HOMOLOGY DECOMPOSITIONS FOR CLASSIFYING SPACES OF FINITE GROUPS ASSOCIATED TO MODULAR REPRESENTATIONS

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Abstract. For a prime p, a homology decomposition of a the classifying space BG of a finite group G consist of a functor $F : \mathbf{D} \longrightarrow \mathbf{spaces}$ from a small category into the category of spaces and a map hocolim $F \longrightarrow BG$ from the homotopy colimit to BG which induces an isomorphism in mod-p homology. Associated to a modular representation $G \longrightarrow Gl(n; \mathbb{F}_p)$ we construct a family of subgroups closed under conjugation, which gives rise to three different homology decompositions, the so called subgroup, centralizer and normalizer decomposition. For an action of G on a $\mathbb{F}_p$-vector space V, this collection consists of the isotropy groups of all nontrivial proper subspaces of V with nontrivial p-Sylow subgroup. These decomposition formulas connect the modular representation theory of G with the homotopy theory of BG.

1.Introduction.

For a finite group G and a prime p, a homology decomposition or homology approximation of the classifying space BG is a a way to glue spaces together such that this new space allows a map into BG which induces an isomorphism in mod-p homology or cohomology. More explicit, it is given by a functor $F : \mathbf{D} \to \mathbf{spaces}$ from a small, a discrete or, even better, a finite category $\mathbf{D}$ into the category of spaces and a mod-p homology isomorphism

$$\operatorname*{hocolim}_{\mathbf{D}} F \to BG$$

from the homotopy colimit of F to the classifying space BG of G. In practice, the category $\mathbf{D}$ is very often given by a full subcategory of the orbit category or of the category of subgroups of G. Homology decompositions of classifying spaces became an important tool in the analysis of the homotopy theory of these spaces. They can be used to make explicit calculations for G or to prove general theorems via an induction of the order of G. In the latter case one needs that the values of the functor F are homotopy equivalent to classifying spaces of smaller groups or of proper subgroups of G. And this turns out to be the case for the known decomposition results. An (incomplete) list of the usefulness and of applications of homology decompositions is given in [3].

In [3], Dwyer described many different homology decomposition formulas in terms of a single invariant: an associated poset of subgroups of G (see Section

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2). For each of these posets he constructed three different homology decompositions, the centralizer, the subgroup and the normalizer decomposition. Some of them were already known some of them gave new decompositions [3] [4].

The goal of this paper is to produce further homology decompositions of BG which are associated to modular representations. We hope to build a bridge between modular representation theory of a finite group and the homotopy theory of the classifying space at a fixed prime. This might give the chance to produce suitable decomposition formulas for questions about the homotopy theory of classifying spaces and/or invariant theory. The method of producing these new homology decompositions were already used with success in [8] and [9], where the author constructed spaces with polynomial mod-p cohomology and proved uniqueness results for them.

Before we formulate our results we first have to recall Dwyer's concepts of sharp homology decompositions [4].

Given a (covariant) functor $F : \mathbf{D} \to \mathbf{spaces}$, $\mathbf{D}$ a discrete category, the standard method to calculate the mod-p homology of $\operatorname{hocolim}_{\mathbf{D}} F$ is given by the Bousfield-Kan spectral sequence. It has the form

$$E_2^{i,j} \cong \varinjlim_{\mathbf{D}}{}^i H_j(F; \mathbb{F}_p) \quad \Longrightarrow \quad H_{i+j}(\operatorname{hocolim}_{\mathbf{D}} F; \mathbb{F}_p)$$

and converges to the mod-p homology of the homotopy colimit. Here, $\varinjlim{}^i$ denotes the i-th left derived functor of the functor $\varinjlim$ which maps diagrams of abelian groups to the direct limit of the diagram.

If there exists a mod-p equivalence $\phi : \operatorname{hocolim}_{\mathbf{D}} F \to BG$, i.e. ϕ induces an isomorphism in mod-p homology, then this map is called a *homology decomposition*. We call this decomposition *sharp*, if

$$\varinjlim_{\mathbf{D}}{}^i H_*(F; \mathbb{F}_p) = \begin{cases} 0 & \text{for } \; i \geq 1 \\ H_*(BG; \mathbb{F}_p) & \text{for } \; i = 0 \end{cases}$$

where, for $i = 0$, the isomorphism is induced by ϕ.

For a collection $\mathcal{C}$ of subgroups of G, i.e. a family which is closed under conjugation, Dwyer constructed three different decompositions, the subgroup, the centralizer and the normalizer decomposition. The first is based on full subcategory of the orbit category and has the form

$$b_{\mathcal{C}} : \operatorname{hocolim}_{\mathbf{O}_{\mathcal{C}}(G)} \beta_{\mathcal{C}} \to BG \ ,$$

the second is based on a full subcategory of the category of subgroups and has the form

$$a_{\mathcal{C}} : \operatorname{hocolim}_{\mathbf{A}_{\mathcal{C}}} \alpha_{\mathcal{C}} \to BG \ ,$$

and the last one is based on the category of orbit simplices for the G-action on a complex $K_{\mathcal{C}}$ associated to $\mathcal{C}$ and has the form

$$d_{\mathcal{C}} : \operatorname{hocolim}_{\bar{s}\mathbf{S}_{\mathcal{C}}} \delta_{\mathcal{C}} \to BG \ .$$

For details see Section 2.

1.1 Definition. We call a collection $\mathcal{C}$ *subgroup sharp* if the map $b_{\mathcal{C}}$ gives a sharp decomposition, *centralizer sharp* if $a_{\mathcal{C}}$ gives a sharp decomposition and *normalizer sharp* if $d_{\mathcal{C}}$ does so.

We will construct collections of subgroups associated to modular representations which are subgroup, centralizer and normalizer sharp. Given a modular representation $\rho : G \to GL(V)$, i.e. V is a vector space over $\mathbb{F}_p$ and p divides the order of G, we consider the collection $\mathcal{I}(\rho)$ of the isotropy groups of all nontrivial proper subspaces of V whose order is divisible by p. The analogue definition we can make for any G-set or G-space X. In this case we denote the collections by $\mathcal{I}(X)$. We say that G acts effectively on X if $gx = x$ for all $x \in X$ then $g = 1$.

1.2 Theorem. *Let G be a finite group such that p divides the order of G.*
(i) Let $\rho : G \to Gl(V)$ be a faithful modular representation. Then, the collection $\mathcal{I}(\rho)$ is subgroup, centralizer and normalizer sharp.
(ii) Let X be a G-space with an effective action. Then, the collection $\mathcal{I}(X)$ is subgroup, centralizer and normalizer sharp.

If a collection $\mathcal{C}$ contains the group G itself, we do not get interesting decomposition formulas. For example, in this case, the orbit category $\mathbf{O}_{\mathcal{C}}(G)$ contains G/G which is a terminal object. Therefore the subgroup decomposition contains BG as a piece and is sharp by trivial reasons. Let P be a p-Sylow subgroup of G. If $V^G \neq V^P$ then $G \notin \mathcal{I}(\rho)$. In particular, since V^P is always a nontrivial subspace, this is true if the representation is fixed point free. The same holds for $\mathcal{I}(X)$ if $X^G \neq X^P$.

Usually one is interested to choose the collection as small as possible. As the proofs will show (see in Remark 4.3), we only have to take into account the isotropy groups of the fixed point sets of nontrivial p-subgroups of G.

The method of proof for Theorem 1.3 can also be used to reconstruct the Jackowski-McClure decomposition via centralizers of nontrivial elementary abelian p-subgroups [7]. Let $\mathcal{E}$ denote the collection of all nontrivial elementary abelian p-subgroups of G and $\mathcal{A}$ the collection of all nontrivial abelian p-subgroups. Let $\mathcal{C}en(\mathcal{E}) := \{C_G(E) | E \in \mathcal{E}\}$ and $\mathcal{C}en(\mathcal{A}) := \{C_G(A) | A \in \mathcal{A}\}$ denote the associated collection of centralizers.

1.3 Theorem. *The collections $\mathcal{C}en(\mathcal{E})$ and $\mathcal{C}en(\mathcal{A})$ are subgroup and normalizer sharp.*

Since the collection $\mathcal{E}$ is not subgroup sharp, one cannot expect that these collections are centralizer sharp.

The Jackowski-McClure decomposition via centralizers of elementary abelian p-subgroups is described via a functor defined on the Quillen-category of all nontrivial elementary abelian p-subgroups. This is a subcategory of the category of subgroups. The subgroup decompositions associated to $\mathcal{C}en(\mathcal{E})$ comes from a functor defined on a subcategory of the orbit category, but one can show that both give the same decomposition formulas.

The proof of the theorems is based on the following idea. Let $\mathcal{C}$ be a collection of subgroups and denote by $\mathcal{P}$ the collection of all nontrivial p-subgroups. Then we have inclusions

$$\mathcal{C} \to \mathcal{C} \cup \mathcal{P} \leftarrow \mathcal{P}$$

where $\mathcal{C} \cup \mathcal{P}$ denotes the union of both collections. We will compare the decompositions associated to these three collections.

1.4 Definition. We say that $\mathcal{C}$ is a *minimalization* of $\mathcal{P}$ if for every $P \in \mathcal{P}$ there exists a unique minimal element $H_P \in \mathcal{C}$ such that $P \subset H_P$, i.e. if $P \subset H$ and $H \in \mathcal{C}$ then $H_P \subset H$.

1.5 Proposition. *If $\mathcal{C}$ is a minimalization of $\mathcal{P}$ then $\mathcal{C}$ is subgroup sharp.*

Actually, this statement should be read as, if $\mathcal{C}$ is a minimalization of $\mathcal{P}$ and one of the collection is subgroup sharp, then so is the other And that is the way we will prove this theorem. Because of Proposition 1.5 the proof of subgroup sharpness for $\mathcal{I}(\rho)$ and $\mathcal{I}(X)$ boils down to the fact that both are minimalizations of $\mathcal{P}$ (Lemma 4.1). The analogue result of 1.5 for centralizer and normalizer sharpness we only can prove for $\mathcal{I}(\rho)$ and $\mathcal{I}(X)$.

In the case of the collections $\mathcal{C}en(\mathcal{E})$ and $\mathcal{C}en(\mathcal{A})$ one has to replace $\mathcal{P}$ by the collection $\mathcal{P}^c$ of all p-centric p-subgroups of G. A p-subgroup $P \subset G$ is called *p-centric* if the center $Z(P) \subset C_G(P)$ of P is a p-Sylow subgroup of the centralizer $C_G(P)$. Then the same ideas apply.

As already mentioned, the subgroup decomposition for $\mathcal{C}en(\mathcal{E})$ is closely related to the Jackowski-McClure decomposition. Actually, it turns out that the decomposition given by p-centric p-subgroups and by centralizers of elementary abelian subgroups are equivalent; i.e. if one the underlying diagrams gives a homology decomposition then so does the other. We intend to analyze this connection in detail for compact Lie groups as well as for p-compact groups in a future paper.

The paper is organized as follows: In the next section we recall material from from [3] and [4]. In particular we explain the concept of subgroup, centralizer and normalizer decompositions. In Section 3 we compare different collection with respect to there ampleness and sharpness properties. We set up the background for the proofs of the theorem. The proofs themselves are worked out in Section 4. And the last section is devoted to a discussion of some examples.

The paper is written with the convention that "spaces" means "simplicial sets". And a map $X \to Y$ between spaces is a (weak) homotopy equivalence, if it induces a (weak) homotopy equivalence between the geometric realizations.

Throughout the paper p is a fixed prime, and $\mathbb{F}_p$ denotes the finite field of p-element. A map between spaces is called a mod-p or $\mathbb{F}_p$-equivalence if it induces an isomorphism in mod-p homology.

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2. The centralizer, the subgroup and the normalizer decomposition.

For each collection $\mathcal{C}$ of subgroups of a finite group G, Dwyer constructed three different decomposition, the centralizer , the subgroup and the normalizer decomposition. Moreover, Dwyer identified the Bousfield-Kan spectral sequence of the decompositions with the equivariant Bredon homology of certain G-spaces. We recall these constructions. All the material in this section is taken from [3] and [4].

Let EG denote the universal cover of BG. For a G-space X we denote by $X_{hG} := (X \times EG)/G$ the Borel construction of G.

Let $\tilde{\gamma}$ be a functor from a small category $\mathbf{D}$ into transitive G-sets. Every G-set can be considered as a discrete G-space. Then, the definition $X := \operatorname{hocolim}_{\mathbf{D}} \tilde{\gamma}$ makes sense and X is a G-space. Moreover, X can be identified with the nerve of the category whose objects are pairs (x, d) such that $x \in \tilde{\gamma}(d)$ and whose morphisms are maps $d \to d'$ such that $\tilde{\gamma}(d \to d')(x) = x'$.

Let $\gamma := \tilde{\gamma}_{hG}$. The natural maps $\tilde{\gamma}(d) \to *$ are G-equivariant and induce maps $\gamma(d) \to BG$ which are compatible with the morphisms in $\mathbf{D}$, and therefore a map $\operatorname{hocolim}_{\mathbf{D}} \gamma \to BG$. If this map is a mod-$p$ equivalence, then γ provides a homology decomposition of BG.

A coefficient functor for G is a functor Φ from the category of $\mathbb{F}_p[G]$-modules to the category of $\mathbb{F}_p$-vector spaces which preserves arbitrary direct sums. If K is a subgroup of G, $\Phi|_K$ denotes the coefficient functor for K defined by $\Phi|_K(A) := \Phi(\mathbb{F}_p[G] \otimes_{\mathbb{F}_p[K]} A)$. Examples are given by twisted homology groups of G, namely $H_*(G; M \otimes -)$ where M is a $\mathbb{F}_p[G]$-module. In this case , by Shapiro's lemma, the restriction to K is isomorphic to $H_*(K; M \otimes -)$.

If X is a G-space and Φ a coefficient functor for G, then we can construct a chain complex $C_n^G(X; \Phi) := \Phi(\mathbb{F}_p[X_n])$ where X_n denotes the set of n-simplices of X. The boundary map is given by the alternating sum of the face maps in X. The Bredon homology groups $H_*^G(X; \Phi)$ are defined to be the homology groups of this chain complex. By construction, Bredon homology is a functor from the category of G-spaces to the category of abelian groups. A G-map $f : X \to Y$ is called a Φ-equivalence if it induces an isomorphism $H_*^G(f; \Phi)$ for Bredon homology. And a G-space X is called acyclic with respect to Φ if the map $X \to *$ induces a Φ-equivalence.

The E^2-term of the Bousfield-Kan spectral sequence of $\operatorname{hocolim}_{\mathbf{D}} \gamma$ can be identified with the Bredon homology groups of $\operatorname{hocolim}_{\mathbf{D}} \tilde{\gamma}$ with coefficient functor $H_*(G; -)$ [4; §3 and 4.4]. And $\operatorname{hocolim}_{\mathbf{D}} \gamma \to BG$ is a sharp decomposition if and only if $\operatorname{hocolim}_{\mathbf{D}} \tilde{\gamma}$ is acyclic with respect to $H_*(G; -)$. This is also true if we work with twisted coefficients, i.e. if we take a $\mathbb{F}_p[G]$-module M and consider $H_*(-; M)$-homology. For details see [4; §3, 4.4 and 6.2].

In the following, $\mathcal{C}$ is a collection of subgroups. We can associate to $\mathcal{C}$ a simplicial complex $K_{\mathcal{C}}$ [1; 6.2]. The n-simplices of $K_{\mathcal{C}}$ are the subsets $\{H_i\} \subset \mathcal{C}$ of cardinality $n + 1$ which are totally ordered by proper inclusions. The group G acts on $\mathcal{C}$ and, since this action respects inclusions, it also acts on $K_{\mathcal{C}}$.

Before we explain the different decompositions associated to the collection $\mathcal{C}$, we notice that, for every transitive G-set X considered as a discrete G-space, the Borel construction X_{hG} produces a space homotopy equivalent to BG_x for some $x \in X$, where G_x is the stabilizer group of x.

2.1 The subgroup decomposition. We denote by $\mathbf{O}_{\mathcal{C}}(G) \subset \mathbf{O}(G)$ the full subcategory of the orbit category whose objects are given by the homogeneous spaces G/H where $H \in \mathcal{C}$. As usual the morphisms are given by G-equivariant maps. Considering transitive G-sets as discrete G-spaces defines a functor $\tilde{\beta}_{\mathcal{C}} : \mathbf{O}_{\mathcal{C}}(G) \to G - \mathbf{spaces}$. Composing $\tilde{\beta}_{\mathcal{C}}$ with the Borel construction $(-)_{hG}$ gives a functor

$$\beta_{\mathcal{C}} : \mathbf{O}_{\mathcal{C}}(G) \to \mathbf{spaces}$$

whose value at G/H has the homotopy type of BH. The natural maps $\beta_{\mathcal{C}}(G/H) = (G/H)_{hG} \to BG$ are compatible with the morphisms of the orbit category and induce a map

$$b_{\mathcal{C}} : \operatorname*{hocolim}_{\mathbf{O}_{\mathcal{C}}(G)} \beta_{\mathcal{C}} \to BG.$$

This is a homology decomposition if $b_{\mathcal{C}}$ is a mod-p equivalence.

The space $X_{\mathcal{C}}^{\beta} := \operatorname{hocolim}_{\mathbf{O}_{\mathcal{C}}(G)} \tilde{\beta}_{\mathcal{C}}$ carries a G-action and is the nerve of the category $\mathbf{X}_{\mathcal{C}}^{\beta}$ whose objects are pairs $(gH, G/H)$ and whose morphism are G-maps $G/H \to G/H'$ which map gH on $g'H'$. The decomposition is sharp if and only if $(X_{\mathcal{C}}^{\beta})$ is acyclic with respect to $H_*(G; -)$.

2.2 The centralizer decomposition. The $\mathcal{C}$-conjugacy category $\mathbf{A}_{\mathcal{C}}$ is the category in which the objects are pairs (H, Θ) where H is a group and Θ a conjugacy class of monomorphisms $i : H \to G$ with $i(H) \in \mathcal{C}$. A morphism $(H, \Theta) \to (H', \Theta')$ is a group homomorphism $j : H \to H'$ (actually a monomorphism) which under composition carries Θ' into Θ. Restricting H for instance by a requiring that H is a subgroup of G or that $|H| \leq |G|$, makes $\mathbf{A}_{\mathcal{C}}$ into a finite category.

We define a (covariant) functor $\tilde{\alpha}_{\mathcal{C}} : \mathbf{A}_{\mathcal{C}}^{op} \to G-\mathbf{spaces}$ by assigning to each object (H, Θ) the transitive G-set Θ. Composing $\tilde{\alpha}_{\mathcal{C}}$ with the Borel construction yields a functor

$$\alpha_{\mathcal{C}} : \mathbf{A}_{\mathcal{C}}^{op} \to \mathbf{spaces} ,$$

whose value at (H, Θ) is homotopy equivalent to $BC_G(i(H))$ for some $i \in \Theta$. The natural transformation from $\tilde{\alpha}_{\mathcal{C}}$ to the constant functor with value a point, establishes a natural transformation from $\alpha_{\mathcal{C}}$ to the constant functor whose value is BG. Hence, we get a map

$$a_{\mathcal{C}} : \operatorname{hocolim}_{\mathbf{A}_{\mathcal{C}}^{op}} \alpha_{\mathcal{C}} \to BG ,$$

If $a_{\mathcal{C}}$ is a mod-p equivalence we get a decomposition formula.

The space $X_{\mathcal{C}}^{\alpha} := \operatorname{hocolim}_{\mathbf{A}_{\mathcal{C}}^{op}} \tilde{\alpha}_{\mathcal{C}}$ carries a G-action and is the nerve of the category $\mathbf{X}_{\mathcal{C}}^{\alpha}$ whose objects are pairs (i, H) where $i \in \Theta$ and whose morphisms are maps $\rho : H \to H'$ such that $i'\rho = i$. The centralizer decomposition is sharp if and only if $X_{\mathcal{C}}^{\alpha}$ is acyclic with respect to $H_*(G; -)$.

For later purpose we need a smaller category. Let $\hat{\mathbf{A}}_{\mathcal{C}}$ denote the category whose object are given by subgroups $i_H : H \hookrightarrow G$ and whose morphisms are homomorphisms $H \to H'$ equal to a conjugation by an element of G. There exists a functor $\hat{\mathbf{A}}_{\mathcal{C}} \to \mathbf{A}_{\mathcal{C}}$ defined by $H \mapsto (H, \Theta_H)$ where Θ_H is the conjugacy class of the inclusion i_H. The values on the morphisms are obvious. This functor is an inclusion. The image is a full subcategory of $\mathbf{A}_{\mathcal{C}}$ and contains an element of each isomorphism class of objects of $\mathbf{A}_{\mathcal{C}}$. Hence, centralizer sharpness can be reduced to a question about the restriction $\alpha_{\mathcal{C}}|_{\hat{\mathbf{A}}_{\mathcal{C}}^{op}}$ of $\alpha_{\mathcal{C}}$.

Analogously as above we can construct a category $\hat{\mathbf{X}}_{\mathcal{C}}^{\alpha}$ whose objects are pairs $(H, gC_G(H)$ and whose morphisms $(H, gC_G(H)) \to (H', g'C_G(H'))$ are described by elements $g_0 \in G$ such that $g_0 H g_0^{-1} \subset H'$ and such that $g' g_0 C_G(H) = gC_G(H)$. This category carries an G-action which comes from the G-action on the homogeneous spaces $G/C_G(H)$. And again, the decomposition $a_{\mathcal{C}} : \operatorname{hocolim}_{\hat{\mathbf{A}}_{\mathcal{C}}} \alpha_{\mathcal{C}} \to BG$ is sharp if and only if $\hat{\mathbf{X}}_{\mathcal{C}}^{\alpha}$ is acyclic with respect to $H_*(G; -)$.

2.3 The normalizer decomposition. Let $\overline{s}\mathbf{S}_{\mathcal{C}}$ denote the category of the orbit simplices for the G-action on $K_{\mathcal{C}}$. That is the objects are the G-orbits $\overline{\sigma}$ of the simplices σ of $K_{\mathcal{C}}$. There is exactly one morphism $\overline{\sigma} \to \overline{\sigma}'$ if there exist $\sigma \in \overline{\sigma}$ and $\sigma' \in \overline{\sigma}'$ such that $\sigma \subset \sigma'$, i.e. σ is a face of σ'. There is a functor $\tilde{\delta}_{\mathcal{C}} : \overline{s}\mathbf{S}_{\mathcal{C}} \to$

$G-$ **spaces** which assigns to each object $\overline{\sigma}$ the transitive G-set $\overline{\sigma}$. Passing to the Borel construction yields a functor

$$\delta_{\mathcal{C}} : \overline{s}\mathbf{S}_{\mathcal{C}} \to \mathbf{spaces} \ ,$$

whose value at $\overline{\sigma}$ is homotopy equivalent to $B(\bigcap_i N_G(H_i))$ where $(H_0 \subset ... \subset H_n) \in \overline{\sigma}$. The natural transformation from $\tilde{\delta}_{\mathcal{C}}$ to the constant functor with value a point establishes a natural transformation from $\delta_{\mathcal{C}}$ to the constant functor with value BG and therefore a map

$$d_{\mathcal{C}} : \operatorname*{hocolim}_{\overline{s}\mathbf{S}_{\mathcal{C}}} \delta_{\mathcal{C}} \to BG \ .$$

If this is a mod-p equivalence then we get a decomposition of BG.

Let $\mathrm{sd}X^{\delta}_{\mathcal{C}} := \operatorname*{hocolim}_{\overline{s}\mathbf{S}_{\mathcal{C}}} \tilde{\delta}_{\mathcal{C}}$. And $X^{\delta}_{\mathcal{C}}$ is the nerve of the category $\mathbf{X}^{\delta}_{\mathcal{C}}$ whose objects are the subgroups $H \in \mathcal{C}$ and where there exists a unique morphism $H \to H'$ if $H \subseteq H'$. Both spaces carry an G action which is induced from the G-action on $\mathcal{C}$. The space $\mathrm{sd}X^{\delta}_{\mathcal{C}}$ is the barycentric subdivision of $X^{\delta}_{\mathcal{C}}$. And $d_{\mathcal{C}}$ is a sharp decomposition if and only if $X^{\delta}_{\mathcal{C}}$ is acyclic with respect to $H_*(G;-)$.

All this three decomposition are closely connected as the next theorem shows. Let $*$ denote the one-point space with trivial G-action.

2.4 Definition. The collection $\mathcal{C}$ is called ample if the G-equivariant map $K_{\mathcal{C}} \to *$ induces a $\mathbb{F}_p$ equivalence $(K_{\mathcal{C}})_{hG} \to BG$.

2.5 Theorem. *(Dwyer [3]) Let G be a finite group and let $\mathcal{C}$ be a collection. If the complex $K_{\mathcal{C}}$ is ample then all three maps $a_{\mathcal{C}}$, $b_{\mathcal{C}}$ and $d_{\mathcal{C}}$ are give homology decompositions. And if one of this three maps is a homology decomposition then $K_{\mathcal{C}}$ is ample.*

2.6 Remark. Let M be an $\mathbb{F}_p[G]$-module. Then, the collection $\mathcal{C}$ is called M-ample if the map $(K_{\mathcal{C}})_{hG} \to BG$ induces an isomorphisms on (twisted) homology with coefficients in M. $\mathcal{C}$ is called M-subgroup sharp if $b_{\mathcal{C}}$ induces a sharp decomposition for the twisted homology groups $H_*(-;M)$. And this is true if and only if the space $X^{\beta}_{\mathcal{C}}$ is acyclic for the coefficient functor $H_*(G; M \otimes -)$. The same holds for the centralizer and the normalizer decompositions.

3. Comparison of collections.

Let $\mathcal{H}$ and $\mathcal{K}$ be two collections of subgroups of a finite group G. We want to show results of the following form. If $\mathcal{K}$ satisfies some ampleness or sharpness properties, then so does $\mathcal{H}$. Of course we have to put some conditions on $\mathcal{H}$. For a subgroup $H \subset G$ we denote by $\mathcal{K}_H := \{K \in \mathcal{K} | K \subset H\}$ the set of all elements in $\mathcal{K}$ which are contained in H. This is a collection of subgroups of H. Actually, we will compare $\mathcal{H}$ with $\mathcal{H} \cup \mathcal{K}$ and $\mathcal{K}$ with $\mathcal{H} \cup \mathcal{K}$. In the latter case we like think of $\mathcal{H} \cup \mathcal{K}$ as an extension of $\mathcal{K}$ and in the first case of $\mathcal{H}$ as a reduction of $\mathcal{H} \cup \mathcal{K}$.

3.1 Definition. Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be three collections of subgroups of G.
(i) The collection $\mathcal{H}$ is called a *sharp reduction* of $\mathcal{L}$ if $\mathcal{H} \subset \mathcal{L}$ and if for for every $L \in \mathcal{L} \setminus \mathcal{H}$ there exists a unique minimal element in $\mathcal{H}$, denoted by H_L, with respect to inclusions such that $L \subset H_L$; i.e. if $L \subset H$ and $H \in \mathcal{H}$, then $H_L \subset H$.
(ii) We say that $\mathcal{L}$ is an *ample extension* of $\mathcal{K}$, if $\mathcal{K} \subset \mathcal{L}$ and if $\mathcal{K}_L$ is an ample collection for all $L \in \mathcal{L} \setminus \mathcal{K}$. The collection is called a *subgroup sharp extension* if, in addition, $\mathcal{K}_L$ is a subgroup sharp collection for L.

3.2 Proposition. *Let $\mathcal{H} \subset \mathcal{L}$ be two collections of subgroups of a finite group G. If $\mathcal{H}$ is a sharp reduction of $\mathcal{L}$, then the following holds:*
(i) The map $K_{\mathcal{H}} \to K_{\mathcal{L}}$ is a homotopy equivalence.
(ii) The maps $X^{\beta}_{\mathcal{H}} \to X^{\beta}_{\mathcal{L}}$, $\hat{X}^{\alpha}_{\mathcal{H}} \to \hat{X}^{\alpha}_{\mathcal{L}}$ and $X^{\delta}_{\mathcal{H}} \to X^{\delta}_{\mathcal{L}}$ are G-equivariant homotopy equivalent.
(iii) In particular $\mathcal{H}$ has all the ampleness and/or sharpness properties which $\mathcal{L}$ has.

3.3 Proposition. *Let $\mathcal{K} \subset \mathcal{L}$ be two collections of subgroups of a finite group G.*
(i) If $\mathcal{L}$ is an ample extension of $\mathcal{K}$, then the map $K_{\mathcal{K}} \to K_{\mathcal{L}}$ is a homotopy equivalence.
(ii) If $\mathcal{L}$ is a subgroup sharp extension of $\mathcal{K}$, then $\mathcal{L}$ is subgroup sharp if and only if $\mathcal{K}$ is.

3.4 Remark. Let M be a $\mathbb{F}_p[G]$-module and let $\mathcal{H}$, $\mathcal{L}$ and $\mathcal{K}$ as in the above propositions. In fact, Proposition 3.1 say that $\mathcal{H}$ has the same M-ampleness and the same M-subgroup, M-centralizer and M-normalizer sharpness properties as $\mathcal{L}$. And, if $\mathcal{K}$ and $\mathcal{K}_L$ are M-subgroup sharp, then so is $\mathcal{L}$. actually, the last claim only follows from the proof of 3.3 (ii).

Proof of Proposition 3.2. For a collection $\mathcal{C}$ we denote by $S_{\mathcal{C}}$ the nerve of the category $\mathbf{S}_{\mathcal{C}}$ which is just the poset $\mathcal{C}$ considered as a category. Then, $K_{\mathcal{C}} \simeq S_{\mathcal{C}}$, since the simplices of $K_{\mathcal{C}}$ are just the nondegenerate simplices of $S_{\mathcal{C}}$.

If $\mathcal{H} \to \mathcal{L}$ is a strong reduction we set $H_L := L$ if $L \in \mathcal{H}$. Then we can define a map $F : \mathcal{L} \to \mathcal{L}$ by $F(L) := H_L$.This is idempotent, respects inclusion, is G-equivariant and maps onto $\mathcal{H}$. The G-equivariance follows, since $gH_Lg^{-1} = H_{gLg^{-1}}$. Actually, $\mathcal{H} \subset \mathcal{L}$ is a strong reduction if and only if such a map exists.

Basically by assumption, the under categories $L \to \mathbf{S}_{\mathcal{H}}$ for the inclusion $\mathbf{S}_{\mathcal{H}} \to \mathbf{S}_{\mathcal{L}}$ have an initial element $L \subset F(L)$. All these under categories are contractible and the inclusion is right cofinal. By Quillens's Theorem A [10], both categories have homotopy equivalent classifying spaces, which shows that $K_{\mathcal{H}} \simeq K_{\mathcal{L}}$.

Next we prove part (ii). The maps are induced by the G-equivariant inclusions $i_\beta : \mathbf{X}^{\beta}_{\mathcal{H}} \subset \mathbf{X}^{\beta}_{\mathcal{L}}$, $i_\alpha : \hat{\mathbf{X}}^{\alpha}_{\mathcal{H}} \subset \hat{\mathbf{X}}^{\alpha}_{\mathcal{L}}$ and $i_\delta : \mathbf{X}^{\delta} \subset \mathbf{X}^{\delta}_{\mathcal{L}}$ of categories. We construct G-equivariant left inverse maps, denoted by s_β, s_α and s_δ and G-equivariant natural transformations $id \to i_\beta s_\beta$, $id \to i_\alpha s_\alpha$ and $id \to i_\delta s_\delta$.

For the subgroup decomposition, s_β is given by $(G/L, gL) \mapsto (G/F(L), gF(L))$. The values on the morphisms are the obvious ones. By the properties of F, this is well defined. And the functor s_β is G-equivariant. The inclusion $L \subset F(L)$ induces a morphisms $(G/L, gL) \to (G/F(L), gF(L))$. The family of these morphisms establishes a natural transformation $id \to i_\beta s_\beta$ which, in addition, is G-equivariant.

For the centralizer decomposition, $(L, gC_G(L) \mapsto (F(L), gC_G(F(L))$ establishes the left inverse. We can also define the values on the morphisms in the obvious way and get a G-equivariant functor. Again the inclusion $L \subset F(L)$ establishes a morphism $(L, gC_G(L) \to (F(L), gC_G(F(L))$ in $\hat{\mathbf{X}}^{\alpha}_{\mathcal{L}}$. And the family of all these morphisms establishes a G-equivariant natural transformation $id \to i_\alpha s_\alpha$.

For the normalizer decomposition, the left inverse is given by $L \mapsto F(L)$ and by the obvious definition for morphisms. This is well defined and G-equivariant. Again the inclusion $L \subset F(L)$ establishes a morphism $L \to F(L)$ and the family of all these morphisms a G-equivariant natural transformation $id \to i_\delta s_\delta$.

The lemma below shows that the construction of the left inverse and the natural transformation is sufficient for proving Part (ii).

Part (iii) is a consequence of (ii). $\square$

3.5 Lemma. *Let $i : \mathbf{C} \to \mathbf{D}$ be a G-equivariant functor between small categories, which has a G-equivariant left inverse $s : \mathbf{D} \to \mathbf{C}$. Let $T : id \to i \circ s$ a G-equivariant natural transformation. Then, the map i induces a G-equivariant homotopy equivalence between the classifying spaces of $\mathbf{C}$ and $\mathbf{D}$.*

Proof. It is sufficient to show that the map i induces a homotopy equivalence between the fixed-point sets of the classifying spaces $B\mathbf{C}$ and $B\mathbf{D}$ of the categories for all subgroups of G. Since the $(B\mathbf{C})^H \simeq B(\mathbf{C}^H)$, passing to fixed-points maintains the situation described in the statement but where we forget the equivariant setting. We only have to prove the nonequivariant version of the statement. But that is obvious since a natural transformation between functors induces a homotopy between the induced maps on the classifying spaces.

Proof of Proposition 3.3. Let $\mathcal{K} \to \mathcal{L}$ be an ample extension. We want to show that $K_{\mathcal{K}}$ is ample if and only $K_{\mathcal{L}}$ is or, equivalently, that $b_{\mathcal{K}} : \operatorname*{hocolim}_{\mathbf{O}_{\mathcal{K}}(G)} \beta_{\mathcal{K}} \to BG$ is a homology decomposition if and only if $b_{\mathcal{L}}$ is one.

For the inclusion $\mathbf{O}_{\mathcal{K}}(G) \to \mathbf{O}_{\mathcal{L}}(G)$, the left Kan-extension of $\beta_{\mathcal{K}}$ is given by $L(\beta_{\mathcal{K}})(G/L) := \operatorname*{hocolim}_{\mathbf{O}_{\mathcal{K}} \to G/L} \beta_{\mathcal{K}}$ and $\operatorname*{hocolim}_{\mathbf{O}_{\mathcal{K}}(G)} \beta_{\mathcal{K}} \simeq \operatorname*{hocolim}_{\mathbf{O}_{\mathcal{L}}(G)} L(\beta_{\mathcal{K}})$ (e.g. see [5; 9.8]). There exists a natural transformation $T : L(\beta_{\mathcal{K}}) \to \beta_{\mathcal{L}} \simeq BL$, since all objects as well as all morphisms in the under category are given by G–maps. There is a functor $\mathbf{O}_{\mathcal{K}_L}(L) \to (\mathbf{O}_{\mathcal{K}} \to G/L)$ which is an equivalence of categories. The inverse is given by mapping $G/K \to G/L$ to the counter image of the orbit $1 \cdot L \in G/L$. By assumption, the collection $\mathcal{K}_L$ is ample which implies that the natural transformation T is an equivalence of functors. Thus, $\operatorname*{hocolim}_{\mathbf{O}_{\mathcal{K}}(G)} \beta_{\mathcal{K}} \simeq \operatorname*{hocolim}_{\mathbf{O}_{\mathcal{L}}(G)} \beta_{\mathcal{L}}$ which proves (i).

Next we consider the case of subgroup sharp extensions. We use the same idea. Let $L(H_*(\beta_{\mathcal{K}}; \mathbb{F}_p) := \varinjlim_{\mathbf{O}_{\mathcal{K}} \to G/L} H_*(\beta_{\mathcal{K}}; \mathbb{F}_p)$ be the left Kan extension of $H_*(\beta_{\mathcal{K}}; \mathbb{F}_p)$ for the inclusion $\mathbf{O}_{\mathcal{K}}(G) \to \mathbf{O}_{\mathcal{L}}(G)$ and denote by L^i the left derived functors of $\varinjlim_{\mathbf{O}_{\mathcal{K}} \to G/L} H_*(\beta_{\mathcal{K}}; \mathbb{F}_p)$. Since we can identify the under category $\mathbf{O}_{\mathcal{K}} \to G/L$ with the orbit category $\mathbf{O}_{\mathcal{K}_L}(L)$ and since , by assumption, $\mathcal{K}_L$ is subgroup sharp, L^i vanishes for $i > 0$ and $L^0 \cong H_*(\beta_{\mathcal{L}}; \mathbb{F}_p)$. An application of the composition of functor spectral sequence (cf. [6; Appendix II, 3.6]) proves the claim. $\square$

4. Proofs of the statements.

In this section we want to prove our results stated in the introduction. We start with

Proof of Proposition 1.5. If $\mathcal{C}$ is a minimalization of $\mathcal{P}$, then, by definition, $\mathcal{C} \subset \mathcal{C} \cup \mathcal{P}$ is a sharp reduction. Moreover, since $\mathcal{P}_H$ is the collection of all nontrivial p-subgroups of H and since such collection are always subgroup sharp [4; 7.4], the claim follows from the propositions 3.2 and 3.3. $\square$

In view of Proposition 1.5, the subgroup sharpness claimed in Theorem 1.3 and Theorem 1.4 is a consequence of the following two lemmas.

4.1 Lemma. *Let G be a finite group.*
(i) Let $\rho : G \to Gl(V)$ be a modular representation. Then $\mathcal{I}(\rho)$ is a minimalization of $\mathcal{P}$.
(ii) Let X be a G-space. Then, $\mathcal{I}(X)$ is a minimalization of $\mathcal{P}$.

4.2 Lemma. *Let G be a finite group such that p divides the order of G. Let $\mathcal{C} := \mathcal{C}en(\mathcal{E})$, $\mathcal{C}en(\mathcal{A})$ denote the collection of centralizers of all nontrivial elementary abelian or of all nontrivial abelian subgroups of G. Then, $\mathcal{C} \subset \mathcal{C} \cup \mathcal{P}^c$ is a strong reduction and $\mathcal{P}^c \subset \mathcal{C} \cup \mathcal{P}^c$ is a subgroup sharp extension.*

Proof of Lemma 4.1. For each subgroup $K \subset G$ we denote by $H_K := Iso_G(V^K)$ the isotropy group of V^K. Then, $K \subset H_K$ and $H_K \subset Iso_G(V^{H_K}) \subset Iso_G(V^K) = H_K$. Hence, in the latter sequence all groups are equal. In particular, $H_{(H_K)} = H_K$. Moreover, if $P \in \mathcal{P}$, $H \in \mathcal{I}(\rho)$ and $P \subset H$, then V^P is a nontrivial proper subspace of V, $Iso_G(H) \subset Iso_G(P)$ and $H_P \subset H$. This shows that $\mathcal{I}(\rho) \subset \mathcal{I}(\rho) \cup \mathcal{P}$ is a minimalization. The proof of (ii) is analogously. $\square$

Proof of Lemma 4.2. For a subgroup $K \subset G$ we denote by $E(K)$ the maximal elementary abelian subgroup of the center $Z(K)$ of K. For $P \in \mathcal{P}$ we set $H_P := C_G(E(P))$ and for $K \in \mathcal{C}en(\mathcal{E}) \setminus \mathcal{P}$, we set $H_K = K$.

If $H := C_G(E) \in \mathcal{C}en(\mathcal{E})$, then $E \subset E(H)$ and $H \subset C_G(E(H)) \subset C_G(E) = H$. Hence, in the latter sequence, all groups are equal. If $P \in \mathcal{P}^c$ and $H := C_G(E) \in \mathcal{C}en(\mathcal{E})$ such that $P \subset H$, then $Z(P) \subset C_G(P) =: C$ is a p-Sylow subgroup and, in particular, $E(H) \subset E(P)$. Thus $H_P \subset C_G(E(H)) = H$. This shows that $\mathcal{C}en(\mathcal{E}) \subset \mathcal{C}en(\mathcal{E}) \cup \mathcal{P}^c$ is a strong reduction.

Moreover, if $H := C_G(E)$ and $P \subset H$ is p-centric in G or H, then $E(H) \subset Z(P)$ and $C_H(P) = C_G(P)$. That is to say that P is a p-centric subgroup of H if and only if it is a p-centric subgroup of G. Therefore, $\mathcal{P}^c_H$ is the collection of all p-centric subgroups of H which is subgroup sharp [4; 10.3]. This proves the second part of statement.

For $\mathcal{C}en(\mathcal{A})$, the proof is analogously. We only have to replace $E(H)$ by the center $Z(H)$ of H. $\square$

4.3 Remark.
(i) As the proofs and the Remark 3.4 shows, $\mathcal{I} := \mathcal{I}(\rho)$, $\mathcal{I}(X)$ is M-subgroup sharp for a $\mathbb{F}_p[G]$-module M if $\mathcal{P}$ as well as $\mathcal{P}_H$ for every $H \in \mathcal{I}$ are M-subgroup sharp. The arguments in [4; 7.4] show that this is true if every element of order p acts trivially on M.
(ii) Again, the arguments in the proofs of both lemmas show that we can choose a smaller collection of isotropy subgroups of G. Let $\mathcal{I}_{\mathcal{P}}(\rho)$ denote the collection of all isotropy subgroups of the subspaces V^P for $P \in \mathcal{P}$. Then, $\mathcal{I}_{\mathcal{P}}(\rho)$ is a minimalization of $\mathcal{P}$ and therefore M-subgroup sharp for M as above. For this collection we can drop the assumption that G acts faithfuly on V. This second remark also holds for a G-space X and the collection $I(X)$.

In the rest of this section we prove centralizer and normalizer sharpness properties for our collections.

4.4 Theorem. *Let G be a finite group such that p divides the order of G.*
(i) Let $\rho : G \to Gl(V)$ be a modular representation. Then, the collection $\mathcal{I}(\rho)$ is normalizer sharp.
(ii) Let X be a G-space. Then, $\mathcal{I}(X)$ is normalizer sharp.

Proof. We use the techniques of [4; §7]. For a subgroup $K \subset G$ we set $H_K := Iso_G(V^K)$. For every nontrivial p-subgroup $Q \subset G$, the fixed-point set $(X^{\delta}_{\mathcal{I}(\rho)})^Q$ is the nerve of the category $\mathbf{X}^{\delta}_{\mathcal{I}(\rho)^Q}$ [4; 7.2]. And $\mathcal{I}(\rho)^Q$ is the full subcategory of all $H \in \mathcal{I}(\rho)$ such that $Q \subset N_G(H)$. Obviously, the isotropy group H_Q is fixed, thus $\mathbf{X}^{\delta}_{\mathcal{I}(\rho)^Q}$ is nonempty. The natural transformations induced by the inclusions $H \subset H_{H\cdot Q} \supset H_Q$ show that the identity is homotopic to the constant functor. Therefore, $\mathbf{X}^{\delta}_{\mathcal{I}(\rho)^Q}$ is contractible.

Moreover, for each simplex $\sigma := (H_0 \subset H_1 \subset \ldots \subset H_n)$ of $X^{\delta}_{\mathcal{I}(\rho)}$, the isotropy group $G_\sigma = \bigcap_i N_G(H_i)$ contains H_0 and therefore an element of order p which acts trivially on $\mathbb{F}_p$. This is sufficient to conclude that $X^{\delta}_{\mathcal{I}(\rho)}$ is acyclic for the functors $H_*(G; -)$ [4; 6.10] and that $\mathcal{I}(\rho)$ is normalizer sharp. $\square$

4.5 Remark. A long the same lines, one can proof that $\mathcal{I}(\rho)$ is subgroup sharp. We use the same notation as above. The fixed-point set $(X^{\beta}_{\mathcal{I}(\rho)})^Q$ is the nerve of the full subcategory of object $(gH, G/H)$ such that $Q \subset gHg^{-1}$. Hence, it is nonempty, and, since there is a unique map $(eH_Q, G/H_Q) \to (gH, G/H)$ for each object, it has an initial element and is therefore contractible.

A typical simplex of $X^{\beta}_{\mathcal{I}(\rho)}$ has as its isotropy group an element of $\mathcal{I}(\rho)$ and contains therefore an element of order p. Analogously as above, this implies that $\mathcal{I}(\rho)$ is subgroup sharp.

4.6 Proposition. *Let G be a finite group such that p divides the order of G.*
(i) Let $\rho : G \to Gl(V)$ be a modular representation. Then, the collection $\mathcal{P} \cup \mathcal{I}(\rho) =: \mathcal{PI}$ is centralizer sharp.
(ii) Let X be G-space. Then, the collection $\mathcal{P} \cup \mathcal{I}(X) =: \mathcal{PI}$ is centralizer sharp.

Proof. Again, H_K denotes $Iso_G(V^K)$ respectively $Iso_G(X^K)$. First we show that for every nontrivial p-subgroup Q the fixed-point sets $(\hat{X}^{\alpha}_{\mathcal{PI}})^Q$ are contractible. This fixed-point set is the nerve of the full subcategory whose objects are pairs $(H, gC_G(H))$ such that $g^{-1}Qg \subset C_G(H)$. The fixed point set is nonempty since $Q \subset C_G(Z(Q))$. The zig-zag $H \subset H_{H\cdot Z(Q)} \supset H_{Z(Q)}$ establishes natural transformation between the associated functors. This shows that $(\hat{X}^{\alpha}_{\mathcal{PF}})^Q$ is contractible.

The same is true for $(\hat{X}^{\alpha}_{\mathcal{P}})^Q$ [4; 7.3]. Therefore, for a p-Sylow subgroup $P \subset G$, the inclusion $j : \hat{X}^{\alpha}_{\mathcal{P}} \to \hat{X}^{\alpha}_{\mathcal{PI}}$ is a weak P-equivalence and induces an isomorphism in P-equivariant Bredon homology for any coefficient functor Φ. Now we can apply a relative version of [4; 6.4] (see the lemma below) to show that j induces an isomorphism in G-equivariant Bredon homology for any coefficient functor for G. Since $\hat{X}^{\alpha}_{\mathcal{P}}$ is acyclic for $H_*(G; -)$ [4; 7.3], this is also true for $X^{\alpha}_{\mathcal{PI}}$. This shows that $\mathcal{PI}$ is centralizer sharp. $\square$

4.7 Lemma. *Let $f : X \to Y$ be a G-map between G-spaces. Let Φ be a coefficient functor for G and let $K \subset G$ be a subgroup of index coprime to p. If f is an $\Phi|_K$-equivalence as a K-map, then it is a Φ-equivalence as a G-map, too.*

Proof. Let C denote the cone of f. Then

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ * \arrow[r] & C \end{tikzcd}$$

is a homotopy push out of G-spaces in the sense of [4; §4]; i.e. for every subgroup $H \subset G$ the associated diagram of H-fixed subspaces is a homotopy pushout of spaces. The long exact sequence for P-equivariant Bredon homology of the above homotopy push out shows that C is acyclic as a K-space for $\Phi|_K$. Hence, by [4; 6.4], the G-space C is acyclic for Φ and f as a G-map is a Φ-equivalence. Again this last step follows from the long exact sequence for G-equivariant Bredon homology. $\square$

4.8 Corollary. *Let G be a finite group such that p divides the order of G.*
(i) Let $\rho : G \to Gl(V)$ be a modular representation. Then, the collection $\mathcal{I}(\rho)$ is centralizer sharp.
(ii) Let X be G-space. Then, the collection $\mathcal{I}(X)$ is centralizer sharp.

Proof. This follows from Lemma 4.1 and its proof, Proposition 3.2 and Proposition 4.6. $\square$

4.9 Remark. The arguments in the proof of Proposition 4.6 and Proposition 3.2 show that the centralizer decomposition for $\mathcal{I}(\rho)$ has the same centralizer sharpness properties as $\mathcal{P}$. If M is $\mathbb{F}_p[G]$-module, such that there exists an element $1 \neq g \in G$ of order p which acts trivially on M, then $\mathcal{P}$ is M-centralizer sharp [4; 7.3] and so is $\mathcal{I}(\rho)$.

Finally we want to show that the collection $\mathcal{C}en(\mathcal{E})$ and $\mathcal{C}en(\mathcal{A})$ are normalizer sharp. This will finish the proofs of everything claimed in the introduction.

4.10 Theorem. *The collections $\mathcal{C}en(\mathcal{E})$ and $\mathcal{C}en(\mathcal{A})$ are normalizer sharp.*

Proof. We only prove this statement for $\mathcal{C} := \mathcal{C}en(\mathcal{E})$. A slight modification of the argument will work for $\mathcal{C}en(\mathcal{A})$.

Let Q be a nontrivial p-subgroup. Then, $(X^\delta_\mathcal{C})^Q$ is the nerve of the full subcategory of $\mathbf{X}^\delta_\mathcal{C}$ whose objects satisfy the condition $Q \subset N_G(C_G(E)$. This category is nonempty since $Q \subset C_G(E(Q)$. Moreover, Q acts on $E' := E(C_G(E))$. The zig-zag

$$C_G(E) \subset C_G((E')^Q) \supset C_G((E')^Q \cdot E(Q)) \subset C_G(E(Q))$$

shows that this fixed point set is contractible. Moreover, $N_G(C_G(E))$ contains an element of order p. Therefore, $(X^\delta_\mathcal{C})$ is acyclic for the coefficient functor $H_*(G; -)$ [4] and $\mathcal{C}$ is normalizer sharp. $\square$

4.11 Remark. Let M be a $\mathbb{F}_p[G]$-module, such that there exists an element $1 \neq g \in G$ of order p which acts trivially on M Then, for every $E \in \mathcal{E}$, there exists an $1 \neq h \in C_G(E) \subset N_G(C_G(E))$ of order p which also acts trivially on M [4; 7.1]. This is sufficient to conclude that $X^\delta_{\mathcal{C}en(\mathcal{E})}$ is acyclic with respect to $H_*(G; M \otimes -)$ [4; 6.10] and that $\mathcal{C}en(\mathcal{E})$ is M-normalizer sharp.

5. Examples.

In this section we discuss some examples. Again, G denotes a finite group.

5.1 Polynomial invariants of finite groups.

This example was used in [8] and [9] to construct spaces with polynomial mod-p cohomology and to prove uniqueness results for these spaces. Actually it was the motivating example for this note.

Let $\mathcal{P}'$ denote the collection of all p-subgroups of G (including the trivial group). This collection is M-subgroup sharp for any $\mathbb{F}_p[G]$-module M [4; 10.1 and the proof], even if we replace homology by cohomology. For cohomology, this is proved in [7].

Let $\rho : G \to Gl(V)$ be a modular representation, let $\mathcal{I}' := \mathcal{I}'(\rho)$ denote the collection of isotropy subgroups of all nontrivial subspaces of V (including V itself) and let $\mathcal{I}_{\mathcal{P}'} := \mathcal{I}_{\mathcal{P}'}(\rho)$ denote the subcollection of all isotropy subgroups of the subspaces V^P for $P \in \mathcal{P}'$. Then, $\mathcal{I}'$ and $\mathcal{I}_{\mathcal{P}'}$ are both M-subgroup sharp for every $\mathbb{F}_p[G]$-module M (for homology as well as for cohomology).

Let $\phi : G \to Gl(W)$ be another representation of G. The the polynomial algebra $A := \mathbb{F}_p[W]$ of formal polynomial function on W carries a G-action. For $\mathcal{J} = \mathcal{I}'$ or $\mathcal{I}_{\mathcal{P}'}$, the functor $H^*_A : \mathbf{O}^{op}_{\mathcal{J}}(G) \to \mathcal{A}b$ defined by $H^*_A(G/H) := H^*(H; A)$ (group cohomology) is isomorphic to $H^*(\beta_{\mathcal{J}}; A)$ (twisted coefficients). We get the algebraic decomposition formula

$$\underset{\overleftarrow{\mathcal{J}}}{\lim}{}^i H^*_A \cong \begin{cases} H^*(G; A) & \text{for} \quad i = 0 \\ 0 & \text{for} \quad i \geq 1 \end{cases}$$

And, specializing to H^0_A, this reads as

$$\underset{\overleftarrow{\mathcal{J}}}{\lim}{}^i \mathbb{F}_p[W]^H \cong \begin{cases} \mathbb{F}_p[W]^G & \text{for} \quad i = 0 \\ 0 & \text{for} \quad i \geq 1 \end{cases}$$

5.2 Decompositions for the symmetric group Σ_n.
The set $X := \{1, ..., n\}$ given by the first n positive integers carries a Σ_n-action. For a p-subgroup $P \subset \Sigma_n$ a P-orbit has length 1 or a power of p. Hence, the fixed-point set X^P is a subset of X of order r such that p divides $n - r$. And the isotropy subgroup of X^P is conjugate to Σ_{n-r}.

Since every subset of X of order $n-r$ such that $p|(n-r)$ appears as the fixed-point set of a p-subgroup, the collection $\mathcal{C}$ of all subgroups conjugate to some $\Sigma_m \subset \Sigma_n$ where $p|m$ is subgroup, centralizer and normalizer sharp.

Let $\Sigma_n \to Gl(n, \mathbb{F}_p)$ denote the permutation representation. And let $B := \{e_1, ..., e_n\}$ denote the standard basis of $\mathbb{F}_p^n =: V$. For a subset $B' \subset B$ we denote by $V(B') \subset V$ the subspace generated by the sum of all elements in B'. The isotropy subgroup of $V(B')$ is conjugate to $\Sigma_r \times \Sigma_{n-r}$ where r is the order of B'. For a nontrivial p-subgroup $P \subset \Sigma_n$ the fixed-point set V^P is the direct sum of subspaces $V(B')$ where B' runs through the P-orbits of B. Since each of these orbits has length a power of p, the isotropy subgroup of V^P is conjugate to a subgroup of the form $\Sigma_{p^{n_1}} \times ... \times \Sigma_{p^{n_k}}$ where $p^{n_1} + ... + p^{n_k} = n$ and where $n_1 \geq 1$. Let $\mathcal{C}$ denote the collection of all subgroups conjugate to a product of symmetric groups as described above. Each of these groups appears as the isotropy subgroup of the fixed-point set of some p-subgroup. Therefore, the collection $\mathcal{C}$ is subgroup, centralizer and normalizer sharp.

5.3 A decomposition for $Gl(n, \mathbb{F}_p)$.
The general linear group $G := Gl(n; \mathbb{F}_p)$ acts on $V := \mathbb{F}_p^n$ by matrix multiplication. Again, we denote by $B := \{e_1, ..., e_n\}$ the standard basis of V. Let $V(r)$ denote the subspace generated by the first r basis vectors. The group G acts transitively on the set of r-dimensional subspaces. Therefore, the isotropy subgroup of a proper nontrivial subspace $V' \subset V$ is conjugate to the isotropy subgroup of $V(r)$, where

$1 \le r = dim_{\mathbb{F}_p} V'$. And this is the subgroup $G(r)$ of all matrices of the form $\begin{pmatrix} 1 & B \\ 0 & D \end{pmatrix}$, where $D \in Gl(n-r, \mathbb{F}_p)$ and where B is a $r \times (n-r)$ matrix over $\mathbb{F}_p$. All these subgroups contain elements of order p. Therefore, the collection of all subgroups conjugate to some $G(r)$, $0 < r < n$ is subgroup, centralizer and normalizer sharp.

5.4 Decompositions for finite groups associated to permutation representation.
Let $H \subset G$ be a subgroup of the finite group G. The G-action of G on the homogeneous space G/H induces an action of G on the vector space $V := \mathbb{F}_p[G/H]$ whose standard basis is given by the elements $gH \in G/H$. As in 5.2, we associate for each subset of $B \subset G/H$ the 1-dimensional subspace $V(B) \subset V$ generated by the sum of all elements in B. For a p-subgroup $P \subset G$ the fixed-point set V^P is the direct sum of all subspaces $V(B)$ where B runs through the P-orbits in G/H. And a subgroup $K \subset G$ with $P \subset K$ is contained in the isotropy subgroup of V^P if the obvious map $P\backslash G/H \to K\backslash G/H$ is a bijection. That is to say that, for all $gH \in G/H$ and for all $k \in K$, $kgH \subset PgH$ or , equivalently, such that $k \in P \cdot gHg^{-1}$. The product is understood as the set of elements ph, $p \in P$ and $h \in gHg^{-1}$, not as the subgroup generated by P and gHg^{-1}. The isotropy group of V^P is therefore the subgroup $H_P := \bigcap_{gH \in G/H} P \cdot gHg^{-1}$. The collection of all subgroups H_P, $P \in \mathcal{P}$ is subgroup, centralizer and normalizer sharp.

5.5 Decomposition associated to collections of subgroups.
Let $X := \mathcal{C}$ be a collection of subgroups of G, such that there exists an element $C \in \mathcal{C}$ whose normalizer contains a p-Sylow subgroup of G. For a p-subgroup $P \subset G$, the isotropy subgroup of X^P is the intersection $\bigcap N_G(C)$ taken over all subgroups C such that $P \subset N_G(C)$. The collection of all subgroups of this form is nonempty and subgroup, centralizer and normalizer sharp.

If we choose $\mathcal{C}$ to be $\mathcal{P}$ itself then the collection of all intersection $N_G(P_1) \cap ... \cap N_G(P_k)$ with nontrivial p-Sylow subgroup is subgroup, centralizer and normalizer sharp.

If we choose $\mathcal{C}$ to be the collection of all p-Sylow subgroups, then the collection of all intersections $N_G(P_1) \cap ... \cap N_G(P_k)$ such that $P_1 \cap ... \cap P_k$ is a nontrivial intersection of p-Sylow subgroups is subgroup, centralizer and normalizer sharp.

If we choose X to be the set of all objects of the category $\mathbf{X}^{\beta}_{\mathcal{C}}$ where $\mathcal{C}$ is again the collection of all p-Sylow subgroups, then the collection of all nontrivial intersections $P_1 \cap ... \cap P_k$, P_i a p-Sylow subgroup for all i, is subgroup centralizer and normalizer sharp.

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