WEAK POSITIVITY AND THE STABILITY OF CERTAIN HILBERT POINTS

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The notion of weakly positive sheaves was originally developed by the author in order to express positivity of direct images of powers of dualizing sheaves, needed to study the generalized Iitaka - conjecture $C^{+}_{n,m}$ (see [17], [18], [20] and the excellent survey articles [2] and [13]). Beside of "weak positivity" applied to families of complex projective varieties over certain projective bundles, in all cases where one was able to prove $C^{+}_{n,m}$ one had to use some moduli theory. For example, in order to show $C^{+}_{n,m}$ for families of manifolds of general type one could use the existence of quasi-projective moduli schemes (as in [18] for curves and surfaces), or local Torelli theorems for cyclic covers ([19] or in a more general situation: Kawamata [9]) or, as Kollár did in [12], Hodge theoretic estimates on the kernel of the multiplication map.

Reconsidering the link between moduli theory and $C^{+}_{n,m}$ for complex manifolds of general type, we tried to use "weak positivity" and other methods from classification theory to construct quasi-projective coarse moduli spaces for certain moduli functors (1.1). This aim is not achieved, due to a technical statement (1.10) which I am not able to prove in a sufficiently general situation. We have to resign ourselves to a partial result, saying that smooth points of the reduced Hilbert scheme of canonically polarized manifolds are stable (in the sense of Mumford [14]) under the usual group action and with respect to some ample sheaf (1.7).

Since we hope that the "gap" 1.10 can be filled some day, may be using more advanced technics from Hodge theory, we formulate our article such that we can state as well:

"An affirmation answer to 1.10 implies that quasi-projective moduli spaces exist for complex canonically polarized manifolds."

Of course our proof is based on Mumford's geometric invariant theory [14].

Sometimes it is easier to obtain coarse moduli spaces M in the category of Moisezon spaces or algebraic spaces (see [10], [14], [15] and [16]). If M happens to be a fine moduli space, our approach to construct an ample invertible sheaf on M becomes quite elementary. The same method works for arbitrary families of Gorenstein varieties of general type provided the map to the moduli functor is finite over some open set (see 1.18 and 1.19 for the exact statements) and it shows the existence of natural sheaves on the base having lots of sections.

As an obvious corollary we obtain an elementary proof of $C^+_{n,m}$ for morphisms whose general fibre is of general type (see 1.20), a result obtained by Kollár [12] before.

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Hélène Esnault had a great influence on the content of this paper. She pointed out several ambiguities in the first version and some of the methods and some improvements are due to her. The approach presented here is partly based on our common work, especially on [3].

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Leitfaden

The proof that certain Hilbert points are stable (1.7) uses only §1, A and B, §3 and §5.

The existence of ample sheaves on the base of a family of certain varieties (1.18 *and* 1.19) *is based on* §1, B *and* C, §2, §3 *and* §4.

The reader just interested in a "simple" proof of $C^+_{n,m}$ for fibre spaces whose general fibre is a manifold of general type (1.20) should read §1, C and D, §2 and §4. The weak positivity results contained in [17] or [18] are strong enough for those applications (see remark 1.21).

§1 Notations and discussion of the main result

All varieties and schemes are supposed to be defined over the field $\mathbb{C}$ of complex numbers. We try to use the notations of [6].

A. Moduli and Hilbert schemes

1.1 Let $h(T)$ be a polynomial of degree n. As in [14] we consider the moduli functor $\mathcal{M}'_h$ of complex projective normal irreducible varieties X with at most rational Gorenstein singularities and with an ample canonical sheaf ω_X satisfying $\chi(X,\omega_X^\nu) = h(\nu)$. In order to have "nice" Hilbert schemes we need further restrictions and therefore we define:

a) If n (which is nothing but $\dim X$ for $X \in \mathcal{M}'_h(\mathbb{C})$) is one or two we define $\mathcal{M}_h = \mathcal{M}'_h$.

b) If $n > 2$ we define $\mathcal{M}_h$ by $\mathcal{M}_h(S) := \{f : \mathfrak{X} \longrightarrow S \in \mathcal{M}'_h(S);$ f smooth or $h^0(X,\omega_X) > 0$ for all fibers X of $f\}$

Results due to Matsusaka, Tankeev and Kollár (see for example [10]) show that the families in $\mathcal{M}_h(S)$ are bounded with respect to the canonical polarization and that for $\nu >> 0$ the ν-canonical embedded $X \in \mathcal{M}_h(\mathbb{C})$ are parametrized by a scheme H:

Theorem 1.2. *Let* $\mathcal{M}_h$ *be as in* 1.1.

i) *There exists some number* ν *such that for all* $X \in \mathcal{M}_h(\mathbb{C})$ ω_X^ν *is very ample.*

ii) *There exists a "Hilbert scheme"* H *and a universal family* $h : \mathfrak{X} \longrightarrow H \in \mathcal{M}_h(H)$ *together with* $i : \mathfrak{X} \longrightarrow \mathbb{P}(h_*\omega_{\mathfrak{X}/H}^\nu) \cong \mathbb{P}^{r-1} \times H$

such that all ν*-canonical embedded*

$$\begin{array}{ccc} \mathfrak{X}_S & \longrightarrow & \mathbb{P}^{r-1} \times S \\ \downarrow f_S & & \downarrow \\ S & = & S \end{array}$$

with $f_S \in \mathcal{M}_h(S)$ *are obtained as pullback of* $\mathfrak{X} \longrightarrow \mathbb{P}^{r-1} \times H$ *under a unique morphism* $S \longrightarrow H$.

iii) *The action of* $Sl(r,\mathbb{C})$ *on* H *corresponding to "change of coordinates in* $\mathbb{P}^{r-1}$" *is proper.*

Notations 1.3. If $f : X \longrightarrow Y$ is a proper flat Gorenstein morphism (respectively a proper surjective morphism between Gorenstein schemes) we denote by $\omega_{X/Y}$ the relative dualizing sheaf (respectively the difference of the dualizing sheaves $\omega_X \otimes f^*\omega_Y{}^{-1}$). If Y is irreducible we will allways try to use:

$$r(\eta) = \text{rank}(f_*\omega_{X/Y}^\eta) \quad \text{and} \quad \lambda_\eta = \det(f_*\omega_{X/Y}^\eta) .$$

1.4. For $\mu >> 0$ the sheaf $\mathcal{L}_0 = \lambda_{\upsilon\cdot\mu}^{r(\upsilon)} \otimes \lambda_{\upsilon}^{-r(\upsilon\cdot\mu)\cdot\mu}$ is ample on the Hilbert scheme H introduced in 1.2. Moreover there exists a $Sl(r(\upsilon),\mathbb{C})$-linearization of $\mathcal{L}_0$ ([14], Def. 1.6).

In fact, $\mathcal{L}_0$ is the ample sheaf arising from the Plücker coordinates on H and μ has to be choosen such that for all $X \in \mathcal{M}_h(\mathbb{C})$ the ideal of X in $\mathbb{P}(H^0(X,\omega_X^{\upsilon}))$ is generated by polynomials of degree μ ([1], in 4.3 we will use a similar construction).

1.5. Mumford introduced in [14] the notion of a stable point under a group action and with respect to any linearized invertible sheaf $\mathcal{L}$ (see 5.3). If we consider the Hilbert scheme and the $Sl(r(\upsilon),\mathbb{C})$ action we will denote the set of stable points by $H(\mathcal{L})^s$. We freely use the notations and results from [14]. Remark, that there it is shown that:

i) $H(\mathcal{L})^s$ *is an open subscheme of* H, *the quotient* $H(\mathcal{L})^s/Sl(r(\upsilon),\mathbb{C})$ *exists as a quasiprojective scheme and the* $Sl(r(\upsilon),\mathbb{C})$*-invariant sections of some power of* $\mathcal{L}$ *define an embedding of this quotient into some projective space*

ii) *If* $H = H(\mathcal{L})^s$ *then* $M = H/Sl(r(\upsilon),\mathbb{C})$ *is a quasiprojective coarse moduli scheme for* $\mathcal{M}_h$.

1.6. Mumford [14] for curves and Gieseker [4] for surfaces verified the stability for all points of H and showed that for $\upsilon,\mu >> 0$ and $\mathcal{L}_0$ as in 1.4 one has $H = H(\mathcal{L}_0)^s$. Regarding Gieseker's proof one finds that

(*) $\mathcal{L}_0$ *is ample on* H *by construction.*

(**) *It is difficult to decide whether a given point lies in* $H(\mathcal{L}_0)^s$.

The approach presented in this paper shifts the difficulty from (**) to (*). We will observe in §5:

Let $\mathcal{L}_\eta = \mathcal{L}_0 \otimes \lambda_\upsilon^\eta$. *Then*

(*) *It is difficult to show that* $\mathcal{L}_\eta$ *is ample on* H.

(**) *If one knows that for* $\eta >> 0$ $\mathcal{L}_\eta$ *is ample on some open* $Sl(r(\upsilon),\mathbb{C})$ *invariant subscheme* H_0 *of* H, *then it is easy to show that* $H_0 \subseteq H\ (\mathcal{L}_\eta)^s$.

The precise statement, which will be shown in §5, is:

Theorem 1.7. *Let* $\mathcal{M}_h$ *be one of the moduli functors considered in 1.1 and let* H *be the Hilbert scheme (for some* $\upsilon > 0$ *as in 1.2).*

a) *Let* $H_0 \subseteq H$ *be the largest open subscheme of* H *such that* $(H_0)_{red}$ *is non singular. Then* $H_0 \cong H_0(\lambda^a_{\upsilon\cdot\mu} \otimes \lambda^b_\upsilon)^s$ *for* $a,b,\mu >> 0$.

b) *An affirmative answer to problem* 1.10 *implies that for* $a,b,\mu >> 0$ *one has* $H = H(\lambda^a_{\upsilon\cdot\mu} \otimes \lambda^b_\upsilon)^s$ *and hence that a coarse quasiprojective moduli scheme exists for* $\mathcal{M}_h$.

B. <u>Weak postivity and some open problems</u>

<u>Definition and Notation 1.8.</u>

i) Let Y be a scheme (or an analytic space) and $U \subseteq Y$ an open subscheme (or Zariski open subspace). We say that an $\mathcal{O}_Y$-module $\mathcal{F}$ is <u>globally generated over</u> U, if the natural map $H^0(Y,\mathcal{F}) \otimes_{\mathbb{C}} \mathcal{O}_Y \longrightarrow \mathcal{F}$ is surjective over U (respectively: $H^0_m(Y,\mathcal{F}) \otimes_{\mathbb{C}} \mathcal{O}_Y \longrightarrow \mathcal{F}$, where H^0_m denotes the meromorphic sections, as in 1.16).

ii) If $\mathcal{F}$ is a coherent sheaf on Y and $i : U \longrightarrow Y$ is maximal open subscheme (or space) where $\mathcal{F}$ is locally free, then we define $S^r(\mathcal{F}) = i_*S^r(i^*\mathcal{F})$, $\Lambda^r(\mathcal{F}) = i_*\Lambda^r(i^*\mathcal{F})$ and $\det(\mathcal{F}) = i_*\det(i^*\mathcal{F})$. In §2 we will introduce tensor-bundles $T(i^*\mathcal{F})$ and then $T(\mathcal{F})$ is supposed to be $i_*T(i^*\mathcal{F})$. For simplicity we write $\mathcal{F}^r$ indstead of $S^r(\mathcal{F})$ when $\mathcal{F}$ is of rank one.

<u>1.9.</u> There are several slightly different definitions of weakly positive sheaves in the literature (See [13] for a discussion). We will return here to the original one and - in order to formulate 1.10 - we have to extend this notion to sheaves on arbitrary reduced quasi projective schemes. In all applications (even when we will sometimes forget to mention it) we will assume that the open set U meets all components of Y and that the sheaf $\mathcal{F}$ is locally free on some neighbourhood of the non normal locus of Y.

<u>Definition</u> Let $\mathcal{F}$ be a coherent torsion free sheaf on a reduced quasiprojective scheme Y and $U \subseteq Y$ be an open subscheme. Let $\mathcal{H}$ be an ample invertible sheaf on Y. Then $\mathcal{F}$ is called <u>weakly positive</u> over U if $\mathcal{F}|_U$ is locally free and if for all $a > 0$ there exists some $b > 0$ such that $S^{ab}(\mathcal{F}) \otimes \mathcal{H}^b$ is globally generated over U.

Obviously this definition is independent of the ample invertible sheaf $\mathcal{H}$ choosen. Moreover, we can say that for all $a > 0$ there exists some $b_0 > 0$ such that $S^{ab}(\mathcal{F}) \otimes \mathcal{H}^b$ is globally generated over U for all $b \geq b_0$ (see [19], 3.2.i). Weakly positive sheaves have properties similar to those of ample sheaves. Some are recalled in §3.

<u>Problem 1.10.</u> Let Y_0 be a reduced quasi projective scheme and $(f_0 : X_0 \longrightarrow Y_0) \in \mathcal{M}_h(Y_0)$ for one of the moduli functors $\mathcal{M}_h$ considered in 1.1. Is for $\upsilon > 1$ the sheaf $f_{0*}\omega^{\upsilon}_{X_0/Y_0}$ weakly positive over Y_0?

In fact, 1.10 should not depend on the ampleness of $\omega_{f^{-1}(y)}$. Moreover the generating sections should come from some compactification. Therefore, being more optimistic, we could ask:

Problem 1.11. Let $f : X \longrightarrow Y$ be a surjective projective flat Gorenstein morphism of reduced quasi projective schemes. Assume that for some $\nu > 0$ $f_*\omega^{\nu}_{X/Y}$ is locally free. Let $Y_0 \subseteq Y$ be an open subscheme, such that for all $y \in Y_0$, $f^{-1}(y)$ is normal with at most rational singularities. Is $f_*\omega^{\nu}_{X/Y}$ weakly positive over Y_0?

As we will show in 3.7 both, 1.10 and 1.11 have an affirmative answer if one assumes in addition that Y_0 is non singular. If moreover $f_0 : X_0 = f^{-1}(Y_0) \longrightarrow Y_0$ is smooth, then this has been obtained in [17] and [18] building up on Kawamata's positivity theorem ($\nu = 1$, see [8]). This last mentioned theorem has also been obtained by Kollár [11] as a corollary of his vanishing theorem for semi ample sheaves. Using his idea and some of [3], one can give now a quite simple "algebraic" proof (see [20], 6 and 8'). Let us remark that 1.11 would follow from 3.7 and an affirmative answer to

Problem 1.12 Let $\mathcal{L}$ be an invertible sheaf on Y and $\tau : Y' \longrightarrow Y$ a desingularization. Assume $\tau^*\mathcal{L}$ is weakly positive over $\tau^{-1}(U)$. Is then $\mathcal{L}$ weakly positive over U?

Several unsucessful attempts to answer this question let me doubt however whether the answer to 1.12 is yes.

1.11 has also an affirmative answer if Y is projective and if the fibres of f are not too bad. In some way our problems have to do with the problem how to find "good" compactifications of morphisms. Instead one could try to compactify the bundles coming from Hodge theory:

1.13. Let $f_0 : X_0 \longrightarrow Y_0$ be a smooth equidimensional projective morphism of quasi projective schemes, let $\tau_0 : Y'_0 \longrightarrow Y_0$ be a desingularization and $i : Y'_0 \longrightarrow Y'$ be a compactification. We call Y' a good compactification if Y' is non singular, projective and if $Y' - Y'_0$ is a normal crossing divisor. If $f'_0 : X'_0 \longrightarrow Y'_0$ is the morphism obtained as pullback of f_0, we assume that (for $k = \dim X_0 - \dim Y_0$) the monodromy of $R^k f'_{0*} \mathbb{C}_{X'_0}$ around the components of $Y' - Y'_0$ is unipotent. By W. Schmid's nilpotent orbit theorem one has a natural locally free sheaf $\mathcal{H}'$ on Y' such that $\mathcal{H}'|_{Y'_0} = (R^k f'_{0*}\mathbb{C}_{X'_0}) \otimes_{\mathbb{C}} \mathcal{O}_{Y'_0}$ and the subbundle $f'_{0*}\omega_{X'_0/Y'_0}$ extends to a subbundle $\mathcal{F}'$ of $\mathcal{H}'$. Both sheaves are compatible with further blowing ups of Y'.

Problem Can one find a compactification Y of Y_0 and a locally free sheaf $\mathcal{F}$ on Y (or even a locally free sheaf $\mathcal{H}$ on Y) such that for any good compactification Y' of Y'_0 with a morphism $\tau : Y' \longrightarrow Y$ one has $\mathcal{F}' = \tau^*\mathcal{F}$ (or $\mathcal{H}' = \tau^*\mathcal{H}$)?

In 3.12 we will indicate how an affirmative answer to 1.13 implies one to 1.11 and 1.10, at least under the additional assumption that f_0 is smooth. Studying base change properties of powers of dualizing sheaves more carefully than we will do, one should also be able to deduce 1.10 and 1.11 as stated.

Convention 1.14. Throughout this article we formulate the proofs such that an affirmative answer to 1.11 allows to erase the words "non singular locus" in all statements of the form "... is weakly positive over the non singular locus of ..." or "... is ample with respect to the non singular locus of ..." Especially this holds for 1.18 and 1.19.

C. Application to fibre spaces

Convention 1.15

i) All analytic spaces Z occuring should be Zariski open subspaces of a reduced irreducible separated compact analytic space $\overline{Z}$.

ii) All coherent sheaves $\mathcal{F}$ on Z should extend as coherent sheaves to $\overline{Z}$. Thereby it makes sense to talk about meromorphic sections of $\mathcal{F}$ and those are denoted by $H^0_m(Z,\mathcal{F})$.

iii) Each morphism between analytic spaces should extend to some compactification.

<u>Definition 1.16.</u> Let Y be an analytic space, $U \subseteq Y$ be a Zariski open subspace and $\mathcal{L}$ be a coherent torsionfree sheaf of rank 1 on Y. We call $\mathcal{L}$ <u>ample with respect to</u> U if $\mathcal{L}|U$ is invertible and if for some $a > 0$ there exists a finite dimensional subspace $V \subseteq H^0_m(Y,\mathcal{L}^a)$ such that $V \otimes_{\mathbb{C}} \mathcal{O}_Y \longrightarrow \mathcal{L}^a$ is surjective over U and the natural morphism $U \longrightarrow \mathbb{P}(V)$ an embedding.

Of course, if $U \neq \emptyset$, this implies that Y is Moisezon and U quasiprojective. As promised in the introduction we will show that a fine moduli space carries a rank one sheaf, ample with respect to its non-singular points. In fact, we can weaken the assumptions:

<u>Assumptions 1.17.</u> Let $f : X \longrightarrow Y$ be a flat surjective projective Gorenstein morphism of analytic spaces (remember 1.15) and $Y_0 \subseteq Y$ be a non empty Zariski open subspace. Assume that:

i) All fibres of $f_0 : f^{-1}(Y_0) = X_0 \longrightarrow Y_0$ are irreducible normal varieties of general type with at most rational singularities.

ii) For all $y \in Y_0$, there exists only a finite set of $y' \in Y_0$ such that $f^{-1}(y')$ is birational to $f^{-1}(y)$.

iii) Y is normal (or at least $f_*\omega^{\nu}_{X/Y}$ and the sheaf $\mathcal{G}$ in 1.19 are both locally free in some neighbourhood of the non normal locus of Y).

<u>Theorem 1.18.</u> *Under the assumptions made in 1.17 assume that for some* $\nu > 1$ *the sheaf* ω^{ν}_F *is very ample for each fibre* F *of* $f_0 : X_0 \longrightarrow Y_0$. *Then for some* $a,b,\mu > 0$ *the sheaf* $\mathcal{L} = \det(f_*\omega^{\nu\cdot\mu}_{X/Y})^a \otimes \det(f_*\omega^{\nu}_{X/Y})^b$ *is ample with respect to the non singular locus of* Y_0.

<u>Theorem 1.19.</u> *Under the assumptions made in 1.17 assume that* $f_*\omega^{\nu}_{X/Y} \otimes \mathbb{C}(y) \subset H^0(f^{-1}(y), \omega^{\nu}_{f^{-1}(y)})$ *defines a birational map of* $f^{-1}(y)$ *for all* $y \in Y_0$. *Then for some* $a,b,\mu > 0$ *and* $\mathcal{G} = \mathrm{Im}(S^{\mu}(f_*\omega^{\nu}_{X/Y}) \longrightarrow f_*\omega^{\nu\cdot\mu}_{X/Y})$ *the sheaf* $\mathcal{L} = \det(\mathcal{G})^a \otimes \det(f_*\omega^{\nu}_{X/Y})^b$ *is ample with respect to the non singular locus of the open subset* $U \subseteq Y_0$ *where both* $f_*\omega^{\nu}_{X/Y}$ *and* $\mathcal{G}$ *are locally free.*

Of course 1.18 is just a special case of 1.19. In fact, since ω_{X_0/Y_0} is ample on each fibre of f_0 one can use the Grauert-Riemenschneider vanishing theorem (see [6]) to show that $f_{0*}\omega^{\nu}_{X_0/Y_0}$ is locally free for $\nu > 1$. Moreover, for $\mu >> 0$ the multiplication map is surjective over Y_0. Therefore we can assume that the inclusion $\mathcal{G} \subset f_*\omega^{\nu\cdot\mu}_{X/Y}$ is an isomorphism over Y_0 and choose $U = Y_0$ in 1.19.

The reader interested in stability of Hilbert points and familiar with [14] can use the proof of theorem 1.19 in §4 as an illustration how the proof of the stability theorem 1.7 will work. In some way, the Reynolds-operator used in [14] can be replaced by the splitting obtained in 2.6 and in 4.5 the Hilbert-Mumford criterium is hidden behind the curtain (see remark 4.4 and 4.6).

D. Proof of $C^+_{n,m}$ for certain fibre spaces

We will use the notations coming from classification theory and the reader not familar with this theory should consult the excellent survey articles [2] and [13] for the exact definitions, references and historical remarks.

Theorem 1.20. (Kollár [12], Kawamata and myself, under more restrictive assumptions [9], [19]).

Let $f : X \longrightarrow Y$ *be a morphism of projective manifolds with an irreducible general fibre* X_w *of general type.*

i) $(C^+_{n,m})$ *If* $\kappa(Y) \geq 0$ *then*

$$\kappa(X) \geq \mathrm{Max}\{\kappa(Y) + \kappa(X_w), \mathrm{Var}\ (f) + \kappa(X_w)\}$$

ii) *If* $\mathrm{Var}(f) = \dim Y$, *then for* $\mu,\nu >> 0$ *the sheaf* $S^\mu(f_*\omega^\nu_{X/Y})$ *contains an ample subscheaf of full rank.*

iii) *If* Var(f) = dim Y, *then for some* $\eta >> 0$ *the reflexive hull of* $f_*\omega^{\eta}_{X/Y}$ *contains an ample invertible sheaf.*

<u>Proof.</u> As explained in [19], 3.4 ii) and iii) are equivalent. In [18] it was shown that ii) implies $C^+_{n,m}$ for the corresponding type of morphism. To prove ii) we use 1.19 together with the constructions developped in [18]:

We can replace Y by the complement of a codimension two subvariety (as in 3.3.d) and thereby we may assume that $f : X \longrightarrow Y$ satisfies the assumptions made in 1.17. As in [18], 6.1 we can make semistable reduction in codimension one and, leaving again out a codimension two subvariety, we may assume that f is semi-stable. 1.19 tells us that for some $\upsilon,\mu,a,b > 0$ the sheaf $\mathcal{L} = \det(\mathcal{G})^a \otimes \det(f_*\omega^{\upsilon}_{X/Y})^b$ has maximal Iitaka dimension $\kappa(\mathcal{L})$. Therefore, choosing a and b big enough, we can assume $\mathcal{L}$ to contain an ample invertible sheaf. Since $\det(\mathcal{G})$ is contained in some wedge product of $f_*\omega^{\upsilon\cdot\mu}_{X/Y}$ we may find some η_1 and η_2 such that $\mathcal{L}$ lies in

$$(\otimes^{\eta_1} f_*\omega^{\upsilon\cdot\mu}_{X/Y}) \otimes (\otimes^{\eta_2} f_*\omega^{\upsilon}_{X/Y})$$

(see §2, for example, or [7]).

By [18], 3.4 and 3.5. $\overset{a}{\otimes} f_*\omega^{\upsilon}_{X/Y}$ is nothing but $f^{(a)}_*\omega^{\upsilon}_{X^{(a)}/Y}$ where $X^{(a)}$ is a desingularization of the a-fold fibre product $X \times_Y X \times_Y \ldots \times_Y X$. The product map for $X^{(\eta_2)}$ induces $S^{\mu} \otimes^{\eta_2} f_* \omega^{\upsilon}_{X/Y} \longrightarrow \otimes^{\eta_2} f_* \omega^{\upsilon \cdot \mu}_{X/Y}$ and therefore $\mathcal{L}^{\mu}$ is a subsheaf of $\otimes^{\eta} f_* \omega^{\upsilon \cdot \mu}_{X/Y}$ for $\eta = \mu \cdot \eta_1 + \eta_2$. The equivalence of ii) and iii), applied to the fibre space $X^{(\eta)}$ shows that for some $\gamma >> 0$ $S^{\gamma} \otimes^{\eta} (f_* \omega^{\upsilon \cdot \mu}_{X/Y})$ contains an ample subsheaf of full rank. Then the same must hold for the quotient $S^{\gamma \cdot \eta} (f_* \omega^{\upsilon \cdot \mu}_{X/Y})$.

<u>Remark 1.21.</u> The proof of $C^+_{n,m}$ given above does not use anymore analytic methods from Hodge theory, except of the degeneration of the Hodge-Deligne spectral sequence, hidden behind the vanishing theorem of Kollár (see [3], §3). Since the degeneration of this spectral sequence has been shown by Deligne and Illusie using characteristic p methods we can say that the proof of $C^+_{n,m}$ for families of manifolds of general type, presented here, is algebraic and "easier" than the ones given before.

If Y is a curve, the necessary tools from "weak positivity" are quite trivial, and the proof of $C^+_{n,1}$ obtained from 1.19 is quite simple.

§2 Tensor bundles

2.1. Throughout this section we consider an algebraic scheme X or an analytic space X together with a locally free sheaf $\mathcal{E}$ which is of rank r on all components of X. As described in [7] for example, a finite dimensional representation $T : Gl(r,\mathbb{C}) \longrightarrow Gl(n,\mathbb{C})$ gives rise to a bundle $T(\mathcal{E})$.

Definition. We call $T(\mathcal{E})$ the tensor bundle (of T). If T is an irreducible representation we call $T(\mathcal{E})$ an irreducible tensor bundle.

2.2. Let T be an irreducible representation. Then the irreducible tensor bundle $T(\mathcal{E})$ is, up to isomorphism, uniquely determined by the "upper weight" $c(T) - (n_1,\dots,n_r)$. This, as well as the following construction of $c(T)$, can be found in [7], A.6: Let P be the group of upper triangular matrices. There is a unique one dimensional subspace of $\mathbb{C}^n$ consisting of eigenvectors of $T|P$. If $\lambda : P \longrightarrow \mathbb{C}^*$ is the corresponding character, then λ applied to a diagonal matrix (h_{ii}) gives $\prod_i h_{ii}^{n_i}$, if $c(T) = (n_1,\dots,n_r)$. One has $n_1 \geq \dots \geq n_r$.

Definition. We call $c(T)$ the upper weight of the irreducible tensor bundle $T(\mathcal{E})$. $T(\mathcal{E})$ is called positive if $n_r \geq 0$ and $n_r > 0$ (for all irreducible summands if T is reducible).

2.3. Examples of tensor bundles are the symmetric products $S^{\upsilon}(\mathcal{E})$, the tensor products $\otimes^{\upsilon}(\mathcal{E})$. If $T_i(\mathcal{E})$ are tensorbundles, for $i = 1,2$, then the same holds for $T_1(\mathcal{E}) \oplus T_2(\mathcal{E})$ and $T_1(\mathcal{E}) \otimes T_2(\mathcal{E})$. The determinant $\det(\mathcal{E})$, as well as $\det(\mathcal{E})^{\eta}$ for $\eta \in \mathbb{Z}$, are irreducible tensor bundles of upper weight $(\eta,\ldots,\eta)$.

Lemma 2.4. *Let* $\rho : \det(\mathcal{E}) \longrightarrow \overset{r}{\otimes}\, \mathcal{E}$ *be the map* $\rho(x_1\wedge\ldots\wedge x_r) = \frac{1}{r!} \sum_{\sigma\in\mathcal{S}_r} \operatorname{sign}(\sigma) X_{\sigma(1)}\otimes\ldots\otimes X_{\sigma(r)}$. *Then for all* $\eta \in \mathbb{N}$ *the image of* ρ^{η} *embedds* $\det(\mathcal{E})^{\eta}$ *as direct summand in* $\overset{r}{\otimes}\, S^{\eta}(\mathcal{E})$.

Proof. If $e_1,\ldots,e_r$ is a basis of $\mathbb{C}^r$ then, with respect to the standard representation,

$u = \frac{1}{r!} \sum_{\sigma\in\mathcal{S}_r} \operatorname{sign}(\sigma) e_{\sigma(1)}\otimes\ldots\otimes e_{\sigma(r)} \in \overset{r}{\otimes}\, \mathbb{C}^r$ is an eigenvector of P. The induced irreducible subrepresentation has upper weight $(1,\ldots,1)$. As in [7], p. 75, let u^{η} be the image of $u^{\otimes\eta}$ under the multiplication map $\overset{\eta}{\otimes}\overset{r}{\otimes}\, \mathbb{C}^r \longrightarrow \overset{r}{\otimes}\, S^{\eta}\mathbb{C}^r$. Then u^r is again an eigenvector, and the corresponding upper weight is $(\eta,\ldots,\eta)$. Since the irreducible subrepresentation is uniquely determined by the upper weight it must be $\det(\mathbb{C}^r)^{\eta}$.

2.5. A more geometric interpretation of the map ρ from (2.4) can be given by considering the projective bundle $\mathbb{P} = \mathbb{P}(\overset{r}{\oplus}\, \mathcal{E}^{\vee}) \xrightarrow{\pi} X$ of $\overset{r}{\oplus}\, \mathcal{E}^{\vee} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \overset{r}{\oplus}\, \mathcal{O}_X)$. We have

$$\pi_{*}\mathcal{O}_{\mathbb{P}}(\upsilon) = S^{\upsilon}(\overset{r}{\oplus}\, \mathcal{E}^{\vee}) = \oplus \overset{r}{\underset{i=1}{\otimes}} S^{\mu_i}(\mathcal{E}^{\vee}),$$

where the direct sum is taken over all $(\mu_1,\ldots,\mu_r)$ with $\sum_{i=1}^{r} \mu_i = \upsilon$. Therefore the map ρ gives rise to $\rho : \det(\mathcal{E})^{-1} \longrightarrow \overset{r}{\otimes}\, \mathcal{E}^{\vee} \longrightarrow \pi_{*}\mathcal{O}_{\mathbb{P}}(r)$. We denote the induced section of $\mathcal{O}_{\mathbb{P}}(r) \otimes \pi^{*}\det(\mathcal{E})$ again by ρ and write D for its zero divisor. On $\mathbb{P}$ we have the universal map $\pi^{*} \overset{r}{\oplus}\, \mathcal{E}^{\vee} \twoheadrightarrow \mathcal{O}_{\mathbb{P}}(1)$ or, taking its dual, a "universal basis"

$$(s_1,\ldots,s_r) : \mathcal{O}_{\mathbb{P}}(-1) \longrightarrow \overset{r}{\oplus}\, \pi^{*}\mathcal{E}.$$

The wedge product of $s_1,\ldots,s_r$ factors over

$$\mathcal{O}_{\mathbb{P}}(-r) \longrightarrow S^{r}(\overset{r}{\oplus}\, \pi^{*}\mathcal{E}) \twoheadrightarrow \overset{r}{\otimes}\, \pi^{*}\mathcal{E} \twoheadrightarrow \det(\pi^{*}\mathcal{E}).$$

Since this is just the dual of ρ we find D to be the degeneration locus of $s_1,\ldots,s_r$. Altogether we obtain:

Lemma 2.6. *Let* $\underline{s} : \bigoplus^{r} \mathcal{O}_{\mathbb{P}}(-1) \longrightarrow \pi^{*}\mathcal{E}$ *be the universal basis and* D *the degeneration locus of* $\underline{s}$. *Then the corresponding section* $\rho^{\eta} : \mathcal{O}_{\mathbb{P}} \longrightarrow \mathcal{O}_{\mathbb{P}}(\eta\cdot D) = \mathcal{O}_{\mathbb{P}}(r\cdot\eta) \otimes \pi^{*}\det(\mathcal{E})^{\eta}$ *gives rise to a direct summand* $\mathcal{O}_{X} \longrightarrow \pi_{*}\mathcal{O}_{\mathbb{P}}(\eta\cdot D)$.

The following proposition is the key of the proof of theorem 1.18 and 1.19 in §4. We remind the reader of the convention 1.15 on analytic spaces and of definition 1.16.

Proposition 2.7. *Let* X *be an analytic space,* $\mathcal{L}$ *be a coherent torsionfree rank one sheaf on* X *and* $U \subseteq X$ *be a Zarisky open subspace. Let* $\tau : \mathbb{P}' \longrightarrow \mathbb{P}$ *be a proper modification of* $\mathbb{P}$ *with center in* D *(we keep the notations from 2.5) and* $\pi' = \pi\circ\tau$. *Let* D' *be an effective divisor of* $\mathbb{P}'$ *with support in* $\tau^{-1}(D)$. *Assume that* $\mathcal{L}' = \pi'^{*}\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}'}(D')$ *is ample with respect to* $\pi'^{-1}(U)$. *Then* $\mathcal{L}$ *is ample with respect to* U.

Proof. For $\upsilon > 0$ we can find some η such that $0 \leq \upsilon D' \leq \tau^{*}(\eta\cdot D)$ and such that one has an inclusion $\tau_{*}\mathcal{O}_{\mathbb{P}'}(\upsilon D') \longrightarrow \mathcal{O}_{\mathbb{P}}(\eta\cdot D)$. By 2.5 we obtain $\mathcal{O}_{X} \longrightarrow \pi'_{*}\mathcal{O}_{\mathbb{P}'}(\upsilon D')$ as a direct summand. The inclusion $\pi'^{*}\mathcal{L}^{\upsilon} \longrightarrow \mathcal{L}'^{\upsilon}$ gives thereby rise to a natural splitting of

$$H^{0}_{\mathfrak{m}}(X,\mathcal{L}^{\upsilon}) \longrightarrow H^{0}_{\mathfrak{m}}(\mathbb{P}',\mathcal{L}'^{\upsilon}) \ .$$

"Natural" means:

If Z is a (not nessarily reduced) analytic subspace of X and $\mathbb{P}'_Z$ the proper transform of Z in $\mathbb{P}'$, the splitting of

$$H^0_m(Z,\mathcal{L}^\upsilon|Z) \longrightarrow H^0_m(\mathbb{P}'_Z,\mathcal{L}'^\upsilon|\mathbb{P}'_Z)$$

is compatible with the one given above.

We have a commutative diagram

$$\begin{array}{ccc} H^0_m(\mathbb{P}',\mathcal{L}'^\upsilon) & \twoheadrightarrow & H^0_m(X,\mathcal{L}^\upsilon) \\ \downarrow \alpha' & & \downarrow \alpha \\ H^0_m(\mathbb{P}'_Z,\mathcal{L}'^\upsilon|\mathbb{P}'_Z) & \twoheadrightarrow & H^0_m(Z,\mathcal{L}^\upsilon) \end{array}$$

If we take $Z = x \cup y$, for two points $x,y \in U$, a finite dimensional subspace of $H^0_m(\mathbb{P}',\mathcal{L}'^\upsilon)$ embedds a neighbourhood of $\mathbb{P}'_Z$ in a projective space. Choosing υ bigger we may assume that α' is surjective. Then α is surjective as well and we find $V_\upsilon \subseteq H^0(X,\mathcal{L}^\upsilon)$ generating $\mathcal{L}^\upsilon$ in x and y and seperating the points x and y.

In the same way we can take Z to be the subspace defined by the square of the ideal of x, in order to see that (for $\upsilon >> 0$) some subspace V_υ seperates the tangent directions in x. If we define $V_{\eta\cdot\upsilon}$ for $\eta \in N$ to be the subspace spanned by monomials in elements of V_υ, the same holds for $V_{\eta \cdot \upsilon}$. Since U is compact in the Zariski topology we can find for $\eta >> 0$ some lager finite dimensional space $V' \subseteq H^0_m(X, \mathcal{L}^{\upsilon \cdot \eta})$ giving an embedding of U.

§3 Weak postivity, revisited

Since the notation of "weakly positive sheaves over a given open subscheme" introduced in 1.9 is central for this article we recall and extend the properties of weakly positive sheaves (see also [17], [18], [19] and [13]). Some of them will be needed in 3.7, where we verify 1.10 and 1.11 under the additional assumption that Y_0 is smooth.

Assumptions 3.1. Y is a reduced quasiprojective scheme, $U \subseteq Y$ an open subscheme and $\mathcal{H}$ is an ample invertible sheaf on Y. We assume that U meets all components of Y. Let $\mathcal{F}$ be a coherent torsion free sheaf which is locally free in some neighbourhood of the non normal locus of Y.

Lemma 3.2. *Assume that $\mathcal{F}$ is of rank one. Then $\mathcal{F}$ is weakly positive over U if and only if $\mathcal{F}^a \otimes \mathcal{H}$ is ample with respect to U for all $a > 0$.*

Proof. If $\mathcal{F}$ is weakly positive over U then $\mathcal{F}^{2ab} \otimes \mathcal{H}^b$ is globally generated over U for $b >> 0$, and therefore $\mathcal{H}^b$ is a subsheaf of $\mathcal{F}^{2ab} \otimes \mathcal{H}^{2b}$, isomorphic over U. This implies that $\mathcal{F}^a \otimes \mathcal{H}$ is ample with respect to U. The other direction is obvious.

3.3. Some simple properties

a) $\mathcal{F}$ is weakly positive over U if and only if each $u \in U$ has a neighbourhood V(u) such that $\mathcal{F}$ is weakly positive over V(u).

b) If $\mathcal{F}$ and $\mathcal{F}'$ satisfy the properties asked for in (3.1), and if both are weakly positive over U then $\mathcal{F} \oplus \mathcal{F}'$ is weakly positive over U.

c) If $\mathcal{F} \longrightarrow \mathcal{F}'$ is surjective over U and $\mathcal{F}$ weakly positive over U, then $\mathcal{F}'$ is weakly positive over U as well.

d) If $U \subseteq Y' \subseteq Y$ are open subschemes and $\mathrm{depth}_{Y-Y'}\mathcal{O}_Y \geq 2$ then $\mathcal{F}$ is weakly positive over U if and only if $\mathcal{F}|_{Y'}$ is weakly positive over U. Especially we can leave out subvarieties of Y of codimension bigger than or equal to two, as long as they do not meet the non normal locus of Y, and thereby we may - whenever it is convenient - assume that $\mathcal{F}$ is locally free.

e) If, for some $\eta > 0$, $S^{\eta}(\mathcal{F})$ or $\otimes^{\eta}(\mathcal{F})$ is weakly positive over U, then the same holds for $\mathcal{F}$.

Proof. a,b and c follow directly from the definition. d) follows from [5], 1.9 and 3.8 and to verify e) one just has to use the natural maps $\otimes^{\eta}\mathcal{F} \longrightarrow S^{\eta}(\mathcal{F})$ and $S^a S^{\eta}(\mathcal{F}) \longrightarrow S^{a\cdot\eta}(\mathcal{F})$, which are both surjective over U.

3.4. Functorial properties

a) If $\tau : Y' \longrightarrow Y$ is a morphism such that $\tau^{-1}(U)$ meets all components of Y', and if $\mathcal{F}$ is weakly positive over U, then $\tau^{*}\mathcal{F}$ is weakly positive over $\tau^{-1}(U)$.

b) Let $\tau : Y' \longrightarrow Y$ be a projective surjective morphism such that $\tau^{-1}(U)$ meets all components of Y' and such that $\tau^{-1}(U) \longrightarrow U$ is finite. Assume moreover that $\mathcal{O}_U \longrightarrow \tau_{*}\mathcal{O}_{\tau^{-1}(U)}$ is a direct summand (for example this holds if U is normal). Then $\mathcal{F}$ is weakly positive over U if and only if $\tau^{*}\mathcal{F}$ is weakly positive over $\tau^{-1}(U)$.

c) $\mathcal{F}$ is weakly positive over U if and only if there exists some $\mu > 0$ such that for all finite surjective morphisms $\tau : Y' \longrightarrow Y$ and for all ample invertible sheaves $\mathcal{H}'$ on Y' the sheaf $\tau^{*}\mathcal{F} \otimes \mathcal{H}'^{\mu}$ is weakly positive over $\tau^{-1}(U)$.

d) Assume that $\mathcal{F}$ is locally free, and let $\pi : \mathbb{P}(\mathcal{F}) \longrightarrow Y$ be the projective bundle of $\mathcal{F}$. Then $\mathcal{F}$ is weakly positive if and only if $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$ is weakly positive over $\pi^{-1}(U)$.

e) Assume that the non singular locus of U is compact. Let $\tau : Y' \longrightarrow Y$ be a surjective projective generically finite morphism. Then $\mathcal{F}$ is weakly positive over U if and only if $\tau^{*}\mathcal{F}$ is weakly positive over $\tau^{-1}(U)$.

Proof.

a) is obvious. Using it together with 3.3d we may assume that $\mathcal{F}$ is locally free in the sequel.

b) The "only if" follow from a). Let us assume $\tau^*\mathcal{F}$ to be weakly positive over $\tau^{-1}(U)$. We may choose $\mathcal{H}$ such that $\tau_*\mathcal{O}_{Y'} \otimes \mathcal{H}^\eta$ is generated by its global sections for all $\eta >> 0$. By a) we can replace Y' by any blow up and hence we may find an effective divisor E such that $\mathcal{O}_{Y'}(-E)$ is relative ample for τ and such that $\tau^{-1}(Y-U) = E_{red}$. Moreover we may assume $\tau^*\mathcal{H}(-E)$ to be ample on Y'. By assumption $\mathcal{O}_Y \longrightarrow \tau_*\mathcal{O}_{Y'}$ splits over U. Therefore for $b >> 0$ we obtain a map $\rho : \tau_*\mathcal{O}_{Y'}(-b\cdot E) \longrightarrow \mathcal{O}_Y$ surjective over U. For given a and $b >> 0$, we have a map

$$\oplus\ \mathcal{O}_{Y'} \longrightarrow S^{2\cdot a\cdot b}(\tau^*\mathcal{F}) \otimes \tau^*\mathcal{H}(-E)^b\ ,$$

surjective over $\tau^{-1}(U)$. Then the induced map

$$\oplus\ \tau_*\mathcal{O}_{Y'} \otimes \mathcal{H}^b \longrightarrow S^{2\cdot a\cdot b}(\mathcal{F}) \otimes \mathcal{H}^{2\cdot b} \otimes \tau_*\mathcal{O}_{Y'}(-bE) \xrightarrow{\ \rho\ }$$

$$\longrightarrow S^{2\cdot a\cdot b}(\mathcal{F}) \otimes \mathcal{H}^{2b}$$

is also surjective over U.

c) The "only if" part follows from a) and the obvious fact, that a weakly positive sheaf keeps this property if it is tensorised by an ample sheaf. For the other direction we choose $\mathcal{H}$ to be very ample on Y and $Y \longrightarrow \mathbb{P}^N$ to be the corresponding embedding. For a given and $d = 1+2a\cdot\mu$ we choose a non singular finite cover $\pi : Z \longrightarrow \mathbb{P}^N$ such that $\pi^*\mathcal{O}_{\mathbb{P}^N}(1) = \mathcal{H}'^d$. If we take $Y' = \pi^{-1}(Y) \xrightarrow{\tau} Y$, then $\mathcal{O}_Y \longrightarrow \tau_*\mathcal{O}_{Y'}$ splits. By assumption, for $b >> 0$,

$$S^{2\cdot a\cdot b}(\tau^*\mathcal{F} \otimes \mathcal{H}'^{\mu}) \otimes \mathcal{H}'^{b} = \tau^* S^{2ab}(\mathcal{F}) \otimes \tau^*\mathcal{H}^{b}$$

is globally generated over $\tau^{-1}(U)$. The same argument as in b) finishes the proof.

d) Since $\mathcal{O}(1) := \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$ is a quotient of $\pi^*\mathcal{F}$ the "only if" is obvious. For the other direction we choose $\mathcal{H}$ such that $\tau^*\mathcal{H} \otimes \mathcal{O}(1)$ is very ample. For given a we find some b such that $\mathcal{O}(2\cdot b(a-1)) \otimes \mathcal{O}(b) \otimes \tau^*\mathcal{H}^b$ is globally generated over $\tau^{-1}(U)$. Then $\mathcal{O}(2\cdot a\cdot b) \otimes \tau^*\mathcal{H}^{2b}$ will have enough global sections to embed all fibres of π over U into some projective space. Therefore for $\eta >> 0$ the multiplication map

$$\pi_* S^{\eta}(\oplus\ \mathcal{O}) \longrightarrow \pi_*\mathcal{O}(2\cdot a\cdot b\cdot\eta) \otimes \mathcal{H}^{2\cdot b\cdot\eta}$$

is surjective over U.

e) By part d) we only have to consider an invertible sheaf $\mathcal{F}$. Using b) we may assume that the singular locus of Y lies in U. Moreover we may assume $\tau : Y' \longrightarrow Y$ to be a desingularization (The general case then follows from b)). Let us consider first the case where τ is an isomorphism outside of the singular locus S of Y. Since $S \subseteq U$, the invertible sheaf $\mathcal{F}|_S$ is numerically effective and (using Seshadri's criterion) $\mathcal{F}^a \otimes \mathcal{H}|_S$ is ample for all $a > 0$ and all ample invertible sheaves $\mathcal{H}$.

As in b) let E be an effective exceptional divisor such that $\tau^*\mathcal{H}(-E)$ as well as $\tau^*\mathcal{H}(E) \otimes \omega_{Y'}^{-1}$ are ample and such that $\tau_*\tau^*\mathcal{O}_{Y'}(-\nu E) \longrightarrow \mathcal{O}_Y$ for all ν. We claim that for some $\nu > 0$, independent of a, and all $b \geq 0$ the sheaf $\tau^*(\mathcal{F}^a \otimes \mathcal{H}^{\nu+b}) \otimes \mathcal{O}_{Y'}(-\nu E)$ is globally generated over $\tau^{-1}(U)$, that $R^i\tau_*\mathcal{O}_{Y'}(-\nu E) = 0$, for $i > 0$, and that one has a surjection

$$H^0(Y, \mathcal{F}^a \otimes \mathcal{H}^{\nu+b}) \longrightarrow\!\!\!\!\rightarrow H^0(Y, \mathcal{C} \otimes \mathcal{F}^a \otimes \mathcal{H}^{\nu+b})$$

for $\mathcal{C} = \mathrm{coker}(\tau_*\mathcal{O}_{Y'}(-\nu E) \longrightarrow \mathcal{O}_Y)$.

These three statements follow easily from the vanishing theorem for integral parts of $\mathbb{Q}$-divisors (see for example [3] 2.13) applied to some compactification of Y'. Replacing a and ν by a sufficiently high multiple, we may assume that $\mathcal{C} \otimes \mathcal{F}^a \otimes \mathcal{H}^\nu$ is globally generated over S and hence that $\mathcal{F}^a \otimes \mathcal{H}^\nu$ is globally generated over U.

Let Y be non singular and $\tau : Y' \longrightarrow Y$ be an arbitrary blowing up. By induction, we may assume τ to be the blowing up along a non singular center. We take for E the reduced exceptional locus. Then $\tau^*\mathcal{H}(-E)$ can be assumed to be ample and $R^{\bullet}_{\tau_*}(\tau^*\mathcal{H}(-E) \otimes \omega_{Y'}) = \mathcal{H} \otimes \omega_Y$. Using again the vanishing theorem for integral parts of $\mathbb{Q}$-divisors, one obtains that $\mathcal{F}^a \otimes \mathcal{H}^{\dim Y+1} \otimes \omega_Y$ is globally generated over U.

Remark 3.5. Obviously the proof of e) shows that an affirmative answer to Problem 1.12 would imply 3.4e without the assumption on the compactness of the non singular locus. If this holds theorem 3.7 and the usual base change arguments ([3], §3 for example) would imply affirmative answers to 1.11 and 1.10.

Lemma 3.6. *Let $\mathcal{F}$ be weakly positive over U, and let $T(\mathcal{F})$ be any tensor bundle (see 1.8). If $T(\mathcal{F})$ is a positive tensor bundle (see 2.2) then $T(\mathcal{F})$ is weakly positive over U.*

Proof. (see also [19], 3.2) Let $\mathcal{H}$ be ample on Y. By 3.4,c it is enough to show that $T(\mathcal{F} \otimes \mathcal{H})$ is weakly positive over U. By [7], 5.1, $S^{\eta}(T(\mathcal{F} \otimes \mathcal{H}))$ will be a direct summand of

$$S^{\nu_1}(\mathcal{F} \otimes \mathcal{H}) \otimes \ldots \otimes S^{\nu_t}(\mathcal{F} \otimes \mathcal{H})$$

for some ν_i which are growing like η. Therefore $S^{\eta}(T(\mathcal{F} \otimes \mathcal{H}))$ will be globally generated over U for $\eta >> 0$ and 3.3e implies the weak positivity of $T(\mathcal{F} \otimes \mathcal{H})$.

Examples of positive tensor bundles are: $\det(\mathcal{F})$, $S^{\nu}(\mathcal{F})$ and $\Lambda^{\nu}(\mathcal{F})$. Especially, if r is the rank of $\mathcal{F}$ and $\mathcal{F}^{\vee} = \mathcal{H}om(\mathcal{F}, \mathcal{O}_Y)$ then $\Lambda^{r-1}(\mathcal{F}) = \mathcal{F}^{\vee} \otimes \det(\mathcal{F})$ is a positive tensor bundle. Using 3.3b and the equality

$$S^2(\mathcal{F} \oplus \mathcal{F}') = S^2(\mathcal{F}) \oplus S^2(\mathcal{F}') \oplus \mathcal{F} \otimes \mathcal{F}'$$

one sees that weak positivity is compatible with tensor products.

Theorem 3.7. *Let $\nu > 0$ and $f : X \longrightarrow Y$ be a surjective projective flat Gorenstein morphism of reduced quasi projective schemes. Assume that $f_*\omega^{\nu}_{X/Y}$ is locally free. Let $Y_0 \subseteq Y$ be an open subscheme meeting all components of Y such that $f^{-1}(Y_0)$ is normal with at most rational singularities, for $y_0 \in Y_0$. Then $f_*\omega^{\nu}_{X/Y}$ is weakly positive over the non singular locus of Y_0.*

Of course 3.7 will be shown by reducing it to the case $\nu = 1$, where it is nothing but the positivity theorem of Kawamata & Fujita (dim Y = 1). Since we really need to keep track of the locus where the sheaves are weakly positive we sketch the proof:

Proof. We may assume Y_0 to be non singular. Let $\tau : Y' \longrightarrow Y$ be a morphism. We write $f' : X' \longrightarrow Y'$ for the fibre product $X \times_Y Y' \longrightarrow Y'$ and $Y_0' = \tau^{-1}(Y_0)$.

Claim 3.8. We may assume that Y is normal.

Proof. Let $Y' \longrightarrow Y$ be the normalization and $\mathcal{J}$ an ideal sheaf such that $\tau_* \mathcal{J} \subset \mathcal{O}_Y$ and such that the support S of the quotient does not meet Y_0. Using 3.4,a and b, we are allowed to replace Y by a blowing up with center in S and hence we may assume $\mathcal{J}$ to be invertible. By flat base change [6], one has

$$pr_{1*} f'^{*} \mathcal{J} \subset \mathcal{O}_X .$$

This implies that $\tau_*((f'_* \omega^{\nu}_{X'/Y'}) \otimes \mathcal{J})$ is contained in $f_* \omega^{\nu}_{X'/Y'}$. Let $f^s : X^s = X \times_Y \ldots \times_Y X \longrightarrow Y$ be the s-fold fibre product. f^s is again a Gorenstein morphism and $f^s_* \omega^{\nu}_{X^s/Y} = \overset{s}{\otimes} f_* \omega^{\nu}_{X/Y}$ ([18], 3.4, for example). Repeating our calculation for X^s instead of X, we obtain $\tau_*((\overset{s}{\otimes} f'_* \omega^{\nu}_{X'/Y'}) \otimes \mathcal{J})$ as a subsheaf of $\overset{s}{\otimes} f_* \omega^{\nu}_{X/Y}$. The same holds for S^s instead of $\overset{s}{\otimes}$. Choose the ample sheaf $\mathcal{H}$ on Y and $\mathcal{J}$ such that $\tau^* \mathcal{H} \otimes \mathcal{J}$ is ample and $\tau_* \mathcal{O}_{Y'} \otimes \mathcal{H}^b$ generated by its global sections for all $b > 0$. If 3.7 holds for $f' : X' \longrightarrow Y'$, then $S^{2a \cdot b}(f'_* \omega^{\nu}_{X'/Y'}) \otimes \tau^* \mathcal{H}^b \otimes \mathcal{J}^b$ is globally generated over Y_0' for some $b >> 0$. Then $S^{2a \cdot b}(f_* \omega^{\nu}_{X/Y}) \otimes \mathcal{H}^{2b}$ is as well globally generated over Y_0.

<u>Claim 3.9.</u> Let Y be normal, Y' non singular and $\tau : Y' \longrightarrow Y$ a projective generically finite morphism. Assume that 3.7 holds for f'. Then the base change map ([6], III, 9.3.1) $\rho : \tau^* f_* \omega^{\nu}_{X/Y} \longrightarrow f'_* \omega^{\nu}_{X'/Y'}$ is an isomorphism over Y'_0. Moreover 3.7 holds for f.

<u>Proof.</u> Since $f_* \omega^{\nu}_{X/Y}$ is locally free and τ generically flat ρ is injective. If ρ were not surjective over Y'_0 we could find some effective divisor F meeting Y'_0 such that $\tau^* \det(f_* \omega^{\nu}_{X/Y}) \otimes \mathcal{O}_{Y'}(F) = \det(f'_* \omega^{\nu}_{X'/Y'})$. Since F must be an exceptional divisor this contradicts the weak positivity of $\det(f'_* \omega^{\nu}_{X'/Y'})$ over Y'_0. In order to see that 3.7 holds for f we just remark that $f_* \omega^{\nu}_{X/Y}$ is a direct summand of $\tau_* \tau^* f_* \omega^{\nu}_{X/Y} = \tau_* f'_* \omega^{\nu}_{X'/Y'}$. The weak positivity of $f_* \omega^{\nu}_{X/Y}$ over Y_0 follows as in 3.4.e.

Due to 3.8 and 3.9 we may assume Y to be non singular. Moreover, whenever it is convenient, we may replace Y by a generically finite cover. Let $\delta : Z \longrightarrow X$ be a desingularization and $g = f \circ \delta : Z \longrightarrow Y$. Since $f^{-1}(Y_0)$ has rational Gorenstein singularities $\delta_* \omega^{\nu}_Z \longrightarrow \omega^{\nu}_X$ is an isomorphism over $f^{-1}(Y_0)$ and $g_* \omega^{\nu}_{Z/Y} \longrightarrow f_* \omega^{\nu}_{X/Y}$ is an isomorphism over Y_0. g is no longer flat. Nevertheless we get:

<u>Claim 3.10.</u> Let $\tau : Y' \longrightarrow Y$ be either a finite cover or a blowing up. Let $g' : Z' \longrightarrow Y'$ be a desingularization of X'. Then we have an inclusion $\rho : g'_* \omega^{\upsilon}_{Z'/Y'} \longrightarrow \tau^* g_* \omega^{\upsilon}_{Z/Y}$ isomorphic over Y'_0.

<u>Proof.</u> The existence of ρ has been shown in [17] 1.8 and [18], 3.2. ρ is an isomorphism over Y'_0 by 3.9.

For $\upsilon = 1$ 3.7 follows from Kawamata's positivity theorem ([8] or [11]). It says that $g_* \omega_{Z/Y}$ is weakly positive over Y, if there exists some $U \subseteq Y$ such that:

i) $Y - U$ is a normal crossing divisor.
ii) $g^{-1}(U) \longrightarrow U$ smooth
iii) For $k = \dim Z - \dim Y$ the monodromy of $R^k g_* \mathbb{C}_{g^{-1}(U)}$ around the components of $Y - U$ is unipotent.

Those three conditions hold if one replaces Y by a finite cover of a blowing up, and 3.7 follows from 3.10 and 3.4.e.

For $\upsilon > 1$ we have to argue as in [18] §5:

<u>Claim 3.11.</u> Assume that $S^{\mu}(f_* \omega^{\upsilon}_{X/Y} \otimes \mathcal{H}^{\upsilon})$ is globally generated over Y_0 for some $\mu >> 0$. Then $f_* \omega^{\upsilon}_{X/Y} \otimes \mathcal{H}^{\upsilon-1}$ is weakly positive

If $\mathcal{H}$ is any ample sheaf on Y one obtains, as in [18] 5.3, that $f_* \omega_{X/Y}^{\upsilon} \otimes \mathcal{H}^{\upsilon^2-\upsilon}$ is weakly positive over Y_0. This holds as well for the pullback morphism f', if $\tau : Y' \longrightarrow Y$ is a finite cover. By 3.4,c, we finished the proof of 3.7.

Proof of 3.11. (see [17]) If $\tau : Y' \longrightarrow Y$ is generically finite 3.9 tells us that $S^{\mu}(f'_* \omega_{X'/Y'}^{\upsilon} \otimes \tau^* \mathcal{H}^{\upsilon})$ is globally generated over Y'_0. Moreover, adding $\tau^* \mathcal{H}^{\upsilon-1}$ does not change the argument indicated in 3.9 and the weak positivity of $f'_* \omega_{X'/Y'}^{\upsilon} \otimes \tau^* \mathcal{H}^{\upsilon-1}$ implies that of $f_* \omega_{X/Y}^{\upsilon} \otimes \mathcal{H}^{\upsilon-1}$. Therefore again we can replace Y by a generically finite cover, whenever we want to do so.

Let $\mathcal{L} = \omega_{X/Y} \otimes f^* \mathcal{H}$ and $\mathcal{M}$ the subsheaf of $\mathcal{L}$ generated by global sections. Let $\delta : Z \longrightarrow X$ be a desingularization such that $\mathcal{M}' = \delta^* \mathcal{M}$/torsion is invertible and such that for $\mathcal{L}' = \tau^* \mathcal{L}$ and an effective normal crossing divisor D one has $\mathcal{O}_Z(D) = \mathcal{L}'^{\upsilon \cdot \mu} \otimes \mathcal{M}'^{-1}$. $\mathcal{M}'$ again is generated by its global sections. Let $g' : Z' \longrightarrow Y$ be the cyclic cover obtained by taking the $\upsilon \cdot \mu$-th root out of a general section of $\mathcal{M}'$. As, for example, in [3] §2 or [18] §5

$$g_*(\mathcal{L}'^{\upsilon-1}(-[\frac{(\upsilon-1)\cdot D}{\upsilon \cdot \mu}]) \otimes \omega_{Z/Y})$$

is a direct summand of $g'_* \omega_{Z'/Y}$.

Replacing Y by some generically finite cover, we may again assume that g' satisfies the assumptions of Kawamata's positivity theorem, and therefore that $g_*(\mathcal{L}'^{\upsilon-1}(-[\frac{(\upsilon-1)\cdot D}{\upsilon\cdot\mu}])\otimes\omega_{Z/Y})$ is weakly positive over Y. By the choice of $\mathcal{M}$ the map $f^*f_*\mathcal{M}\longrightarrow\mathcal{M}$ is surjective over $f^{-1}(Y_0)$. We have inclusions

$$\mathcal{M}\longrightarrow\delta_*\mathcal{M}'\longrightarrow\delta_*\mathcal{L}'^{\upsilon-1}(-[\frac{(\upsilon-1)\cdot D}{\upsilon\cdot\mu}])\otimes\omega_{Z/Y}\otimes f^*\mathcal{H}.$$

Then the natural inclusion

$$g_*(\mathcal{L}'^{\upsilon-1}(-[\frac{(\upsilon-1)\cdot D}{\upsilon\cdot\mu}])\otimes\omega_{Z/Y})\longrightarrow f_*\mathcal{L}^{\upsilon-1}\otimes\omega_{X/Y}=f_*\omega^{\upsilon}_{X/Y}\otimes\mathcal{H}^{\upsilon-1}$$

is an isomorphism over Y_0 and we obtain 3.11.

<u>Remark 3.12.</u> Let us assume in addition to the assumptions made in 3.7 that $X_0=f^{-1}(Y_0)\longrightarrow Y_0$ is smooth. Then an affirmative answer to problem 1.13 implies that $f_*\omega^{\upsilon}_{X/Y}$ is weakly positive over Y_0.

<u>"Proof."</u> Even if Y_0 is not normal we can find a finite cover $\tau: Y_0'\longrightarrow Y_0$ such that $\mathcal{O}_{Y_0}$ is a direct summand of $\tau_*\mathcal{O}_{Y_0'}$ and such that the morphism $X_0'\longrightarrow Y_0'$ obtained as pullback of f satisfies the assumptions made in 1.13. By 3.4.6 we may assume that $f_0=f|_{X_0}: X_0\longrightarrow Y_0$ satisfies those assumptions. Blowing up the boundary $Y-Y_0$ and using 1.13 we can extend $f_*\omega_{X_0/Y_0}$ to a locally free sheaf $\mathcal{F}$ on

some compactification $\overline{Y}$ of Y_0. If $\overline{Y}'$ is a good desingularization of $\overline{Y}$ the pullback $\mathcal{F}'$ of $\mathcal{F}$ to $\overline{Y}'$ is weakly positive over $\overline{Y}'$. Then by 3.4.e $\mathcal{F}$ is weakly positive over $\overline{Y}$. It is well known that $\mathcal{F}'$ is the direct image of the relative dualizing sheaf of some desingularization of $\overline{Y}' \times_Y X$. Therefore one has a natural map from $\mathcal{F}$ to $f_*\omega_{X/Y'}$ isomorphic over Y_0. If $\upsilon > 1$ one has to repeat the arguments used in 3.11.

A similar argument should work as well if $X_0 \longrightarrow Y_0$ is not smooth. However one has to try to study the necessary base change properties more carefully. Since, anyway, we do not know an answer to 1.13 we do not insist on this implication.

§4 Fibre spaces

We want to prove 1.19. As we have seen already in §1, C 1.19 implies 1.18 as well.

4.1. Let $f : X \longrightarrow Y$ and $Y_0 \subseteq Y$ satisfy the assumptions made in 1.17 and 1.19. As in 3.3,d it is easy to see that we can assume that $f_* \omega^{\nu}_{X/Y}$ is locally free. Let $\mathcal{E} = f_* \omega^{\nu}_{X/Y}$ and r be the rank of $\mathcal{E}$. Let $\pi : \mathbb{P}(\mathcal{E}) \longrightarrow Y$ be the projective bundle and $\rho : X \longrightarrow \mathbb{P}(\mathcal{E})$ the induced rational map.

For $y \in Y_0$ we have assumed that $\rho|_{f^{-1}(y)}$ is birational. If $\mathcal{J}$ is the ideal sheaf of $\rho(X)$ we can find some $\mu >> 0$ such that $\pi^* \pi_* (\mathcal{J} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(\mu)) \longrightarrow \mathcal{J} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(\mu)$ is surjective. For simplicity we assume that r divides μ. Let us consider $m : S^{\mu}(f_* \omega^{\nu}_{X/Y}) \longrightarrow f_* \omega^{\nu \cdot \mu}_{X/Y}$. By our assumption $\mathcal{G} = \mathrm{Im}(m)$ is locally free over U, and, leaving out some codimension two subspace of $Y - U$, we may again assume that $\mathcal{G}$ is locally free over Y. For simplicity we write $U = Y_0$ and assume Y_0 to be already non singular.

4.2. Recall that we have to show that $\mathcal{L} = \det(\mathcal{G})^a \otimes \det(\mathcal{E})^b$ is ample with respect to Y_0 for some $a,b > 0$.

Let $r' = \mathrm{rank}(\mathcal{G})$ and consider

$$\Lambda^{r'}m : \Lambda^{r'}S^{\mu}(f_*\omega^{\nu}_{X/Y}) \longrightarrow \Lambda^{r'}f_*\omega^{\nu\cdot\mu}_{X/Y}.$$

The image of $\Lambda^{r'}m$ is $\det(\mathcal{G})$. For
$\mathcal{L}_0 = \det(\mathcal{G}) \otimes \det(\mathcal{E})^{-\frac{r'\cdot\mu}{r}}$ we obtain from $\Lambda^{r'}m$ a surjection

$$\tau : \Lambda^{r'}S^{\mu}(\mathcal{E}) \otimes \det(\mathcal{E})^{-\frac{r'\mu}{r}} \longrightarrow \mathcal{L}_0 .$$

Let us return to the construction made in 2.5, i.e.: Let
$\pi:\mathbb{P} = \mathbb{P}(\overset{r}{\oplus}\,\mathcal{E}^{\vee}) \longrightarrow Y$ be the projective bundle.
$\underline{s} : \overset{r}{\oplus}\,\mathcal{O}_{\mathbb{P}}(-1) \longrightarrow \pi^*\mathcal{E}$ the universal basis and D the degeneration locus of $\underline{s}$.

<u>Claim 4.3.</u> $\pi^*\mathcal{L}_{0|\mathbb{P}-D}$ is ample with respect to
$\pi^{-1}(Y_0) \cap \mathbb{P} - D$.

<u>Proof.</u> By definitions $\underline{s}|_{\mathbb{P}-D}$ is an isomorphism. Therefore
$\mathcal{O}_{\mathbb{P}}(-r)|_{\mathbb{P}-D} = \pi^*\det(\mathcal{E})|_{\mathbb{P}-D}$. $\pi^*(\tau)$ is a surjection

$$\Lambda^{r'}S^{\mu}(\mathbb{C}^r) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}-D} \longrightarrow\!\!\!\!\rightarrow \pi^*\mathcal{L}_0|_{\mathbb{P}-D}.$$

This implies that $\pi^*\mathcal{L}_0|_{\mathbb{P}-D}$ is globally generated over $\mathbb{P}-D$.
Let $\mathcal{H}ilb(\mathbb{P}^{r-1})$ be the Hilbert scheme of subschemes of $\mathbb{P}^{r-1}$.
Since $\pi^*\mathcal{E}|_{\mathbb{P}-D}$ is a direct sum of r copies of $\mathcal{O}_{\mathbb{P}-D}(-1)$
the ν-canonical rational map gives rise to a morphism

$h : \pi^{-1}(Y_0) \cap \mathbb{P}\text{-}D \longrightarrow \mathcal{H}ilb(\mathbb{P}^{r-1})$. Let H be the component containing the image of h. H can be embedded in a projective space by the Plücker coordinates ([1], 2.6 for example). Composing with h we obtain a rational map $\tilde{h}$ from $\mathbb{P} - D$ in some projective space. By [1], 2.6, this map is given by the surjection $\pi^*(\tau)$. Especially $\tilde{h} : \mathbb{P} - D \longrightarrow \mathbb{P}(\Lambda^{r'} S^{\mu}(\mathbb{C}^r))$ is a morphism and $\tilde{h}^* \mathcal{O}(1) = \pi^* \mathcal{L}_0|_{\mathbb{P}-D}$. We have assumed that only finitely many fibres $f^{-1}(y)$ are birational, for $y \in Y_0$. Since the map $X \times_Y \mathbb{P} \longrightarrow \mathbb{P}(\pi^* \mathcal{E}) \simeq \mathbb{P}^{r-1} \times \mathbb{P}$ is given by a universal basis this implies that the fibres of $h : (\mathbb{P} - D) \cap \pi^{-1}(Y_0) \longrightarrow H$ are finite. Then $\pi^* \mathcal{L}_0|_{\mathbb{P}-D}$ must be ample with respect to $(\mathbb{P} - D) \cap \pi^{-1}(Y_0)$.

<u>Remark 4.4.</u> If $\mathcal{G} = f_* \omega_{X/Y}^{\upsilon \cdot \mu}$ the sheaf $\mathcal{L}_0^r$ is just the same as the sheaf considered in 1.4. If $Y = Y_0$, in addition, the map $\tilde{h}$ already appeared in 1.2 and 1.3. The bundle $\mathbb{P} - D \longrightarrow Y$ has $\mathbb{P}Gl(r,\mathbb{C})$ as fibres. If one considers the corresponding group action the invariant sections of $\pi^* \mathcal{L}_0$ are those coming from Y.

<u>4.5.</u> Let $\tau : \mathbb{P}' \longrightarrow \mathbb{P}$ be a proper modification with center in D such that the rational map $h' = \tilde{h} \circ \tau : \mathbb{P}' \longrightarrow \mathbb{P}(\overset{r}{\Lambda} S^{\mu}(\mathbb{C}^r))$ is a morphism. We can as well assume that there exists an exceptional divisor F of τ and a divisor E supported outside of $\pi'^{-1}(Y_0)$, where $\pi' = \pi \circ \tau$, such that $\mathcal{O}_{\mathbb{P}'}(-E-F)$ is relatively ample for h'. If we write $(\tau^* D)_{red} = \sum D_i'$. We have $F_{red} \leq \sum D_i'$. Therefore

we can find some $a > 0$ and $\gamma_i \in \mathbb{Z}$ such that $\pi'^*\mathcal{L}_0^a \otimes \mathcal{O}_{\mathbb{P}'}(\sum \gamma_i D_i' - E)$ is ample on $\mathbb{P}'$.

Remark 4.6. Under the assumptions made in 4.4, the Hilbert-Mumford criterium ([14], Ch. II, §1) for the $\mathbb{P}Gl(r,\mathbb{C})$ action on $\mathbb{P} - D$ seems to say that one can choose $\gamma_i > 0$. Since we can not verify this criterium we just add some effective divisor supported in $\sum D_i'$ and use "weak positivity" to show that this does not effect "ampleness".

4.7. By 3.7 $\mathcal{E}$ is weakly positive over Y_0 and from 3.4 a) and 3.6 it follows that $\overset{r}{\oplus} \pi'^*(\mathcal{E}^\vee \otimes \det(\mathcal{E}))$ is weakly positive over $\pi'^{-1}(Y_0)$. Then the quotient sheaf $\tau^*\mathcal{O}_{\mathbb{P}}(1) \otimes \pi'^*\det(\mathcal{E})$ and its r-th power are again weakly positive over $\pi'^{-1}(Y_0)$. By definition of D (see 2.5) this is nothing but

$$\mathcal{O}_{\mathbb{P}'}(\tau^*D) \otimes \pi'^*\det(\mathcal{E})^{r-1}.$$

For some $\eta \gg 0$ the divisor $\eta \cdot \tau^*D + \sum \gamma_i D_i' = D'$ will be effective. By 3.2

$$\pi'^*\mathcal{L}_0^a \otimes \mathcal{O}_{\mathbb{P}'}(\sum \gamma_i D_i' - E) \otimes (\mathcal{O}_{\mathbb{P}'}(\tau^*D) \otimes \pi'^*\det(\mathcal{E})^{r-1})^\eta$$

$$= \pi'^*(\mathcal{L}_0^a \otimes \det(\mathcal{E})^{(r-1)\cdot\eta}) \otimes \mathcal{O}_{\mathbb{P}'}(D') \otimes \mathcal{O}_{\mathbb{P}'}(-E)$$

is ample with respect to $\pi'^{-1}(Y_0)$. For $b = -\frac{a \cdot r' \cdot \mu}{r} + (r-1)\cdot\eta$ this sheaf is $\pi'^*(\mathcal{L}) \otimes \mathcal{O}_{\mathbb{P}'}(D') \otimes \mathcal{O}_{\mathbb{P}'}(-E)$. Since E

is not meeting Y_0 the sheaf $\pi'^*(\mathcal{L}) \otimes \mathcal{O}_{\mathbb{P}'}(D')$ is also ample with respect to $\pi'^{-1}(Y_0)$. We can apply 2.7 and we find $\mathcal{L}$ to be ample with respect to U.

Remark 4.8. Some ingrediants used before in the proofs of $C^+_{n,m}$ are reappearing in this chapter: The Hilbert-Mumford criterium and stability (see 4.6) which was used in [18]. The multiplication map, used by Kollár in [12]. The "mysterious covering trick" used in [12] and [19] is of course hidden in the construction of $\mathbb{P}$ and 2.7.

§5 Stability of certain Hilbert points

In this section we want to prove 1.7. Recall that for each of our moduli functors $\mathcal{M}_h$ we have some $\upsilon > 0$ and the Hilbert scheme H of υ-canonical embedded varieties of $\mathcal{M}_h$. We have a universal family $h : \mathfrak{X} \longrightarrow H$. $\mathcal{E} := h_* \omega^{\upsilon}_{\mathfrak{X}/H}$ is a direct sum of r copies of an invertible sheaf $\mathcal{N}$ and $\lambda_{\upsilon} = \det(\mathcal{E}) = \mathcal{N}^r$ (see 1.2). Moreover the sheaf $\mathcal{L}_0 = \lambda^{r(\upsilon)}_{\upsilon\cdot\mu} \otimes \lambda^{-r(\upsilon\cdot\mu)\cdot\mu}_{\upsilon}$ introduced in 1.4 is ample on H.

Similar to our arguments in §4 we will use the results on weak positivity (§3) to show that:

Replacing $\mathcal{L}_0$ by $\mathcal{L}_{\eta} = \mathcal{L}_0 \otimes \lambda^{\eta}_{\upsilon}$ for $\eta >> 0$ we get enough sections of $\mathcal{L}_{\eta}$, "positive" at the boundary of an orbit, and this implies that all points, where $\mathcal{L}_{\eta}$ is ample, are stable with respect to the $G = Sl(r,\mathbb{C})$ action on H.

Due to our "gap" 1.10 we are only able to prove that $\mathcal{L}_{\eta}$ is ample with respect to the largest open subscheme H_0 of H with $(H_0)_{red}$ smooth.

Before we recall the tools needed from Mumford's geometric invariant theory [14] we have to prove an unpleasent technical lemma:

__Lemma 5.1.__ *Let* $x \in H$ *be given and* H_x *the corresponding G-orbit. Then there exists a quasi projective scheme* H' *containing* H *as an open subscheme and a prolongation* $h' : \mathfrak{X}' \longrightarrow H' \in \mathcal{M}_h(H')$ *of* h *such that the closure* H'_x *of* H_x *in* H' *is projective and* $h'^{-1}(H'_x) \longrightarrow H'_x$ *is isomorphic to* $H'_x \times h^{-1}(x) \longrightarrow H'_x$.

__Proof:__ Let us start with an arbitrary projective H' such that $H'-H$ is a divisor. Let $\mathcal{E}'$ be a coherent extension of $\mathcal{E}$ to H' , which we can assume to be locally free (blowing up H' a little bit). Since $h^{-1}(H_x) \longrightarrow H_x$ is trivial $\mathcal{E}|_{H_x} \cong \overset{r}{\oplus}\, \mathcal{O}_{H_x} = \mathcal{O}_{H_x} \otimes_{\mathbb{C}} H^0(h^{-1}(x), \omega^{\upsilon}_{h^{-1}(x)})$. Let $s_1 \dots s_r$ be the "trivial sections" of $\mathcal{E}|_{H_x}$ coming from a basis of $H^0(h^{-1}(x), \omega^{\upsilon}_{h^{-1}(x)})$. Adding some multiple of $H'-H$ we may assume that $s_1 \dots s_r$ give rise to sections of $\mathcal{E}'|_{H'_x}$, which we denote again by s_i . Let $\mathcal{A}$ be a very ample sheaf on H' such that $\mathcal{E}' \otimes \mathcal{A} \otimes \mathcal{J}$ is generated by its sections and has no higher cohomology, where $\mathcal{J}$ is the ideal sheaf of H'_x .

Let $A_1, \dots, A_\ell$ be divisors of $\mathcal{A}$ in general position such that $\cap A_j = \emptyset$. For each j we have sections $s_i^{(j)} \in H^0(H'_x, \mathcal{E}' \otimes \mathcal{O}_{H'_x}(A_j \cap H'_x))$ and we can find a tuple of sections

$$\underline{s}_i^{(j)} : \oplus\, \mathcal{O}_{H'} \longrightarrow \mathcal{E}' \otimes \mathcal{O}_{H'}(A_j)$$

such that each of the sections, restricted to H'_x, gives $s_i^{(j)}$ and such that $\underline{s}_i^{(j)}$ is surjective over $H' - H'_x$. We denote the induced map $\oplus\, \mathcal{A}^{-1} \longrightarrow \mathcal{E}'$ again by $\underline{s}_i^{(j)}$. Adding up over all i and j we obtain $\underline{s} : \bigoplus_{i,j} \oplus\, \mathcal{A}^{-1} \longrightarrow \mathcal{E}'$, surjective over $H' - H'_x$, and $\mathrm{Im}(\underline{s}|_{H'_x})$ is just the subsheaf generated by $s_1, \ldots, s_r$. Blowing up with center in $H'_x - H_x$, we obtain a similar map where the image is locally free.

Therefore we may assume that we have choosen $\mathcal{E}'$ from the beginning to be a locally free sheaf such that $s_1, \ldots, s_r$ generate $\mathcal{E}'|_{H'_x}$. We have a diagram

$$\begin{array}{ccccc} \mathfrak{X} & \longrightarrow & \mathbb{P}(\mathcal{E}) & \longrightarrow & \mathbb{P}(\mathcal{E}') \\ \downarrow & & \downarrow & & \downarrow \\ H & = & H & \longrightarrow & H' . \end{array}$$

We choose $h' : \mathfrak{X}' \longrightarrow H'$ by taking the compactification of $\mathfrak{X}$ in $\mathbb{P}(\mathcal{E}')$. h' need not be flat, but (as we did in [18] p. 345) we can blow up H' with center in $H' - H$ such that the dominating component of the pullback family becomes flat. So we may assume that $\mathfrak{X}'$ is a subscheme of $\mathbb{P}(\mathcal{E}')$, flat over H'.

Restricting everything to H'_x we have

$$\begin{array}{ccc} h'^{-1}(H'_x) & \longrightarrow & \mathbb{P}(\mathcal{E}'|_{H'_x}) = \mathbb{P}^{r-1} \times H'_x \\ \downarrow & & \downarrow \\ H'_x & = & H'_x \end{array}$$

and - over H_x the family, as well as the embedding, is constant. Then $h'^{-1}(H'_x)$ must be the product of H'_x with $h^{-1}(x)$.

The condition that a fibre of h' belongs to $\mathcal{M}_h(\mathbb{C})$ is open (otherwise we would not have a Hilbert scheme, see [10]). Therefore, replacing H' by some open neighbourhood of H'_x which contains H we are done.

5.2. Let $G = Sl(r,\mathbb{C})$ be acting on an algebraic scheme X and let $\mathcal{L}$ be an invertible G-linearized sheaf on X ([14], Def.: 1.6).

Definition 5.2. ([14], Def. 1.7)

(i) A geometric point $x \in X$ is called stable (with respect to $\mathcal{L}$) if there exists, for some $N > 0$, a G-invariant section $s \in H^0(X,\mathcal{L}^N)$ such that: $s(X) \neq 0$, $X_s = X$ - (zero set of s) is affine and the action of G on X_s is closed.

(ii) We write $X(\mathcal{L})^s$ for the set of stable points with finite stabelizers.

Remark 5.4. $X(\mathcal{L})^s$ is an open subscheme and $\mathcal{L}|_{X(\mathcal{L})^s}$ is ample. In [14] $X(\mathcal{L})^s$ is denoted by $X^s_{(0)}$ (We changed the notation since $H^s_{(0)}(\mathcal{L})$ looks too much like a cohomology group). The subscheme $X(\mathcal{L})^s$ is independent of the G-linearization choosen ([14], Cor. 1.17).

Lemma 5.5 *Assume that* X *is a projective variety containing a dense orbit* X_0 *on which* G *acts with finite stabelizers. Assume moreover that the* N*-th power of the sheaf* $\mathcal{L}$ *has a section* s *with zero set* D *such that* $X - X_0 = D_{red}$. *Then* $X(\mathcal{L})^s \supset X_0$.

Proof. D_{red} is invariant under G. Therefore D has only finitely many conjugates under G and taking the product of the corresponding sections we may assume D to be G-invariant. Therefore we have one G-linearization of $\mathcal{L}^N = \mathcal{O}_X(D)$ such that s is a G-invariant section. By [14], Prop. 1.4, there exists at most one G-linearization of $\mathcal{L}^N$. So we found a G-invariant section s with $X_0 = X_s$. Since G is affine and acts on X_0 with finite stabelizer X_0 is affine.

Proposition 5.6. ([14] Prop. 1.18, 1.16 and Thm. 1.19) *Let* $i : Y \longrightarrow X$ *be a G-linear embedding. Then:*

a) $Y \cap X(\mathcal{L})^s \subset Y(i^*\mathcal{L})^s$

b) *If* $Y = X_{red}$, *then* $(X(\mathcal{L})^{s})_{red} = Y(i^{*}\mathcal{L})^{s}$

c) *If* Y *is proper and* $\mathcal{L}$ *ample, then* $Y \cap X(\mathcal{L})^{s} = Y(i^{*}\mathcal{L})^{s}$

Remark. The open subscheme $X(\mathcal{L})^{s}$ of X depends on the G-linearized sheaf $\mathcal{L}$ choosen. By [14], converse 1.13, one knows however that for an open G-invariant subscheme $U \subseteq X$ one has an equivalence of:

(i) For some G-linearized invertible sheaf $\mathcal{L}'$ one has $U \subseteq X(\mathcal{L}')^{s}$

(ii) The action of G on U is proper and a geometric quotient of U by G exists as a quasi projective scheme.

5.7. Let us return to the notations introduced in §1, A) and recalled in the beginning of this chapter. We know from 3.7, 3.5 and 3.2 that for all $\eta \geq 0$ the sheaf $\mathcal{L}_{\eta} = \mathcal{L}_{0} \otimes \lambda_{\upsilon}^{\eta}$ is ample with respect to $(H_{0})_{red}$.

It is only here that we need $(H_{0})_{red}$ to be smooth. In order, not to distinguish between part a and b of 1.7 we write $H = H_{0}$ in case a) and leave H unchanged in case b). Then $\mathcal{L}_{\eta}$ is ample with respect to $(H)_{red}$. We have seen that stability is a Zariski-open condition. Therefore 1.7 follows if we show that:

<u>Claim 5.8.</u> For a given point $x \in H$ there exists some η_0, such that $x \in H(\mathcal{L}_\eta)^s$ for all $\eta \geq \eta_0$.

<u>Proof.</u> By 5.6,b we can replace H by $(H)_{red}$ and - again by abuse of notations - we assume $\mathfrak{X}$ and H to be reduced. The orbit H_x is a quotient of the $Sl(r,\mathbb{C})$ by the stabelizer of x or of $\mathbb{P}Gl(r,\mathbb{C})$. $\mathbb{P}Gl(r,\mathbb{C})$ can be compactified by $\mathbb{P} = \mathbb{P}(\overset{r}{\oplus}\, \mathbb{C}^r)$.

If H' and H'_x are choosen as in 5.1, we can blow up the boundary $\mathbb{P} - \mathbb{P}Gl(r,\mathbb{C})$ to obtain another compactification $\mathbb{P}'$ of $\mathbb{P}Gl(r,\mathbb{C})$ and a finite map $\tau : \mathbb{P}' \longrightarrow H'_x$. The sheaf $\mathcal{L}_0$ is ample on H and $\mathcal{L}_0|_{H_x} = \mathcal{O}_{H_x}$. Blowing H' up we may assume that $\mathcal{L}_0^\alpha$ extends to an ample sheaf $\mathcal{L}_0^{(\alpha)}$ on H' and that

$$\tau^*\mathcal{L}_0^{(\alpha)} = \mathcal{O}_{\mathbb{P}'}(\sum \eta_i D_i),$$

with $\eta_i \in \mathbb{Z}$ and $\sum D_i = \mathbb{P}' - \mathbb{P}Gl(r,\mathbb{C})$.

By 3.7 the sheaf $\mathcal{E}' = h'_*\omega^\nu_{\mathfrak{X}/H'}$ is weakly positive over the non singular locus of H'.

If (1.10) holds for $X \longrightarrow Y \in \mathcal{M}_h(Y)$ then $\mathcal{E}'$ is weakly positive over H'.

Since we replaced H by its non singular locus in case a) we may assume H' to be non singular in this case and $\mathcal{E}'$ to be weakly positive over H' as well.

Remember that $\mathcal{E} = \mathcal{E}'|_H \simeq \bigoplus^r \mathcal{N}$. We can choose an extension $\mathcal{N}'$ of $\mathcal{N}$ such that this isomorphism gives an inclusion $\underline{s} : \bigoplus^r \mathcal{N}' \longrightarrow \mathcal{E}'$ or $\mathcal{N}' \longrightarrow \bigoplus^r \mathcal{E}'$. Blowing up centers in H'-H , we may assume that $\mathcal{N}'$ is a subbundle. As in 2.5 we obtain natural maps

$$\mathcal{N}'^r \longrightarrow S^r(\bigoplus^r \mathcal{E}') \longrightarrow\!\!\!\!\rightarrow \bigotimes^r \mathcal{E}' \longrightarrow\!\!\!\!\rightarrow \det(\mathcal{E}') \ .$$

Let Δ be the divisor of the corresponding section of $\det(\mathcal{E}') \otimes \mathcal{N}'^{-r}$.

Regarding the dual construction we get surjections

$S^r(\bigoplus^r \mathcal{E}'^{\vee}) \longrightarrow \mathcal{N}'^{-r}$ and

$S^r(\bigoplus^r \mathcal{E}'^{\vee} \otimes \det \mathcal{E}) \longrightarrow \mathcal{N}'^{-r} \otimes \det(\mathcal{E}')^r = \mathcal{O}_{H'}(\Delta) \otimes \det(\mathcal{E}')^{r-1}$.

From 3.5 and 3.3 c) we find that $\mathcal{O}_{H'}(\Delta) \otimes \det(\mathcal{E}')^{r-1}$ is weakly positive over H'.

If r-1 divides α we write

$$\mathcal{L}_\eta^{(\alpha)} = \mathcal{L}_0^{(\alpha)} \otimes (\mathcal{O}_{H'}(\Delta) \otimes \det(\mathcal{E}')^{r-1})^{\frac{\alpha \cdot \eta}{r-1}}.$$

One has $\mathcal{L}_\eta^{(\alpha)}|_{H_0} = \mathcal{L}_\eta^\alpha$. As a tensor product of an ample and a weakly positive sheaf $\mathcal{L}_\eta^{(\alpha)}$ is ample (see 3.2).

<u>Claim 5.9.</u> There exists some η_0 such that, for all $\eta > \eta_0$, $\tau^*\mathcal{L}_\eta^{(\alpha)} = \mathcal{O}_{\mathbb{P}'}(D')$ for an effective divisor D' with $(D')_{red} = \mathbb{P}' - \mathbb{P}GL(r,\mathbb{C})$.

<u>Proof.</u> Since $\tau^*\mathcal{L}_0^{(\alpha)} = \mathcal{O}_{\mathbb{P}'}(\sum \eta_i D_i)$, for $\eta_i \in \mathbb{Z}$ and $\sum D_i = \mathbb{P}' - \mathbb{P}Gl(r,\mathbb{C})$, we just have to verify that $(\tau^*\Delta)_{red} = \mathbb{P}' - \mathbb{P}Gl(r,\mathbb{C})$. This is however contained in 2.5: The inclusion $\tau^*\mathcal{N}' \longrightarrow \overset{r}{\oplus}\, \tau^*\mathcal{E}' = \overset{r}{\oplus}\, \mathbb{C}^r \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}'}$ was induced by $\underline{s} : \overset{r}{\oplus}\, \tau^*\mathcal{N}' \longrightarrow \mathbb{C}^r \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}'}$. Restricted to $\mathbb{P}Gl(r,\mathbb{C})$ $\underline{s}$ is just given by the action of $\mathbb{P}Gl(r,\mathbb{C})$ on $\mathbb{P}(H^0(h^{-1}(x),\omega^\nu_{h^{-1}(x)}))$. Therefore $\underline{s}$ coincides with the universal basis considered in 2.5 and $\tau^*\mathcal{N}'$ is the pullback of $\mathcal{O}_{\mathbb{P}}(-1)$ to $\mathbb{P}'$. Therefore $\tau^*\Delta$ is the pullback of the degeneration locus of $\underline{s}$ and $\mathbb{P}' - (\tau^*\Delta)_{red} = \mathbb{P}Gl(r,\mathbb{C})$.

<u>Now we can finish the proof of 5.8:</u>

$\tau^*\mathcal{L}_\eta^{(\alpha)}$ has a section s whose zero divisor is supported exactly on $\mathbb{P}' - \mathbb{P}Gl(r,\mathbb{C})$. Since $\tau : \mathbb{P}' \longrightarrow H'_x$ is finite we may assume that $\mathcal{L}_\eta^{(\alpha)}|H'_x$ has a section s' whose zero set D' satisfies $(D')_{red} = H'_x - H_x$ (replacing α by some multiple, if necessary) and we may assume $\mathcal{L}_\eta^{(\alpha)}$ to be very ample. $\mathcal{L}_\eta^{(\alpha)}$ extends to a very ample invertible sheaf on some compactification $\overline{H}'$ of H'. By Serre's vanishing theorem some power of the section s' of $\mathcal{L}_\eta^{(\alpha)}|H'_x$ is the image of a section $\sigma' \in H^0(\overline{H}',\mathcal{L}_\eta^{(\alpha)\cdot N})$. Again we may assume that $N = 1$.

Since $H^0(\overline{H}',\mathcal{L}_\eta^{(\alpha)}) \subseteq H^0(H,\mathcal{L}_\eta^{(\alpha)})$ we can use [14], Ch. I, §1, to find a G-invariant finite dimensional subspace V containing $H^0(\overline{H}',\mathcal{L}_\eta^{(\alpha)})$. Let $i : H \longrightarrow \mathbb{P}(V)$ the corresponding embedding and $\overline{H}$ the closure of $i(H)$. We have a birational map $\rho : \overline{H}' \longrightarrow \overline{H}$. We can blow up $\overline{H}'$ along centers in $\overline{H}' - H$ to get $\delta : \overline{H}'' \longrightarrow \overline{H}'$ and a morphism $\rho \circ \delta$. Choosing α big enough we will still get a section σ'' of an ample subsheaf of $\delta^*\mathcal{L}_\eta^{(\alpha)}$ with a zero divisor B such that H_x is closed in $\overline{H}'' - (B)_{red}$. Hence, to simplify notations, we may assume ρ to be a morphism.

Let $\mathcal{O}_{\overline{H}}(1) = \mathcal{O}_{\mathbb{P}(V)}(1)|_{\overline{H}}$. Since $\mathcal{L}_\eta^{(\alpha)} \longrightarrow \rho^*\mathcal{O}_{\overline{H}}(1)$ the section σ' is the pullback of some section σ of $\mathcal{O}_{\overline{H}}(1)$. By construction G operates on $\mathbb{P}(V)$, on $\overline{H}$ and on the closure $\overline{H}_x$ of H_x in $\overline{H}$. Both $\mathcal{O}_{\mathbb{P}(V)}(1)$ and $\mathcal{O}_{\overline{H}}(1)$ are G-linearised. Since the zero locus of $\sigma|_{\overline{H}_x}$ is exactly supported on $\overline{H}_x - H_x$ we can apply 5.5 and find $H_x \subset \overline{H}_x(\mathcal{O}_{\overline{H}}(1)|_{\overline{H}_x})^s$.

By 5.6,c

$$\overline{H}_x(\mathcal{O}_{\overline{H}}(1)|_{\overline{H}_x})^s = \overline{H}_x \cap \overline{H}(\mathcal{O}_{\overline{H}}(1))^s$$

and $H_x \subseteq \overline{H}(\mathcal{O}_{\overline{H}}(1))^s$. On the other hand 5.6, a tells us that

$$H_x \subseteq H \cap \overline{H}(\mathcal{O}_{\overline{H}}(1))^s \subseteq H(\mathcal{O}_{\overline{H}}(1)|_H)^s.$$

Since $\mathcal{O}_{\overline{H}}(1)|_H = \mathcal{L}_\eta^\alpha$ our given point x lies in $H(\mathcal{L}_\eta^\alpha)^s = H(\mathcal{L}_\eta)^s$ for all $\eta > \eta_0$.

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