Some congruences related to modular forms

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ABSTRACT. Let f be a modular form of weight k for a congruence subgroup $\Gamma \subset SL_2(\mathbb{Z})$, and t a weight 0 modular function for Γ. Assume that near $t = 0$ we can write an expansion $f = \sum_{n\geq 0} b_n t^n$, $b_n \in \mathbb{Z}$. Let $L(\tau)$ be a weight $k+2$ modular form with q-expansion $\sum \gamma_n q^n$, such that the Mellin transform of L can be expressed as an Euler product. Then we show that if $f\frac{q\,dt/dq}{t} = a_1 L(d_1\tau) + \cdots a_m L(d_m\tau)$ for some integers a_i, d_i, then there is a congruence relation between the b_n and the γ_n.

Stienstra ([S]) proves a congruence result relating the coefficients of the L-series of a fibred variety to the coefficients of an expansion of the solution of the Picard-Fuchs equation. Stienstra's result is proved for certain double covers of projective space, and it is stated to hold also for complete intersections, and may also hold in other cases.

Stienstra's result means that if you have a variety, and calculate the L-series, this may enable you to deduce information about the solution of the Picard-Fuchs equation, or, conversely, a solution of the Picard-Fuchs equation may allow one to find the L-series. In some situations e.g., when working with Calabi-Yau type varieties, such as elliptic curves, K3-surfaces, and Calabi-Yau three-folds, we may expect that the L-series is modular (has the same coefficients as the q-expansion of some modular form), and also that the Picard-Fuchs differential equation may be solved in terms of modular forms (for examples, see [SB], [P], [V1], [V2].)

The aim of this paper is to produce a practical result which can be applied to use Stienstra's result in examples where the L-series and solution of Picard-Fuchs equations are in terms of modular forms. The main result, which is based on Theorem 4 of [B] is as follows:

THEOREM 0.1. *Let Γ be a level N congruence subgroup of $SL_2(\mathbb{Z})$. Let t be a weight 0 modular function for Γ (so t is non holomorphic). Let $f \in M_k(\Gamma)$ be a weight k modular form, and Let $L \in M_{k+2}(\Gamma)$ be a weight $k+2$ modular form for Γ, which has an Euler product expansion e.g., a Hecke eigen form. Write*

$$f = \sum b_n t^n, \quad L(q) = \sum \gamma_n q^n$$

then if for some integers a_d, $d|M$ for some integer M, we have the following equality

$$f(q)\frac{q\frac{dt}{dq}}{t} = \sum_{d|M} a_d L(q^d) \tag{0.1}$$

then for prime $p \not| NM$, and integers m, r,

$$b_{mp^r} - \gamma_p b_{mp^{r-1}} + \varepsilon_p p^{k+1} b_{mp^{r-2}} \equiv 0 \bmod p^r, \tag{0.2}$$

where ε_p is the character of L; explicitly, we assume there is an expression

$$\sum_{n\geq 1} \frac{\gamma_n}{n^s} = \prod_{p\not| N} \frac{1}{1-\gamma_p p^s + \varepsilon_p p^{k+1} p^{-s}} \prod_{p|N} E_p,$$

where E_p is the Euler factor for the primes not dividing N.

In particular, if $b_1 = 1$ then $b_p \equiv \gamma_p \bmod p$

The proof of this result can be found in section 1. This result will also be of interest independently to the geometrical interpretation, and is related to extensions of the congruences for Apéry numbers given in [B]. For example it may be related to the examples in the paper [AO], which corresponds to the case $k = 0$ in the above theorem. The b_n in Theorem 0.1 also satisfy a recurrence relation of the form

$$P_0(n)b_n + P_1(n)b_{n-1} + \cdots + P_k(n)b_{n-k} = 0 \tag{0.3}$$

where $P_i(n)$ are polynomials in n. More details of this congruence are given in section 0.4. In this paper, we do not look at the explicit solutions of such congruences, i.e., formula for b_n, though these are given in several of the examples. Finding solutions to such recurrence is difficult in general. In [Z], integral solutions of certain recurrence relations are given, and in [AO], solutions for other cases are given. When we use both kinds of relation, (0.2) and (0.3) we can obtain congruences, such as those related to the Apéry numbers. It would be interesting to do this for all k, investigating the interplay between the two relationships, though in this paper, we only concentrate on one kind of congruence, Theorem 0.1. The examples given are generally for the case $k = 2$.

In applications to geometry, the condition $b_p \equiv \gamma_p \bmod p$ is easy to check and can be very useful. The following example of Theorem 0.1, is geometrically motivated, as explained below. Note, $\eta(\tau)$ is the Dedekind eta function, $\eta(\tau) = q^{1/24}\prod_{n\geq 1}(1-q^n)$, where $\tau \in \mathcal{H}$, the upper-half complex plane, and $q = e^{2\pi i\tau}$.

Theorem 0.2. *Let $\Gamma = \Gamma_0(12)$, and define t, f, L by*

$$t = \frac{\eta(3\tau)^4\eta(12\tau)^8\eta(2\tau)^{12}}{\eta(\tau)^4\eta(4\tau)^8\eta(6\tau)^{12}} \in M_0(\Gamma),$$

$$f = \frac{\eta(2\tau)^4\eta(6\tau)^4}{\eta(\tau)^2\eta(3\tau)^2} \in M_2(\Gamma),$$

and

$$L = \eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^2 \in M_4(\Gamma).$$

Write

$$f = \sum b_n t^n = x - 2x^2 + 15x^3 - 124x^4 + 1181x^5 - 12126x^6 + 131059x^7 ...$$

$$L(q) = \sum \gamma_n q^n = q - 2q^2 - 3q^3 + 4q^4 + 6q^5 + 6q^6 - 16q^7 ...$$

then for primes $p > 3$, and integers $m, r > 0$ $b_p \equiv \gamma_p \bmod p$, and moreover,

$$b_{mp^r} - \gamma_p b_{mp^{r-1}} + p^3 b_{mp^{r-2}} \equiv 0 \bmod p^r$$

Proof. We want to apply Theorem 0.1, which we can do by showing that

$$f(q)\frac{q\frac{dt}{dq}}{t} = L(\tau) + 8L(2\tau).$$

Since $\frac{q\frac{dt}{dq}}{t}$ is a holomorphic modular form, by Lemma 0.3 below, the right and left hand sides of the expression are both holomorphic modular forms for $\Gamma_0(12)$, and so by checking enough terms of the q-expansions we can show they are equal. We also need to know that $L(\tau)$ has an Euler product expansion, which is because it is a Hecke eigen form, since it is in the space of cusp forms for $\Gamma_0(6)$, and this space is one dimensional. □

Since in the examples in § 2 the equality

$$f(q)\frac{q\frac{dt}{dq}}{t} = \sum_{d|M} a_d L(q^d)$$

will be checked in exactly the same way as in the above proof, we should explicitly state a simple lemma:

Lemma 0.3. *If $t(\tau) = \prod_{n\geq 1}(1-q^n)^{c(n)}$ is a modular function (weight 0, not required to be holomorphic at cusps), where $c(n) \in \mathbb{Z}$ just depends on $n \bmod M$ for some $M \in \mathbb{Z}$, then $\frac{q\frac{dt}{dq}}{t}$ is a holomorphic modular form of weight 2.*

Proof. Since there are only a finite number of possible $c(n)$ we can simply differentiate and write out $\frac{q\frac{dt}{dq}}{t}$ as a linear combination of Eisenstein series of the form $\sum_{n\geq 1}\sum_{d|n}\chi(d)dq^n$, where χ is some character mod M, and it is known that these are holomorphic weight 2 modular forms. □

The geometrical interpretation of Theorem 0.2 is as follows. Let $\mathcal{Z}_{A_3}$ be a certain Calabi-Yau three fold, described in [V2]. There is a map

$$\phi : \mathcal{Z}_{A_3} \to \mathbb{P}^1$$

such that $\phi^{-1}(t)$ is a smooth K3 surface for general t. Then $f(t) = \sum b_n t^n$ is a solution of the Picard-Fuchs differential equation of this family, which in this case is an ordinary linear third order differential equation:

$$\mathcal{F} : (t^2-14t+1)(t+1)^2\Theta^3 + 3(t^2-6t+1)(t+1)(t-1)\Theta^2 +$$
$$(3t^4-10t^3-10t^2-10t+3)\Theta + (t+1)(t-1)^3,$$

where $\Theta = t\frac{d}{dt}$. In [V2] Lemma 3.2 it is shown that the monodromy group of this equation is $\Gamma_0(12)+12$ (a certain subgroup of the normalizer of $\Gamma_0(12)$ in $SL_2(\mathbb{R})$). When lifted to the universal cover, $\mathcal{H}$, where we take

$$t(\tau) = -\frac{\eta(12\tau)^4\eta(2\tau)^2}{\eta(6\tau)^2\eta(4\tau)^4},$$

the equation $\mathcal{F}$ has solution space

$$f(\tau)(\mathbb{C} \oplus \tau\mathbb{C} \oplus \tau^2\mathbb{C}),$$

where $f(\tau)$ is a weight 2 modular forms for $\Gamma(12)+12$ given by

$$f(\tau) = -\frac{(\eta(2\tau)\eta(6\tau))^4}{(\eta(\tau)\eta(3\tau))^2}.$$

By Theorem 3.14 [V2], the modular form $L(\tau)$ is (up to primes of bad reduction) the Mellin transform of the L-series of the middle cohomology of $\mathcal{Z}_{A_3}$.

In [V2], the proof that (up to primes of bad reduction) the L-series of $H^3(\mathcal{Z}_{A_3}, \mathbb{Q}_\ell)$ has Mellin transform $\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^2$, used counting points, and an application of Livné's result (Theorem 4.3 [L]).

However, there should be another way to prove this was the L-series, which would involve the following steps:

• Extend Stienstra's result [S], to show that for $\mathcal{Z}_{A_3}$, if an expansion of the solution of the Picard-Fuchs differential equation is given by $f(t) = \sum b_n t^n$, and the L-series of the middle cohomology is two dimensional with $\mathrm{Tr}\,\mathrm{Frob}_p = \gamma_p$, then

$$b_{mp^r} - \gamma_p b_{mp^{r-1}} + p^3 b_{mp^{r-2}} \equiv 0 \bmod p^r. \tag{0.4}$$

• Apply Theorem 0.1, which in this case, by Theorem 0.2, says that the modular form $\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^2 = \sum \tilde{\gamma}_n$ satisfies the congruence

$$b_{mp^r} - \tilde{\gamma}_p b_{mp^{r-1}} + p^3 b_{mp^{r-2}} \equiv 0 \bmod p^r. \tag{0.5}$$

where b_n are the coefficients of the solution of the Picard-Fuchs differential equation.

• Use a uniqueness result, Theorem 1.8: we would like to show that if we have two 2-dimensional Galois representations ρ_1, ρ_2, with $\mathrm{Tr}\,\rho_1(\mathrm{Frob}_p) = \gamma_p$ and $\mathrm{Tr}\,\rho_2(\mathrm{Frob}_p)\tilde{\gamma}_p$, then given that for p a prime of good reduction, γ_p satisfies (0.4) and $\tilde{\gamma}_p$ satisfies (0.5), then we can conclude that $\gamma_p = \tilde{\gamma}_p$ for all primes p with good reduction.

If we have results of this nature, then just finding the solution of the Picard-Fuchs equation would tell us what the L-series is. To actually use this method to obtain a proof that the L-series of $\mathcal{Z}_{A_3}$ has Mellin transform $\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^2$, we need to extend Stienstra's result so that it applies to $\mathcal{Z}_{A_3}$. In the example of § 2.3 a congruence relation was found experimentally, relating some modular forms. It was noticed that the L-series coincided with the L-series of certain rigid Calabi-Yau three-folds, and so this immediately allowed us to find the Picard-Fuchs equation of a certain fibration of those varieties, and the solution in terms of modular forms.

Notation: In the following, we always write $q = e^{2\pi i\tau}$, for $\tau \in \mathcal{H}$, the upper-half complex plane. When b_k, $k \in \mathbb{Z}$ are used to denote coefficients of some expansion, for $k \notin \mathbb{Z}$, set $b_k = 0$. For $k > 0$, by $M_k(\Gamma)$ we mean holomorphic modular forms on $\mathcal{H}$ for the congruence subgroup Γ. When we write $M_0(\Gamma)$, we mean functions on $\mathcal{H}$ invariant under the action of Γ, which may have poles. Generally, in this paper, such functions will have q-expansion of the form $\sum_{n\geq 1} a_n q^n$, with $a_n \in \mathbb{Z}$, and $a_1 = 1$.

0.1. Differential equations for modular forms, congruences, and recurrence relations. As well as the congruence of theorem 0.1, the coefficients b_n also satisfy a recurrence relation. This is because of the following result:

THEOREM 0.4. *Let Γ be some discrete subgroup of $SL_2(\mathbb{R})$ commensurable with $SL_2(\mathbb{Z})$. If f is a weight k modular form for Γ, and t is a weight zero modular form for Γ, then if we can locally write an expression $f = \sum b_n t^n$, near $t = 0$. Then there is a linear, order $k+1$ differential equation satisfied by $g(x) = \sum_{n\geq 0} b_n x^n$ of the form*

$$P_{k+1}(x)\frac{d^{k+1}g}{dx^{k+1}} + P_k(x)\frac{d^k g}{dx^k} + \cdots + P_0(x)g = 0,$$

where $P_i(x)$ are algebraic functions in x.

PROOF. This is proved in [St] §5, Theorem 5.1. □

One could also use the method of [P] §6.1, and view this differential equation is as coming from a certain integrable connection. The main point is to consider the action of Γ on $f(\tau)(1, \tau, \tau^2, \dots, \tau^k)$.

Generally, we will take t to generate the space of modular functions for Γ, so that the $P_i(x)$ can be written as polynomials in x.

In practice, given f and t, we can find an expansion $f = \sum b_n t^n$, and then, since we know this satisfies an order k differential equation, we can experimentally find the equation, if the degree of the polynomials are small enough. Suppose we "guess" that the polynomials are of degree m, then we look at the expansions in t of $t^i \frac{d^j f}{dt^j}$, for $0 \leq i \leq m$, $0 \leq j \leq k$, and try to solve a linear equation, to find some linear combination of these such that the coefficient of t^i vanishes for sufficiently many i. If this is not possible, then we have to try larger m. If it is possible, then we have some $a_{i,j}$, not all zero, such that

$$\sum a_{i,j} t^i \frac{d^j f}{dt^j}$$

vanishes to high order at $t = 0$, which means it must be identically zero, and this gives us the differential equation.

For convenience we state the well known result:

PROPOSITION 0.5. *Given $b_n \in k$, $n \in \mathbb{Z}$, for k some field of characteristic zero, and a formal power series $f = \sum b_n t^n$, then for $P_i(X) \in k[X]$, and $\Theta = t\frac{d}{dt}$, the following are equivalent:*
(A) The b_n satisfy a recurrence relation

$$\forall n,\ b_n P_0(n) + b_{n-1} P_1(n) + \cdots b_{n-m} P_m(n) = 0$$

(B) The formal power series f satisfies the differential equation:

$$[P_0(\Theta) + tP_1(\Theta + 1) + \cdots + t^m P_m(\Theta + m)]f = 0$$

PROOF. This follows from the fact that $\Theta t^n = nt^n$. □

The above results mean that whenever we write a weight k modular form f in terms of modular function t, as $f = \sum b_n t^n$, there will be a recurrence relation for the b_n of the form

$$b_n P_0(n) + b_{n-1} P_1(n) + \cdots b_{n-k} P_k(n) = 0$$

where $P_i(n)$ are polynomials in n. Sometimes such recurrence relations can be solved to give the b_n in terms of binomial type expressions, though this in general is difficult or impossible to do. See for example [Z] for a example where some solutions of certain order two differential equations are given in terms of modular forms, and coefficients are given explicitly by combinatorial expressions.

We are more likely to have a simple combinatorial expression for the coefficients b_n if there is some simple and explicit geometry involved. In particular, the following result is useful in examples

PROPOSITION 0.6. *If a family of varieties, parameterized by $t \in \mathbb{P}^1$ is given in $\mathbb{P}^n \times \mathbb{P}^1$ by $f(x_1, \dots x_n)t = x_1..x_n$, where f is a homogeneous polynomial of degree n, then the Picard-Fuchs differential equation for the family has a solution $\sum b_k t^k$, where b_k is the coefficient of $(x_1 \dots x_n)^k$ in $[f(x_i, ..., x_n)]^k$.*

Proof. This is the same as the proof in [SB] §10, but in a higher dimension. □

Putting together all the results gives interesting examples, similar to congruences for Apéry numbers. Some examples will be given in § 2.

0.1.1. *Example 1.* In the example of Theorem 0.2, by [V2] Lemma 3.2, the b_n can also be given as

$$b_n = \sum_{m=0}^{n-1} \sum_{p+q+r+s=m} (-1)^m \binom{n+m}{2m+1} \binom{m}{p,q,r,s}^2,$$

so Theorem 0.2 gives a congruence relation between coefficients of the modular form $\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^2 = \sum \gamma_n q^n$, and a combinatorial expression, ie, we have

$$\gamma_p \equiv \sum_{m=0}^{n-1} \sum_{p+q+r+s=m} (-1)^m \binom{n+m}{2m+1} \binom{m}{p,q,r,s}^2 \bmod p$$

(as well as the congruence to higher order).

0.1.2. *Example 2.* In [B] Beukers proved that for the Apéry numbers,

$$a_n = \sum \binom{n}{k}^2 \binom{n+k}{k}^2,$$

are related to the coefficients of the q-expansion of $L = \eta(2\tau)^4\eta(4\tau)^4$. The geometric interpretation of this example is given in [PS], where the recurrence relation for the Apéry numbers is shown to come from the Picard-Fuchs equation of a certain variety. In [V2] it is shown that the L-series of the variety (there called $\mathcal{X}_{A_1^3}$) has Mellin transform $\eta(2\tau)^4\eta(4\tau)^4$. The congruence relation for this example was proved in [B], and the proof of Theorem 0.1 in this paper uses the same method as in [B].

In terms of Theorem 0.1, the congruence of [B] is because if we define the modular forms t, f:

$$t = \frac{\eta(2\tau)^6\eta(12\tau)^6}{\eta(4\tau)^6\eta(6\tau)^6}, \quad f = \eta(2\tau)\eta(4\tau)\eta(6\tau)\eta(12\tau)$$

Then we have

$$f = \sum a_n t^{2n+1}, \text{ and } f(q)\frac{q\frac{dt}{dq}}{t} = L(\tau) - 9L(3\tau).$$

So we can apply theorem 0.1 to obtain the congruence between the coefficients of the L-series, and the Apéry numbers:

$$a_{\frac{p-1}{2}} \equiv \gamma_p \bmod p$$

where $\eta(2\tau)^4\eta(4\tau)^4 = \sum_{n\geq 1} \gamma_n q^n$. (Note, a change of variables has been made to obtain the f and t here from the functions given in [B], so that Theorem 0.1 can be applied directly.)

1. Proof of Theorem 0.2

The following is the main result, which shows how a congruence relation is equivalent to a relation for a certain differential form. We use $p^{v(n)}$ to denote the highest power of p dividing n.

PROPOSITION 1.1. *Given a sequence of integers $\{a_n\}$, and a power series $\Omega(x) = \sum a_n x^{n-1}dx$ Then the relation:*

$$a_n + c_p a_{n/p} + c_{p^2} a_{n/p^2} + \dots c_{p^m} a_{n/p^m} \equiv 0 \bmod p^{v(n)} \; n \in \mathbb{Z}$$

for some constants $c_p \in \mathbb{Z}$, is equivalent with the relation:

$$\Omega(x) + \frac{c_p}{p}\Omega(x^p) + \frac{c_{p^2}}{p^2}\Omega(x^{p^2}) + \dots + \frac{c_{p^m}}{p^m}\Omega(x^{p^m}) = d\theta(x)$$

for some $\theta(x) \in \mathbb{Z}_p[[x]]$.

PROOF. Since

$$\Omega(x^{p^m}) = \sum a_n x^{p^m(n-1)} dx^{p^m} = p^m \sum a_n x^{p^m n-1} dx = p^m \sum a_{n/p^m} x^{n-1} dx$$

the first expression is the the coefficient of x^n in the second expression, so since this is divisible by n p-adically, we can integrate in $Z_p[[x]]$ to obtain θ Conversely, if we can integrate, then the coefficient of x^n must be divisible by $p^{v(n)}$. $\square$

LEMMA 1.2. *If $L(\tau)$ is a weight $k+2$, level N modular form with an Euler product:*

$$\sum_{n\geq 1} \frac{\gamma_n}{n^s} = \prod_{p\not| N} \frac{1}{1-\gamma_p p^s + \varepsilon_p p^{k+1} p^{-s}} \prod_{p|N} E_p,$$

and with q-expansion

$$L(q) = \sum_{n\geq 1} \gamma_n q^n$$

and given some fixed integers a_d, $\tilde{\gamma}_n$ are defined by the q-expansion

$$\sum_{d|M} a_d L(q^d) = \sum_{n\geq 1} \tilde{\gamma}_n q^n,$$

then for prime $p \not| N$, and positive integers m, r, we have

$$\tilde{\gamma}_{mp^r} - \gamma_p \tilde{\gamma}_{mp^{r-1}} + \varepsilon p^{k+1} \tilde{\gamma}_{mp^{r-2}} \equiv 0 \bmod p^r$$

PROOF. Because of the Euler product expansion for L, we have

$$\sum \frac{\tilde{\gamma}_n}{n^s} = \sum \frac{\sum_{d|M} a_d \gamma_{n/d}}{n^s} = \sum_{d|M} \left[\frac{a_d}{d^s} \sum \frac{\gamma_n}{n^s}\right]$$

$$= \sum_{d|M} \frac{a_d}{d^s} \prod_{p\not| N} \frac{1}{1-\gamma_p p^{-s} + \varepsilon_p p^{k+1} p^{-2s}} \prod_{p|N} E_p$$

Hence for a prime $q \not| N$ we have

$$(1-\gamma_q q^{-s} + \varepsilon_q q^{k+1} q^{-2s}) \sum \frac{\tilde{\gamma}_n}{n^s} = \sum_{d|M} \frac{a_d}{d^s} \prod_{p\not| qN} \frac{1}{1-\gamma_p p^{-s} + \varepsilon_p p^{k+1} p^{-2s}} \prod_{p|N} E_p$$

The LHS of the above is

$$\sum \frac{\tilde{\gamma}_n}{n^s} - \sum \frac{\gamma_q \tilde{\gamma}_n}{(nq)^s} + \sum \frac{\varepsilon_q q^{k+1} \tilde{\gamma}_n}{(nq^2)^s}$$

But in the RHS, there is no Euler factor for the prime q. So, equating the coefficient of n^{-s}, for $n = mq^r$, with $(m,q) = 1$, and $r > 0$, we get

$$\tilde{\gamma}_{mq^r} - \gamma_q \tilde{\gamma}_{mq^{r-1}} + \varepsilon_q q^{k+1} \tilde{\gamma}_{mq^{r-2}} = 0$$

$\square$

LEMMA 1.3. *Let L be a level N, weight $k+2$ modular form with an Euler product expansion, and q-expansion $\sum \gamma_n q^n$, and for fixed integers a_d let $\tilde{\gamma}_n$ be such that $\sum_{d|M} a_d L(d\tau) = \sum_{n\geq 1} \tilde{\gamma}_n q^n$, as in Lemma 1.2. Let f be a weight k modular form satisfying*

$$f\frac{qdt/dq}{t} = \sum_{d|M} a_d L(d\tau).$$

Define the differential form $\tilde{\omega}(t)$ by

$$\tilde{\omega}(t) = \frac{f}{t}dt,$$

so, writing $\tilde{\omega}(q)$ to denote $\tilde{\omega}(t(q))$, we have $\tilde{\omega}(q) = \sum_{n\geq 1} \tilde{\gamma}_n q^{n-1}$. Then

$$\tilde{\omega}(q) - \frac{\gamma_p}{p}\tilde{\omega}(q^p) + \frac{\varepsilon_p p^{k+1}}{p^2}\tilde{\omega}(q^{p^2}) = d\phi(q)$$

where $\phi(q) \in \mathbb{Z}_p[[q]]$.

PROOF. This is just an application of Proposition 1.1 to the recurrence relation of Lemma 1.2. □

We would like to change the above $\tilde{\omega}(q^{p^m})$ to $\omega(t^{p^m})$, and then via Proposition 1.1 we will obtain the recurrence we want. The following lemmas show how ω and $\tilde{\omega}$ are related.

LEMMA 1.4. *If t has a q-expansion $t(q) = \sum_{k\geq 1} d_k q^k$, for some $d_n \in \mathbb{Z}$, with $d_1 = 1$, then when this is inverted to give $q(t)$ in terms of t, we have*

$$q(t)^{np^m} = q(t^{p^m})^n + npf_n(t)$$

for some $f(t) \in \mathbb{Z}_p[[t]]$. (Note, f depends on p, n, m.)

PROOF. When the series for t in terms of q is inverted, we have $q(t) = \sum_{k\geq 1} c_k t^k$ for some $c_n \in \mathbb{Z}$, with $c_1 = 1$. So

$$q(t)^p = \left(\sum c_k t^k\right)^p \equiv \sum c_k^p t^{kp} \equiv \sum c_k t^{kp} = q(t^p) \bmod p$$

This is because for $r > 0, p|\binom{p}{r}$, and $c_k^p \equiv c_k \bmod p$.

Hence

$$q(t)^p = q(t^p) + pf_1(t) \tag{1.1}$$

for some for some $f_1(t) \in \mathbb{Z}_p[[t]]$.

For $m > 1$, we obtain the result by induction. Suppose we have

$$q(t)^{p^{m-1}} = q(t^{p^{m-1}}) + pf(t),$$

Then we have

$$q(t)^{p^m} = (q(t^{p^{m-1}}) + pf(t))^p = q(t^{p^{m-1}})^p + pf(t).$$

Since t has a q-expansion with coefficient of q^0 equal to 1, then this is also the case for t^{m-1}, and so, applying equation 1.1 with t^{m-1} instead of t, we have

$$q(t^{p^{m-1}})^p = q(t^{p^m}) + pf(t).$$

So, by substitution, we have the result of the lemma in the case $n = 1$:

$$q(t)^{p^m} = q(t^{p^m}) + pf(t).$$

Taking the n^{th} power of both sides of this, we get

$$q(t)^{p^m n} = (q(t^{p^m}) + pf(t))^n = q(t^{p^m})^n + pnf_n(t)$$

for some $f_n(t) \in \mathbb{Z}_p[[t]]$. If $p \not| n$ then this is just because $\frac{1}{n} \in \mathbb{Z}_p$. If $p|n$, then we have

$$q(t)^{p^m n} = q(t^{p^m})^n + \sum_{k=1}^{n} \binom{n}{k} p^k f^{n-k}(t)$$

and then we need to know that for $k \geq 1$, $p^k \binom{n}{k}$ is divisible by pn in $\mathbb{Z}_p$, and this is because $p^k \binom{n}{k} = n\frac{p^k}{k}\binom{n-1}{k-1}$, and we always have $k = v(p^k) > v(k)$ for $k > 0$. □

Lemma 1.5. *With $\omega(t)$ and $\tilde{\omega}(q)$ as in Lemma 1.3,*

$$\tilde{\omega}(q(t)^{p^m}) = \tilde{\omega}(q(t^{p^m})) + pd\phi(t)$$

Proof. By Lemma 1.4

$$\tilde{\omega}(q^{p^m}) = d\left(\sum \frac{\tilde{\gamma}_n}{n} q^{np^m}\right)$$

$$= d\left(\sum \left[\frac{\tilde{\gamma}_n}{n}(q(t^{p^m}))^n + pn\phi(t)\right]\right)$$

$$= \omega(t^{p^m}) + pd\phi(t)$$

□

Lemma 1.6. *With $\omega(t)$ and γ_p as in Lemma 1.3,*

$$\omega(t) - \frac{\gamma_p}{p}\omega(t^p) + \frac{p^3}{p^2}\omega(t^{p^2}) = d\psi(t)$$

where $\psi(t) \in \mathbb{Z}_p[[t]]$.

Proof. We convert the statement about $\tilde{\omega}(t)$ in Lemma 1.3 to the above statement about $\omega(t)$, by applying Lemma 1.5. □

Corollary 1.7. *Let L be a weight $k+2$ modular form of level N, with an Euler product expansion, as before. Given a weight k modular form f satisfying $f\frac{qdt/dq}{t} = \sum_{d|M} a_d L(d\tau)$, for some integers a_d, and a modular function t, define b_n to be the coefficients of the local expansion*

$$f = \sum_{n\geq 0} b_n t^n.$$

Then for primes $p \not| N$,

$$b_{mp^r} - \gamma_p b_{mp^{r-1}} + \varepsilon_p p^{k+1} b_{mp^{r-2}} \equiv 0 \bmod p^r$$

In particular, if $b_1 = 1$, then

$$b_p \equiv \gamma_p \bmod p$$

Proof. Note that we have a t-expansion

$$\omega(t) = \frac{f}{t} dt \sum b_n t^{n-1}$$

Then the result is obtained by application of Proposition 1.1 to the relation for $\omega(t)$ obtained in Lemma 1.6. When $b_1 = 1$, then applying this congruence with $m = r = 1$ gives $b_p \equiv \gamma_p \bmod p$. □

1.1. Uniqueness. For geometrical applications, to deduce the L-series from a solution of the Picard-Fuchs equation, we will also need to use a uniqueness result.

Theorem 1.8. *Given two 2-dimensional Galois representations ρ_1, ρ_2, with $\mathrm{Tr}\,\rho_1(\mathrm{Frob}_p) = \gamma_p$ and $\mathrm{Tr}\,\rho_2(\mathrm{Frob}_p) = \tilde{\gamma}_p$, let N be an integer divisible by all the primes of bad reduction of ρ_1 and ρ_2. Assume that there are infinitely many primes p with $\gamma_p \not\equiv 0 \bmod p$. If there is a sequence of integers b_n with $b_1 = 1$, then if*

$$b_{mp^r} - \gamma_p b_{mp^{r-1}} + p^3 b_{mp^{r-2}} \equiv 0 \bmod p^r. \tag{1.2}$$

and

$$b_{mp^r} - \tilde{\gamma}_p b_{mp^{r-1}} + p^3 b_{mp^{r-2}} \equiv 0 \bmod p^r. \tag{1.3}$$

then

$$\gamma_p = \tilde{\gamma}_p \quad \forall p \not| N$$

Proof. Take some $p \not| N$. Suppose $\gamma_p \not\equiv 0 \bmod p$. We have

$$b_{p^r} \equiv \gamma_p b_{p^{r-1}} \bmod p$$

So, since $b_1 = 1$, $b_{p^r} \equiv (\gamma_p)^r \bmod p$ for all integers $r \geq 1$. Subtracting (1.2) from (1.3), and setting $m = 1$, we have

$$b_{p^{r-1}}(\gamma_p - \tilde{\gamma}_p) \equiv 0 \bmod p^r$$

Since $b_{p^{r-1}} \equiv (\gamma_p)^r \not\equiv 0 \bmod p$, we have $\gamma_p \equiv \tilde{\gamma}_p \bmod p^r$ for all $r > 1$, which means that $\gamma_p = \gamma_p$. A similar argument works if $\tilde{\gamma}_p \not\equiv 0 \bmod p$. Given the assumption that for infinitely many primes p we have that $\gamma_p \not\equiv 0 \bmod p$, then we have enough $\gamma_p = \tilde{\gamma}_p$ for this equality to hold for all p not of bad reduction. □

The assumption that for infinitely many primes $\gamma_p \not\equiv 0 \bmod p$ will in practice not be necessary, since we can find sufficiently many γ_p which are not $0 \bmod p$, and then use Theorem 4.3 [L] to conclude that since for these p, γ_p and $\tilde{\gamma}_p$ are equal, then γ_p and $\tilde{\gamma}_p$ are equal, for all primes p of good reduction. However, we will will make the following:

Conjecture 1.9. If f is a non zero, weight 4 Hecke eigen form for some congruence subgroup Γ, and $f(q) = \sum_{n\geq 1} a_n q^n$, with $a_n \in \mathbb{Z}$, then for primes p, we have $a_p \not\equiv 0 \bmod p$ for infinitely many p.

One might conjecture further, that we only have $a_p \equiv 0 \bmod p$ but $a_p \neq 0$ for finitely many p, and the p for which this happens either divide the level, or else are primes for which the image of the corresponding representation is small. This result is probably also true for weight greater than 4.

2. Examples

To look for explicit examples, we simply need to find modular forms t, f, L of weights $0, k, k+2$ for a congruence subgroup Γ, and some integers a_d satisfying

$$f(q)\frac{q\frac{dt}{dq}}{t} = \sum_{d|M} a_d L(d\tau)$$

In the simplest case, we can just look for examples where the q-expansions satisfy

$$f(q)\frac{q\frac{dt}{dq}}{t} = L(\tau)$$

To simplify matters further, we will restrict attention to modular forms for groups $\Gamma_0(N)$, possibly with a character, and ask that the weight 2 modular form $\frac{q\frac{dt}{dq}}{t}$ should be holomorphic, which means that t should be given by some product, for example, something of the form $\prod_{i=0}^{n} \eta(a_i\tau)^{m_i}$, for some integers a_i, m_i. In this case, $\frac{q\frac{dt}{dq}}{t}$ will in the space of weight 2 modular forms for Γ spanned by Eisenstein series. So, all we need to do is check if L is divisible by any such expression. For example, if $k = 2$, and we are looking for a weight 2 modular form to divide a weight 4 Hecke eigen form, then if the genus of Γ is 0, all weight 2 modular forms are given by Eisenstein series, and so any factorization of L into two weight 2 pieces should give an example. Many explicit examples of modular forms given in terms of eta products and eta quotients can be found in [CN] and [M].

Notation. In the following for a congruence subgroup of level N, and $d|N$, we use h_d to denote the Eisenstein series

$$h_d(\tau) = g_2(d\tau) - \frac{N}{d} g_2(N\tau)$$

where $g_2(\tau)$ is given by

$$g_2(\tau) = -1 + 24\sum \sigma_1(n)q^n = -1 + 24q + 72q^2 + 96q^3 + \dots$$

We will use the fact that if $t = \prod_{d|N} \eta(d\tau)^{m_d}$, then $\frac{qdt/dq}{t} = -\frac{1}{24}\sum_{d|N, d\neq N} dm_d h_d$.

2.1. example for $\Gamma_0(5)$.

Let

$$t = \frac{\eta(5\tau)^6}{\eta(\tau)^6}, \quad f = \frac{\eta(5\tau)^5}{\eta(\tau)}, \quad L = \sum_{n=1}^{\infty}\left(\sum_{d|n} \chi\left(\frac{n}{d}\right) d^3\right) q^n,$$

where $\chi(m) = \left(\frac{m}{5}\right)$, the Kronecker symbol. Then, computing coefficients of the q-expansions, we show that

$$f\frac{qdt/fq}{t} = L$$

So if we write $f = \sum b_n t^n$, then we have a congruence relation between the coefficients of L and the b_n:

$$b_{mp^r} - \left(p^3 + \left(\frac{p}{5}\right)\right) b_{mp^{r-1}} + \left(\frac{p}{5}\right) p^3 b_{mp^{r-2}} \equiv 0 \bmod p^r, \;\; m, r \in \mathbb{Z}_{>0}$$

The result holds since there is an Euler product

$$\sum_{n=1}^{\infty} \frac{\sum_{d|n}\left(\frac{n/d}{5}\right) d^3}{n^s} = \prod_p \left(1 - \left(p^3 + \left(\frac{p}{5}\right)\right) p^{-s} + p^3\left(\frac{p}{5}\right) p^{-2s}\right)^{-1},$$

so Theorem 0.1 applys.

The b_n also satisfy a recurrence:

$$(n-1)^3 b_n + (2n-3)(11n^2 - 33n + 27)b_{n-1} + 125(n-2)^3 b_{n-2} = 0$$

Although this recurrence is not simple to solve directly, expressions can be obtained relating the b_n to the Apéry numbers, since if we take variables r, g with $\frac{1}{r} - 11 - r = \frac{1}{t}$, and $rg^2 = f$, then $g = \sum a_n r^n$, where $a_n = \sum \binom{n}{k}^2 \binom{n+k}{k}$ are the Apéry numbers. This relationship is obtained from the explicit expressions in [Z]. See also [B2] for details of the relationship between the Apéry numbers and $\Gamma_0(5)$.

2.2. Example for $\Gamma_0(6)$. The group $M_2(\Gamma_0(6))$ is spanned by h_1, h_2, h_3, and $S_4(\Gamma_0(6))$ is spanned by $f = \eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^2$. There are exactly three ways of writing f as a product of weight 2 modular forms. This is because we can factor f as

$$f = \frac{1.6^6}{2^2 3^3} \cdot \frac{3^6 2}{1^2 6^3} \cdot \frac{2^6 3}{1^3 6^2} \cdot \frac{1^6 6}{2^3 3^2},$$

where a^b means $\eta(a\tau)^b$. This notation is used throughout this section.

Each of the factors is a weight 1 modular form for $\Gamma_0(6)$, with an order 2 character. So, the product of any two is a modular form of weight 2 for $\Gamma_0(6)$. Since $\Gamma_0(6)$ has genus 0, it is spanned by Eisenstein series, and so each of these products can also be expressed in terms of Eisenstein series, and then logarithmically integrated to get a Hauptmodul for $\Gamma_0(6)$.

We will label these weight one modular forms a, b, c, d as follows:

$$\begin{array}{rclcrclc} a &=& \frac{1.6^6}{2^2 3^3} & & ; & b &=& \frac{3^6 2}{1^2 6^3} & = w_2 a \\ c &=& \frac{2^6 3}{1^3 6^2} & = w_3 a & ; & d &=& \frac{1^6 6}{2^3 3^2} & = w_6 a \end{array}$$

Here, w_n are the Atkin-Lehner involutions.

The products can be written as Eisenstein series as follows:

$$\begin{array}{rclclclcl} ab &=& \frac{3^3 6^3}{1.2} &=& \frac{h_1 - 2h_2 - h_3}{24} &=& -\frac{1}{3}[\frac{1^3 6}{3.2^3}]' &=& q + q^2 + 3q^3 + q^4 + \dots \\ cd &=& \frac{1^3 2^3}{3.6} &=& \frac{-h_1 + 2h_2 + 9h_3}{8} &=& [\frac{1^3 6^9}{2^3 3^9}]' &=& 1 - 3q - 3q^2 + 15q^3 - 3q^4 + \dots \\ ac &=& \frac{6^4 2^4}{1^2 3^2} &=& \frac{h_1 - h_2 - 3h_3}{24} &=& -\frac{1}{2}[\frac{1^2 6}{3^2 2}]' &=& q + 2q^2 + q^3 + 4q^4 + \dots \\ bd &=& \frac{1^4 3^4}{2^2 6^2} &=& \frac{-h_1 + 4h_2 + 3h_3}{6} &=& [\frac{1^4 6^8}{3^4 2^8}]' &=& 1 - 4q + 4q^2 - 4q^3 + 20q^4 + \dots \\ ad &=& \frac{6^7 1^7}{2^5 3^5} &=& \frac{h_1 - 10h_2 + 15h_3}{24} &=& [\frac{2^5 6}{1.3^5}]' &=& q - 7q^2 + 19q^3 - 23q^4 + \dots \\ bc &=& \frac{2^7 3^7}{6^5 1^5} &=& \frac{5h_1 - 2h_2 + 3h_3}{24} &=& [\frac{2.6^5}{1^5 3}]' &=& 1 + 5q + 13q^2 + 23q^3 + 29q^4 + \dots \end{array}$$

Here, $[t]'$ means $\frac{q\,dt/dq}{t}$.

From the above, we see we can write f as follows:

$$\begin{array}{rcr} (h_1 - 4h_2 - 3h_3)(h_1 - h_2 - 3h_3) &=& -144f \\ (h_1 - 2h_2 - 9h_3)(h_1 - 2h_2 - h_3) &=& -192f \\ (h_1 - 10h_2 + 15h_3)(5h_1 - 2h_2 + 3h_3) &=& 576f \end{array}$$

$t = \frac{1^3 6^9}{2^3 3^9}$	$g = ab = \frac{3^3 6^3}{1.2} = \frac{h_1 - 2h_2 - h_3}{24}$	$c_n = \sum_{k+l=n-1} \sum_{i=0}^{k} \sum_{j=0}^{l} \binom{k}{i}^3 \binom{l}{j}^3$
$t = q - 3q^2 + 3q^3 + 5q^4 - 18q^5 + 15q^6 + 24q^7 - 75q^8 + \dots$ $g = t + 4t^2 + 24t^3 + 152t^4 + 1016t^5 + 7008t^6 + 49432t^7 + \dots = \sum c_n t^n$ $(8t-1)^2(t+1)^2 g''' + (8t^2+1)(8t-1)(t+1)g'' +$ $(-64t^4 - 40t^3 + 3t^2 + 5t - 1)g' + (8t^2+1)(8t^2+4t-1)g = 0$		
$t = \frac{1^4 6^8}{3^4 2^8}$	$g = ac = \frac{6^4 2^4}{1^2 3^2} = \frac{h_1 - h_2 - 3h_3}{24}$	$c_n = \sum_{\substack{n_1+n_1 \\ =n-1}} \sum_{k_1=0}^{n_1} \binom{n_1}{k_1}^2 \binom{2k_1}{k_1} \sum_{k_2=0}^{n_2} \binom{n_2}{k_2}^2 \binom{2k_2}{k_2}$
$t = q - 4q^2 + 10q^3 - 20q^4 + 39q^5 - 76q^6 + 140q^7 - 244q^8 + \dots$ $g = -8t - 48t^2 - 312t^3 - 2208t^4 - 16488t^5 - 127440t^6 + \dots = \sum c_n t^n$ $(t-1)^2(9t-1)^2 g''' + 3(3t-1)(3t+1)(t-1)(9t-1)g'' +$ $(27t^2 + 2t + 3)(9t^2 - 8t + 11)g' + (3t+1)(-3t+1)^3 g = 0$		
$t = \frac{2.6^5}{1^5 3}$	$g = ad = \frac{6^7 1^7}{2^5 3^5} = \frac{h_1 - 10h_2 + 15h_3}{24} = \frac{2^5 6}{1.3^5}$	$c_n = \sum_{n_1+n_2=n}$ $\left[\sum_{k_1=0}^{n_1} \binom{n_1}{k_2} (-8)^{k_1} \sum_{j_1=0}^{n_1-k_1} \binom{n_1 - k_1}{j_1}^3 \sum_{k_2=0}^{n_2} \binom{n_2}{k_2} (-8)^{k_2} \sum_{j_2=0}^{n_2-k_2} \binom{n_2 - k_2}{j_2}^3\right]$
$t = q + 5q^2 + 19q^3 + 61q^4 + 174q^5 + 455q^6 + 1112q^7 + 2573q^8 + \dots$ $g = t - 12t^2 + 120t^3 - 1128t^4 + 10296t^5 - 92448t^6 + \dots = \sum c_n t^n$ $(8t-1)^2(t+1)^2 g''' + (8t^2+1)(8t-1)(t+1)g'' +$ $(-64t^4 - 40t^3 + 3t^2 + 5t - 1)g' + (8t^2+1)(8t^2+4t-1)g = 0$		

TABLE 1. Examples of order three differential equations for weight 2 modular forms g for $\Gamma_0(6)$, in terms of modular functions t. In each case, the differential g' means $t\frac{dg}{dt}$.

Each of these factorizations gives rise to a differential equation, as shown in Table 1. In Table 1, explicit descriptions of the coefficients of the solutions of the differential equations are given. These are found by finding differential equations for the weight one modular forms, as in Table 2. Content-Length: 20670 X-Lines: 519 Status: RO

In each of the three cases, in Table 1, we have a third order differential equation, $\mathcal{F}$. These are obtained from symmetric squares of differential equations for the second order differential equations $\mathcal{D}$ in Table 2. These second order equations appear in [Z], where the explicit expressions for the coefficients of the solutions are given. Geometrically, this corresponds to having a family of elliptic curves with Picard-Fuchs equation $\mathcal{D}$, and obtaining $\mathcal{F}$ as the Picard-Fuchs equation of the fibre product of the family of elliptic curves with itself. For the first two examples,

$t = \frac{1^3 6^9}{2^3 3^9}$	$c_n = \sum_{k=0}^n \binom{n}{k}^3$
$t = \quad q - 3q^2 + 3q^3 + 5q^4 - 18q^5 + 15q^6 + 24q^7 - 75q^8 + \dots$ $a = t + 2t^2 + 10t^3 + 56t^4 + 346t^5 + 2252t^6 + \dots$ $b = 1 + 2t + 10t^2 + 56t^3 + 346t^4 + 2252t^5 + \dots \quad = \sum c_n t^n$ $(8t-1)(t+1)b'' + t(16t+7)b' + 2t(4t+1)b = 0$ $8(n-1)^2 c_{n-2} + (7n^2 - 7n + 2)c_{n-1} - n^2 c_n = 0$	
$t = \frac{1^4 6^8}{3^4 2^8}$	$c_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k}$
$t = \quad q - 4q^2 + 10q^3 - 20q^4 + 39q^5 - 76q^6 + 140q^7 - 244q^8 + \dots$ $a = t + 3t^2 + 15t^3 + 93t^4 + 639t^5 + 4653t^6 + \dots$ $c = 1 + 3t + 15t^2 + 93t^3 + 639t^4 + 4653t^5 + \dots \quad = \sum c_n t^n$ $(t-1)(9t-1)c'' + 2t(9t-5)c' + 3t(3t-1)c = 0$ $9(n-1)^2 c_{n-2} - (10n^2 - 10n + 3)c_{n-1} + n^2 c_n = 0$	
$t = \frac{2.6^5}{1^5 3}$	$c_n = \sum_{k=0}^n \binom{n}{k}(-8)^k \sum_{j=0}^{n-k} \binom{n-k}{j}^3$
$t = \quad q + 5q^2 + 19q^3 + 61q^4 + 174q^5 + 455q^6 + 1112q^7 + 2573q^8 + \dots$ $a = t - 6t^2 + 42t^3 - 312t^4 + 2394t^5 - \dots$ $d = 1 - 6t + 42t^2 - 312t^3 + 2394t^4 + \dots \quad = \sum c_n t^n$ $(8t+1)(9t+1)d'' + t(144t+17)d' + 6t(12t+1)d = 0$ $72(n-1)^2 c_{n-2} + (17n^2 - 17n + 6)c_{n-1} + n^2 c_n = 0$	

Table 2. Examples of order 2 differential equations for weight 1 modular forms a, b, c, d, in terms of modular functions t, for $\Gamma_0(6)$, together with the corresponding recurrence relations. In each case x' means $x\frac{dx}{dt}$. Recall, the eta products a, b, c, d are given by $\frac{1.6^6}{2^2 3^3}, \frac{3^6 2}{1^2 6^3}, \frac{2^6 3}{1^3 6^2}, \frac{1^6 6}{2^3 3^2}$ respectively.

corresponding families of elliptic curves are given in [SB] as examples $\mathcal{B}$ and $\mathcal{C}$. For the third example, we can easily construct a family of elliptic curves with this Picard-Fuchs equation. Since this is a differential equation with monodromy group $\Gamma_0(6)$, and four singular points, we expect that by [Bv], any corresponding family of elliptic curves should be of the form

$$(x+y)(y+z)(z+x) - \alpha xyz = \lambda xyz,$$

where $\lambda = \frac{1}{t}$ is the parameter of the family, and α is fixed. (Luckily it turns out that we don't need a more complicated change of variables than just altering α to get the different families here.) We find that the first two cases correspond to $\alpha = 0$

and $\alpha = -1$, and the third to $\alpha = 8$. So, the coefficient c_n in the third example is the coefficient of $(xyz)^n$ in

$$(x^2y + y^2x + y^2z + z^2y + z^2x + x^2z - 6xyz)^n.$$

We can write these coefficients as

$$c_n = \sum_{k=0}^{n} \binom{n}{k} (-8)^k \sum_{j=0}^{n-k} \binom{n-k}{j}^3$$

Now we have the corollary:

COROLLARY 2.1. *If* $f = (\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau))^6 = \sum \gamma_n q^n$, *then (as well as congruences to higher powers of* p*),*

$$\begin{aligned} \gamma_p &\equiv \sum_{\substack{n_1+n_1\\=n-1}} \sum_{k_1=0}^{n_1} \binom{n_1}{k_1}^2 \binom{2k_1}{k_1} \sum_{k_2=0}^{n_2} \binom{n_2}{k_2}^2 \binom{2k_2}{k_2} \bmod p \\ &\equiv \textstyle\sum_{k+l=p-1} \sum_{i=0}^{k} \sum_{j=0}^{l} \binom{k}{i}^3 \binom{l}{j}^3 \bmod p \\ &\equiv \textstyle\sum_{r+s+t+u+v+w=n} \frac{(-8)^{n-t-w} n!}{r!s!t!u!v!w!} \binom{r+s}{r}^2 \binom{u+v}{u}^2 \binom{n}{r+s+t}^{-1} \bmod p \end{aligned}$$

PROOF. This follows from the information given in Table 1, and Theorem 0.1 □

In each of the three cases, we just have a different parameterization for the same pencil of elliptic curves, and so the fibres products in all three cases are isomorphic, and all have L-series $\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^2$, as shown in [V3]. So, these examples are all expressions of the congruence relating L-series and Picard-Fuchs equations in [S].

2.3. Example for $\Gamma_0(9)$.

THEOREM 2.2. *Define modular forms* t, f, L *for* $\Gamma_0(9)$ *by*

$$t = \frac{\eta(9\tau)^3}{\eta(\tau)^3}, \quad f = \frac{\eta(9\tau)^3\eta(\tau)^3}{\eta(3\tau)^2}, \quad L = \eta(3\tau)^8.$$

Define b_n, γ_n *by*

$$L(q) = \sum \gamma_n q^n = q - 8q^4 + 20q^7 - 70q^{13} + ...$$

$$f(t) = \sum b_n t^n = t - 6t^2 + 27t^3 - 96t^4 + 225t^5 + 162t^6 - 6021t^7 + 45360t^8 + ...$$

Then

$$b_{mp^r} - \gamma_p b_{mp^{r-1}} + p^3 b_{mp^{r-2}} \equiv 0 \bmod p^r$$

PROOF. Note that $\eta(3\tau)^8$ is a Hecke eigen form for $\Gamma_0(8)$. Comparing enough terms of the q-expansions of the modular forms $\eta(3\tau)^8$ and $f\frac{qdt/dq}{t}$, we see that they are equal. Hence we can apply Theorem 0.1 to obtain the result. □

In [V3] it is shown that $\eta(3\tau)^8$ is the L-series of two Calabi-Yau three folds, birational to

$$rst(x^3 + y^3 + z^3) = (r^3 + s^3 + t^3)xyz$$

and

$$rst(x^2y + y^2z + z^2x) = (r^2s + s^2t + t^2r)xyz$$

where these are the equations in $\mathbb{P}^2 \times \mathbb{P}^2$, with homogeneous coordinates x, y, z and r, s, t. This leads us to expect that the Picard-Fuchs equation in these cases might in fact be the equation for f in terms of t:

$$[(27t^2+9t+1)^2\Theta^3 + (27t^2+9t+1)(27t^2-1)\Theta^2 + (729t^4+189t^3+27t^2+7t+11)\Theta \\ +(27t^2-1)(27t^2+6t+1)]f = 0$$

To show this is the case, we let $f_1 = \sqrt{\frac{f}{t}} = \frac{\eta(\tau)^3}{\eta(3\tau)} = 1 - 3t + 9t^2 - 21t^3 + 9t^4 + 297t^5 + \cdots$. Then we have a differential equation for f_1 given by:

$$[(27t^2+9t+1)\Theta^2 + (6t+1)t\Theta + (9x+1)x]f_1 = 0$$

Note that the series for f_1 is just a twist of the solution of a certain differential equation in [Z] (case B). (I.e., one series is obtained from the other by a change of variables $t \mapsto -t$). In [Z], the coefficients c_n, where $f_1 = \sum c_n t^n$, are given by:

$$c_n = \sum_{k=0}^{[n/3]} (-1)^k 3^{n-3k} \binom{n}{3k}\binom{3k}{k}\binom{2k}{k}$$

Bearing in mind the description of coefficients of the Picard-Fuchs differential equation given in Proposition 0.6, we rewrite this as

$$c_n = \sum_{k=0}^{[n/3]} (-1)^k 3^{n-3k} \frac{n!}{k!k!k!(n-3k)!}.$$

Then immediately we see that up to sign, this is the coefficient of $x^n y^n z^n$ in

$$(x^3 + y^3 + z^3 - 3xyz)^n,$$

and in fact, up to sign, it is also the coefficient of $x^n y^n z^n$ in $(x^2y+y^2z+z^2x-3xyz)^n$. Hence by Theorem 0.6, the two families of elliptic curves

$$(x^2y + y^2z + z^2x - 3xyz)\lambda = xyz,$$
$$(x^3 + y^3 + z^3 - 3xyz)\lambda = xyz,$$

have monodromy group $\Gamma_0(9)$, and solution of the Picard-Fuchs differential equation given in terms of the above modular forms. Note, the fact that t is the reciprocal of the Hauptmodul for $\Gamma_0(9)$ was read off from Table 3 in [CN]. Note, we already know from [Bv] that these families of elliptic curves can be given as modular families of elliptic curves for $\Gamma(3)$ and $\Gamma_0(9) \cap \Gamma_0^0(3)$.

Up to a change of parameter, these families are exactly the families of elliptic curves the Calabi-Yau three folds with L-series $\eta(3\tau)^8$ in [V3] are obtained from. In [Sch] it is shown that these fibre products of elliptic curves are rigid Calabi-Yau three-folds. In [P] §5 it is shown how to obtain the Picard-Fuchs equation for the the fibre product of the family of elliptic curves from the Picard-Fuchs equation of the family of elliptic curves, and the result in this case gives the above modular forms.

We end up with an example of the kind of congruence relation between the L-series and Picard-Fuchs equation of a family of K3 surfaces, as predicted by Stienstra's results. In this case the fibrations are given in $\mathbb{P}^2 \times \mathbb{P}^2$ by

$$rst(x^3 + y^3 + z^3 - 3xyz) = (r^3 + s^3 + t^3 - 3rst)xyz = \lambda$$

and

$$rst(x^2y + y^2z + z^2x - 3xyz) = (r^2s + s^2t + t^2r - 3rst)xyz = \lambda$$

where $\lambda \in \mathbb{P}^1$ is the parameter of the family.

The b_n can be expressed in terms of the c_m; however, these are not such pretty expressions. However, we can show that

$$f_1 \frac{q dt/dq}{t} = \frac{\eta(3\tau)^9}{\eta(\tau)^3},$$

where $\frac{\eta(3\tau)^9}{\eta(\tau)^3}$ is a weight 3 Hecke eigen form by [M], and so has an Euler product expression, though it is not a cusp form. So by Theorem 0.1, if

$$\frac{\eta(3\tau)^9}{\eta(\tau)^3} = q + 3q^2 + 9q^3 + 13q^4 + 24q^5 + \cdots = \sum \gamma_n q^n,$$

then we have

$$\sum_{k=0}^{[p-1/3]} (-1)^k 3^{p-1-3k} \binom{p-1}{3k} \binom{3k}{k} \binom{2k}{k} \equiv \gamma_p \mod p$$

2.4. Example for $\Gamma_0(14)$. Let t and f be defined by

$$t = \frac{\eta(2\tau)^3 \eta(14\tau)^3}{\eta(\tau)^3 \eta(7\tau)^3} \quad f = \eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau).$$

Then $f \frac{q dt/dq}{t}$ is a weight 4 level 14 Hecke eigen form. Note that $f = t f_1^2$, where $f_1 = \frac{\eta(\tau)^2 \eta(7\tau)^2}{\eta(2\tau)\eta(14\tau)}$. We have a differential equation satisfied by f_1:

$$(8t^2 + 5t + 1)(t + 1)(8t + 1)\Theta^2 f_1 + (128t^3 + 168t^2 + 61t + 7)t\Theta f_1 +$$

$$2(8t^2 + 7t + 1)(4t + 1)t f_1 = 0.$$

Here Θ means $t\frac{d}{dt}$. It would be interesting to know if this is the Picard-Fuchs equation for some family of elliptic curves.

2.5. Example for $\Gamma_0(22)$. Let $f_1, f_2 \in S_2(\Gamma_0(22)$ be given by

$$f_1 = \eta(11\tau)^2 \eta(\tau)^2, \ \ f_2 = \eta(22\tau)^2 \eta(2\tau)^2.$$

Then the products $(f_1 - 2f_2)(h_1 - 2h_2 + 11h_{11})$, and $(f_1 + 2f_2)(h_1 - 2h_2 + 11h_{11})$, are both weight 4, level 22 Hecke eigen forms.

The form $h_1 - 2h_2 + 11h_{11}$, on logarithmically integrating, gives $t = \left(\frac{\eta(2\tau)\eta(22\tau)}{\eta(\tau)\eta(11\tau)}\right)^2$. By Table 3 in [CN], this is the reciprocal of the Hauptmodul for $\Gamma_0(22) + 11$.

In both cases, we can write $f_1 \pm 2f_2 = \sum b_n t^n$ for some integers b_n. The coefficients b_n satisfy a recurrence with 10 terms, the coefficients being polynomials of degree 3. This corresponds to the differential equations being of order 3, with coefficients polynomials of degree 10.

There is no reason to expect this example will be related to geometry. It would be nice to have a result to say definitely whether or not it is, and to answer the same question for any other example constructed like these.

2.6. An example with $k = 0$. The congruences we have looked at so far are for $k > 0$. For $k = 0$, we don't have quite the same geometrical situation, since we can't have interesting fibres of dimension 0. However, we can still look for congruences. For example, take modular forms for $\Gamma_0(12)$,

$$L = \eta(2\tau)\eta(4\tau)\eta(6\tau)\eta(12\tau) = \sum_{n\geq 1}\gamma_n q^n, \quad t = \frac{\eta(2\tau)^2\eta(12\tau)^4}{\eta(4\tau)^4\eta(6\tau)^2}$$

Then we can try and write $g := L/[\frac{qdt/dq}{t}]$ in terms of t. We obtain

$$g = \frac{t}{\sqrt{(t^2-1)(9t^2-1)}} = \sum_{N\geq 0}\sum_{n+m=N}\binom{2n}{n}\binom{2m}{m}\frac{9^m}{2^{2N}}t^{2N+1}$$

$$= \sum_{n\geq 0} b_n t^{n+1} = t + 5t^3 + 33t^5 + 245t^7 + 1921t^9 + \dots$$

So, with

$$b_{2N+1} = \sum_{n+m=N}\binom{2n}{n}\binom{2m}{m}\frac{9^m}{2^{2N}}$$

and γ_n given by the coefficients of the q-expansion of $\eta(2\tau)\eta(4\tau)\eta(6\tau)\eta(12\tau)$, we have for primes $p \neq 2, 3$, and integers $m, r > 0$,

$$b_{mp^r} - \gamma_p b_{mp^{r-1}} + p b_{mp^{r-2}} \equiv 0 \bmod p^r.$$

3. An anomaly

In the above examples, we have been trying to find congruence relations between modular forms for the same group. However in [SB], there is an example, $\mathcal{B}$, which is a K3 surface with an elliptic fibration, given in $\mathbb{P}^2 \times \mathbb{P}^1$ by:

$$\mathcal{B}:\ xyz = t^2(x+y)(y+z)(z+x)$$

If we write $\tau = t^2$, the Picard-Fuchs equation is given by

$$\mathcal{F}: \tau(\tau-1)(8\tau+1)\frac{d^2}{d\tau^2} + (24\tau^2 - 14\tau - 1)\frac{d}{d\tau} + (8\tau + 2)$$

If f is a solution, $\mathcal{F}f = 0$, with a Taylor expansion about $t = 0$, with $f = \sum b_n\tau^n$, then the b_n are given by

$$b_n = \sum_{k=0}^{n}\binom{n}{k}^3,$$

and satisfy a recurrence

$$8(n-1)^2 b_{n-2} + (7n^2 - 7n + 2)b_{n-1} - n^2 b_n = 0.$$

(See [SB] §11). This is exactly the same as the first recurrence in Table 2. This means $\mathcal{F}$ can be solved in terms of modular forms for $\Gamma_0(6)$. To solve the Picard-Fuchs equation for $\mathcal{B}$, we need to make a change or variables $\tau = t^2$, which means the solution

$$f(t) = 1 + 2t^2 + 10t^4 + 56t^6 + 346t^8 + 2252t^{10} + \dots$$

can be given in terms of a modular forms for an index 2 subgroup of $\Gamma_0(6)$. This subgroup is the monodromy group of the given fibration of $\mathcal{B}$.

But, in [SB] §14, it is shown that the Mellin transform of the L-series of this variety is

$$L = \eta(\tau)^2\eta(2\tau)\eta(4\tau)\eta(8\tau)^2 = \sum \gamma_n q^n$$

which is a weight 4 cusp form for $\Gamma_0(8)$. Since this variety is a double cover of $\mathbb{P}^1$ with two branch points, Stienstra's result [S] applies, and for primes $p > 2$ we do obtain a congruence:

$$b_{\frac{p-1}{2}} \equiv \gamma_p \bmod p.$$

However, since the modular forms involved are different levels, this result can not be obtained by application of Theorem 0.1.

We might expect there to be a different fibration of $\mathcal{B}$, which does have Picard-Fuchs equation soluble in terms of modular forms for $\Gamma_0(8)$. In fact, note that if $f \in M_1(\Gamma_0(8))$ and $t \in M_0(\Gamma_0(8))$ are given by

$$t = \frac{\eta(2\tau)^2\eta(8\tau)^4}{\eta(\tau)^4\eta(4\tau)^2}, \quad f = \frac{\eta(8\tau)^6\eta(\tau)^6}{\eta(2\tau)^5\eta(4\tau)^5} = t - 10t^2 + 82t^3 - 636t^4 + 4850t^5 + \dots$$

Then by comparing q-expansions, we can show that

$$f\frac{qdt/dq}{t} = L(q) = q - 2q^2 - 2q^3 + 4q^4 + 4q^6 - 8q^8 - 5q^9 + \dots$$

We find that f satisfies a differential equation:

$$(4t+1)^2(8t+1)^2t^2\frac{d^2f}{dt^2} + (4t+1)(8t+1)t(96t^2+12t-1)\frac{df}{dt}$$
$$+(1024t^4+320t^3+44t+10t+1)f = 0$$

Solving the indicial equation at the singularities, we find that if we replace f by $f_1 = (8t+1)^{1/2}(4t+1)^{1/2}t^{-1}f = \frac{\eta(\tau)^4}{\eta(2\tau)^2}$, then we obtain a differential equation

$$(4t+1)(8t+1)t\frac{d^2f_1}{dt^2} + (96t^2+24t+1)\frac{df_1}{dt} + 4(8t+1)f_1 = 0$$

One might hope that this is the Picard-Fuchs equation of a family of elliptic curves, and we might expect the L-series of the total space to have Mellin transform $\eta(\tau)^2\eta(2\tau)\eta(4\tau)\eta(8\tau)^2$. If this is the case, then this family has the same L-series as $\mathcal{B}$, so there should be a geometrical relationship between the two families, and it would be interesting to know what it is.

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