi = \mathbf{t}$. The Killing fields X with

$$\int_M f = 0 \quad \left(\iff \int_P \theta X = 0\right)$$

form another ideal of $\mathbf{f}$, which is clearly complementary to $\mathbf{t}$:

Let Y be such a field. Then

$$L_X\theta Y - \theta[X,Y] = L_X\theta \cdot Y = 0$$

and

$$\int_P \theta[X,Y] = \int_P L_X\theta Y = \int_P \theta Y = 0$$

by the very definition of the Lie derivative. In this sense, $\mathbf{f} = \mathbf{t} \oplus i(M)$, and this splitting is compatible with the biinvariant metric $\int_P \mathbf{g}(X,Y)$ on $i(P)$. □

Remark. Indeed, for any compact Lie group with biinvariant metric

$$\mathbf{g} = \mathbf{z} \oplus \mathbf{z}^\perp = \mathbf{z} \oplus [\mathbf{g},\mathbf{g}],$$

where $\mathbf{z}$ is the centre of $\mathbf{g}$. [Bes, 11.53], [Hel, p. 132].

2.5 F in terms of $T \hookrightarrow I^0(M)$. For any Riemannian submersion, $P \xrightarrow{\pi} M$ induces a homomorphism

$$\pi\colon F \to I^0(M), \qquad \text{given by} \quad (\pi a)(\pi p) = \pi(ap).$$

Thus $\ker\pi$ is a normal subgroup of F, and so is $T = (\ker\pi)^0$. So we define a homomorphism

$$F \to \operatorname{Aut}(T) \qquad \text{by}$$

$$\phi \mapsto \left(a \mapsto \phi a \phi^{-1}\right).$$

Observe that F is compact and $\operatorname{Aut}(T)$ is discrete. By the latter,

$$\phi \in F^0 \quad \Longrightarrow \quad \phi \in I^0(M) \ \text{ and } \ \phi a = a\phi \ \forall a \in I^0(P).$$

Thus

F^0 is the centralizer of T in $I^0(P)$,

since the latter is connected (cf. [Hel, p.287] or [Bes, 8.19]).

Remark. *F is the normalizer of T in $I^0(P)$.*

2.6 Example. Let $T \to P \to M$ be the Hopf fibration $S^1 \to S^{2m+1} \to \mathbb{C}P^m$, with the standard metrics. Then $I^0(M) = PSU(m+1)$. Choose $\theta = \langle ip, \rangle$. Then η is a multiple of the Kähler form and $I^0(P) = SO(2m+2)$.

Looking at $(a \mapsto \phi a \phi^{-1})$ with $a = i$ we see that

$$F^0 = \{\, \phi \mid \phi a = a\phi \,\} = U(m+1), \qquad \text{and}$$

F is generated by F^0 and complex conjugation.

Let $A \in u(m+1)$, and $X(p) = Ap$ be the corresponding Killing field. Then

$$\int_P \theta X = -\int_{S^{2m+1}} \langle Aip, p \rangle = -\operatorname{vol}(S^{2m+1}) \operatorname{trace}_{\mathbb{R}} Ai = 0$$

$$\iff \qquad A \in su(m+1).$$

2.7 Example. Same situation as above, but now choose $\langle u, v \rangle_{\mathbf{t}} = c^2 \cdot uv$ with $0 < c < 1$, obtaining a Berger sphere. If c is small enough, the fibres are shortest simply closed geodesics [Wa-Zi, chap.3], and are therefore mapped to fibres. Thus,

$$I^0(P) = F = U(m+1).$$

(See also [Bes, 7.13].)

2.8 Moral. Example 2.6 shows that $F \neq I^0(P)$ in general. Furthermore F^0 and $I^0(M) \times T$ are only related up to the finite coverings

$$SU(m+1) \times S^1 \longrightarrow G \longrightarrow PSU(m+1) \times S^1,$$

where G stands for $U(m+1)$ resp. $PSU(m+1) \times S^1$, and the arrows denote the obvious covering maps.

2.9 A cohomogeneity formula. Still the knowledge of Killing fields is sufficient to calculate the orbits of F^0: The isometries in question are members of one-parameter subgroups, and these are projected and lifted by

$$\mathbf{f} \xrightarrow{\pi} i(M) \hookrightarrow i(M) \oplus \mathbf{t} \xrightarrow{\cong} \mathbf{f}.$$

The projection $F^0 \xrightarrow{\pi} I^0(M)$ is well defined by $(\pi a)(\pi p) = \pi(ap)$, whereas lifting $I^0(M) \to F^0$ involves a choice of the one-parameter subgroup through $\check{a}$. But then, if $\check{a}\pi p = \check{q}$, we obtain at least $ap \in P_{\check{q}}$, and with the torus action on the fibre we obtain that orbits are just

$$F^0 \cdot p = P_{I^0(M) \cdot \pi p}.$$

If we denote the cohomogeneity for an "arbitrary" group action $G \times M \to M$ by

$$\operatorname{coh}(M, G) := \inf_{p \in M} \operatorname{codim} G \cdot p,$$

we finally obtain

$$\operatorname{coh}(P, F^0) = \operatorname{coh}(M). \tag{2.9}$$

2.10. For trivial bundles the situation is much easier, even in a more general setting:

Theorem. Let M, N be compact Riemannian manifolds. Then the homomorphism

$$\times : I^0(M) \times I^0(N) \longrightarrow I^0(M \times N), \quad (a,b) \mapsto a \times b$$

is an isomorphism.

Proof. π_1 and π_2 are Riemannian submersions. First we show that every Killing field X is projectable under $\pi_1 : M \times N \to M$. Let c be a geodesic in the fibre $\{m\} \times N$. We have to show that $d\pi_1 \, X \circ c$ is constant. Let $u \in T_m M$ be fixed and look at the vector field

$$\tilde{u}(m,n) = (u,0) \qquad \text{on} \quad \{m\} \times N.$$

Clearly

$$d\pi_1 \, \tilde{u} = u, \qquad d\pi_2 \, \tilde{u} = 0, \qquad \tilde{u} \circ c \text{ is parallel.}$$

Consider the function

$$h(s) = \langle d\pi_1 \, X \circ c, u \rangle = \langle d\pi_1 \, X \circ c, d\pi_1 \, \tilde{u} \circ c \rangle = \langle X \circ c, \tilde{u} \circ c \rangle.$$

Since $X \circ c$ is a Jacobi field, and $\tilde{u} \circ c$ is parallel,

$$h''(s) = \langle R_{c'} \, X \circ c, \tilde{u} \circ c \rangle = 0$$

by the decomposition of the curvature tensor ($d\pi_2 \, c' = 0$, $d\pi_2 \, \tilde{u} = 0$). Hence $h(s) = as + b$, but since $|h(s)| \leq \|X\|_\infty |u|$ for all s, h is constant. This shows

$$I^0(M \times N) = F_1 = F_1 \cap F_2,$$

where F_1 and F_2 are now the subgroups of $I^0(M \times N)$ which preserve the fibres of the submersions π_1 and π_2. Hence the homomorphism

$$(\pi_1, \pi_2) : F_1 \cap F_2 = I^0(M \times N) \longrightarrow I^0(M) \times I^0(N)$$

is defined, and clearly inverse to $\times$. □

Remark. This shows $I^0(T) = T$, which has been used above, for the product metric. For any other biinvariant metric the isotropy group of any point must preserve the harmonic forms dx_i, and the result follows again.

Remark. A similar result holds if the manifolds are complete and simply connected. Then effects like $SO(2) \subset I^0(\mathbb{R}^2)$ make the formulation slightly more complicated. See [Ko-No 1, p. 240].

2.11 How exceptional is $F \neq I^0(P)$? Example 2.7 tells us, that $F \neq I^0(P)$ should only happen in very special situations. E.g. in example 2.6 $I^0(P)/F$ is a symmetric space (of type D III, see [Bes, p.312]). Additional, if $I^0(P)$ is semisimple, $I^0(P)/F$ is homogeneous Kählerian and algebraic [Bor, Thm 2].

But this knowledge does not help too much unless $I^0(P)$ is known. On S^3, $\mathbf{m} := \mathbf{f}^\perp$ consists of horizontal Killing fields. With our simple minded methods we can only show, that these make up an $\mathrm{ad}\,\mathbf{f}$-invariant subspace of $i(P)$:

Theorem. Let $\mathbf{m}$ *be the space of horizontal Killing fields. Then*

(1)	$[i(P),\mathbf{t}] \subset \mathbf{m}$,	$i(P) \neq \mathbf{f} \Longrightarrow \mathbf{m} \neq \mathbf{0}$
(2)	$[\mathbf{f},\mathbf{m}] \subset \mathbf{m}$,	$\mathbf{t} \perp \mathbf{m}$

If η is nondegenerate,

(3) $\mathbf{f} \cap \mathbf{m} = \{\mathbf{0}\}$

Proof. As before, $L_X\theta = -i_X\Omega + d\,\theta X$ is horizontal even for $X \in i(P)$. Therefore, if $U \in \mathbf{t}$,

$$0 = (L_X\theta)\cdot U = \underbrace{L_X\,\theta U}_{0} - \theta[X,U].$$

This shows the first part of (1). If $X \notin \mathbf{f} =$ centralizer of $\mathbf{t}$, choose an $u \in \mathbf{t}$ with $[X,U] \neq 0$.

In order to prove (2), let $X \in \mathbf{f}$, $Y \in \mathbf{m}$. Then

$$\theta[X,Y] = L_X(\theta Y) = 0.$$

Now let $X \in \mathbf{f} \cap \mathbf{m}$, Y be a basic field. Then

$$0 = (L_X\theta)\cdot Y = \theta[X,Y] = \Omega(X,Y) = \eta\,(\check{X},\check{Y})\,,$$

which proves (3). □

3. Isometries of certain projective bundles

Now we come back to the metrics of Koiso and Sakane. We shall see that the isometry group is simpler than in the case of Kaluza-Klein metrics, although the metric is more complicated.

Observe that $H^1(M,\mathbb{R}) = 0$ holds (easy, cf. [Bes, 6.56]), and even that $\pi_1(M) = 1$ (difficult, cf. [Bes, 11.26]).

3.1 $i(\hat{P}) = \mathbf{f}$. Due to the complex structure, we can show that $i(\hat{P}) = \mathbf{f}$. Indeed, let X be an arbitrary Killing field on $\hat{P}$. ω is harmonic, hence

$$0 = L_X\omega = (L_X g)(J\ ,\ ) + g(L_X J\ ,\ ),$$

which implies $L_X J = 0$, i.e. X is holomorphic. Thus every isometry $a \in I^0(\hat{P})$ is holomorphic. Using the topology of $I^0(\hat{P})$ and Liouville's theorem for the holomorphic map

$$\underbrace{\hat{P}_x}_{\text{compact}} \xrightarrow{a} \underbrace{\hat{P}_U}_{\text{open}} \xrightarrow{\pi} U \overset{\text{chart}}{\longleftrightarrow} \mathbb{C}^m$$

you see that F is also open in $I^0(\hat{P})$. This proof is due to Blanchard [Bla]. $\square$

3.2 Remark. Since fibres are mapped to fibres by $I^0(\hat{P})$, $\hat{P}$ is inhomogeneous. This might have been observed earlier, since the fibres are totally geodesic [Ko-No 2, p.60].

3.3 Properties. We consider the metric

$$g = \pi^* g - U\pi^* B + U'\langle\theta\ ,\theta\ \rangle_{\mathbb{C}}$$

Let X be a (projectable) Killing field. By 1.6 we know that $L_X U = L_X U' = 0$. Thus

$$\textbf{(3.3)}\qquad 0 = L_X g = \pi^* L_{\check{X}}\check{g} - U\pi^* L_{\check{X}} B + U'\big(\langle L_X\theta\ ,\theta\ \rangle + \langle\theta\ , L_X\theta\ \rangle\big).$$

But furthermore, if we write $\mathcal{V}X$ as a linear combination of $\hat{\mathbf{1}}$ and $\hat{\mathbf{i}}$, the coefficient of $\hat{\mathbf{1}}$ is zero. Since $\mathcal{V}X$ and $\hat{\mathbf{i}}$ are Killing fields along the fibres, $\mathcal{V}X$ is a *constant* multiple of $\hat{\mathbf{i}}$. Thus

(0) θX is constant along each fibre and in $i\mathbb{R}$.

(1) Lifting two vectors $\check{Y}$, $\check{Z}$ horizontally to $\nu^{-1}(0)$ and $\nu^{-1}(\infty)$, evaluating and solving for $L_{\check{X}}\check{g}$ and $L_{\check{X}}B$ (remember that U takes values ± 1, $U' = 0$ there) we find

$$L_{\check{X}}\check{g} = L_{\check{X}}B = 0, \qquad \text{i.e. } \check{X} \text{ is a Killing field that preserves } B.$$

(2) and ...

(3) are as before (on $P \subset \hat{P}$, where $U' > 0$).

Thus X is projectable Killing iff

$$L_{\check{X}}\check{g} = 0, \quad L_{\check{X}}B = 0, \quad L_X\theta = 0, \quad \text{and } L_XU = L_XU' = 0.$$

3.4 Existence. Let $\check{X}$ be a Killing field on M. ρ^∇ has constant trace and thus is harmonic [Bes, 2.33]. Hence

$$L_{\check{X}}B = \underbrace{L_{\check{X}}\rho^\nabla}_{0}(\ ,J\,) + \rho^\nabla(\ ,L_{\check{X}}J\,) = 0.$$

Again we find a function $f\colon M \to i\mathbb{R}$ with $i_{\check{X}}\eta = df$. (Here $\eta = i\rho^\nabla$, hence again $\Omega = \pi^*\eta$.) Once f is chosen, there is a unique lift X with $\theta X = \pi^* f$. This is the desired Killing field.

3.5 Further properties. $\check{X} = 0$ implies $f = \text{const}$, hence $\ker\pi = \mathbb{R}\hat{\mathbf{i}}$.

Let μ and $\check{\mu}$ denote the volume forms. By (1.4a),

$$\mu = U'Q(U)\,\pi^*\check{\mu} \;\wedge\; (\operatorname{Re}\theta)\wedge(\operatorname{Im}\theta).$$

Again, product integration of

$$\theta X\,\mu = U'Q(U)\,\pi^*(f\check{\mu}) \;\wedge\; (\operatorname{Re}\theta)\wedge(\operatorname{Im}\theta)$$

yields that

$$\int_M f = 0 \quad\Longleftrightarrow\quad \int_P \theta X = 0.$$

The Killing fields X with $\int_P \theta X = 0$ form a complementary ideal to $\mathbf{t}$ as before. In this sense, $\mathbf{f} = i\mathbb{R} \oplus i(M)$, and this splitting is compatible with the biinvariant metric $\int_P \mathbf{g}(X,Y)$ on $i(\hat{P})$. □

Remark. Obviously, $\mathbb{R}$ is the center of $\mathbf{f} = i(\hat{P})$, so $i(\hat{P})$ cannot be semisimple. Thus, the claim in [Bes, 11.55] is false. But [Bes, 8.88] again shows that $\hat{P}$ is inhomogeneous.

3.6 Digression. Let $\check{X}$ be a Killing field. Because of $d\,i_{\check{X}}\check{\omega} = L_{\check{X}}\check{\omega} = 0$, there is a so-called "moment map" $\mu_{\check{X}}$ satisfying $d\mu_{\check{X}} = i_{\check{X}}\omega$. $\check{X}$ may be recovered by $\mu_{\check{X}}$, since ω is nondegenerate. The moment map for the lifted Killing field X is just

$$\mu_X = \pi^*\mu_{\check{X}} - U\pi^*(\operatorname{Im} f)\,,$$

where $d(\operatorname{Im} f) = i_{\check{X}}\rho^\nabla$ is as before.

3.7 The cohomogeneity formula. Since $F \cdot p \subset \nu^{-1}(\nu(p))$, we look at $\nu^{-1}(t)$. This is either $\cong M$, or a S^1-bundle with Kaluza-Klein metric. By the same proof as in 2.9 we obtain

$$\operatorname{coh}(\hat{P}) = \operatorname{coh}(M) + 1.$$

3.8 Infinitesimal automorphisms. Let $a(M)$ denote the Lie algebra of (real) holomorphic vector fields. Koiso and Sakane use the well-known relationship [Bes, 11.52]

$$a(M) = i(M) \oplus Ji(M) \qquad \text{(valid for Kähler manifolds with } \rho = \omega)$$

and some complex analysis to prove the cohomogeneity formula under further assumptions (e.g., that M is a product manifold), see [Ko-Sa 2, prop. 6.3].

The other way round, the result of Matsushima quoted above and our description of $i(\hat{P})$ yield

$$a(\hat{P}) = a(M) \oplus \mathbb{R}\hat{\mathbf{1}} \oplus \mathbb{R}\hat{\mathbf{i}}.$$

4. Examples

In order to give an example one "only" has to give a Hermitian holomorphic line bundle over a Kähler-Einstein manifold (with $\tilde{\rho} = \tilde{\omega}$), such that

(a) the eigenvalues λ_i of the curvature form are constant in $]-1, 1[$ and

(b) $\int_{-1}^{1} s \prod_{i=1}^{m} (1 - \lambda_i s)\, ds = 0$.

4.1. If (b) does not hold, the following trick will work: Look at

$$M \times M \xrightarrow{\pi_i} M, \qquad L_1 = \pi_1^* L, \qquad L_2 = \pi_2^* L^* = \pi_2^* \left(L^{-1}\right).$$

Then

L_1	has Eigenvalues	$\lambda_1, \ldots, \lambda_m, \; 0 \;, \ldots, \; 0$
L_2	has Eigenvalues	$0 \;, \ldots, \; 0 \;, -\lambda_1, \ldots, -\lambda_m$
$L_1 \otimes L_2$	has Eigenvalues	$\lambda_1, \ldots, \lambda_m, -\lambda_1, \ldots, -\lambda_m.$

Hence (b) is satisfied.

Since the tautological bundle $\tau \longrightarrow \mathbb{C}P^m$ has all eigenvalues equal to $-\frac{1}{m+1}$, $L = \tau^k$ gives examples for $0 < k < m + 1$. These were the first examples of Koiso and Sakane.

4.2 Even-dimensional examples. Look at the special Milnor hypersurfaces

$$H := H_{mm} := \left\{ \, [z_0 : \ldots : z_m], [w_0 : \ldots : w_m] \mid \sum z_i w_i = 0 \right\} \subset \mathbb{C}P^m \times \mathbb{C}P^m.$$

for $m \geq 2$. If $(\;,\;)$ denotes the Hermitian form on $\mathbb{C}^{m+1}$, the equation may also be written as $(z, \overline{w}) = 0$. $SU(m+1)$ acts transitively on H by

$$A \cdot (z, w) = \left(Az, \overline{A\overline{w}}\right),$$

since it does on the Stiefel manifold of unitary 2-frames.

Although compact homogeneous Kähler manifolds are well-known as abstract manifolds [Bes, chap. 8], and $\mathbb{C}z \subset \ker(\;, \overline{w})$ clearly describes a flag manifold, we continue with the description of this embedding. We will show that the induced metric is an Einstein metric, and that the bundles $L := L_1 \otimes L_2$ still yield examples when restricted to H.

I apologize for introducing local coordinates.

$$\begin{aligned} \mathbb{C}^m &\to \mathbb{C}P^m \\ p &\mapsto [1 : p_0 : \ldots : p_m] \end{aligned}$$

is a coordinate chart for $\mathbb{C}P^m$. The standard metric (with curvature between 1 and 4) is then given by

$$g_p(u,v) = \mathrm{Re}\left(\frac{1}{l(p)}(u,v) - \frac{1}{l^2(p)}(u,p)\overline{(v,p)}\right)$$

with $l(p) := 1 + |p|^2$. The Christoffel symbols are given by

$$\Gamma_p(u,v) = -\frac{1}{l(p)}((u,p)v + (v,p)u)$$

(cf. (5.1a)). In the product coordinate system of $\mathbb{C}P^m \times \mathbb{C}P^m$

$$(p,q) \longmapsto ([1:p_0:\ldots:p_m],[1:q_0:\ldots:q_m])$$

the Milnor Hypersurface H is given by the equation

$$1 + (p,q) = 0.$$

Let f be the real part of this function. Then — at least on H and TH —

$$\begin{aligned} \operatorname{grad} f(p,q) &= (\quad l(p)(-p+\overline{q}) \quad , \quad l(q)(-q+\overline{p})) \\ D_{(u,v)} \operatorname{grad} f(p,q) &= (\overline{(u,p)}(-p+\overline{q}) + l(p)\overline{v}, \quad \ldots \quad) \end{aligned}$$

Now let us describe the embedding in terms of the unit normal vector field

$$N := \operatorname{grad} f / |\operatorname{grad} f|$$

and the shape operator

$$(u,v) \longmapsto \mathcal{V}\, D_{(u,v)} N,$$

where $\mathcal{V}$ denotes the orthogonal projection onto TH. When computing $|\operatorname{grad} f|$, observe that *each component of* $\operatorname{grad} f$ *has*

$$|(\operatorname{grad} f)_i|^2 = l(p)l(q).$$

Since H is homogeneous, it is sufficient to calculate the Ricci tensor at one point, e.g. at

$$(p,q) = (e,-e) \qquad \text{with} \quad e = (1,0,\ldots,0).$$

Now, for any complex hypersurface M in a Kähler manifold $\tilde{M}$ the Ricci endomorphism is

$$r = \tilde{r} + \mathcal{V}\,\tilde{R}(N,JN)J - 2S^2,$$

where each endomorphism above is symmetric and does not depend on the choice of N.(Confer 5.5f.)

Now let $\mathbb{C}P^m \times \mathbb{C}P^m$ be denoted by $\tilde{M}$. With the above information about $S = \mathcal{V}DN$ and the well-known curvature tensor of $\mathbb{C}P^m$ [e.g. Kar, 6.4.2] you may verify the following eigenvectors and eigenvalues:

eigenvector	eigenvalue of $-2S^2$	eigenvalue of $R_{\tilde{M}}(N, JN)J$
(e, e)	0	-2
(ie, ie)	0	-2
$\{ u_1 = v_1 = 0 \}$	-1	-1

Since $r_{\tilde{M}}$ has eigenvalues $2m + 2$, we see that

$$r_H = 2m\, g = \tfrac{2m}{2m+2}\, r_{\tilde{M}}.$$

Now consider the bundle $L|H$, where

$$L = \pi_1^* \tau^{-k} \otimes \pi_2 \tau^k.$$

The curvature of L on $\tilde{M}$ is given by

$$B(u, v) = g\big(\alpha \times (-\alpha)\, u, v\big)\ .$$

Here $\alpha = k/(m + 1)$ is a number between 0 and 1, and $\alpha \times (-\alpha)$ denotes the endomorphism field

$$\alpha \times (-\alpha)(u) = \alpha \times (-\alpha)(u_1, u_2) = (\alpha u_1, -\alpha u_2).$$

Since B_H is just B restricted to TH, the calculation of its eigenvalues becomes a problem of linear algebra: The endomorphism associated to B_H is just

$$\mathcal{V}(\alpha \times (-\alpha))|TH.$$

Since the components of the unit normal vector $N = (N_1, N_2)$ *have equal length*, $(N_1, -N_2) \in TH$. Therefore B_H has the following eigenspaces and eigenvalues:

eigenspace	$\dim_{\mathbb{C}}$	eigenvalue
$\mathbb{C}\,(N_1, N_2)$	1	0
$\{ (u, v) \in TH \mid v = 0 \}$	$m - 1$	α
$\{ (u, v) \in TH \mid u = 0 \}$	$m - 1$	$-\alpha$

Remember that these eigenvalues are w.r.t. the metric $r_{\tilde{M}}$. Hence B_H has eigenvalues $\pm k/m$ w.r.t. the metric r_H. Thus we obtain new examples, if $0 < k < m$.

Remark. H is irreducible by theorems of Hano and Matsushima [Ha-Ma, Thm 4, Thm 5]. Therefore these even-dimensional examples are also irreducible [Ko-Sa 1, 5.6].

4.3 Further examples. Koiso and Sakane developed a method to construct suitable bundles over homogeneous Kähler manifolds [Ko-Sa 2]. I did not check whether the above examples are easier understood in their description.

4.4 Higher cohomogeneity. Since a large part of this paper is devoted to the calculation of cohomogeneity, there should be given some examples with large cohomogeneity. There are two ideas:

(a) Let M_1 be of cohomogeneity 1. Then $\underbrace{M_1 \times \ldots \times M_1}_{d}$ is of cohomogeneity d by 2.10.

(b) Iteration gives $M_d \xrightarrow{\mathbb{C}P^1} \ldots \xrightarrow{\mathbb{C}P^1} M_1 \xrightarrow{\mathbb{C}P^1} M_0$, where M_i is of cohomogeneity i.

Unfortunately, (a) is too silly, while (2) cannot be carried out, since you loose control over the holomorphic vector bundles over the more and more complicated M_i.

Fortunately, at least the tautological bundle of a projective bundle is well understood. Take for example $M_1 = \hat{P} \to \mathbb{C}P^m \times \mathbb{C}P^m$, where L is as in 4.1 with $0 < k < m+1$. Let $E = 1 \oplus L$. Then

$$\begin{aligned} M_1 &= P \times_{\mathbb{C}^*} \mathbb{C}P^1 \\ &= P \times_{\mathbb{C}^*} P(\mathbb{C} \oplus \mathbb{C}) = P\big(P \times_{\mathbb{C}^*} (\mathbb{C} \oplus \mathbb{C})\big) \\ &= P(E) \end{aligned}$$

is a projective bundle, and

$$K_{M_1} = \pi^* \left(K_M \otimes \det(E^*)\right) \otimes \tau_E^{-2}$$

by the dualization of [Gri, (2.38)]. (*Remark*: $-2 = -\operatorname{rank} E$, check the correctness of the sign by the assumption $M = \{\mathrm{p}\}$.)

Let $g = c_1(\tau^{-1}) \in H^2(\mathbb{C}P^m, \mathbb{Z})$ be the generator. Now

$$c_1(M_1) = c_1\left(K_{M_1}\right) = \pi^*(c_1(M) + c_1(E^*)) - 2c_1\left(\tau_E\right).$$

Since M and E^* are direct sums, by standard calculations,

$$c_1\left(M_1\right) = \pi^*\Big(\pi_1^*((m+1-k)g) + \pi_2^*((m+1+k)g)\Big) - 2c_1\left(\tau_E\right).$$

Thus, for $0 < m+1-k$ even, the bundle

$$L_1 := \pi^*\left(\pi_1^* H^{(m+1-k)/2} \otimes \pi_2^* H^{(m+1+k)/2}\right) \otimes \tau_E^{-1}$$

has $\left[\rho^\nabla\right] = \frac{1}{2}\left[\rho_{M_1}\right]$ on the cohomology level. But, by [Bes, 2.110], you can modify a chosen fibre metric h on L_1 to obtain also $\rho^\nabla = \frac{1}{2}\rho_{M_1}$, as follows:

if $\rho^\nabla(h) - \frac{1}{2}\rho = dd^c\Phi$, **then** $h_1 := e^{2\Phi}h$ **has** $\rho^\nabla(h_1) = \frac{1}{2}\rho$.

These building blocks are due to Koiso and Sakane and may be used to construct manifolds of arbitrary cohomogeneity (with large dimensions, of course).

Example. Let

$$\begin{aligned} M_2 &= M_1 \times \ldots \times M_1, \qquad (l \text{ copies}) \\ L_2 &= \pi_1^* L_1 \otimes \ldots \otimes \pi_l^* L_1, \end{aligned}$$

and perform the trick described in 4.1. Then $\operatorname{coh}(\hat{P}) = 2l + 1$.

Example. Build up your example from

$$\begin{aligned} L_1 &\to M_1 && l \text{ times} \quad (l \geq 1) \\ L_1^{-1} &\to M_1 && (l-1) \text{ times} \\ T^{(\dim M_1 + 1)/2} &\to \mathbb{C}P^{\dim M_1} && 1 \text{ times} \end{aligned}$$

to obtain each eigenvalue $\pm 1/2$ $(l+1)$ times on

$$\underbrace{M_1 \times \cdots \times M_1}_{l} \times \underbrace{M_1 \times \cdots \times M_1}_{l-1} \times \mathbb{C}P^{\dim M_1}.$$

Then $\operatorname{coh}(\hat{P}) = 2l + 2$.

Remark. Thus there are positive Kähler-Einstein manifolds with arbitrary cohomogeineity. Since the smallest dimension of M_1 is $\dim_{\mathbb{R}} M_1 = 10$, the quotient

$$\operatorname{coh} / \dim_{\mathbb{R}}$$

is asymptotically $\leq 1/10$. Therefore one should mention, that there are also completely different positive Kähler-Einstein manifolds ***with finite isometry group*** due to Tian [Tia].

4.5 Final Remark. Up to now, all known examples have even polynomial Q, and therefore $\operatorname{trace} B = 0$.

5. Appendix

Here are some short proofs for well-known facts.

5.1 Proposition. *Let M be a complex manifold with Hermitian metric. In a complex coordinate chart the following conditions are equivalent*

(a) *The Christoffel tensor $\Gamma(X,Y) = D_X Y - \partial_X Y$ is linear over* $\mathbb{C}$ *(in the second argument)*

(b) $DJ = 0$ (the metric is a Kähler metric)

(c) $d\omega = 0$ (the Kähler form is closed).

Here ∂ denotes the local connection of the chart.

Remark. (c)⇒(b) is often pretended to be "delicate", e.g. by [Bes, 2.29].

Proof.

$(a) \Rightarrow (b)$

$$\begin{aligned} D_X\, JY &= \partial_X\, JY + \Gamma(X, JY) \\ &= J\partial_X Y + J\Gamma(X,Y) = JD_X Y \end{aligned}$$

$(b) \Rightarrow (c)$ $\quad DJ = 0 \quad \Longrightarrow \quad D\omega = 0 \quad \Longrightarrow \quad d\omega = 0.$

$(c) \Rightarrow (a)$ Here we use the well-known formula for $2g(X,\Gamma(Y,Z))$ twice:

$$\begin{aligned} &2g(X,\Gamma(Y,JZ)) + 2g(JX,\Gamma(Y,Z)) \\ &= -\partial_X g\,(Y,JZ) \;+\partial_Y g\,(JZ,X) \;+\partial_{JZ} g\,(X,Y) \\ &\quad -\partial_{JX} g\,(Y,Z) \;+\partial_Y g\,(Z,JX) \;+\partial_Z g\,(JX,Y) \\ &= \;\partial_X \omega\,(Y,Z) \;+\partial_Y \omega\,(Z,X) \;-\partial_{JZ} \omega\,(JX,Y) \\ &\quad -\partial_{JX} \omega\,(Y,JZ) - \partial_Y \omega\,(JZ,JX) + \partial_Z \omega\,(X,Y) \\ &= 0, \end{aligned}$$

since each cyclic sum of $\partial\omega$ over (X,Y,Z) and (JX,Y,JZ) yields 0. □

5.2 Proposition (cf. [Ko-No 2, p.59]). *Let M be a totally geodesic submanifold of a Riemannian manifold $\tilde{M}$, X be a Killing field on $\tilde{M}$. Then the tangential component νX is a Killing field on M.*

Proof. Let c be a geodesic in M (and $\tilde{M}$). Then

$$\langle D_{c'}\, \nu X, c'\rangle = d_{c'}\langle \nu X, c'\rangle = d_{c'}\langle X, c'\rangle = \left\langle \tilde{D}_{c'} X, c'\right\rangle = 0.$$ □

5.3 Ricci curvature of complex hypersurfaces. Let $(\tilde{M}, \langle\ ,\ \rangle)$ be a Kähler manifold and $M \subset \tilde{M}$ be a complex submanifold of real codimension 2. By the last proposition, M is a Kähler manifold. Figuratively spoken, the following computation of the Ricci curvature of M lies somewhere between [Ko-No 2] (arbitrary codimension) and [Smy], [No-Sm] (who assume $\tilde{M}$ to be a complex space form and are therefore led to inappropriate proofs).

5.4. Let us start with a unit normal vector field N. Then

$$Su := \mathcal{V}\, D_u N$$

is the corresponding *shape operator*, and

$$b(u,v) := \langle Su, v\rangle$$

is the correscponding *second fundamental form.*

Since $N_2 := JN$ is another unit normal vector field, we get another set of data N_2, S_2, and b_2. We show

$$S_2 = JS = -SJ,$$
$$b(Ju, Jv) = -b(u,v).$$

Proof.

$$\langle S_2 u, v\rangle = \langle D_u JN, v\rangle = \langle JS\, u, v\rangle, \quad \text{also } S_2 = JS.$$
$$S_2 = S_2^* = S(-J)$$
$$b(Ju, Jv) = \langle SJ\, u, v\rangle = -\langle JS\, u, Jv\rangle = -b(u,v).$$

□

5.5 Lemma. *$-S^2$ and* $\operatorname{trace} S = 0$ *do not depend on the choice of N.*

Proof. Let $\tilde{N}$ be another unit normal vectorfield. There are real-valued functions a_1 and a_2, such that

$$\tilde{N} = a_1 N + a_2 JN \qquad \text{and} \quad a_1^2 + a_2^2 = 1.$$

Hence

$$\begin{aligned}\tilde{S}\cdot u &= \mathcal{V}(D_u \tilde{N})\\ &= a_1 S\cdot u + a_2 JS \cdot u\end{aligned}$$

and

$$\begin{aligned}\tilde{S}^2 &= a_1^2 S^2 + a_1 a_2 SJS + a_2 a_1 \underbrace{JS}_{-SJ} S + a_2^2 \underbrace{JS}_{-SJ} JS\\ &= S^2.\end{aligned}$$

Finally, $\operatorname{trace} S = \langle -JS_2, id\rangle = \langle S_2, J\rangle = 0.$ □

Remark. $N \wedge JN$ does not depend on the choice of N. The matrix of S is diag $(\lambda_1, -\lambda_1, \ldots, \lambda_m, -\lambda_m)$ with respect to a suitable adapted orthonormal frame $e_1, Je_1, \ldots, e_m, Je_m$.

5.6. Now we express $\tilde{r}$ in terms of r, $\tilde{R}$, and S. Remember the *Gauß equation*

$$\begin{aligned} R(u,v,w,t) = \tilde{R}(u,v,w,t) &+\langle Su,w\rangle\langle Sv,t\rangle \quad -\langle Su,t\rangle\langle Sv,w\rangle \\ &+\langle S_2u,w\rangle\langle S_2v,t\rangle -\langle S_2u,t\rangle\langle S_2v,w\rangle \end{aligned}$$

Let $e_1, \ldots, e_{2m}$ be an orthonormal frame on TM. Add N, JN in order to obtain an orthonormal frame of $T\tilde{M}|M$. By contraction over $e_1, \ldots, e_{2m}$, the first term $\tilde{R}(u,v,w,t)$ yields

$$\tilde{r}(u,w) - \tilde{R}(u,N,w,N) - \tilde{R}(u,JN,w,JN).$$

By the symmetries of the curvature tensor of a Kähler manifold, and by the algebraic Jacobi identity, this may be simplified to

$$\tilde{r}(u,w) - \tilde{R}(u,Jw,N,JN).$$

The other terms yield

$$\begin{aligned} \sum_i \big(\langle Su,w\rangle\langle Se_i,e_i\rangle - \langle Su,e_i\rangle\langle Se_i,w\rangle\big) &= \langle Su,w\rangle \operatorname{trace} S - \langle Su,Sw\rangle \\ &= \big\langle -S^2u,w\big\rangle = \big\langle -S_2^2u,w\big\rangle. \end{aligned}$$

Therefore

$$r(u,v) \quad \doteq \quad \tilde{r}(u,v) - \tilde{R}(u,Jv,N,JN) - 2g(S^2u,v),$$

or, as endomorphisms of TM

$$r \quad = \quad \tilde{r} + \mathcal{V}\,\tilde{R}(N,JN)J - 2S^2,$$

where each endomorphism above is symmetric and does not depend on the choice of N.

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