ON THE DYNAMICAL YANG-BAXTER EQUATION

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To Mira

1. Introduction

This talk is inspired by two previous ICM talks, [Dr] and [Fe]. Namely, one of the main ideas of [Dr] is that the quantum Yang-Baxter equation (QYBE), which is an important equation arising in quantum field theory and statistical mechanics, is best understood within the framework of Hopf algebras, or quantum groups. On the other hand, in [Fe], it is explained that another important equation of mathematical physics, the star-triangle relation, may (and should) be viewed as a generalization of QYBE, in which solutions depend on additional "dynamical" parameters. It is also explained there that to a solution of the quantum dynamical Yang-Baxter equation one may associate a kind of quantum group. These ideas gave rise to a vibrant new branch of "quantum algebra", which may be called the theory of dynamical quantum groups. My goal in this talk is to give a bird's eye review of some aspects of this theory and its applications.

Remark. Because of size restrictions, the list of references below is not complete. The reader is referred to original papers and review [ES2] for more references.

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2. The quantum dynamical Yang-Baxter equation

2.1. **The equation.** The quantum dynamical Yang-Baxter equation (QDYBE), proposed by Gervais, Neveu, and Felder, is an equation with respect to a (meromorphic) function $R : \mathfrak{h}^* \to \mathrm{End}_{\mathfrak{h}}(V \otimes V)$, where $\mathfrak{h}$ is a commutative finite dimensional Lie algebra over $\mathbb{C}$, and V is a semisimple $\mathfrak{h}$-module. It reads

$$R^{12}(\lambda - h^3)R^{13}(\lambda)R^{23}(\lambda - h^1) = R^{23}(\lambda)R^{13}(\lambda - h^2)R^{12}(\lambda)$$

on $V \otimes V \otimes V$. Here h^i is the *dynamical notation*, to be extensively used below: for instance, $R^{12}(\lambda - h^3)$ is defined by the formula $R^{12}(\lambda - h^3)(v_1 \otimes v_2 \otimes v_3) := \big(R^{12}(\lambda - \mu)(v_1 \otimes v_2)\big) \otimes v_3$ if v_3 is of weight μ under $\mathfrak{h}$.

It is also useful to consider QDYBE with spectral parameter. In this case the unknown function R depends meromorphically on an additional complex variable u. Let $u_{ij} = u_i - u_j$. Then the equation reads

$$R^{12}(u_{12}, \lambda - h^3)R^{13}(u_{13}, \lambda)R^{23}(u_{23}, \lambda - h^1) = R^{23}(u_{23}, \lambda)R^{13}(u_{13}, \lambda - h^2)R^{12}(u_{12}, \lambda).$$

Sometimes it is necessary to consider QDYBE with step $\gamma \in \mathbb{C}^*$, which differs from the usual QDYBE by the replacement $h^i \to \gamma h^i$. Clearly, $R(\lambda)$ satisfies QDYBE iff $R(\lambda/\gamma)$ satisfies QDYBE with step γ.

Invertible solutions of QDYBE are called quantum dynamical R-matrices (with or without spectral parameter, and with step γ if needed). If $\mathfrak{h} = 0$, QDYBE turns into the usual quantum Yang-Baxter equation $R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$.

2.2. **Examples of solutions of QDYBE.** Let V be the vector representation of $sl(n)$, and $\mathfrak{h}$ the Lie algebra of traceless diagonal matrices. In this case $\lambda \in \mathfrak{h}^*$ can be written as $\lambda = (\lambda_1, ..., \lambda_n)$, where $\lambda_i \in \mathbb{C}$. Let v_a, $a = 1, ..., n$ be the standard basis of V. Let E_{ab} be the matrix units given by $E_{ab}v_c = \delta_{bc}v_a$.

We will now give a few examples of quantum dynamical R-matrices. The general form of the R-matrices will be

$$R = \sum_a E_{aa} \otimes E_{aa} + \sum_{a \neq b} \alpha_{ab} E_{aa} \otimes E_{bb} + \sum_{a \neq b} \beta_{ab} E_{ab} \otimes E_{ba}, \tag{1}$$

where α_{ab}, β_{ab} are functions which will be given explicitly in each example.

Example 1. The basic rational solution. Let $\beta_{ab} = \frac{1}{\lambda_b - \lambda_a}$, $\alpha_{ab} = 1 + \beta_{ab}$. Then $R(\lambda)$ is a dynamical R-matrix (with step 1).

Example 2. The basic trigonometric solution. Let $\beta_{ab} = \frac{q-1}{q^{\lambda_b - \lambda_a} - 1}$, $\alpha_{ab} = q + \beta_{ab}$. Then $R(\lambda)$ is a dynamical R-matrix (also with step 1).

Example 3. The basic elliptic solution with spectral parameter (Felder's solution). Let $\theta(u, \tau) = -\sum_{j \in \mathbb{Z}+1/2} e^{\pi i (j^2\tau + 2j(u+1/2))}$ be the standard theta-function (for brevity will write it as $\theta(u)$). Let $\beta_{ab} = \frac{\theta(u - \lambda_b + \lambda_a)\theta(\gamma)}{\theta(u-\gamma)\theta(\lambda_b - \lambda_a)}$, $\alpha_{ab} = \frac{\theta(\lambda_a - \lambda_b + \gamma)\theta(u)}{\theta(\lambda_a - \lambda_b)\theta(u-\gamma)}$. These functions can be viewed as functions of $z = e^{2\pi i u}$. Then $R(u, \lambda)$ is a quantum dynamical R-matrix with spectral parameter and step γ. One may also define the basic trigonmetric and basic rational solution with spectral parameter, which differ from the elliptic solution by replacement of $\theta(x)$ by $\sin(x)$ and x, respectively.

Remark 1. In examples 1 and 2, the dynamical R-matrix satisfies the *Hecke condition* $(PR - 1)(PR + q) = 0$, with $q = 1$ in example 1 (where P is the permutation on $V \otimes V$), and in example 3 the unitarity condition $R(u, \lambda)R^{21}(-u, \lambda) = 1$.

Remark 2. The basic trigonometric solution degenerates into the basic rational solution as $q \to 1$. Also, the basic elliptic solution with spectral parameter can be degenerated into the basic trigonometric and rational solutions with spectral parameter by renormalizing variables and sending one, respectively both periods of theta functions to infinity (see [EV2]). Furthermore, the basic trigonometric and rational solutions with spectral parameter can, in turn, be degenerated into the solutions of Examples 1,2, by sending the spectral parameter to infinity. Thus, in essense, all the examples we gave are obtained from Felder's solution.

2.3. **The tensor category of representations associated to a quantum dynamical R-matrix.** Let R be a quantum dynamical R-matrix with spectral parameter. According to [Fe], a representation of R is a semisimple $\mathfrak{h}$-module W and an invertible meromorphic function $L = L_W : \mathbb{C} \times \mathfrak{h}^* \to \mathrm{End}_{\mathfrak{h}}(V \otimes W)$, such that

$$R^{12}(u_{12}, \lambda - h^3)L^{13}(u_{13}, \lambda)L^{23}(u_{23}, \lambda - h^1) = L^{23}(u_{23}, \lambda)L^{13}(u_{13}, \lambda - h^2)R^{12}(u_{12}, \lambda). \tag{2}$$

(In the case of step γ, h^i should be replaced by γh^i).

For example: $(\mathbb{C}, 1)$ (trivial representation) and (V, R) (vector representation).

A morphism $f : (W, L_W) \to (W', L_{W'})$ is a meromorphic function $f : \mathfrak{h}^* \to \mathrm{End}_{\mathfrak{h}}(W)$ such that $(1 \otimes f(\lambda))L_W(u,\lambda) = L_W(u,\lambda)(1 \otimes f(\lambda - \gamma h^1))$. With this definition, representations form an (additive) category $\mathrm{Rep}(R)$. Moreover, it is a tensor category [Fe]: given (W, L_W) and (U, L_U), one can form the tensor product representation $(W \otimes U, L_{W\otimes U})$, where $L_{W\otimes U}(u,\lambda) = L_W^{12}(u, \lambda - \gamma h^3)L_U^{13}(u,\lambda)$; tensor product of morphisms is defined by $(f \otimes g)(\lambda) = f(\lambda - \gamma h^2) \otimes g(\lambda)$.

In absence of spectral parameter, one should use the same definitions without u.

2.4. **Gauge transformations and classification.** There exists a group of rather trivial transformations acting on quantum dynamical R-matrices with step γ. They are called *gauge transformations*. If $\mathfrak{h}$ and V are as in the previous section, then gauge transformations (for dynamical R-matrices without spectral parameter) are:

1. Twist by a closed multiplicative 2-form ϕ: $\alpha_{ab} \to \alpha_{ab}\phi_{ab}$, where $\phi_{ab} = \phi_{ba}^{-1}$, and $\phi_{ab}(\lambda)\phi_{bc}(\lambda)\phi_{ca}(\lambda) = \phi_{ab}(\lambda-\gamma\omega_c)\phi_{bc}(\lambda-\gamma\omega_a)\phi_{ca}(\lambda-\gamma\omega_b)$ ($\omega_i = \mathrm{weight}(v_i)$).
2. Permutation of indices $a = 1, ..., n$; $\lambda \to \lambda - \nu$; $R \to cR$.

In presence of the spectral parameter u, the constant in c in 2 may depend on u, and there are the following additional gauge transformations:

3. Given meromorphic $\psi : \mathfrak{h}^* \to \mathbb{C}$, $\alpha_{ab} \to e^{u(\psi(\lambda)-2\psi(\lambda-\gamma\omega_a)+\psi(\lambda-\gamma\omega_a-\gamma\omega_b))}\alpha_{ab}$, $\beta_{ab} \to e^{u(\psi(\lambda)-\psi(\lambda-\gamma\omega_a)-\psi(\lambda-\gamma\omega_b)+\psi(\lambda-\gamma\omega_a-\gamma\omega_b))}\beta_{ab}$.
4. Multiplication of u by a constant.

Theorem 2.1. [EV2] *Any quantum dynamical R-matrix for $\mathfrak{h}, V$ satisfying the Hecke condition with $q = 1$ (respectively, $q \neq 1$) is a gauge transformation of the basic rational (respectively, trigonometric) solution, or a limit of such R-matrices.*

In the spectral parameter case, a similar result is known only under rather serious restrictions (see [EV2], Theorem 2.5). For more complicated configurations $(\mathfrak{h}, V)$, classification is not available.

Remark. Gauge transformations 2-4 do not affect the representation category of the R-matrix. Gauge transformation 1 does not affect the category if the closed form ϕ is exact: $\phi = d\xi$, i.e. $\phi_{ab}(\lambda) = \xi_a(\lambda)\xi_b(\lambda - \gamma\omega_a)\xi_a(\lambda - \gamma\omega_b)^{-1}\xi_b(\lambda)^{-1}$, where $\xi_a(\lambda)$ is a collection of meromorphic functions. We note that this is a very mild condition: for example, if γ is a formal paremeter and we work with analytic functions of λ in a simply connected domain, then a multiplicative 2-form is closed iff it is exact ("multiplicative Poincare lemma").

2.5. **Dynamical quantum groups.** Equation 2 may be regarded as a set of defining relations for an associative algebra A_R (see [EV2] for precise definitions). This algebra is a dynamical analogue of the quantum group attached to an R-matrix defined in [FRT], and representations of R are an appropriate class of representations of this algebra. The algebra A_R is called the dynamical quantum group attached to R. If R is the basic elliptic solution then A_R is the elliptic quantum group of [Fe]. The structure and representations of A_R are studied in many papers (e.g. [Fe, EV2, FV1, TV]).

To keep this paper within bounds, we will not discuss A_R in detail. However, let us mention ([EV2]) that A_R is a bialgebroid with base $\mathfrak{h}^*$. This corresponds to the fact that the category $\mathrm{Rep}(R)$ is a tensor category. Moreover, if R satisfies an additional rigidity assumption (valid for example for the basic rational and trigonometric solutions) then the category $\mathrm{Rep}(R)$ has duality, and A_R is a Hopf algebroid, or a quantum groupoid (i.e. it has an antipode).

Remark. For a general theory of bialgebroids and Hopf algebroids the reader is referred to [Lu]. However, let us mention that bialgebroids with base X correspond to pairs (tensor category, tensor functor to $O(X)$-bimodules), similarly to how bialgebras correspond to pairs (tensor category, tensor functor to vector spaces) (i.e. via Tannakian formalism).

2.6. **The classical dynamical Yang-Baxter equation.** Recall that if $R = 1 - \hbar r + O(\hbar^2)$ is a solution QYBE, then r satisfies the classical Yang-Baxter equation (CYBE), $[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0$, and that r is called the classical limit of R, while R is called a quantization of r. Similarly, let $R(\lambda, \hbar)$ be a family of solutions of QDYBE with step $\hbar$ given by a series $1 - \hbar r(\lambda) + O(\hbar^2)$. Then it is easy to show that $r(\lambda)$ satisfies the following differential equation, called the *classical dynamical Yang-Baxter equation* (CDYBE):

$$(3)\quad \sum_i \left(x_i^{(1)} \frac{\partial r^{23}}{\partial x_i} - x_i^{(2)} \frac{\partial r^{13}}{\partial x_i} + x_i^{(3)} \frac{\partial r^{12}}{\partial x_i} \right) + [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0,$$

(where x_i is a basis of $\mathfrak{h}$). The function $r(\lambda)$ is called the classical limit of $R(\lambda, \hbar)$, and $R(\lambda, \hbar)$ is called a quantization of $r(\lambda)$.

Define a classical dynamical r-matrix to be a meromorphic function $r : \mathfrak{h}^* \to \mathrm{End}_{\mathfrak{h}}(V \otimes V)$ satisfying CDYBE.

Conjecture 2.2. Any classical dynamical r-matrix can be quantized.

This conjecture is known in the non-dynamical case ([EK]), and was proved in [Xu] in the dynamical case for skew-symmetric solutions ($r^{21} = -r$) satisfying additional technical assumptions. A general proof for the skew-symmetric case is recently proposed in a preprint by T.Mochizuki. However, the most interesting non-skew-symmetric case is still open.

Similarly, the classical limit of QDYBE with spectral parameter is CDYBE with spectral parameter. It is an equation with respect to $r(u, \lambda)$ and differs from the usual CDYBE by insertion of u_{ij} as an additional argument of r^{ij}.

Remark 1. Similarly to CYBE, CDYBE makes sense for functions with values in $\mathfrak{g} \otimes \mathfrak{g}$, where $\mathfrak{g}$ is a Lie algebra containing $\mathfrak{h}$.

Remark 2. The classical limit of the notion of a quantum groupoid is the notion of a Poisson groupoid, due to Weinstein. By definition, a Poisson groupoid is a groupoid G which is also a Poisson manifold, such that the graph of the multiplication is coisotropic in $G \times G \times \bar{G}$ (where $\bar{G}$ is G with reversed sign of Poisson bracket). Such a groupoid can be attached ([EV1]) to a classical dynamical r-matrix $r : \mathfrak{h}^* \to \mathfrak{g} \otimes \mathfrak{g}$, such that $r^{21} + r$ is constant and invariant (i.e. r is a "dynamical quasitriangular structure" on $\mathfrak{g}$). This is the classical limit of the assignment of a quantum groupoid to a quantum dynamical R-matrix ([EV2]).

2.7. **Examples of solutions of CDYBE.** We will give examples of solutions of CDYBE in the case when $\mathfrak{g}$ is a finite dimensional simple Lie algebra, and $\mathfrak{h}$ is its Cartan subalgebra. We fix an invariant inner product on $\mathfrak{g}$, It restricts to a nondegenerate inner product on $\mathfrak{h}$. Using this inner product, we identify $\mathfrak{h}^*$ with $\mathfrak{h}$ ($\lambda \in \mathfrak{h}^* \to \bar{\lambda} \in \mathfrak{h}$), which yields an inner product on $\mathfrak{h}^*$. The normalization of the inner product is chosen so that short roots have squared length 2. Let x_i be an orthonormal basis of $\mathfrak{h}$,and let e_α, $e_{-\alpha}$ denote positive (respectively, negative) root elements of $\mathfrak{g}$, such that $(e_\alpha, e_{-\alpha}) = 1$.

Example 1. The basic rational solution $r(\lambda) = \sum_{\alpha > 0} \frac{e_\alpha \wedge e_{-\alpha}}{(\lambda, \alpha)}$.

Example 2. The basic trigonometric solution $r(\lambda) = \frac{\Omega}{2} + \sum_{\alpha>0} \frac{1}{2}\mathrm{cotanh}(\frac{(\lambda,\alpha)}{2})$, where $\Omega \in S^2\mathfrak{g}$ is the inverse element to the inner product on $\mathfrak{g}$.

Example 3. The basic elliptic solution with spectral parameter (Felder's solution) $r(u,\lambda) = \frac{\theta'(u)}{\theta(u)} \sum_i x_i \otimes x_i + \sum_\alpha \frac{\theta(u+(\lambda,\alpha))\theta'(0)}{\theta((\lambda,\alpha))\theta(u)} e_\alpha \otimes e_{-\alpha}$.

Remark 1. One says that a classical dynamical r-matrix r has coupling constant ε if $r + r^{21} = \varepsilon\Omega$. If r is with spectral parameter, one says that it has coupling constant ε if $r(u,\lambda) + r^{21}(-u,\lambda) = 0$, and $r(u,\lambda)$ has a simple pole at $u = 0$ with residue $\varepsilon\Omega$ (these are analogs of the Hecke and unitarity conditions in the quantum case). With these definitions, the basic rational solution has coupling constant 0, while the trigonometric and elliptic solutions have coupling constant 1.

Remark 2. As in the quantum case, Example 3 can be degenerated into its trigonometric and rational versions where $\theta(x)$ is replaced by $\sin(x)$ and x, respectively, and Examples 1 and 2 can be obtained from Example 3 by a limit.

Remark 3. The classical limit of the basic rational and trigonometric solutions of QDYBE (modified by $\lambda \to \lambda/\hbar$) is the basic rational, respectively trigoinometric, solutions of CDYBE for $\mathfrak{g} = \mathfrak{sl}_n$ (in the trigonometric case we should set $q = e^{-\hbar/2}$). The same is true for the basic elliptic solution (with $\gamma = \hbar$).

Remark 4. These examples make sense for any reductive Lie algebra $\mathfrak{g}$.

2.8. **Gauge transformations and classification of solutions for CDYBE.** It is clear from the above that it is interesting to classify solutions of CDYBE. As in the quantum case, it should be done up to gauge transformations. These transformations are classical analogs of the gauge transformations in the quantum case, and look as follows.

1. $r \to r + \omega$, where $\omega = \sum_{i,j} C_{ij}(\lambda) x_i \wedge x_j$ is a meromorphic closed differential 2-form on $\mathfrak{h}^*$.
2. $r(u,\lambda) \to ar(a\lambda - \nu)$; Weyl group action.

In the case of spectral parameter, there are additional transformations:

3. $u \to bu$.
4. Let $r = \sum_{i,j} S_{ij} x_i \otimes x_j + \sum_\alpha \phi_\alpha e_\alpha \otimes e_{-\alpha}$. The transformation is $S_{ij} \to S_{ij} + u\frac{\partial^2\psi}{\partial x_i \partial x_j}$, $\phi_\alpha \to \phi_\alpha e^{u\partial_\alpha \psi}$, where ψ is a function on $\mathfrak{h}^*$ with meromorphic $d\psi$.

Theorem 2.3. [EV1] *(i) Any classical dynamical r-matrix with zero coupling constant is a gauge transformation of the basic rational solution for a reductive subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$, or its limiting case.*

(ii) Any classical dynamical r-matrix with nonzero coupling constant is a gauge transformation of the basic trigonometric solution for $\mathfrak{g}$, or its limiting case.

(iii) Any classical dynamical r-matrix with spectral parameter and nonzero coupling constant is a gauge transformation of the basic elliptic solution for $\mathfrak{g}$, or its limiting case.

Remark. One may also classify dynamical r-matrices with nonzero coupling constant (without spectral parameter) defined on $\mathfrak{l}^*$ for a Lie subalgebra $\mathfrak{l} \subset \mathfrak{h}$, on which the inner product is nondegenerate ([Sch]). Up to gauge transformations they are classified by generalized Belavin-Drinfeld triples, i.e. triples (Γ_1, Γ_2, T), where Γ_i are subdiagrams of the Dynkin diagram Γ of $\mathfrak{g}$, and $T : \Gamma_1 \to \Gamma_2$ is a bijection perserving the inner product of simple roots (so this classification is a dynamical analog of the Belavin-Drinfeld classification of r-matrices on simple Lie algebras, and the classification of [EV1] is the special case $\Gamma_1 = \Gamma_2 = \Gamma$, $T = 1$). The formula for a classical dynamical r-matrix corresponding to such a triple given in [Sch] works for any Kac-Moody algebra, and in the case of affine Lie algebras yields classical dynamical r-matrices with spectral parameter (however, the classification is this case has not been worked out). Explicit quantization of the dynamical r-matrices from [Sch] (for any Kac-Moody algebra) is given in [ESS].

3. The fusion and exchange construction

Unlike QYBE, interesting solutions of QDYBE may be obtained already from classical representation theory of Lie algebras. This can be done through the fusion and exchange construction, see [Fa, EV3].

3.1. **Intertwining operators.** Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra over $\mathbb{C}$, with polar decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$. For any $\mathfrak{g}$-module V, we write $V[\nu]$ for the weight subspace of V of weight $\nu \in \mathfrak{h}^*$. Let M_λ denote the Verma module over $\mathfrak{g}$ with highest weight $\lambda \in \mathfrak{h}^*$, x_λ its highest weight vector, and x_λ^* the lowest weight vector of the dual module. Let V be a finite dimensional representation of $\mathfrak{g}$. Consider an intertwining operator $\Phi : M_\lambda \to M_\mu \otimes V$. The vector $x_\mu^*(\Phi x_\lambda) \in V[\lambda - \mu]$ is called the expectation value of Φ, and denoted $\langle\Phi\rangle$.

Lemma 3.1. *If M_μ is irreducible (i.e. for generic μ), the map $\Phi \to \langle\Phi\rangle$ is an isomorphism* $\mathrm{Hom}_{\mathfrak{g}}(M_{\mu+\nu}, M_\mu \otimes V) \to V[\nu]$.

Lemma 3.1 allows one to define for any $v \in V[\nu]$ (and generic λ) the intertwining operator $\Phi_\lambda^v : M_\lambda \to M_{\lambda-\nu} \otimes V$, such that $\langle\Phi_\lambda^v\rangle = v$.

3.2. **The fusion and exchange operators.** Now let V, W be finite dimensional $\mathfrak{g}$-modules, and $v \in V, w \in W$ homogeneous vectors, of weights $\mathrm{wt}(v), \mathrm{wt}(w)$. Consider the composition of two intertwining operators

$$\Phi_\lambda^{w,v} := (\Phi^w_{\lambda-\mathrm{wt}(v)} \otimes 1)\Phi_\lambda^v : M_\lambda \to M_{\lambda-\mathrm{wt}(v)-\mathrm{wt}(w)} \otimes W \otimes V.$$

The expectation value of this composition, $\langle\Phi_\lambda^{w,v}\rangle$, is a bilinear function of w and v. Therefore, there exists a linear operator $J_{WV}(\lambda) \in \mathrm{End}(W \otimes V)$ (of weight zero, i.e. commuting with $\mathfrak{h}$), such that $\langle\Phi_\lambda^{w,v}\rangle = J_{WV}(\lambda)(w \otimes v)$. In other words, we have $(\Phi^w_{\lambda-\mathrm{wt}(v)} \otimes 1)\Phi_\lambda^v = \Phi_\lambda^{J_{WV}(\lambda)(w\otimes v)}$. The operator $J_{WV}(\lambda)$ is called the *fusion operator* (because it tells us how to "fuse" two intertwining operators).

The fusion operator has a number of interesting properties, which we discuss below. In particular, it is lower triangular, i.e. has the form $J = 1 + N$, where N is a sum of terms which have strictly positive weights in the second component. Consequently, N is nilpotent, and J is invertible.

Define also the *exchange operator* $R_{VW}(\lambda) := J_{VW}^{-1}(\lambda) J_{WV}^{21}(\lambda) : V \otimes W \to V \otimes W$. This operator tells us how to exchange the order of two intertwining operators, in the sense that if $R_{WV}(\lambda)(w \otimes v) = \sum_i w_i \otimes v_i$ (where w_i, v_i are homogeneous), then $\Phi_\lambda^{w,v} = P \sum_i \Phi_\lambda^{v_i,w_i}$ (where P permutes V and W).

Theorem 3.2. [EV3] *$R_{VV}(\lambda)$ is a solution of QDYBE.*

3.3. **Fusion and exchange operators for quantum groups.** The fusion and exchange constructions generalize without significant changes to the case when the Lie algebra $\mathfrak{g}$ is replaced by the quantum group $U_q(\mathfrak{g})$, where q is not zero or a root of unity. The only change that needs to be made is in the definition of the exchange operator: namely, one sets $R(\lambda) = J_{VW}^{-1}(\lambda)\mathcal{R}^{21} J_{WV}^{21}(\lambda)$, where $\mathcal{R}$ is the universal R-matrix of $U_q(\mathfrak{g})$. This is because when changing the order of intertwining operators, we must change the order of tensor product of representations $V \otimes W$, which in the quantum case is done by means of the R-matrix.

Example 3.3. Let $\mathfrak{g} = \mathfrak{sl}_n$, and V be the vector representation of $U_q(\mathfrak{g})$. Then the exchange operator has the form $R = q^{1-1/n}\tilde{R}$, where $\tilde{R}$ is given by (1), with $\beta_{ab} = \frac{q^{-2}-1}{q^{2(\lambda_a-\lambda_b-a+b)}-1}$, $\alpha_{ab} = q^{-1}$ for $a < b$, and $\alpha_{ab} = \frac{(q^{2(\lambda_b-\lambda_a+a-b)}-q^{-2})(q^{2(\lambda_b-\lambda_a+a-b)}-q^{2})}{q(q^{2(\lambda_b-\lambda_a+a-b)}-1)^2}$ if $a > b$. The exchange operator for the vector representation of $\mathfrak{g}$ is obtained by passing to the limit $q \to 1$; i.e. it is given by (1), with $\beta_{ab} = \frac{1}{\lambda_b-\lambda_a-b+a}$, $\alpha_{ab} = 1$ for $a < b$, and $\alpha_{ab} = \frac{(\lambda_b-\lambda_a+a-b-1)(\lambda_b-\lambda_a+a-b+1)}{(\lambda_b-\lambda_a+a-b)^2}$ if $a > b$. It is easy to see that these exchange operators are gauge equivalent to the basic rational and trigonometric solutions of QDYBE, respectively.

3.4. **The ABRR equation.** The fusion operator is not only a tool to define the exchange operator satisfying QDYBE, but is an interesting object by itself, which deserves a separate study; so we will briefly discuss its properties.

Let ρ the half-sum of positive roots. Let $\Theta(\lambda) \in U(\mathfrak{h})$ be given by $\Theta(\lambda) = \bar{\lambda} + \bar{\rho} - \frac{1}{2}\sum x_i^2$. Then $\Theta(\lambda)$ defines an operator in any $U_q(\mathfrak{g})$-module with weight decomposition. Let $\mathcal{R}_0 = \mathcal{R}q^{-\sum x_i \otimes x_i}$ be the unipotent part of the universal R-matrix.

Theorem 3.4. *(ABRR equation,* [ABRR, JKOS]*). For $q \neq 1$, the fusion operator is a unique lower triangular zero weight operator, which satisfies the equation:*

$$J(\lambda)(1 \otimes q^{2\Theta(\lambda)}) = \mathcal{R}_0^{21}(1 \otimes q^{2\Theta(\lambda)})J(\lambda). \tag{4}$$

For $q = 1$, the fusion operator satisfies the classical limit of this equation:

$$[J(\lambda), 1 \otimes \Theta(\lambda)] = (\sum_{\alpha>0} e_{-\alpha} \otimes e_\alpha)J(\lambda), \tag{5}$$

(Here for brevity we have dropped the subscripts W and V, with the understanding that both sides are operators on $W \otimes V$).

3.5. **The universal fusion operator.** Using the ABRR equation, we can define the universal fusion operator, living in a completion of $U_q(\mathfrak{g})^{\otimes 2}$, which becomes $J_{WV}(\lambda)$ after evaluating in $W \otimes V$. Namely, the universal fusion operator $J(\lambda)$ is the unique lower triangular solution of the ABRR equation in a completion of $U_q(\mathfrak{g})^{\otimes 2}$. This solution can be found in the form of a series $J = \sum_{n\geq 0} J_n$, $J_0 = 1$, where $J_n \in U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ has zero weight and its second component has degree n in principal gradation; so J_n are computed recursively.

This allows one to compute the universal fusion operator quite explicitly. For example, if $q = 1$ and $\mathfrak{g} = \mathfrak{s}l_2$, then the universal fusion operator is given by the formula $J(\lambda) = \sum_{n\geq 0} \frac{(-1)^n}{n!} f^n \otimes (\lambda - h + n + 1)^{-1} \dots (\lambda - h + 2n)^{-1} e^n$.

3.6. **The dynamical twist equation.** Another important property of the fusion operator is the dynamical twist equation (which is a dynamical analog of the equation for a Drinfeld twist in a Hopf algebra).

Theorem 3.5. *The universal fusion operator $J(\lambda)$ satisfies the dynamical twist equation $J^{12,3}(\lambda)J^{1,2}(\lambda - h^{(3)}) = J^{1,23}(\lambda)J^{2,3}(\lambda)$.*

Here the superscripts of J stand for components where the first and second component of J acts; for example $J^{1,23}$ means $(1 \otimes \Delta)(J)$, and $J^{1,2}$ means $J \otimes 1$.

3.7. **The fusion operator for affine algebras.** The fusion and exchange construction can be generalized to the case when the simple Lie algebra $\mathfrak{g}$ is replaced by any Kac-Moody Lie algebra. This generalization is especially interesting if $\mathfrak{g}$ is replaced with the affine Lie algebra $\hat{\mathfrak{g}}$, and V, W are finite dimensional representations of $U_q(\hat{\mathfrak{g}})$ (where q is allowed to be 1). In this case, for each $z \in \mathbb{C}^*$ we have an outer automorphism $D_z : U_q(\hat{\mathfrak{g}}) \to U_q(\hat{\mathfrak{g}})$, which preserves the Chevalley generators q^h and e_i, f_i, $i > 0$, of $U_q(\hat{\mathfrak{g}})$, while multiplying e_0 by z and f_0 by z^{-1}. Define the shifted representation $V(z)$ by $\pi_{V(z)}(a) = \pi_V(D_z(a))$. Let $\hat{\lambda} = (\lambda, k)$ be a weight for $\hat{\mathfrak{g}}$ (k is the level). Then similarly to the finite dimensional case one can define the intertwining operator $\Phi^v_{\lambda,k}(z) : M_{\hat{\lambda}} \to M_{\hat{\lambda}-\mathrm{wt}(v)} \hat{\otimes} V(z)$ (to a completed tensor product); it can be written as an infinite series $\sum_{n\in\mathbb{Z}} \Phi^v_{\lambda,k}[n] z^{-n}$, where $\Phi^v_{\lambda,k}[n] : M_{\hat{\lambda}} \to M_{\hat{\lambda}-\mathrm{wt}(v)} \otimes V$ are linear operators. Furthermore, the expectation value of the composition $\langle (\Phi^w_{\lambda-\mathrm{wt}(v),k}(z_1) \otimes 1)\Phi^v_{\lambda,k}(z_2)\rangle$ is defined as a Taylor series in z_2/z_1. Thus, one can define the fusion operator $J_{WV}(z) \in \mathrm{End}_{\mathfrak{h}}(W \otimes V)[[z]]$ such that this expectation value is equal to $J_{WV}(z)(w \otimes v)$.

3.8. **Fusion operator and correlators in the WZW model.** One may show (see [FR, EFK]) that the series $J_{WV}(z)$ is convergent in some neighborhood of 0 to a holomorphic function. This function has a physical interpretation. Namely, if $q = 1$, the operators $\Phi^v_{\lambda,k}(z)$ are, essentially, vertex operators (primary fields) for the Wess-Zumino-Witten conformal field theory, and the function $J_{WV}(z)(w \otimes v)$ is a 2-point correlation function of vertex operators. If $q \neq 1$, this function is a 2-point correlation function of q-vertex operators, and has a similar interpretation in terms of statistical mechanics.

3.9. **The ABRR equation in the affine case as the KZ (q-KZ) equation.** The ABRR equation has a straightforward generalization to the affine case, which also has a physical interpretation. Namely, in the case $q = 1$ it coincides with the (trigonometric) Knizhnik-Zamolodchikov (KZ) equation for the 2-point correlation function, while for $q \neq 1$ it coincides with the quantized KZ equation for the 2-point function of q-vertex operators, derived in [FR] (see also [EFK]).

One may also define (for any Kac-Moody algebra) the *multicomponent universal fusion operator* $J^{1\dots N}(\lambda) = J^{1,2\dots N}(\lambda)\dots J^{2,3\dots N}(\lambda)J^{N-1,N}(\lambda)$. It satisfies a multicomponent version of the ABRR equation. In the affine case, $J^{1\dots N}(\lambda)$ is interpreted as the N-point correlation function for vertex (respectively, q-vertex) operators, while the multicomponent ABRR equation is interpreted as the KZ (respectively, qKZ) equation for this function. See [EV5] for details.

3.10. **The exchange operator for affine algebras and monodromy of KZ (q-KZ) equations.** The generalization of the exchange operator to the affine case is rather tricky, and there is a serious difference between the classical ($q = 1$), and quantum ($q \neq 1$) case. The naive definition would be $R_{VW}(u, \hat{\lambda}) = J_{VW}(z, \hat{\lambda})^{-1}\mathcal{R}^{21}|_{V\otimes W(z)} J^{21}_{WV}(z^{-1}, \hat{\lambda})$ (where $z = e^{2\pi i u}$). However, this definition does not immediately make sense, since we are multiplying infinite Taylor series in z by infinite Taylor series in z^{-1}. To make sense of such product, consider the cases $q = 1$ and $q \neq 1$ separately.

If $q = 1$, the functions $J_{VW}(z, \hat{\lambda})$ and $J^{21}_{WV}(z^{-1}, \hat{\lambda})$ are both solutions of the Knizhnik-Zamolodchikov differential equation, one regular near 0 and the other near ∞. The equation is nonsingular outside $0, \infty$, and 1. Thus, the series $J_{VW}(z, \hat{\lambda})$

is convergent for $|z| < 1$. On the other hand, for $0 < z < 1$ define the function $A^{\pm}(J^{21}_{WV}(z^{-1}, \hat{\lambda}))$ to be the analytic continuation of $J^{21}_{WV}(z^{-1}, \hat{\lambda})$ from the region $z > 1$ along a curve passing 1 from above (for +) and below (for -). Then the function $R^{\pm}_{VW}(u, \hat{\lambda}) := J_{VW}(z, \hat{\lambda})^{-1} A^{\pm}(J^{21}_{WV}(z^{-1}, \hat{\lambda}))$ is of zero weight, and satisfies the QDYBE with spectral parameter (for $V = W$). The operator $R^{\pm}_{VW}(u, \hat{\lambda})$ is the appropriate analog of the exchange operator (depending on a choice of sign).

If $q \neq 1$, the functions $J_{VW}(z, \hat{\lambda})$ and $\mathcal{R}^{21}|_{V\otimes W(z)} J^{21}_{WV}(z^{-1}, \hat{\lambda})$ are both solutions of the quantized Knizhnik-Zamolodchikov equation, which is a difference equation with multiplicative step $p = q^{-2m(k+g)}$, where m is the ratio of squared norms of long and short roots, k the level of $\hat{\lambda}$, and g the dual Coxeter number of $\mathfrak{g}$. Therefore, if $|p| \neq 1$, these functions admit a meromorphic continuation to the whole $\mathbb{C}^*$ (unlike the $q = 1$ case, they are now single-valued), and the naive formula for $R_{VW}(u, \hat{\lambda})$ makes sense. As in the $q = 1$ case, this function is of zero weight and satisfies the QDYBE with spectral parameter (for $V = W$); it is the appropriate generalization of the exchange operator (see [EFK] for details).

The operators $R^{\pm}_{VW}(z, \hat{\lambda})$, essentially, represent the monodromy of the KZ differential equation. In particular, $R^{\pm}_{VW}$ is "almost constant" in u: its matrix elements in a homogeneous basis under $\mathfrak{h}$ are powers of $e^{2\pi i u}$ (in fact, $R^{\pm}_{VV}(u, \hat{\lambda})$ is gauge equivalent, in an appropriate sense, to a solution of QDYBE without spectral parameter). Similarly, the operator $R_{VW}(z, \hat{\lambda})$ for $q \neq 1$ represents the q-monodromy (Birkhoff's connection matrix) of the q-KZ equation. In particular, the matrix elements of $R_{VW}(u, \hat{\lambda})$ are quasiperiodic in u with period $\tau = \log p/2\pi i$. Since they are also periodic with period 1 and meromorphic, they can be expressed rationally via elliptic theta-functions.

Example 3.6. Let $\mathfrak{g} = \mathfrak{sl}_n$, and V the vector representation of $U_q(\hat{\mathfrak{g}})$. If $q \neq 1$, the exchange operator $R_{VV}(u, \hat{\lambda})$ is a solution of QDYBE, gauge equivalent to the basic elliptic solution with spectral parameter (see [Mo] and references therein, and also [FR, EFK]). The gauge transformation involves an exact multiplicative 2-form, which expresses via q-Gamma functions with q = p. Similarly, if $q = 1$, the exchange operators $R^{\pm}_{VV}(u, \hat{\lambda})$ are gauge equivalent to the basic trigonometric solution without spectral parameter, with $q = e^{\pi i/m(k+g)}$; the gauge transformation involves an exact multiplicative 2-form expressing via classical Gamma-functions. This is obtained by sending q to 1 in the result of [Mo].

Remark. Note that the limit $q \to 1$ in this setting is rather subtle. Indeed, for $q \neq 1$ the function $J_{VW}(u, \lambda)$ has an infinite sequence of poles in the z-plane, which becomes denser as q approaches 1 and eventually degenerates into a branch cut; i.e. this single valued meromorphic function becomes multivalued in the limit.

3.11. **The quantum Kazhdan-Lusztig functor.** Let $\mathfrak{g} = \mathfrak{sl}_n$, $q \neq 1$. Example 3.6 allows one to construct a tensor functor from the category $\mathrm{Rep}_f(U_q(\hat{\mathfrak{g}}))$ of finite dimensional $U_q(\hat{\mathfrak{g}})$-modules, to the category $\mathrm{Rep}_f(R)$ of finite dimensional representations of the basic elliptic solution R of QDYBE with spectral parameter (i.e. to the category of finite dimensional representations of Felder's elliptic quantum group). Namely, let V be the vector representation, and for any finite dimensional representation W of $U_q(\hat{\mathfrak{g}})$, set $L_W = R_{VW}$. Then (W, L_W) is a representation of the dynamical R-matrix R_{VV}. This defines a functor $F : \mathrm{Rep}_f(U_q(\hat{\mathfrak{g}})) \to \mathrm{Rep}_f(R_{VV})$. Moreover, this functor is a tensor functor: the tensor structure $F(W) \otimes F(U) \to$

$F(W\otimes U)$ is given by the fusion operator $J_{WU}(\hat\lambda)$ (the axiom of a tensor structure follows from the dynamical twist equation for J). On the other hand, since R_{VV} and R are gauge equivalent by an exact form, their representation categories are equivalent, so one may assume that F lands in $\mathrm{Rep}_f(R)$.

If the scalars for $\mathrm{Rep}_f(U_q(\hat{\mathfrak g}))$ are taken to be the field of periodic functions of $\hat\lambda$ (in particular, k is regarded as a variable), then F is fully faithful; it can be regarded as a q-analogue of the Kazhdan-Lusztig functor (see [EM] and references therein). It generalizes to infinite dimensional representations, and allows one to construct elliptic deformations of all evaluation representations of $U_q(\hat{\mathfrak g})$) (which was done for finite dimensional representations in [TV]). The versions of this functor without spectral parameter, from representations of $\mathfrak g$ ($U_q(\mathfrak g)$) to representations of the basic rational (trigonometric) solution of QDYBE can be found in [EV3].

4. Traces of intertwining operators and Macdonald functions

In this section we discuss a connection between dynamical R-matrices and certain integrable systems and special functions (in particular, Macdonald functions).

4.1. **Trace functions.** Let V be a finite dimensional representation of $U_q(\mathfrak g)$ ($q\neq 1$), such that $V[0]\neq 0$. Recall that for any $v\in V[0]$ and generic μ, one can define an intertwining operator Φ_μ^v such that $\langle\Phi_\mu^v\rangle=v$. Following [EV4] set $\Psi^v(\lambda,\mu)=\mathrm{Tr}|_{M_\mu}(\Phi_\mu^v q^{2\bar\lambda})$. This is an infinite series in the variables $q^{-(\lambda,\alpha_i)}$ (where α_i are the simple roots) whose coefficients are rational functions of $q^{(\mu,\alpha_i)}$ (times a common factor $q^{2(\lambda,\mu)}$). For generic μ this series converges near 0, and its matrix elements belong to $q^{2(\lambda,\mu)}(\mathbb C(q^{(\lambda,\alpha_i)})\otimes\mathbb C(q^{(\mu,\alpha_i)}))$.

Let $\Psi_V(\lambda,\mu)$ be the $\mathrm{End}(V[0])$-valued function, such that $\Psi_V(\lambda,\mu)v=\Psi^v(\lambda,\mu)$. The function Ψ_V has remarkable properties and in a special case is closely related to Macdonald functions. To formulate the properties of Ψ_V, we will consider a renormalized version of this function. Namely, let $\delta_q(\lambda)$ be the Weyl denominator $\prod_{\alpha>0}(q^{(\lambda,\alpha)}-q^{-(\lambda,\alpha)})$. Let also $Q(\mu)=\sum S^{-1}(b_i)a_i$, where $\sum a_i\otimes b_i=J(\mu)$ is the universal fusion operator (this is an infinite expression, but it makes sense as a linear operator on finite dimensional representations; moreover it is of zero weight and invertible). Define the *trace function* $F_V(\lambda,\mu)=\delta_q(\lambda)\Psi_V(\lambda,-\mu-\rho)Q(-\mu-\rho)^{-1}$.

4.2. **Commuting difference operators.** For any finite dimensional $U_q(\mathfrak g)$-module W, we define a difference operator $\mathcal D_W$ acting on functions on $\mathfrak h^*$ with values in $V[0]$. Namely, we set $(\mathcal D_W f)(\lambda)=\sum_{\nu\in\mathfrak h^*}\mathrm{Tr}|_W(R_{WV}(-\lambda-\rho))f(\lambda+\nu)$. These operators are dynamical analogs of transfer matrices, and were introduced in [FV2]. It can be shown that $\mathcal D_{W_1\otimes W_2}=\mathcal D_{W_1}\mathcal D_{W_2}$; in particular, $\mathcal D_W$ commute with each other, and the algebra generated by them is the polynomial algebra in $\mathcal D_{W_i}$, where W_i are the fundamental representations of $U_q(\mathfrak g)$.

4.3. **Difference equations for the trace functions.** It turns out that trace functions $F_V(\lambda,\mu)$, regarded as functions of λ, are common eigenfunctions of $\mathcal D_W$.

Theorem 4.1. [EV4] *One has*

$$\mathcal D_W^{(\lambda)}F_V(\lambda,\mu)=\chi_W(q^{-2\bar\mu})F_V(\lambda,\mu), \tag{6}$$

where $\chi_W(x)=\mathrm{Tr}|_W(x)$ is the character of W.

In fact, it is easy to deduce from this theorem that if v_i is a basis of $V[0]$ then $F_V(\lambda,\mu)v_i$ is a basis of solutions of (6) in the power series space. Thus, trace functions allow us to integrate the quantum integrable system defined by the commuting operators $\mathcal{D}_{W_i}$.

Theorem 4.2. [EV4] *The function F_V is symmetric in λ and μ in the following sense: $F_{V^*}(\mu,\lambda) = F_V(\lambda,\mu)^*$.*

This symmetry property implies that F_V also satisfies "dual" difference equations with respect to μ: $\mathcal{D}_W^{(\mu)} F_V(\lambda,\mu)^* = \chi_W(q^{-2\bar{\lambda}})F_V(\lambda,\mu)^*$.

4.4. **Macdonald functions.** An important special case, worked out in [EKi], is $\mathfrak{g} = \mathfrak{sl}_n$, and $V = L_{kn\omega_1}$, where ω_1 is the first fundamental weight, and k a nonnegative integer. In this case, $\dim V[0] = 1$, and thus trace functions can be regarded as scalar functions. Furthermore, it turns out ([FV2]) that the operators $\mathcal{D}_W$ can be conjugated (by a certain explicit product) to Macdonald's difference operators of type A, and thus the functions $F_V(\lambda,\mu)$ are Macdonald functions (up to multiplication by this product). One can also obtain Macdonald's polynomials by replacing Verma modules M_λ with irreducible finite dimensional modules L_λ; see [EV4] for details. In this case, Theorem 4.2 is the well known Macdonald's symmetry identity, and the "dual" difference equations are the recurrence relations for Macdonald's functions.

Remark 1. The dynamical transfer matrices $\mathcal{D}_W$ can be constructed not only for the trigonometric but also for the elliptc dynamical R-matrix; in the case $\mathfrak{g} = \mathfrak{sl}_n$, $V = L_{kn\omega_1}$ this yields the Ruijsenaars system, which is an elliptic deformation of the Macdonald system.

Remark 2. If $q = 1$, the difference equations of Theorem 4.1 become differential equations, which in the case $\mathfrak{g} = \mathfrak{sl}_n$, $V = L_{kn\omega_1}$ reduce to the trigonometric Calogero-Moser system. In this limit, the symmetry property is destroyed, but the "dual" difference equations remain valid, now with the exchange operator for $\mathfrak{g}$ rather than $U_q(\mathfrak{g})$. Thus, both for $q = 1$ and $q \neq 1$, common eigenfunctions satisfy additional difference equations with respect to eigenvalues – the so called bispectrality property.

Remark 3. Apart from trace Ψ^v of a single intertwining operator multiplied by $q^{2\bar{\lambda}}$, it is useful to consider the trace of a product of several such operators. After an appropriate renormalization, such multicomponent trace function (taking values in $\mathrm{End}((V_1 \otimes \ldots \otimes V_N)[0])$ satisfies multicomponent analogs of (6) and its dual version, as well as the symmetry. Furthermore, it satisfies an additional quantum Knizhnik-Zamolodchikov-Bernard equation, and its dual version (see [EV4]).

Remark 4. The theory of trace functions can be generalized to the case of affine Lie algebras, with V being a finite dimensional representation of $U_q(\hat{\mathfrak{g}})$. In this case, trace functions will be interesting transcendental functions. In the case $\mathfrak{g} = \mathfrak{sl}_n$, $V = L_{kn\omega_1}$, they are analogs of Macdonald functions for affine root systems. It is expected that for $\mathfrak{g} = \mathfrak{sl}_2$ they are the elliptic hypergeometric functions studied in [FV3]. This is known in the trigonometric limit ([EV4]) and for $q = 1$.

Remark 5. The theory of trace functions, both finite dimensional and affine, can be generalized to the case of any generalized Belavin-Drinfeld triple; see [ES1].

Remark 6. Trace functions $F_V(\lambda,\mu)$ are not Weyl group invariant. Rather, the diagonal action of the Weyl group multiplies them by certain operators, called the *dynamical Weyl group* operators (see [EV5]). These operators play an important role in the theory of dynamical R-matrices and trace functions, but are beyond the scope of this paper.

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