ELLIPTIC STRUCTURES ON WEIGHTED THREE-DIMENSIONAL FANO HYPERSURFACES

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ABSTRACT. Let X be a sufficiently general quasismooth hypersurface in the weighted projective space $\mathbb{P}(1, a_1, a_2, a_3, a_4)$ of degree $d = \sum_{i=1}^{4} a_i$ such that the hypersurface X has terminal singularities, where $a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$. Then the hypersurface X is a birationally rigid Fano 3-fold, and there are exactly 95 possibilities for the five-tuple (d, a_1, a_2, a_3, a_4). In the given paper we classify all birational transformations of the hypersurface X into elliptic fibrations.

INTRODUCTION.

Let X be a quasismooth hypersurface in $\mathbb{P}(1, a_1, a_2, a_3, a_4)$ of degree d, such that

$$-K_X \sim_{\mathbb{Q}} \mathcal{O}_{\mathbb{P}(1,\, a_1,\, a_2,\, a_3,\, a_4)}(1)$$

and X has terminal singularities, where $a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$. Then X is a Fano 3-fold, the singularities of the hypersurface X are $\mathbb{Q}$-factorial, the hypersurface X is rationally connected, and there are exactly 95 possibilities for the five-tuple (d, a_1, a_2, a_3, a_4), which are found in [7] by means of computer computations[1]. Let n be a number of X in the notations of the appendix D.

Remark 0.1. In the case $n = 1$ the hypersurface X is a smooth quartic 3-fold, in the case $n = 3$ the hypersurface X is a double cover of $\mathbb{P}^3$ branched over a smooth sextic, in all other cases the hypersurface X is singular.

Suppose in addition that X is sufficiently general. The following result is due to [6].

Theorem 0.2. *The 3-fold X is birationally rigid*[2].

Moreover, it is proved in [6] that for every possible value of n there are explicitly constructed birational involutions $\tau_1, \ldots, \tau_{k_n}$ of X such that there is an exact sequence of groups

$$1 \to \Gamma \to \mathrm{Bir}(X) \to \mathrm{Aut}(X) \to 1,$$

where Γ is the group generated by the involutions $\tau_1, \ldots, \tau_k$ and

- $k_n = 5$ when $n = 7$;
- $k_n = 3$ when $n \in \{4, 9, 17, 20, 27\}$;
- $k_n = 2$ when $n \in \{6, 12, 13, 15, 23, 25, 30, 31, 33, 36, 38, 40, 41, 42, 44, 58, 61, 68, 76\}$;
- $k_n = 1$ when $n \in \{2, 8, 16, 18, 24, 26, 32, 43, 45, 46, 47, 48, 54, 56, 60, 65, 69, 74, 79\}$;
- $k_n = 0$ in other cases.

Remark 0.3. In the case when $k_n > 0$ and $n \notin \{7, 20\}$ the hypersurface X is birationally equivalent to an elliptic fibration that induces the birational involutions $\tau_1, \ldots, \tau_{k_n}$. The latter is used in the paper [3] to describe the relations between the birational involutions $\tau_1, \ldots, \tau_{k_n}$.

Therefore, it is natural to try to classify all possible birational transformations of the hypersurface X into elliptic fibration. In the case $n = 3$ the hypersurface X is not birational to any elliptic fibration (see [1]), but in the case $n = 1$ all birational transformations of X into elliptic fibrations are induced by projections from lines (see [2] and Theorem 2.1)).

[1] The completeness of the list obtained in [7] is proved in [8].

[2] Let V be a Fano variety of Picard rank 1 having terminal $\mathbb{Q}$-factorial singularities. Then V is called birationally rigid if the following conditions hold: the variety V is not birational to any variety Y such that there is a surjective morphism $\xi : Y \to Z$ such that a general fiber of ξ has negative Kodaira dimension and $\dim(Y) \neq \dim(Z) \neq 0$; the variety V is not birational to a Fano variety Y of Picard rank 1 having terminal $\mathbb{Q}$-factorial singularities that is not biregular to V. The variety V is birationally superrigid if it is birationally rigid and $\mathrm{Bir}(V) = \mathrm{Aut}(V)$.

Lemma 0.4. *Suppose that $n \notin \{1,2,3,7,11,19,60,75,84,87,93\}$. Let $\psi : X \dashrightarrow \mathbb{P}(1,a_1,a_2)$ be a projection. Then the normalization of a general fiber of ψ is an elliptic curve.*

Proof. Let C be a general fiber of ψ. Then C is not rational by Theorem 0.2, and C is a hypersurface of degree d in $\mathbb{P}(1,a_3,a_4) \cong \mathrm{Proj}(\mathbb{C}[x,t,w])$, where $\mathrm{wt}(x)=1$, $\mathrm{wt}(t)=a_3$, $\mathrm{wt}(w)=a_4$.

Suppose that $n \notin \{18,32,43,45,69\}$. Then either $\lceil \frac{d}{a_3} \rceil \leqslant 3$, or $\lceil \frac{d}{a_3} \rceil \leqslant 4$ and $d = 2a_4$.

Let V be an affine subset of $\mathbb{P}(1,a_3,a_4)$ that is given by the equation $x \neq 0$, and Z be an affine curve $V \cap C$. Then $V \cong \mathbb{C}^2$ and

- either $\lceil \frac{d}{a_3} \rceil \leqslant 3$, and the curve Z is a plane cubic,
- or $\lceil \frac{d}{a_3} \rceil \leqslant 4$, $d = 2a_4$, and Z is a double cover of $\mathbb{C}$ branched over at most 4 points.

Let C be a normalization of Z. Then C is either rational or elliptic. The birational rigidity of the hypersurface X implies that C is an elliptic curve.

We may assume that $n \in \{18,32,43,45,69\}$. We consider only the case $n = 18$, because other cases are similar. Thus, the variety X is a hypersurface in $\mathbb{P}(1,2,2,3,5)$ of degree 12, and there is a commutative diagram

$$\begin{tikzcd}
& & W \arrow[lld, "\gamma"'] \arrow[rd, "\alpha"] & & \\
Y \arrow[rd, "\eta"'] & & & X \arrow[lld, dashed, "\xi"'] \arrow[rd, dashed, "\psi"] & \\
& \mathbb{P}(1,2,2,3) \arrow[rrr, dashed, "\chi"'] & & & \mathbb{P}(1,2,2),
\end{tikzcd}$$

where χ and ξ are projections, α is a weighted blow up with weights $(1,2,3)$ of the singular point of the hypersurface X that is a quotient singularity of type $\frac{1}{5}(1,2,3)$, γ is a birational morphism, and η is a double cover. Thus, the variety Y is a hypersurface in $\mathbb{P}(1,2,2,3,7)$ of degree 14 with canonical singularities. Now the arguments used in the previous case imply that the normalization of a general fiber of $\chi \circ \eta$ is an elliptic curve, which implies that the normalization of a general fiber of the rational map ψ is an elliptic curve. $\square$

Lemma 0.5. *Suppose that $n \in \{7,11,19\}$. Then X is birational to an elliptic fibration.*

Proof. We consider only the case $n = 19$, because other cases are similar. Then X is a hypersurface in $\mathbb{P}(1,2,3,3,4)$ of degree 12 that can be given by the equation

$$wf_8(x,y,z,t,w) + zf_3(z,t) + yf_{10}(x,y,z,t,w) + xf_{11}(x,y,z,t,w) = 0 \subset \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x)=1$, $\mathrm{wt}(y)=2$, $\mathrm{wt}(z)=\mathrm{wt}(t)=3$, $\mathrm{wt}(w)=4$, and f_i is a quasihomogeneous polynomial of degree i. Let $\mathcal{H}$ and $\mathcal{B}$ be pencils of surfaces that are cut on X by the equations

$$\lambda x^2 + \mu y = 0, \ \ \delta x^3 + \gamma z = 0$$

respectively, where $(\lambda : \mu) \in \mathbb{P}^1$ and $(\delta : \gamma) \in \mathbb{P}^1$. Then the pencils $\mathcal{H}$ and $\mathcal{B}$ induces a rational map $\rho : X \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. Let C be a general fiber of ρ. Then C is a hypersurface in $\mathbb{P}(1,3,4)$ of degree 12 that passes through the point $(0:1:0)$, which implies that an affine part of C is a cubic curve in $\mathbb{C}^2$. The curve C is not rational, because X is birationally rigid, which implies that the normalization of the curve C is an elliptic curve. $\square$

Let $\Sigma = \{3,60,75,84,87,93\}$. Then Lemmas 0.4 and 0.5 imply that X is birational to an elliptic fibration if $n \notin \Sigma$. The following result is due to [3] (see Lemma 1.13).

Theorem 0.6. *The hypersurface X is birational to an elliptic fibration if and only if $n \notin \Sigma$.*

Let $\Omega = \{1,2,7,9,11,17,19,20,26,30,36,44,49,51,64\}$. Then one can easily check that there are many birational transforms of X into elliptic fibrations whenever $n \in \Omega$. Namely, one can show that in the case $n \in \Omega$ there are rational maps $\alpha : X \dashrightarrow \mathbb{P}^2$ and $\beta \dashrightarrow \mathbb{P}^2$ such that the normalizations of general fibers of the rational maps α and β are irreducible elliptic curves, but there is no commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\sigma"] \arrow[d, dashed, "\alpha"'] & X \arrow[d, dashed, "\beta"] \\ \mathbb{P}^2 \arrow[r, dashed, "\zeta"'] & \mathbb{P}^2, \end{tikzcd}$$

where σ and ζ are birational maps.

In the given paper we classify all birational transformation of the hypersurface X into elliptic fibrations in the case $n \in \Omega$ and prove the following result.

Proposition 0.7. *Let $\rho : X \dashrightarrow \mathbb{P}^2$ be a rational map such that the normalization of a general fiber of ρ is an elliptic curve. Suppose that $n \notin \Omega \cup \Sigma$. Then $\rho = \phi \circ \psi$, where $\psi : X \dashrightarrow \mathbb{P}(1, a_1, a_2)$ is the natural projection, and $\phi : \mathbb{P}(1, a_1, a_2) \dashrightarrow \mathbb{P}^2$ is a birational map.*

The claim of Proposition 0.7 is proved in [13] in the case $n = 5$, and in the paper [3] in the case when $n \in \{14, 22, 28, 34, 37, 39, 52, 53, 57, 59, 66, 70, 72, 73, 78, 81, 86, 88, 89, 90, 92, 94, 95\}$.

To illustrate our technique let us prove the following result.

Proposition 0.8. *The claim of Proposition 0.7 holds for $n = 14$.*

Proof. Suppose that $n = 14$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1, 1, 1, 4, 6)$ of degree 12, the singularities of the hypersurface X consists of a single point, which is a quotient singularity of type $\frac{1}{2}(1, 1, 1)$. Let P be a singular point of the hypersurface X, and $\psi : X \dashrightarrow \mathbb{P}^2$ be the natural projection. Then a general fiber of ψ is an elliptic curve. Let $\pi : U \to X$ be a weighted blow up of the singular point P with weights $(1, 1, 1)$. Then $\psi \circ \pi$ is a morphism.

Let $\rho : X \dashrightarrow \mathbb{P}^2$ be a rational map such that the normalization of a general fiber of the rational map ρ is an irreducible elliptic curve. We must show that $\rho = \phi \circ \psi$ for some $\phi \in \mathrm{Bir}(\mathbb{P}^2)$.

Let us consider commutative diagram

$$\begin{tikzcd} & V \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X \arrow[rr, dashed, "\rho"'] & & \mathbb{P}^2, \end{tikzcd}$$

where V is smooth, the morphism α is birational, and β is a morphism such that the general fiber of β is an elliptic curve. Let $\mathcal{B}$ be the linear system $|\beta^*(\mathcal{O}_{\mathbb{P}^2}(1))|$, and $\mathcal{M}$ be a proper transform of the linear system $\mathcal{B}$ on the hypersurface X. Then to conclude the proof we must show that the proper transform of the linear system $\mathcal{M}$ on the variety U lies in the fibers of the elliptic fibration $\psi \circ \pi$.

There is a natural number $k > 0$ such that surfaces of the linear system $\mathcal{M}$ are rationally equivalent to the divisor $-kK_X$. Then

$$K_V + \frac{1}{k}\mathcal{B} \sim_{\mathbb{Q}} \alpha^*(K_X + \frac{1}{k}\mathcal{M}) + \sum_{i=1}^{\delta} a_i E_i \sim_{\mathbb{Q}} \sum_{i=1}^{\delta} a_i E_i,$$

where E_i is an α-exceptional divisor, a_i is a rational number, and δ is the number of exceptional divisors of the morphism α. Let C be a sufficiently general fiber of the morphism β. Then C is a smooth elliptic curve that is not contained in the support of the divisor $\cup_{i=1}^{\delta} E_i$. In particular, the equality $K_V \cdot C = 0$ holds, but the curve C does not intersect a general surface of the linear system $\mathcal{B}$. Therefore, we proved that

$$\sum_{i=1}^{\delta} a_i E_i \cdot C = 0, \tag{0.9}$$

which easily implies that there is an index j such that $a_j \leqslant 0$. In other words, the singularities of the log pair $(X, \frac{1}{k}\mathcal{M})$ are not terminal (see Theorem B.4). Indeed, suppose that $a_i > 0$ for all possible values of the index i. Then the equality 0.9 implies that the equality $E_i \cdot C = 0$ holds for all possible values of the index i. Therefore, the projection formula implies $-K_X \cdot \alpha(C) = 0$, which contradicts the ampleness of the anticanonical divisor $-K_X$, because the curve C is not contracted by the birational morphism α.

The singularities of $(X, \frac{1}{k}\mathcal{M})$ are not terminal, but X is birationally superrigid, which implies that the singularities of $(X, \frac{1}{k}\mathcal{M})$ are canonical (see Corollary B.2). Hence, the inequality $a_i \geqslant 0$ holds for every index i, and there is an index j such that $a_j = 0$.

Let $Z_j = \alpha(E_j)$. The there are following possibilities:

- the subvariety Z_j is a smooth point of the hypersurface X;
- the subvariety Z_j is a curve on the hypersurface X;
- the subvariety Z_j is a singular point of the hypersurface X.

Let S_1 and S_2 be general surfaces of the linear system $\mathcal{M}$.

The linear system $\mathcal{M}$ is not composed from a pencil[3]. Thus, there is no proper Zariski closed subset $\Sigma \subset X$ such that the support of the cycle $S_1 \cdot S_2$ is contained in Σ. Indeed, the support of the effective cycle $S_1 \cdot S_2$ contains the proper transform of the fiber of the morphism β over the point $\rho(S_1) \cap \rho(S_2)$, which implies that the support of the cycle $S_1 \cdot S_2$ spans a Zariski dense subset of the hypersurface X when we vary the surfaces S_1 and S_2 in the linear system $\mathcal{M}$.

Suppose that Z_j is a smooth point of the hypersurface X. Then

$$\mathrm{mult}_{Z_j}(S_1 \cdot S_2) \geqslant 4k^2$$

by Theorem C.15. On the other hand, the linear system $|-4K_X|$ does not have base points and induces the natural double cover $X \to \mathbb{P}(1,1,1,4)$. Hence, we have

$$2k^2 = H \cdot S_1 \cdot S_2 \geqslant \mathrm{mult}_{Z_j}(S_1 \cdot S_2) \geqslant 4k^2,$$

where H is a sufficiently general divisor in the linear system $|-4K_X|$ that passes through the point Z_j, which is a contradiction.

Suppose that Z_j is an irreducible curve. Let H be a general divisor in $|-4K_X|$. Then

$$2k^2 = H \cdot S_1 \cdot S_2 \geqslant \mathrm{mult}_{Z_j}(S_1 \cdot S_2) H \cdot Z_j \geqslant k^2 H \cdot Z_j,$$

because $\mathrm{mult}_{Z_j}(S_1) = \mathrm{mult}_{Z_j}(S_2) = k$. Therefore, the inequality $-K_X \cdot Z_j \leqslant 1/2$ holds. However, the divisor $-2K_X$ is a Cartier divisor, which implies that $-K_X \cdot Z_j = 1/2$, and the support of the effective cycle $S_1 \cdot S_2$ is contained in the curve Z_j, which is a contradiction.

Therefore, the equality $a_j = 0$ implies that Z_j is the singular point of the hypersurface X.

Let $\mathcal{D}$ be a proper transform of the linear system $\mathcal{M}$ on the variety U. Then

$$\mathcal{D} \sim_{\mathbb{Q}} \pi^*(-kK_X) + mG,$$

where G is the exceptional divisor of the morphism π, and m is a positive rational number. We may assume that there is a birational morphism $\omega : V \to U$ such that the diagram

$$\begin{tikzcd}
U \arrow[dr, "\pi"'] & & V \arrow[ll, "\omega"'] \arrow[dl, "\alpha"] \arrow[dr, "\beta"] & \\
& X \arrow[rr, dashed, "\rho"'] & & \mathbb{P}^2
\end{tikzcd}$$

is commutative. The inequality $m \leqslant 1/k$ holds, because

$$K_U \sim_{\mathbb{Q}} \pi^*(K_X) + \frac{1}{k}G.$$

Suppose that the inequality $m < 1/k$ holds. Then

$$\mathcal{D}|_G \sim_{\mathbb{Q}} -mG|_G,$$

where $G \cong \mathbb{P}^2$ and $G|_G \sim \mathcal{O}_{\mathbb{P}^2}(-2)$. Let P a point of the surface G, and D be a sufficiently general divisor of the linear system $\mathcal{D}$. Intersecting the divisor $D|_G$ with a general line on the surface $G \cong \mathbb{P}^2$ passing through P we see that $\mathrm{mult}_P(D) < k$, but $\mathrm{mult}_{\omega(Z_j)}(\mathcal{D}) \geqslant k$.

Therefore, the equivalence $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ holds, which implies that the linear system $\mathcal{D}$ lies in the fibers of the elliptic fibration $\psi \circ \pi$, which concludes the proof. $\square$

[3]The linear system $\mathcal{M}$ on the variety X is not composed from a pencil if $\dim(\phi_{\mathcal{M}}(X)) \neq 1$.

The classification of the birational transformations into elliptic fibration is based on *submaximal* singularity (see Theorem B.4). However, submaximal singularities can be also induced by by Fano varieties with canonical singularities and by fibrations into surfaces of Kodaira dimension zero (see Theorems B.6 and B.7). The purpose of the given paper is to classify all birational transformation of the hypersurface X into elliptic fibrations. Therefore, we can consider submaximal singularities not induced by elliptic fibrations as *bad* submaximal singularities.

The only way to obtain the classification of birational transformations into elliptic fibration is to exclude submaximal singularities that do not exists and to exclude bad submaximal singularities. The proof of Proposition 0.8 is rather simple, because in the case $n = 14$ the indeterminacy of the natural projection $X \dashrightarrow \mathbb{P}(1, a_1, a_2)$ can be resolved by one weighted blow up of the unique singular point of the hypersurface X. On the other hand, in many cases the indeterminacy of a projection $X \dashrightarrow \mathbb{P}(1, a_1, a_2)$ can be resolved by a composition of several weighted blow ups, which implies that submaximal singularities must be studied on every *floor* of this composition. To handle the letter problem we use Theorem A.15 and Lemmas A.16 and B.5.

To exclude bad submaximal singularities induced by fibrations into surfaces of zero Kodaira dimension we use Lemmas A.17 and A.20 together with Corollary A.19. We can exclude such bad submaximal singularities without their classification. A priori, one can use Theorem B.6 to classify birational transformations of X into fibrations into surfaces of Kodaira dimension zero, which is done in some cases in [1], [13], [2], [14].

The exclusion of bad submaximal singularities induced by Fano varieties with canonical singularities uses Lemma B.5. However, one can easily see that it is impossible to exclude such bad submaximal singularities and skip their classification. Namely, the classification of birational transformations of X into elliptic fibrations implies the classification of birational transformations of X into Fano varieties with canonical singularities. For example, one can easily see that the hypersurface X is the only Fano variety that is birational to X in the case $n = 14$, but in most of cases the latter is not true.

In certain sense, the complexity of the classification of birational transformations of X into elliptic fibrations is determined by the number and types of all possible birational transformations of the hypersurface X into Fano varieties with canonical singularities. From this point of view the most complicated cases are $n = 25$, $n = 43$, $n = 47$, $n = 56$, $n = 62$, $n = 82$, because in these cases there are birational transformations of the hypersurface X into Fano varieties with canonical singularities whose constructions nontrivially involve antiflip[4].

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1. Classification of elliptic structures.

Let X be a general quasismooth hypersurface in $\mathbb{P}(1, a_1, a_2, a_3, a_4)$ of degree $d = \sum_{i=1}^{4} a_i$, where $a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$, and the singularities of X are terminal. Let n be a number of the hypersurface of X in the notations of appendix D.

Suppose that we have a set Ξ such that every element of the set Ξ is an explicitly constructed rational map $\xi : X \dashrightarrow S$, whose general fiber is birational to a smooth irreducible elliptic curve, and we want to show that the maps of the set Ξ exhausts all possibilities of birational transformations of the hypersurface X into elliptic fibrations.

Example 1.1. Let X be a quartic hypersurface in $\mathbb{P}^3$, and Ξ be a set of all projections from lines contained in X.

Example 1.2. Let $n = 3$, and Ξ be an empty set.

[4]The case $n = 60$ is an exceptional case among all cases when $n \in \Sigma$, because in the case $n = 60$ the hypersurface X is not birational to an elliptic fibration, but the hypersurface X can be birationally transformed into a Fano variety with canonical singularities that is not biregular to the hypersurface X.

Suppose that there is a birational map $\rho : X \dashrightarrow V$ and a morphism $\nu : V \to \mathbb{P}^2$ such that the variety V is smooth, and a general fiber of ν is an elliptic curve. We want to show that for some element $\xi : X \dashrightarrow S$ of the set Ξ there is a commutative diagram

(1.3)
$$\begin{tikzcd} X \arrow[r, dashed, "\sigma"] \arrow[d, dashed, "\xi"'] & X \arrow[r, dashed, "\rho"] & V \arrow[d, "\nu"] \\ S \arrow[rr, dashed, "\zeta"'] & & \mathbb{P}^2, \end{tikzcd}$$

where σ and ζ are birational maps, which implies that $\Xi \neq \varnothing$. The commutative diagram 1.3 implies the existence of the commutative diagram

(1.4)
$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\xi"'] & V \arrow[d, "\nu"] \\ S \arrow[r, dashed, "\zeta"'] & \mathbb{P}^2 \end{tikzcd}$$

if for every birational automorphism σ of the hypersurface X and every rational map ξ of the set Ξ we have $\xi \circ \sigma = \chi \circ \xi$ for some birational automorphism $\chi \in \mathrm{Bir}(S)$.

Example 1.5. Let $\psi : X \dashrightarrow \mathbb{P}(1, a_1, a_2)$ be a projection, and σ be any birational automorphism of X. Suppose that $n \notin \{1, 2, 3, 7, 11, 19, 20, 36, 60, 75, 84, 87, 93\}$. Then it follows from [6] that there is a birational automorphism χ of $\mathbb{P}(1, a_1, a_2)$ such that $\psi \circ \sigma = \chi \circ \psi$.

Let $\mathcal{M}$ be a proper transform of the linear system $|\nu^*(\mathcal{O}_{\mathbb{P}^2}(1))|$ on X. Then $\mathcal{M} \sim_{\mathbb{Q}} -kK_X$ for some natural k, but the singularities of the log pair $(X, \frac{1}{k}\mathcal{M})$ are not terminal by Theorem B.4.

Remark 1.6. It follows from Theorem 0.2 and Corollary B.3 that there is a birational automorphism σ of the hypersurface X such that the singularities of the log pair $(X, \frac{1}{k}\sigma(\mathcal{M}))$ are canonical. Therefore, we can replace the birational map ρ by the composition $\rho \circ \sigma$ and assume that the singularities of the log pair $(X, \frac{1}{k}\mathcal{M})$ are canonical.

The following result is due to [6].

Theorem 1.7. *The set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ does not contain smooth points of X if $n \neq 1$ and $n \neq 2$.*

On the other hand, the following simple result holds.

Lemma 1.8. *Let C be a curve on X such that $C \in \mathbb{CS}(X, \frac{1}{k}\mathcal{M})$. Then $-K_X \cdot C < -K_X^3$.*

Proof. Let H be a general divisor in $|-rK_X|$ for $r \gg 0$. Then
$$-rk^2K_X^3 = H \cdot D_1 \cdot D_2 \geqslant \mathrm{mult}_C(D_1 \cdot D_2)H \cdot C \geqslant \mathrm{mult}_C^2(\mathcal{M})H \cdot C \geqslant -rk^2K_X \cdot C,$$
where D_1 and D_2 are general surfaces in $\mathcal{M}$. Therefore, the inequality $-K_X \cdot C \leqslant -K_X^3$ holds, but in the case when $-K_X \cdot C = -K_X^3$ the support of the effective cycle $D_1 \cdot D_2$ is contained in the curve C, which is impossible by Lemma A.17. □

Corollary 1.9. *Suppose that $n \geqslant 6$. Then the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ does not contain curves that do not contain singular points of the hypersurface X.*

In particular, the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains a singular point of the hypersurface X in the case when $n \geqslant 6$ by Theorem A.15.

Proposition 1.10. *Suppose that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains a singular point O of the hypersurface X such that O is a quotient singularity of type $\frac{1}{r}(1, r-a, a)$, where a and r are coprime natural numbers and $r > a$. Let $\pi : Y \to X$ be a weighted blow up of the singular point O with weights $(1, a, r-a)$. Then $-K_Y^3 \geqslant 0$.*

Proof. Suppose that $-K_Y^3 < 0$. Let E be an exceptional divisor of the morphism π, and $\mathcal{B}$ be a proper transform of the linear system $\mathcal{M}$ on the variety Y. Then
$$K_Y \sim_{\mathbb{Q}} \pi^*(K_X) + \frac{1}{r}E,$$

which implies that $-K_Y^3 = -K_X^3 - 1/(ra(r-a))$, but $\mathcal{B} \sim_{\mathbb{Q}} -kK_Y$ by Theorem A.15.

Let $\overline{\mathbb{NE}}(Y)$ be a closure in $\mathbb{R}^2$ of the cone generated by effective one-dimensional cycles of the variety Y. Then $-E \cdot E$ generates the extremal ray of the cone $\overline{\mathbb{NE}}(Y)$. Moreover, it follows from Corollary 5.4.6 in [6] that there are integers numbers $b > 0$ and $c \geqslant 0$ such that the cycle

$$-K_Y \cdot (-bK_Y + cE)$$

is numerically equivalent to an effective, irreducible and reduced curve Γ on the variety Y that generates the extremal ray of the cone $\overline{\mathbb{NE}}(Y)$ different from the ray generated by $-E \cdot E$.

Let S_1 and S_2 be general surfaces of the linear system $\mathcal{B}$. Then $S_1 \cdot S_2 \in \overline{\mathbb{NE}}(Y)$, but

$$S_1 \cdot S_2 \equiv k^2 K_Y^2,$$

which implies that the cycle $S_1 \cdot S_2$ generates an extremal ray of the cone $\overline{\mathbb{NE}}(Y)$ that contains the curve Γ. Moreover, for every effective cycle $C \in \mathbb{R}^{+}\Gamma$ we have

$$\mathrm{Supp}(C) = \mathrm{Supp}(S_1 \cdot S_2),$$

because $S_1 \cdot \Gamma < 0$ and $S_2 \cdot \Gamma < 0$, which contradicts Lemma A.17. $\square$

The following result is implied by Proposition 1.10.

Proposition 1.11. *The claim of Proposition 0.7 holds for all values of n that is contained in the set* $\{14, 22, 28, 34, 37, 39, 52, 53, 57, 59, 66, 70, 72, 73, 78, 81, 86, 88, 89, 90, 92, 94, 95\}$.

Proof. We must show the existence of the commutative diagram

(1.12)
$$\begin{tikzcd}
X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & V \arrow[d, "\nu"] \\
\mathbb{P}(1, a_1, a_2) \arrow[r, dashed, "\phi"'] & \mathbb{P}^2,
\end{tikzcd}$$

where ψ is the natural projection, and ϕ is a birational map.

It follows from Theorem 1.7, Lemma1.8 and Theorem A.15 that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains a singular point P of X that is a quotient singularity of type $\frac{1}{r}(1, a, r-a)$, where a and r are coprime natural numbers and $r > a$.

Let $\pi : Y \to X$ be a weighted blow up of P with weights $(1, a, r-a)$, E be the exceptional divisor of the birational morphism π, and $\mathcal{B}$ be the proper transform of the linear system $\mathcal{M}$ on variety Y. Then the equivalence $\mathcal{B} \sim_{\mathbb{Q}} -kK_Y$ holds by Theorem A.15. Moreover, one can easily check that either $-K_Y^3 < 0$, or $-K_Y^3 = 0$.

The equality $-K_Y^3 = 0$ holds by Proposition 1.10. The linear system $|-rK_Y|$ does not have base points for $r \gg 0$ and induces the morphism $\eta : Y \to \mathbb{P}(1, a_1, a_2)$ such that the diagram

$$\begin{tikzcd}
& Y \arrow[dl, "\pi"'] \arrow[dr, "\eta"] & \\
X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1, a_1, a_2)
\end{tikzcd}$$

is commutative. Let C be a general fiber of the elliptic fibration η, and S is a general surface of the linear system $\mathcal{M}$. Then $S \sim_{\mathbb{Q}} -kK_Y$ by Theorem A.15. Thus, the equality $S \cdot C = 0$ holds, which implies the existence of the commutative diagram 1.12. $\square$

The following result implies Theorem 0.6.

Lemma 1.13. *The existence of ρ and ν implies that $n \notin \{3, 60, 75, 84, 87, 93\}$.*

Proof. It follows from Proposition 1.10 that $n \notin \{75, 84, 87, 93\}$.

Suppose that $n = 3$. Then the hypersurface X is smooth, and the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains an irreducible curve Γ such that $-K_X \cdot \Gamma = 1$ by Lemma 1.8. In particular, the curve Γ is smooth.

Let $\gamma : \bar{X} \to X$ be a blow up of the curve Γ, and G be the exceptional divisor of the birational morphism γ. Then the divisor $\gamma^*(-3K_X) - G$ is nef and big, but

$$\Big(\eta^*(-3K_X) - G\Big) \cdot \bar{S}_1 \cdot \bar{S}_1 = 0,$$

where S_1 and S_2 are proper transforms on X of general surfaces of the linear system $\mathcal{M}$, which is impossible by Corollary A.19.

Thus, we have $n = 60$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,4,5,6,9)$ of degree 24, and it follows from Proposition 1.10 that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains the singular point O of the hypersurface X that is a quotient singularity of type $\frac{1}{9}(1,4,5)$. It is easy to check (see [13]) that

$$-K_X \cdot C > -K_X^3 = \frac{1}{45},$$

where C is a curve on X, which implies that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{O\}$ by Lemma 1.8.

Let $\pi : Y \to X$ be a weighted blow up of the point O with weights $(1,4,5)$, and $\mathcal{D}$ be a proper transform of the linear system $\mathcal{M}$ on the variety Y. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_Y$ by Theorem A.15, but the set $\mathbb{CS}(Y, \frac{1}{k}\mathcal{D})$ is not empty by Lemma B.5. Let P and Q be singular points of Y that are quotient singularities of types $\frac{1}{4}(1,1,3)$ and $\frac{1}{5}(1,1,4)$ contained in the exceptional divisor of the morphism π respectively. Then it follows from Lemma A.16 that

$$\mathbb{CS}(Y, \frac{1}{k}\mathcal{D}) \cap \{P, Q\} \neq \varnothing.$$

Suppose that the set $\mathbb{CS}(Y, \frac{1}{k}\mathcal{D})$ contains the point Q. Let $\alpha : U \to Y$ be a weighted blow up of the point Q with weights $(1,1,4)$, and $\mathcal{B}$ be a proper transform of the linear system $\mathcal{M}$ on the variety U. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_U$ by Theorem A.15, the linear system $|-4K_U|$ is a proper transform of the pencil $|-4K_X|$, and the base locus of the pencil $|-4K_U|$ consists of an irreducible reduced curve Z on the variety U such that the curve $\pi \circ \alpha(Z)$ is a base curve of the pencil $|-4K_X|$.

Let H be a sufficiently general surface of the pencil $|-4K_U|$. Then the equality

$$Z^2 = -K_U^3 = -\frac{1}{30}$$

holds on the surface H, but $\mathcal{B}|_H \sim_{\mathbb{Q}} kZ$. Therefore, it follows from Lemma A.21 that

$$\mathrm{Supp}(D) \cap \mathrm{Supp}(H) = \mathrm{Supp}(Z),$$

where D is a sufficiently general surface of the linear system $\mathcal{B}$, which contradicts Lemma A.20.

Therefore, the set $\mathbb{CS}(Y, \frac{1}{k}\mathcal{D})$ contains the singular point P.

The hypersurface X can be given by a quasihomogeneous equation of degree 24

$$w^2t + wf_{15}(x,y,z,t) + f_{24}(x,y,z,t) = 0 \subset \mathbb{P}(1,4,5,6,9) \cong \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 4$, $\mathrm{wt}(z) = 5$, $\mathrm{wt}(t) = 6$, $\mathrm{wt}(w) = 9$, $f_i(x,y,z,t)$ is a quasihomogeneous polynomial of degree i. Let D be a general surface of the linear system $|-5K_X|$, and S be the unique surface of the linear system $|-K_X|$. Then D is cut on X by the equation

$$\lambda x^5 + \delta xy + \mu z = 0,$$

where $(\lambda, \delta, \mu) \in \mathbb{P}^2$, and S is cut on X by the equation $x = 0$. Moreover, the base locus of the linear system $|-5K_X|$ consists of the irreducible curve C that is cut on the hypersurface X by the equations $x = z = 0$. In particular, we have $D \cdot S = C$.

In the neighborhood of the point O the monomials x, y and z can be considered as weighted local coordinates on X such that $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 4$ and $\mathrm{wt}(z) = 5$. In the neighborhood of the point P the birational morphism π can be given by the equations

$$x = \tilde{x}\tilde{y}^{\frac{1}{9}},\ y = \tilde{y}^{\frac{4}{9}},\ z = \tilde{z}\tilde{y}^{\frac{5}{9}},$$

where $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ are weighted local coordinated on the variety Y in the neighborhood of the point P such that $\mathrm{wt}(\tilde{x}) = 1$, $\mathrm{wt}(\tilde{y}) = 3$ and $\mathrm{wt}(\tilde{z}) = 1$. Let $\tilde{D}$, $\tilde{S}$ and $\tilde{C}$ be proper transforms on the variety Y of the surface D, the surface S and the curve C respectively, and E be the exceptional divisor of π. Then in the neighborhood of the point P the surface E is given by the equation $\tilde{y} = 0$, the surface $\tilde{D}$ is given by the equation

$$\lambda\tilde{x}^5 + \delta\tilde{x} + \mu\tilde{z} = 0,$$

and the surface S is given by the equation $\tilde{x} = 0$. Hence, we have

$$\tilde{D} \sim_{\mathbb{Q}} \pi^*(-5K_X) - \frac{5}{9}E \sim_{\mathbb{Q}} 5\tilde{S} \sim_{\mathbb{Q}} -5K_Y,$$

the curve C is the intersection of the surfaces D and S, the linear system $|-5K_Y|$ is a proper transform of $|-5K_X|$, and the base locus of the linear system $|-5K_Y|$ consists of the curve $\tilde{C}$.

Let $\beta: W \to Y$ be a weighted blow up of the point P with weights $(1,1,3)$, $\bar{D}$, $\bar{S}$ and $\bar{C}$ be proper transforms on the variety W of the surface D, the surface S and the curve C respectively, and F be an exceptional divisor of the morphism β. Then the surface F is a weighted projective space $\mathbb{P}(1,1,3)$, and in the neighborhood of the singular point of the surface F the birational morphism β can be given by the equations

$$\tilde{x} = \bar{x}\bar{y}^{\frac{1}{4}},\ \tilde{y} = \bar{y}^{\frac{3}{4}},\ \tilde{z} = \bar{z}\bar{y}^{\frac{1}{4}},$$

where $\bar{x}$, $\bar{y}$ and $\bar{z}$ are weighted local coordinates on the variety W in the neighborhood of the singular point of F such that $\mathrm{wt}(\bar{x}) = 1$, $\mathrm{wt}(\bar{y}) = 2$ and $\mathrm{wt}(\bar{z}) = 1$. In particular, the exceptional divisor F is given by the equation $\bar{y} = 0$, the surface $\bar{D}$ is given by the equation

$$\lambda\bar{x}^5\bar{y} + \delta\bar{x} + \mu\bar{z} = 0,$$

and the surface $\bar{S}$ is given by the equation $\bar{x} = 0$. Therefore, we have

$$\bar{D} \sim_{\mathbb{Q}} \beta^*(\tilde{D}) - \frac{1}{4}F \sim_{\mathbb{Q}} (\pi\circ\beta)^*(-5K_X) - \frac{5}{9}\beta^*(E) - \frac{1}{4}F,\ \bar{S} \sim_{\mathbb{Q}} \beta^*(\bar{S}) - \frac{1}{4}F \sim_{\mathbb{Q}} -K_W,$$

and the curve $\bar{C}$ is the intersection of the surfaces $\bar{D}$ and $\bar{S}$. Let $\mathcal{P}$ be a proper transform of the linear system $|-5K_X|$ on W. Then $\bar{D}$ is a general surface of the linear system $\mathcal{P}$, the base locus of the linear system $\mathcal{P}$ consists of the curve $\bar{C}$, and the equalities

$$\bar{D}\cdot\bar{C} = \bar{D}\cdot\bar{D}\cdot\bar{S} = \frac{1}{3}$$

holds. Thus, the divisor $\bar{D}$ is nef and big, because $\bar{D}^3 = 2$. On the other hand, we have

$$\bar{D}\cdot B_1\cdot B_2 = \Big(\beta^*(-5K_Y) - \frac{1}{4}F\Big)\cdot\Big(\beta^*(-kK_Y) - \frac{k}{4}F\Big)^2 = 0,$$

where B_1 and B_2 are general divisors of $\mathcal{B}$, which is impossible by Corollary A.19. □

Now we prove the following very simple result.

Proposition 1.14. *Suppose that $n = 11$. Let Ξ be the set of maps $\xi_i : X \dashrightarrow \mathbb{P}(1,1,2)$ that is implicitly constructed in the proof of Lemma 0.5, where $i = 1,2,3,4,5$. Then the commutative diagram 1.4 exists for some $\xi \in \Xi$.*

Proof. The variety X is a hypersurface in $\mathbb{P}(1,1,2,2,5)$ of degree 10, the hypersurface X is birationally superrigid, the singularities of X consist of points P_1, P_2, P_3, P_4 and P_5 that are quotient singularities of types $\frac{1}{2}(1,1,1)$, and the equality $-K_X^3 = 1/2$ holds.

The proof of Lemma 0.5 implies that there is a commutative diagram

$$\begin{tikzcd} & U_i \arrow[dl, "\pi_i"'] \arrow[dr, "\eta_i"] & \\ X \arrow[rr, dashed, "\xi_i"'] & & \mathbb{P}(1,1,2), \end{tikzcd}$$

where π_i is the weighted blow up of P_i with weights $(1,1,1)$, and η_i is an elliptic fibration.

It follows from Theorem 1.7, Lemma 1.8 and Theorem A.15 follows that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains the singular point P_i, where $i \in \{1,2,3,4,5\}$. Let $\mathcal{D}_i$ be the proper transform of the linear system $\mathcal{M}$ on the variety U_i. Then $\mathcal{D}_i \sim_{\mathbb{Q}} -kK_{U_i}$ by Theorem A.15, which easily implies the existence of the commutative diagram 1.4. □

The following result is due to [13].

Theorem 1.15. *Suppose that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains an irreducible curve C on the hypersurface X, but $n \neq 1$ and $n \neq 2$. Then $\mathrm{Supp}(C) \subset \mathrm{Supp}(S_1\cdot S_2)$, where S_1 and S_2 are different surfaces of the linear system $|-K_X|$.*

Thus, the claim of Theorem 1.15 implies the following result.

Lemma 1.16. *Suppose that $a_3 \neq 1$. Then the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ does not contains curves.*

Proof. Suppose that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains a curve C. Then the claim of Theorem 1.15 implies that there are surfaces S_1 and S_2 of the pencil $|-K_X|$ such that the curve C is an irreducible component of the curve $S_1 \cap S_2$, which is reduced and irreducible. Therefore, the equality

$$-K_X \cdot C = -K_X^3$$

holds, which is impossible by Lemma 1.8. □

Let us prove the following result, which is proved in [13].

Proposition 1.17. *The claim of Proposition 0.7 holds for $n = 5$.*

Proof. Let $n = 5$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,1,1,2,3)$ of degree 7, the singularities of the hypersurface X consist of points P and Q such that the point P is a quotient singularity of type $\frac{1}{2}(1,1,1)$, and the point Q is a quotient singularity of type $\frac{1}{3}(1,1,2)$, and $-K_X^3 = 7/6$.

The hypersurface X can be given by the equation

$$w^2 f_1(x,y,z) + f_4(x,y,z,t)w + f_7(x,y,z,t) = 0 \subset \mathbb{P}(1,1,1,2,3) \cong \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = \mathrm{wt}(z) = 1$, $\mathrm{wt}(t) = 2$, $\mathrm{wt}(w) = 3$, and f_i is a quasihomogeneous polynomial of degree i. Hence, the point P is given by the equations $x = y = z = w = 0$, and the point Q is given by the equations $x = y = z = t = 0$. There is a commutative diagram

$$\begin{tikzcd}
 & & W \arrow[dll, "\gamma"'] \arrow[d, "\alpha"] & & \\
Y \arrow[dr, "\eta"'] & & X \arrow[dl, dashed, "\xi"'] \arrow[dr, dashed, "\psi"] & & U \arrow[ull, "\beta"'] \arrow[dl, "\omega"] \\
 & \mathbb{P}(1,1,1,2) \arrow[rr, dashed, "\chi"'] & & \mathbb{P}^2, &
\end{tikzcd}$$

where χ and ξ are the natural projections, the morphism ω is an elliptic fibration, the morphism α is a weighted blow up of the point Q with weights $(1,1,2)$, the morphism γ is the birational morphism that contracts 14 smooth irreducible rational curves $C_1, \ldots, C_{14}$ into 14 isolated ordinary double points $P_1, \ldots, P_{14}$ of the variety Y respectively, the morphism η is a double cover branched over the surface $R \subset \mathbb{P}(1,1,1,2)$ that is given by the equation

$$f_4(x,y,z,t)^2 - 4f_1(x,y,z)f_7(x,y,z,t) = 0 \subset \mathbb{P}(1,1,1,2) \cong \mathrm{Proj}\big(\mathbb{C}[x,y,z,t]\big)$$

and has 14 isolated ordinary double points $\eta(P_1), \ldots, \eta(P_{14})$, and β is the composition of the weighted blow ups with the weights $(1,1,1)$ of two singular points of the variety W that are quotient singularities of types $\frac{1}{2}(1,1,1)$ such that one of them is the proper transform of the point P, but another one is contains in the exceptional divisor of the morphism α.

Let G be an exceptional divisor of the morphism α, $\bar{P}$ be the proper transform of the singular point P on the variety W, and O be the singular point of the variety W that is different from the point $\bar{P}$. Then G is a weighted projective space $\mathbb{P}(1,1,2)$ that can be identified with a quadratic cone, the point O is a singular point of G, and the singular points of the surface R are contained in the divisor $\eta \circ \gamma(G)$, which is given by the equation $f_1(x,y,z) = 0$ in $\mathbb{P}(1,1,1,2)$.

It follows from Theorem 1.15 and the generality of X that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ does not contain curves (see the proof of Proposition 3.2). The set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains a singular point of X.

Suppose that $Q \in \mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ and $P \notin \mathbb{CS}(X, \frac{1}{k}\mathcal{M})$. Let $\mathcal{D}$ be the proper transform of the linear system $\mathcal{M}$ on the variety W. It follows from Theorem A.15 and Lemmas B.5 and A.16 that the set $\mathbb{CS}(W, \frac{1}{k}\mathcal{D})$ contains the singular point O. Let $\mathcal{B}$ be the proper transform of the linear system $\mathcal{D}$ on the variety Y. Then the set $\mathbb{CS}(Y, \frac{1}{k}\mathcal{B})$ contains an irreducible curve C such that the equality $-K_Y \cdot C = 1/2$ holds, and $\chi \circ \eta(C)$ is a point.

The variety $\mathbb{P}(1,1,1,2)$ is a cone over the Veronese surface. Hence, the curve $\eta(C)$ is a ruling of the cone $\mathbb{P}(1,1,1,2)$, and the point $\gamma(\bar{P}) = \gamma(O)$ is a vertex of $\mathbb{P}(1,1,1,2)$. The generality of the hypersurface X implies the existence of the irreducible curve Z on the variety Y such that the curve Z is different from the curve C, but $\eta(Z) = \eta(C)$, and $\gamma(\bar{P}) \in Z$.

We have $Z \not\in \mathbb{CS}(Y, \frac{1}{k}\mathcal{B})$. Let S be a general surface in $|-K_Y|$ containing C. Then the strict inequality $Z^2 < 0$ holds on S, but $\mathcal{B}|_S \sim_{\mathbb{Q}} kC + kZ$, which easily implies that $Z \in \mathbb{CS}(Y, \frac{1}{k}\mathcal{B})$.

Thus, the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ consists of the singular points P and Q. Let $\mathcal{H}$ be the proper transform of the linear system $\mathcal{M}$ on the variety U, and E and F be exceptional divisors of the birational morphism β such that $\beta(E) = \bar{P}$ and $\beta(F) = O$. Then the surfaces E and F are sections of the elliptic fibration ω, and it follows from Theorem A.15 that

$$\mathcal{H} \sim_{\mathbb{Q}} -kK_U + \Big(\frac{k}{2} - \mathrm{mult}_O(\mathcal{D})\Big)F,$$

where $\mathcal{D}$ is the proper transform of the linear system $\mathcal{M}$ on W. Then $\mathrm{mult}_O(\mathcal{D}) \geqslant k/2$ implies the claim of Proposition 0.7.

Therefore, we may assume that $\mathrm{mult}_O(\mathcal{D}) < k/2$. It follows from Lemmas B.5 and A.16 that the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{H})$ contains a curve L that is a line on the surface $E \cong \mathbb{P}^2$. Intersecting a general surface of the linear system $\mathcal{H}$ with a fiber of the elliptic fibration ω over a general point of the line $\omega(L) \subset \mathbb{P}^2$, we obtain a contradiction. □

In the rest of the paper we continue the classification of the birational transformations of the hypersurface X into elliptic fibrations for all remaining values of the index n.

2. Smooth quartic threefold.

Let X be a smooth hypersurface in $\mathbb{P}^4$ of degree 4. Then X contains a one-dimensional family of lines. Moreover, the following result holds (see [2]).

Theorem 2.1. *Let $\xi : X \dashrightarrow \mathbb{P}^2$ be a rational map such that the normalization of a general fiber of ξ is an elliptic curve. Then there is a line $L \subset X$ such that $\xi = \sigma \circ \psi$, where $\psi : X \dashrightarrow \mathbb{P}^2$ is the projection from the line L, and σ is a birational automorphism of $\mathbb{P}^2$.*

In the rest of the chapter we prove Theorem 2.1. Let us use the notations and assumptions of chapter 1. We must show that there is a line $L \subset X$ such that there is a commutative diagram

(2.2)
$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & V \arrow[d, "\nu"] \\ \mathbb{P}^2 \arrow[r, dashed, "\sigma"'] & \mathbb{P}^2, \end{tikzcd}$$

where ψ is the projection from the L, and σ is a birational map.

Lemma 2.3. *The set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ does not contain points of the quartic X*

Proof. Suppose that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains a point P of X. Then $\mathrm{mult}_P(\mathcal{M}) > k$, which implies that there are finite number of lines on X that pass through the point P. Let H be a general hyperplane section of X passing through the point P. Then it follows from Theorem C.15 that

$$4k^2 \leqslant \mathrm{mult}_P(D_1 \cdot D_2) \leqslant D_1 \cdot D_2 \cdot H = 4k^2,$$

where D_1 and D_2 are general surfaces of the linear system $\mathcal{M}$. Therefore, the support of the effective one-dimensional cycle $D_1 \cdot D_2$ is contained in the union of a finite number of lines on the quartic X that pass through the point P, which contradicts Lemma A.17. □

Therefore, the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains a curve C. Thus, the inequality $\mathrm{mult}_C(\mathcal{M}) \geqslant k$ holds, but it follows from Lemma 1.8 that $\deg(C) \leqslant 3$.

Lemma 2.4. *The curve C is contained in two-dimensional linear subspace of $\mathbb{P}^4$.*

Proof. Suppose that C is not contained in any plane in $\mathbb{P}^4$. Then C is either a smooth curve of degree 3 or 4, or a rational curve of degree 4 having one double point.

Suppose that C is smooth. Let $\alpha : U \to X$ be the blow up of the curve C, F be the the exceptional divisor of α, and $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on the variety U. Then the base locus of the linear system $|\alpha^*(-\deg(C)K_X) - F|$ does not contain curves, but

$$\Big(\alpha^*(-\deg(C)K_X) - F\Big) \cdot D_1 \cdot D_2 < 0,$$

where D_1 and D_2 are general surfaces of the linear system $\mathcal{D}$, which is a contradiction.

Thus, the curve C is a quartic curve with a double point P. Let $\beta : W \to X$ be a composition of the blow up of P with the blow up of he proper transform of C. Let G and E be the exceptional divisors of β such that $\beta(E) = C$ and $\beta(G) = P$. Then the base locus of the linear system $|\beta^*(-4K_X) - E - 2G|$ does not contain curves, but

$$\Big(\beta^*(-4K_X) - E - 2G\Big) \cdot D_1 \cdot D_2 < 0$$

where D_1 and D_2 are general surfaces of the linear system $\mathcal{D}$, which is a contradiction. $\square$

Lemma 2.5. *The curve C is a line.*

Proof. Suppose that $\deg(C) \neq 1$. Then we have the following possibilities:

- the curve C is a smooth conic;
- the curve C is a smooth plane cubic;
- the curve C is a singular plane cubic.

Suppose that C is smooth. Let $\alpha : U \to X$ be a blow up of the curve C, F be the exceptional divisor of the morphism α, and $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on the variety U. Then one can easily check that the base locus of the linear system $|\alpha^*(-\deg(C)K_X) - F|$ does not contain curves. Therefore, the divisor $\alpha^*(-\deg(C)K_X) - F$ is nef and big, but

$$\Big(\alpha^*(-\deg(C)K_X) - F\Big) \cdot D_1 \cdot D_2 = 0,$$

where D_1 and D_2 are general surfaces of $\mathcal{D}$, which contradicts Corollary A.19.

Hence, the curve C is a plane cubic with a double point P. Let $\beta : W \to X$ be a composition of the blow up of P with the blow up of the proper transform of C. Let G and E be the exceptional divisors of the morphism β such that $\beta(E) = C$ and $\beta(G) = P$. Then the base locus of the linear system $|\beta^*(-3K_X) - E - 2G|$ does not contain curves, which implies that the divisor $\beta^*(-3K_X) - E - 2G$ is nef and big. On the other hand, the inequality

$$\Big(\beta^*(-3K_X) - E - 2G\Big) \cdot D_1 \cdot D_2 \leqslant 0$$

holds, where D_1 and D_2 are general surfaces of $\mathcal{D}$, which is impossible by Corollary A.19. $\square$

Let $\pi : Y \to X$ be the blow up of C, $\psi : X \dashrightarrow \mathbb{P}^2$ be the projection from C, D be the proper transform of a general divisor of the linear system $\mathcal{M}$ on the variety Y, and Z be a general fiber of the morphism $\psi \circ \pi$. Then $D \cdot Z = 0$, because $\mathrm{mult}_C(D) = k$, which implies the existence of the commutative diagram 2.2, where $L = C$.

3. Case $n = 2$, hypersurface of degree 5 in $\mathbb{P}(1,1,1,1,2)$.

We use the notations and assumptions of chapter 1. Let $n = 2$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,1,1,1,2)$ of degree 5, the equality $-K_X^3 = 5/2$ holds, and the singularities of the hypersurface X consist of a point O that is a quotient singularity of type $\frac{1}{2}(1,1,1)$.

The hypersurface X can be given by the equation

$$w^2 f_1(x,y,z,t) + f_3(x,y,z,t)w + f_5(x,y,z,t) = 0 \subset \mathbb{P}(1,1,1,1,2) \cong \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = \mathrm{wt}(z) = \mathrm{wt}(t) = 1$ and $\mathrm{wt}(w) = 2$, $f_i(x,y,z,t)$ is a homogeneous polynomial of degree i, and the point O is given by the equations $x = y = z = t = 0$.

Let $\psi : X \dashrightarrow \mathbb{P}^3$ be the natural projection. Then there is a commutative diagram

$$\begin{tikzcd}
 & Y \arrow[dl, "\pi"'] \arrow[dr, "\gamma"] & & W_i \arrow[ll, "\alpha_i"'] \arrow[dr, "\omega_i"] & \\
X \arrow[dr, dashed, "\psi"'] & & Z \arrow[dl, "\eta"] & & U_i \arrow[ll, "\beta_i"'] \arrow[dl, "\xi_i"] \\
 & \mathbb{P}^3 \arrow[rr, dashed, "\chi_i"'] & & \mathbb{P}^2 &
\end{tikzcd}$$

where π is the weighted blow up of the point O with weights $(1,1,1)$, the morphism γ is the birational morphism that contracts 15 smooth rational curves $C_1,\ldots,C_{15}$ to 15 isolated ordinary double points $P_1,\ldots,P_{15}$ of the variety Z respectively, η is a double cover branched over the surface $R\subset\mathbb{P}^3$ of degree 6 that is given by the equation

$$f_3(x,y,z,t)^2-4f_1(x,y,z,t)f_5(x,y,z,t)=0\subset\mathbb{P}^3\cong\mathrm{Proj}\big(\mathbb{C}[x,y,z,t]\big)$$

and has 15 isolated ordinary double points $\eta(P_1),\ldots,\eta(P_{15})$, α_i is a blow up of C_i, β_i is a blow up of the point P_i, w_i is a birational morphism, χ_i is a projection from $\eta(P_i)$, and ξ_i is an elliptic fibration. Moreover, the points $\eta(P_1),\ldots,\eta(P_{15})$ are given by the equations $f_3=f_1=f_5=0$.

Remark 3.1. Let τ be the birational involution of the hypersurface X induced by the double cover η. Then it follows from [6] that the group $\mathrm{Bir}(X)$ is generated by the involution τ and biregular automorphisms of X. Moreover, a general fiber of the rational map $\chi_i\circ\psi$ is left invariant by the birational involution τ.

In the rest of the chapter we prove the following result.

Proposition 3.2. *There is a commutative diagram*

$$\begin{tikzcd} U_i \arrow[r, dashed, "\rho"] \arrow[d, "\xi_i"'] & V \arrow[d, "\nu"] \\ \mathbb{P}^2 \arrow[r, dashed, "\phi"'] & \mathbb{P}^2 \end{tikzcd} \tag{3.3}$$

for some $i\in\{1,\ldots,15\}$, where ϕ is a birational map.

Let $\mathcal{B}_i$ be the proper transform of the linear system $\mathcal{M}$ on the variety W_i. Then in order to prove the existence of the commutative diagram 3.3 it is enough to show that $\mathcal{B}_i$ lies in the fibers of the elliptic fibration $\xi_i\circ\omega_i$, which is implied by the equivalence $\mathcal{B}_i\sim_{\mathbb{Q}}-kK_{W_i}$.

Lemma 3.4. *Suppose that $\mathbb{CS}(X,\frac{1}{k}\mathcal{M})$ contains a smooth point of X. Then the commutative diagram 3.3 exists for some $i\in\{1,\ldots,15\}$.*

Proof. Suppose that $\mathbb{CS}(X,\frac{1}{k}\mathcal{M})$ contains a smooth point P of X. Let S be a sufficiently general surface in $|-K_X|$ that passes through P. In the case when $P\not\in\cup_{i=1}^{15}\pi(C_i)$, the surface S does not contain irreducible components of the effective cycle $D_1\cdot D_2$, where D_1 and D_2 are general surfaces of the linear system $\mathcal{M}$. Therefore, in the latter case we have

$$\mathrm{mult}_P(D_1\cdot D_2)\leqslant D_1\cdot D_2\cdot S=-k^2K_X^3=\frac{5}{2}k^2,$$

which contradicts Theorem C.15. Thus, the point P is contained in $\pi(C_i)$ for some i.

Let us use the arguments of the paper [6]. Put $C=\pi(C_i)$ and

$$\mathcal{M}|_S=\mathcal{L}+\mathrm{mult}_C(\mathcal{M})C,$$

where $\mathcal{L}$ is a linear system on the surface S without fixed components. Then it follows from the claim of Theorem C.10 that the point P is contained in the set

$$\mathbb{LCS}\Big(S,\frac{1}{k}\mathcal{L}+\frac{\mathrm{mult}_C(\mathcal{M})}{k}C\Big),$$

but the surface S is smooth in the point P. Therefore, we have

$$\mathrm{mult}_P(L_1\cdot L_2)\geqslant 4\Big(1-\frac{\mathrm{mult}_C(\mathcal{M})}{k}\Big)k^2$$

by Theorem C.13, where L_1 and L_2 are general curves in $\mathcal{L}$. On the other hand, the equality

$$L_1\cdot L_2=\frac{5}{2}k^2-\mathrm{mult}_C(\mathcal{M})k-\frac{3}{2}\mathrm{mult}_C^2(\mathcal{M})$$

holds on the surface S, because $C^2=-3/2$. Hence, we have

$$\frac{5}{2}k^2-\mathrm{mult}_C(\mathcal{M})k-\frac{3}{2}\mathrm{mult}_C^2(\mathcal{M})\geqslant 4\Big(1-\frac{\mathrm{mult}_C(\mathcal{M})}{k}\Big)k^2,$$

which implies $\mathrm{mult}_C(\mathcal{M}) = k$. Thus, the curve $\pi(C_i)$ is contained in $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$, and the equivalence $\mathcal{B}_i \sim_{\mathbb{Q}} -kK_{W_i}$ follows from Theorem A.15, which concludes the proof. $\square$

We may assume that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ does not contain smooth points of X.

Lemma 3.5. *Let C be a curve on X such that $C \cap \mathrm{Sing}(X) = \varnothing$. Then $C \not\in \mathbb{CS}(X, \frac{1}{k}\mathcal{M})$.*

Proof. Suppose that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains the curve C. Then $\mathrm{mult}_C(\mathcal{M}) = k$. Let H be a very ample divisor on X. Then $H \sim_{\mathbb{Q}} -\lambda K_X$ holds for a natural number λ. Thus, we have

$$\frac{5\lambda k^2}{2} = -\lambda k^2 K_X^3 = H \cdot S_1 \cdot S_2 \geqslant \mathrm{mult}_C^2(\mathcal{M}) H \cdot C \geqslant -\lambda k^2 K_X \cdot C,$$

where S_1 and S_2 are general surfaces in $\mathcal{M}$. Therefore, we have the following possibilities:

- the equality $-K_X \cdot C = 1$ holds, and the curve C is smooth and rational;
- the equality $-K_X \cdot C = 2$ holds, and the curve C is smooth and rational;
- the equality $-K_X \cdot C = 2$ holds, and the arithmetic genus of the curve C is 1.

Let $\sigma : \check{X} \to X$ be the blow up of the ideal sheaf of the curve C, and G be the exceptional divisor of the birational morphism σ. Then the variety $\check{X}$ is smooth in the neighborhood of the divisor G whenever the curve C is smooth. Moreover, in the case when the curve C has an ordinary double point the singularities of $\check{X}$ in the neighborhood of the divisor G consist of a single isolated ordinary double point. In the case when the curve C has a cuspidal singularity the singularities of the variety $\check{X}$ in the neighborhood of G consist of an isolated double point such that $\check{X}$ in the neighborhood of this point is locally isomorphic to the hypersurface

$$x_1^2 + x_2^2 + x_3^2 + x_4^3 = 0 \subset \mathbb{C} \cong \mathrm{Spec}(\mathbb{C}[x_0, x_1, x_2, x_3]).$$

Let $\check{S}_1$ and $\check{S}_2$ be the proper transforms of the surfaces S_1 and S_2 on the variety $\check{X}$ respectively.

Suppose that $-K_X \cdot C = 1$. Then the curve C is cut in the set-theoretic sense by the surfaces of the linear system $|-2K_X|$ that pass through C. Moreover, the scheme-theoretic intersection of two general surfaces of the linear system $|-2K_X|$ passing through C is reduced in a general point of C. Thus, the divisor $\sigma^*(-2K_X) - G$ is nef and big (see Lemma 5.2.5 in [6]), but

$$\Big(\sigma^*(-2K_X) - G\Big) \cdot \check{S}_1 \cdot \check{S}_2 = 0,$$

which is impossible by Corollary A.19.

Suppose that $-K_X \cdot C = 2$, and C is smooth and rational. Then $\sigma^*(-2K_X) - G$ is nef, because the curve C is cut in the set-theoretic sense by the surfaces of the linear system $|-2K_X|$ that pass through the curve C, but the scheme-theoretic intersection of two general surfaces of the linear system $|-2K_X|$ passing through$C$ is reduced in a general point of C. We have

$$0 > -3k^2 = \Big(\sigma^*(-2K_X) - G\Big) \cdot \check{S}_1 \cdot \check{S}_2 \geqslant 0.$$

Hence, the arithmetic genus of the curve C is 1 and $-K_X \cdot C = 2$. The curve C is a set-theoretic intersection of the surfaces in $|-4K_X|$ that pass through C. Moreover, the scheme-theoretic intersection of two general surfaces of the linear system $|-4K_X|$ passing through C is reduced in a general point C. Hence, the divisor $\sigma^*(-4K_X) - G$ is nef and big, but

$$\Big(\sigma^*(-4K_X) - G\Big) \cdot \check{S}_1 \cdot \check{S}_2 = 0,$$

which contradicts Corollary A.19. $\square$

It follows from Theorem A.15 that $O \in \mathbb{CS}(X, \frac{1}{k}\mathcal{M})$. Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on the variety Y. Then Theorem A.15 implies that $\mathcal{D} \sim_{\mathbb{Q}} -kK_Y$. The existence of the commutative diagram 3.3 is clear, if the set $\mathbb{CS}(Y, \frac{1}{k}\mathcal{D})$ contains the curve C_i. Therefore, we may assume that the set $\mathbb{CS}(Y, \frac{1}{k}\mathcal{D})$ does not contain the curve C_i for every $i \in \{1, \ldots, 15\}$.

Lemma 3.6. *The set $\mathbb{CS}(Y, \frac{1}{k}\mathcal{D})$ does not contain smooth points of the variety Y.*

Proof. The set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ does not contain smooth points of X. Therefore, to conclude the proof it is enough to show that the set $\mathbb{CS}(Y, \frac{1}{k}\mathcal{D})$ does not contain points of the exceptional divisor of the morphism π, which is implied by Lemma A.16. $\square$

Put $\mathcal{H} = \gamma(\mathcal{D})$. Then $\mathcal{H} \sim_{\mathbb{Q}} -kK_Z$, the singularities of the log pair $(Z, \frac{1}{k}\mathcal{H})$ are canonical, and the claim of Lemma B.5 implies that the set $\mathbb{CS}(Z, \frac{1}{k}\mathcal{H})$ is not empty.

Lemma 3.7. *The set $\mathbb{CS}(Z, \frac{1}{k}\mathcal{H})$ does not contain points of the variety Z.*

Proof. It follows from Lemma 3.6 that smooth points of the variety Z are not contained in the set $\mathbb{CS}(Z, \frac{1}{k}\mathcal{H})$. The condition $P_i \in \mathbb{CS}(Z, \frac{1}{k}\mathcal{H})$ implies that the set $\mathbb{CS}(Y, \frac{1}{k}\mathcal{D})$ contains either the curve C_i, or a point on the curve C_i, which is impossible. □

Thus, there is a curve Γ on Z that is contained in the set $\mathbb{CS}(Z, \frac{1}{k}\mathcal{H})$, and $\mathrm{mult}_{\Gamma}(\mathcal{H}) = k$.

Lemma 3.8. *The equality $-K_Z \cdot \Gamma = 1$ holds.*

Proof. Let H be a general divisor of the linear system $|-K_Z|$. Then

$$2k^2 = H \cdot D_1 \cdot D_2 \geqslant \mathrm{mult}_{\Gamma}(D_1 \cdot D_2) H \cdot \Gamma \geqslant -k^2 K_Z \cdot \Gamma,$$

where D_1 and D_2 are sufficiently general surfaces of the linear system $\mathcal{H}$. Therefore, the inequality $-K_Z \cdot \Gamma \leqslant 2$ holds. Moreover, the equality $-K_Z \cdot \Gamma = 2$ implies that the support of the effective cycle $D_1 \cdot D_2$ coincides with the curve Γ, which contradicts Lemma A.17. □

The curve $\eta(\Gamma)$ is a line in $\mathbb{P}^3$, and the restriction $\eta|_{\Gamma} : \Gamma \to \eta(\Gamma)$ is an isomorphism, and the proof of Lemma 3.8 implies that $\mathbb{CS}(Z, \frac{1}{k}\mathcal{H}) = \{\Gamma\}$, and the inequality

$$\mathrm{mult}_{\Gamma}(D_1 \cdot D_2) < 2k^2$$

holds, where D_1 and D_2 are sufficiently general surfaces of the linear system $\mathcal{H}$.

Lemma 3.9. *The line $\eta(\Gamma)$ is contained in the ramification surface R of the double cover η.*

Proof. Suppose that the line $\eta(\Gamma)$ is not contained in the surface R. Let S be a general surface of the linear system $|-K_Z|$ that passes through the curve Γ. Then

$$\mathcal{H}|_S = \mathrm{mult}_{\Gamma}(\mathcal{H})\Gamma + \mathrm{mult}_{\Omega}(\mathcal{H})\Omega + \mathcal{L},$$

where $\mathcal{L}$ is a linear system on the surface S that does not have fixed components, and Ω is a smooth rational curve on the variety Z such that $\eta(\Omega) = \eta(\Gamma)$, but $\Omega \neq \Gamma$. We have

$$\mathrm{Sing}(Z) \cap \Gamma = \{P_{i_1}, \ldots, P_{i_r}\} \subsetneq \{P_1, \ldots, P_{15}\} = \mathrm{Sing}(Z),$$

but P_{i_j} is an ordinary double point of the surface S. Moreover, the equality

$$\Gamma^2 = \Omega^2 = -2 + \frac{r}{2}$$

holds on the surface S, but $r \leqslant 3$. Hence, the inequality $\Omega^2 < 0$ holds on the surface S, and

$$\Big(k - \mathrm{mult}_{\Omega}(\mathcal{H})\Big)\Omega^2 = \Big(\mathrm{mult}_{\Gamma}(\mathcal{H}) - k\Big)\Gamma \cdot \Omega + L \cdot \Omega = L \cdot \Omega \geqslant 0,$$

where L is a general curve in the linear system $\mathcal{L}$. Therefore, the inequality $\mathrm{mult}_{\Omega}(\mathcal{H}) \geqslant k$ holds, which implies that $\Omega \in \mathbb{CS}(Z, \frac{1}{k}\mathcal{H})$, which is impossible. □

Let H be a general hyperplane in $\mathbb{P}^3$ passing through the line $\eta(\Gamma)$. Then the curve

$$\Delta = H \cap R = \eta(\Gamma) \cup \Upsilon$$

is reduced and $\eta(\Gamma) \not\subset \mathrm{Supp}(\Upsilon)$, where Υ is a plane curve of degree 5. Moreover, the reducible curve Δ is singular in every singular point $\eta(P_i)$ of the surface R that lies on the line $\eta(\Gamma)$, but the set $\eta(\Gamma) \cap \Upsilon$ contains at most 5 points. On the other hand, we have

$$\mathrm{Sing}(\Delta) \cap \eta(\Gamma) = \Upsilon \cap \eta(\Gamma),$$

which implies that $|\mathrm{Sing}(Z) \cap \Gamma| \leqslant 5$. Moreover, the surface R is given by the equation

$$f_3(x, y, z, t)^2 = 4f_1(x, y, z, t)f_5(x, y, z, t) \subset \mathbb{P}^3 \cong \mathrm{Proj}\big(\mathbb{C}[x, y, z, t]\big),$$

and singular points of R are given by the equations $f_1 = f_3 = f_5 = 0$. We may assume that the equations $f_1 = f_3 = 0$ and $f_1 = f_5 = 0$ defines irreducible curves in $\mathbb{P}^3$, which implies that at most 3 points of the subset $\mathrm{Sing}(R) \subset \mathbb{P}^3$ can lie on a single line. Therefore, the Bertini theorem implies that $\eta(\Gamma) \cap \Upsilon$ contains different points O_1 and O_2 that are not contained in $\mathrm{Sing}(R)$.

Remark 3.10. The hyperplane H tangents the surface R in the points O_1 and O_2.

Let L_j be a general line on the plane H that passes through the point O_j, $\tilde{O}_j$ be a proper transform of the point O_j on the variety Z, and $\tilde{L}_j$ be the proper transform of the line L_j on the variety Z. Then L_j tangents the surface R in the point O_j, and the curve $\tilde{L}_j$ is irreducible and singular in the point $\tilde{O}_j$, but $-K_Z \cdot \tilde{L}_j = 2$. Let $\tilde{H}$ be the proper transform of H on the variety Z, and D be a general surface of the linear system $\mathcal{H}$. Then

$$D|_{\tilde{H}} = \mathrm{mult}_{\Gamma}(\mathcal{H})\Gamma + \Psi,$$

where Ψ is an effective divisor on $\tilde{H}$ such that $\Gamma \not\subset \mathrm{Supp}(\Psi)$. Let $\Lambda_j = (D\backslash\Gamma) \cap \tilde{L}_j$. Then

$$2k = D\cdot\tilde{L}_j \geqslant \mathrm{mult}_{\tilde{O}_j}(\tilde{L}_j)\mathrm{mult}_{\Gamma}(D) + \sum_{P\in\Lambda_j} \mathrm{mult}_P(D)\cdot\mathrm{mult}_P(\tilde{L}_j) \geqslant 2k + \sum_{P\in\Lambda_j} \mathrm{mult}_P(D)\cdot\mathrm{mult}_P(\tilde{L}_j),$$

which implies that $D \cap \tilde{L}_j \subset \Gamma$. On the other hand, when we vary the lines L_1 and L_2 on the plane H the curves $\tilde{L}_1$ and $\tilde{L}_2$ span two different pencils on the surface $\tilde{H}$, whose base locus consist of the points $\tilde{O}_1$ and $\tilde{O}_1$ respectively. Hence, we have

$$\mathrm{Supp}(D) \cap \mathrm{Supp}(\tilde{H}) = \mathrm{Supp}(\Gamma),$$

where $\tilde{H}$ is a general divisor in $|-K_Z|$ that passes through the curve Γ, and D is a general divisor in $\mathcal{H}$, which contradicts Lemma A.20. The claim of Proposition 3.2 is proved[5].

4. Case $n = 4$, hypersurface of degree 6 in $\mathbb{P}(1, 1, 1, 2, 2)$.

We use the notations and assumptions of chapter 1. Let $n = 4$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1, 1, 1, 2, 2)$ of degree 6, the equality $-K_X^3 = 3/2$ holds, and the singularities of the hypersurface X consist of points P_1, P_2, P_3 that are quotient singularities of types $\frac{1}{2}(1, 1, 1)$.

Let $\psi : X \dashrightarrow \mathbb{P}^2$ be the natural projection. Then a general fiber of the rational map ψ is an elliptic curve, and the composition $\psi \circ \eta$ is a morphism, where $\eta : Y \to X$ is a composition of the weighted blow ups of the singular points P_1, P_2 and P_3 with weights $(1, 1, 1)$.

Proposition 4.1. *The claim of Proposition 0.7 holds for the hypersurface X.*

Let us prove Proposition 4.1. We must show the existence of the commutative diagram

(4.2)
$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & V \arrow[d, "\nu"] \\ \mathbb{P}^2 \arrow[r, dashed, "\phi"'] & \mathbb{P}^2, \end{tikzcd}$$

where $\phi \in \mathrm{Bir}(\mathbb{P}^2)$. Let Z be an element of $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$. We have the following possibilities:

- the subvariety Z is a curve that is contained in $X \setminus \mathrm{Sing}(X)$;
- the subvariety Z is a curve that contains a singular point of the hypersurface X;
- the subvariety Z is a singular point of the hypersurface X.

Suppose that Z is an irreducible curve such that Z does not contain singular points of the hypersurface X. Then the equality $-K_X \cdot Z = 1$ holds by Lemma 1.8, which implies that the curve Z is smooth. Let $\gamma : W \to X$ be the blow up of the curve Z, and G be the exceptional divisor of the morphism γ. Then the divisor $\gamma^*(-4K_X) - G$ is nef, which implies that

$$(\gamma^*(-4K_X) - G) \cdot \bar{S}_1 \cdot \bar{S}_2 \geqslant 0,$$

where $\bar{S}_1$ and $\bar{S}_2$ are proper transforms on the variety W of sufficiently general surfaces of the linear system $\mathcal{M}$. We have $0 \leqslant (\gamma^*(-4K_X) - G) \cdot \bar{S}_1 \cdot \bar{S}_2 = -k^2$, which is a contradiction.

Suppose that Z is an irreducible curve that passes through some singular point of the hypersurface X. Then the inequality $-K_X \cdot Z \leqslant 1$ holds by Lemma 1.8. The curve C is contracted by the rational map ψ to a point, and either $-K_X \cdot Z = 1/2$, or $-K_X \cdot Z = 1$.

[5] It is easy to see that the given proof implies the claim of Proposition 3.2 under the weaker assumption that the hypersurface X is quasismooth, and the projection $\psi : X \dashrightarrow \mathbb{P}^3$ contracts 15 different curves.

Let F be a sufficiently general surface of the linear system $|-K_X|$ that passes through the curve Z. Then the surface F is smooth outside the points P_1, P_2 and P_3, which are isolated ordinary double points of F. Let $\tilde{Z}$ be a fiber of ψ over the point $\psi(Z)$. Then the generality of the hypersurface X implies that the curve Z is an irreducible component of $\tilde{Z}$.

Suppose that $\tilde{Z}$ consists of irreducible curves Z and $\bar{Z}$. Then the inequality $\bar{Z}^2<0$ holds on the surface F, but $\mathcal{M}|_F \sim_{\mathbb{Q}} kZ+k\bar{Z}$. On the other hand, we have

$$\mathcal{M}|_F = \mathrm{mult}_Z(\mathcal{M})Z+\mathrm{mult}_{\bar{Z}}(\mathcal{M})\bar{Z}+\mathcal{F},$$

where $\mathcal{F}$ is a linear system on the surface F that does not have fixed components, but

$$\big(k-\mathrm{mult}_{\bar{Z}}(\mathcal{M})\big)\bar{Z}\sim_{\mathbb{Q}}\mathcal{F},$$

which implies that $\mathrm{mult}_{\bar{Z}}(\mathcal{M})=k$, and the support of $S_1\cdot S_2$ is contained in $Z\cup\bar{Z}$, which is impossible by Lemma A.17.

Suppose that the fiber $\tilde{Z}$ consists of irreducible curves Z, $\hat{Z}$ and $\check{Z}$. Then

$$-K_X\cdot\hat{Z}=-K_X\cdot\check{Z}=-K_X\cdot Z=\frac{1}{2},$$

but the intersection form of the curves $\hat{Z}$ and $\check{Z}$ on the surface F is negatively defined, which implies that the support of $S_1\cdot S_2$ is contained in $Z\cup\hat{Z}\cup\check{Z}$, where S_1 and S_2 are general surfaces of the linear system $\mathcal{M}$, which is impossible by Lemma A.17.

Hence, we have $\mathbb{CS}(X,\frac{1}{k}\mathcal{M})\subseteq\{P_1,P_2,P_3\}$.

Let $\pi:U\to X$ be a composition of the weighted blow ups with weights $(1,1,1)$ of the singular points of the hypersurface X that are contained in the set $\mathbb{CS}(X,\frac{1}{k}\mathcal{M})$, and $\mathcal{D}$ be the proper transform of the linear system $\mathcal{M}$ on the variety U. Then it follows from Theorem A.15 that the equivalence $\mathcal{D}\sim_{\mathbb{Q}}-kK_U$ holds, but the divisor $-K_U$ is nef.

Suppose that $\mathbb{CS}(U,\frac{1}{k}\mathcal{D})$ contains a subvariety $\Delta\subset U$. Then Δ is contained in some exceptional divisor of π. Let G be an exceptional divisor of π that contains Δ. Then Δ is a line on the surface $G\cong\mathbb{P}^2$ by Lemma A.16. The linear system $|\pi^*(-2K_X)-G|$ does not have base points, and the divisor $\pi^*(-2K_X)-G$ is nef and big. It follows from Lemma 0.3.3 in [10] that there is a proper Zariski closed subset $\Lambda\subset U$ such that Λ contains all curves on U having trivial intersections with the divisor $\pi^*(-2K_X)-G$. We have

$$2k^2=\Big(\pi^*(-2K_X)-G\Big)\cdot\tilde{S}_1\cdot\tilde{S}_2\geqslant\mathrm{mult}^2_{\Delta}(\mathcal{D})\Big(\pi^*(-2K_X)-G\Big)\cdot\Delta=2k^2,$$

where $\tilde{S}_1$ and $\tilde{S}_2$ are the proper transforms of S_1 and S_2 on U respectively. Thus, the support of the effective cycle $\tilde{S}_1\cdot\tilde{S}_2$ is contained in $\Lambda\cup\Delta$, which is impossible by Lemma A.17.

Hence, the set $\mathbb{CS}(U,\frac{1}{k}\mathcal{D})$ is empty.

Let $-K_U^3>0$. Then the divisor K_U is nef and big, which is impossible by Lemma B.5.

Hence, the equality $-K_U^3=0$ holds. Then $\pi=\eta$ and

$$-K_U\sim(\psi\circ\pi)^*\big(\mathcal{O}_{\mathbb{P}^2}(1)\big),$$

which implies that $\mathcal{D}$ lies in the fibers of the elliptic fibration $\psi\circ\pi$. Therefore, the commutative diagram 4.2 exists. The claim of Proposition 4.1 is proved.

5. Case $n=6$, hypersurface of degree 8 in $\mathbb{P}(1,1,1,2,4)$.

We use the notations and assumptions of chapter 1. Let $n=6$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,1,1,2,4)$ of degree 8, the equality $-K_X^3=1$ holds, and the singularities of the hypersurface X consists of points P_1 and P_2 that are quotient singularities of type $\frac{1}{2}(1,1,1)$.

There is a commutative diagram

(5.1)
$$\begin{tikzcd} & U \arrow[dl, "\pi"'] \arrow[dr, "\eta"] & \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}^2, \end{tikzcd}$$

where ψ is the natural projection, π is a composition of the weighted blow ups of the singular points P_1 and P_2 with weights $(1,1,1)$, and η is an elliptic fibration.

Proposition 5.2. *The claim of Proposition 0.7 holds for the hypersurface X.*

Let us prove Proposition 5.2. It follows from Theorem 1.7 that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ does not contain smooth points of the hypersurface X. Therefore, the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains a singular point of the hypersurface X by Corollary 1.9 and Theorem A.15.

Remark 5.3. Suppose that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains both points P_1 and P_2. Then it easily follows from Theorem A.15 that the claim of the Proposition 0.7 holds for the hypersurface X.

Hence, to conclude the proof we may assume that P_1 is contained in the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$, but the point P_2 is not contained in the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$.

Lemma 5.4. *The set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ does not contain curves.*

Proof. Suppose that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains a curve C. Then it follows from Lemma 1.8 that the equality $-K_X \cdot C = 1/2$ holds.

Let $\check{C}$ be the proper transform of the curve C on the variety U. Then $-K_U \cdot \check{C} = 0$, which implies that the curve $\check{C}$ is a component of a fiber of η. Therefore, the curve C is contracted by the rational map $\psi : X \dashrightarrow \mathbb{P}^2$ to a point. In particular, the curve C is smooth and rational.

Let S be a general surface of the linear system $|-K_X|$ that contains the curve C. Then the surface S is smooth outside the points P_1 and P_2, which are isolated ordinary double points on the surface S. Let F be a fiber of the rational map ψ over the point $\psi(C)$. Then F consists of two irreducible components such that the curve C is one of them. Let Z be the component of the curve F that is different from the curve C. Then $Z^2 < 0$ on the surface S, but

$$\mathcal{M}|_S = \mathrm{mult}_C(\mathcal{M})C + \mathrm{mult}_Z(\mathcal{M})Z + \mathcal{L},$$

where $\mathcal{L}$ is a linear system on the surface S that does not have fixed components. However, the equivalence $(k - \mathrm{mult}_Z(\mathcal{M}))Z \sim_{\mathbb{Q}} \mathcal{L}$ holds, which implies that $\mathrm{mult}_Z(\mathcal{M}) = k$.

It follows from Theorem A.15 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains P_2, which is a contradiction. □

Thus, the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ consists of the point P_1. Let $\zeta : Y \to X$ be the weighted blow up of the point P_1 with weights $(1, 1, 1)$, and $\mathcal{D}$ be the proper transform of the linear system $\mathcal{M}$ on the variety Y. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_Y$ by Theorem A.15, and $\mathbb{CS}(Y, \frac{1}{k}\mathcal{D}) \neq \varnothing$ by Lemma B.5.

Let T be a subvariety of Y that is contained in $\mathbb{CS}(Y, \frac{1}{k}\mathcal{D})$. Then $\zeta(T) \in \mathbb{CS}(X, \frac{1}{k}\mathcal{M})$, which implies that $T \subset G$, where G is the exceptional divisor of ζ. The subvariety T is a line on the surface $G \cong \mathbb{P}^2$ by Lemma A.16. Let D be a general surface in $\mathcal{D}$, and Z be a general fiber of the rational map $\psi \circ \zeta$ that intersects T. Then the Z is not contained in $\mathrm{Supp}(D)$. We have

$$\frac{k}{2} = D \cdot F \geqslant \mathrm{mult}_T(D)\mathrm{mult}_T(Z) \geqslant k,$$

which is a contradiction. Thus, the claim of Proposition 5.2 is proved.

6. Case $n = 7$, hypersurface of degree 8 in $\mathbb{P}(1, 1, 2, 2, 3)$.

We use the notations and assumptions of chapter 1. Let $n = 7$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1, 1, 2, 2, 3)$ of degree 8, the equality $-K_X^3 = 2/3$ holds, and the singularities of the hypersurface X consist of the points P_1, P_2, P_3 and P_4 that are quotient singularities of type $\frac{1}{2}(1, 1, 1)$, and the point Q that is a quotient singularity of type $\frac{1}{3}(1, 1, 2)$.

The hypersurface X can be given quasihomogeneous equation of degree 8

$$w^2 f(x, y, z, t) + wg(x, y, z, t) + xa(x, y, z, t) + yb(x, y, z, t) + h(z, t) = 0 \subset \mathrm{Proj}\big(\mathbb{C}[x, y, z, t, w]\big),$$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = 1$, $\mathrm{wt}(z) = \mathrm{wt}(t) = 2$, $\mathrm{wt}(w) = 3$, and f, g, a, b and h are sufficiently general quasihomogeneous polynomials. Then the points P_1, P_2, P_3 and P_4 are given by the equations $x = y = h(z, t) = w = 0$, and Q is given by the equations $x = y = z = t = 0$.

We may assume that $h(z, t) = z\bar{h}(z, t)$, where $\bar{h}(z, t)$ is a polynomial of degree 3, and the point P_1 is given by the equations $x = y = z = w = 0$. Let $\xi_1 : X \dashrightarrow \mathbb{P}(1, 1, 2)$ be a projection induced by the embedding of the graded algebras $\mathbb{C}[x, y, z] \subset \mathbb{C}[x, y, z, t, w]$. Then ξ_1 is not defined in the points P_1 and Q, and the normalization of a general fiber of ξ_1 is a smooth elliptic curve (see the proof of Lemma 0.5).

Similarly, we can construct a rational map $\xi_i : X \dashrightarrow \mathbb{P}(1,1,2)$ such that ξ_i is not defined in the singular points P_i and Q, and the normalization of a general fiber of the map ξ_i is a smooth elliptic curve. Moreover, there is a commutative diagram

$$\begin{tikzcd}
 & & Y_i \arrow[dll, "\beta_i"'] \arrow[dd, "\eta_i"] \\
U_i \arrow[dr, "\alpha_i"'] & & \\
 & X \arrow[r, dashed, "\xi_i"'] & \mathbb{P}(1,1,2),
\end{tikzcd}$$

where α_i is the weighted blow up of the point P_i with weights $(1,1,1)$, β_i is the weighted blow up of the proper transform of the point Q on the variety U_i with weights $(1,1,2)$, and η_i is an elliptic fibration, which is a morphism induced by the linear system $|-2K_{Y_i}|$.

Let $\xi_0 : X \dashrightarrow \mathbb{P}^3$ be a rational map that is induced by the linear subsystem of the linear system $|-2K_X|$ consisting of the divisors that are cut on the hypersurface X by the equation

$$\lambda_0 f(x,y,z,t) + \lambda_1 x^2 + \lambda_2 xy + \lambda_3 y^2 = 0,$$

where $(\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in \mathbb{P}^3$. Then ξ_0 is not defined in the point Q, the normalization of a general fiber of the map ξ_0 is an elliptic curve, and there is a commutative diagram

$$\begin{tikzcd}
U_0 \arrow[d, "\alpha_0"'] & Y_0 \arrow[l, "\beta_0"'] \arrow[dr, "\eta_0"] & \\
X \arrow[rr, dashed, "\xi_0"'] & & \mathbb{P}(1,1,2),
\end{tikzcd}$$

where α_0 is the weighted blow up of the point Q with weights $(1,1,2)$, β_0 is the weighted blow up with weights $(1,1,1)$ of the singular point of U_0 that dominates the point Q, and η_0 is an elliptic fibration.

Proposition 6.1. *There is a commutative diagram*

(6.2)
$$\begin{tikzcd}
X \arrow[r, dashed, "\sigma"] \arrow[d, dashed, "\xi_i"'] & X \arrow[r, dashed, "\rho"] & V \arrow[d, "\nu"] \\
\mathbb{P}(1,1,2) \arrow[rr, dashed, "\phi"'] & & \mathbb{P}^2
\end{tikzcd}$$

for some $i \in \{0,1,2,3,4\}$, where σ and ϕ are birational maps.

Let us prove Proposition 6.1. The set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ does not contains smooth points of the hypersurface X, and it follows from Lemma 1.16 that

$$\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) \subseteq \{P_1, P_2, P_3, P_4, Q\}.$$

Lemma 6.3. *The set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ does not contain two points of the set $\{P_1, P_2, P_3, P_4\}$.*

Proof. Suppose that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains the points P_1 and P_2. Let $\pi : W \to X$ be the composition of the weighted blow ups of P_1 and P_2 with weights $(1,1,1)$, and $\mathcal{B}$ be the proper transform of the linear system $\mathcal{M}$ on the variety W. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ by Theorem A.15.

The base locus of $|-K_W|$ consists of an irreducible curve C such that $C^2 = -1/3$ on the surface S, which implies that $\pi(C)$ is contained in $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$, which is a contradiction. ◻

Let $\mathcal{D}_i$ be the proper transform of the linear system $\mathcal{M}$ on the variety U_i.

Remark 6.4. The set $\mathbb{CS}(U_0, \frac{1}{k}\mathcal{D}_0)$ is not empty by Lemma B.5, if the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains the point Q, because the divisor $-K_{U_0}$ is nef and big. Similarly, the set $\mathbb{CS}(U_i, \frac{1}{k}\mathcal{D}_i)$ is not empty, if the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains the point P_i, because the divisor $-K_{U_i}$ is nef and big.

Lemma 6.5. *The set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ does not consists of the point P_i.*

Proof. Suppose that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_i\}$. Then the set $\mathbb{CS}(U_i, \frac{1}{k}\mathcal{D}_i)$ contains an irreducible subvariety Z of the variety U_i by Lemma B.5.

Let E_i be the exceptional divisor of the morphism α_i. Then $Z \subsetneq E_i$, and Z is a line on the surface $E_i \cong \mathbb{P}^2$ by Lemma A.16. Let D be a general surface of $\mathcal{D}_i$, and C be a general fiber of η_i such that $\beta_i(C) \cap Z \neq \varnothing$. Then $\beta_i(C)$ is not contained in D, but $D \cdot \beta_i(C) = k/3$, which contradicts the equality $\mathrm{mult}_Z(D) = k$. □

Therefore, the condition $P_i \in \mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ implies that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_i, Q\}$, which easily implies that the proper transform of the linear system $\mathcal{M}$ on the variety Y_i lies in the fibers of the elliptic fibration η_i. The latter implies the existence of the commutative diagram 6.2.

We may assume that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ consists of the point Q.

Let O be a singular point of the variety U_0 that is contained in the exceptional divisor of the morphism α_0. Then O is contained in $\mathbb{CS}(U_0, \frac{1}{k}\mathcal{D}_0)$ by Lemma A.16, which implies the existence of the commutative diagram 6.2 by Theorem A.15. The claim of Proposition 6.1 is proved.

7. Case $n = 8$, hypersurface of degree 9 in $\mathbb{P}(1,1,1,3,4)$.

We use the notations and assumptions of chapter 1. Let $n = 8$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,1,1,3,4)$ of degree 9, the equality $-K_X^3 = 3/4$ holds, and the singularities of the hypersurface X consist of the singular point O that is a quotient singularity of type $\frac{1}{4}(1,1,3)$.

Proposition 7.1. *The claim of Proposition 0.7 holds for the hypersurface X.*

Let us prove Proposition 7.1. The hypersurface X can be given by the equation

$$w^2 f_1(x,y,z) + f_5(x,y,z,t)w + f_9(x,y,z,t) = 0 \subset \mathbb{P}(1,1,1,3,5) \cong \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = \mathrm{wt}(z) = 1$, $\mathrm{wt}(t) = 3$, $\mathrm{wt}(w) = 4$, and f_i is a quasihomogeneous polynomial of degree i. Thus, the singular point O is given by the equations $x = y = z = t = 0$.

There is a commutative diagram

$$\begin{tikzcd}
 & & W \arrow[dll, "\sigma"'] \arrow[d, "\alpha"] & U \arrow[l, "\beta"'] & \\
Y \arrow[dr, "\omega"'] & & X \arrow[dl, dashed, "\xi"'] \arrow[dr, dashed, "\psi"] & & Z \arrow[ul, "\gamma"'] \arrow[dl, "\eta"] \\
 & \mathbb{P}(1,1,1,3) \arrow[rr, dashed, "\chi"'] & & \mathbb{P}^2, &
\end{tikzcd}$$

where rational maps ξ, ψ and χ are the natural projections, α is the weighted blow up of the singular point O with weights $(1,1,3)$, β is the weighted blow up with weights $(1,1,2)$ of the singular point of the variety W that is a quotient singularity of type $\frac{1}{3}(1,1,2)$, γ is the weighted blow up with weights $(1,1,1)$ of the singular point of the variety U that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, η is an elliptic fibration, σ is a morphism that contracts 15 smooth rational curves to 15 isolated ordinary double points $P_1, \ldots, P_{15}$ of the variety Y respectively, ω is a double cover branched over the surface $R \subset \mathbb{P}(1,1,1,3)$ that is given by the equation

$$f_5(x,y,z,t)^2 - 4f_1(x,y,z)f_9(x,y,z,t) = 0 \subset \mathbb{P}(1,1,1,3) \cong \mathrm{Proj}\big(\mathbb{C}[x,y,z,t]\big)$$

and has 15 isolated ordinary double points $\omega(P_1), \ldots, \omega(P_{15})$ given by $f_5 = f_1 = f_9 = 0$.

Let G be the exceptional divisor of the morphism α, F be the exceptional divisor of the morphism β, P be the singular point of W, Q be the singular point of U, $\mathcal{D}$ be the proper transform of the linear system $\mathcal{M}$ on the variety W, $\mathcal{H}$ be the proper transform of the linear system $\mathcal{M}$ on the variety U, $\mathcal{P}$ be the proper transform of $\mathcal{M}$ on the variety Z, and $\mathcal{B}$ be the proper transform of the linear system $\mathcal{M}$ on the variety Y.

Remark 7.2. The divisors $-K_W$ and $-K_U$ are nef and big, and surface $\omega \circ \sigma(G)$ is given by the equation $f_1(x,y,z) = 0$.

It follows from Theorem 1.7 that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ does not contain smooth points of the hypersurface X, and it follows from Lemma 1.9 and Theorem A.15 that $O \in \mathbb{CS}(X, \frac{1}{k}\mathcal{M})$.

Lemma 7.3. *The set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ does not contain curves.*

Proof. The required claim follows from Lemma 1.16 and the proof of Proposition 3.2. □

Therefore, the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ consists of the singular point O. Hence, the claim of Theorem A.15 implies that $\mathcal{D} \sim_{\mathbb{Q}} -kK_W$, but $P \in \mathbb{CS}(W, \frac{1}{k}\mathcal{D})$ by Lemmas B.5 and A.16.

Lemma 7.4. *Suppose that the set $\mathbb{CS}(W, \frac{1}{k}\mathcal{D})$ does not contain subvarieties of W that are different from the point P. Then the claim of Proposition 0.7 holds for the hypersurface X.*

Proof. The equivalence $\mathcal{H} \sim_{\mathbb{Q}} -kK_U$ holds by Theorem A.15. Hence, the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{H})$ contains the point Q by Lemmas B.5 and A.16. Thus, the linear system $\mathcal{P}$ lies in the fibers of the elliptic fibration η by Theorem A.15, which concludes the proof. □

Suppose that $\mathbb{CS}(W, \frac{1}{k}\mathcal{D})$ contains a subvariety Z different from P. Then Z is contained in the surface G, which is a cone over the smooth rational curve of degree 3. Moreover, it follows from Lemma A.16 that Z is a ruling of G. Put $C = \sigma(Z)$. Then $C \in \mathbb{CS}(Y, \frac{1}{k}\mathcal{B})$ and $-K_Y \cdot C = 1/3$.

Remark 7.5. The variety $\mathbb{P}(1,1,1,3)$ is a cone over the Veronese surface, whose vertex is the point $\omega \circ \sigma(P)$. Thus, the curve $\omega(C)$ is a ruling of the cone $\mathbb{P}(1,1,1,3)$.

There is the irreducible smooth rational curve L on the variety Y such that L is different from the curve C, and $\omega(Z) = \omega(C)$. The curve L is not contained in $\mathbb{CS}(Y, \frac{1}{k}\mathcal{B})$. Let S be a general surface of the linear system $|-K_Y|$ that contains C. Then S is smooth outside of the points

$$\sigma(P) \cup \Big(\{P_1, \ldots, P_{15}\} \cap (C \cup L)\Big),$$

the point $\sigma(P)$ is a Du Val singular point of type $\mathbb{A}_2$ on the surface S, but the other singular points of the surface S are isolated ordinary double points. The equality

$$L^2 = -4/3 + r/2$$

holds on the surface S, where r is the number of ordinary double points of the surface S contained in the curve L. The generality of the polynomials f_5 and f_9 implies that the curve $\omega(L)$ does not contains more than one singular point of R. Therefore, the inequality $L^2 < 0$ holds on the surface S, but the equivalence $\mathcal{B}|_S \sim_{\mathbb{Q}} kC + kL$ holds. We have

$$\mathcal{B}|_S = \mathrm{mult}_C(\mathcal{B})C + \mathrm{mult}_L(\mathcal{B})L + \mathcal{L},$$

where $\mathcal{L}$ is a linear system on S without fixed components. Thus, we have $(k - \mathrm{mult}_L(\mathcal{B}))L \sim_{\mathbb{Q}} \mathcal{L}$, which implies $L \in \mathbb{CS}(Y, \frac{1}{k}\mathcal{B})$, which is a contradiction. The claim of Proposition 7.1 is proved.

8. Case $n = 9$, hypersurface of degree 9 in $\mathbb{P}(1,1,2,3,3)$.

We use the notations and assumptions of chapter 1. Let $n = 9$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,1,2,3,3)$ of degree 9, the equality $-K_X^3 = 1/2$ holds, and the singularities of the hypersurface X consist of the point O that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, and the points P_1, P_2 and P_3 that are quotient singularities of type $\frac{1}{3}(1,1,2)$.

The hypersurface X can be given by the quasihomogeneous equation of degree 9

$$f(t,w) + \sum_{i=1}^{4} z^i g_i(x,y,t,w) + xa(x,y,t,w) + yb(x,y,t,w) = 0 \subset \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = 1$, $\mathrm{wt}(z) = 2$, $\mathrm{wt}(t) = \mathrm{wt}(w) = 3$, f, g_i, a and b are quasihomogeneous polynomials, the points P_1, P_2 and P_3 are given by the equations $x = y = z = f(t,w) = 0$, and the point Q is given by the equations $x = y = t = w = 0$.

Let $\xi : X \dashrightarrow \mathbb{P}(1,1,2)$ be the natural projection. Then the rational map ξ is not defined in the points P_1, P_2 and P_3, and a general fiber of ξ is an elliptic curve. Moreover, there is a commutative diagram

$$\begin{tikzcd}
 & U_{ij} \arrow[dl, "\beta_{ij}"'] & & Y \arrow[ll, "\gamma_{ij}"'] \arrow[dd, "\eta"] \\
U_i \arrow[dr, "\alpha_i"'] & & & \\
 & X \arrow[rr, dashed, "\xi"'] & & \mathbb{P}(1,1,2),
\end{tikzcd}$$

where α_i is the weighted blow up of the point P_i with weights $(1,1,2)$, β_{ij} is the weighted blow up with weights $(1,1,2)$ of the proper transform of the point P_j on the variety U_i, γ_{ij} is the weighted blow up with weights $(1,1,2)$ of the proper transform of P_k on the U_{ij}, and η is an elliptic fibration, where $i \neq j$ and $k \notin \{i,j\}$.

Remark 8.1. The fibration η is a morphism induced by the linear system $|-2K_Y|$, and the divisors $-K_{U_i}$ and $-K_{U_{ij}}$ are nef and big.

Let $\chi : X \dashrightarrow \mathbb{P}^4$ be the rational map that is given by the linear system of divisors

$$\lambda_0 z g_4(x,y) + \lambda_1 x^3 + \lambda_2 x^2 y + \lambda_3 x y^2 + \lambda_4 y^3 = 0,$$

where $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{P}^4$. Then χ is not defined in the point Q, the closure of the image of the rational map χ is a cone over the rational curve in $\mathbb{P}^3$ of degree 3, the normalization of a general fiber of the rational map χ is an elliptic curve, and there is a commutative diagram

$$\begin{tikzcd}
 & Z \arrow[dl, "\pi"'] \arrow[dr, "\sigma"] & \\
X \arrow[rr, dashed, "\chi"'] & & \mathbb{P}(1,1,3),
\end{tikzcd}$$

where π is the weighted blow up of Q with weights $(1,1,1)$, and σ is an elliptic fibration.

Proposition 8.2. *Either there is a commutative diagram*

(8.3)
$$\begin{tikzcd}
X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\xi"'] & V \arrow[d, "\nu"] \\
\mathbb{P}(1,1,2) \arrow[r, dashed, "\phi"'] & \mathbb{P}^2,
\end{tikzcd}$$

or there is a commutative diagram

(8.4)
$$\begin{tikzcd}
X \arrow[r, dashed, "\omega"] \arrow[d, dashed, "\chi"'] & X \arrow[r, dashed, "\rho"] & V \arrow[d, "\nu"] \\
\mathbb{P}(1,1,3) \arrow[rr, dashed, "\upsilon"'] & & \mathbb{P}^2,
\end{tikzcd}$$

where ϕ, ω and υ are birational maps.

Let us prove Proposition 8.2.

Lemma 8.5. *Suppose that $Q \in \mathbb{CS}(X, \frac{1}{k}\mathcal{M})$. Then there is a commutative diagram 8.4.*

Proof. Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on Z. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_X$ by Theorem A.15, which implies the equality $S \cdot C$, where S is a general surface of the linear system $\mathcal{M}$, and C is a general fiber of the morphism σ. Hence, the linear system $\mathcal{B}$ lies in the fibers of the morphism σ, which implies the existence of the commutative diagram 8.4. $\square$

We may assume that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) \subseteq \{P_1, P_2, P_3\}$ by Theorem 1.7 and Lemma 1.16.

Lemma 8.6. *The commutative diagram 8.3 exists whenever* $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_1, P_2, P_3\}$.

Proof. See the proof of Lemma 8.5. $\square$

To conclude the proof of Proposition 8.2, we may assume that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ does not contain the point P_3 and contains the point P_1.

Lemma 8.7. *The set* $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ *contains the point* P_2.

Proof. Suppose that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ consists of the point P_1. Let $\mathcal{B}$ be the proper transform of the linear system $\mathcal{M}$ on the variety U_1. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_{U_1}$ by Theorem A.15, and Lemma B.5 implies that the set $\mathbb{CS}(U_1, \frac{1}{k}\mathcal{B})$ is not empty.

Let Z be an element of the set $\mathbb{CS}(U_1, \frac{1}{k}\mathcal{B})$, and G be the exceptional divisor of the birational morphism α_1. Then G is a cone over the rational curve of degree 3, and Lemma A.16 implies that Z is either a vertex of the cone G, or a ruling of the cone G.

Suppose that Z is a ruling of the cone G. Let C be a sufficiently general fiber of the elliptic fibration η such that $\beta_{12} \circ \gamma_{12}(C) \cap Z \neq \varnothing$, and S be a sufficiently general surface of the linear system $\mathcal{B}$. Then the curve $\beta_{12} \circ \gamma_{12}(C)$ is not contained in the surface S. Thus, we have

$$\frac{2k}{3} = S \cdot \beta_{12} \circ \gamma_{12}(C) \geqslant \mathrm{mult}_Z(S) \geqslant k,$$

which is a contradiction.

Therefore, the subvariety Z is a singular point of the surface G, which is a quotient singularity of type $\frac{1}{2}(1,1,1)$ on the variety U_1.

Let $\delta : W \to U_1$ be the weighted blow up of the point Z with weights $(1,1,1)$, $\mathcal{D}$ be the proper transform of the linear system $\mathcal{M}$ on the variety W, and F be a general surface of the linear system $|-K_W|$. Then the inequality $\Delta^2 = -1/2$ holds on the surface F, but

$$\mathcal{D}|_F \sim_{\mathbb{Q}} -kK_{U_1}|_F \sim_{\mathbb{Q}} k\Delta$$

by Theorem A.15, which is impossible by Lemmas A.21 and A.20. $\square$

Thus, the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ consists of the points P_1 and P_2. Now we can apply the arguments of the proof of Lemma 8.7 to the variety U_{12} and proper transform of the linear system $\mathcal{M}$ on the variety U_{12} to get a contradiction. The claim of Proposition 8.2 is proved.

9. Case $n = 10$, hypersurface of degree 10 in $\mathbb{P}(1,1,1,3,5)$.

We use the notations and assumptions of chapter 1. Let $n = 10$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,1,1,3,5)$ of degree 10, the equality $-K_X^3 = 2/3$ holds, the singularities of the hypersurface X consist of the point O that is a quotient singularity of type $\frac{1}{3}(1,1,2)$.

It should be pointed out that X is birationally superrigid.

Proposition 9.1. *The claim of Proposition 0.7 holds for the hypersurface* X.

There is a commutative diagram

$$\begin{tikzcd}
 & & W \arrow[dl, "\alpha"'] & \\
 & X \arrow[dl, "\xi"'] \arrow[drr, dashed, "\psi"] & & Y \arrow[ul, "\beta"'] \arrow[d, "\eta"] \\
\mathbb{P}(1,1,1,3) \arrow[rrr, dashed, "\chi"'] & & & \mathbb{P}^2,
\end{tikzcd}$$

where ξ, ψ and χ are projections, α is the weighted blow up of O with weights $(1,1,1)$, β is the weighted blow up with weights $(1,1,1)$ of the singular point of the variety W, and η is an elliptic fibration.

In the rest of the chapter we prove Proposition 9.1. Let Q be the unique singular point of the variety W, $\mathcal{D}$ be the proper transforms of the linear system $\mathcal{M}$ on the variety W, and $\mathcal{H}$ be the proper transforms of the linear system $\mathcal{M}$ on the variety Y.

Lemma 9.2. *The set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ does not contains curves.*

Proof. Suppose that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains a curve C. Then $-K_X \cdot C = 1/3$ by Lemma 1.8, which implies that the curve C is contracted by the rational map ψ to a point.

The variety $\mathbb{P}(1,1,1,3)$ is a cone, whose vertex is the point $\xi(O)$. Thus, the curve $\xi(C)$ is a ruling of the cone $\mathbb{P}(1,1,1,3)$. Moreover, the generality of the hypersurface X implies that the curve $\xi(C)$ is not contained in the ramification divisor of ξ. There is the irreducible curve Z on the variety X such that Z is different from C, but $\xi(Z) = \xi(C)$. Moreover, it follows from the arguments of the proof of Lemma 1.8 that Z is not contained in $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$.

Let S be a sufficiently general surface of the linear system $|-K_X|$ that passes through the curves C and Z. Then S is smooth outside of the point O, and the point O is a Du Val singularity of type $\mathbb{A}_2$ on the surface S. The equality $Z^2 = -4/3$ holds on the surface S, but

$$\mathcal{M}|_S = \mathrm{mult}_C(\mathcal{M})C + \mathrm{mult}_Z(\mathcal{M})Z + \mathcal{L} \sim_{\mathbb{Q}} kC + kZ,$$

where $\mathcal{L}$ is a linear system on S without fixed components. Therefore, the curve Z is contained in the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$, which is a contradiction. $\square$

The set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ consists of the point O by Theorem 1.7, the set $\mathbb{CS}(W, \frac{1}{k}\mathcal{D})$ contains the point Q by Lemmas B.5 and A.16, but $\mathcal{H} \sim_{\mathbb{Q}} -kK_Y$ by Theorem A.15. Therefore, the linear system $\mathcal{H}$ is contained in the fibers of the elliptic fibration η.

The claim of Proposition 9.1 is proved.

10. Case $n = 12$, hypersurface of degree 10 in $\mathbb{P}(1,1,2,3,4)$.

We use the notations and assumptions of chapter 1. Let $n = 12$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,1,2,3,4)$ of degree 10, the equality $-K_X^3 = 5/12$ holds, and the singularities of the hypersurface X consist of the points P_1 and P_2 that are quotient singularities of type $\frac{1}{2}(1,1,1)$, the point P_3 that is a quotient singularity of type $\frac{1}{3}(1,1,2)$, the point P_4 that is a quotient singularity of type $\frac{1}{4}(1,1,3)$.

Proposition 10.1. *The claim of Proposition 0.7 holds for the hypersurface X.*

There is a commutative diagram

$$\begin{tikzcd}
 & & Y \arrow[dl, "\gamma_5"'] \arrow[dr, "\gamma_3"] \arrow[drrr, "\eta"] & & & \\
 & U_{34} \arrow[dl, "\beta_4"'] \arrow[dr, "\beta_3"] & & U_{45} \arrow[dl, "\beta_5"] & & \mathbb{P}(1,1,2), \\
U_3 \arrow[dr, "\alpha_3"'] & & U_4 \arrow[dl, "\alpha_4"] & & & \\
 & X \arrow[uurrrr, dashed, "\psi"'] & & & &
\end{tikzcd}$$

where ψ is a projection, α_3 is the weighted blow up of P_3 with weights $(1,1,2)$, α_4 is the weighted blow up of P_4 with weights $(1,1,3)$, β_4 is the weighted blow up with weights $(1,1,3)$ of the proper transform of the point P_4 on U_3, β_3 is the weighted blow up with weights $(1,1,2)$ of the proper transform of the point P_3 on U_4, β_5 is the weighted blow up with weights $(1,1,2)$ of the singular point of U_4 that is contained in the exceptional divisor of α_4, γ_3 is the weighted blow up with weights $(1,1,2)$ of the proper transform of the point P_3 on the variety U_{45}, γ_5 is the weighted blow up with weights $(1,1,2)$ of the singular point of U_{34} that is contained in the exceptional divisor of the morphism β_4, and η is an elliptic fibration.

Remark 10.2. The divisors $-K_{U_3}$, $-K_{U_4}$, $-K_{U_{34}}$ and $-K_{U_{45}}$ are nef and big.

Let us prove Proposition 10.1. It follows from Theorem 1.7 and Lemma 1.16 that

$$\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) \subseteq \{P_1, P_2, P_3, P_4\}.$$

Lemma 10.3. *The set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ does not contain P_1 and P_2.*

Proof. Suppose that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains the point P_1. Then to conclude the proof, it is enough to get a contradiction.

Let $\pi : W \to X$ be the weighted blow up of the point P_1 with weights $(1,1,1)$, $\mathcal{B}$ be the proper transform of the linear system $\mathcal{M}$ on the variety W, S be a general surface of the linear system $|-K_W|$, and Z be the base curve of $|-K_W|$. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ by Theorem A.15, the surface S has Du Val singularities, the surface S smooth outside of the proper transforms of the points P_2, P_3 and P_4 on the variety W, and the curve Z irreducible, smooth and rational.

The equality $Z^2 = -1/24$ holds on the surface S, but

$$\mathcal{B}|_S \sim_{\mathbb{Q}} S|_S \sim_{\mathbb{Q}} kZ,$$

which is impossible by Lemmas A.21 and A.20. □

Let $\mathcal{D}_3$, $\mathcal{D}_4$, $\mathcal{D}_{34}$ and $\mathcal{D}_{45}$ be the proper transforms of $\mathcal{M}$ on U_3, U_4, U_{34} and U_{45} respectively, then it follows from Lemma B.5 that the set $\mathbb{CS}(U_\mu, \frac{1}{k}\mathcal{D}_\mu)$ is not empty whenever $\mathcal{D}_\mu \sim_{\mathbb{Q}} -kK_{U_\mu}$.

Lemma 10.4. *Suppose that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains the point P_4. Let $\bar{P}_3$ be the proper transform of the point P_3 on the variety U_4, and $\bar{P}_5$ be the singular point of the variety U_4 that dominates the point P_4. Then $\mathcal{D}_4 \sim_{\mathbb{Q}} -kK_{U_4}$ and $\mathbb{CS}(U_4, \frac{1}{k}\mathcal{D}_4) \subseteq \{\bar{P}_3, \bar{P}_5\}$.*

Proof. We must show that $\mathbb{CS}(U_4, \frac{1}{k}\mathcal{D}_4) \subseteq \{\bar{P}_3, \bar{P}_5\}$, because $\mathcal{D}_4 \sim_{\mathbb{Q}} -kK_{U_4}$ by Theorem A.15.

Suppose that $\mathbb{CS}(U_4, \frac{1}{k}\mathcal{D}_4) \not\subseteq \{\bar{P}_3, \bar{P}_5\}$. Let C be a subvariety of the variety U_4 that is contained in the set $\mathbb{CS}(U_4, \frac{1}{k}\mathcal{D}_4)$ and different from $\bar{P}_3$ and $\bar{P}_5$. Then $\alpha_4(C) = P_4$.

Let G be the exceptional divisor of the morphism α_4. Then G is a cone over the smooth rational curve in $\mathbb{P}^3$ of degree 3. It follows from Lemma A.16 that C is a ruling of G.

The hypersurface X can be given by the equation

$$w^2z + f_6(x,y,z,t)w + f_{10}(x,y,z,t) = 0 \subset \mathbb{P}(1,1,2,3,4) \cong \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = 1$, $\mathrm{wt}(z) = 2$, $\mathrm{wt}(t) = 3$, $\mathrm{wt}(w) = 4$, and f_i is a quasihomogeneous polynomial of degree i. Moreover, there is a commutative diagram

$$\begin{tikzcd}
 & U_4 \arrow[r, "\alpha_4"] \arrow[dl, "\pi"'] & X \arrow[ddl, dashed, "\chi"'] \arrow[ddr, dashed, "\psi"] & \\
W \arrow[dr, "\omega"'] & & & \\
 & \mathbb{P}(1,1,2,3) \arrow[rr, dashed, "\xi"'] & & \mathbb{P}(1,1,2),
\end{tikzcd}$$

where ξ and χ are the natural projections, π is a birational morphism that contracts 20 smooth rational curves to isolated ordinary double points of the variety W, and ω is a double cover branched over the surface $R \subset \mathbb{P}(1,1,2,3)$ that is given by the equation

$$f_6(x,y,z,t)^2 - 4zf_{10}(x,y,z,t) = 0 \subset \mathbb{P}(1,1,2,3) \cong \mathrm{Proj}\big(\mathbb{C}[x,y,z,t]\big)$$

and has 20 ordinary double points, which are given by the equations $z = f_6 = f_{10} = 0$.

Let $\mathcal{B}$ be the proper transform of the linear system $\mathcal{M}$ on the variety W, and Z be the proper transform of C on W. There is an irreducible curve $L \subset W$ such that $\omega(L) = \omega(Z)$, but L is different from Z. The log pair $(W, \frac{1}{k}\mathcal{B})$ has canonical singularities, the set $\mathbb{CS}(W, \frac{1}{k}\mathcal{B})$ does not contain the curve L, the curve L contains $\pi(\bar{P}_3)$, and Z contains $\pi(\bar{P}_5)$.

Let S be a general surface in $|-2K_W|$ containing Z. Then S is smooth outside of the points

$$\pi(\bar{P}_3) \cup \pi(\bar{P}_5) \cup \Big(\Lambda \cap (Z \cup L)\Big),$$

where Λ is the set of ordinary double points of the variety W. The equality

$$L^2 = -5/3 + \frac{|\Lambda \cap L|}{2}$$

holds on S, but the generality of f_6 and f_{10} implies that the curve $\omega(L)$ does not contain more than two singular points of the surface R. Therefore, the inequality $L^2 < 0$ holds on S, but

$$\mathcal{B}|_S = \mathrm{mult}_Z(\mathcal{B})Z + \mathrm{mult}_L(\mathcal{B})L + \mathcal{L} \sim_{\mathbb{Q}} kZ + kL,$$

where $\mathcal{L}$ is a linear system on the surface S that does not have fixed components. Therefore, the equivalence $(k - \mathrm{mult}_L(\mathcal{B}))L \sim_{\mathbb{Q}} \mathcal{L}$ implies that the set $\mathbb{CS}(W, \frac{1}{k}\mathcal{B})$ contains the curve L, which is a contradiction. $\square$

Lemma 10.5. *Suppose that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains the point P_3. Let $\bar{P}_4$ be the proper transform of the point P_4 on the variety U_3. Then $\mathcal{D}_3 \sim_{\mathbb{Q}} -kK_{U_3}$ and $\mathbb{CS}(U_3, \frac{1}{k}\mathcal{D}_3) = \{\bar{P}_4\}$.*

Proof. See the proof of Lemma 8.7. $\square$

It follows from Theorem A.15 that either $\mathcal{D}_{34} \sim_{\mathbb{Q}} -kK_{U_{34}}$ or $\mathcal{D}_{45} \sim_{\mathbb{Q}} -kK_{U_{45}}$. Let $\mathcal{D}$ be the proper transform of the linear system $\mathcal{M}$ on the variety Y.

Lemma 10.6. *Suppose that $\mathcal{D}_{34} \sim_{\mathbb{Q}} -kK_{U_{34}}$. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_Y$.*

Proof. Let F be the exceptional divisor of the morphism β_3, G be the exceptional divisor of the morphism β_4, $\check{P}_5$ be the singular point of the surface G, and $\check{P}_6$ be the singular point of the surface F. Then it follows from Theorem A.15 that the equivalence $\mathcal{D} \sim_{\mathbb{Q}} -kK_Y$ holds, if the set $\mathbb{CS}(U_{34}, \frac{1}{k}\mathcal{D}_{34})$ contains the point $\check{P}_5$.

Suppose that $\check{P}_5 \notin \mathbb{CS}(U_{34}, \frac{1}{k}\mathcal{D}_{34})$. It follows from Lemmas B.5 and A.16 that

$$\check{P}_6 \in \mathbb{CS}(U_{34}, \frac{1}{k}\mathcal{D}_{34}).$$

Let $\pi : W \to U_{34}$ be the weighted blow up of the point $\check{P}_6$ with weights $(1,1,1)$, $\mathcal{B}$ be the proper transform of the linear system $\mathcal{M}$ on the variety W, and S be a general surface of the linear system $|-K_W|$. Then the equivalence $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ holds by Theorem A.15, the base locus of the linear system $|-K_W|$ consists of the irreducible curve Δ such that

$$\mathcal{B}|_S \sim_{\mathbb{Q}} S|_S \sim_{\mathbb{Q}} k\Delta,$$

but the inequality $\Delta^2 < 0$ holds on S, which is impossible by Lemmas A.21 and A.20. $\square$

The proof of Lemma 10.6 implies that $\mathcal{D} \sim_{\mathbb{Q}} -kK_Y$ whenever $\mathcal{D}_{45} \sim_{\mathbb{Q}} -kK_{U_{45}}$. However, if follows from the equivalence $\mathcal{D} \sim_{\mathbb{Q}} -kK_Y$ that the linear system $\mathcal{D}$ lies in the fibers of the elliptic fibrations η. Hence, the claim of Proposition 10.1 is proved.

11. Case $n = 13$, hypersurface of degree 11 in $\mathbb{P}(1,1,2,3,5)$.

We use the notations and assumptions of chapter 1. Let $n = 13$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,1,2,3,5)$ of degree 11, the singularities of the hypersurface X consist of the point P_1 that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, the point P_2 that is a quotient singularity of type $\frac{1}{3}(1,1,2)$, the point P_3 that is a quotient singularity of type $\frac{1}{5}(1,2,3)$, and the equality $-K_X^3 = 11/30$ holds. There is a commutative diagram

$$\begin{tikzcd}
 & & Y \arrow[dl, "\gamma_4"'] \arrow[dr, "\gamma_2"] \arrow[drrr, "\eta"] & & & \\
 & U_{23} \arrow[dl, "\beta_3"'] \arrow[dr, "\beta_2"] & & U_{34} \arrow[dl, "\beta_4"] & & \mathbb{P}(1,1,2), \\
U_2 \arrow[dr, "\alpha_2"'] & & U_3 \arrow[dl, "\alpha_3"] & & & \\
 & X \arrow[uurrrr, dashed, "\psi"'] & & & &
\end{tikzcd}$$

where ψ is a projection, α_2 is the weighted blow up of P_2 with weights $(1,1,2)$, α_3 is the weighted blow up of P_3 with weights $(1,2,3)$, β_3 is the weighted blow up with weights $(1,2,3)$ of the proper transform of the point P_3 on the variety U_2, β_2 is the weighted blow up with weights $(1,1,2)$ of the proper transform of P_2 on the variety U_3, β_4 is the weighted blow up with weights $(1,1,2)$ of the singular point of the variety U_3 that is the quotient singularity of type $\frac{1}{3}(1,1,2)$ contained in the exceptional divisor of α_3, γ_2 is the weighted blow up with weights $(1,1,2)$ of the proper transform of P_2 on the variety U_{34}, γ_4 is the weighted blow up with weights $(1,1,2)$ of the singular point of the variety U_{23} that is the quotient singularity of type $\frac{1}{3}(1,1,2)$ contained in the exceptional divisor of the morphism β_3, and η is an elliptic fibration.

Proposition 11.1. *The claim of Proposition 0.7 holds for the hypersurface X.*

Let us prove Proposition 11.1. It follows from Theorem 1.7, Lemma 1.16 and Proposition 1.10 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) \subseteq \{P_2, P_3\}$.

Lemma 11.2. *The set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains the point P_3.*

Proof. Suppose that $P_3 \notin \mathbb{CS}(X, \frac{1}{k}\mathcal{M})$. Then $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_2\}$. Let $\mathcal{D}_2$ be the proper transform of $\mathcal{M}$ on the variety U_2. Then it follows from Theorem A.15 that $\mathcal{D}_2 \sim_{\mathbb{Q}} -kK_{U_2}$, but the set $\mathbb{CS}(U_2, \frac{1}{k}\mathcal{D}_2)$ is not empty by Lemma B.5.

Let E be the exceptional divisor of the morphism α_2. Then E can be identified with a cone over the smooth rational curve in $\mathbb{P}^3$ of degree 3. Let Z be a subvariety of the variety U_2 that is contained in the set $\mathbb{CS}(U_2, \frac{1}{k}\mathcal{D}_2)$. Then it follows from Lemma A.16 that either Z is the vertex of the cone E, or Z is a ruling of the cone E.

Suppose that Z is a ruling of the cone E. Let D be a sufficiently general divisor of the linear system $\mathcal{D}_2$, and C be a general fiber of the fibration η such that the curve $\beta_3 \circ \gamma_4(C)$ intersects the curve C. Then $\mathrm{Supp}(C) \not\subset \mathrm{Supp}(D)$, which implies that

$$\frac{2k}{5} = C \cdot D \geqslant \mathrm{mult}_Z(D) \geqslant k,$$

which is a contradiction.

The subvariety Z is the vertex of E and a quotient singularity of type $\frac{1}{2}(1,1,1)$ on U_2.

Let $\pi : W \to U_2$ be the blow up of the point Z with weights $(1,1,1)$, and S be a sufficiently general surface of the pencil $|-K_W|$. Then S is irreducible and normal, the base locus of the pencil $|-K_W|$ consists of the irreducible curve Δ contained in S. Moreover, the equality

$$\Delta^2 = -K_W^3 = -\frac{3}{10}$$

holds on the surface S. On the other hand, the equivalence $\mathcal{B}|_S \sim_{\mathbb{Q}} k\Delta$ holds, where $\mathcal{B}$ is the proper transform of $\mathcal{M}$ on the variety W. Therefore, we have $\mathcal{B}|_S = k\Delta$, which implies that

$$\mathrm{Supp}(S) \cap \mathrm{Supp}(D) = \mathrm{Supp}(\Delta),$$

where D is a general surface in $\mathcal{B}$, which is impossible by Lemma A.20. $\square$

Lemma 11.3. *The set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains the point P_2.*

Proof. Suppose that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ does not contain the point P_2. Then $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ consists of the point P_3. Let $\mathcal{D}_3$ be the proper transform of $\mathcal{M}$ on U_3. Then $\mathcal{D}_3 \sim_{\mathbb{Q}} -kK_{U_3}$ by Theorem A.15, and the set $\mathbb{CS}(U_3, \frac{1}{k}\mathcal{D}_3)$ is not empty by Lemma B.5.

Let G be the exceptional divisor of the morphism α_3, P_4 be the singular point of G that is a quotient singularity of type $\frac{1}{3}(1,1,2)$ on the variety U_3, and P_5 be the singular point of the surface G that is a quotient singularity of type $\frac{1}{2}(1,1,1)$ on U_3. Then $G \cong \mathbb{P}(1,2,3)$, and it follows from Lemma A.16 that either $\mathbb{CS}(U_3, \frac{1}{k}\mathcal{D}_3) = \{P_4\}$, or $P_5 \in \mathbb{CS}(U_3, \frac{1}{k}\mathcal{D}_3)$.

Suppose that the set $\mathbb{CS}(U_3, \frac{1}{k}\mathcal{D}_3)$ contains the point P_5. Let $\pi : W \to U_3$ be the weighted blow up of P_5 with weights $(1,1,1)$, $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on W, L be the curve on the surface G that is contained in the linear system $|\mathcal{O}_{\mathbb{P}(1,2,3)}(1)|$, $\bar{L}$ be the proper transform of the curve L on the variety W, and S be a general surface of the linear system $|-K_W|$. Then

$$\mathcal{B}|_S \sim_{\mathbb{Q}} k\Delta + k\bar{L},$$

the equivalence $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ holds by Theorem A.15, and the base locus of $|-K_W|$ consists of the curve $\bar{L}$ and the irreducible curve Δ such that the curve $\alpha \circ \pi(\Delta)$ is the base curve of the pencil $|-K_X|$. Moreover, the equalities

$$\Delta^2 = -5/6,\ \bar{L}^2 = -4/3,\ \Delta \cdot \bar{L} = 1$$

hold on the surface S, which imply that the intersection form of the curves Δ and $\bar{L}$ on the surface S is negatively defined. Therefore, the support of the effective cycle $S \cdot D$ is contained in the union $\Delta \cup \bar{L}$ by Lemma A.21, where D is a general surface of the linear system $\mathcal{B}$, which is impossible by Lemma A.20.

Hence, the set $\mathbb{CS}(U_3, \lambda\mathcal{D}_3)$ consists of the point P_4. Let D_{34} be the proper transform of the linear system $\mathcal{M}$ on the variety U_{34}. Then $\mathbb{CS}(U_{34}, \frac{1}{k}\mathcal{D}_{34}) \neq \varnothing$ by Lemma B.5, because the equivalence $D_{34} \sim_{\mathbb{Q}} -kK_{U_{34}}$ holds by Theorem A.15.

Let E be the exceptional divisor of the morphism β_4, and P_6 be the singular point of the surface E. Then $E \cong \mathbb{P}(1,1,3)$, the point P_6 is a quotient singularity of type $\frac{1}{2}(1,1,1)$ on the variety U_{34}, and $P_6 \in \mathbb{CS}(U_{34}, \frac{1}{k}\mathcal{D}_{34})$ by Lemma A.16.

Let $\zeta : Z \to U_{34}$ be the weighted blow up of P_6 with weights $(1,1,1)$, $\mathcal{H}$ be the proper transform of $\mathcal{M}$ on Z, and F be a general surface of the pencil $|-K_Z|$. Then the base locus of the pencil $|-K_Z|$ consists of irreducible curves $\check{L}$ and $\check{\Delta}$ such that the equalities

$$\check{\Delta}^2 = -5/6,\ \check{L}^2 = -3/2,\ \check{\Delta} \cdot \check{L} = 1$$

hold on F. Thus, the intersection form of the curves $\check{\Delta}$ and $\check{L}$ on the surface F is negatively defined. However, the equivalence $\mathcal{H}|_F \sim_{\mathbb{Q}} k\check{\Delta} + k\check{L}$ holds by Theorem A.15. Thus, if follows from Lemma A.21 that the support of the effective cycle $F \cdot H$ is contained in the union of the curves $\check{\Delta}$ and $\check{L}$, where H is a general surface in $\mathcal{H}$, which contradicts Lemma A.20. □

Therefore, we have $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_2, P_3\}$. Let $\mathcal{D}_{23}$ be the proper transform of the linear system $\mathcal{M}$ on the variety U_{23}. Then the set $\mathbb{CS}(U_{23}, \frac{1}{k}\mathcal{D}_{23})$ is not empty by Lemma B.5, because the equivalence $\mathcal{D}_{23} \sim_{\mathbb{Q}} -kK_{U_{23}}$ holds by Theorem A.15.

Remark 11.4. The proof of Lemma 11.2 implies that the set $\mathbb{CS}(U_{23}, \frac{1}{k}\mathcal{D}_{23})$ does no contain the singular point of the variety U_{23} that is contained in the exceptional divisor of the morphism β_2, but the proof of Lemma 11.3 implies that the set $\mathbb{CS}(U_{23}, \frac{1}{k}\mathcal{D}_{23})$ does not contain the singular point of the variety U_{23} that is a quotient singularity of type $\frac{1}{2}(1,1,1)$ contained in the exceptional divisor of the morphism β_3.

Therefore, the set $\mathbb{CS}(U_{23}, \frac{1}{k}\mathcal{D}_{23})$ contains the singular point the variety U_{23} that is a quotient singularity of type $\frac{1}{3}(1,1,2)$ contained in the exceptional divisor of the morphism β_3. Now the claim of Theorem A.15 implies that the proper transform of $\mathcal{M}$ on the variety Y is contained in the fibers of the elliptic fibration η.

The claim of Proposition 11.1 is proved.

12. Case $n = 15$, hypersurface of degree 12 in $\mathbb{P}(1,1,2,3,6)$.

We use the notations and assumptions of chapter 1. Let $n = 15$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,1,2,3,6)$ of degree 12, the equality $-K_X^3 = 1/3$ holds, and the singularities of the hypersurface X consist of the points P_1 and P_2 that are quotient singularities of type $\frac{1}{2}(1,1,1)$, and the points P_3 and P_4 that are quotient singularities of type $\frac{1}{3}(1,1,2)$.

Proposition 12.1. *The claim of Proposition 0.7 holds for the hypersurface X.*

Proof. There is a commutative diagram

$$\begin{tikzcd}
& Y \arrow[dl, "\gamma"'] \arrow[dr, "\delta"] \arrow[drrr, "\eta"] & & & \\
U \arrow[dr, "\alpha"'] & & W \arrow[dl, "\beta"] & & \mathbb{P}(1,1,2), \\
& X \arrow[urrr, dashed, "\psi"'] & & &
\end{tikzcd}$$

where ψ is a projection, α is the weighted blow up of the point P_3 with weights $(1,1,2)$, β is the weighted blow up of P_4 with weights $(1,1,2)$, γ is the weighted blow up with weights $(1,1,2)$ of the proper transform of P_4 on the variety U, δ is the weighted blow up with weights $(1,1,2)$ of the proper transform of the point P_3 on the variety W, and η is an elliptic fibration.

It follows from Theorem 1.7, Proposition 1.10 and Lemma 1.16 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) \subseteq \{P_3, P_4\}$. Let $\mathcal{H}$ be the proper transform of the linear system $\mathcal{M}$ on the variety Y. To conclude the proof we must show that $\mathcal{H}$ lies in the fibers of the morphism η, which easily follows from the condition $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_3, P_4\}$ by Theorem A.15. Therefore, without loss of generality, we may assume that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains P_3 and does not contain P_4.

Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U, and Q be the singular point of U that is contained in the exceptional divisor of α. Then the equivalence $\mathcal{B} \sim_{\mathbb{Q}} -kK_U$ holds by Theorem A.15, but the point Q is a quotient singularity of type $\frac{1}{2}(1,1,1)$ on the variety U. In particular, it follows from Lemmas B.5 and A.16 that the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{B})$ contains the singular point Q.

Let $\zeta : Z \to U$ be the weighted blow up of the point Q with weights $(1,1,3)$, $\mathcal{D}$ be the proper transform of the linear system $\mathcal{M}$ on the variety Z, and S be a sufficiently general surface of the pencil $|-K_Z|$. Then the equivalence $\mathcal{D} \sim_{\mathbb{Q}} -kK_Z$ holds by Theorem A.15, but the base locus of the pencil $|-K_Z|$ consists of an irreducible curve Δ such that $\Delta^2 < 0$ on the surface S, but the equivalence $\mathcal{D}|_S \sim_{\mathbb{Q}} k\Delta$ holds, which is impossible by Lemmas A.21 and A.20. □

13. Case $n = 16$, hypersurface of degree 12 in $\mathbb{P}(1,1,2,4,5)$.

We use the notations and assumptions of chapter 1. Let $n = 16$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,1,2,4,5)$ of degree 12, the equality $-K_X^3 = 3/10$ holds, and the singularities of the hypersurface X consist of the points P_1, P_2 and P_3 that are quotient singularities of type $\frac{1}{2}(1,1,1)$, and the point P_4 that is a quotient singularity of type $\frac{1}{5}(1,1,4)$.

Proposition 13.1. *The claim of Proposition 0.7 holds for the hypersurface X.*

Let us prove Proposition 13.1. There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] & & Y \arrow[ll, "\gamma"'] \arrow[d, "\eta"] \\ X \arrow[rrr, dashed, "\psi"'] & & & \mathbb{P}(1,1,2) \end{tikzcd}$$

where ψ is the natural projection, α is the weighted blow up of P_4 with weights $(1,1,4)$, β is the weighted blow up with weights $(1,1,3)$ of the singular point of the variety U that is contained in the exceptional divisor of α, γ is the weighted blow up with weights $(1,1,2)$ of the singular point of W that is contained in the exceptional divisor of β, and η is an elliptic fibration.

Remark 13.2. The divisors $-K_U$ and $-K_W$ are nef and big, and the morphism η are induced by the complete linear system $|-2K_Y|$.

It follows from Theorem 1.7, Lemma 1.16 and Proposition 1.10 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ consists of the point P_4. Let G be the exceptional divisor of the morphism α, $\bar{P}_5$ be the singular point of the surface G, and $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on the variety U. Then G is a cone over a smooth rational curve of degree 4, amd the point $\bar{P}_5$ is a quotient singularity of type $\frac{1}{4}(1,1,3)$ on the variety U. Moreover, the equivalence $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ holds by Theorem A.15, and the set of centers of canonical singularities $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ is not empty by Lemma B.5.

Lemma 13.3. *The set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ consists of the point $\bar{P}_5$.*

Proof. Suppose that the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ contains a subvariety C of the variety U that is different from the point $\bar{P}_5$. Then it follows from Lemma A.16 that C is a ruling of the cone G.

The hypersurface X can be given by the equation

$$w^2z + f_7(x,y,z,t)w + f_{12}(x,y,z,t) = 0 \subset \mathbb{P}(1,1,2,4,5) \cong \operatorname{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x)=\mathrm{wt}(y)=1$, $\mathrm{wt}(z)=2$, $\mathrm{wt}(t)=4$, $\mathrm{wt}(w)=5$, and f_i is a quasihomogeneous polynomial of degree i. Moreover, there is commutative diagram

$$\begin{tikzcd} & & U \arrow[dll, "\pi"'] \arrow[rr, "\alpha"] & & X \arrow[dll, dashed, "\chi"'] \arrow[drr, dashed, "\psi"] & & \\ Z \arrow[drr, "\omega"'] & & & & & & \\ & & \mathbb{P}(1,1,2,4) \arrow[rrrr, dashed, "\xi"'] & & & & \mathbb{P}(1,1,2), \end{tikzcd}$$

where ξ and χ are the natural projections, π is a birational morphism that contracts 21 smooth rational curves to isolated ordinary double points of the variety Z, and ω is a double cover branched over the surface $R\subset\mathbb{P}(1,1,2,4)$ that is given by the equation

$$f_7(x,y,z,t)^2-4zf_{12}(x,y,z,t)=0\subset\mathbb{P}(1,1,2,4)\cong\mathrm{Proj}\big(\mathbb{C}[x,y,z,t]\big)$$

and has 21 isolated ordinary double points, which are given by the equations $z=f_7=f_{12}=0$.

Let $\mathcal{B}$ be the proper transform of the linear system $\mathcal{M}$ on the variety Z, and Δ be the proper transform of the curve C on the variety Z. Then there is the smooth rational curve Γ on the variety Z such that $\omega(\Gamma)=\omega(\Delta)$, but the curve Γ is different from the curve Δ. Moreover, the log pair $(Z,\frac{1}{k}\mathcal{B})$ has canonical singularities, the set $\mathbb{CS}(Z,\frac{1}{k}\mathcal{B})$ contains the curve Δ and does not contains the curve Γ, the curves Γ and Δ contains $\pi(\bar{P}_5)$, and the curves Γ and Δ are contacted by the rational map $\xi\circ\omega$ to a nonsingular point of the surface $\mathbb{P}(1,1,2)$.

Let S be a sufficiently general surface of the linear system $|-2K_Z|$ that contains the irreducible curve Δ. Then the surface S contains the curve Γ. Moreover, the inequality $\Gamma^2<0$ holds on the surface S. On the other hand, the equivalence $\mathcal{B}|_S\sim_{\mathbb{Q}}k\Gamma+k\Delta$ holds and

$$\mathcal{B}|_S=\mathrm{mult}_{\Gamma}(\mathcal{B})\Gamma+\mathrm{mult}_{\Delta}(\mathcal{B})\Delta+\mathcal{L},$$

where $\mathcal{L}$ is a linear system on the surface S that does not have fixed components. However, it easily follows from the equivalence $(k-\mathrm{mult}_{\Gamma}(\mathcal{B}))\Gamma\sim_{\mathbb{Q}}\mathcal{L}$ that the set $\mathbb{CS}(Z,\frac{1}{k}\mathcal{B})$ contains the curve Γ, which is a contradiction. $\square$

Let $\mathcal{H}$ be the proper transform of $\mathcal{M}$ on W. Then it follows from Theorem A.15 and Lemmas B.5 and A.16 that the set $\mathbb{CS}(W,\frac{1}{k}\mathcal{H})$ contains the singular point of the variety W that is contained in the exceptional divisor of the morphism β. Therefore, the proper transform of the linear system $\mathcal{M}$ on the variety Y lies in the fibers of the morphism η by Theorem A.15.

The claim of Proposition 13.1 is proved.

14. Case $n=17$, hypersurface of degree 12 in $\mathbb{P}(1,1,3,4,4)$.

We use the notations and assumptions of chapter 1. Let $n=17$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,1,3,4,4)$ of degree 12, the equality $-K_X^3=1/4$ holds, and the singularities of the hypersurface X consist of the points P_1, P_2 and P_3 that are quotient singularities of type $\frac{1}{4}(1,1,3)$. There is a commutative diagram

$$\begin{tikzcd} & Z \arrow[dl, "\pi"'] \arrow[dr, "\omega"] & \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,3), \end{tikzcd}$$

where ψ is a projection, π is a composition of the weighted blow ups of P_1, P_2 and P_3 with weights $(1,1,3)$, and ω is and elliptic fibration.

The hypersurface X can be given by the quasihomogeneous equation of degree 12

$$wf(t,w)+xa(x,y,z,t,w)+yb(x,y,z,t,w)+zc(x,y,z,t,w)=0\subset\mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x)=\mathrm{wt}(y)=1$, $\mathrm{wt}(z)=3$, $\mathrm{wt}(t)=\mathrm{wt}(w)=4$, and f, a, b and c are quasihomogeneous polynomials of appropriate degrees. Moreover, we may assume that the point P_1 is given by the equations $x=y=z=w=0$. Let $\xi_1 : X \dashrightarrow \mathbb{P}^5$ be the rational map that is given by the linear subsystem of the linear system $|-4K_X|$ consisting of divisors

$$\lambda_0 w + \lambda_1 x^4 + \lambda_2 x^3 y + \lambda_3 x^2 y^2 + \lambda_4 x y^3 + \lambda_5 y^4 = 0,$$

where $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \in \mathbb{P}^5$. Then the rational map ξ_1 is not defined in the point P_1, the closure of the image of the rational map ξ_1 is the surface $\mathbb{P}(1,1,4)$, and the normalization of a general fiber of the rational map ξ_1 is an elliptic curve. Similarly, we can construct the rational maps $\xi_2 : X \dashrightarrow \mathbb{P}(1,1,4)$ and $\xi_3 : X \dashrightarrow \mathbb{P}(1,1,4)$ such that the rational map ξ_i is not defined in the point P_i, and the normalization of a general fiber of the map ξ_i is an elliptic curve.

Proposition 14.1. *Either there is a commutative diagram*

(14.2)
$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & V \arrow[d, "\nu"] \\ \mathbb{P}(1,1,3) \arrow[r, dashed, "\phi"'] & \mathbb{P}^2, \end{tikzcd}$$

or there is a commutative diagram

(14.3)
$$\begin{tikzcd} X \arrow[r, dashed, "\sigma"] \arrow[d, dashed, "\xi_i"'] & X \arrow[r, dashed, "\rho"] & V \arrow[d, "\nu"] \\ \mathbb{P}(1,1,4) \arrow[rr, dashed, "\upsilon"'] & & \mathbb{P}^2 \end{tikzcd}$$

for some $i \in \{1,2,3\}$, where ϕ, σ and υ are birational maps.

Let us prove Proposition 14.1. The singularities of the log pair $(X, \frac{1}{k}\mathcal{M})$ are canonical, but it follows from Theorem 1.7 and Lemma 1.16 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) \subseteq \{P_1, P_2, P_3\}$.

Remark 14.4. It easily follows from Theorem A.15 that the commutative diagram 14.2 exists, if the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains the point P_1, P_2 and P_3.

Therefore, we may assume that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains the point P_1 and does not contain the point P_3. There is a commutative diagram

$$\begin{tikzcd} & Y \arrow[dl, "\beta"'] \arrow[ddr, "\eta"] & \\ U \arrow[dr, "\alpha"'] & & \\ & X \arrow[r, dashed, "\xi_1"'] & \mathbb{P}(1,1,4), \end{tikzcd}$$

where α is the weighted blow up of the point P_1 with weights $(1,1,3)$, β is the weighted blow up with weights $(1,1,2)$ of the singular point the variety U that is contained in the exceptional divisor of the morphism α, and η is an elliptic fibration.

Remark 14.5. The morphism η is induced by the complete linear system $|-4K_X|$, which implies that the divisor $-K_U$ is nef and big.

Let $\mathcal{D}$ be the proper transform of the linear system $\mathcal{M}$ on the variety U, $\bar{P}_2$ be the proper transform of the point P_2 on the variety U, and $\bar{P}_4$ be the singular point of the variety U that is contained in the exceptional divisor of α. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem A.15, and it follows from Lemma B.5 that the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ is not empty.

Remark 14.6. It easily follows from Theorem A.15 that the commutative diagram 14.3 exists, if the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ contains the point $\bar{P}_4$.

Therefore, we may assume that $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ does not contain the point $\bar{P}_4$. Hence, it follows from the proof of Lemma 8.7 that $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ does not contains subvarieties of U that are contained in the exceptional divisor of α. Thus, the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ contains $\bar{P}_3$.

Let $\gamma: W \to U$ be the weighted blow up of the point P^3 with weights $(1,1,3)$, $\mathcal{B}$ is the proper transform of $\mathcal{M}$ on the variety W, S be a general surface of in $|-K_W|$, and C be the base curve of the pencil $|-K_W|$. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ by Theorem A.15, the curve C is irreducible, the inequality $C^2 = -1/24$ holds on the normal surface S, and the equivalence $\mathcal{B}|_S \sim_{\mathbb{Q}} kC$ holds, which is impossible by Lemmas A.21 and A.20.

The claim of Proposition 14.1 is proved.

15. Case $n = 18$, hypersurface of degree 12 in $\mathbb{P}(1,2,2,3,5)$.

We use the notations and assumptions of chapter 1. Let $n = 18$. Then X is a general hypersurface in $\mathbb{P}(1,2,2,3,5)$ of degree 12, the equality $-K_X^3 = 1/5$ holds, and the singularities of the hypersurface X consist of the points O_1, O_2, O_3, O_4, O_5 and O_6 that are quotient singularities of type $\frac{1}{2}(1,1,1)$, and the point P that is a quotient singularity of type $\frac{1}{5}(1,2,3)$.

There is a commutative diagram

$$\begin{tikzcd} U & & W \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[u, leftarrow, "\alpha"'] \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,2,2), \end{tikzcd}$$

where ψ is a projection, α is the weighted blow up of the point P with weights $(1,2,3)$, β is the weighted blow up with weights $(1,1,2)$ of the singular point of the variety U that is a quotient singularity of type $\frac{1}{3}(1,1,2)$, and η is an elliptic fibration.

Proposition 15.1. *The claim of Proposition 0.7 holds for the hypersurface X.*

Proof. It follows from Theorem 1.7, Lemma 1.16 and Proposition 1.10 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P\}$, and the singularities of the log pair $(X, \frac{1}{k}\mathcal{M})$ are canonical.

Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on U. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem A.15, and the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ is not empty by Lemma B.5. Let G be the exceptional divisor of the birational morphism α, and Q and O be the singular points of the surface G that are quotient singularities of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{2}(1,1,1)$ on the variety U respectively. Then it follows from Lemma A.16 that the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ contains either the point Q, or the point O.

Suppose that the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ contains the point O. Let $\pi: Y \to U$ be the weighted blow up of O with weights $(1,1,1)$, F be the π-exceptional divisor, and $\mathcal{H}$ and $\mathcal{P}$ be the proper transforms of $\mathcal{M}$ and $|-3K_U|$ on Y respectively. Then $\mathcal{H} \sim_{\mathbb{Q}} -kK_Y$ by Theorem A.15, but

$$\mathcal{P} \sim_{\mathbb{Q}} \pi^*(-3K_U) - \frac{1}{2}F,$$

and the base locus of the linear system $\mathcal{P}$ consists of the irreducible curve Z such that $\alpha \circ \pi(Z)$ is the base curve of the linear system $|-3K_X|$. Moreover, for a general surface S of the linear system $\mathcal{P}$, the inequality $S \cdot Z > 0$ holds, which implies that the divisor $\pi^*(-6K_U) - F$ is nef and big. On the other hand, for general surfaces D_1 and D_2 of the linear system $\mathcal{H}$, we have

$$\Big(\pi^*(-6K_U) - F\Big) \cdot D_1 \cdot D_2 = \Big(\pi^*(-6K_U) - F\Big) \cdot \Big(\pi^*(-kK_U) - \frac{k}{2}F\Big)^2 = 0,$$

which contradicts Corollary A.19.

Therefore, the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ contains the point Q. Let $\mathcal{B}$ be the proper transform of the linear system $\mathcal{M}$ on the variety W. Then the equivalence $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ holds by Theorem A.15, which easily implies that the claim of Proposition 0.7 holds for the hypersurface X. ☐

16. Case $n = 19$, hypersurface of degree 12 in $\mathbb{P}(1,2,3,3,4)$.

We use the notations and assumptions of chapter 1. Let $n = 19$. Then X is a general hypersurface in $\mathbb{P}(1,2,3,3,4)$ of degree 12, the singularities of the hypersurface X consist of the points O_1, O_2, O_3 that are quotient singularities of type $\frac{1}{2}(1,1,1)$, the points P_1, P_2, P_3, P_4 that are quotient singularities of type $\frac{1}{3}(1,1,2)$, and the equality $-K_X^3 = 1/6$ holds.

The proof of Lemma 0.5 implies that for every $i \in \{1,2,3,4\}$ there is a commutative diagram

$$\begin{tikzcd} & U_i \arrow[dl, "\pi_i"'] \arrow[dr, "\eta_i"] & \\ X \arrow[rr, dashed, "\xi_i"'] & & \mathbb{P}(1,2,3), \end{tikzcd}$$

where ξ_i is a rational map, which is not defined in the point P_i, π_i is the weighted blow up of the point P_i with weights $(1,1,2)$, and η_i is an elliptic fibration.

Proposition 16.1. *There is a commutative diagram*

$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\xi_i"'] & V \arrow[d, "\nu"] \\ \mathbb{P}(1,2,3) \arrow[r, dashed, "\sigma"'] & \mathbb{P}^2 \end{tikzcd} \tag{16.2}$$

for some $i \in \{1,2,3,4\}$, where σ is a birational map.

Proof. It follows from Theorem 1.7, Corollary 1.9, Theorem A.15 and Proposition 1.10 that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains the point P_i for some $i \in \{1,2,3,4\}$. Let $\mathcal{D}$ be the proper transform of the linear system $\mathcal{M}$ on the variety U_i. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_{U_i}$ by Theorem A.15, which implies that the linear system $\mathcal{D}$ is contained in the fibers of the elliptic fibration η_i, which implies the existence of the commutative diagram 16.2. $\square$

17. Case $n = 20$, hypersurface of degree 13 in $\mathbb{P}(1,1,3,4,5)$.

We use the notations and assumptions of chapter 1. Let $n = 20$. Then X is a general hypersurface in $\mathbb{P}(1,1,3,4,5)$ of degree 13, the equality $-K_X^3 = 13/60$ holds, and the singularities of the hypersurface X consist of the point P_1 that is a quotient singularity of type $\frac{1}{3}(1,1,2)$, the point P_2 that is a quotient singularity of type $\frac{1}{4}(1,1,3)$, and the point P_3 that is a quotient singularity of type $\frac{1}{5}(1,1,4)$. There is a commutative diagram

$$\begin{tikzcd} & & Y \arrow[dl, "\gamma_4"'] \arrow[dr, "\gamma_2"] \arrow[rrrd, "\eta"] & & & \\ & U_{23} \arrow[dl, "\beta_3"'] \arrow[dr, "\beta_2"] & & U_{34} \arrow[dl, "\beta_4"] & & \mathbb{P}(1,1,3), \\ U_2 \arrow[dr, "\alpha_2"'] & & U_3 \arrow[dl, "\alpha_3"] & & & \\ & X \arrow[rrrruu, dashed, "\psi"'] & & & & \end{tikzcd}$$

where ψ is a projection, α_2 is the weighted blow up of P_2 with weights $(1,1,3)$, α_3 is the weighted blow up ofP_3 with weights $(1,1,4)$, β_3 is the weighted blow up with weights $(1,1,4)$ of the proper transform of the point P_3 on U_2, β_2 is the weighted blow up with weights $(1,1,3)$ of the proper transform of the point P_2 on the variety U_3, β_4 is the weighted blow up with weights $(1,1,3)$ of the singular point of U_3 that is contained in the exceptional divisor of α_3, γ_2 is the weighted blow up with weights $(1,1,3)$ of the proper transform of P_2 on U_{34}, γ_4 is the weighted blow up with weights $(1,1,3)$ of the singular point of the variety U_{23} that is contained in the exceptional divisor of the morphism β_3, and η is an elliptic fibration.

Remark 17.1. The divisors $-K_{U_2}$, $-K_{U_3}$, $-K_{U_{23}}$ and $-K_{U_{34}}$ are nef and big.

The hypersurface X can be given by the quasihomogeneous equation of degree 13

$$z^3 f_4(x,y,t) + z^2 f_7(x,y,t,w) + z f_{10}(x,y,t,w) + f_{13}(x,y,t,w) = 0 \subset \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = 1$, $\mathrm{wt}(z) = 3$, $\mathrm{wt}(t) = 4$, $\mathrm{wt}(w) = 5$, and f_i is a general quasihomogeneous polynomial of degree i. Let $\xi : X \dashrightarrow \mathbb{P}^5$ be the rational map that is given by the linear subsystem of the linear system $|-4K_X|$ consisting of the divisors

$$\lambda_0 t + \lambda_1 x^4 + \lambda_2 x^3 y + \lambda_3 x^2 y^2 + \lambda_4 x y^3 + \lambda_5 y^4 = 0,$$

where $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \in \mathbb{P}^5$. Then ξ is not defined in the points P_1 and P_3, the closure of the image of the rational map ξ is the surface $\mathbb{P}(1,1,4)$, and the normalization of a general fiber of the map ξ is an elliptic curve. Moreover, there is a commutative diagram

$$\begin{tikzcd}
 & Y \arrow[dl, "\gamma_3"'] \arrow[dr, "\gamma_1"] \arrow[rrr, "\omega"] & & & \mathbb{P}(1,1,4), \\
U_1 \arrow[dr, "\alpha_1"'] & & U_3 \arrow[dl, "\alpha_3"] & & \\
 & X \arrow[rrruu, dashed, "\xi"'] & & &
\end{tikzcd}$$

where α_1 is the weighted blow up of the point P_1 with weights $(1,1,2)$, α_3 is the weighted blow up of the point P_3 weights $(1,1,4)$, γ_3 is the weighted blow up with weights $(1,1,4)$ of the proper transform of P_3 on the variety U_1, γ_1 is the weighted blow up with weights $(1,1,2)$ of the proper transform of the point P_1 on the variety U_3, and ω is an elliptic fibration.

Remark 17.2. The divisor $-K_{U_1}$ is nef and big.

In the rest of the chapter we prove the following result holds.

Proposition 17.3. *Either there is a commutative diagram*

(17.4)
$$\begin{tikzcd}
X \arrow[r, dashed, "\sigma"] \arrow[d, dashed, "\psi"'] & X \arrow[r, dashed, "\rho"] & V \arrow[d, "\nu"] \\
\mathbb{P}(1,1,3) \arrow[rr, dashed, "\upsilon"'] & & \mathbb{P}^2,
\end{tikzcd}$$

or there is a commutative diagram

(17.5)
$$\begin{tikzcd}
X \arrow[r, dashed, "\sigma"] \arrow[d, dashed, "\xi"'] & X \arrow[r, dashed, "\rho"] & V \arrow[d, "\nu"] \\
\mathbb{P}(1,1,4) \arrow[rr, dashed, "\zeta"'] & & \mathbb{P}^2,
\end{tikzcd}$$

where σ, υ and ζ are birational maps.

It follows from Theorem 1.7 and Lemma 1.16 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) \subseteq \{P_1, P_2, P_3\}$.

Lemma 17.6. *Suppose that $\{P_1, P_3\} \subseteq \mathbb{CS}(X, \frac{1}{k}\mathcal{M})$. Then the commutative diagram 17.5 exists.*

Proof. Let $\mathcal{B}$ be the proper transform of the linear system $\mathcal{M}$ on the variety Y, and S be a sufficiently general surface of the linear system $\mathcal{B}$. Then $S \sim_{\mathbb{Q}} -kK_Y$ by Theorem A.15, which implies the equality $S \cdot C = 0$, where C is a general fiber of the morphism ω. Therefore, the linear system $\mathcal{B}$ lies in the fibers of the elliptic fibration ω, which implies the existence of the commutative diagram 17.5. $\square$

Lemma 17.7. *The set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ does not contain the set $\{P_1, P_2\}$.*

Proof. Suppose that $\{P_1, P_3\} \subseteq \mathbb{CS}(X, \frac{1}{k}\mathcal{M})$. Let $\pi : W \to U_1$ be the weighted blow up with weights $(1,1,3)$ of the proper transform of the point P_2 on the variety U_1, and $\mathcal{B}$ be the proper transform of the linear system $\mathcal{M}$ on the variety W. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ by Theorem A.15.

The linear system $|-K_W|$ is a pencil, the base locus of the pencil $|-K_W|$ is the irreducible curve Δ such that the curve $\alpha_1 \circ \pi(C)$ is cut on the hypersurface X by the equations $z = y = 0$.

Let S be a sufficiently general surface of the linear system $|-K_W|$, $\bar{P}_3$ be the proper transform of the singular point P_3 on the variety W, and $\bar{P}_5$ and $\bar{P}_6$ be other singular points of W such that $\alpha_1 \circ \pi(\bar{P}_5) = P_1$ and $\alpha_1 \circ \pi(\bar{P}_6) = P_2$. Then $\bar{P}_5$ is a quotient singularity of type $\frac{1}{2}(1,1,1)$ on the variety W, the point $\bar{P}_6$ is a quotient singularity of type $\frac{1}{3}(1,1,2)$ on the variety W, the surface S is smooth outside of the points $\bar{P}_3$, $\bar{P}_5$ and $\bar{P}_6$, the singularities of the surface S in the points $\bar{P}_3$, $\bar{P}_5$ and $\bar{P}_6$ are Du Val singularities of types $\mathbb{A}_4$, $\mathbb{A}_1$ and $\mathbb{A}_2$ respectively.

The equality $\Delta^2 = -1/30$ holds on the surface S, but the equivalence $\mathcal{B}|_S \sim_{\mathbb{Q}} k\Delta$ holds, which implies that $\mathcal{B}|_S = k\Delta$. The generality of the surface S implies that $\mathrm{mult}_{\Delta}(\mathcal{B}) = k$, hence, the curve $\alpha_1 \circ \pi(\Delta)$ is contained in the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$, which is a contradiction. □

Lemma 17.8. *The set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ is not consists of the point P_i.*

Proof. Suppose that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_i\}$. Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on U_i. Then the equivalence $\mathcal{D} \sim_{\mathbb{Q}} -kK_{U_i}$ holds by Theorem A.15. Moreover, it follows from Lemma B.5 and the proof of Lemma 8.7 that the set $\mathbb{CS}(U_i, \frac{1}{k}\mathcal{D})$ contains the singular point of U_i that is the singular point of the exceptional divisor of the birational morphism α_i.

Let $\pi : W \to U_i$ be the weighted blow up with weights $(1,1,i)$ of the singular point of the variety U_i that is contained in the exceptional divisor of the morphism α_i, and $\mathcal{B}$ be the proper transform of the linear system $\mathcal{M}$ on the variety W. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ by Theorem A.15.

Let S be a sufficiently general surface of the pencil $|-K_W|$, and Δ be the unique base curve of the pencil $|-K_W|$. Then the surface S is normal, but the curve Δ is irreducible, rational and smooth. Moreover, simple computations imply that the equality

$$\Delta^2 = \begin{cases} -9/20 & i = 1, \\ -1/30 & i = 2, \\ 0 \quad i = 3, \end{cases}$$

holds on the surface S. However, we have the equivalence $\mathcal{B}|_S \sim_{\mathbb{Q}} k\Delta$, which implies (see the proof of Lemma 17.7) that the curve $\alpha_i \circ \pi(\Delta)$ is contained in the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ if $i \neq 3$.

Therefore, the equality $i = 3$ holds.

Let G be the exceptional divisor of the birational morphism α_3, and $\bar{P}_4$ be the singular point of the variety U_3 that is contained in divisor G. Then the surface G can be identified with a cone over a smooth rational curve $\mathbb{P}^5$ of degree 5, and the point $\bar{P}^4$ is a quotient singularity of type $\frac{1}{4}(1,1,3)$ on the variety U_3.

The point $\bar{P}_4$ is contained in the set $\mathbb{CS}(U_3, \frac{1}{k}\mathcal{D})$. Suppose that the set $\mathbb{CS}(U_3, \frac{1}{k}\mathcal{D})$ contains a subvariety $Z \subset U_3$ that is different from $\bar{P}_4$. Then it follows from Lemma A.16 that Z is a ruling of the cone G. On the other hand, the restriction

$$\omega \circ \gamma_1^{-1}|_G : G \dashrightarrow \mathbb{P}(1,1,4)$$

is biregular. Let D be a sufficiently general surface of the linear system $\mathcal{D}$, and C be the fiber of the elliptic fibration ω over a general point on the curve $\omega \circ \gamma_1^{-1}(Z)$. Then the curve $\gamma_1(C)$ is not contained in the surface D. Hence, we have

$$\frac{2k}{3} = \gamma_1(C) \cdot D \geqslant \mathrm{mult}_Z(D) = k,$$

which is a contradiction. Therefore, the set $\mathbb{CS}(U_3, \frac{1}{k}\mathcal{D})$ consists of the singular point $\bar{P}_4$.

The variety W is the variety U_{34}, and π is the morphism β_4. Thus, the divisor $-K_W$ is nef and big. Therefore, it follows from Lemmas B.5 and A.16 that the set $\mathbb{CS}(W, \frac{1}{k}\mathcal{B})$ contains the singular point of the variety W that is contained in the exceptional divisor of the π.

Let $\mu : Z \to W$ be the weighted blow up weights $(1,1,2)$ of the singular point of the variety W that is contained in the exceptional divisor of the morphism π, and $\mathcal{P}$ be the proper transform of the linear system $\mathcal{M}$ on the variety Z. Then the equivalence $\mathcal{P} \sim_{\mathbb{Q}} -kK_Z$ holds by Theorem A.15. Let F be a general surface of the pencil $|-K_Z|$, and Γ be the unique base curve of the pencil $|-K_Z|$. Then the surface F is irreducible and normal, but the curve Γ is irreducible, rational and smooth.

The equality $\Gamma^2 = -1/24$ holds on the surface F, but $\mathcal{P}|_S \sim_{\mathbb{Q}} k\Gamma$, which implies that

$$\alpha_3 \circ \pi \circ \mu(\Gamma) \in \mathbb{CS}(X, \frac{1}{k}\mathcal{M}),$$

but the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ does not contain curves. □

Hence, to conclude the proof of Proposition 17.3 we may assume that

$$\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_2, P_3\}.$$

Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on U_{23}. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_{U_{23}}$ by Theorem A.15, and it follows from Lemmas B.5 and A.16 that the set $\mathbb{CS}(U_{23}, \frac{1}{k}\mathcal{D})$ contains either the singular point of the variety U_{23} that is contained in the exceptional divisor of the morphism β_3, or the singular point of the variety U_{23} that is contained in the exceptional divisor of β_2.

Lemma 17.9. *The set $\mathbb{CS}(U_{23}, \frac{1}{k}\mathcal{D})$ does not contain the singular point of the variety U_{23} that is contained in the exceptional divisor of the morphism β_2.*

Proof. Let E be the exceptional divisor of the morphism β_2, and $\bar{P}_6$ be the singular point of the surface E. Then $\bar{P}_6$ is a quotient singularity of type $\frac{1}{3}(1,1,2)$ on U_{23}.

Suppose that the set $\mathbb{CS}(U_{23}, \frac{1}{k}\mathcal{D})$ contains the point $\bar{P}_6$.

Let $\pi : W \to U_{23}$ be the weighted blow up of the point $\bar{P}_6$ with weights $(1,1,2)$, $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on W, S be a general surface of the pencil $|-K_W|$, and Δ be the base curve of the pencil $|-K_W|$. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ by Theorem A.15, the surface S is irreducible and normal, and the curve Δ is irreducible. Moreover, the equivalence $\mathcal{B}|_S \sim_{\mathbb{Q}} k\Delta$ holds, but the equality $\Delta^2 = -1/24$ holds on the surface S, which contradicts Lemmas A.21 and A.20. □

Hence, the set $\mathbb{CS}(U_{23}, \frac{1}{k}\mathcal{D})$ contains the singular point of the variety U_{23} that is contained in the exceptional divisor of the morphism β_3. Thus, the existence of the commutative diagram 17.5 is easily implied by Theorem A.15. The claim of Proposition 17.3 is proved.

18. Case $n = 23$, hypersurface of degree 14 in $\mathbb{P}(1,2,3,4,5)$.

We use the notations and assumptions of chapter 1. Let $n = 23$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,2,3,4,5)$ of degree 14, the singularities of the hypersurface X consist of the points P_1, P_2, and P_3 that are quotient singularities of type $\frac{1}{2}(1,1,1)$, the point P_4 that is a quotient singularity of type $\frac{1}{3}(1,1,2)$, the point P_5 that is a quotient singularity of type $\frac{1}{4}(1,1,3)$, the point P_6 that is quotient singularity of type $\frac{1}{5}(1,2,3)$, and the equality $-K_X^3 = 7/60$ holds.

There is a commutative diagram

$$\begin{tikzcd}
 & Y \arrow[dl, "\beta_6"'] \arrow[dr, "\beta_5"] \arrow[drrr, "\eta"] & & & \\
U_5 \arrow[dr, "\alpha_5"'] & & U_6 \arrow[dl, "\alpha_6"] & & \mathbb{P}(1,2,3), \\
 & X \arrow[urrr, dashed, "\psi"'] & & &
\end{tikzcd}$$

where ψ is the natural projection, α_5 is the weighted blow up of the singular point P_5 with weights $(1,1,3)$, α_6 is the weighted blow up of P_6 with weights $(1,2,3)$, β_5 is the weighted blow up with weights $(1,1,3)$ of the proper transform of P_5 on the variety U_6, β_6 is the weighted blow up with weights $(1,2,3)$ of the proper transform of P_6 on U_5, and η is an elliptic fibration.

In the rest of the chapter we prove the following result.

Proposition 18.1. *The claim of Proposition 0.7 holds for the hypersurface X.*

It follows from Theorem 1.7, Lemma 1.16 and Proposition 1.10 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) \subseteq \{P_5, P_6\}$.

Lemma 18.2. *The set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains the point P_6.*

Proof. Suppose that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ does not contain P_6. Then the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ consists of the point P_5. Let $\mathcal{D}_5$ be the proper transform of $\mathcal{M}$ on U_5. Then $\mathcal{D}_5 \sim_{\mathbb{Q}} -kK_{U_5}$ by Theorem A.15, but the set $\mathbb{CS}(U_5, \frac{1}{k}\mathcal{D}_5)$ is not empty by Lemma B.5.

Let $\bar{P}_7$ be the singular point of the variety U_5 that is contained in the exceptional divisor of the morphism α_5. Then the point $\bar{P}_7$ is a quotient singularity of type $\frac{1}{3}(1,1,2)$ on U_5, and it follows from Lemma A.16 that $\mathbb{CS}(U_5, \frac{1}{k}\mathcal{D}_5)$ contains $\bar{P}_7$. Let $\pi : U \to U_5$ be the weighted blow up of the point P_7 with weights $(1,1,3)$. Then the linear system $|-2K_U|$ is a proper transform of the pencil $|-2K_X|$, and the base locus of the pencil $|-2K_U|$ consists of a single irreducible curve Z such that $\alpha_5\circ\pi(Z)$ is the unique base curve of the pencil $|-2K_X|$.

Let S be a sufficiently general surface of the pencil $|-2K_U|$. Then the surface S is normal, the surface S contains the curve Z, and the inequality $Z^2<0$ holds on the surface S, because the inequality $-K_U^3<0$ holds. However, the equivalence $\mathcal{B}|_S\sim_{\mathbb{Q}} kZ$ holds by Theorem A.15, where $\mathcal{B}$ is the proper transform of $\mathcal{M}$ on W. It follows from Lemma A.21 that

$$\mathrm{Supp}(S)\cap\mathrm{Supp}(D)=\mathrm{Supp}(Z),$$

where D is a sufficiently general surface of the linear system $\mathcal{B}$, which contradicts Lemma A.20, because the linear system $\mathcal{B}$ is not composed from a pencil. $\square$

It easily follows from Theorem A.15 that the claim of Proposition 0.7 holds for X whenever the set $\mathbb{CS}(X,\frac{1}{k}\mathcal{M})$ contains the points P_5 and P_6. So, we may assume that $P_5\not\in\mathbb{CS}(X,\frac{1}{k}\mathcal{M})$.

Let $\mathcal{D}_6$ be the proper transform of $\mathcal{M}$ on U_6. Then $\mathcal{D}_6\sim_{\mathbb{Q}}-kK_{U_6}$ by Theorem A.15, which implies that the set $\mathbb{CS}(U_6,\frac{1}{k}\mathcal{D}_6)$ is not empty by Lemma B.5. Let $\bar{P}_7$ and $\bar{P}_8$ be the singular points of the variety U_6 that are quotient singularities of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{2}(1,1,1)$ contained in the exceptional divisor of the morphism α_6 respectively. Then the set $\mathbb{CS}(U_6,\frac{1}{k}\mathcal{D}_6)$ contains either the point $\bar{P}_7$, or the point $\bar{P}_8$ by Lemma A.16.

Lemma 18.3. *The set $\mathbb{CS}(U_6,\frac{1}{k}\mathcal{D}_6)$ does not contain the point $\bar{P}_7$.*

Proof. Suppose that $\bar{P}_7\in\mathbb{CS}(U_6,\frac{1}{k}\mathcal{D}_6)$. Let $\gamma: W\to U_6$ be the weighted blow up of $\bar{P}_7$ with weights $(1,1,2)$, and S be a general surface of in $|-2K_W|$. Then the surface S is irreducible and normal, the linear system $|-2K_W|$ is the proper transform of the pencil $|-2K_X|$, and the base locus of the pencil $|-2K_W|$ consists of the irreducible curve Δ such that the equality

$$\Delta^2=-2K_W^3=-\frac{1}{6}$$

holds on the surface S. Moreover, the equivalence $\mathcal{B}|_S\sim_{\mathbb{Q}}k\Delta$ holds, where $\mathcal{B}$ is the proper transform of $\mathcal{M}$ on W, which contradicts Lemmas A.21 and A.20. $\square$

Therefore, the set $\mathbb{CS}(U_6,\frac{1}{k}\mathcal{D}_6)$ contains the point $\bar{P}_8$.

Remark 18.4. The linear system $|-3K_{U_6}|$ is the proper transform of the linear system $|-3K_X|$, the base locus of the linear system $|-3K_{U_6}|$ consists of the irreducible fiber of $\psi\circ\alpha_6$ that passes through the singular point $\bar{P}_8$.

Let $\pi: U\to U_6$ be the weighted blow up of the point $\bar{P}_8$ with weights $(1,1,1)$, F be the exceptional divisor of the morphism π, $\mathcal{D}$ be the proper transform of the linear system $\mathcal{M}$ on the variety U, and $\mathcal{H}$ be the proper transform of the linear system $|-3K_{U_6}|$ on U. Then

$$\mathcal{D}\sim_{\mathbb{Q}}-kK_U\sim_{\mathbb{Q}}\pi^*(-kK_{U_6})-\frac{k}{2}F$$

by Theorem A.15. The simple computations imply that

$$\mathcal{H}\sim_{\mathbb{Q}}\pi^*(-3K_{U_6})-\frac{1}{2}F,$$

and the base locus of $\mathcal{H}$ consists of the irreducible curve Z such that $\alpha_6\circ\pi(Z)$ is the base curve of the linear system $|-3K_X|$. Moreover, the equality $S\cdot Z=1/12$ holds, where S is a general surface of the linear system $\mathcal{H}$.

Let D_1 and D_2 be general surfaces of the linear system $\mathcal{D}$. Then

$$-k^2/2=\Big(\pi^*(-3K_{U_6})-\frac{1}{2}F\Big)\cdot\Big(\pi^*(-kK_{U_6})-\frac{k}{2}F\Big)^2=\Big(\pi^*(-3K_{U_6})-\frac{1}{2}F\Big)\cdot D_1\cdot D_2\geqslant 0,$$

which is a contradiction. Hence, the claim of Proposition 18.1 is proved.

19. Case $n = 25$, hypersurface of degree 15 in $\mathbb{P}(1,1,3,4,7)$.

We use the notations and assumptions of chapter 1. Let $n = 25$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,1,3,4,7)$ of degree 15, the equality $-K_X^3 = 5/28$ holds, and the singularities of X consist of the point P_1 that is a quotient singularity of type $\frac{1}{4}(1,1,3)$, and the point P_2 that is a quotient singularity of type $\frac{1}{7}(1,3,4)$.

Proposition 19.1. *The claim of Proposition 0.7 holds for the hypersurface X.*

There is a commutative diagram

$$\begin{tikzcd}
 & & Y \arrow[dl, "\gamma_3"'] \arrow[dr, "\gamma_1"] \arrow[drrr, "\eta"] & & & \\
 & U_{12} \arrow[dl, "\beta_2"'] \arrow[dr, "\beta_1"] & & U_{23} \arrow[dl, "\beta_3"] & & \mathbb{P}(1,1,3), \\
U_1 \arrow[dr, "\alpha_1"'] & & U_2 \arrow[dl, "\alpha_2"] & & & \\
 & X \arrow[uurrrr, dashed, "\psi"'] & & & &
\end{tikzcd}$$

where ψ is a projection, α_1 is the weighted blow up of P_1 with weights $(1,1,3)$, α_2 is the weighted blow up of P_2 with weights $(1,3,4)$, β_2 is the weighted blow up with weights $(1,3,4)$ of the proper transform of P_2 on U_1, β_1 is the weighted blow up with weights $(1,1,3)$ of the proper transform of the point P_1 on the variety U_2, β_3 is the weighted blow up with weights $(1,1,3)$ of the singular point of the variety U_2 that is a quotient singularity of type $\frac{1}{4}(1,1,3)$ contained in the exceptional divisor of the morphism α_2, γ_1 is the weighted blow up with weights $(1,1,3)$ of the proper transform of P_1 on U_{23}, γ_3 is the weighted blow up with weights $(1,1,3)$ of the point of the variety U_{12} that is a quotient singularity of type $\frac{1}{4}(1,1,3)$ contained in the exceptional divisor of the morphism β_2, and η is an elliptic fibration.

Remark 19.2. The divisors $-K_{U_1}$, $-K_{U_2}$, $-K_{U_{12}}$ and $-K_{U_{23}}$ are nef and big.

Let us prove Proposition 19.1. It follows from Theorem 1.7, Lemma 1.16 and Proposition 1.10 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) \subseteq \{P_1, P_2\}$. To conclude the proof of Proposition 19.1, we may assume that the singularities of the log pair $(X, \frac{1}{k}\mathcal{M})$ are canonical (see Remark 1.6).

Lemma 19.3. *The set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains the point P_2.*

Proof. Suppose that the $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ does not contain the point P_2. Let $\mathcal{D}_1$ be the proper transform of the $\mathcal{M}$ on the variety U_1. Then $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_1\}$, and the set $\mathbb{CS}(U_1, \frac{1}{k}\mathcal{D}_1)$ is not empty by Lemma B.5, because the equivalence $\mathcal{D}_1 \sim_{\mathbb{Q}} -kK_{U_1}$ holds by Theorem A.15.

Let P_5 be the singular point of the variety U_1 that is contained in the exceptional divisor of the morphism α_1. Then the point P_5 is a quotient singularity of type $\frac{1}{3}(1,1,2)$ on U_1, and it follows from Lemma A.16 that $\mathbb{CS}(U_1, \frac{1}{k}\mathcal{D}_1)$ contains the point P_5.

Let $\pi : W \to U_1$ be the blow up of the point P_5 with weights $(1,1,2)$, and S be a sufficiently general surface of the pencil $|-K_W|$. Then the surface S is irreducible and normal, and the base locus of the pencil $|-K_W|$ consists of the irreducible curve Δ such that

$$\Delta^2 = -K_W^3 = -\frac{1}{14}$$

on the surface S, but $\mathcal{B}|_S \sim_{\mathbb{Q}} k\Delta$, where $\mathcal{B}$ is the proper transform of the linear system $\mathcal{M}$ on the variety W. Therefore, we have $\mathcal{B}|_S = k\Delta$, which implies that

$$\mathrm{Supp}(S) \cap \mathrm{Supp}(D) = \mathrm{Supp}(\Delta),$$

where D is a general surface in $\mathcal{B}$. The latter contradicts Lemma A.20. $\square$

Let G be the exceptional divisor of α_2, $\mathcal{D}_2$ be the proper transform of the linear system $\mathcal{M}$ on the variety U_2, $\bar{P}_1$ be the proper transform of P_1 on U_2, and $\bar{P}_3$ and $\bar{P}_4$ are the singular points of the variety U_2 that are quotient singularities of types $\frac{1}{4}(1,1,3)$ and $\frac{1}{3}(1,1,2)$ contained in the exceptional divisor G respectively. Then $G \cong \mathbb{P}(1,3,4)$, the points $\bar{P}_3$ and $\bar{P}_4$ are singular points of the surface G, and $\mathcal{D}_2 \sim_{\mathbb{Q}} -kK_{U_2}$ by Theorem A.15. Hence, the set $\mathbb{CS}(U_2, \frac{1}{k}\mathcal{D}_2)$ is not empty by Lemma B.5. Moreover, the proof of Lemma 19.3 implies that $\mathbb{CS}(U_2, \frac{1}{k}\mathcal{D}_2) \neq \{\bar{P}_1\}$.

Lemma 19.4. *The set $\mathbb{CS}(U_2, \frac{1}{k}\mathcal{D}_2)$ does not contain both points $\bar{P}_3$ and $\bar{P}_4$.*

Proof. Suppose that $\{\bar{P}_3, \bar{P}_4\} \subseteq \mathbb{CS}(U_2, \frac{1}{k}\mathcal{D}_2)$. Let $\pi : W \to U_2$ be a composition of the weighted blow ups of the points $\bar{P}_3$ and $\bar{P}_4$ with weights $\frac{1}{4}(1,1,3)$ and $\frac{1}{3}(1,1,2)$ respectively, and $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on the variety W. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ by Theorem A.15.

Let S be a general surface of the pencil $|-K_W|$. Then the surface S is irreducible and normal, but the base locus of the pencil $|-K_W|$ consists of the irreducible curves C and L such that the curve $\alpha_2 \circ \pi(C)$ is the unique base curve of the pencil $|-K_X|$, the curve $\pi(L)$ is contained in the surface G, and $\pi(L)$ is the unique curve in $|\mathcal{O}_{\mathbb{P}(1,3,4)}(1)|$. We have

$$\mathcal{B}|_S \sim_{\mathbb{Q}} -kK_W|_S \sim_{\mathbb{Q}} kS|_S = kC + kL,$$

but the intersection form of L and C on S is negatively defined, and Lemma A.21 implies that

$$\mathrm{Supp}(S) \cap \mathrm{Supp}(D) = \mathrm{Supp}(C) \cup \mathrm{Supp}(L),$$

where D is a general surface in $\mathcal{B}$, which is impossible by Lemma A.20. □

Thus, we have $\mathbb{CS}(U_2, \frac{1}{k}\mathcal{D}_2) \subsetneq \{\bar{P}_1, \bar{P}_3, \bar{P}_4\}$ by Lemma A.16. .

Lemma 19.5. *The set $\mathbb{CS}(U_2, \frac{1}{k}\mathcal{D}_2)$ contains either the point $\bar{P}_1$, or the point $\bar{P}_4$.*

Proof. Suppose that the set $\mathbb{CS}(U_2, \frac{1}{k}\mathcal{D}_2)$ does not contain neither the singular point $\bar{P}_1$, nor the singular point $\bar{P}_4$. Then the set $\mathbb{CS}(U_2, \frac{1}{k}\mathcal{D}_2)$ contains the point $\bar{P}_3$.

The linear system $|-K_{U_2}|$ is the proper transform of the pencil $|-K_X|$, and the base locus of the pencil $|-K_{U_2}|$ consists of the irreducible curves L and Δ such that $\alpha_2(\Delta)$ is the unique base curve of the pencil $|-K_X|$, the curve L is contained in the divisor G, the curve L is the unique curve of the linear system $|\mathcal{O}_{\mathbb{P}(1,3,4)}(1)|$.

Let $\tilde{P}_6$ be the singular points of the variety U_{23} that is contained in the exceptional divisor of β_3, and $\mathcal{D}_{23}$ be the proper transform of $\mathcal{M}$ on U_{23}. Then $\mathcal{D}_{23} \sim_{\mathbb{Q}} -kK_{U_{23}}$ by Theorem A.15, and it follows from Lemmas B.5 and A.16 the the set $\mathbb{CS}(U_{23}, \frac{1}{k}\mathcal{D}_{23})$ contains the point $\tilde{P}_6$ that is a quotient singularity of type $\frac{1}{3}(1,1,2)$ on the variety U_{23}.

Let $\pi : W \to U_{23}$ be the weighted blow up of $\tilde{P}_6$ with weights $(1,1,2)$, $\mathcal{D}$ be the proper transform of the linear system $\mathcal{M}$ on the variety W, and $\bar{L}$ and $\bar{\Delta}$ be the proper transforms of the curves L and Δ on W respectively. Then the equivalence $\mathcal{D} \sim_{\mathbb{Q}} -kK_W$ holds by Theorem A.15, the linear system $|-K_W|$ is the proper transform of the pencil $|-K_{U_2}|$, and the base locus of the pencil $|-K_W|$ consists of the curves $\bar{L}$ and $\bar{\Delta}$.

Let S be a general surface of the pencil $|-K_W|$. Then the surface S is irreducible and normal, the equivalence $\mathcal{D}|_S \sim_{\mathbb{Q}} k\bar{\Delta} + k\bar{L}$ holds, but the equalities

$$\bar{\Delta}^2 = -7/12,\ \bar{L}^2 = -5/6,\ \bar{\Delta} \cdot \bar{L} = 2/3$$

hold on the surface S. Therefore, the intersection form of the curves $\bar{\Delta}$ and $\bar{L}$ on normal the surface S is negatively defined, which contradicts Lemmas A.21 and A.20. □

The hypersurface X can be given by the equation

$$w^2y + wt^2 + wtf_4(x,y,z) + wf_8(x,y,z) + tf_{11}(x,y,z) + f_{15}(x,y,z) = 0 \subset \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 1$, $\mathrm{wt}(z) = 3$, $\mathrm{wt}(t) = 4$, $\mathrm{wt}(w) = 7$, and $f_i(x,y,t)$ is a sufficiently general quasihomogeneous polynomial of degree i.

Remark 19.6. Suppose that the set $\mathbb{CS}(U_2, \frac{1}{k}\mathcal{D}_2)$ contains both points $\bar{P}_1$ and $\bar{P}_3$. Then the claim of Theorem A.15 easily implies the existence of the commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & V \arrow[d, "\nu"] \\ \mathbb{P}(1,1,3) \arrow[r, dashed, "\zeta"'] & \mathbb{P}^2, \end{tikzcd}$$

where ζ is a birational map.

Therefore, we may assume that set $\mathbb{CS}(U_2, \frac{1}{k}\mathcal{D}_2)$ does not contains both points $\bar{P}_1$ and $\bar{P}_3$.

Lemma 19.7. *The set* $\mathbb{CS}(U_2, \frac{1}{k}\mathcal{D}_2)$ *contains the point* $\bar{P}_1$.

Proof. Suppose that $\bar{P}_1 \notin \mathbb{CS}(U_2, \frac{1}{k}\mathcal{D}_2)$. Then $\mathbb{CS}(U_2, \frac{1}{k}\mathcal{D}_2) = \{\bar{P}_4\}$.

Let $\pi : W \to U_2$ be the weighted blow up of the point $\bar{P}_4$ with weights $(1,1,2)$, E be the exceptional divisor of the morphism π, and $\bar{G}$ and $\mathcal{B}$ be proper transforms of the divisor G and the linear system $\mathcal{M}$ on the variety W respectively. Then it follows from Theorem A.15 that the equivalence $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ holds, but the proof of Lemma 19.5 implies that the set $\mathbb{CS}(W, \frac{1}{k}\mathcal{B})$ does not contain the singular point of the variety W that is contained in the exceptional divisor of the morphism π. Therefore, the singularities of the log pair $(W, \frac{1}{k}\mathcal{B})$ are terminal by Lemma A.16.

Let S_x, S_y, S_z, S_t and S_w be proper transforms on the variety W of the surfaces that are cut on the variety X by the equations $x = 0$, $y = 0$, $z = 0$, $t = 0$ and $w = 0$ respectively. Then

$$\begin{cases} S_x \sim_{\mathbb{Q}} (\alpha_2 \circ \pi)^*(-K_X) - \dfrac{3}{7}E - \dfrac{1}{7}\bar{G}, \\ S_y \sim_{\mathbb{Q}} (\alpha_2 \circ \pi)^*(-K_X) - \dfrac{10}{7}E - \dfrac{8}{7}\bar{G}, \\ S_z \sim_{\mathbb{Q}} (\alpha_2 \circ \pi)^*(-3K_X) - \dfrac{2}{7}E - \dfrac{3}{7}\bar{G}, \\ S_t \sim_{\mathbb{Q}} (\alpha_2 \circ \pi)^*(-4K_X) - \dfrac{5}{7}E - \dfrac{4}{7}\bar{G}, \\ S_w \sim_{\mathbb{Q}} (\alpha_2 \circ \pi)^*(-7K_X). \end{cases} \tag{19.8}$$

The base locus of the pencil $|-K_Z|$ consists of the irreducible curves C and L such that the curve $\alpha_2 \circ \pi(C)$ is cut by the equations $x = y = 0$ on the hypersurface X, the curve $\pi(L)$ is contained in the surface G, and the curve $\pi(L)$ is contained in the linear system $|\mathcal{O}_{\mathbb{P}(1,3,4)}(1)|$.

The equivalences 19.8 implies that the rational functions y/x, zy/x^4, ty/x^5 and wy^3/x^{10} are contained in the linear system $|aS_x|$, where $a = 1$, 4, 5 and 10 respectively. Therefore, the linear system $|-20K_W|$ induces the birational map $\chi : W \dashrightarrow X'$, where X' is a hypersurface with canonical singularities in $\mathbb{P}(1,1,4,5,10)$ of degree 20. In particular, the divisor $-K_W$ is big.

It follows from [15] that there is a composition of antiflips $\zeta : W \dashrightarrow Z$ such that the rational map ζ is regular outside of $C \cup L$, and the divisor divisor $-K_Z$ is nef. Let $\mathcal{P}$ be the proper transform of the linear system $\mathcal{M}$ on Z. Then the singularities of log pair $(Z, \frac{1}{k}\mathcal{P})$ are terminal, because the rational map ζ is a log-flop with respect to the log pair $(W, \frac{1}{k}\mathcal{B})$, which has terminal singularities. However, it follows from Lemma B.5 that the log pair $(Z, \frac{1}{k}\mathcal{P})$ does not have terminal singularities, which is a contradiction. □

Hence, the set $\mathbb{CS}(U_2, \frac{1}{k}\mathcal{D}_2)$ consists of the points $\bar{P}_1$ and $\bar{P}_4$.

Let $\pi : W \to U_2$ be a composition of the weighted blow ups of the points $\bar{P}_1$ and $\bar{P}_4$ with weights $(1,1,3)$ and $(1,1,2)$ respectively, $\bar{G}$ and $\mathcal{B}$ be the proper transforms of G and $\mathcal{M}$ on the variety W respectively, and F and E be exceptional divisors of the morphism π that dominates the points $\bar{P}_1$ and $\bar{P}_4$ respectively. Then the equivalence $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ holds by Theorem A.15, but it follows from the proof of Lemma 19.5 that the singularities of $(W, \frac{1}{k}\mathcal{B})$ are terminal.

Let S_x, S_y, S_z, S_t and S_w be the proper transforms on the variety W of the surfaces that are cut on X by the equations $x=0$, $y=0$, $z=0$, $t=0$ and $w=0$ respectively. Then

$$\begin{cases} S_x \sim_{\mathbb{Q}} (\alpha_2\circ\pi)^*(-K_X) - \frac{3}{7}E - \frac{1}{7}\bar{G} - \frac{1}{4}F \sim_{\mathbb{Q}} -K_W,\\ S_y \sim_{\mathbb{Q}} (\alpha_2\circ\pi)^*(-K_X) - \frac{10}{7}E - \frac{8}{7}\bar{G} - \frac{1}{4}F,\\ S_z \sim_{\mathbb{Q}} (\alpha_2\circ\pi)^*(-3K_X) - \frac{2}{7}E - \frac{3}{7}\bar{G} - \frac{3}{4}F,\\ S_t \sim_{\mathbb{Q}} (\alpha_2\circ\pi)^*(-4K_X) - \frac{5}{7}E - \frac{4}{7}\bar{G},\\ S_w \sim_{\mathbb{Q}} (\alpha_2\circ\pi)^*(-7K_X) - \frac{11}{4}F, \end{cases}$$

which imply that the rational functions y/x, zy/x^4, twy^4/x^{15} and wy^3/x^{10} are contained in the linear system $|aS_x|$, where $a=1$, 4, 15 and 10 respectively. The linear system $|-60K_W|$ induces the birational map $\chi : W \dashrightarrow X'$ such that the variety X' is a hypersurface in $\mathbb{P}(1,1,4,10,15)$ of degree 30. In particular, the divisor $-K_W$ is big. Now we can obtain a contradiction in the same was as in the proof of Lemma 19.7. The claim of Proposition 19.1 is proved.

20. Case $n=26$, hypersurface of degree 15 in $\mathbb{P}(1,1,3,5,6)$.

We use the notations and assumptions of chapter 1. Let $n=26$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,1,3,5,6)$ of degree 15, the equality $-K_X^3=1/6$ holds, and the singularities of the hypersurface X consist of the points P_1 and P_2 that are quotient singularities of type $\frac{1}{3}(1,1,2)$, and the point P_3 that is a quotient singularity of type $\frac{1}{6}(1,1,5)$.

There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] & Y \arrow[l, "\gamma"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,3) \end{tikzcd}$$

where ψ is a projection, α is the weighted blow up of P_3 with weights $(1,1,5)$, β is the weighted blow up with weights $(1,1,4)$ of the singular point of the variety U that is contained in the exceptional divisor of the morphism α, γ is the weighted blow up with weights $(1,1,3)$ of the singular point of the variety W that is contained in the exceptional divisor of the morphism β, and η is an elliptic fibration.

The hypersurface X can be given by the equation of degree 15

$$z^3w + z^2g(x,y,t,w) + zh(x,y,t,w) + q(x,y,t,w) = 0 \subset \operatorname{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x)=\mathrm{wt}(y)=1$, $\mathrm{wt}(z)=3$, $\mathrm{wt}(t)=5$, $\mathrm{wt}(w)=6$, and f, g, h and q are quasihomogeneous polynomials. We may assume that P_1 is given by the equations $x=y=t=w=0$.

Let $\xi_1 : X \dashrightarrow \mathbb{P}^7$ be the rational map that is given by the linear system of the divisors

$$\mu w + \sum_{i=0}^{6}\lambda_i x^i y^{6-i} = 0,$$

where $(\mu,\lambda_0,\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6)\in\mathbb{P}^7$. Then ξ_1 is not defined in the point P_1, the closure of the image of the rational map ξ_1 is the surface $\mathbb{P}(1,1,6)$, and the normalization of a general fiber of the rational map ξ_1 is an elliptic curve.

Similarly, we can construct the rational map $\xi_2 : X \dashrightarrow \mathbb{P}(1,1,6)$ such that ξ_2 is not defined in the point P_2, and the normalization of a general fiber of the map ξ_2 is an elliptic curve.

There is a commutative diagram

$$\begin{tikzcd}
& U_i \arrow[dl, "\sigma_i"'] \arrow[dr, "\omega_i"] & \\
X \arrow[rr, dashed, "\xi_i"'] & & \mathbb{P}(1,1,6),
\end{tikzcd}$$

where σ_i is the weighted blow up of the point P_i with weights $(1,1,2)$, and ω_i is an elliptic fibration, which is induced by the linear system $|-6K_{U_i}|$.

Remark 20.1. The divisors $-K_U$ and $-K_W$ are nef and big, the morphism η is induced by the linear system $|-3K_Y|$, and the group $\mathrm{Bir}(X)$ is generated by biregular automorphisms of the hypersurface X and a birational involution $\tau \in \mathrm{Bir}(X)$ such that $\psi \circ \tau = \psi$ and $\xi_1 \circ \tau = \xi_2$.

In the rest of the chapter we prove the following result.

Proposition 20.2. *Either there is a commutative diagram*

(20.3)
$$\begin{tikzcd}
X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & V \arrow[d, "\nu"] \\
\mathbb{P}(1,1,3) \arrow[r, dashed, "\phi"'] & \mathbb{P}^2,
\end{tikzcd}$$

or there is a commutative diagram

(20.4)
$$\begin{tikzcd}
X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\xi_i"'] & V \arrow[d, "\nu"] \\
\mathbb{P}(1,1,6) \arrow[r, dashed, "\sigma"'] & \mathbb{P}^2,
\end{tikzcd}$$

where ϕ and σ are birational maps, and $i = 1$ or $i = 2$.

It follows from Theorem 1.7 and Lemma 1.16 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) \subseteq \{P_1, P_2, P_3\}$.

Lemma 20.5. *Suppose that $P_i \in \mathbb{CS}(X, \frac{1}{k}\mathcal{M})$. Then the commutative diagram 20.4 exists.*

Proof. Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U_i. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_{U_i}$ by Theorem A.15, which implies that the linear system $\mathcal{B}$ lies in the fibers of the morphism ω_i, which implies the existence of the commutative diagram 20.4. $\square$

Therefore, we may assume that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ consists of the point P_3.

Let $\mathcal{D}$ be the proper transform of the linear system $\mathcal{M}$ on the variety U, G be the exceptional divisor of α, and $\bar{P}_4$ be the singular point of G. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem A.15, and the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ is not empty by Lemma B.5, but the surface G is a cone over a smooth rational curve of degree 5, and the point $\bar{P}_4$ is a quotient singularity of type $\frac{1}{5}(1,1,4)$ on the variety U.

Lemma 20.6. *The set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ consists of the point $\bar{P}_4$.*

Proof. Suppose that the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ contains a subvariety C of the variety U that is different from the point $\bar{P}_4$. Then it follows from Lemma A.16 that C is a ruling of the cone G.

The hypersurface X can be given by the equation

$$w^2z + f_9(x,y,z,t)w + f_{15}(x,y,z,t) = 0 \subset \mathbb{P}(1,1,3,5,6) \cong \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = 1$, $\mathrm{wt}(z) = 3$, $\mathrm{wt}(t) = 5$, $\mathrm{wt}(w) = 6$, and f_i is a quasihomogeneous polynomial of degree i. Moreover, there is a commutative diagram

$$\begin{tikzcd}
 & U \arrow[rr, "\alpha"] \arrow[dl, "\pi"'] & & X \arrow[ddl, dashed, "\chi"'] \arrow[ddr, dashed, "\psi"] & \\
Z \arrow[drr, "\omega"'] & & & & \\
 & & \mathbb{P}(1,1,3,5) \arrow[rr, dashed, "\zeta"'] & & \mathbb{P}(1,1,3),
\end{tikzcd}$$

where ζ and χ are the natural projections, π is a birational morphism that contracts 27 smooth rational curves into isolated ordinary double points of the variety Z, and ω is a double cover branched over the surface $R \subset \mathbb{P}(1,1,3,5)$ that is given by equation

$$f_9(x,y,z,t)^2 - 4zf_{15}(x,y,z,t) = 0 \subset \mathbb{P}(1,1,3,5) \cong \mathrm{Proj}\big(\mathbb{C}[x,y,z,t]\big)$$

and has 27 isolated ordinary double points, which are given by the equations $z = f_9 = f_{15} = 0$.

Let $\mathcal{B}$ be the proper transform of the linear system $\mathcal{M}$ on the variety Z, and Δ be the proper transform of the curve C on the variety Z. Then there is a smooth rational curve Γ on the variety Z such that $\omega(\Gamma) = \omega(\Delta)$, but $\Gamma \neq \Delta$. Therefore, the log pair $(Z, \frac{1}{k}\mathcal{B})$ has canonical singularities, the set $\mathbb{CS}(Z, \frac{1}{k}\mathcal{B})$ contains the curve Δ and does not contain the curve Γ, the curves Γ and Δ pass through the point $\pi(\bar{P}_4)$, and the curves Γ and Δ are contracted by the rational map $\xi \circ \omega$ to a smooth point of the surface $\mathbb{P}(1,1,3)$.

Let S be a general surface in $|-3K_Z|$ that contains the curve Δ. Then the surface S contains the curve Γ, but the inequality $\Gamma^2 < 0$ holds on the surface S, but

$$\mathcal{B}|_S = \mathrm{mult}_\Gamma(\mathcal{B})\Gamma + \mathrm{mult}_\Delta(\mathcal{B})\Delta + \mathcal{L} \sim_{\mathbb{Q}} k\Gamma + k\Delta,$$

where $\mathcal{L}$ is a linear system on the surface S that does not have fixed components. Hence, the equality $\mathrm{mult}_\Gamma(\mathcal{B}) = k$ holds by Lemma A.21, but $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ does not contain curves. $\square$

Let $\mathcal{H}$ be the proper transform of $\mathcal{M}$ on W. Then $\mathcal{H} \sim_{\mathbb{Q}} -kK_W$ by Theorem A.15, but it follows from Lemmas B.5 and Lemma A.16 that $\mathbb{CS}(W, \frac{1}{k}\mathcal{H})$ contains the singular point of the variety W that is contained in the exceptional divisor of β. It follows from Theorem A.15 that the proper transform of the linear system $\mathcal{M}$ on the variety Y lies in the fibers of the elliptic fibration η, which implies the existence of the the commutative diagram 20.3.

The claim of Proposition 20.2 is proved.

21. Case $n = 27$, hypersurface of degree 15 in $\mathbb{P}(1,2,3,5,5)$.

We use the notations and assumptions of chapter 1. Let $n = 27$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,2,3,5,5)$ of degree 15, the equality $-K_X^3 = 1/10$ holds, and the singularities of X consist of the point O that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, and the points P_1, P_2 and P_3 that are quotient singularities of type $\frac{1}{5}(1,2,3)$.

Let $\psi : X \dashrightarrow \mathbb{P}(1,2,3)$ be the natural projection. Then the map ψ is not defined in the points P_1, P_2 and P_3. There is a commutative diagram

$$\begin{tikzcd}
 & U_{ij} \arrow[dl, "\beta_{ij}"'] & Y \arrow[l, "\gamma_{ij}"'] \arrow[dd, "\eta"] \\
U_i \arrow[dr, "\alpha_i"'] & & \\
 & X \arrow[r, dashed, "\psi"'] & \mathbb{P}(1,2,3),
\end{tikzcd}$$

where α_i is the weighted blow up of the point P_i with weights $(1,2,3)$, β_{ij} is the weighted blow up with weights $(1,2,3)$ of the proper transform of the point P_j on the variety U_i, γ_{ij} is the weighted blow up with weights $(1,2,3)$ of the proper transform of P_k on U_{ij}, and η is and elliptic fibration, where $i \neq j$ and $k \notin \{i,j\}$.

Proposition 21.1. *The claim of Proposition 0.7 holds for the hypersurface X.*

Let us prove Proposition 21.1.

Remark 21.2. In the case $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_1, P_2, P_3\}$, the claim of Proposition 0.7 holds for the hypersurface X by Theorem A.15.

It follows from Theorem 1.7, Lemma 1.16 and Proposition 1.10 that

$$\varnothing \neq \mathbb{CS}(X, \frac{1}{k}\mathcal{M}) \subseteq \{P_1, P_2, P_3\},$$

which implies that we may assume that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains point P_1 and does not contain the point P_3.

Lemma 21.3. *The set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains the point P_2.*

Proof. Suppose that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ does not contains the point P_2. Then $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ consists of the point P_1. Let $\mathcal{D}_1$ be the proper transform of $\mathcal{M}$ on U_1. Then Theorem A.15 implies that the equivalence $\mathcal{D}_1 \sim_{\mathbb{Q}} -kK_{U_1}$ holds. The set $\mathbb{CS}(U_1, \frac{1}{k}\mathcal{D}_1)$ is not empty by Lemma B.5.

Let G be is the exceptional divisor of α_1, and O and Q are the singular points of G that are quotient singularities of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{2}(1,1,1)$ on U_1 respectively. Then it follows from the claim of Lemma A.16 that $\mathbb{CS}(U_1, \frac{1}{k}\mathcal{D}_1)$ contains either the point O, or the point Q.

The linear system $|-2K_{U_1}|$ is a pencil, and the base locus of the pencil $|-2K_{U_1}|$ consists of the irreducible curve C such that the curve C passes through the point O, and C is contracted by the rational map $\psi \circ \alpha_1$ to a singular point of the surface $\mathbb{P}(1,2,3)$.

Suppose that the set $\mathbb{CS}(U_1, \frac{1}{k}\mathcal{D}_1)$ contains the point O. Let $\pi : W \to U_1$ be the weighted blow up of O with weights $(1,1,2)$, $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on W, and $\bar{C}$ be the proper transform of C on W. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ by Theorem A.15, the linear system $|-2K_W|$ is the proper transform of the pencil $|-2K_{U_1}|$, and the base locus of $|-2K_W|$ consists of $\bar{C}$.

Let S be a general surface in $|-2K_W|$. Then $\mathcal{B}|_S \sim_{\mathbb{Q}} k\bar{C}$, but on the surface S, the strict inequality $\bar{C}^2 < 0$ holds. Therefore, we have

$$\mathrm{Supp}(S) \cap \mathrm{Supp}(D) = \mathrm{Supp}(\bar{C}),$$

where D is a general surface in $\mathcal{B}$, which is impossible by Lemma A.20.

Hence, the set $\mathbb{CS}(U_1, \frac{1}{k}\mathcal{D}_1)$ contains the point Q.

Let $\zeta : U \to U_1$ be the weighted blow up of the point Q with weights $(1,1,1)$, F be the exceptional divisor of ζ, $\mathcal{H}$ be the proper transform of $\mathcal{M}$ on U, and $\mathcal{P}$ be the proper transform of the linear system $|-3K_{U_1}|$ on U. Then $\mathcal{H} \sim_{\mathbb{Q}} -kK_U$ by Theorem A.15, but

$$\mathcal{P} \sim_{\mathbb{Q}} \zeta^*(-3K_{U_1}) - \frac{1}{2}F,$$

and the base locus of $\mathcal{P}$ consists of the irreducible curve Z such that the curve $\alpha_1 \circ \zeta(Z)$ is the unique base curve of the linear system $|-3K_X|$. Therefore, we have

$$\Big(\zeta^*(-3K_{U_1}) - \frac{1}{2}F\Big) \cdot Z = \frac{1}{10},$$

which implies that

$$-\frac{3k^2}{10} = \Big(\zeta^*(-3K_{U_1}) - \frac{1}{2}F\Big) \cdot \Big(\zeta^*(-kK_{U_1}) - \frac{k}{2}F\Big)^2 = \Big(\zeta^*(-3K_{U_1}) - \frac{1}{2}F\Big) \cdot H_1 \cdot H_2 \geqslant 0,$$

where H_1 and H_2 are general surfaces of the linear system $\mathcal{H}$. □

We have $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_1, P_2\}$. Now we can apply the proof of Lemma 21.3 to the proper transform of $\mathcal{M}$ on U_{12} to get a contradiction. The claim of Proposition 21.1 is proved.

22. Case $n = 29$, hypersurface of degree 16 in $\mathbb{P}(1,1,2,5,8)$.

We use the notations and assumptions of chapter 1. Let $n = 29$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,1,2,5,8)$ of degree 16, the equality $-K_X^3 = 1/5$ holds, and the singularities of the hypersurface X consist of the points O_1 and O_2 that are quotient singularities of type $\frac{1}{2}(1,1,1)$, and the point P that is a quotient singularity of type $\frac{1}{5}(1,2,3)$.

There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & W \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,2), \end{tikzcd}$$

where ψ is the natural projection, α is the weighted blow up of P with weights $(1,2,3)$, β is the weighted blow up with weights $(1,1,2)$ of the singular point of the variety U that is a quotient singularity of type $\frac{1}{3}(1,1,2)$, and η is an elliptic fibration.

It should be pointed out that the hypersurface X is birationally superrigid.

Proposition 22.1. *The claim of Proposition 0.7 holds for the hypersurface X.*

Proof. It follows from Theorem 1.7, Lemma 1.16 and Proposition 1.10 that

$$\varnothing \neq \mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P\}.$$

Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on U. Then

$$\mathcal{D} \sim_{\mathbb{Q}} -kK_U \sim_{\mathbb{Q}} \alpha^*(-kK_X) - \frac{k}{5}G,$$

by Theorem A.15, where G is the exceptional divisor of the morphism α. Moreover, it follows from Lemma B.5 that the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ is not empty. The exceptional divisor G is the weighted projective space $\mathbb{P}(1,2,3)$.

Let Q and O be the singular points of the exceptional divisor G that are quotient singularities of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{2}(1,1,1)$ on U respectively. Then it follows from Lemma A.16 that either the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ consists of the point O, or the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ contains the point Q.

Suppose that the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ contains the point O. Let $\pi : Y \to U$ be the weighted blow up of O with weights $(1,1,1)$, $\mathcal{H}$ be the proper transform of $\mathcal{M}$ on Y, L be the curve on G that is contained in $|\mathcal{O}_{\mathbb{P}(1,2,3)}(1)|$, $\bar{L}$ be the proper transform of L on Y, and S be a general surface of the linear system $|-K_Y|$. Then $\mathcal{H} \sim_{\mathbb{Q}} -kK_Y$ by Theorem A.15, and the base locus of the pencil $|-K_Y|$ consists of the curve $\bar{L}$ and the irreducible curve Δ such that $\alpha \circ \pi(\Delta)$ is the base locus of the pencil $|-K_X|$. Moreover, the equalities

$$\Delta^2 = -1, \ \bar{L}^2 = -4/3, \ \Delta \cdot \bar{L} = 1$$

holds on the surface S. Thus, the intersection form of the curves Δ and $\bar{L}$ on the surface S is negatively defined. On the other hand, we have

$$\mathcal{H}|_S \sim_{\mathbb{Q}} kS|_S \sim_{\mathbb{Q}} k\Delta + k\bar{L},$$

which implies (see Lemma A.21) that

$$\mathrm{Supp}(S) \cap \mathrm{Supp}(D) = \mathrm{Supp}(\Delta) \cup \mathrm{Supp}(\bar{L}),$$

where D is a general surface in $\mathcal{H}$, which is impossible by Lemma A.20.

Therefore, the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ contains the point Q. Let $\mathcal{B}$ be the proper transform of the linear system $\mathcal{M}$ on the variety W. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ by Theorem A.15, which implies that the linear system $\mathcal{B}$ is contained in the fibers of the morphism η. $\square$

We use the notations and assumptions of chapter 1. Let $n = 30$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,1,3,4,8)$ of degree 16, the equality $-K_X^3 = 1/6$ holds, and the singularities of X consist of the point O that is a quotient singularity of type $\frac{1}{3}(1,1,2)$, and the points P_1 and P_2 that are quotient singularities of type $\frac{1}{4}(1,1,3)$.

There is a commutative diagram

$$\begin{tikzcd}
 & Y \arrow[ld, "\gamma"'] \arrow[rd, "\delta"] \arrow[rrr, "\eta"] & & & \mathbb{P}(1,1,3), \\
U \arrow[rd, "\alpha"'] & & W \arrow[ld, "\beta"] & & \\
 & X \arrow[rrruu, dashed, "\psi"'] & & &
\end{tikzcd}$$

where ψ is the natural projection, α is the weighted blow up of P_1 with weights $(1,1,3)$, β is the weighted blow up of P_2 with weights $(1,1,3)$, γ is the weighted blow up with weights $(1,1,3)$ of the proper transform of P_2 on U, δ is the weighted blow up with weights $(1,1,3)$ of the proper transform of the point P_1 on the variety W, and η is and elliptic fibration.

The hypersurface X can be given by the equation of degree 16

$$z^4t + z^2g(x,y,t,w) + zh(x,y,t,w) + q(x,y,t,w) = 0 \subset \operatorname{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = 1$, $\mathrm{wt}(z) = 3$ and $\mathrm{wt}(t) = 4$, $\mathrm{wt}(w) = 8$, f, g, h and q are quasihomogeneous polynomials. Let $\xi : X \dashrightarrow \mathbb{P}^5$ be the rational map that is given by the linear subsystem of the linear system $|-4K_X|$ consisting of the divisors

$$\mu t + \sum_{i=0}^{4} \lambda_i x^i y^{4-i} = 0,$$

where $(\mu, \lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{P}^5$. Then ξ is not defined in the point O, the closure of the image of the rational map ξ is the surface $\mathbb{P}(1,1,4)$, and the normalization of a general fiber of the map ξ is an elliptic curve curve. There is a commutative diagram

$$\begin{tikzcd}
 & Z \arrow[ld, "\zeta"'] \arrow[rd, "\omega"] & \\
X \arrow[rr, dashed, "\xi"'] & & \mathbb{P}(1,1,4),
\end{tikzcd}$$

where ζ is the weighted blow up of O with weights $(1,1,2)$, and ω is an elliptic fibration.

Proposition 23.1. *Either there is a commutative diagram*

(23.2)
$$\begin{tikzcd}
X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & V \arrow[d, "\nu"] \\
\mathbb{P}(1,1,3) \arrow[r, dashed, "\phi"'] & \mathbb{P}^2,
\end{tikzcd}$$

or there is a commutative diagram

(23.3)
$$\begin{tikzcd}
X \arrow[r, dashed, "\theta"] \arrow[d, dashed, "\xi"'] & X \arrow[r, dashed, "\rho"] & V \arrow[d, "\nu"] \\
\mathbb{P}(1,1,4) \arrow[rr, dashed, "\sigma"'] & & \mathbb{P}^2,
\end{tikzcd}$$

where ϕ, θ and σ are birational maps.

Proof. It follows from Theorem 1.7, Proposition 1.10 and Lemma 1.16 that

$$\varnothing \neq \mathbb{CS}\Big(X, \frac{1}{k}\mathcal{M}\Big) \subseteq \{O, P_1, P_2\}.$$

Suppose that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains the point O. Let $\mathcal{P}$ be the proper transform of the linear system $\mathcal{M}$ on the variety Z. Then $\mathcal{P} \sim_{\mathbb{Q}} -kK_Z$ by Theorem A.15, which implies that the linear system $\mathcal{P}$ is contained in the fibers of the elliptic fibration ω, which implies the existence of the the commutative diagram 23.3.

Therefore, we may assume that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) \subseteq \{P_1, P_2\}$. The existence of the commutative diagram 23.2 follows from $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_1, P_2\}$ by Theorem A.15. Therefore, to conclude the proof, we may assume that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_1\}$.

Let $\mathcal{B}$ be the proper transform of the linear system $\mathcal{M}$ on the variety U, and Q be the singular point of U that is contained in the exceptional divisor of α. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_U$ by Theorem A.15, and it follows from Lemmas B.5 and A.16 that the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{B})$ contains the point Q.

Let $\upsilon : \bar{U} \to U$ be the weighted blow up of the point Q with weights $(1,1,2)$, $\mathcal{D}$ be the proper transform of the linear system $\mathcal{M}$ on the variety $\bar{U}$, and S be a sufficiently general surface of the pencil $|-K_{\bar{U}}|$. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_{\bar{U}}$ by Theorem A.15, the surface S is normal, and the base locus of the pencil $|-K_{\bar{U}}|$ consists of the irreducible curve Δ such that $\alpha \circ \upsilon(\Delta)$ is the unique base curve of the pencil $|-K_X|$. Moreover, the inequality $\Delta^2 < 0$ holds on the surface S, but the equivalence $\mathcal{D}|_S \sim_{\mathbb{Q}} k\Delta$ holds, which contradicts Lemmas A.21 and A.20. $\square$

24. Case $n = 31$, hypersurface of degree 16 in $\mathbb{P}(1,1,4,5,6)$.

We use the notations and assumptions of chapter 1. Let $n = 31$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,1,2,3,4)$ of degree 10, the equality $-K_X^3 = 2/15$ holds, and the singularities of X consist of the point P_1 that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, the point P_2 that is a quotient singularity of type $\frac{1}{5}(1,1,4)$, and the point P_3 that is a quotient singularity of type $\frac{1}{6}(1,1,5)$.

Proposition 24.1. *The claim of Proposition 0.7 holds for the hypersurface X.*

Let us prove Proposition 24.1. There is a commutative diagram

$$\begin{tikzcd}
& & Y \arrow[dl, "\gamma_4"'] \arrow[dr, "\gamma_2"] \arrow[rrrd, "\eta"] & & & \\
& U_{23} \arrow[dl, "\beta_3"'] \arrow[dr, "\beta_2"] & & U_{34} \arrow[dl, "\beta_4"] & & \mathbb{P}(1,1,4), \\
U_2 \arrow[dr, "\alpha_2"'] & & U_3 \arrow[dl, "\alpha_3"] & & & \\
& X \arrow[rrrruu, dashed, "\psi"'] & & & &
\end{tikzcd}$$

where ψ is a projection, α_2 is the weighted blow up of P_2 with weights $(1,1,4)$, α_3 is the weighted blow up of P_3 with weights $(1,1,5)$, β_3 is the weighted blow up with weights $(1,1,5)$ of the proper transform of P_3 on U_2, β_2 is the weighted blow up with weights $(1,1,4)$ of the proper transform of the point P_2 on the variety U_3, β_4 is the weighted blow up with weights $(1,1,4)$ of the singular point of the variety U_3 that is contained in the exceptional divisor of the morphism α_3, γ_2 is the weighted blow up with weights $(1,1,4)$ of the proper transform of P_2 on U_{34}, γ_4 is the weighted blow up with weights $(1,1,4)$ of the singular point of the variety U_{23} that is contained in the exceptional divisor of the morphism β_3, and η is an elliptic fibration.

It follows from Theorem 1.7, Lemma 1.16 and Proposition 1.10 that

$$\varnothing \neq \mathbb{CS}(X, \frac{1}{k}\mathcal{M}) \subseteq \{P_2, P_3\}.$$

Let $\mathcal{D}_2$, $\mathcal{D}_3$, $\mathcal{D}_{23}$ and $\mathcal{D}_{34}$ be the proper transforms of $\mathcal{M}$ on U_2, U_3, U_{23} and U_{34} respectively, then it follows from Lemma B.5 that the set $\mathbb{CS}(U_\mu, \frac{1}{k}\mathcal{D}_\mu)$ is not empty, if $\mathcal{D}_\mu \sim_{\mathbb{Q}} -kK_{U_\mu}$.

Lemma 24.2. *Suppose that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains the point P_3. Let $\bar{P}_2$ be the proper transform of the point P_2 on the variety U_3, and $\bar{P}_4$ be the singular point of U_3 that is contained in the exceptional divisor of α_3. Then $\mathcal{D}_3 \sim_{\mathbb{Q}} -kK_{U_3}$ and $\mathbb{CS}(U_3, \frac{1}{k}\mathcal{D}_3) \subseteq \{\bar{P}_2, \bar{P}_4\}$.*

Proof. The equivalence $\mathcal{D}_3 \sim_{\mathbb{Q}} -kK_{U_3}$ follows from Theorem A.15. Suppose that

$$\mathbb{CS}(U_3, \frac{1}{k}\mathcal{D}_3) \not\subseteq \{\bar{P}_2, \bar{P}_4\},$$

and let G be the exceptional divisor of α_3. Then $G \cong \mathbb{P}(1,1,5)$, and it follows from Lemma A.16 that there is a curve $C \subset G$ of the linear system $|\mathcal{O}_{\mathbb{P}(1,\,1,\,5)}(1)|$ that is contained in $\mathbb{CS}(U_3, \frac{1}{k}\mathcal{D}_3)$.

The hypersurface X can be given by the quasihomogeneous equation of degree 16

$$w^2z + f_{10}(x,y,z,t)w + f_{16}(x,y,z,t) = 0 \subset \mathbb{P}(1,1,4,5,6) \cong \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = 1$, $\mathrm{wt}(z) = 4$, $\mathrm{wt}(t) = 5$, $\mathrm{wt}(w) = 6$, and f_i is a quasihomogeneous polynomial of degree i. There is a commutative diagram

$$\begin{tikzcd}
& U_3 \arrow[r, "\alpha_3"] \arrow[dl, "\pi"'] & X \arrow[ddl, dashed, "\chi"'] \arrow[ddr, dashed, "\psi"] & \\
W \arrow[dr, "\omega"'] & & & \\
& \mathbb{P}(1,1,4,5) \arrow[rr, dashed, "\xi"'] & & \mathbb{P}(1,1,4),
\end{tikzcd}$$

where ξ and χ are projections, π is a birational morphism, which contracts 90 irreducible, smooth and rational curves to isolated ordinary double points of the variety W, and ω is a double cover branched over the the surface $R \subset \mathbb{P}(1,1,4,5)$ that is given by the equation

$$f_{10}(x,y,z,t)^2 - 4zf_{16}(x,y,z,t) = 0 \subset \mathbb{P}(1,1,4,5) \cong \mathrm{Proj}\big(\mathbb{C}[x,y,z,t]\big)$$

and has 90 isolated ordinary double points, which are given by the equations

$$z = f_{10}(x,y,z,t) = f_{16}(x,y,z,t) = 0 \subset \mathbb{P}(1,1,4,5) \cong \mathrm{Proj}\big(\mathbb{C}[x,y,z,t]\big).$$

Let $\mathcal{B}$ be the proper transform of the linear system $\mathcal{M}$ on the variety W, and Z be the proper transform of the curve C on the variety W. Then the generality of $f_{10}(x,y,z,t)$ and $f_{16}(x,y,z,t)$ implies that there is the smooth rational curve L on the variety W such that $\omega(L) = \omega(Z)$, but the curve L is different from the curve Z.

The singularities of the log pair $(W, \frac{1}{k}\mathcal{B})$ are canonical, the set $\mathbb{CS}(W, \frac{1}{k}\mathcal{B})$ does not contain the curve L, the curve L contains $\pi(\bar{P}_2)$, the curve Z contains $\pi(\bar{P}_4)$, the curve Z does not contain more than two isolated ordinary double points of the variety W, and the curve L does not contain more than two isolated ordinary double points of the variety W.

Let S be a general surface in $|-4K_W|$ that contains the curve Z. Then $L^2 < 0$ on S, but

$$\mathcal{B}|_S = \mathrm{mult}_Z(\mathcal{B})Z + \mathrm{mult}_L(\mathcal{B})L + \mathcal{L} \sim_{\mathbb{Q}} kZ + kL$$

where $\mathcal{L}$ is a linear system on the surface S that does not have fixed components. Let D be a general surface of the linear system $\mathcal{B}$. Then it follows from Lemma A.21 that the support of the cycle $S \cdot D$ is contained in the union $Z \cup L$, which is impossible by Lemma A.20. $\square$

Lemma 24.3. *Suppose that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains the point P_2. Let $\bar{P}_3$ be the proper transform of the singular point P_3 on U_2. Then $\mathcal{D}_2 \sim_{\mathbb{Q}} -kK_{U_2}$ and $\mathbb{CS}(U_2, \frac{1}{k}\mathcal{D}_2) = \{\bar{P}_3\}$.*

Proof. The equivalence $\mathcal{D}_2 \sim_{\mathbb{Q}} -kK_{U_2}$ is implied by Theorem A.15. Suppose that the set of centers of canonical singularities $\mathbb{CS}(U_2, \frac{1}{k}\mathcal{D}_2)$ does not consists of the point $\bar{P}_3$. Let $\bar{P}_5$ be the singular point of U_2 that is contained in the exceptional divisor of α_2. Then $\bar{P}_5$ is a quotient singularity of type $\frac{1}{3}(1,1,3)$ on U_2, and $\mathbb{CS}(U_2, \frac{1}{k}\mathcal{D}_2)$ contains $\bar{P}_5$ by Lemma A.16.

Let $\pi : W \to U_2$ be the weighted blow up of the point $\bar{P}_5$ with weights $(1,1,3)$, $\mathcal{B}$ be the proper transform of the linear system $\mathcal{M}$ on the variety W, and S be a sufficiently general surface of the pencil $|-K_W|$. Then S is normal, the base locus of the pencil $|-K_W|$ consists of the irreducible curve Δ such that $\mathcal{B}|_S \sim_{\mathbb{Q}} k\Delta$, and $\Delta^2 < 0$ on S, which contradicts Lemma A.20. $\square$

Therefore, it follows from Theorem A.15 that either the equivalence $\mathcal{D}_{23} \sim_{\mathbb{Q}} -kK_{U_{23}}$ holds, or the equivalence $\mathcal{D}_{34} \sim_{\mathbb{Q}} -kK_{U_{34}}$ holds.

Let $\mathcal{D}$ be the proper transform of the linear system $\mathcal{M}$ on the variety Y.

Lemma 24.4. *Suppose that $\mathcal{D}_{23} \sim_{\mathbb{Q}} -kK_{U_{23}}$. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_Y$.*

Proof. Let F be the exceptional divisor of β_2, G be the exceptional divisor of β_3, $\check{P}_4$ be the singular point of G, and $\check{P}_5$ be the singular point of F. Then the proof of Lemma 24.3 implies that $\mathbb{CS}(U_{23}, \frac{1}{k}\mathcal{D}_{23})$ does not contain the point $\check{P}_5$. Hence, it follows from Lemmas A.16 and B.5 that the set $\mathbb{CS}(U_{23}, \frac{1}{k}\mathcal{D}_{23})$ contains $\check{P}_4$, which implies $\mathcal{D} \sim_{\mathbb{Q}} -kK_Y$ by Theorem A.15. □

Lemma 24.5. *Suppose that $\mathcal{D}_{34} \sim_{\mathbb{Q}} -kK_{U_{34}}$. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_Y$.*

Proof. Let G be the exceptional divisor of the morphism β_4, $\check{P}_2$ be the proper transform of the point P_2 on the variety U_{34}, and $\check{P}_6$ be the singular point of the surface G. Then G is a cone over the smooth rational cubic curve, and $\check{P}_6$ is a quotient singularity of type $\frac{1}{3}(1,1,2)$ on U_{34}.

The set $\mathbb{CS}(U_{23}, \frac{1}{k}\mathcal{D}_{23})$ is not empty by Lemma B.5, but the equivalence $\mathcal{D} \sim_{\mathbb{Q}} -kK_Y$ follows from Theorem A.15, if the set $\mathbb{CS}(U_{23}, \frac{1}{k}\mathcal{D}_{23})$ contains the point $\check{P}_2$. Therefore, we may assume that the set $\mathbb{CS}(U_{23}, \frac{1}{k}\mathcal{D}_{23})$ contains the point $\check{P}_6$ by Lemma A.16.

Let $\pi : W \to U_{34}$ be the weighted blow up of the point $\check{P}_6$ with weights $(1,1,3)$, $\mathcal{B}$ be the proper transform of the linear system $\mathcal{M}$ on the variety W, and S be a sufficiently general surface of the pencil $|-K_W|$. Then the base locus of the pencil $|-K_W|$ consists of the irreducible the curve Δ such that the equivalence $\mathcal{B}|_S \sim_{\mathbb{Q}} k\Delta$ holds, and the inequality $\Delta^2 < 0$ holds on the surface S. It follows from Lemma A.21 that $\mathcal{B}|_S = k\Delta$, which is impossible by Lemma A.20. □

Hence, the equivalence $\mathcal{D} \sim_{\mathbb{Q}} -kK_Y$ holds, which implies that the linear system $\mathcal{D}$ is contained in the fibers of the fibration η. The claim of Proposition 24.1 is proved.

25. Case $n = 32$, hypersurface of degree 16 in $\mathbb{P}(1,2,3,4,7)$.

We use the notations and assumptions of chapter 1. Let $n = 32$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,2,3,4,6)$ of degree 16, the equality $-K_X^3 = 2/21$ holds, and the singularities of the hypersurface X consist of the points P_1, P_2, P_3 and P_4 that are quotient singularities of type $\frac{1}{2}(1,1,1)$, the point P_5 that is a quotient singularity of type $\frac{1}{3}(1,1,2)$, and the point P_6 that is a quotient singularity of type $\frac{1}{7}(1,3,4)$.

Proposition 25.1. *The claim of Proposition 0.7 holds for the hypersurface X.*

In the rest of the chapter we prove Proposition 25.1. There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & Y \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,2,3), \end{tikzcd}$$

where ψ is a projection, α is the weighted blow up of P_6 with weights $(1,3,4)$, β is the weighted blow up with weights $(1,1,3)$ of the singular point of the variety U that is a quotient singularity of type $\frac{1}{4}(1,1,3)$ contained in the exceptional divisor of α, and η is an elliptic fibration.

It follows from Theorem 1.7, Lemma 1.16 and Proposition 1.10 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_6\}$.

Let E be the exceptional divisor of the morphism α, and $\mathcal{D}$ be the proper transform of the linear system $\mathcal{M}$ on the variety U. Then E is the weighted projective space $\mathbb{P}(1,3,4)$, and it follows from Theorem A.15 that

$$\mathcal{D} \sim_{\mathbb{Q}} -kK_U \sim_{\mathbb{Q}} \alpha^*(-kK_X) - \frac{k}{7}E,$$

but the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ is not empty by Lemma B.5, because the divisor $-K_U$ is nef and big.

Let P_7 and P_8 be the singular points of U that are quotient singularities of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{4}(1,1,3)$ contained in the surface E respectively. Then it follows from Lemma A.16 that either the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ contains the point P_8, or $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ consists of the point P_7.

Lemma 25.2. *Suppose that $P_8 \in \mathbb{CS}(X, \frac{1}{k}\mathcal{M})$. Then there is a commutative diagram*

$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & V \arrow[d, "\nu"] \\ \mathbb{P}(1,2,3) \arrow[r, dashed, "\zeta"'] & \mathbb{P}^2, \end{tikzcd} \tag{25.3}$$

where ζ is a birational map.

Proof. Let $\mathcal{H}$ be the proper transform of the linear system $\mathcal{M}$ on the variety Y. Then it follows from Theorem A.15 that the equivalence $\mathcal{H} \sim_{\mathbb{Q}} -kK_Y$ holds. Hence, the linear system $\mathcal{H}$ lies in the fibers of the fibration η, which implies the existence of the commutative diagram 25.3. □

To conclude the proof of Proposition 25.1, we may assume that $\mathbb{CS}(U, \frac{1}{k}\mathcal{D}) = \{P_7\}$.

Let $\gamma : W \to U$ be the weighted blow up of the point P_7 with weights $(1,1,2)$, F be the exceptional divisor of the morphism γ, $\bar{E}$ be the proper transform of the surface E on the variety W, and $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on W. Then the surface F is the weighted projective space $\mathbb{P}(1,1,2)$, and it follows from Theorem A.15 that

$$\mathcal{B} \sim_{\mathbb{Q}} -kK_W \sim_{\mathbb{Q}} \gamma^*(-kK_U) - \frac{k}{3}F \sim_{\mathbb{Q}} (\alpha \circ \gamma)^*(-kK_X) - \frac{k}{7}\bar{E} - \frac{3k}{7}F.$$

The hypersurface X can be given by the quasihomogeneous equation

$$w^2y + wf_9(x,y,z,t) + f_{16}(x,y,z,t) = 0 \subset \mathbb{P}(1,2,3,4,7) \cong \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 2$, $\mathrm{wt}(z) = 3$, $\mathrm{wt}(t) = 4$, $\mathrm{wt}(w) = 7$, and f_9 and f_{16} are quasihomogeneous polynomials of degree 9 and 16 respectively. Let S be the unique surface of the linear system $|-K_X|$, and D be a general surface of the pencil $|-2K_X|$. Then the surface S is cut on the hypersurface X by the equation $x = 0$, and D is cut on the hypersurface X by the equation

$$\lambda x^2 + \mu y = 0,$$

where $(\lambda, \mu) \in \mathbb{P}^1$. In particular, the surface D is irreducible and normal, and the base locus of the linear system $|-2K_X|$ consists of the curve C that is cut on X by the equations $x = y = 0$, which implies that $C = D \cdot S$.

In the neighborhood of the point P_6, the monomials x, z and t can be considered as a weighted local coordinates on X such that $\mathrm{wt}(x) = 1$, $\mathrm{wt}(z) = 3$ and $\mathrm{wt}(z) = 4$. Then in the neighborhood of the singular point P_6, the surface D can be given by equation

$$\lambda x^2 + \mu\big(\epsilon_1 x^9 + \epsilon_2 zx^6 + \epsilon_3 z^2x^3 + \epsilon_4 z^3 + \epsilon_5 t^2x + \epsilon_6 tx^5 + \epsilon_7 tzx^2 + h_{16}(x,z,t) + \Delta(x,z,t)\big) = 0,$$

where $\epsilon_i \in \mathbb{C}$, $h_{16}(x,z,t)$ is a quasihomogeneous polynomial of degree 16, and $\Delta(x,z,t)$ is a power series such that the quasihomogeneous degree of every monomial in $\Delta(x,z,t)$ is greater than 16, which implies that the geometry of the log par (X, D) does not depend on the properties of the power series $\Delta(x,z,t)$ in the neighborhood of the point P_6. In the neighborhood of the singular point P_7, the morphism α can be given by the equations

$$x = \tilde{x}\tilde{z}^{\frac{1}{7}},\ z = \tilde{z}^{\frac{3}{7}},\ t = \tilde{t}\tilde{z}^{\frac{4}{7}},$$

where $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ are weighted local coordinates on the variety U in the neighborhood of the singular point P_7 such that $\mathrm{wt}(\tilde{x}) = 1$, $\mathrm{wt}(\tilde{z}) = 2$ and $\mathrm{wt}(\tilde{t}) = 1$. Let $\tilde{D}$, $\tilde{S}$ and $\tilde{C}$ be the proper transforms on U of the surface D, the surface S and the curve C respectively, and E be the exceptional divisor of the morphism α. Then in the neighborhood of the singular point P_7 the surface E is given by the equation $\tilde{z} = 0$, the surface $\tilde{D}$ is given by the vanishing of the function

$$\lambda\tilde{x}^2+\mu\Big(\epsilon_1\tilde{x}^9\tilde{z}+\epsilon_2\tilde{z}\tilde{x}^6+\epsilon_3\tilde{z}\tilde{x}^3+\epsilon_4\tilde{z}+\epsilon_5\tilde{t}^2\tilde{x}\tilde{z}+\epsilon_6\tilde{t}\tilde{x}^5\tilde{z}+\epsilon_7\tilde{t}\tilde{z}\tilde{x}^2+\frac{h_{16}(\tilde{x}\tilde{z}^{\frac{1}{7}},\tilde{z}^{\frac{3}{7}},\tilde{t}\tilde{z}^{\frac{4}{7}})}{\tilde{z}^{\frac{2}{7}}}+\frac{\Delta(\tilde{x}\tilde{z}^{\frac{1}{7}},\tilde{z}^{\frac{3}{7}},\tilde{t}\tilde{z}^{\frac{4}{7}})}{\tilde{z}^{\frac{2}{7}}}\Big),$$

and the surface S is given by the equation $\tilde{x} = 0$.

In the neighborhood of the singular point of F, the morphism β can be given by the equations

$$\tilde{x} = \bar{x}\bar{z}^{\frac{1}{3}},\ \tilde{z} = \bar{z}^{\frac{2}{3}},\ \tilde{t} = \bar{t}\bar{z}^{\frac{1}{3}},$$

where $\bar{x}$, $\bar{z}$ and $\bar{t}$ are weighted local coordinates on the variety W in the neighborhood of the singular point of the surface F such that $\mathrm{wt}(\bar{x}) = \mathrm{wt}(\bar{z}) = \mathrm{wt}(\bar{t}) = 1$. The surface F is given by the equation $\bar{z} = 0$, the proper transform of the surface D on the variety W is given by the vanishing of the analytical function

$$\lambda\bar{x}^2+\mu\Big(\epsilon_1\bar{x}^9\bar{z}^3+\epsilon_2\bar{z}^2\bar{x}^6+\epsilon_3\bar{z}\bar{x}^3+\epsilon_4+\epsilon_5\bar{t}^2\bar{x}\bar{z}+\epsilon_6\bar{t}\bar{x}^5\bar{z}^2+\epsilon_7\bar{t}\bar{z}\bar{x}^2+\frac{h_{16}(\bar{x}\bar{z}^{\frac{3}{7}},\bar{z}^{\frac{2}{7}},\bar{t}\bar{z}^{\frac{5}{7}})}{\bar{z}^{\frac{4}{21}}}+\frac{\Delta(\bar{x}\bar{z}^{\frac{3}{7}},\bar{z}^{\frac{2}{7}},\bar{t}\bar{z}^{\frac{5}{7}})}{\bar{z}^{\frac{4}{21}}}\Big),$$

the proper transform of the surface S on the variety W is given by the equation $\bar{x} = 0$, and the proper transform of the surface E of the variety W is given by the equation $\bar{z} = 0$.

Let $\mathcal{P}$, $\bar{D}$, $\bar{S}$ and $\bar{C}$ be the proper transforms on the variety W of the pencil $|-2K_X|$, the surface D, the surface S and the curve C respectively, and $\bar{H}$ be the proper transform on the variety W of the surface that is cut on X by the equation $y = 0$. Then the surface $\bar{D}$ is a general surface of the pencil $\mathcal{P}$. Moreover, we have

$$(25.4)\qquad \begin{cases} \bar{E} \sim_{\mathbb{Q}} \gamma^*(E) - \dfrac{2}{3}F,\\ \bar{D} \sim_{\mathbb{Q}} (\alpha\circ\gamma)^*(-2K_X) - \dfrac{2}{7}\gamma^*(E) - \dfrac{2}{3}F \sim_{\mathbb{Q}} (\alpha\circ\gamma)^*(-2K_X) - \dfrac{2}{7}\bar{E} - \dfrac{6}{7}F,\\ \bar{S} \sim_{\mathbb{Q}} (\alpha\circ\gamma)^*(-K_X) - \dfrac{1}{7}\gamma^*(E) - \dfrac{1}{3}F \sim_{\mathbb{Q}} (\alpha\circ\gamma)^*(-K_X) - \dfrac{1}{7}\bar{E} - \dfrac{3}{7}F,\\ \bar{H} \sim_{\mathbb{Q}} \gamma^*\Big(\alpha^*(-2K_X) - \dfrac{9}{7}E\Big) \sim_{\mathbb{Q}} (\alpha\circ\gamma)^*(-2K_X) - \dfrac{9}{7}\bar{E} - \dfrac{6}{7}F \sim_{\mathbb{Q}} 2\bar{S} - \bar{E}. \end{cases}$$

The curve $\bar{C}$ is contained in the base locus of the pencil $\mathcal{P}$, but the curve $\bar{C}$ is not the only curve in the base locus of the pencil $\mathcal{P}$. Namely, let L be the curve on the surface E that is contained in the linear system $|\mathcal{O}_{\mathbb{P}(1,3,4)}(1)|$, which means that L is the curve given locally by the equations $\tilde{x} = \tilde{z} = 0$, and $\bar{L}$ be the proper transform of L on W. Then the curve $\bar{L}$ is contained in the base locus of the pencil $\mathcal{P}$ as well. Moreover, it follows from the local computations that the base locus of the pencil $\mathcal{P}$ does not contain curves outside of the union of $\bar{C}\cup\bar{L}$.

The curve $\bar{C}$ is the intersection of the divisors $\bar{S}$ and $\bar{H}$, and the curve $\bar{L}$ is the intersections of the divisors $\bar{S}$ and $\bar{E}$. Moreover, we have $2\bar{C} = \bar{D}\cdot\bar{H}$, $\bar{C}+\bar{L} = \bar{S}\cdot\bar{D}$ and $2\bar{L} = \bar{D}\cdot\bar{E}$.

The curves $\bar{C}$ and $\bar{L}$ can be considered as divisors on the normal surface $\bar{D}$. Then it follows from the equivalences 25.4 that

$$(25.5)\qquad \begin{cases} \bar{L}\cdot\bar{L} = \dfrac{\bar{E}\cdot\bar{E}\cdot\bar{D}}{4} = -\dfrac{5}{8},\\ \bar{C}\cdot\bar{C} = \dfrac{\bar{H}\cdot\bar{H}\cdot\bar{D}}{4} = \bar{S}\cdot\bar{S}\cdot\bar{D} - \bar{S}\cdot\bar{E}\cdot\bar{D} - \dfrac{3}{2}\bar{E}\cdot\bar{E}\cdot\bar{D} = -\dfrac{7}{24},\\ \bar{C}\cdot\bar{L} = \dfrac{\bar{H}\cdot\bar{E}\cdot\bar{D}}{4} = \dfrac{\bar{S}\cdot\bar{E}\cdot\bar{D}}{2} - \dfrac{\bar{E}\cdot\bar{E}\cdot\bar{D}}{4} = \dfrac{3}{8}, \end{cases}$$

which implies that the intersection forms of $\bar{C}$ and $\bar{L}$ on $\bar{D}$ is negatively definite.

Let G be a sufficiently general surface of the linear system $\mathcal{B}$. Then

$$G|_{\bar{D}} \sim_{\mathbb{Q}} -kK_W|_{\bar{D}} \sim_{\mathbb{Q}} k\bar{S}|_{\bar{D}} \sim_{\mathbb{Q}} k\bar{C} + k\bar{L},$$

but it follows from Lemma A.21 that

$$\mathrm{Supp}(G\cap\bar{D}) = \mathrm{Supp}(\bar{C})\cup\mathrm{Supp}(\bar{L}),$$

which is impossible by Lemma A.20. The claim of Proposition 25.1 is proved.

Remark 25.6. Let $\overline{\mathbb{NE}}(W)$ be a closure in $\mathbb{R}^3$ of the cone that is generated by the effective one-dimensional cycles of the variety W. Then the negative definiteness of the intersection form of the curves $\bar{C}$ and $\bar{L}$ on the surface $\bar{D}$ implies that the curves $\bar{C}$ and $\bar{L}$ generates two-dimensional face of the cone $\overline{\mathbb{NE}}(W)$ that does not contain irreducible curves except $\bar{C}$ and $\bar{L}$, which can be deduced directly from the equivalences 25.4.

We use the notations and assumptions of chapter 1. Let $n = 36$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,1,4,6,7)$ of degree 18, the equality $-K_X^3 = 3/28$ holds, and the singularities of X consist of the point P_1 that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, the point P_2 that is a quotient singularity of type $\frac{1}{4}(1,1,3)$, and the point P_3 that is a quotient singularity of type $\frac{1}{7}(1,1,6)$. There is a commutative diagram

$$\begin{tikzcd} U_3 \arrow[d, "\alpha_3"'] & U_{34} \arrow[l, "\beta_4"'] & Y \arrow[l, "\gamma_5"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,4) \end{tikzcd}$$

where ψ is a projection, α_3 is the weighted blow up of P_3 with weights $(1,1,7)$, β_4 is the weighted blow up with weights $(1,1,6)$ of the singular point of the variety U_3 that is contained in the exceptional divisor of the morphism α_3, γ_5 is the weighted blow up with weights $(1,1,4)$ of the singular point of the variety U_{34} that is contained in the exceptional divisor of the birational morphism β_4, and η is an elliptic fibration.

Remark 26.1. The divisors $-K_{U_3}$ and $-K_{U_{34}}$ are nef and big.

The hypersurface X can be given by the quasihomogeneous equation of degree 18

$$z^3t + z^2g(x,y,t,w) + zh(x,y,t,w) + q(x,y,t,w) = 0 \subset \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = 1$, $\mathrm{wt}(z) = 4$, $\mathrm{wt}(t) = 6$, $\mathrm{wt}(w) = 7$, and f, g, h and q are quasihomogeneous polynomials. Let $\xi : X \dashrightarrow \mathbb{P}^7$ be a map that is given by the linear system of divisors that are cut on the hypersurface X by the equations $\mu t + \sum_{i=0}^{6} \lambda_i x^i y^{6-i} = 0$, where μ and λ_i are complex numbers. Then there is a commutative diagram

$$\begin{tikzcd} & W \arrow[dl, "\beta_3"'] \arrow[dr, "\beta_2"] \arrow[rrr, "\omega"] & & & \mathbb{P}(1,1,6), \\ U_2 \arrow[dr, "\alpha_2"'] & & U_3 \arrow[dl, "\alpha_3"] & & \\ & X \arrow[rrruu, dashed, "\xi"'] & & & \end{tikzcd}$$

where α_2 is the weighted blow up of the singular point P_2 with weights $(1,1,3)$, α_3 is the weighted blow up of P_3 with weights $(1,1,6)$, β_2 is the weighted blow up with weights $(1,1,3)$ of the proper transform of P_2 on U_3, β_3 is the weighted blow up with weights $(1,1,6)$ of the proper transform of P_3 on U_2, and ω is an elliptic fibration.

Remark 26.2. The divisor $-K_{U_2}$ is nef and big.

In the rest of the chapter we prove the following result.

Proposition 26.3. *Either there is a commutative diagram*

(26.4)
$$\begin{tikzcd} X \arrow[r, dashed, "\zeta"] \arrow[d, dashed, "\psi"'] & X \arrow[r, dashed, "\rho"] & V \arrow[d, "\nu"] \\ \mathbb{P}(1,1,4) \arrow[rr, dashed, "\phi"'] & & \mathbb{P}^2, \end{tikzcd}$$

or there is a commutative diagram

(26.5)
$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\xi"'] & V \arrow[d, "\nu"] \\ \mathbb{P}(1,1,6) \arrow[r, dashed, "\sigma"'] & \mathbb{P}^2, \end{tikzcd}$$

where ζ, ϕ and σ are birational maps.

It follows from Theorem 1.7, Proposition 1.10 and Lemma 1.16 that

$$\varnothing \neq \mathbb{CS}(X, \frac{1}{k}\mathcal{M}) \subseteq \{P_2, P_3\},$$

and the existence of the commutative diagram 20.4 is obvious, if $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_2, P_3\}$.

Lemma 26.6. *The set* $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ *contains point* P_3.

Proof. Suppose that $P_3 \notin \mathbb{CS}(X, \frac{1}{k}\mathcal{M})$. Then $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_2\}$. Let $\mathcal{D}_2$ be the proper transform of $\mathcal{M}$ on U_2, and P_6 be the singular point of U_2 that is contained in the exceptional divisor of α_2. Then $\mathcal{D}_2 \sim_{\mathbb{Q}} -kK_{U_2}$ by Theorem A.15, the point P_6 is a quotient singularity of type $\frac{1}{3}(1,1,2)$ on U_2, and it follows from Lemmas B.5 and A.16 that $P_6 \in \mathbb{CS}(U_2, \frac{1}{k}\mathcal{D}_2)$.

Let $\pi : Z \to U_2$ be the weighted blow up of P_6 with weights $(1,1,3)$, and $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on Z, and S be a general surface of $|-K_Z|$. Then S is normal, and the base locus of the $|-K_Z|$ consists of the irreducible curve Δ such that $\mathcal{B}|_S \sim_{\mathbb{Q}} k\Delta$.

The equality $\Delta^2 = 1/7$ holds on the surface S, which contradicts Lemmas A.20 and A.21. ☐

To conclude the proof of the Proposition 26.3, we may assume that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_3\}$.

Let $\mathcal{D}_3$ be the proper transform of $\mathcal{M}$ on U_3, and P_4 be the singular point of U_3 that is contained in the exceptional divisor of the α_3. Then $\mathcal{D}_3 \sim_{\mathbb{Q}} -kK_{U_3}$, and P_4 is a quotient singularity of type $\frac{1}{6}(1,1,5)$ on U_3 that is contained in $\mathbb{CS}(U_3, \frac{1}{k}\mathcal{D}_3)$ by Lemmas B.5 and A.16.

Lemma 26.7. *The set* $\mathbb{CS}(U_3, \frac{1}{k}\mathcal{D}_3)$ *consists of the point* P_4.

Proof. Suppose that the set $\mathbb{CS}(U_3, \frac{1}{k}\mathcal{D}_3)$ contains a subvariety C of the variety U_3 that is different from the point P_4. Let G be the exceptional divisor of the morphism β_4. Then G is a cone over a smooth rational curve of degree 6, and C is a ruling of G by Lemma A.16.

The proper transform of G on W is a section of the elliptic fibration ω. Let D be a general surface of the linear system $\mathcal{D}_3$, and Z be a sufficiently general fiber of the fibration ω such that the curve $\beta_2(Z)$ intersects the curve C. Then $\beta_2(Z)$ is not contained in D. Thus, we have

$$\frac{k}{2} = \beta_2(Z) \cdot D \geqslant \mathrm{mult}_C(D) \geqslant k,$$

which is a contradiction. ☐

Hence, the set $\mathbb{CS}(U_3, \frac{1}{k}\mathcal{D}_3)$ consists of the point P_4. Let $\mathcal{D}_{34}$ be the proper transform of the linear system $\mathcal{M}$ on the variety U_{34}, and P_5 be the singular point of the variety U_{34} that is contained in exceptional divisor of the morphism β_4. Then $\mathcal{D}_{34} \sim_{\mathbb{Q}} -kK_{U_{34}}$ by Theorem A.15, the point P_5 is a quotient singularity of type $\frac{1}{5}(1,1,4)$ on U_{34}, but $\mathbb{CS}(U_{34}, \frac{1}{k}\mathcal{D}_{34})$ contains the point P_5 by Lemmas B.5 and A.16. It follows from Theorem A.15 that the proper transform of the linear system $\mathcal{M}$ on Y is contained in fibers of the elliptic fibration η, which implies the existence of the commutative diagram 26.4. The claim of Proposition 26.3 is proved.

27. Case $n = 38$, hypersurface of degree 18 in $\mathbb{P}(1,2,3,5,8)$.

We use the notations and assumptions of chapter 1. Let $n = 38$. Then X is a hypersurface of degree 18 in $\mathbb{P}(1,2,3,5,8)$, the singularities of X consist of the points P_1 and P_2 that are quotient singularities of type $\frac{1}{2}(1,1,1)$, the point P_3 that is a quotient singularity of type $\frac{1}{5}(1,2,3)$, and the point P_4 that is a quotient singularity of type $\frac{1}{8}(1,3,5)$, and the equality $-K_X^3 = 3/40$ holds.

Proposition 27.1. *The claim of Proposition 0.7 holds for the hypersurface* X.

Proof. It follows from Theorem 1.7, Lemma 1.16 and Proposition 1.10 that

$$\varnothing \neq \mathbb{CS}(X, \frac{1}{k}\mathcal{M}) \subseteq \{P_3, P_4\},$$

but the proof of Proposition 21.1 implies that $P_4 \in \mathbb{CS}(X, \frac{1}{k}\mathcal{M})$.

There is a commutative diagram

$$\begin{tikzcd}
 & & Y \arrow[dl, "\gamma_5"'] \arrow[dr, "\gamma_3"] \arrow[drr, "\eta"] & & \\
 & U_{34} \arrow[dl, "\beta_4"'] \arrow[dr, "\beta_3"] & & U_{45} \arrow[dl, "\beta_5"] & \mathbb{P}(1,2,3), \\
U_3 \arrow[dr, "\alpha_3"'] & & U_4 \arrow[dl, "\alpha_4"] & & \\
 & X \arrow[uurrr, dashed, "\psi"'] & & &
\end{tikzcd}$$

where ψ is a projection, α_3 is the weighted blow up of P_3 with weights $(1,2,3)$, α_4 is the weighted blow up of P_4 with weights $(1,3,5)$, β_4 is the weighted blow up with weights $(1,3,5)$ of the proper transform of P_4 on U_3, β_3 is the weighted blow up with weights $(1,2,3)$ of the proper transform of the point P_3 on U_4, β_5 is the weighted blow up with weights $(1,2,3)$ of the singular point of the variety U_4 that is a quotient singularity of type $\frac{1}{5}(1,2,3)$ contained in the exceptional divisor of the morphism α_4, γ_3 is the weighted blow up with weights $(1,2,3)$ of the proper transform of the point P_3 on the U_{45}, γ_5 is the weighted blow up with weights $(1,2,3)$ of the singular point of the variety U_{34} that is a quotient singularity of type $\frac{1}{5}(1,2,3)$ contained in the exceptional divisor of the morphism β_4, and η is an elliptic fibration.

Let $\mathcal{D}_4$ be the proper transform of the linear system $\mathcal{M}$ on the variety U_4, $\bar{P}_3$ be the proper transform of the point P_3 on U_4, and P_5 and P_6 be the singular points of the variety U_4 that are quotient singularities of types $\frac{1}{5}(1,2,3)$ and $\frac{1}{3}(1,1,2)$ contained in exceptional divisor of the morphism α_4 respectively. Then the arguments of the proof of Proposition 25.1 imply that

$$\mathbb{CS}(U_4, \frac{1}{k}\mathcal{D}_4) \cap \{\bar{P}_3, P_5\} \neq \varnothing.$$

Suppose that $\bar{P}_3 \in \mathbb{CS}(U_4, \frac{1}{k}\mathcal{D}_5)$. Then the proofs of Propositions 21.1 and 25.1 implies that the set $\mathbb{CS}(U_4, \frac{1}{k}\mathcal{D}_5)$ contains the singular point P_5. Therefore, the claim of Theorem A.15 implies that the claim of Proposition 0.7 holds for the hypersurface X.

We may assume that the set $\mathbb{CS}(U_4, \frac{1}{k}\mathcal{D}_5)$ contains the point P_5.

Let $\mathcal{D}_{45}$ be the proper transform of $\mathcal{M}$ on U_{45}, and P_7 and P_8 be the singular points of the variety U_{45} that are quotient singularities of types $\frac{1}{2}(1,1,1)$ and $\frac{1}{3}(1,1,2)$ contained in the exceptional divisor of β_5 respectively. Then if follows from Lemma B.5 that

$$\mathbb{CS}(U_{45}, \frac{1}{k}\mathcal{D}_{45}) \neq \varnothing,$$

and Theorem A.15 implies that the claim of Proposition 0.7 holds for the hypersurface X in the case when the set $\mathbb{CS}(U_{45}, \frac{1}{k}\mathcal{D}_{45})$ contains the proper transform of the point P_3 on U_{45}.

Therefore, it follows from Lemma A.16 that to conclude the proof, we may assume that the set $\mathbb{CS}(U_{45}, \frac{1}{k}\mathcal{D}_{45})$ contains either the point P_7, or the point P_8.

Suppose that $P_7 \in \mathbb{CS}(U_{45}, \frac{1}{k}\mathcal{D}_{45})$. Then considering the proper transform of the complete linear system $|-3K_X|$ on the weighted blow up of the point P_7 with weights $(1,1,1)$, we easily obtain a contradiction as in the proof of Lemma 21.3.

Thus, the set $\mathbb{CS}(U_{45}, \frac{1}{k}\mathcal{D}_{45})$ contains the point P_8.

Let $\pi : W \to U_{45}$ be the weighted blow up of P_8, $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on the variety W, and D be a general surface in $|-2K_W|$. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ by Theorem A.15, the surface D is normal, the pencil $|-2K_W|$ is the proper transform of the pencil $|-2K_X|$, and the base locus of the pencil $|-2K_W|$ consists of the curves C and L such that $\alpha_4 \circ \beta_5 \circ \pi(C)$ is the unique base curve of the pencil $|-2K_X|$, and the curve $\beta_5 \circ \pi(L)$ is contained in the exceptional divisor of the morphism α_4.

The the intersection form of the curves C and L on the surface D is negatively definite, but the equivalence $\mathcal{B}|_D \sim_{\mathbb{Q}} kC + kL$ holds, which is impossible by Lemmas A.21 and A.20. $\square$

We use the notations and assumptions of chapter 1. Let $n = 40$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,3,4,5,7)$ of degree 19, the singularities of the hypersurface X consist of the point P_1 that is a quotient singularity of type $\frac{1}{3}(1,1,2)$, the point P_2 that is a quotient singularity of type $\frac{1}{4}(1,1,3)$, the point P_3 that is a quotient singularity of type $\frac{1}{5}(1,2,3)$, and the point P_4 that is a quotient singularity of type $\frac{1}{7}(1,3,4)$, and $-K_X^3 = 19/420$.

There is a commutative diagram

$$\begin{tikzcd}
& Y \arrow[dl, "\beta_4"'] \arrow[dr, "\beta_3"] \arrow[drrr, "\eta"] & & & \\
U_3 \arrow[dr, "\alpha_3"'] & & U_4 \arrow[dl, "\alpha_4"] & & \mathbb{P}(1,3,4), \\
& X \arrow[urrr, dashed, "\psi"'] & & &
\end{tikzcd}$$

where ψ is the natural projection, α_3 is the weighted blow up of the singular point P_3 with weights $(1,2,3)$, α_4 is the weighted blow up of P_4 with weights $(1,3,4)$, β_3 is the weighted blow up with weights $(1,2,3)$ of the proper transform of P_3 on U_4, β_4 is the weighted blow up with weights $(1,3,4)$ of the proper transform of P_4 on U_3, and η is an elliptic fibration.

In the rest of the chapter we prove the following result.

Proposition 28.1. *The claim of Proposition 0.7 holds for the hypersurface X.*

It follows from Theorem 1.7, Lemma 1.16 and Proposition 1.10 that

$$\varnothing \neq \mathbb{CS}(X, \frac{1}{k}\mathcal{M}) \subseteq \{P_3, P_4\}.$$

Let $\mathcal{D}_3$ and $\mathcal{D}_4$ be the proper transforms of the linear system $\mathcal{M}$ on U_3 and U_4 respectively.

Lemma 28.2. *Suppose that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_3, P_4\}$. Then there is a commutative diagram*

$$\begin{tikzcd}
X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & V \arrow[d, "\nu"] \\
\mathbb{P}(1,3,4) \arrow[r, dashed, "\zeta"'] & \mathbb{P}^2,
\end{tikzcd} \tag{28.3}$$

where ζ is a birational map.

Proof. Let $\mathcal{H}$ be the proper transform of the linear system $\mathcal{M}$ on the variety Y. Then it follows from Theorem A.15 that the equivalence $\mathcal{H} \sim_{\mathbb{Q}} -kK_Y$ holds, which implies that $\mathcal{H}$ lies in the fibers of the elliptic fibration η, which implies the existence of the commutative diagram 28.3. $\square$

Let P_5 and P_6 be the singular points of the variety U_3 that are contained in the exceptional divisor of α_3 such that P_5 and P_6 are quotient singularities of types $\frac{1}{2}(1,1,1)$ and $\frac{1}{3}(1,1,2)$ respectively, and P_7 and P_8 are the singular points of the variety U_4 that are quotient singularities of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{4}(1,1,3)$ contained in the exceptional divisor of the morphism α_4 respectively. Then it follows from Theorem A.15 and Lemmas B.5 and Lemma A.16 that

$$\begin{cases} P_4 \notin \mathbb{CS}(X, \frac{1}{k}\mathcal{M}) \Rightarrow \mathbb{CS}(U_3, \frac{1}{k}\mathcal{D}_3) \cap \{P_5, P_6\} \neq \varnothing, \\ P_3 \notin \mathbb{CS}(X, \frac{1}{k}\mathcal{M}) \Rightarrow \mathbb{CS}(U_4, \frac{1}{k}\mathcal{D}_4) \cap \{P_7, P_8\} \neq \varnothing. \end{cases}$$

Lemma 28.4. *Suppose that $P_4 \notin \mathbb{CS}(X, \frac{1}{k}\mathcal{M})$. Then $P_5 \notin \mathbb{CS}(U_3, \frac{1}{k}\mathcal{D}_3)$.*

Proof. Suppose that the set $\mathbb{CS}(U_3, \frac{1}{k}\mathcal{D}_3)$ contains the point P_5. Let $\gamma : W \to U_3$ be the weighted blow up of P_5 with weights $(1,1,1)$, F be the exceptional divisor of γ, $\mathcal{D}$ be the proper transform of the linear system $\mathcal{M}$ on the variety W, $\mathcal{H}$ be the proper transform of the pencil $|-3K_X|$ on the variety W, $\bar{D}$ be a sufficiently general surface of the linear system $\mathcal{D}$, and $\bar{H}$ be a sufficiently general surface of the pencil $\mathcal{H}$. Then we have the equivalence

$$\bar{H} \sim_{\mathbb{Q}} (\alpha_3 \circ \gamma)^*(-3K_X) - \frac{3}{5}\gamma^*(E) - \frac{1}{3}F,$$

but the base locus of the pencil $\mathcal{H}$ consists of the curve $\bar{C}$ such that $\alpha_3 \circ \gamma(C)$ is the unique base curve of the pencil $|-3K_X|$. On the other hand, it follows from Theorem A.15 that

$$\bar{D} \sim_{\mathbb{Q}} -kK_W \sim_{\mathbb{Q}} \gamma^*(-kK_{U_3}) - \frac{k}{3}F.$$

The equivalence $\bar{D}|_{\bar{H}} \sim_{\mathbb{Q}} k\bar{C}$ and the equality $\bar{C}^2 = -13/28$ hold on $\bar{H}$. Therefore, it follows from Lemma A.21 that the support of $\bar{H} \cdot \bar{D}$ consists of $\bar{C}$, which contradicts Lemma A.20. □

Lemma 28.5. *Suppose that $P_4 \notin \mathbb{CS}(X, \frac{1}{k}\mathcal{M})$. Then $P_6 \notin \mathbb{CS}(U_3, \frac{1}{k}\mathcal{D}_3)$.*

Proof. Suppose that $P_6 \in \mathbb{CS}(U_3, \frac{1}{k}\mathcal{D}_3)$. Let $\gamma : W \to U_3$ be the weighted blow up of P_6 with weights $(1,1,2)$, F and G be the exceptional divisors of α_3 and γ respectively, $\mathcal{B}$ and $\mathcal{D}$ be the proper transforms of $\mathcal{M}$ and $|-7K_X|$ on the variety W respectively, and D be a sufficiently general surface of the linear system $\mathcal{D}$. Then it follows from Theorem A.15 that

$$\mathcal{B} \sim_{\mathbb{Q}} -kK_W \sim_{\mathbb{Q}} (\alpha_3 \circ \gamma)^*(-kK_X) - \frac{k}{5}F - \frac{k}{2}G,$$

but the base locus of the linear system $\mathcal{D}$ does not contain curves. Moreover, we have

$$D \sim_{\mathbb{Q}} (\alpha_3 \circ \gamma)^*(-7K_X) - \frac{2}{5}F - G,$$

the divisor D is nef, but the explicit calculations imply that

$$D \cdot B_1 \cdot B_2 = \Big((\alpha_3 \circ \gamma)^*(-7K_X) - \frac{2}{5}F - G\Big)\Big((\alpha_3 \circ \gamma)^*(-kK_X) - \frac{k}{5}F - \frac{k}{2}G\Big)^2 = -\frac{3}{4}k^2,$$

where B_1 and B_2 are general surfaces in $\mathcal{B}$, which is a contradiction. □

Lemma 28.6. *Suppose that $P_3 \notin \mathbb{CS}(X, \frac{1}{k}\mathcal{M})$. Then $P_7 \notin \mathbb{CS}(U_4, \frac{1}{k}\mathcal{D}_4)$.*

Proof. Suppose that the set $\mathbb{CS}(U_4, \frac{1}{k}\mathcal{D}_4)$ contains the point P_7. Let $\gamma : W \to U_4$ be the weighted blow up of P_7 with weights $(1,1,2)$, F be the exceptional divisor of γ, $\mathcal{D}$ be the proper transform of the linear system $\mathcal{M}$ on W, $\mathcal{H}$ be the proper transform of $|-4K_X|$ on W, $\bar{D}$ be a general surface of the linear system $\mathcal{D}$, and $\bar{H}$ be a general surface of the linear system $\mathcal{H}$. Then

$$\bar{H} \sim_{\mathbb{Q}} (\alpha_3 \circ \gamma)^*(-4K_X) - \frac{4}{5}\gamma^*(E) - \frac{1}{3}F,$$

and the base locus of $\mathcal{H}$ consists of the irreducible curve $\bar{C}$ such that $\alpha_3 \circ \gamma(C)$ is the unique base curve of the linear system $|-4K_X|$. On the other hand, it follows from Theorem A.15 that the equivalence $\bar{D} \sim_{\mathbb{Q}} -kK_W$ holds. Moreover, the equality $\bar{C}^2 = -1/6$ holds on $\bar{H}$, which implies that the support of $\bar{H} \cdot \bar{D}$ consists of $\bar{C}$, because $\bar{D}|_{\bar{H}} \sim_{\mathbb{Q}} k\bar{C}$, which contradicts Lemma A.20. □

Lemma 28.7. *Suppose that $P_3 \notin \mathbb{CS}(X, \frac{1}{k}\mathcal{M})$. Then $P_8 \notin \mathbb{CS}(U_4, \frac{1}{k}\mathcal{D}_4)$.*

Proof. Suppose that the set $\mathbb{CS}(U_4, \frac{1}{k}\mathcal{D}_4)$ contains the point P_8. Let $\gamma : W \to U_4$ be the weighted blow up of P_8 with weights $(1,1,3)$, $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on W, $\bar{H}$ be a general surface of the pencil $|-3K_W|$, and $\bar{D}$ be a general surface in $\mathcal{D}$. Then $\bar{D} \sim_{\mathbb{Q}} -kK_W$, but the base locus of the pencil $|-3K_W|$ consists of the irreducible curve $\bar{C}$ such that $\alpha_4 \circ \gamma(C)$ is the base curve of $|-3K_X|$. Moreover, the equality $\bar{C}^2 = -1/20$ holds on $\bar{H}$, but $\bar{D}|_{\bar{H}} \sim_{\mathbb{Q}} k\bar{C}$, which implies that the support of $\bar{H} \cdot \bar{D}$ consists of $\bar{C}$, which is impossible by Lemma A.20. □

The claim Proposition 28.1 is proved.

29. Case $n = 43$, hypersurface of degree 20 in $\mathbb{P}(1,2,4,5,9)$.

We use the notations and assumptions of chapter 1. Let $n = 43$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,2,4,5,9)$ of degree 20, the equality $-K_X^3 = 1/18$ holds, and the singularities of the hypersurface X consist of the points P_1, P_2, P_3, P_4 and P_5 that are quotient singularities of type $\frac{1}{2}(1,1,1)$, and the point P_6 that is a quotient singularity of type $\frac{1}{9}(1,4,5)$.

Proposition 29.1. *The claim of Proposition 0.7 holds for the hypersurface X.*

In the rest of the chapter we prove Proposition 29.1. It follows from Theorem 1.7, Lemma 1.16 and Proposition 1.10 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_6\}$. There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & Y \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,2,4), \end{tikzcd}$$

where ψ is a projection, α is the weighted blow up of P_6 with weights $(1,4,5)$, β is the weighted blow up with weights $(1,1,4)$ of the singular point of the variety U that is a quotient singularity of type $\frac{1}{5}(1,1,4)$ contained in the exceptional divisor of α, and η is an elliptic fibration.

Let $\mathcal{D}$ be the proper transform of the linear system $\mathcal{M}$ on the variety U, and P_7 and P_8 be the singular points of U that are quotient singularities of types $\frac{1}{4}(1,1,3)$ and $\frac{1}{5}(1,1,4)$ contained in the exceptional divisor of α respectively. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem A.15.

We must show that the proper transform of the linear system $\mathcal{M}$ on the variety Y is contained in the fibers of the η, which is implied by Theorem A.15, if $P_8 \in \mathbb{CS}(U, \frac{1}{k}\mathcal{D})$. Thus, to conclude the proof of Proposition 29.1 we may assume that $P_8 \notin \mathbb{CS}(U, \frac{1}{k}\mathcal{D})$.

Remark 29.2. The set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ contains P_7 by Lemma A.16, because $-K_U$ is nef and big.

Let $\gamma : W \to U$ be the weighted blow up of the singular point P_7 with weights $(1,1,3)$, $\mathcal{B}$ be the proper transform of the linear system $\mathcal{M}$ on the variety W, and P_9 be the singular point of the variety W that is a quotient singularity of type $\frac{1}{3}(1,1,2)$ contained in the exceptional divisor of the morphism γ. Then the equivalence $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ holds by Theorem A.15.

Lemma 29.3. *The set $\mathbb{CS}(W, \frac{1}{k}\mathcal{B})$ does not contains the point P_9.*

Proof. Suppose that the set $\mathbb{CS}(W, \frac{1}{k}\mathcal{B})$ contains the point P_9. Let $\pi : Z \to W$ be the weighted blow up of the singular point P_9 with weights $(1,1,2)$, $\mathcal{H}$ be the proper transform of the linear system $\mathcal{M}$ on the variety Z, and $\mathcal{P}$ be the proper transform of the linear system $|-5K_X|$ on the variety Z. Then $\mathcal{H} \sim_{\mathbb{Q}} -kK_Z$ by Theorem A.15, but the base locus of $\mathcal{P}$ consists of the irreducible curve Γ such that $\alpha \circ \gamma \circ \pi(\Gamma)$ is the base curve in of $|-5K_X|$.

Let H_1 and H_2 be general surfaces of the linear system $\mathcal{H}$, and D be general surface of the linear system $\mathcal{P}$. Then $D \cdot \Gamma = 1$ and $D^3 = 6$. Therefore, the divisor D is nef and big, but the elementary computations imply that $D \cdot H_1 \cdot H_2 = 0$, which is impossible by Corollary A.19. □

Therefore, the claim of Lemma A.16 implies the following corollary.

Corollary 29.4. *The singularities of the log pair $(W, \frac{1}{k}\mathcal{B})$ are terminal.*

The hypersurface X can be given by the quasihomogeneous equation of degree 20

$$w^2y + wf_{11}(x,y,z,t) + f_{20}(x,y,z,t) = 0 \subset \mathbb{P}(1,2,4,5,9) \cong \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 2$, $\mathrm{wt}(z) = 4$, $\mathrm{wt}(t) = 5$, $\mathrm{wt}(w) = 9$, and $f_i(x,y,z,t)$ is a quasihomogeneous polynomial of degree i. Let D be a general surface in $|-2K_X|$, and S be a surface that is cut on the hypersurface X by the equation $x = 0$. Then D is cut on X by the quasihomogeneous equation $\lambda x^2 + \mu y = 0$, where $(\lambda, \mu) \in \mathbb{P}^1$, and the base locus of $|-2K_X|$ consists of the irreducible curve C that is cut on the hypersurface X by the equations $x = y = 0$.

In the neighborhood of the point P_6, the monomials x, z and t can be considered as weighted local coordinates on X such that $\mathrm{wt}(x) = 1$, $\mathrm{wt}(z) = 4$ and $\mathrm{wt}(t) = 5$. Then in the neighborhood of the singular point P_7, the weighted blow up α is given by the equations

$$x = \tilde{x}\tilde{z}^{\frac{1}{9}},\ z = \tilde{z}^{\frac{4}{9}},\ t = \tilde{t}\tilde{z}^{\frac{5}{9}},$$

where $\tilde{x}$, $\tilde{z}$ and $\tilde{t}$ are weighted local coordinated on the variety U in the neighborhood of the singular point P_7 such that $\mathrm{wt}(\tilde{x}) = 1$, $\mathrm{wt}(\tilde{z}) = 3$ and $\mathrm{wt}(\tilde{t}) = 1$.

Let E be the exceptional divisor of the morphism α, and $\tilde{D}$, $\tilde{S}$ and $\tilde{C}$ be the proper transforms on the variety U of the surface D, the surface S and the curve C respectively. Then E is given by the equation $\tilde{z} = 0$, and the surface $\tilde{S}$ is given by the equation $\tilde{x} = 0$. Moreover, it follows from the local equation of the surface $\tilde{D}$ that $\tilde{D} \cdot \tilde{S} = \tilde{C} + 2\tilde{L}_1$, where $\tilde{L}_1$ is the curve that is locally given by the equations $\tilde{z} = \tilde{x} = 0$. Moreover, the surface $\tilde{D}$ is not normal in a general point of the curve $\tilde{L}_1$. Nevertheless, we have the equivalences

$$\tilde{D} \sim_{\mathbb{Q}} 2\tilde{S} \sim_{\mathbb{Q}} \alpha^*(-2K_X) - \frac{2}{9}E.$$

In the neighborhood of the point P_9 the morphism γ is given by the equations

$$\tilde{x} = \bar{x}\bar{z}^{\frac{1}{4}},\ \tilde{z} = \bar{z}^{\frac{3}{4}},\ \tilde{t} = \bar{t}\bar{z}^{\frac{1}{4}},$$

where $\bar{x}$, $\bar{z}$ and $\bar{t}$ are weighted local coordinates on the variety W in the neighborhood of the point P_9 such that $\mathrm{wt}(\bar{x}) = 1$, $\mathrm{wt}(\bar{z}) = 2$ and $\mathrm{wt}(\bar{t}) = 1$. In particular, the exceptional divisor of the morphism γ is given by the equation $\bar{z} = 0$, and the proper transform of the surface S on the variety W is given by the equation $\bar{x} = 0$.

Let F be the exceptional divisor of the morphism γ, and $\bar{D}$, $\bar{S}$, $\bar{E}$, $\bar{C}$ and $\bar{L}_1$ be the proper transforms on the variety W of the surface D, the surface S, the surface E, the curve C and the curve $\tilde{L}_1$ respectively. Then we the equivalence

$$(29.5)\qquad \begin{cases} \bar{S} \sim_{\mathbb{Q}} -K_W \sim_{\mathbb{Q}} (\alpha\circ\gamma)^*(-K_X) - \frac{1}{9}\gamma^*(E) - \frac{1}{4}F, \\ \bar{D} \sim_{\mathbb{Q}} -2K_W \sim_{\mathbb{Q}} (\alpha\circ\gamma)^*(-2K_X) - \frac{2}{9}\gamma^*(E) - \frac{1}{2}F, \\ \bar{E} \sim_{\mathbb{Q}} \gamma^*(E) - \frac{3}{4}F. \end{cases}$$

Let $\bar{L}_2$ be the curve on the variety W that is given by the equation $\bar{z} = \bar{x} = 0$. Then

$$\bar{D} \cdot \bar{S} = \bar{C} + 2\bar{L}_1 + \bar{L}_2,\ \bar{D} \cdot \bar{E} = 2\bar{L}_1,\ \bar{D} \cdot F = 2\bar{L}_2,$$

but the base locus of $|-2K_W|$ consists of the curves $\bar{C}$, $\bar{L}_1$ and $\bar{L}_2$. The equivalences 29.5 imply

$$\bar{D} \cdot \bar{C} = 0,\ \bar{D} \cdot \bar{L}_1 = -\frac{2}{5},\ \bar{D} \cdot \bar{L}_2 = \frac{2}{3}.$$

Let $\bar{H}$ and $\bar{T}$ be the proper transforms on the variety W of the surfaces that are cut on the hypersurface X by the equations $y = 0$ and $t = 0$ respectively. Then

$$(29.6)\qquad \begin{cases} \bar{T} \sim_{\mathbb{Q}} (\alpha\circ\gamma)^*(-5K_X) - \frac{5}{9}\gamma^*(E) - \frac{1}{4}F \sim_{\mathbb{Q}} (\alpha\circ\gamma)^*(-5K_X) - \frac{5}{9}\bar{E} - \frac{2}{3}F, \\ \bar{H} \sim_{\mathbb{Q}} (\alpha\circ\gamma)^*(-2K_X) - \frac{11}{9}\gamma^*(E) - \frac{3}{2}F \sim_{\mathbb{Q}} (\alpha\circ\gamma)^*(-2K_X) - \frac{11}{9}\bar{E} - \frac{5}{3}F, \end{cases}$$

which implies that

$$-14K_W \sim_{\mathbb{Q}} 14\bar{D} \sim_{\mathbb{Q}} 2\bar{T} + 2\bar{H} + 2\bar{E},$$

and the support of the cycle $\bar{T} \cdot \bar{H}$ does not contain the curves $\bar{L}_2$ and $\bar{C}$. Therefore, the base locus of the linear system $|-14K_W|$ does not contain curves except the curve $\bar{L}_1$.

The singularities of the mobile log pair $(W, \lambda|-14K_W|)$ are log-terminal for some rational number $\lambda > 1/14$, but the divisor $K_W + \lambda|-14K_W|$ has non-negative intersection with all curves on the variety W except the curve $\bar{L}_1$. It follows from [15] that the log-flip $\zeta : W \dashrightarrow Z$ in the curve $\bar{L}_1$ with respect to the log pair $(W, \lambda|-14K_W|)$ exists.

Let $\mathcal{P}$ be the proper transform of the linear system $\mathcal{M}$ on the variety Z. Then the singularities of the log pair $(Z, \frac{1}{k}\mathcal{P})$ are terminal, because the singularities of the log pair $(W, \frac{1}{k}\mathcal{B})$ are terminal, but the rational map ζ is a log flop with respect to the log pair $(W, \frac{1}{k}\mathcal{B})$, but $-K_Z$ is numerically effective, because the base locus of the linear system $|-14K_W|$ does not contain curves outside the curve $\bar{L}_1$, and the inequality $-K_W \cdot \bar{L}_1 < 0$ holds.

In the rest of the chapter we show that $-K_Z$ is big, which contradicts Lemma B.5.

The rational functions y/x^2 and ty/x^7 are contained in $|2S|$ and $|7S|$ respectively, but the equivalences 29.6 implies that y/x^2 and ty/x^7 are contained in $|2\bar{S}|$ and $|7\bar{S}|$ respectively.

Let $\bar{Z}$ be the proper transform on the variety W of the irreducible surface that is cut on the hypersurface X by the equation $z = 0$. Then the equivalences

$$\bar{Z} \sim_{\mathbb{Q}} (\alpha \circ \gamma)^*(-4K_X) - \frac{4}{9}\gamma^*(E) \sim_{\mathbb{Q}} (\alpha \circ \gamma)^*(-4K_X) - \frac{4}{9}\bar{E} - \frac{1}{3}F$$

hold, which imply that $-6K_W \sim_{\mathbb{Q}} \bar{Z} + \bar{H} + \bar{E}$. Thus, the rational function zy/x^6 is contained in the linear system $|6\bar{S}|$. Thus, the linear system $|-42K_W|$ maps the variety W dominantly on some three-dimensional variety, which implies that the divisor $-K_Z$ is big.

The claim of Proposition 29.1 is proved.

Remark 29.7. The rational function wy^3/x^{15} is contained in the linear system $|15\bar{S}|$, which implies that the linear system $|-210K_W|$ induces the birational map $W \dashrightarrow X'$ such that the variety X' is a hypersurface in $\mathbb{P}(1,2,6,7,15)$ of degree 30 with canonical singularities.

30. Case $n = 44$, hypersurface of degree 20 in $\mathbb{P}(1,2,5,6,7)$.

We use the notations and assumptions of chapter 1. Let $n = 44$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,2,5,6,7)$ of degree 20, the equality $-K_X^3 = 1/21$ holds, and the singularities of the hypersurface X consist of the points P_1, P_2 and P_3 that are quotient singularities of type $\frac{1}{2}(1,1,1)$, the point P_4 that is a quotient singularity of type $\frac{1}{6}(1,1,5)$, and the point P_5 that is a quotient singularity of type $\frac{1}{7}(1,2,5)$.

There is a commutative diagram

$$\begin{tikzcd}
 & Y \arrow[dl, "\beta_5"'] \arrow[dr, "\beta_4"] \arrow[rrrd, "\eta"] & & & \\
U_4 \arrow[dr, "\alpha_4"'] & & U_5 \arrow[dl, "\alpha_5"] & & \mathbb{P}(1,2,5), \\
 & X \arrow[rrru, dashed, "\psi"'] & & &
\end{tikzcd}$$

where ψ is a projection, α_4 is the weighted blow up of P_4 with weights $(1,1,5)$, α_5 is the weighted blow up of P_5 with weights $(1,2,5)$, β_4 is the weighted blow up with weights $(1,1,5)$ of the proper transform of P_4 on U_5, β_5 is the weighted blow up with weights $(1,2,5)$ of the proper transform of the point P_5 on the variety U_4, and η is an elliptic fibration.

The hypersurface X can be given by the quasihomogeneous equation of degree 20

$$w^2t + wf(x,y,z,t) + g(x,y,t) = 0 \subset \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 2$, $\mathrm{wt}(z) = 5$, $\mathrm{wt}(t) = 6$, $\mathrm{wt}(w) = 7$, and f, g and h are quasihomogeneous polynomials. Let $\xi : X \dashrightarrow \mathbb{P}^4$ be the rational map that is given by the linear subsystem of the linear system $|-6K_X|$ consisting of divisors

$$\lambda_0 t + \lambda_1 y^3 + \lambda_2 y^2 x^2 + \lambda_3 y x^4 + \lambda_4 x^6 = 0,$$

where $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{P}^4$. Then the rational map ξ is not defined in the point P_5, and the closure of the image of the rational map ξ is the surface $\mathbb{P}(1,1,3)$, which can be identified with a cone over a smooth rational curve in $\mathbb{P}^3$ of degree 3. Moreover, the normalization of a general fiber of the rational map ξ is an elliptic curve. There is a commutative diagram

$$\begin{tikzcd}
U_5 \arrow[d, "\alpha_5"'] & W \arrow[l, "\beta_6"'] \arrow[d, "\omega"] \\
X \arrow[r, dashed, "\xi"'] & \mathbb{P}(1,1,3),
\end{tikzcd}$$

where β_6 is the weighted blow up with weights $(1,2,3)$ of the singular point of U_5 that is a quotient singularity of type $\frac{1}{5}(1,2,3)$ contained in the exceptional divisor of α_5, and ω is an elliptic fibration.

In the rest of the chapter we prove the following result.

Proposition 30.1. *Either there is a commutative diagram*

(30.2)
$$\begin{tikzcd}
X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & V \arrow[d, "\nu"] \\
\mathbb{P}(1,2,5) \arrow[r, dashed, "\phi"'] & \mathbb{P}^2,
\end{tikzcd}$$

or there is a commutative diagram

(30.3)
$$\begin{tikzcd}
X \arrow[r, dashed, "\zeta"] \arrow[d, dashed, "\xi"'] & X \arrow[r, dashed, "\rho"] & V \arrow[d, "\nu"] \\
\mathbb{P}(1,1,3) \arrow[rr, dashed, "\sigma"'] & & \mathbb{P}^2
\end{tikzcd}$$

where ϕ, ζ and σ are birational maps.

It follows from Theorem 1.7, Lemma 1.16 and Proposition 1.10 that

$$\varnothing \neq \mathbb{CS}(X, \frac{1}{k}\mathcal{M}) \subseteq \{P_4, P_5\},$$

but the proof of Lemma 18.2 implies that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains the point P_5.

The existence of the commutative diagram 30.2 easily follows from Theorem A.15 in the case when $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_4, P_5\}$. Thus, to conclude the proof of Proposition 30.1, we may assume that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ consists of the point P_5.

In the rest of the chapter we prove the existence of the commutative diagram 30.3.

Let $\mathcal{D}_5$ be the proper transform of $\mathcal{M}$ on U_5. Then $\mathcal{D}_5 \sim_{\mathbb{Q}} -kK_{U_5}$ by Theorem A.15, which implies that the set $\mathbb{CS}(U_5, \frac{1}{k}\mathcal{D}_5)$ is not empty by Lemma B.5. Let G be the exceptional divisor of the morphism α_5, and $\bar{P}_6$ and $\bar{P}_7$ are the singular points of G that are quotient singularities of types $\frac{1}{5}(1,2,3)$ and $\frac{1}{2}(1,1,1)$ on U_5 respectively. Then it follows from Lemma A.16 that the set $\mathbb{CS}(U_5, \frac{1}{k}\mathcal{D}_5)$ contains either the point $\bar{P}_6$, or the point $\bar{P}_7$.

In the case when the set $\mathbb{CS}(U_5, \frac{1}{k}\mathcal{D}_5)$ contains the point $\bar{P}_6$, the existence of the commutative diagram 30.3 follows from Theorem A.15. Therefore, to conclude the proof of Proposition 30.1, we may assume that the set $\mathbb{CS}(U_5, \frac{1}{k}\mathcal{D}_5)$ contains the point $\bar{P}_7$.

In the rest of the chapter we obtain a contradiction.

Remark 30.4. The linear system $|-5K_{U_5}|$ is a proper transform of $|-5K_X|$, and the base locus of the linear system $|-5K_{U_5}|$ consists of the irreducible curve that is the fiber of the rational map $\psi \circ \alpha_5$ passing through the point $\bar{P}_7$.

Let $\pi : U \to U_5$ be the weighted blow up of $\bar{P}_7$ with weights $(1,1,1)$, F be the exceptional divisor of π, $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on the variety U, and $\mathcal{H}$ be the proper transform of the linear system $|-5K_{U_5}|$ on the variety U. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem A.15, but

$$\mathcal{H} \sim_{\mathbb{Q}} \pi^*(-5K_{U_5}) - \frac{1}{2}F,$$

and the base locus of $\mathcal{H}$ consists of the irreducible curve Z such that $\alpha_5 \circ \pi(Z)$ is the unique curve in the base locus of the linear system $|-5K_X|$.

Let S be a general surface of the linear system $\mathcal{H}$. Then the equality $S \cdot Z = 1/3$ holds, which implies that the divisor $\pi^*(-10K_{U_5}) - F$ is nef. Let D_1 and D_2 be general surfaces of the linear system $\mathcal{D}$. Then we have

$$-\frac{2k^2}{3} = \Big(\pi^*(-10K_{U_5}) - F\Big) \cdot \Big(\pi^*(-kK_{U_5}) - \frac{k}{2}F\Big)^2 = \Big(\pi^*(-10K_{U_5}) - F\Big) \cdot D_1 \cdot D_2 \geqslant 0,$$

which is a contradiction.

The claim of Proposition 30.1 is proved.

31. Case $n = 47$, hypersurface of degree 21 in $\mathbb{P}(1,1,5,7,8)$.

We use the notations and assumptions of chapter 1. Let $n = 47$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,1,5,7,8)$ of degree 21, the equality $-K_X^3 = 3/40$ holds, and the singularities of X consist of the point P_1 that is a quotient singularity of type $\frac{1}{5}(1,2,3)$, and the point P_2 that is a quotient singularity of type $\frac{1}{8}(1,1,7)$.

Proposition 31.1. *The claim of Proposition 0.7 holds for the hypersurface X.*

In the rest of the chapter we prove Proposition 31.1. It follows from Theorem 1.7, Lemma 1.16 and Proposition 1.10 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) \subseteq \{P_1, P_2\}$. Moreover, to conclude the proof of Proposition 31.1 we may assume that the singularities of $(X, \frac{1}{k}\mathcal{M})$ are canonical (see Remark 1.6).

The hypersurface X can be given by the equation

$$w^2z + \sum_{i=0}^{2} wz^i g_{13-5i}(x,y,t) + \sum_{i=0}^{3} z^i g_{21-5i}(x,y,t) = 0 \subset \operatorname{Proj}\bigl(\mathbb{C}[x,y,z,t,w]\bigr),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 1$, $\mathrm{wt}(z) = 5$, $\mathrm{wt}(t) = 7$, $\mathrm{wt}(w) = 8$, and $g_i(x,y,t)$ is a quasihomogeneous polynomial of degree i. There is a commutative diagram

$$\begin{tikzcd}
U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] & Y \arrow[l, "\gamma"'] \arrow[d, "\eta"] \\
X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,5),
\end{tikzcd}$$

where α is the weighted blow up of the point P_2 with weights $(1,1,7)$, β is the weighted blow up with weights $(1,1,6)$ of the singular point of the variety U that is a quotient singularity of type $\frac{1}{7}(1,1,6)$, γ is the weighted blow up with weights $(1,1,5)$ of the singular point W that is a quotient singularity of type $\frac{1}{6}(1,1,5)$, and η is and elliptic fibration.

Lemma 31.2. *The set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains the point P_2.*

Proof. Suppose that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ does not contains P_2. Then $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_1\}$.

Let $\pi : Z \to X$ be the weighted blow up of P_1 with weights $(1,2,3)$, E be the exceptional divisor of π, and $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on the variety Z. Then E is the weighted projective space $\mathbb{P}(1,2,35)$, and the equivalence $\mathcal{B} \sim_{\mathbb{Q}} -kK_Z$ holds by Theorem A.15.

Let $\bar{P}_3$ and $\bar{P}_4$ be the singular points of the variety Z that are contained in the divisor E such that $\bar{P}_3$ and $\bar{P}_4$ are quotient singularities of types $\frac{1}{2}(1,1,1)$ and $\frac{1}{3}(1,1,2)$ respectively. Then the proof of Proposition 22.1 implies that

$$\mathbb{CS}(Z, \frac{1}{k}\mathcal{B}) \cap \{\bar{P}_3, \bar{P}_4\} = \varnothing,$$

and it follows from Lemma A.16 that the singularities of $(Z, \frac{1}{k}\mathcal{B})$ are terminal.

The base locus of the pencil $|-K_Z|$ consists of the irreducible curves C and L such that the curve $\pi(C)$ is cut by the equations $x = y = 0$ on the hypersurface X, the curve L is contained in the divisor E, and L is contained in $|\mathcal{O}_{\mathbb{P}(1,2,3)}(1)|$. The inequalities

$$-K_Z \cdot C < 0, \ -K_Z \cdot L > 0$$

hold, which imply that the curve C is the only curve on the variety Z that has negative intersection with the divisor $-K_Z$.

It follows from [15] that the antiflip $\zeta : Z \dashrightarrow \bar{Z}$ in the curve C exists, and $-K_{\bar{Z}}$ is nef.

Let $\mathcal{P}$ be the proper transform of the linear system $\mathcal{M}$ on the variety $\bar{Z}$. Then the singularities of the log pair $(\bar{Z}, \frac{1}{k}\mathcal{P})$ are terminal, because the singularities of the log pair $(Z, \frac{1}{k}\mathcal{B})$ are terminal, and the antiflip ζ is a log flop with respect to the log pair $(Z, \frac{1}{k}\mathcal{B})$.

One can easily check that the rational functions y/x, zy/x^6, ty/x^8 and yw/x^9 are contained in the linear system $|-aK_Z|$, where $a = 1$, 6, 8 and 9 respectively. Therefore, the complete linear system $|-72K_Z|$ induces the birational map $\chi : Z \dashrightarrow \bar{X}$ such that $\bar{X}$ is a hypersurface of degree 24 in $\mathbb{P}(1,1,6,8,9)$. Hence, the divisor $-K_{\bar{Z}}$ is big, which contradicts Lemma B.5. $\square$

Let G be the exceptional divisor of the morphism α, $\mathcal{D}$ be the proper transform of the linear system $\mathcal{M}$ on the variety U, $\bar{P}_1$ be the proper transform of P_1 on U, and $\bar{P}_5$ be the singular point of the variety U that is contained in G. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem A.15.

Lemma 31.3. *The set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ contains the point $\bar{P}_5$.*

Proof. Suppose that $\bar{P}_5 \notin \mathbb{CS}(U, \frac{1}{k}\mathcal{D})$. Then it follows from Lemmas B.5 and A.16 that

$$\mathbb{CS}(U, \frac{1}{k}\mathcal{D}) = \{\bar{P}_1\},$$

because the divisor $-K_U$ is nef and big.

Let $\pi : Z \to U$ be the weighted blow up of the point $\bar{P}_1$ with weights $(1,2,3)$, E be the exceptional divisor of the morphism π, and $\bar{G}$ and $\mathcal{B}$ be the proper transforms of G and $\mathcal{M}$ on the variety Z respectively. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_Z$ by Theorem A.15, but the proof of Lemma 31.2 implies that the singularities of the log pair $(Z, \frac{1}{k}\mathcal{B})$ are terminal.

Let S_x, S_y, S_z, S_t and S_w be the proper transforms on the variety Z of the surfaces that are cut on X by the equations $x = 0$, $y = 0$, $z = 0$, $t = 0$ and $w = 0$ respectively. Then

$$\text{(31.4)} \qquad \begin{cases} S_x \sim_{\mathbb{Q}} (\alpha \circ \pi)^*(-K_X) - \frac{1}{5}E - \frac{1}{8}\bar{G}, \\ S_y \sim_{\mathbb{Q}} (\alpha \circ \pi)^*(-K_X) - \frac{6}{5}E - \frac{1}{8}\bar{G}, \\ S_z \sim_{\mathbb{Q}} (\alpha \circ \pi)^*(-5K_X) - \frac{1}{5}E - \frac{13}{8}\bar{G}, \\ S_t \sim_{\mathbb{Q}} (\alpha \circ \pi)^*(-7K_X) - \frac{2}{5}E - \frac{7}{8}\bar{G}, \\ S_w \sim_{\mathbb{Q}} (\alpha \circ \pi)^*(-8K_X) - \frac{3}{5}E. \end{cases}$$

The base locus of the pencil $|-K_Z|$ consists of the irreducible curves C and L such that the curve $\alpha \circ \pi(C)$ is cut by the equation $x = y = 0$ on the hypersurface X, the curve L is contained in the divisor E, and the curve L is the unique curve of the linear system $|\mathcal{O}_{\mathbb{P}(1,2,3)}(1)|$.

It follows from [15] that there is the antiflip $\zeta : Z \dashrightarrow \bar{Z}$ in C such that $-K_{\bar{Z}}$ is nef.

Let $\mathcal{P}$ be the proper transform of the linear system $\mathcal{M}$ on the variety $\bar{Z}$. Then the singularities of the log pair $(\bar{Z}, \frac{1}{k}\mathcal{P})$ are terminal.

The equivalences 31.4 imply that the functions y/x, zy/x^6, ty/x^8 and wzy^2/x^{15} are contained in the complete linear system $|aS_x|$, where $a = 1$, 6, 8 and 15 respectively. Hence, the linear system $|-120K_Z|$ induces the birational map $\chi : Z \dashrightarrow \bar{X}$ such that $\bar{X}$ is a hypersurface of degree 30 in $\mathbb{P}(1,1,6,8,15)$, which implies that $-K_{\bar{Z}}$ is big, which contradicts Lemma B.5. $\square$

Remark 31.5. It follows from the proof of Proposition 13.1 that the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ does not contain the subvarieties of the surface G that are different from the point $\bar{P}_5$.

Let $\mathcal{H}$ be the proper transform of the linear system $\mathcal{M}$ on the variety W, F be the exceptional divisor of β, $\tilde{P}_1$ be the proper transform of P_1 on W, and $\tilde{P}_6$ be the singular point of W that is contained in F. Then $\mathcal{H} \sim_{\mathbb{Q}} -kK_W$ by Theorem A.15, and the divisor $-K_W$ is nef and big, which implies that the set $\mathbb{CS}(W, \frac{1}{k}\mathcal{H})$ is not empty by Lemma B.5.

Lemma 31.6. *Suppose that $\tilde{P}_6 \in \mathbb{CS}(W, \frac{1}{k}\mathcal{H})$. Then there is a commutative diagram*

$$\text{(31.7)} \qquad \begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & V \arrow[d, "\nu"] \\ \mathbb{P}(1,1,5) \arrow[r, dashed, "\zeta"'] & \mathbb{P}^2, \end{tikzcd}$$

where ζ is a birational map.

Proof. Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on Y. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_Y$ by Theorem A.15, which implies that the linear system $\mathcal{B}$ lies in the fibers of the morphism η, which implies the existence of the commutative diagram 31.7. $\square$

To conclude the proof of Proposition 31.1, we may assume that the set $\mathbb{CS}(W, \frac{1}{k}\mathcal{H})$ does not contains the point $\tilde{P}_6$. In particular, the set $\mathbb{CS}(W, \frac{1}{k}\mathcal{H})$ consists of the point $\tilde{P}_1$ by Lemma A.16.

In the rest of the chapter we obtain a contradiction.

Let $\pi : Z \to W$ be the weighted blow up of $\tilde{P}_1$ with weights $(1, 2, 3)$, E be the exceptional divisor of π, $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on Z, and $\tilde{G}$ and $\tilde{F}$ be the proper transforms on the variety Z of the surfaces G and F respectively. Then it follows from the proof of Lemma 31.2 that the singularities of the log pair $(Z, \frac{1}{k}\mathcal{B})$ are terminal, but it follows from Theorem A.15 that the equivalence $\mathcal{B} \sim_{\mathbb{Q}} -kK_Z$ holds.

Let S_x, S_y, S_z, S_t and S_w be proper transforms on the variety Z of the surfaces that are cut on the variety X by the equations $x = 0$, $y = 0$, $z = 0$, $t = 0$ and $w = 0$ respectively. Then

$$\begin{cases} S_x \sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-K_X) - \frac{1}{5}E - \frac{1}{8}\tilde{G} - \frac{1}{4}\tilde{F}, \\ S_y \sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-K_X) - \frac{6}{5}E - \frac{1}{8}\tilde{G} - \frac{1}{4}\tilde{F}, \\ S_z \sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-5K_X) - \frac{1}{5}E - \frac{13}{8}\tilde{G} - \frac{9}{4}\tilde{F}, \\ S_t \sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-7K_X) - \frac{2}{5}E - \frac{7}{8}\tilde{G} - \frac{3}{4}\tilde{F}, \\ S_w \sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-8K_X) - \frac{3}{5}E. \end{cases} \tag{31.8}$$

The equivalences 31.8 imply that the functions y/x, zy/x^6, tzy^2/x^{14} and wz^2y^3/x^{21} are contained in the linear systems $|S_x|$, $|6S_x|$, $|14S_x|$ and $|21S_x|$ respectively. Therefore, the complete linear system $|-42K_Z|$ induces the birational map $\chi : Z \dashrightarrow \bar{X}$ such that the variety $\bar{X}$ is a hypersurface in $\mathbb{P}(1, 1, 6, 14, 21)$ of degree 42 having canonical singularities.

The base locus of the linear system $|-42K_Z|$ consists of the irreducible curve C such that the curve $\alpha \circ \beta \circ \pi(C)$ is cut on X by the equations $x = y = 0$. Therefore, the existence of the antiflip $\zeta : Z \dashrightarrow \bar{Z}$ in the curve C follows from [15], which implies that $-K_{\bar{Z}}$ is nef and big.

The rational map ζ is a log flip with respect to the log pair $(Z, \frac{1}{k}\mathcal{B})$. Therefore, we see that the singularities of the mobile log pair $(\bar{Z}, \frac{1}{k}\mathcal{P})$ are terminal, where $\mathcal{P}$ is the proper transform of the linear system $\mathcal{M}$ on the variety Z, which is impossible by Lemma B.5.

The claim of Proposition 31.1 is proved.

Remark 31.9. The proof of Proposition 31.1 gives the birational transformations of X into hypersurfaces in $\mathbb{P}(1, 1, 6, 8, 9)$, $\mathbb{P}(1, 1, 6, 8, 15)$ and $\mathbb{P}(1, 1, 6, 14, 21)$ of degrees 24, 30 and 42 respectively. The anticanonical models of the varieties U and W are hypersurfaces in $\mathbb{P}(1, 1, 5, 7, 13)$ and $\mathbb{P}(1, 1, 5, 12, 18)$ of degrees 26 and 36 respectively. It follows from the proof of Proposition 31.1 that up to to the action of the group $\mathrm{Bir}(X)$ there are no other non-trivial birational transformations of the hypersurface X into Fano varieties with canonical singularities.

32. Case $n = 48$, hypersurface of degree 21 in $\mathbb{P}(1, 2, 3, 7, 9)$.

We use the notations and assumptions of chapter 1. Let $n = 48$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1, 2, 3, 7, 9)$ of degree 21, the equality $-K_X^3 = 1/18$ holds, and the singularities of X consist of the point P_1 that is a quotient singularity of type $\frac{1}{2}(1, 1, 1)$, the points P_2 and P_3 that are quotient singularity of type $\frac{1}{3}(1, 1, 2)$, and the point P_4 that is a quotient singularity of type $\frac{1}{9}(1, 2, 7)$. There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] & Y \arrow[l, "\gamma"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1, 2, 3), \end{tikzcd}$$

where ψ is a projection, α is the weighted blow up of P_4 with weights $(1, 2, 7)$, β is the weighted blow up with weights $(1, 2, 5)$ of the singular point of the variety U that is a quotient singularity of type $\frac{1}{7}(1, 2, 5)$ contained in the exceptional divisor of the morphism α, γ is the weighted blow up with weights $(1,2,3)$ of the singular point of the variety W that is a quotient singularity of type $\frac{1}{5}(1,2,3)$ contained in the exceptional divisor of β, and η is an elliptic fibration.

Proposition 32.1. *The claim of Proposition 0.7 holds for the hypersurface X.*

In the rest of the chapter we prove Proposition 32.1. It follows from Theorem 1.7, Lemma 1.16 and Proposition 1.10 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_4\}$.

Let E be the exceptional divisor of the morphism α, $\mathcal{D}$ be the proper transform of the linear system $\mathcal{M}$ on the variety U, and P_5 and P_6 be the singular points of U that are quotient singularities of types $\frac{1}{2}(1,1,1)$ and $\frac{1}{7}(1,2,5)$ contained in E respectively. Then E is the weighted projective space $\mathbb{P}(1,2,7)$, and the equivalences

$$\mathcal{D} \sim_{\mathbb{Q}} -kK_U \sim_{\mathbb{Q}} \alpha^*(-kK_X) - \frac{k}{9}E$$

hold by Theorem A.15. It follows from Lemmas B.5 and A.16 that either $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ contains the point P_5, or $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ consists of the point P_6.

Lemma 32.2. *The set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ does not contain the point P_5.*

Proof. Suppose that the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ contains the point P_5. Let $\pi : Z \to U$ be the weighted blow up of P_5 with weights $(1,1,1)$, G be the exceptional divisor of π, and $\mathcal{B}$ and $\mathcal{P}$ be the proper transforms of the linear systems $\mathcal{M}$ and $|-7K_X|$ on the variety Z respectively. Then

$$\mathcal{B} \sim_{\mathbb{Q}} -kK_Z \sim_{\mathbb{Q}} (\alpha \circ \pi)^*(-kK_X) - \frac{k}{9}\pi^*(E) - \frac{k}{2}G$$

by Theorem A.15, but the base locus of the linear system $\mathcal{P}$ does not contain curves. Let H be a general divisor of the linear system $\mathcal{P}$. Then the divisor H is numerically effective, but

$$H \cdot B_1 \cdot B_2 = \Big((\alpha \circ \pi)^*(-kK_X) - \frac{k}{9}\pi^*(E) - \frac{k}{2}G\Big)^2 \Big((\alpha \circ \pi)^*(-7K_X) - \frac{7}{9}\pi^*(E) - \frac{1}{2}G\Big) = -\frac{1}{6}k^2,$$

where B_1 and B_2 are general surfaces of the linear system $\mathcal{B}$, which is a contradiction. □

Hence, the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ consists of the point P_6.

Let F be the exceptional divisor of the morphism β, $\mathcal{H}$ be the proper transform of the linear system $\mathcal{M}$ on the variety W, and P_7 and P_8 be the singular points of W that are quotient singularities of types $\frac{1}{2}(1,1,1)$ and $\frac{1}{5}(1,2,3)$ contained in F respectively. Then F is the weighted projective space $\mathbb{P}(1,2,5)$, but we have the equivalence

$$\mathcal{H} \sim_{\mathbb{Q}} -kK_W \sim_{\mathbb{Q}} (\alpha \circ \beta)^*(-kK_X) - \frac{k}{9}\beta^*(E) - \frac{k}{7}F$$

by Theorem A.15. The divisor $-K_W$ is nef and big, and it follows from Lemmas B.5 and A.16 that the set $\mathbb{CS}(W, \frac{1}{k}\mathcal{H})$ contains either the point P_7, or the point P_8.

Lemma 32.3. *Suppose that $P_8 \in \mathbb{CS}(W, \frac{1}{k}\mathcal{H})$. Then there is a commutative diagram*

(32.4)
$$\begin{tikzcd}
X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & V \arrow[d, "\nu"] \\
\mathbb{P}(1,2,3) \arrow[r, dashed, "\zeta"'] & \mathbb{P}^2,
\end{tikzcd}$$

where ζ is a birational map.

Proof. Let S be the proper transform on the variety Y of a sufficiently general surface of the linear system $\mathcal{M}$, and Γ be a general fiber of η. Then $S \sim_{\mathbb{Q}} -kK_Y$ by Theorem A.15, which implies that $S \cdot \Gamma = 0$. Therefore, the surface S lies in the fibers of the fibration η, which implies the existence of the commutative diagram 32.4. □

To conclude the proof of Proposition 32.1, we may assume that $P_7 \in \mathbb{CS}(W, \frac{1}{k}\mathcal{H})$.

Let $\pi : Z \to W$ be the weighted blow up of the point P_7 with weights $(1,1,1)$, G be the exceptional divisor of π, and $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on Z. Then

$$\mathcal{B} \sim_{\mathbb{Q}} -kK_Z \sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-kK_X) - \frac{k}{9}(\beta \circ \pi)^*(E) - \frac{k}{7}(\pi)^*(F) - \frac{k}{2}G$$

by Theorem A.15.

The hypersurface X can be given by the equation

$$w^2z + wf_{12}(x,y,z,t) + f_{21}(x,y,z,t) = 0 \subset \mathbb{P}(1,2,3,7,9) \cong \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 2$, $\mathrm{wt}(z) = 3$, $\mathrm{wt}(t) = 7$, $\mathrm{wt}(w) = 9$, and $f_i(x,y,z,t)$ is a quasihomogeneous polynomial of degree i.

Let $\mathcal{P}$ be the proper transform on the variety Z of the pencil of surfaces that are cut on the hypersurface X by the equations $\lambda x^3 + \mu z = 0$, where $(\lambda, \mu) \in \mathbb{P}^1$. Then the base locus of the pencil $\mathcal{P}$ consists of the irreducible curves C, L_1 and L_2 such that $\alpha \circ \beta \circ \pi(C)$ is the curve that is cut on the hypersurface X by the equations $x = z = 0$, the curve $\beta \circ \pi(L_1)$ is contained in the exceptional divisor E, the curve $\beta \circ \pi(L_1)$ is the unique curve in the base locus of the linear system $|\mathcal{O}_{\mathbb{P}(1,\,2,\,7)}(1)|$, the curve $\pi(L_2)$ is contained in F, and the curve $\pi(L_2)$ is the unique curve of the linear system $|\mathcal{O}_{\mathbb{P}(1,\,2,\,5)}(1)|$.

Let D be a general surface of the pencil $\mathcal{P}$, $\bar{E}$ and $\bar{F}$ be the proper transforms of the exceptional divisors E and F on the variety Z respectively, and S be the proper transform on Z of the surface that is cut on the hypersurface X by the equation $x = 0$. Then

$$S \cdot D = C + L_1 + L_2,\ \ \bar{E} \cdot D = 3L_1,\ \ \bar{F} \cdot D = 3L_2,$$

the surface D is normal, and

$$\begin{cases} \bar{F} \sim_{\mathbb{Q}} \pi^*(F) - \dfrac{1}{2}G, \\ \bar{E} \sim_{\mathbb{Q}} (\beta \circ \pi)^*(E) - \dfrac{5}{7}\pi^*(F) - \dfrac{1}{2}G, \\ D \sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-3K_X) - \dfrac{3}{9}(\beta \circ \pi)^*(E) - \dfrac{3}{7}\pi^*(F) - \dfrac{3}{2}G, \\ S \sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-K_X) - \dfrac{1}{9}(\beta \circ \pi)^*(E) - \dfrac{1}{7}\pi^*(F) - \dfrac{1}{2}G. \end{cases} \tag{32.5}$$

Consider the curves C, L_1 and L_2 as divisors on $\bar{D}$. The equivalences 32.5 imply that

$$C \cdot C = L_1 \cdot L_1 = -\frac{1}{2},\ \ L_2 \cdot L_2 = -\frac{2}{5},\ \ C \cdot L_1 = C \cdot L_2 = L_1 \cdot L_2 = 0,$$

which implies that the intersection form of the curves C, L_1 and L_2 on the normal surface D is negatively defined. On the other hand, we have

$$B|_D \sim_{\mathbb{Q}} -kK_Z|_D \sim_{\mathbb{Q}} kS|_D \sim_{\mathbb{Q}} kC + kL_1 + kL_2,$$

where B is a general surface of the linear system $\mathcal{B}$, which contradicts Lemmas A.21 and A.20.

The claim of Proposition 32.1 is proved.

33. Case $n = 49$, hypersurface of degree 21 in $\mathbb{P}(1,3,5,6,7)$.

We use the notations and assumptions of chapter 1. Let $n = 49$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,3,5,6,7)$ of degree 21, the equality $-K_X^3 = 1/30$ holds, and the singularities of the hypersurface X consist of the points P_1, P_2 and P_3 that are quotient singularities of type $\frac{1}{3}(1,1,2)$, the point P_4 that is a quotient singularity of type $\frac{1}{5}(1,2,3)$, and the point P_5 that is a quotient singularity of type $\frac{1}{6}(1,1,5)$.

There is a commutative diagram

$$\begin{tikzcd} & Y \arrow[dl, "\alpha"'] \arrow[dr, "\eta"] & \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,3,5), \end{tikzcd}$$

where ψ is a projection, α is the weighted blow up of P_5 with weights $(1,1,5)$, and η is an elliptic fibration.

The hypersurface X can be given equation of degree 21

$$z^3t + z^2g(x,y,t,w) + zh(x,y,t,w) + q(x,y,t,w) = 0 \subset \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 3$, $\mathrm{wt}(z) = 5$ and $\mathrm{wt}(t) = 6$, $\mathrm{wt}(w) = 7$, and f, g, h and q are quasihomogeneous polynomials. Let $\xi : X \dashrightarrow \mathbb{P}^3$ be the rational map that is given by the linear subsystem of the linear system $|-6K_X|$ consisting of the divisors

$$\lambda_0 t + \lambda_1 x^6 + \lambda_2 x^3 y + \lambda_3 y^2 = 0,$$

where $(\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in \mathbb{P}^3$. Then the closure of the image of the rational map ξ is the quadric cone $Q \subset \mathbb{P}^3$, and there is a commutative diagram

$$\begin{tikzcd} & U \arrow[dl, "\beta"'] \arrow[dr, "\omega"] & \\ X \arrow[rr, dashed, "\xi"'] & & Q, \end{tikzcd}$$

where β is the weighted blow up of P_4 with weights $(1,2,3)$, and ω is an elliptic fibration.

Proposition 33.1. *Either there is a commutative diagram*

(33.2)
$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & V \arrow[d, "\nu"] \\ \mathbb{P}(1,3,5) \arrow[r, dashed, "\phi"'] & \mathbb{P}^2, \end{tikzcd}$$

or there is a commutative diagram

(33.3)
$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\xi"'] & V \arrow[d, "\nu"] \\ Q \arrow[r, dashed, "\sigma"'] & \mathbb{P}^2, \end{tikzcd}$$

where ϕ and σ are birational maps.

Proof. The singularities of $(X, \frac{1}{k}\mathcal{M})$ are canonical by Corollary B.2, because X is birationally superrigid, but it follows from Theorem 1.7, Proposition 1.10 and Lemma 1.16 that

$$\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) \subseteq \{P_4, P_5\}.$$

Suppose that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains point P_4. Let $\mathcal{D}$ be the proper transform of the linear system $\mathcal{M}$ on the variety U. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem A.15. Intersecting a general surface of the linear system $\mathcal{D}$ with a general fiber of ω, we see that $\mathcal{D}$ lies in the fibers of the elliptic ω, which implies the existence of the commutative diagram 33.3.

Similarly, the commutative diagram 33.2 exists when $P_4 \in \mathbb{CS}(X, \frac{1}{k}\mathcal{M})$. $\square$

34. Case $n = 51$, hypersurface of degree 22 in $\mathbb{P}(1,1,4,6,11)$.

We use the notations and assumptions of chapter 1. Let $n = 51$. Then X is a hypersurface of degree 22 in $\mathbb{P}(1,1,4,6,11)$, the equality $-K_X^3 = 1/12$ holds, and the singularities of X consist of the point P_1 that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, the point P_2 that is a quotient singularity of type $\frac{1}{3}(1,1,4)$, and the point P_3 that is a quotient singularity of type $\frac{1}{6}(1,1,5)$.

There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & Y \arrow[l, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[r, dashed, "\psi"'] & \mathbb{P}(1,1,4), \end{tikzcd}$$

where ψ is a projection, α is the weighted blow up of P_3 with weights $(1,1,5)$, β is the weighted blow up with weights $(1,1,4)$ of the singular point of the variety U that is contained in the exceptional divisor of the birational morphism α, and η is an elliptic fibration.

The hypersurface X can be given equation of degree 22

$$z^4t + z^2g(x,y,t,w) + zh(x,y,t,w) + q(x,y,t,w) = 0 \subset \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = 1$, $\mathrm{wt}(z) = 4$ and $\mathrm{wt}(t) = 6$, $\mathrm{wt}(w) = 11$, and f, g, h and q are quasihomogeneous polynomials. Let $\xi : X \dashrightarrow \mathbb{P}^7$ be the rational map that is induced by the linear systems of surfaces that are cut on the hypersurface X by the equations

$$\lambda_0 t + \sum_{i=1}^{7} \lambda_i x^i y^{6-i} = 0,$$

where $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7) \in \mathbb{P}^7$. Then the closure of the image of the rational map ξ is the surface $\mathbb{P}(1,1,6)$, and the normalization of a general fiber of ξ is an elliptic curve.

There is a commutative diagram

$$\begin{tikzcd} & W \arrow[dl, "\gamma"'] \arrow[dr, "\omega"] & \\ X \arrow[rr, dashed, "\xi"'] & & \mathbb{P}(1,1,6), \end{tikzcd}$$

where γ is the weighted blow up of P_2 with weights $(1,1,3)$, and ω is an elliptic fibration.

It should be pointed out that the hypersurface X is birationally superrigid.

Proposition 34.1. *Either there is a commutative diagram*

(34.2)
$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & V \arrow[d, "\nu"] \\ \mathbb{P}(1,1,4) \arrow[r, dashed, "\phi"'] & \mathbb{P}^2, \end{tikzcd}$$

or there is a commutative diagram

(34.3)
$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\xi"'] & V \arrow[d, "\nu"] \\ \mathbb{P}(1,1,6) \arrow[r, dashed, "\sigma"'] & \mathbb{P}^2, \end{tikzcd}$$

where ϕ and σ are birational maps.

Proof. It follows from Theorem 1.7, Proposition 1.10 and Lemma 1.16 that

$$\varnothing \neq \mathbb{CS}(X, \frac{1}{k}\mathcal{M}) \subseteq \{P_2, P_3\}.$$

Suppose that P_2 is contained in $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$. Let $\mathcal{D}$ be the proper transform of the linear system $\mathcal{M}$ on the variety W. Then the equivalence $\mathcal{D} \sim_{\mathbb{Q}} -kK_W$ holds by Theorem A.15, which implies that the linear system $\mathcal{D}$ lies in the fibers of the fibration ω, which implies the existence of the commutative diagram 34.3.

We may assume that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ consists of the point P_3.

Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_U$ by Theorem A.15, but the divisor $-K_U$ is nef and big. Therefore, it follows from Lemmas B.5 and A.16 that the set of centers of canonical singularities $\mathbb{CS}(U, \frac{1}{k}\mathcal{B})$ contains the singular point of the variety U that is contained in the exceptional divisor of the birational morphism α.

Let $\mathcal{H}$ be the proper transform of $\mathcal{M}$ on Y. Then $\mathcal{H} \sim_{\mathbb{Q}} -kK_Y$ by Theorem A.15.

Intersecting a general surface of the linear system $\mathcal{H}$ with a general fiber of the fibration η, we see that the linear system $\mathcal{H}$ lies in the fibers of the elliptic fibration η, which implies the existence of the commutative diagram 34.2. $\square$

We use the notations and assumptions of chapter 1. Let $n = 56$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,2,3,8,11)$ of degree 24, the equality $-K_X^3 = 1/22$ holds, and the singularities of the hypersurface X consist of the points P_1, P_2 and P_3 that are quotient singularities of type $\frac{1}{2}(1,1,1)$, and the point P_4 that is a quotient singularity of type $\frac{1}{11}(1,3,8)$.

There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] & & Y \arrow[ll, "\gamma"'] \arrow[d, "\eta"] \\ X \arrow[rrr, dashed, "\psi"'] & & & \mathbb{P}(1,2,3), \end{tikzcd}$$

where ψ is a projection, α is the weighted blow up of P_4 with weights $(1,3,8)$, β is the weighted blow up with weights $(1,3,5)$ of the point of U that is a quotient singularity of type $\frac{1}{8}(1,3,5)$ contained in the exceptional divisor of α, γ is the weighted blow up with weights $(1,2,3)$ of the point of W that is a quotient singularity of type $\frac{1}{5}(1,2,3)$ contained in the exceptional divisor of the morphism β, and η is an elliptic fibration.

In the rest of the chapter we prove the following result.

Proposition 35.1. *The claim of Proposition 0.7 holds for the hypersurface X.*

It follows from Theorem 1.7, Lemma 1.16 and Proposition 1.10 that

$$\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_4\},$$

and we may assume that the singularities of $(X, \frac{1}{k}\mathcal{M})$ are canonical (see Remark 1.6).

Let E be the exceptional divisor of the morphism α, $\mathcal{D}$ be the proper transform of the linear system $\mathcal{M}$ on the variety U, and P_5 and P_6 be the singular point of the variety U that are quotient singularities of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{8}(1,3,5)$ contained in E respectively. Then E is the weighted projective space $\mathbb{P}(1,3,8)$, the equivalence $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ holds by Theorem A.15, and the divisor $-K_U$ is nef and big.

It follows from Lemmas B.5 and A.16 that we have the following possibilities:

- the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ contains the point P_5;
- the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ consists of of the point P_6.

Lemma 35.2. *The set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ does not contain the point P_5.*

Proof. Suppose that the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ contains the point P_5. Let $\pi : Z \to U$ be the weighted blow up of P_5 with weights $(1,1,2)$, G be the exceptional divisor of π, and $\mathcal{B}$ and $\mathcal{P}$ be the proper transforms of $\mathcal{M}$ and $|-8K_X|$ on the variety Z respectively. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_Z$, but the base locus of the linear system $\mathcal{P}$ does not contain curves.

Let H be a general surface in $\mathcal{P}$. Then the divisor H is nef and big. In particular, we have

$$H \cdot B_1 \cdot B_2 = \Big((\alpha \circ \pi)^*(-kK_X) - \frac{k}{11}\pi^*(E) - \frac{k}{3}G\Big)^2 \Big((\alpha \circ \pi)^*(-8K_X) - \frac{8}{11}\pi^*(E) - \frac{2}{3}G\Big) = 0,$$

where B_1 and B_2 are general surfaces in $\mathcal{B}$, which contradicts Corollary A.19. $\square$

Hence, the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ consists of the singular point P_6.

Let F be the exceptional divisor of the morphism β, $\mathcal{H}$ be the proper transform of the linear system $\mathcal{M}$ on the variety W, and P_7 and P_8 be the singular points of W that are quotient singularities of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{5}(1,2,3)$ contained in F respectively. Then F is the weighted projective space $\mathbb{P}(1,3,5)$, the claim of Theorem A.15 implies that

$$\mathcal{H} \sim_{\mathbb{Q}} -kK_W \sim_{\mathbb{Q}} (\alpha \circ \beta)^*(-kK_X) - \frac{k}{11}\beta^*(E) - \frac{k}{8}F,$$

and it follows from Lemmas B.5 and A.16 that either $\mathbb{CS}(W, \frac{1}{k}\mathcal{H}) = \{P_7\}$, or $P_8 \in \mathbb{CS}(W, \frac{1}{k}\mathcal{H})$.

The proof of Lemma 32.3 implies the existence of the commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & V \arrow[d, "\nu"] \\ \mathbb{P}(1,2,3) \arrow[r, dashed, "\zeta"'] & \mathbb{P}^2 \end{tikzcd}$$

in the case when the set $\mathbb{CS}(W, \frac{1}{k}\mathcal{H})$ contains the point P_8, where ζ is a birational map.

To conclude the proof of Proposition 35.1, we may assume that $\mathbb{CS}(W, \frac{1}{k}\mathcal{H}) = \{P_7\}$.

Let $\pi : Z \to W$ be the weighted blow up of P_7 with weights $(1,1,2)$, G be the exceptional divisor of π, and $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on Z. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_Z$ by Theorem A.15.

Lemma 35.3. *The singularities of the log pair $(Z, \frac{1}{k}\mathcal{B})$ are terminal.*

Proof. Suppose that $\mathbb{CS}(Z, \frac{1}{k}\mathcal{B}) \neq \varnothing$. Let P_9 be the singular point of G. Then P_9 is a quotient singularity of type $\frac{1}{2}(1,1,1)$ on Z, the set $\mathbb{CS}(Z, \frac{1}{k}\mathcal{B})$ contains P_9 by Lemma A.16, and G is the weighted projective space $\mathbb{P}(1,1,2)$.

It follows from the local computations that there is a Weil divisor D on Z, such that

$$D \sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-21K_X) - \frac{21}{11}(\beta \circ \pi)^*(E) - \frac{5}{8}\pi^*(F) - \frac{2}{3}G,$$

the base locus of $|D|$ does not contain curves, and divisors of the linear system $|D|$ cut on the surface G the complete linear system $|\mathcal{O}_{\mathbb{P}(1,1,2)}(2)|$.

Let $\bar{\pi} : \bar{Z} \to Z$ be the weighted blow up of the point P_9 with weights $(1,1,1)$, and $\bar{G}$ be the exceptional divisor of $\bar{\pi}$. Then it follows from the properties of the linear system $|D|$ that there is a Weil divisor $\bar{D}$ on the variety $\bar{Z}$ such that

$$\bar{D} \sim_{\mathbb{Q}} \bar{\pi}^*(2D) - \bar{G},$$

but the base locus of the linear system $|\bar{D}|$ does not contain curves. Hence, the divisor $\bar{D}$ is nef and big. On the other hand, the equality $\bar{D} \cdot \bar{H}_1 \cdot \bar{H}_2 = 0$ holds, where $\bar{H}_1$ and $\bar{H}_2$ are the proper transforms on $\bar{Z}$ of general surfaces in $\mathcal{M}$, which is impossible by Corollary A.19. $\square$

The hypersurface X can be given by the equation

$$w^2y + wf_{13}(x,y,z,t) + f_{24}(x,y,z,t) = 0 \subset \mathbb{P}(1,2,3,8,11) \cong \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 2$, $\mathrm{wt}(z) = 3$, $\mathrm{wt}(t) = 8$, $\mathrm{wt}(w) = 11$, and $f_i(x,y,z,t)$ is a sufficiently general quasihomogeneous polynomial of degree i.

Let $\bar{E}$ and $\bar{F}$ be the proper transforms on the variety Z of the divisors E and F respectively, and S_1, S_2, S_3 and S_8 be the proper transforms on the variety Z of the surfaces that are cut on the hypersurface X by the equations $x = 0$, $y = 0$, $z = 0$ and $t = 0$ respectively. Then

$$\begin{cases} \bar{F} \sim_{\mathbb{Q}} \pi^*(F) - \frac{1}{3}G, \\ \bar{E} \sim_{\mathbb{Q}} (\beta \circ \pi)^*(E) - \frac{5}{8}\pi^*(F) - \frac{2}{3}G, \\ S_1 \sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-K_X) - \frac{1}{11}(\beta \circ \pi)^*(E) - \frac{1}{8}\pi^*(F) - \frac{1}{3}G, \\ S_2 \sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-2K_X) - \frac{13}{11}(\beta \circ \pi)^*(E) - \frac{5}{8}\pi^*(F) - \frac{2}{3}G, \\ S_3 \sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-3K_X) - \frac{3}{11}(\beta \circ \pi)^*(E) - \frac{3}{8}\pi^*(F) - \frac{1}{3}G, \\ S_8 \sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-8K_X) - \frac{8}{11}(\beta \circ \pi)^*(E). \end{cases} \tag{35.4}$$

The base locus of $|-2K_Z|$ consists of the irreducible curves C, L_1, L_2 and L_3 such that the curve $\alpha \circ \beta \circ \pi(C)$ is cut on X by the equations $x = y = 0$, the curve $\beta \circ \pi(L_1)$ is contained in the divisor E, the curve $\beta \circ \pi(L_1)$ is contained in $|\mathcal{O}_{\mathbb{P}(1,3,8)}(1)|$, the curve $\pi(L_2)$ is contained in the divisor $\bar{F}$, the curve $\pi(L_2)$ is contained in $|\mathcal{O}_{\mathbb{P}(1,3,5)}(1)|$, the curve L_3 is contained in the divisor G, the curve L_3 is contained in $|\mathcal{O}_{\mathbb{P}(1,1,2)}(1)|$. Moreover, we have

$$S_1 \cdot D = C + 2L_1 + 2L_2 + L_3,\ \bar{E} \cdot D = 2L_1,\ \bar{F} \cdot D = 2L_2,\ G \cdot D = 2L_3,$$

where D is a general surface of the pencil $|-2K_Z|$. In particular, it follows from the equivalences 35.4 that

$$-K_Z \cdot C = \frac{1}{10},\ -K_Z \cdot L_1 = -\frac{1}{3},\ -K_Z \cdot L_2 = -\frac{1}{10},\ -K_Z \cdot L_3 = \frac{1}{2},$$

which implies that the curves L_1 and L_2 are the only curves on the variety Z that have negative intersection with the divisor $-K_Z$.

The singularities of the log pair $(Z, \lambda|-2K_Z|)$ are log-terminal for some rational $\lambda > 1/2$, but the divisor $K_Z + \lambda|-2K_Z|$ has nonnegative intersection with all curves on the variety Z except the curves L_1 and L_2. It follows from [15] that there is a composition of antiflips $\zeta : Z \dashrightarrow \bar{Z}$ such that the divisor $-K_{\bar{Z}}$ is numerically effective.

Let $\mathcal{P}$ be the proper transform of the linear system $\mathcal{M}$ on the variety $\bar{Z}$. Then the singularities of the log pair $(\bar{Z}, \frac{1}{k}\mathcal{P})$ are terminal, because the singularities of the log pair $(Z, \frac{1}{k}\mathcal{B})$ are terminal, and the rational map ζ is a log flop with respect to the log pair $(Z, \frac{1}{k}\mathcal{B})$.

It follows from the equivalences 35.4 that $-K_Z \sim_{\mathbb{Q}} S_1$ and

$$\begin{cases} S_1 \sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-K_X) - \frac{1}{11}\bar{E} - \frac{2}{11}\bar{F} - \frac{5}{11}G, \\ S_2 \sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-2K_X) - \frac{13}{11}\bar{E} - \frac{15}{11}\bar{F} - \frac{21}{11}G, \\ S_3 \sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-3K_X) - \frac{3}{11}\bar{E} - \frac{6}{11}\bar{F} - \frac{4}{11}G, \\ S_8 \sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-8K_X) - \frac{8}{11}\bar{E} - \frac{5}{11}\bar{F} - \frac{7}{11}G. \end{cases} \tag{35.5}$$

The equivalences 35.5 implies that the rational functions y/x^2, zy/x^5 and ty^3/x^{14} are contained in the linear systems $|2S_1|$, $|5S_1|$ and $|14S_1|$ respectively. In particular, the complete linear system $|-70K_Z|$ induces the dominant rational map $Z \dashrightarrow \mathbb{P}(1,2,5,14)$, which implies that the anticanonical divisor $-K_{\bar{Z}}$ is nef and big.

Remark 35.6. The rational function wy^5/x^{21} is contained in the linear system $|21S_1|$, which implies that $|-210K_{\bar{Z}}|$ induces the birational morphism $\bar{Z} \to X'$, where X' is a hypersurface of degree 42 in $\mathbb{P}(1,2,5,14,21)$ that has canonical singularities.

Hence, the singularities of the log pair $(\bar{Z}, \frac{1}{k}\mathcal{P})$ are terminal, but the divisor $-K_{\bar{Z}}$ is nef and big, which contradicts Lemma B.5. The claim of Proposition 35.1 is proved.

36. Case $n = 58$, hypersurface of degree 24 in $\mathbb{P}(1,3,4,7,10)$.

We use the notations and assumptions of chapter 1. Let $n = 58$. Then X is a hypersurface of degree 24 in $\mathbb{P}(1,3,4,7,10)$, the equality $-K_X^3 = 1/35$ holds, and the singularities of X consist of the point P_1 that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, the point P_2 that is a quotient singularity of type $\frac{1}{7}(1,3,4)$, and the point P_3 that is a quotient singularity of type $\frac{1}{10}(1,3,7)$.

There is a commutative diagram

$$\begin{tikzcd}
& & Y \arrow[dl, "\gamma_4"'] \arrow[dr, "\gamma_2"] \arrow[rrrd, "\eta"] & & & \\
& U_{23} \arrow[dl, "\beta_3"'] \arrow[dr, "\beta_2"] & & U_{34} \arrow[dl, "\beta_4"] & & \mathbb{P}(1,3,4), \\
U_2 \arrow[dr, "\alpha_2"'] & & U_3 \arrow[dl, "\alpha_3"] & & & \\
& X \arrow[rrrruu, dashed, "\psi"'] & & & &
\end{tikzcd}$$

where ψ is a projection, α_2 is the weighted blow up of P_2 with weights $(1,3,4)$, α_3 is the weighted blow up of P_3 with weights $(1,3,7)$, β_3 is the weighted blow up with weights $(1,3,7)$ of the proper transform of P_3 on U_2, β_2 is the weighted blow up with weights $(1,3,4)$ of the proper transform of the point P_2 on U_2, β_4 is the weighted blow up with weights $(1,3,4)$ of the singular point of the variety U_3 that is a quotient singularity of type $\frac{1}{7}(1,3,4)$ contained in the exceptional divisor of the morphism α_3, γ_2 is the weighted blow up with weights $(1,3,4)$ of the proper transform of the point P_2 on U_{34}, γ_4 is the weighted blow up with weights $(1,3,4)$ of the singular point of the variety U_{23} that is a quotient singularity of type $\frac{1}{7}(1,3,4)$ contained in the exceptional divisor of the morphism β_3, and η is an elliptic fibration.

In the rest of the chapter we prove the following result.

Proposition 36.1. *The claim of Proposition 0.7 holds for the hypersurface X.*

It follows from Theorem 1.7, Lemma 1.16 and Proposition 1.10 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) \subseteq \{P_2, P_3\}$.

Lemma 36.2. *The set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains the point P_3.*

Proof. Suppose that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ does not contain the point P_3. Let $\mathcal{D}_2$ be the proper transform of the linear system $\mathcal{M}$ on the variety U_2. Then the equivalence $\mathcal{D}_2 \sim_{\mathbb{Q}} -kK_{U_2}$ holds by Theorem A.15, and the set $\mathbb{CS}(U_2, \frac{1}{k}\mathcal{D}_2)$ is not empty by Lemma B.5.

Let O and Q be the singular points of the variety U_2 that are contained in the exceptional divisor of the morphism α_2 such that the singular points O and Q are quotient singularities of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{4}(1,1,3)$ respectively. Then it follows from Lemma A.16 that the set of centers of canonical singularities $\mathbb{CS}(U_2, \frac{1}{k}\mathcal{D}_2)$ contains either the point O, or the point Q.

Suppose that the set $\mathbb{CS}(U_2, \frac{1}{k}\mathcal{D}_2)$ contains the point O. Let $\pi : W \to U_2$ be the weighted blow up of the point O with weights $(1,1,2)$, $\mathcal{B}$ and $\mathcal{P}$ be the proper transforms of the linear system $\mathcal{M}$ and $|-4K_X|$ on the variety W respectively, and S be a general surface of the linear system $\mathcal{P}$. Then the base locus of the linear system $\mathcal{P}$ consists of the irreducible curve C such that the curve $\alpha_2(C)$ is the base curve of the linear system $|-4K_X|$.

One can easily check that S is normal, the inequality $C^2 < 0$ holds on the surface S, and the equivalence $\mathcal{B}|_S \sim_{\mathbb{Q}} kC$ holds. Therefore, it follows from Lemma A.21 that

$$\mathrm{Supp}(S) \cap \mathrm{Supp}(B) = \mathrm{Supp}(C),$$

where B is a general surface in $\mathcal{B}$, which is impossible by Lemma A.20.

Thus, the set $\mathbb{CS}(U_2, \frac{1}{k}\mathcal{D}_2)$ contains the point Q.

Let $\zeta : U \to U_2$ be the weighted blow up of the singular point Q with weights $(1,1,3)$, $\mathcal{H}$ be the proper transform of the linear system $\mathcal{M}$ on the variety U, H be a sufficiently general surface of the linear system $\mathcal{H}$, and D be a general surface in $|-3K_U|$. Then D is normal, and the base locus of the pencil $|-3K_U|$ consists of the irreducible curve Z such that $\alpha_2(Z)$ is the unique base curve of the pencil $|-3K_X|$.

The equivalence $\mathcal{H}|_D \sim_{\mathbb{Q}} kZ$ holds by Theorem A.15, and the inequality $Z^2 < 0$ holds on the surface D. Therefore, it follows from Lemma A.21 that the support of the cycle $D \cdot H$ consists of the curve Z, which contradicts Lemma A.20. □

Let $\mathcal{D}_2$ and $\mathcal{D}_{23}$ be the proper transforms of $\mathcal{M}$ on U_2 and U_{23} respectively. Then the arguments of the proof of Lemma 36.2 implies the following corollaries.

Corollary 36.3. *Suppose that $\mathcal{D}_2 \sim_{\mathbb{Q}} -kK_{U_2}$. Then the set $\mathbb{CS}(U_2, \frac{1}{k}\mathcal{D}_2)$ does not contain subvarieties of the variety U_2 that are contained in the exceptional divisor of the morphism α_2.*

Corollary 36.4. *Suppose that $\mathcal{D}_{23} \sim_{\mathbb{Q}} -kK_{U_{23}}$. Then the set $\mathbb{CS}(U_{23}, \frac{1}{k}\mathcal{D}_{23})$ does not contain subvarieties of the variety U_{23} that are contained in the exceptional divisor of the morphism β_2.*

Let $\mathcal{D}_3$ and $\mathcal{D}_{23}$ be the proper transforms of $\mathcal{M}$ on U_3 and U_{34} respectively. Then we have the equivalence $\mathcal{D}_3 \sim_{\mathbb{Q}} -kK_{U_3}$ by Theorem A.15.

Lemma 36.5. *The set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains the point P_2.*

Proof. Suppose that $P_2 \notin \mathbb{CS}(X, \frac{1}{k}\mathcal{M})$. Then $\mathcal{D}_3 \sim_{\mathbb{Q}} -kK_{U_2}$ by Theorem A.15, which implies that the set $\mathbb{CS}(U_3, \frac{1}{k}\mathcal{D}_3)$ is not empty by Lemma B.5.

Let P_4 and P_5 be the singular points of the variety U_3 that are contained in the exceptional divisor of the birational morphism α_3 such that the points P_4 and P_5 are quotient singularities of types $\frac{1}{7}(1,3,4)$ and $\frac{1}{3}(1,1,2)$ respectively. Then the set $\mathbb{CS}(U_3, \frac{1}{k}\mathcal{D}_3)$ contains either the singular point P_4, or the singular point P_5 by Lemma A.16.

It follows from Lemma 36.2 that the set $\mathbb{CS}(U_3, \frac{1}{k}\mathcal{D}_3)$ does not contain the point P_5, which implies that $\mathbb{CS}(U_3, \frac{1}{k}\mathcal{D}_3)$ contains P_4. Hence, the equivalence $\mathcal{D}_{34} \sim_{\mathbb{Q}} -kK_{U_{34}}$ holds by Theorem A.15, and the set $\mathbb{CS}(U_{34}, \frac{1}{k}\mathcal{D}_{34})$ is not empty by Lemma B.5.

Let P_6 and P_7 be the singular points of the variety U_{34} that are contained in the exceptional divisor of the birational morphism β_4 such that the points P_6 and P_7 are quotient singularities of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{4}(1,1,3)$ respectively. Then the set $\mathbb{CS}(U_{34}, \frac{1}{k}\mathcal{D}_{34})$ contains either the singular point P_6, or the singular point P_7 by Lemma A.16.

Suppose that the set $\mathbb{CS}(U_{34}, \frac{1}{k}\mathcal{D}_{34})$ contains the point P_7. Let $\zeta : U \to U_{34}$ be the weighted blow up of P_7 with weights $(1,1,3)$, $\mathcal{H}$ be the proper transform of $\mathcal{M}$ on U, H be a general surface of $\mathcal{H}$, and D be a general surface of the pencil $|-3K_U|$. Then D is normal, and the base locus of the pencil $|-3K_U|$ consists of the irreducible curve Z such that $\alpha_3 \circ \beta_4(Z)$ is the unique base curve of the pencil $|-3K_X|$. Moreover, the equivalence $\mathcal{H}|_D \sim_{\mathbb{Q}} kZ$ holds, and the inequality $Z^2 < 0$ holds on the surface D, which is impossible by Lemmas A.20 and A.21.

Therefore, the set $\mathbb{CS}(U_{34}, \frac{1}{k}\mathcal{D}_{34})$ contains the point P_6.

The hypersurface X can be given by the quasihomogeneous equation

$$w^2z + wf_{14}(x,y,z,t) + f_{24}(x,y,z,t) = 0 \subset \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 3$, $\mathrm{wt}(z) = 4$, $\mathrm{wt}(t) = 7$, $\mathrm{wt}(w) = 10$, and $f_i(x,y,z,w)$ is a quasihomogeneous polynomial of degree i. Let $\mathcal{P}$ be a pencil consisting of the surfaces that are cut on the hypersurface X by the equations $\lambda x^4 + \mu z = 0$, where $(\lambda, \mu) \in \mathbb{P}^1$. Then the base locus of the pencil $\mathcal{P}$ consists of the irreducible curve that are cut on X by the equations $x = z = 0$.

Let $\pi : W \to U_{34}$ be the weighted blow up of P_6 with weights $(1,1,2)$, $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on W, $\mathcal{H}$ be the proper transform of $\mathcal{P}$ on W, H be a general surface of the pencil $\mathcal{H}$, and E, F and G be the exceptional divisors of α_3, β_4 and π respectively. Then

$$H \sim_{\mathbb{Q}} (\alpha_3 \circ \beta_4 \circ \pi)^*(-4K_X) - \frac{1}{5}(\beta_4 \circ \pi)^*(E) - \frac{4}{7}(\pi)^*(F) - \frac{1}{3}G,$$

the surface H normal, and the base locus of the pencil $\mathcal{H}$ consists of the curves C and L such that $\alpha_3 \circ \beta_4 \circ \pi(C)$ is the unique base curve of the pencil $\mathcal{P}$, and the curve $\beta_4 \circ \pi(L)$ is the unique curve on the surface $E \cong \mathbb{P}(1,3,7)$ that is contained in the linear system $|\mathcal{O}_{\mathbb{P}(1,3,7)}(1)|$.

Let S be the surface of the linear system $|-K_W|$, and $\bar{E}$ be the proper transform of the surface E on the variety W. Then $S \cdot H = C + L$ and $\bar{E} \cdot H = 4L$, but

$$\bar{E} \sim_{\mathbb{Q}} (\beta_4 \circ \pi)^*(E) - \frac{4}{7}(\pi)^*(F) - \frac{1}{3}G,$$

which implies that the intersection form of the curves L and C on the surface H is negatively defined. The equivalence $\mathcal{B}|_H \sim_{\mathbb{Q}} kC + kL$ holds, which contradicts Lemmas A.21 and A.20. $\square$

Hence, we have $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_2, P_3\}$, and $\mathcal{D}_{23} \sim_{\mathbb{Q}} -kK_{U_{23}}$ by Theorem A.15.

It easily follows from Lemmas B.5 and A.16, the proof of Lemma 36.5 and Corollary 36.4 that the set $\mathbb{CS}(U_{23}, \frac{1}{k}\mathcal{D}_{23})$ contains the singular point of the variety U_{23} that is a quotient singularity of type $\frac{1}{7}(1,3,4)$ contained in the exceptional divisor of β_3, which implies that the proper transform of the linear system $\mathcal{M}$ on the variety Y lies in the fibers of the fibration η.

The claim of Proposition 36.1 is proved.

37. Case $n = 64$, hypersurface of degree 26 in $\mathbb{P}(1,2,5,6,13)$.

We use the notations and assumptions of chapter 1. Let $n = 64$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,2,5,6,13)$ of degree 26, the equality $-K_X^3 = 1/30$ holds, and the singularities of the hypersurface X consist of the points P_1, P_2, P_3 and P_4 that are quotient singularities of type $\frac{1}{2}(1,1,1)$, the point P_5 that is a quotient singularity of type $\frac{1}{5}(1,2,3)$, and the point P_6 that is a quotient singularity of type $\frac{1}{6}(1,1,5)$.

There is a commutative diagram

$$\begin{tikzcd}
& Y \arrow[dl, "\alpha"'] \arrow[dr, "\eta"] & \\
X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,2,5),
\end{tikzcd}$$

where ψ is a projection, α is the weighted blow up of P_6 with weights $(1,1,5)$, and η is an elliptic fibration.

The hypersurface X can be given by the equation of degree 26

$$z^4t + z^2g(x,y,t,w) + zh(x,y,t,w) + q(x,y,t,w) = 0 \subset \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x)=1$, $\mathrm{wt}(y)=2$, $\mathrm{wt}(z)=5$ and $\mathrm{wt}(t)=6$, $\mathrm{wt}(w)=13$, and f, g, h and q are general quasihomogeneous polynomials. Let $\xi : X \dashrightarrow \mathbb{P}^3$ be the rational map that is induced by the linear subsystem of the linear system $|-6K_X|$ consisting of the divisors

$$\lambda_0 t + \lambda_1 x^6 + \lambda_2 x^4 y + \lambda_3 y^3 = 0,$$

where $(\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in \mathbb{P}^3$. Then ξ is not defined in the point P_5, the closure of the image of the rational map ξ is the quadric cone $Q \subset \mathbb{P}^3$, and the normalization of a general fiber of the rational map ξ is an elliptic curve. There is a commutative diagram

$$\begin{tikzcd}
& U \arrow[dl, "\beta"'] \arrow[dr, "\omega"] & \\
X \arrow[rr, dashed, "\xi"'] & & Q,
\end{tikzcd}$$

where β is the weighted blow up of P_5 with weights $(1,2,3)$, and ω is an elliptic fibration.

Proposition 37.1. *Either there is a commutative diagram*

(37.2)
$$\begin{tikzcd}
X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & V \arrow[d, "\nu"] \\
\mathbb{P}(1,2,5) \arrow[r, dashed, "\phi"'] & \mathbb{P}^2,
\end{tikzcd}$$

or there is a commutative diagram

(37.3)
$$\begin{tikzcd}
X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\xi"'] & V \arrow[d, "\nu"] \\
Q \arrow[r, dashed, "\sigma"'] & \mathbb{P}^2,
\end{tikzcd}$$

where ϕ and σ are birational maps.

Proof. See the proof of Proposition 33.1. □

38. Case $n = 65$, hypersurface of degree 27 in $\mathbb{P}(1,2,5,9,11)$.

We use the notations and assumptions of chapter 1. Let $n = 65$. Then X is a hypersurface of degree 27 in $\mathbb{P}(1,2,5,9,11)$, the equality $-K_X^3 = 3/110$ holds, and the singularities of X consist of the point P_1 that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, the point P_2 that is a quotient singularity of type $\frac{1}{5}(1,1,4)$, and the point P_3 that is a quotient singularity of type $\frac{1}{11}(1,2,9)$.

There is a commutative diagram

$$\begin{tikzcd}
U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] & Y \arrow[l, "\gamma"'] \arrow[d, "\eta"] \\
X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,2,5),
\end{tikzcd}$$

where ψ is a projection, α is the weighted blow up of P_3 with weights $(1,2,9)$, β is the weighted blow up with weights $(1,2,7)$ of the singular point of the variety U that is a quotient singularity of type $\frac{1}{9}(1,2,7)$ contained in the exceptional divisor of the morphism α, γ is the weighted blow up with weights $(1,2,5)$ of the singular point of the variety W that is a quotient singularity of type $\frac{1}{7}(1,2,5)$ contained in the exceptional divisor of β, and η is an elliptic fibration.

Proposition 38.1. *The claim of Proposition 0.7 holds for the hypersurface X.*

Let us prove Proposition 38.1. It follows from Theorem 1.7, Lemma 1.16 and Proposition 1.10 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_3\}$.

Let E be the exceptional divisor of the morphism α, $\mathcal{D}$ be the proper transform of the linear system $\mathcal{M}$ on the variety U, and P_4 and P_5 be the singular points of U that are quotient singularities of types $\frac{1}{2}(1,1,1)$ and $\frac{1}{9}(1,2,7)$ contained in E respectively. Then E is the weighted projective space $\mathbb{P}(1,2,9)$, and the equivalences

$$\mathcal{D} \sim_{\mathbb{Q}} -kK_U \sim_{\mathbb{Q}} \alpha^*(-kK_X) - \frac{k}{11}E$$

hold by Theorem A.15.

It follows from Lemmas B.5 and A.16 and the proof of Lemma 32.2 that $\mathbb{CS}(U, \frac{1}{k}\mathcal{D}) = \{P_5\}$.

Let F be the exceptional divisor of the morphism β, $\mathcal{H}$ be the proper transform of the linear system $\mathcal{M}$ on the variety W, and P_6 and P_7 be the singular points of W that are quotient singularities of types $\frac{1}{2}(1,1,1)$ and $\frac{1}{7}(1,2,5)$ contained in F respectively. Then F is the weighted projective space $\mathbb{P}(1,2,7)$, it follows from Theorem A.15 that

$$\mathcal{H} \sim_{\mathbb{Q}} -kK_W \sim_{\mathbb{Q}} (\alpha\circ\beta)^*(-kK_X) - \frac{k}{11}\beta^*(E) - \frac{k}{9}F,$$

and it follows from Lemmas B.5 and A.16 that the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ contains either the singular point P_6, or the singular point P_7, because the divisor $-K_W$ is nef and big.

Remark 38.2. In the case when the set $\mathbb{CS}(W, \frac{1}{k}\mathcal{H})$ contains the point P_7, it follows from Theorem A.15 that there is a commutative diagram

$$\begin{tikzcd}
X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & V \arrow[d, "\nu"] \\
\mathbb{P}(1,2,5) \arrow[r, dashed, "\zeta"'] & \mathbb{P}^2,
\end{tikzcd}$$

where ζ is a birational map.

To conclude the proof of Proposition 38.1, we may assume that $P_6 \in \mathbb{CS}(W, \frac{1}{k}\mathcal{H})$.

Let $\pi : Z \to W$ be the weighted blow up of P_6 with weights $(1,1,1)$, G be the exceptional divisor of π, and $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on the variety Z. Then $G \cong \mathbb{P}^2$, and the equivalence $\mathcal{B} \sim_{\mathbb{Q}} -kK_Z$ holds by Theorem A.15.

The hypersurface X can be given by equation

$$w^2z + wf_{16}(x,y,z,t) + f_{27}(x,y,z,t) = 0 \subset \mathbb{P}(1,2,5,9,11) \cong \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 2$, $\mathrm{wt}(z) = 5$, $\mathrm{wt}(t) = 9$, $\mathrm{wt}(w) = 11$, and $f_i(x,y,z,t)$ is a quasihomogeneous polynomial of degree i. Let $\bar{E}$ and $\bar{F}$ be the proper transforms of the exceptional divisors E and F on the variety Z respectively, and $\mathcal{P}$ be the proper transform on Z of the pencil of surfaces that are cut on the hypersurface X by the equations

$$\lambda x^5 + \mu z = 0,$$

where $(\lambda,\mu) \in \mathbb{P}^1$. Then one can easily check that the base locus of the pencil $\mathcal{P}$ consists of the irreducible curves C, L_1, L_2, Δ_1, Δ_2 and Δ such that the curve $\alpha\circ\beta\circ\pi(C)$ is cut on the hypersurface X by the equations $x = z = 0$, the curve $\beta\circ\pi(L_1)$ is contained in the exceptional divisor E, the curve $\beta\circ\pi(L_1)$ is the unique curve in $|\mathcal{O}_{\mathbb{P}(1,\,2,\,9)}(1)|$, the curve $\pi(L_1)$ is contained in the divisor F, the curve $\pi(L_1)$ is contained in $|\mathcal{O}_{\mathbb{P}(1,\,2,\,7)}(1)|$, the curves Δ_1 and Δ_2 are the lines on G that are cut by $\bar{E}$ and $\bar{F}$ respectively, and the curve Δ is a line on G, which is different from the lines Δ_1 and Δ_2.

Let D be a general surface of the pencil $\mathcal{P}$, and S be the proper transform on Z of the surface that is cut on the hypersurface X by the equation $x = 0$. Then

$$S \cdot D = C + L_1 + L_2,\ \bar{E} \cdot D = 5L_1 + \Delta_1,\ \bar{F} \cdot D = 5L_2 + \Delta_2,$$

the surface D is normal, and D is smooth in the neighborhood of G. In particular, it follows from the local computations and the adjunction formula that the equalities

$$\Delta_1 \cdot \Delta_2 = \Delta_1 \cdot L_2 = \Delta_2 \cdot L_1 = 1\ \Delta_1 \cdot C = \Delta_2 \cdot C = 0,\ \Delta_1^2 = \Delta_2^2 = -4 \tag{38.3}$$

hold on the surface D. However, we have

$$\begin{cases} \bar{F} \sim_{\mathbb{Q}} \pi^*(F) - \frac{1}{2}G, \\ \bar{E} \sim_{\mathbb{Q}} (\beta \circ \pi)^*(E) - \frac{7}{9}\pi^*(F) - \frac{1}{2}G, \\ D \sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-5K_X) - \frac{5}{11}(\beta \circ \pi)^*(E) - \frac{5}{9}\pi^*(F) - \frac{3}{2}G, \\ S \sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-K_X) - \frac{1}{11}(\beta \circ \pi)^*(E) - \frac{1}{9}\pi^*(F) - \frac{1}{2}G. \end{cases} \tag{38.4}$$

It follows from the equalities 38.3 and equivalences 38.4 that the equalities

$$C \cdot C = L_1 \cdot L_1 = -\frac{1}{2},\ L_2 \cdot L_2 = -\frac{3}{7},\ C \cdot L_1 = C \cdot L_2 = L_1 \cdot L_2 = 0$$

hold on the surface D. Therefore, the intersection form of the curves C, L_1 and L_2 on the surface D is negatively defined. On the other hand, we have

$$\mathcal{B}|_D \sim_{\mathbb{Q}} -kK_Z|_D \sim_{\mathbb{Q}} kS|_D \sim_{\mathbb{Q}} kC + kL_1 + kL_2,$$

which contradicts Lemmas A.21 and A.20. The claim of Proposition 38.1 is proved.

39. Case $n = 68$, hypersurface of degree 28 in $\mathbb{P}(1, 3, 4, 7, 14)$.

We use the notations and assumptions of chapter 1. Let $n = 68$. Then X is a hypersurface of degree 28 in $\mathbb{P}(1, 3, 4, 7, 14)$, the singularities of the X consist of the point P_1 that is a quotient singularity of type $\frac{1}{2}(1, 1, 1)$, the point P_2 that is a quotient singularity of type $\frac{1}{3}(1, 1, 2)$, the points P_3 and P_4 that are quotient singularities of type $\frac{1}{7}(1, 3, 4)$, and $-K_X^3 = 1/42$.

Proposition 39.1. *The claim of Proposition 0.7 holds for the hypersurface X.*

Proof. It follows from Theorem 1.7, Lemma 1.16 and Proposition 1.10 that

$$\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) \subseteq \{P_5, P_6\},$$

but there is a commutative diagram

$$\begin{tikzcd} & Y \arrow[dl, "\gamma"'] \arrow[dr, "\delta"] \arrow[drrr, "\eta"] & & & \\ U \arrow[dr, "\alpha"'] & & W \arrow[dl, "\beta"] & & \mathbb{P}(1,2,3), \\ & X \arrow[urrr, dashed, "\psi"'] & & & \end{tikzcd}$$

where ψ is a projection, α is the weighted blow up of P_3 with weights $(1, 3, 4)$, β is the weighted blow up of the point P_4 with weights $(1, 3, 4)$, γ is the weighted blow up with weights $(1, 3, 4)$ of the proper transform of the singular point P_4 on the variety U, δ is the weighted blow up with weights $(1, 3, 4)$ of the proper transform of P_3 on W, and η is an elliptic fibration.

The required claim is implied by Theorem A.15 if $\{P_3, P_4\} \subset \mathbb{CS}(X, \frac{1}{k}\mathcal{M})$. Hence, we may assume that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains the point P_3 and does not contain the point P_4.

Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U, and P_5 and P_6 be the singular points of U that are quotient singularities of types $\frac{1}{3}(1, 1, 2)$ and $\frac{1}{4}(1, 1, 3)$ contained in the exceptional divisor of the morphism α respectively. Then it follows from Lemma A.16 that the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{B})$ contains either the point P_5, or the point P_6.

Suppose that $P_6 \in \mathbb{CS}(U, \frac{1}{k}\mathcal{B})$. Let $\pi : W \to U$ be the weighted blow up of the point P_6 with weights $(1,1,3)$, and $\mathcal{B}$ and $\mathcal{P}$ be the proper transforms of $\mathcal{M}$ and the pencil $|-3K_X|$ on the variety W respectively. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ by Theorem A.15, and the base locus of $\mathcal{P}$ consists of the irreducible curve Z such that $\alpha \circ \pi(Z)$ is the base curve of the pencil $|-3K_X|$.

Let S be a general surface in $\mathcal{P}$, and B be a general surface in $\mathcal{B}$. Then the surface S is normal, the inequality $Z^2 < 0$ holds on the surface S, but $\mathcal{B}|_S \sim_{\mathbb{Q}} kZ$. Therefore, the support of the cycle $B \cdot S$ is contained in Z by Lemma A.21, which is impossible by Lemma A.20.

Hence, the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{B})$ contains the point P_5.

Let $\zeta : Z \to U$ be the weighted blow up of P_5 with weights $(1,1,2)$, and $\mathcal{D}$ and $\mathcal{H}$ be the proper transforms of $\mathcal{M}$ and $|-4K_X|$ on Z respectively. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_Z$ by Theorem A.15, and the base locus of $\mathcal{H}$ consists of the irreducible curve C such that $\alpha \circ \zeta(C)$ is the unique base curve of the linear system $|-4K_X|$.

Let H be a general surface in $\mathcal{H}$. Then $H \cdot C = 0$ and $H^3 > 0$. Thus, the divisor H is nef and big. On the other hand, the equality $H \cdot D_1 \cdot D_2 = 0$ holds, where D_1 and D_2 are general surfaces of the linear system $\mathcal{D}$, which is impossible by Corollary A.19. □

40. Case $n = 74$, hypersurface of degree 30 in $\mathbb{P}(1,3,4,10,13)$.

We use the notations and assumptions of chapter 1. Let $n = 74$. Then X is a hypersurface of degree 30 in $\mathbb{P}(1,3,4,10,13)$, the equality $-K_X^3 = 1/52$ holds, and the singularities of X consist of the point P_1 that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, the point P_2 that is a quotient singularity of type $\frac{1}{4}(1,1,3)$, and the point P_3 that is a quotient singularity of type $\frac{1}{13}(1,3,10)$.

There is a commutative diagram

$$\begin{tikzcd}
U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] & & Y \arrow[ll, "\gamma"'] \arrow[d, "\eta"] \\
X \arrow[rrr, dashed, "\psi"'] & & & \mathbb{P}(1,3,4),
\end{tikzcd}$$

where ψ is a projection, α is the weighted blow up of P_3 with weights $(1,3,10)$, β is the weighted blow up with weights $(1,3,7)$ of the singular point of the variety U that is a quotient singularity of type $\frac{1}{10}(1,3,7)$ contained in the exceptional divisor of the morphism α, γ is the weighted blow up with weights $(1,3,4)$ of the singular point of the variety W that is a quotient singularity of type $\frac{1}{7}(1,3,4)$ contained in the exceptional divisor of β, and η is an elliptic fibration.

In the rest of the chapter we prove the following result.

Proposition 40.1. *The claim of Proposition 0.7 holds for the hypersurface X.*

It follows from Theorem 1.7, Lemma 1.16 and Proposition 1.10 $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_3\}$.

Let E be the exceptional divisor of the morphism α, $\mathcal{D}$ be the proper transform of the linear system $\mathcal{M}$ on the variety U, and P_4 and P_5 be the singular points of U that are quotient singularities of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{10}(1,3,7)$ contained in the divisor E respectively. Then the equivalence $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ holds by Theorem A.15.

Lemma 40.2. *The set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ does not contain the point P_4.*

Proof. Suppose that the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ contains the point P_4. Let $\pi : Z \to U$ be the weighted blow up of P_4 with weights $(1,1,2)$, G be the exceptional divisor of π, and $\mathcal{B}$ and $\mathcal{P}$ be the proper transforms of $\mathcal{M}$ and $|-10K_X|$ on Z respectively. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_Z$ by Theorem A.15.

One can explicitly check that the base locus of the linear system $\mathcal{P}$ does not contain curves, but the equalities

$$H \cdot B_1 \cdot B_2 = \Big((\alpha \circ \pi)^*(-kK_X) - \frac{k}{13}\pi^*(E) - \frac{k}{3}G\Big)^2 \Big((\alpha \circ \pi)^*(-10K_X) - \frac{10}{13}\pi^*(E) - \frac{1}{3}G\Big) = 0$$

hold, where H is a general surface of the linear system $\mathcal{P}$, and B_1 and B_2 are general surfaces of the linear system $\mathcal{B}$. The latter is impossible by Corollary A.19. □

Corollary 40.3. *The set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ consists of the point P_5 by Lemmas B.5 and A.16.*

Let F be the exceptional divisor of the morphism β, $\mathcal{H}$ be the proper transform of the linear system $\mathcal{M}$ on the variety W, and P_6 and P_7 be the singular points of W that are quotient singularities of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{7}(1,3,4)$ contained in F respectively. Then $\mathcal{H} \sim_{\mathbb{Q}} -kK_W$.

Suppose that the set $\mathbb{CS}(W, \frac{1}{k}\mathcal{H})$ contains the point P_7. Then it easily follows from the proof of Lemma 32.3 that the claim of Proposition 0.7 holds for the hypersurface X. Hence, we may assume that the set $\mathbb{CS}(W, \frac{1}{k}\mathcal{H})$ does not contain the point P_7. Thus, the set $\mathbb{CS}(W, \frac{1}{k}\mathcal{H})$ consists of the singular point P_6 by Lemmas B.5 and A.16

Let $\pi : Z \to W$ be the weighted blow up of the point P_6 with weights $(1,1,2)$, G be the exceptional divisor of the birational morphism π, and $\mathcal{B}$ be the proper transform of the linear system $\mathcal{M}$ on the variety Z. Then the equivalence $\mathcal{B} \sim_{\mathbb{Q}} -kK_Z$ holds by Theorem A.15.

The hypersurface X can be given by the equation

$$w^2z + wf_{17}(x,y,z,t) + f_{30}(x,y,z,t) = 0 \subset \mathbb{P}(1,2,3,8,11) \cong \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 3$, $\mathrm{wt}(z) = 4$, $\mathrm{wt}(t) = 10$, $\mathrm{wt}(w) = 13$, and $f_i(x,y,z,t)$ is a quasi-homogeneous polynomial of degree i. Let $\mathcal{P}$ be the linear system on the hypersurface X that are generated by the monomials x^{24}, y^8, z^6, ty^4x^2 and t^2z. Then the base locus of $\mathcal{P}$ does not contain curves.

Let $\mathcal{R}$ be the proper transform of the linear system $\mathcal{P}$ on the variety Z, and R be a general surface of the linear system $\mathcal{R}$. Then it follows from the local computations that the base locus of the linear system $\mathcal{R}$ does not contain curves, and the equivalence

$$R \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\pi)^*(-24K_X) - \frac{24}{13}(\beta\circ\pi)^*(E) - \frac{14}{10}\pi^*(F) - \frac{2}{3}G$$

holds. However, the equality $R\cdot B_1\cdot B_2 = 0$ holds, where B_1 and B_2 are general surfaces of the linear system $\mathcal{B}$, which is impossible by Corollary A.19.

The claim of the Proposition 40.1 is proved.

41. Case $n = 79$, hypersurface of degree 33 in $\mathbb{P}(1,3,5,11,14)$.

We use the notations and assumptions of chapter 1. Let $n = 79$. Then X is a hypersurface of degree 33 in $\mathbb{P}(1,3,5,11,14)$, the equality $-K_X^3 = 1/70$ holds, and the singularities of X consist of the point P_1 that is a quotient singularity of type $\frac{1}{5}(1,1,4)$, and the point P_2 that is a quotient singularity of type $\frac{1}{14}(1,3,11)$.

Proposition 41.1. *The claim of Proposition 0.7 holds for the hypersurface X.*

Proof. There is a commutative diagram

$$\begin{tikzcd}
U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] & & Y \arrow[ll, "\gamma"'] \arrow[d, "\eta"] \\
X \arrow[rrr, dashed, "\psi"'] & & & \mathbb{P}(1,3,5),
\end{tikzcd}$$

where ψ is a projection, α is the weighted blow up of P_2 with weights $(1,3,11)$, β is the weighted blow up with weights $(1,3,8)$ of the singular point the variety U that is a quotient singularity of type $\frac{1}{11}(1,3,8)$ contained in the exceptional divisor of α, γ is the weighted blow up with weights $(1,3,5)$ of the point of W that is a quotient singularity of type $\frac{1}{8}(1,3,5)$ contained in the exceptional divisor of the morphism β, and η is an elliptic fibration.

It follows from Theorem 1.7, Lemma 1.16 and Proposition 1.10 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_2\}$.

Let E be the exceptional divisor of the morphism α, $\mathcal{D}$ be the proper transform of the linear system $\mathcal{M}$ on U, and P_3 and P_4 be the singular points of U that are quotient singularities of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{11}(1,3,8)$ contained in E respectively. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$, and it follows from Lemmas B.5 and A.16 and the proof of Lemma 40.2 that $\mathbb{CS}(U, \frac{1}{k}\mathcal{D}) = \{P_4\}$.

Let F be the exceptional divisor of the morphism β, $\mathcal{H}$ be the proper transform of the linear system $\mathcal{M}$ on the variety W, and P_5 and P_6 be the singular points of W that are quotient singularities of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{8}(1,3,4)$ contained in F . Then $\mathcal{H} \sim_{\mathbb{Q}} -kK_W$.

In the case when the set $\mathbb{CS}(W, \frac{1}{k}\mathcal{H})$ contains the point P_6, it easily follows from Theorem A.15 that the claim of Proposition 0.7 holds for the hypersurface X. Therefore, we may assume that the set $\mathbb{CS}(W, \frac{1}{k}\mathcal{H})$ consists of the point P_5 by Lemmas B.5 and A.16.

Let $\pi : Z \to W$ be the weighted blow up of the point P_5 with weights $(1,1,2)$, G be the exceptional divisor of the birational morphism π, and $\mathcal{B}$ be the proper transform of the linear system $\mathcal{M}$ on the variety Z. Then the equivalence $\mathcal{B} \sim_{\mathbb{Q}} -kK_Z$ holds by Theorem A.15.

The hypersurface X can be given by the equation

$$w^2z + wf_{19}(x,y,z,t) + f_{33}(x,y,z,t) = 0 \subset \mathbb{P}(1,2,3,8,11) \cong \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 3$, $\mathrm{wt}(z) = 5$, $\mathrm{wt}(t) = 11$, $\mathrm{wt}(w) = 14$, and $f_i(x,y,z,t)$ is a quasihomogeneous polynomial of degree i. Let $\mathcal{P}$ be the linear system on the hypersurface X that is generated by the monomials x^{30}, y^{10}, z^6, t^2x^8, $t^2y^2x^2$, ty^6x and wtz. Then the base locus of the linear system $\mathcal{P}$ does not contain curves.

Let $\mathcal{R}$ be the proper transform of the linear system $\mathcal{P}$ on the variety Z, and R be a general surface of the linear system $\mathcal{R}$. Then it follows from the local computations that the base locus of the linear system $\mathcal{R}$ does not contains curves and

$$R \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\pi)^*(-30K_X) - \frac{30}{11}(\beta\circ\pi)^*(E) - \frac{8}{11}\pi^*(F) - \frac{2}{3}G,$$

which implies that the equality $R \cdot B_1 \cdot B_2 = 0$ holds, where B_1 and B_2 are general surfaces of the linear system $\mathcal{B}$. The latter contradicts Corollary A.19. ◻

42. Case $n = 80$, hypersurface of degree 34 in $\mathbb{P}(1,3,4,10,17)$.

We use the notations and assumptions of chapter 1. Let $n = 80$. Then X is a hypersurface of degree 34 in $\mathbb{P}(1,3,4,10,17)$, the equality $-K_X^3 = 1/60$ holds, and the singularities of X consist of the point P_1 that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, the point P_2 that is a quotient singularity of type $\frac{1}{3}(1,1,2)$, the point P_3 that is a quotient singularity of type $\frac{1}{4}(1,1,3)$, and the point P_4 that is a quotient singularity of type $\frac{1}{10}(1,3,7)$.

Proposition 42.1. *The claim of Proposition 0.7 holds for the hypersurface X.*

Proof. The singularities of the log pair $(X, \frac{1}{k}\mathcal{M})$ are canonical by Corollary B.2, because the hypersurface X is birationally superrigid. Moreover, it follows from Theorem 1.7, Lemma 1.16 and Proposition 1.10 that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ consists of of the point P_4.

There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & Y \arrow[l, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[r, dashed, "\psi"'] & \mathbb{P}(1,3,4), \end{tikzcd}$$

where ψ is a projection, α is the weighted blow up of P_4 with weights $(1,3,7)$, β is the weighted blow up with weights $(1,3,4)$ of the singular point the variety U that is a quotient singularity of type $\frac{1}{7}(1,3,4)$ contained in the exceptional divisor of α, and η is an elliptic fibration.

Let $\mathcal{D}$ be the proper transform of the linear system $\mathcal{M}$ on the variety U, and P_5 and P_6 be the singular points of the variety U that are quotient singularities of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{4}(1,1,3)$ contained in the exceptional divisor of α respectively. Then the equivalence $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$.

It follows from Lemmas B.5 and A.16 that either the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ contains the point P_5, or the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ contains the point P_6. In the latter case it follows from Theorem A.15 that the claim of Proposition 0.7 holds for the hypersurface X. Therefore, we may assume that the set of centers of canonical singularities $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ contains the point P_5.

The hypersurface X can be given by the quasihomogeneous equation

$$t^3z + t^2f_{14}(x,y,z,w) + tf_{24}(x,y,z,w) + f_{34}(x,y,z,w) = 0 \subset \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 3$, $\mathrm{wt}(z) = 4$, $\mathrm{wt}(t) = 10$, $\mathrm{wt}(w) = 17$, and $f_i(x,y,z,w)$ is a general quasihomogeneous polynomial of degree i. Let $\mathcal{P}$ be a pencil consisting of the surfaces that are cut on X by the equations $\lambda x^4 + \mu z = 0$, where $(\lambda, \mu) \in \mathbb{P}^1$. Then the base locus of $\mathcal{P}$ consists of the irreducible curve that is cut on X by the equations $x = z = 0$.

Let $\gamma : W \to U$ be the weighted blow up of P_5 with weights $(1,1,2)$, $\mathcal{B}$ be the proper transform of the linear system $\mathcal{M}$ on the variety W, $\mathcal{H}$ be the proper transform of the pencil $\mathcal{P}$ on the variety W, D be a sufficiently general surface of the pencil $\mathcal{H}$, and E and F be the exceptional divisors of the morphisms α and γ respectively. Then the surface D is normal, the equivalences

$$D \sim_{\mathbb{Q}} -4K_W \sim_{\mathbb{Q}} (\alpha \circ \gamma)^*(-4K_X) - \frac{2}{5}\gamma^*(E) - \frac{4}{3}F$$

hold, and the base locus of the pencil $\mathcal{H}$ consists of the curves C and L such that $\alpha \circ \gamma(C)$ is the base curve of the pencil $\mathcal{P}$, and $\gamma(L)$ is the curve on the surface $E \cong \mathbb{P}(1,3,7)$ that is contained in the linear system $|\mathcal{O}_{\mathbb{P}(1,3,7)}(1)|$.

Let S be the surface of the linear system $|-K_W|$, and $\bar{E}$ be the proper transform of the surface E on the variety W. Then $S \cdot D = C + L$ and $\bar{E} \cdot D = 4L$, but

$$\bar{E} \sim_{\mathbb{Q}} \gamma^*(E) - \frac{2}{3}F,$$

which implies that the equalities

$$C \cdot C = -\frac{1}{3},\ C \cdot L = 0,\ L \cdot L = -\frac{2}{7}$$

hold on D. Therefore, the intersection form of the curves C and L on the surface D is negatively defined, but $\mathcal{B}|_D \sim_{\mathbb{Q}} kC + kL$, which contradicts Lemmas A.21 and A.20. □

43. Case $n = 82$, hypersurface of degree 36 in $\mathbb{P}(1,1,5,12,18)$.

We use the notations and assumptions of chapter 1. Let $n = 82$. Then X is a hypersurface of degree 36 in $\mathbb{P}(1,1,5,12,18)$, the equality $-K_X^3 = 1/30$ holds, and the singularities of X consist of the point P_1 that is a quotient singularity of type $\frac{1}{5}(1,2,3)$, and the point P_2 that is a quotient singularity of type $\frac{1}{6}(1,1,5)$. The hypersurface X is birationally superrigid.

There is a commutative diagram

$$\begin{tikzcd} & U \arrow[dl, "\alpha"'] \arrow[dr, "\eta"] & \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,5), \end{tikzcd}$$

where ψ is a projection, α is the weighted blow up of P_2 with weights $(1,1,6)$, and η is an elliptic fibration. Let $\mathcal{D}$ be the proper transform of the linear system $\mathcal{M}$ on the variety U.

Proposition 43.1. *The claim of Proposition 0.7 holds for the hypersurface X.*

Proof. It follows from Theorem 1.7, Lemma 1.16 and Proposition 1.10 that

$$\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_1, P_2\}.$$

Suppose that $P_2 \in \mathbb{CS}(X, \frac{1}{k}\mathcal{M})$. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem A.15. Intersecting a general surface of the linear system $\mathcal{D}$ with a general fiber of the elliptic fibration η, we see that the linear system $\mathcal{B}$ lies in the fibers of the fibration η. Therefore, there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & V \arrow[d, "\nu"] \\ \mathbb{P}(1,1,5) \arrow[r, dashed, "\sigma"'] & \mathbb{P}^2, \end{tikzcd}$$

where σ is a birational map.

We may assume that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ consists of the point P_1.

Let $\pi : W \to X$ be the weighted blow up of P_1 with weights $(1,2,3)$, and $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on W. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ by Theorem A.15, and the singularities of the exceptional divisor of the morphism π consist of the points Q and O that are quotient singularities of types $\frac{1}{2}(1,1,1)$ and $\frac{1}{3}(1,1,2)$ on the variety W respectively.

Suppose that the set $\mathbb{CS}(W, \frac{1}{k}\mathcal{B})$ is not empty. Then the set $\mathbb{CS}(W, \frac{1}{k}\mathcal{B})$ contains either the point O, or the point Q. On the other hand, it easily follows from the proof of Proposition 22.1 that the set $\mathbb{CS}(W, \frac{1}{k}\mathcal{B})$ does not contain neither the point O, nor the point Q, because the linear system $\mathcal{B}$ is not composed from the pencil.

Therefore, the singularities of the log pair $(W, \frac{1}{k}\mathcal{B})$ are terminal.

It follows from the proof of Proposition 29.1 that there is a birational map $\gamma : W \dashrightarrow Y$ such that γ is an antiflip, the divisor $-K_Y$ is nef and big, and the linear system $|-rK_Y|$ induced a birational map $Y \dashrightarrow X'$ such that X' is a hypersurface in $\mathbb{P}(1,1,6,14,21)$ of degree 42 with canonical singularities (see the proof of Theorem 5.5.1 in [6]), where r is a sufficiently big and divisible natural number.

Let $\mathcal{H}$ be the proper transform of $\mathcal{M}$ on the variety Y. Then $\mathcal{H} \sim_{\mathbb{Q}} -kK_Y$, because γ is an isomorphism in codimension one. On the other hand, the birational map γ is a log flip with respect to the log pair $(W, \lambda\mathcal{B})$, which has terminal singularities, where λ is a rational number such that $\lambda > 1/k$. Therefore, the singularities of the mobile log pair $(Y, \frac{1}{k}\mathcal{H})$ are terminal, which contradicts Lemma B.5. □

44. Case $n \in \{21,24,33,35,41,42,46,50,54,55,61,62,63,67,69,71,76,77,83,85,91\}$.

Now we conclude the proof of Proposition 0.7. We use the notations and assumptions of chapter 1 assuming that $n \in \{21,24,33,35,41,42,46,50,54,55,61,62,63,67,69,71,76,77,83,85,91\}$.

Proposition 44.1. *The claim of Proposition 0.7 holds for the hypersurface X.*

In the rest of the chapter we prove Proposition 44.1. We may assume that the singularities of the mobile log pair $(X, \frac{1}{k}\mathcal{M})$ are canonical (see Remark 1.6). The set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ consists of the finite number of singular points of the hypersurface X by Theorem 1.7 and Lemma 1.16.

We must show that there is a commutative diagram

(44.2)
$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & V \arrow[d, "\nu"] \\ \mathbb{P}(1,a_1,a_2) \arrow[r, dashed, "\sigma"'] & \mathbb{P}^2, \end{tikzcd}$$

where ψ is the natural projection, and σ is a birational map.

`Case` $n = 21$.

The variety X is a hypersurface in $\mathbb{P}(1,1,2,4,7)$ of degree 14, the equality $-K_X^3 = 1/4$ holds, and the singularities of the hypersurface X consist of the points O_1 and O_2 that are quotient singularities of type $\frac{1}{2}(1,1,1)$, and the point P that is a quotient singularity of type $\frac{1}{4}(1,1,3)$.

There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[r, dashed, "\psi"'] & \mathbb{P}(1,1,2), \end{tikzcd}$$

where ψ is a projection, α is the weighted blow up of P with weights $(1,1,3)$, β is the weighted blow up with weights $(1,1,2)$ of the singular point the variety U that is a quotient singularity of type $\frac{1}{3}(1,1,2)$, and η is an elliptic fibration.

It follows from Proposition 1.10 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P\}$.

Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on U. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem A.15, and it follows from Lemmas B.5 and A.16 that the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ contains the singular point of the variety U that is contained in the exceptional divisor of the morphism α.

Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on Y. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_Y$ by Theorem A.15, which implies that the linear system $\mathcal{B}$ lies in the fibers of the elliptic fibration η, which implies the existence of the the commutative diagram 44.2.

Case $n = 24$.

The variety X is a hypersurface in $\mathbb{P}(1,1,2,5,7)$ of degree 15, the equality $-K_X^3 = 3/14$ holds, and the singularities of the hypersurface X consist of the point P_1 that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, and the point P_2 that is a quotient singularity of type $\frac{1}{7}(1,2,5)$.

It follows from Proposition 1.10 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_2\}$.

There is a commutative diagram

$$\begin{tikzcd}
U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] & Y \arrow[l, "\gamma"'] \arrow[d, "\eta"] \\
X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,2),
\end{tikzcd}$$

where α is the weighted blow up of the point P with weights $(1,2,5)$, β is the weighted blow up with weights $(1,2,3)$ of the singular point of U that is a quotient singularity of type $\frac{1}{5}(1,2,3)$, γ is the weighted blow up with weights $(1,1,2)$ of the singular point of the variety W that is a quotient singularity of type $\frac{1}{3}(1,1,2)$, and η is an elliptic fibration.

The proof of Proposition 22.1 implies that $\mathcal{D} \sim_{\mathbb{Q}} -kK_W$, where $\mathcal{D}$ is the proper transform of the linear system $\mathcal{M}$ on the variety W, and the set $\mathbb{CS}(W, \frac{1}{k}\mathcal{D})$ does not contain subvarieties of the variety W that are not contained in the exceptional divisor of the morphism β.

We can apply arguments of the proof of Proposition 22.1 to the log pair $(W, \frac{1}{k}\mathcal{D})$ to prove that the set $\mathbb{CS}(W, \frac{1}{k}\mathcal{D})$ contains the singular point of the variety W that is a quotient singularity of type $\frac{1}{3}(1,1,2)$.The existence of the commutative diagram 44.2 follows from Theorem A.15.

Case $n = 33$.

The variety X is a hypersurface in $\mathbb{P}(1,2,3,5,7)$ of degree 17, the singularities of X consist of the point P_1 that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, the point P_2 that is a quotient singularity of type $\frac{1}{3}(1,1,2)$, the point P_3 that is a quotient singularity of type $\frac{1}{5}(1,2,3)$, and the point P_4 that is a quotient singularity of type $\frac{1}{7}(1,2,5)$. The equality $-K_X^3 = 17/210$ holds.

There is a commutative diagram

$$\begin{tikzcd}
 & & Y \arrow[dl, "\gamma_5"'] \arrow[dr, "\gamma_3"] \arrow[drr, "\eta"] & & \\
 & U_{34} \arrow[dl, "\beta_4"'] \arrow[dr, "\beta_3"] & & U_{45} \arrow[dl, "\beta_5"] & \mathbb{P}(1,2,3), \\
U_3 \arrow[dr, "\alpha_3"'] & & U_2 \arrow[dl, "\alpha_4"] & & \\
 & X \arrow[uurrr, dashed, "\psi"'] & & &
\end{tikzcd}$$

where α_3 is the weighted blow up of the point P_3 with weights $(1,2,3)$, α_4 is the weighted blow up of the point P_4 with weights $(1,2,5)$, β_4 is the weighted blow up with weights $(1,2,5)$ of the proper transform of P_4 on U_3, β_3 is the weighted blow up with weights $(1,2,3)$ of the proper transform of P_3 on U_4, β_5 is the weighted blow up with weights $(1,2,3)$ of the singular point of the variety U_4 that is a quotient singularity of type $\frac{1}{5}(1,2,3)$ contained in the exceptional divisor of the morphism α_4, γ_3 is the weighted blow up with weights $(1,2,3)$ of the proper transform of the point P_3 on the variety U_{45}, γ_5 is the weighted blow up with weights $(1,2,3)$ of the singular point of U_{34} that is a quotient singularity of type $\frac{1}{5}(1,2,3)$ contained in the exceptional divisor of the morphism β_4, and η is an elliptic fibration.

It follows from Proposition 1.10 and the proof of Proposition 18.1 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_3, P_4\}$.

Let $\mathcal{D}_{34}$ be the proper transform of $\mathcal{M}$ on U_{34}, and $\bar{P}_5$ and $\bar{P}_6$ be the singular points of the variety U_{34} that are quotient singularities of types $\frac{1}{5}(1,2,3)$ and $\frac{1}{2}(1,1,1)$ contained in the exceptional divisor of the morphism β_4 respectively. Then $\mathcal{D}_{34} \sim_{\mathbb{Q}} -kK_{U_{34}}$ by Theorem A.15.

It follows from Lemma A.16 and the proof of Proposition 18.1 that $\bar{P}_5 \in \mathbb{CS}(U_{34}, \frac{1}{k}\mathcal{D}_{34})$, and the existence of the commutative diagram 44.2 follows from Theorem A.15.

`Case` $n = 35$.

The variety X is a hypersurface in $\mathbb{P}(1,1,3,5,9)$ of degree 18, the equality $-K_X^3 = 2/15$ holds, and the singularities of the hypersurface X consist of the points P_1 and P_2 that are quotient singularities of type $\frac{1}{3}(1,1,2)$, and the point P_3 that is a quotient singularity of type $\frac{1}{5}(1,1,4)$.

There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & W \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,3), \end{tikzcd}$$

where α is the weighted blow up of P_3 with weights $(1,1,4)$, β is the weighted blow up with weights $(1,1,3)$ of the point of U that is a quotient singularity of type $\frac{1}{4}(1,1,3)$, and η is an elliptic fibration.

It follows from Theorem 1.7, Lemma 1.16 and Proposition 1.10 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_3\}$.

Let $\mathcal{D}$ is the proper transform $\mathcal{M}$ on U. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem A.15, and it follows from Lemma B.5 that the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ is not empty.

It follows from Lemma A.16 that $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ contains the singular point of U that is contained in the exceptional divisor of the morphism α. Now the existence of the commutative diagram 44.2 follows from Theorem A.15.

`Case` $n = 41$.

The variety X is a hypersurface in $\mathbb{P}(1,1,4,5,10)$ of degree 20, the singularities of X consist of the point O that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, the points P_1 and P_2 that are quotient singularities of type $\frac{1}{5}(1,1,4)$. The equality $-K_X^3 = 1/10$ holds.

There is a commutative diagram

$$\begin{tikzcd} & Y \arrow[dl, "\gamma"'] \arrow[dr, "\delta"] \arrow[rrr, "\eta"] & & & \mathbb{P}(1,1,4), \\ U \arrow[dr, "\alpha"'] & & W \arrow[dl, "\beta"] & & \\ & X \arrow[rrruu, dashed, "\psi"'] & & & \end{tikzcd}$$

where α is the weighted blow up of the point P_1 with weights $(1,1,4)$, β is the weighted blow up of the point P_2 with weights $(1,1,4)$, γ is the weighted blow up with weights $(1,1,4)$ of the proper transform of P_2 on U, δ is the weighted blow up with weights $(1,1,4)$ of the proper transform of the point P_1 on the variety W, and η is an elliptic fibration.

It follows from Proposition 1.10 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) \subseteq \{P_1, P_2\}$, and the existence of the commutative diagram 44.2 follows from the proof of Proposition 12.1.

`Case` $n = 42$.

The variety X is a hypersurface in $\mathbb{P}(1,2,3,5,10)$ of degree 20, the singularities of X consist of the points P_1, P_2 and P_3 that are quotient singularities of type $\frac{1}{2}(1,1,1)$, the point P_4 that is a quotient singularity of type $\frac{1}{3}(1,1,2)$, the points P_5 and P_6, that are quotient singularities of types $\frac{1}{5}(1,2,3)$. The equality $-K_X^3 = 1/15$ holds. There is a commutative diagram

$$\begin{tikzcd} & Y \arrow[dl, "\gamma"'] \arrow[dr, "\delta"] \arrow[rrr, "\eta"] & & & \mathbb{P}(1,2,3), \\ U \arrow[dr, "\alpha"'] & & W \arrow[dl, "\beta"] & & \\ & X \arrow[rrruu, dashed, "\psi"'] & & & \end{tikzcd}$$

where α is the weighted blow up of the point P_5 with weights $(1,2,3)$, β is the weighted blow up of the point P_6 with weights $(1,2,3)$, γ is the weighted blow up of the proper transform of the point P_6 on the variety U with weights $(1,2,3)$, δ is the weighted blow up of the proper transform of the point P_5 on the variety W with weights $(1,2,3)$, and η is an elliptic fibration.

It follows from Proposition 1.10 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) \subseteq \{P_5, P_6\}$.

The existence of the commutative diagram 44.2 is obvious, if $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_5, P_6\}$. Therefore, we may assume that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains P_5 and does not contain P_6.

Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U, and O and Q be the singular points of U that are quotient singularities of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{2}(1,1,1)$ contained in the exceptional divisor of the morphism α respectively. Then it follows from Lemma A.16 and the proof of Lemma 21.3 that the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{B})$ consists of of the point Q.

Let $\zeta : Z \to U$ be the weighted blow up of the point Q with weights $(1,1,1)$, $\mathcal{D}$ be the proper transform of the linear system $\mathcal{M}$ on the variety Z, and $\mathcal{H}$ be the proper transform of the linear system $|-3K_X|$ on the variety Z. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_Z$ by Theorem A.15, and the base locus of the linear system $\mathcal{H}$ consists of the irreducible curve C such that $\alpha \circ \zeta(C)$ is the unique curve in the base locus of the linear system $|-3K_X|$.

Let S be a general surface in $\mathcal{H}$. Then S is normal, and the inequality $C^2 < 0$ holds on the surface S, but the equivalence $\mathcal{D}|_S \sim_{\mathbb{Q}} kC$ holds, which contradicts Lemmas A.20 and A.21.

`Case` $n = 45$.

The variety X is a hypersurface in $\mathbb{P}(1,3,4,5,8)$ of degree 20, the equality $-K_X^3 = 1/24$ holds, and the singularities of X consist of the point P_1 that is a quotient singularity of type $\frac{1}{3}(1,1,2)$, the points P_2 and P_3 that are quotient singularities of type $\frac{1}{4}(1,1,3)$, and the point P_4 that is a quotient singularity of type $\frac{1}{8}(1,3,5)$. Moreover, there is a commutative diagram

$$\begin{tikzcd} U & & Y \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[u, leftarrow, "\alpha"'] \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,3,4), \end{tikzcd}$$

where α is the weighted blow up of the point P_4 with weights $(1,3,5)$, β is the weighted blow up with weights $(1,2,3)$ of the singular point of U that is a quotient singularity of type $\frac{1}{5}(1,2,3)$ contained in the exceptional divisor of the morphism α, and η is an elliptic fibration.

It follows from Proposition 1.10 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_4\}$.

Let $\mathcal{B}$ be the proper transform of the linear system $\mathcal{M}$ on the variety U, $\bar{P}_5$ be the singular point of the variety U that is a quotient singularity of type $\frac{1}{5}(1,2,3)$ contained in the exceptional divisor of the morphism α, and $\bar{P}_6$ be the singular point of the the variety U that is a quotient singularity of type $\frac{1}{2}(1,1,1)$ contained in the exceptional divisor of the morphism α.

It follows from Lemma A.16 that

$$\mathbb{CS}(U, \frac{1}{k}\mathcal{B}) \cap \{\bar{P}_5, \bar{P}_6\} \neq \varnothing,$$

but the proof of Proposition 30.1 easily implies that $\bar{P}_5 \in \mathbb{CS}(U, \frac{1}{k}\mathcal{B})$. Hence, the existence of the commutative diagram 44.2 follows from Theorem A.15.

`Case` $n = 46$.

The variety X is a hypersurface in $\mathbb{P}(1,1,3,7,10)$ of degree 21, the singularities of X consist of the point P that is a quotient singularity of type $\frac{1}{10}(1,3,7)$. The equality $-K_X^3 = 1/10$ holds.

There is a commutative diagram

$$\begin{tikzcd} U & W \arrow[l, "\beta"'] & Y \arrow[l, "\gamma"'] \arrow[d, "\eta"] \\ X \arrow[u, leftarrow, "\alpha"'] \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,3), \end{tikzcd}$$

where α is the weighted blow up of the point P with weights $(1,3,7)$, β is the weighted blow up with weights $(1,3,4)$ of the singular point of U that is a quotient singularity of type $\frac{1}{7}(1,3,4)$,

γ is the weighted blow up with weights $(1,1,3)$ of the singular point of the variety W that is a quotient singularity of type $\frac{1}{4}(1,1,3)$, and η is an elliptic fibration.

It follows from Proposition 1.10 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P\}$.

Let $\mathcal{D}$ be the proper transform of the linear system $\mathcal{M}$ on the variety W. Then it follows from the proof of Proposition 22.1 that $\mathcal{D} \sim_{\mathbb{Q}} -kK_W$, and the set $\mathbb{CS}(W, \frac{1}{k}\mathcal{D})$ does not contain subvarieties of the variety W that are not contained in the exceptional divisor of β.

The set $\mathbb{CS}(W, \frac{1}{k}\mathcal{D})$ is not empty, because the divisor $-K_W$ is nef and big. Therefore, we can apply the arguments of the proof of Proposition 22.1 the the log pair $(W, \frac{1}{k}\mathcal{D})$, which implies that the set $\mathbb{CS}(W, \frac{1}{k}\mathcal{D})$ contains the singular point of the variety W that is a quotient singularity of type $\frac{1}{4}(1,1,3)$. Now the existence of the commutative diagram 44.2 is implies by Theorem A.15.

`Case` $n = 50$.

The variety X is a hypersurface in $\mathbb{P}(1,1,3,7,11)$ of degree 22, the singularities of X consist of the point P_1 that is a quotient singularity of type $\frac{1}{3}(1,1,2)$, and the point P_2 that is a quotient singularity of type $\frac{1}{7}(1,3,4)$. The equality $-K_X^3 = 2/21$ holds.

There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & W \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,3), \end{tikzcd}$$

where α is the weighted blow up of the point P_2 with weights $(1,3,7)$, β is the weighted blow up with weights $(1,1,3)$ of the singular point of the variety U that is a quotient singularity of type $\frac{1}{4}(1,1,3)$, and η is an elliptic fibration.

It follows from Proposition 1.10 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_2\}$, and the proof of Proposition 22.1 implies the existence of the commutative diagram 44.2.

`Case` $n = 54$.

The variety X is a hypersurface in $\mathbb{P}(1,1,6,8,9)$ of degree 24, the equality $-K_X^3 = 1/18$ holds, and the singularities of X consist of the point P_1 that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, the point P_2 that is a quotient singularity of type $\frac{1}{3}(1,1,2)$, and the point P_3 that is a quotient singularity of type $\frac{1}{9}(1,1,8)$.

There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] & & Y \arrow[ll, "\gamma"'] \arrow[d, "\eta"] \\ X \arrow[rrr, dashed, "\psi"'] & & & \mathbb{P}(1,1,6), \end{tikzcd}$$

where α is the weighted blow up of the point P_3 with weights $(1,1,8)$, β is the weighted blow up with weights $(1,1,7)$ of the singular point of U that is a quotient singularity of type $\frac{1}{8}(1,1,7)$, γ is the weighted blow up with weights $(1,1,3)$ of the singular point of W that is a quotient singularity of type $\frac{1}{7}(1,1,6)$, and η is an elliptic fibration.

It follows from Proposition 1.10 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_3\}$, and the existence of the commutative diagram 44.2 follows from the proof of Proposition 13.1.

`Case` $n = 55$.

The variety X is a hypersurface in $\mathbb{P}(1,2,3,7,12)$ of degree 24, the singularities of the hypersurface X consist of the points P_1 and P_2 that are quotient singularities of type $\frac{1}{2}(1,1,1)$, the points P_3 and P_4 that are quotient singularities of type $\frac{1}{3}(1,1,2)$, and the point P_5 that is a quotient singularity of type $\frac{1}{7}(1,2,5)$. The equality $-K_X^3 = 1/21$ holds.

There is a commutative diagram

$$\begin{tikzcd} U & & W \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[u, leftarrow, "\alpha"] \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,2,3), \end{tikzcd}$$

where α is the weighted blow up of the point P_5 with weights $(1,2,5)$, β is the weighted blow up with weights $(1,2,3)$ of the singular point of the variety U that is a quotient singularity of type $\frac{1}{5}(1,2,3)$, and η is an elliptic fibration.

It follows from Proposition 1.10 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_5\}$.

Let $\mathcal{B}$ be the proper transform of the linear system $\mathcal{M}$ on the variety U, and P_6 and P_7 be the singular points of U that are quotient singularities of types $\frac{1}{2}(1,1,1)$ and $\frac{1}{5}(1,2,3)$ contained in the exceptional divisor of α respectively. Then it follows from Lemmas B.5 and A.16 that either the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{B})$ contains P_6, or the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{B})$ contains P_7. In the latter case the existence of the commutative diagram 44.2 follows from Theorem A.15. Hence, we may assume that the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{B})$ contains the point P_6.

The hypersurface X can be given by the equation

$$t^3 z + t^2 f_{10}(x,y,z,w) + t f_{17}(x,y,z,w) + f_{24}(x,y,z,w) = 0 \subset \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x)=1$, $\mathrm{wt}(y)=2$, $\mathrm{wt}(z)=3$, $\mathrm{wt}(t)=7$, $\mathrm{wt}(w)=12$, and $f_i(x,y,z,w)$ is a sufficiently general quasihomogeneous polynomial of degree i. Let $\gamma : W \to U$ be the weighted blow up of the point P_6 with weights $(1,1,1)$, $\mathcal{H}$ be the proper transform of the linear system $\mathcal{M}$ on the variety W, $\mathcal{P}$ be the proper transform on the variety W of the pencil of surfaces that are cut on the hypersurface X by the equations $\lambda x^3 + \mu z = 0$, where $(\lambda, \mu) \in \mathbb{P}^1$, and D be a sufficiently general surface of the pencil $\mathcal{P}$. Then the base locus of the pencil $\mathcal{P}$ consists of the irreducible curves C, L and Δ such that $\alpha \circ \gamma(C)$ is the base curve of $|-3K_X|$, the curve $\gamma(L)$ is contained in the exceptional divisor of the morphism α, and the curve Δ is contained in the exceptional divisor of the morphism γ.

The surface D is normal, the surface D is smooth in the neighborhood of the exceptional divisor of the morphism γ, and the equalities

$$C \cdot C = -\frac{1}{3},\ L \cdot L = -\frac{11}{15},\ \Delta \cdot \Delta = -2,\ C \cdot L = \frac{1}{3},\ \Delta \cdot L = 1, \Delta \cdot C = 0$$

on D. Therefore, the intersection form of the curves L and C on the surface D is negatively defined, but $\mathcal{H}|_D \sim_{\mathbb{Q}} kC + kL$, which is impossible by Lemmas A.21 and A.20.

`Case` $n = 61$.

The variety X is a hypersurface in $\mathbb{P}(1,4,5,7,9)$ of degree 25, the singularities of X consist of the point P_1 that is a quotient singularity of type $\frac{1}{4}(1,1,3)$, the point P_2 that is a quotient singularity of type $\frac{1}{7}(1,2,5)$, and the point P_3 that is a quotient singularity of type $\frac{1}{9}(1,4,5)$.

The equality $-K_X^3 = 5/252$ holds, and there is a commutative diagram

$$\begin{tikzcd} & Y \arrow[dl, "\beta_3"'] \arrow[dr, "\beta_2"] \arrow[rrr, "\eta"] & & & \mathbb{P}(1,4,5), \\ U_2 \arrow[dr, "\alpha_2"'] & & U_3 \arrow[dl, "\alpha_3"] & & \\ & X \arrow[rrruu, dashed, "\psi"'] & & & \end{tikzcd}$$

where α_2 is the weighted blow up of the point P_2 with weights $(1,2,5)$, α_3 is the weighted blow up of the point P_3 with weights $(1,4,5)$, β_2 is the weighted blow up with weights $(1,2,5)$ of the proper transform of the point P_2 on U_3, β_3 is the weighted blow up with weights $(1,4,5)$ of the proper transform of P_3 on U_2, and η is an elliptic fibration.

It follows from Proposition 1.10 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) \subseteq \{P_2, P_3\}$, the proof of Lemma 36.2 implies that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_2, P_3\}$, and the existence of the commutative diagram 44.2 immediately follows from Theorem A.15.

`Case` $n = 62$.

The variety X is a hypersurface in $\mathbb{P}(1,1,5,7,13)$ of degree 26, the singularities of X consist of the point P_1 that is a quotient singularity of type $\frac{1}{5}(1,2,3)$, and the point P_2 that is a quotient singularity of type $\frac{1}{7}(1,1,6)$. The equality $-K_X^3 = 2/35$ holds.

There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & Y \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,5), \end{tikzcd}$$

where α is the weighted blow up of the point P_2 with weights $(1,1,6)$, β is the weighted blow up with weights $(1,1,5)$ of the singular point of the variety U that is a quotient singularity of type $\frac{1}{6}(1,1,5)$, and η is an elliptic fibration.

It follows from Proposition 1.10 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) \subseteq \{P_1, P_2\}$. Therefore, the existence of the commutative diagram 44.2 follows from the proof of Proposition 31.1.

`Case` $n = 63$.

The variety X is a hypersurface in $\mathbb{P}(1,2,3,8,13)$ of degree 26, the singularities of the hypersurface X consist of the points P_1, P_2 and P_3 that are quotient singularities of type $\frac{1}{2}(1,1,1)$, the point P_4 that is a quotient singularity of type $\frac{1}{3}(1,1,2)$, and the point P_5 that is a quotient singularity of type $\frac{1}{8}(1,3,5)$. The equality $-K_X^3 = 1/24$ holds.

There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & Y \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,2,3), \end{tikzcd}$$

where α is the weighted blow up of the point P_5 with weights $(1,3,5)$, and β is the weighted blow up with weights $(1,2,3)$ of the singular point of the variety U that is a quotient singularity of type $\frac{1}{5}(1,2,3)$ contained in the exceptional divisor of the birational morphism α, and η is an elliptic fibration.

It follows from Proposition 1.10 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_5\}$.

Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U, and P_6 and P_7 be the singular points of U that are quotient singularities of types $\frac{1}{5}(1,2,3)$ and $\frac{1}{2}(1,1,1)$ contained in the exceptional divisor of the morphism α respectively. Then it follows from Lemma A.16 that the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{B})$ contains either the point P_6, or the point P_7.

Suppose that the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{B})$ contains the point P_7. Let $\gamma : W \to U$ be the weighted blow up of the point P_7 with weights $(1,1,1)$, and $\mathcal{H}$ and $\mathcal{P}$ be the proper transforms of the linear system $\mathcal{M}$ and the pencil $|-2K_X|$ on the variety W respectively. Then the base locus of the pencil $\mathcal{P}$ consists of the irreducible curves C and L such that the curve $\alpha \circ \gamma(C)$ is the unique curve in the base locus of the pencil $|-2K_X|$, and the curve $\gamma(L)$ is contained in the exceptional divisor of the birational morphism α. Moreover, the surface D is normal, and the equalities

$$C \cdot C = -\frac{5}{12},\ L \cdot L = -\frac{7}{20},\ C \cdot L = \frac{1}{4}$$

hold on D. Hence, the intersection form of the curves L and C on the surface D is negatively defined. On the other hand, the equivalence $\mathcal{H}|_D \sim_{\mathbb{Q}} kC + kL$ holds on the surface D, which contradicts Lemmas A.21 and A.20.

Therefore, the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{B})$ contains the point P_6. Now the claim of Theorem A.15 easily implies the existence of the commutative diagram 44.2.

Case $n = 67$.

The variety X is a hypersurface in $\mathbb{P}(1,1,4,9,14)$ of degree 28, and the singularities of the hypersurface X consist of the point P_1 that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, and the point P_2 that is a quotient singularity of type $\frac{1}{9}(1,4,5)$. The equality $-K_X^3 = 1/18$ holds.

It follows from Proposition 1.10 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_2\}$.

There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & W \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,4), \end{tikzcd}$$

where α is the weighted blow up of the point P_2 with weights $(1,4,5)$, β is the weighted blow up with weights $(1,1,4)$ of the singular point of U that is quotient singularity of type $\frac{1}{5}(1,1,4)$, and η is an elliptic fibration.

The existence of the commutative diagram 44.2 follows from the proof of Proposition 22.1.

Case $n = 69$.

The variety X is a hypersurface in $\mathbb{P}(1,4,6,7,11)$ of degree 28, and the singularities of the hypersurface X consist of the points P_1 and P_2 that are quotient singularities of type $\frac{1}{2}(1,1,1)$, and the points P_3 and P_4 that are quotient singularities of types $\frac{1}{6}(1,1,5)$ $\frac{1}{11}(1,4,7)$ respectively.

The equality $-K_X^3 = 1/66$ holds, and there is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & W \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,3,4), \end{tikzcd}$$

where α is the weighted blow up of the point P_4 with weights $(1,4,7)$, β is the weighted blow up with weights $(1,3,4)$ of the singular point of the variety U that is a quotient singularity of type $\frac{1}{7}(1,3,4)$, and η is an elliptic fibration.

It follows from Proposition 1.10 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_4\}$.

Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U, and P_5 and P_6 be the singular points of U that are quotient singularities of types $\frac{1}{7}(1,3,4)$ and $\frac{1}{4}(1,1,3)$ contained in the exceptional divisor of the morphism α respectively. Then it follows from Lemmas B.5 and A.16 that $\mathbb{CS}(U, \frac{1}{k}\mathcal{B})$ contains either the point P_5, or the point P_6.

It follows from the proof of Proposition 15.1 that the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{B})$ does not contain the singular point P_6. Thus, the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{B})$ contains the point P_5. Hence, the existence of the commutative diagram 44.2 follows from Theorem A.15.

Case $n = 71$.

The variety X is a hypersurface in $\mathbb{P}(1,1,6,8,15)$ of degree 30, the singularities of X consist of the point P_1 that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, the point P_2 that is a quotient singularity of type $\frac{1}{3}(1,1,2)$, and the point P_3 that is a quotient singularity of type $\frac{1}{8}(1,1,7)$.

The equality $-K_X^3 = 1/24$ holds, and there is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & W \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,6), \end{tikzcd}$$

where α is the weighted blow up of the point P_3 with weights $(1,1,7)$, β is the weighted blow up with weights $(1,1,6)$ of the singular point of the variety U that is contained in the exceptional divisor of the morphism α, and η is an elliptic fibration.

It follows from Proposition 1.10 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_3\}$.

Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on U. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem A.15, and it follows from Lemmas B.5 and A.16 that the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ contains the singular point of the variety U that is contained in the exceptional divisor of the morphism α. Thus, the existence of the commutative diagram 44.2 is implied by Theorem A.15.

`Case` $n = 76$.

The variety X is a hypersurface in $\mathbb{P}(1,5,6,8,11)$ of degree 25, the singularities of X consist of the point P_1 that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, the point P_2 that is a quotient singularity of type $\frac{1}{8}(1,3,5)$, and the point P_3 that is a quotient singularity of type $\frac{1}{11}(1,5,6)$.

The equality $-K_X^3 = 1/88$ holds, and there is a commutative diagram

$$\begin{tikzcd}
 & Y \arrow[dl, "\beta_3"'] \arrow[dr, "\beta_2"] \arrow[rrr, "\eta"] & & & \mathbb{P}(1,5,6), \\
U_2 \arrow[dr, "\alpha_2"'] & & U_3 \arrow[dl, "\alpha_3"] & & \\
 & X \arrow[rrruu, dashed, "\psi"'] & & &
\end{tikzcd}$$

where α_2 is the weighted blow up of the point P_2 with weights $(1,3,5)$, α_3 is the weighted blow up of the point P_3 with weights $(1,5,6)$, β_2 is the weighted blow up of the proper transform of the point P_2 on the variety U_3 with weights $(1,3,5)$, β_3 is the weighted blow up of the proper transform of the point P_3 on the variety U_2 with weights $(1,5,6)$, and η is an elliptic fibration.

It follows from Proposition 1.10 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) \subseteq \{P_2, P_3\}$. Moreover, it follows from the proof of Lemmas 29.3 and 36.2 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_2, P_3\}$, which implies that the existence of the commutative diagram 44.2 follows from Theorem A.15.

`Case` $n = 77$.

The variety X is a hypersurface in $\mathbb{P}(1,2,5,9,16)$ of degree 32, the singularities of X consist of the points P_1 and P_2 that are quotient singularities of type $\frac{1}{2}(1,1,1)$, the point P_3 that is a quotient singularity of type $\frac{1}{5}(1,1,4)$, the point P_4 that is a quotient singularity of type $\frac{1}{9}(1,2,7)$.

The equality $-K_X^3 = 1/45$ holds, and there is a commutative diagram

$$\begin{tikzcd}
U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] \arrow[d, "\eta"] \\
X \arrow[r, dashed, "\psi"'] & \mathbb{P}(1,2,5),
\end{tikzcd}$$

where α is the weighted blow up of the point P_4 with weights $(1,2,7)$, β is the weighted blow up with weights $(1,2,5)$ of the singular point of the variety U that is a quotient singularity of type $\frac{1}{7}(1,2,5)$, and η is an elliptic fibration.

It follows from Proposition 1.10 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_4\}$.

Let $\mathcal{B}$ be the proper transform the linear system $\mathcal{M}$ on the vareity U, and P_5 and P_6 be the singular points of U that are quotient singularities of types $\frac{1}{7}(1,2,5)$ and $\frac{1}{2}(1,1,1)$ contained in the exceptional divisor of α respectively. Then it follows from Lemmas B.5 and A.16 that

$$\mathbb{CS}(U, \frac{1}{k}\mathcal{B}) \cap \{P_5, P_6\} \neq \varnothing.$$

Suppose that the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{B})$ contains the point P_6. Let $\gamma : W \to U$ be the weighted blow up of the singular point P_5 with weights $(1,1,1)$, $\mathcal{H}$ and $\mathcal{D}$ be the proper transforms of the linear systems $\mathcal{M}$ and $|-16K_X|$ on the variety W respectively, D be a general surface of the linear system $\mathcal{D}$, and H_1 and H_2 be general surfaces of the linear system $\mathcal{H}$. Then the base locus of the linear system $\mathcal{D}$ does not contain curves. In particular, the divisor D is nef, which implies that the inequality $D \cdot H_1 \cdot H_1 \geqslant 0$ holds. On the other hand, it follows from the explicit computations that the inequality $D \cdot H_1 \cdot H_1 < 0$ holds.

Therefore, the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{B})$ contains the point P_5. Now the claim of Theorem A.15 implies the existence of the commutative diagram 44.2.

Case $n = 83$.

The variety X is a general hypersurface in $\mathbb{P}(1,3,4,11,18)$ of degree 36, and the singularities of the hypersurface X consist of the point P_1 that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, the points P_2 and P_3 that are quotient singularities of type $\frac{1}{3}(1,1,2)$, the point P_4 that is a quotient singularity of type $\frac{1}{11}(1,4,7)$.

The equality $-K_X^3 = 1/66$ holds, and there is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & W \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,3,4), \end{tikzcd}$$

where α is the weighted blow up of the point P_4 with weights $(1,4,7)$, β is the weighted blow up with weights $(1,3,4)$ of the singular point of U that is a quotient singularity of type $\frac{1}{7}(1,3,4)$, and η is an elliptic fibration.

It follows from Proposition 1.10 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_4\}$.

Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U, and P_5 and P_6 be the singular points of U that are quotient singularities of types $\frac{1}{7}(1,3,4)$ and $\frac{1}{4}(1,1,3)$ contained in the exceptional divisor of the morphism α respectively. Then it follows from Lemmas B.5 and A.16 that $\mathbb{CS}(U, \frac{1}{k}\mathcal{B})$ contains either the point P_5, or the point P_6.

The proof of Proposition 25.1 easily implies that the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{B})$ does not contain the singular point P_6. Thus, the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{B})$ contains the singular point P_5, and the existence of the commutative diagram 44.2 follows from Theorem A.15.

Case $n = 85$.

The variety X is a hypersurface in $\mathbb{P}(1,3,5,11,19)$ of degree 38, the singularities of X consist of the point P_1 that is a quotient singularity of type $\frac{1}{3}(1,1,2)$, the point P_2 that is a quotient singularity of type $\frac{1}{5}(1,1,4)$, and the point P_3 that is a quotient singularity of type $\frac{1}{11}(1,3,8)$.

The equality $-K_X^3 = 2/165$ holds, and there is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & W \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,3,5), \end{tikzcd}$$

where α is the weighted blow up of the point P_3 with weights $(1,3,8)$, β is the weighted blow up with weights $(1,3,5)$ of the singular point of the variety U that is a quotient singularity of type $\frac{1}{8}(1,3,5)$, and η is an elliptic fibration.

It follows from Proposition 1.10 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_3\}$.

Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U, and P_4 and P_5 be the singular points of U that are quotient singularities of types $\frac{1}{8}(1,3,5)$ and $\frac{1}{3}(1,1,2)$ contained in the exceptional divisor of the morphism α respectively. Then it follows from Lemmas B.5 and A.16 that

$$\mathbb{CS}(U, \frac{1}{k}\mathcal{B}) \cap \{P_4, P_5\} \neq \varnothing.$$

Suppose that the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{B})$ contains the point P_5. Let $\gamma : W \to U$ be the weighted blow up of the singular point P_5 with weights $(1,1,2)$, $\mathcal{H}$ and $\mathcal{D}$ be the proper transforms of the linear systems $\mathcal{M}$ and $|-19K_X|$ on the variety W respectively, D be a general surface of the linear system $\mathcal{D}$, and H_1 and H_2 be general surfaces of the linear system $\mathcal{H}$. Then the base locus of the linear system $\mathcal{D}$ does not contain curves.

The divisor D is nef. In particular, the inequality $D \cdot H_1 \cdot H_1 \geqslant 0$ holds, but it follows from the simple explicit computations that the equality $D \cdot H_1 \cdot H_1 = -2k^2/15$ holds, which is a contradiction.

Hence, the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{B})$ contains the singular point P_4. Thus, the existence of the commutative diagram 44.2 follows from Theorem A.15.

`Case` $n = 91$.

The variety X is a hypersurface in $\mathbb{P}(1,4,5,13,22)$ of degree 44, the singularities of X consist of the point P_1 that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, the point P_2 that is a quotient singularity of type $\frac{1}{5}(1,2,3)$, the point P_3 that is a quotient singularity of type $\frac{1}{13}(1,4,9)$.

The equality $-K_X^3 = 1/130$ holds, and there is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & W \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,4,5), \end{tikzcd}$$

where α is the weighted blow up of the point P_3 with weights $(1,4,9)$, β is the weighted blow up with weights $(1,4,5)$ of the singular point of the variety U that is a quotient singularity of type $\frac{1}{9}(1,4,5)$, and η is an elliptic fibration.

It follows from Proposition 1.10 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_3\}$.

Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U, and P_4 and P_5 be the singular points of U that are quotient singularities of types $\frac{1}{9}(1,4,5)$ and $\frac{1}{4}(1,1,3)$ contained in the exceptional divisor of the morphism α respectively. Then it follows from Lemmas B.5 and A.16 that $\mathbb{CS}(U, \frac{1}{k}\mathcal{B})$ contains either the point P_4, or the point P_5.

It follows from the proof of Proposition 25.1 that the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{B})$ does not contain the singular point P_5. Thus, the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{B})$ contains the singular point P_4, and the existence of the commutative diagram 44.2 follows from Theorem A.15.

The claim of Proposition 44.1 is proved, which concludes the proof of Proposition 0.7.

Many problems of birational algebraic geometry are related to the global and local properties of pairs consisting of a variety and a divisor[6]. Depending on the context, the variety and the divisor must satisfy certain restrictions. For example, the singularities of the variety are usually assumed to be $\mathbb{Q}$-factorial, and the divisor is assumed to be effective. However, sometimes it is convenient and useful to consider pairs consisting of a variety and a linear system that does not have fixed components. In the given chapter we consider properties of such pairs (see [4]).

Definition A.1. A mobile log pair $(X, \lambda\mathcal{M})$ consists of a variety X, linear system $\mathcal{M}$ on the variety X such that $\mathcal{M}$ does not have fixed components, and a rational number $\lambda \geqslant 0$.

Every mobile log pair can always be consider as a usual log pair. In particular, for a mobile log pair $(X, \lambda\mathcal{M})$ the boundary $\lambda\mathcal{M}$ can be considered as an effective divisor. Hence, the intersection of the boundary $\lambda\mathcal{M}$ with curves on the variety X is well defined if the singularities of the variety X are $\mathbb{Q}$-factorial. Therefore, the formal sum $K_X + \lambda\mathcal{M}$ can be considered as a divisor on the variety X, which we call the log canonical divisor of the log pair $(X, \lambda\mathcal{M})$. In the rest of the chapter we assume that log-canonical divisors of all log pairs are $\mathbb{Q}$-Cartier divisors.

Remark A.2. For a log pair $(X, \lambda\mathcal{M})$ the self-intersection $\mathcal{M}^2$ can be considered as an effective one-dimensional cycle if the variety X has $\mathbb{Q}$-factorial singularities. Namely, take sufficiently general divisors S and $\bar{S}$ in the linear system $\mathcal{M}$ and put $\mathcal{M}^2 = S \cdot \bar{S}$.

We say that mobile log pairs $(X, \lambda\mathcal{M})$ and $(Y, \lambda\mathcal{D})$ are birationally equivalent if there is a birational map $\rho : X \dashrightarrow Y$ such that the linear system $\mathcal{D}$ is a proper transform of the linear system $\mathcal{M}$ with respect to the birational map ρ.

Definition A.3. The singularities of the mobile log pair $(X, \lambda\mathcal{M})$ are canonical (terminal respectively) if for every birational morphism $\pi : W \to X$ every rational number $a(X, \lambda\mathcal{M}, E_i)$, defined through the equivalence

$$K_W + \lambda\mathcal{D} \sim_{\mathbb{Q}} \pi^*(K_X + \lambda\mathcal{M}) + \sum_{i=1}^{k} a(X, \lambda\mathcal{M}, E_i)E_i,$$

is not negative (positive respectively), where $\mathcal{D}$ is a proper transform of the linear system $\mathcal{M}$ on the variety W, and E_i is a π-exceptional divisor[7].

Example A.4. Let X be a variety, $\mathcal{M}$ be a linear system on the variety X, and S be a general divisor of the linear system $\mathcal{M}$. Suppose that the dualizing sheaf ω_X is locally free, and the linear system $\mathcal{M}$ does not have fixed components. Then S has canonical singularities if and only if the singularities of $(X, \mathcal{M})$ are canonical (see Theorems 4.5.1 and 7.9 in [11]).

It should be pointed out that every mobile log pair is birationally equivalent to a log pair with terminal singularities.

Remark A.5. Let $(X, \lambda\mathcal{M})$ be a mobile log pair with terminal singularities. Then there is a rational number $\epsilon > \lambda$ such that the singularities of the log pair $(X, \epsilon\mathcal{M})$ are terminal.

Definition A.6. Let $(X, \lambda\mathcal{M})$ be a log pair, and Y be a proper irreducible subvariety of the variety X. Then Y is called a center of canonical singularities of the log pair $(X, \lambda\mathcal{M})$ if there is a birational morphism $\pi : W \to X$ and π-exceptional divisor $E_1 \subset W$ such that

$$K_W + \lambda\mathcal{D} \sim_{\mathbb{Q}} \pi^*(K_X + \lambda\mathcal{M}) + \sum_{i=1}^{k} a(X, \lambda\mathcal{M}, E_i)E_i,$$

[6]We assume that every divisor is a $\mathbb{Q}$-divisor, namely, a formal linear combination of irreducible reduced subvarieties of codimension one with rational coefficients.

[7]The number $a(X, \lambda\mathcal{M}, E_i)$ is called the discrepancy of $(X, \lambda\mathcal{M})$ in the divisor E_i. The number $a(X, \lambda\mathcal{M}, E_i)$ depends on the properties of the log pair $(X, \lambda\mathcal{M})$ and the discrete valuation that is related to the exceptional divisor E_i. Moreover, in order to check the canonicity (terminality respectively) of the log pair $(X, \lambda\mathcal{M})$ one can consider a single birational morphism $\pi : W \to X$ such that the variety W is smooth, and the proper transform of the linear system $\mathcal{M}$ on the variety W does not have base points.

where $a(X, \lambda\mathcal{M}, E_i)$ is a rational number, $\mathcal{D}$ is a proper transform of the linear system $\mathcal{M}$ on the variety W, E_i is a π-exceptional divisor, $\pi(E_1) = Y$ and $a(X, \lambda\mathcal{M}, E_1) \leqslant 0$. The set of all centers of canonical singularities of the log pair $(X, \lambda\mathcal{M})$ is denoted as $\mathbb{CS}(X, \lambda\mathcal{M})$.

Remark A.7. Let $(X, \lambda\mathcal{M})$ be a mobile log pair, H be a sufficiently general hyperplane section of the variety X, and Z be an element of $\mathbb{CS}(X, \lambda\mathcal{M})$. Then every component of $Z \cap H$ is a center of canonical singularities of the log pair $(H, \lambda\mathcal{M}|_H)$.

Remark A.8. Let $(X, \lambda\mathcal{M})$ be a mobile log pair, and Z be a proper irreducible subvariety of the variety X such that X is smooth in the general point of Z. Then $\mathrm{mult}_Z(\mathcal{M}) \geqslant 1/\lambda$ if the subvariety Z is a center of canonical singularities of the log pair $(X, \lambda\mathcal{M})$. Moreover, in the case $\mathrm{codim}(Z \subset X) = 2$ the inequality $\mathrm{mult}_Z(\mathcal{M}) \geqslant 1/\lambda$ implies that $Z \in \mathbb{CS}(X, \lambda\mathcal{M})$.

Definition A.9. Let $(X, \lambda\mathcal{M})$ be a mobile log pair, and $(W, \lambda\mathcal{D})$ be a log pair with canonical singularities that is birationally equivalent to $(X, \lambda\mathcal{M})$. Choose a natural number m such that the divisor $m(K_W + \lambda\mathcal{M})$ is a Cartier divisor. The Kodaira dimension $\kappa(X, \lambda\mathcal{M})$ of the mobile log pair $(X, \lambda\mathcal{M})$ as a maximal dimension of the variety $\phi_{|nm(K_W+\lambda\mathcal{D})|}(W)$ for $n \gg 0$ if at least one linear system $|nm(K_W + \lambda\mathcal{D})|$ is not empty, otherwise we put $\kappa(X, \lambda\mathcal{M}) = -\infty$.

The Kodaira dimension of a mobile log pair is a birational invariant and a nondecreasing function of the coefficients of the boundary. In the case of empty boundary, classical definition of the Kodaira dimension of a variety is coincide with Definition A.9.

Lemma A.10. *The Kodaira dimension of a mobile log pair is well defined.*

Proof. Let $(X, \lambda\mathcal{M})$ be a mobile log pair, and $\rho : Y \dashrightarrow X$ be a birational map such that the singularities of $(Y, \lambda\mathcal{B})$ are canonical, where $\mathcal{B}$ is a proper transform of the linear system $\mathcal{M}$ on the vareity Y. Take a natural m such that $m(K_X + \lambda\mathcal{M})$ and $m(K_Y + \lambda\mathcal{B})$ are Cartier divisors.

We must show that either linear systems $|nm(K_X + \lambda\mathcal{M})|$ and $|nm(K_Y + \lambda\mathcal{B})|$ are empty for all natural numbers n, or $\phi_{|nm(K_X+\lambda\mathcal{M})|}(X) = \phi_{|nm(K_Y+\lambda\mathcal{B})|}(Y)$ for all $n \gg 0$.

Consider a commutative diagram

$$\begin{tikzcd} & W \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X \arrow[rr, dashed, "\rho"'] & & Y, \end{tikzcd}$$

where W is a smooth variety, and α and β are birational morphisms. Then

$$K_W + \lambda\mathcal{D} \sim_{\mathbb{Q}} \alpha^*(K_X + \lambda\mathcal{M}) + \sum_{i=1}^{k} a_i G_i \sim_{\mathbb{Q}} \beta^*(K_Y + \lambda\mathcal{B}) + \sum_{i=1}^{l} b_i F_i,$$

where $\mathcal{D}$ is a proper transform of $\mathcal{M}$ on W, a_i and b_j are rational numbers, and G_i and F_i are exceptional divisors of the birational morphisms α and β respectively. Moreover, the canonicity of the log pairs $(X, \lambda\mathcal{M})$ and $(Y, \lambda\mathcal{B})$ implies that $a_i \geqslant 0$ and $b_j \geqslant 0$. Therefore, it follows from the negativity properties of an exceptional locus (see chapter 1.1 in [15]) that the complete linear systems $|\alpha^*(nm(K_X + \lambda\mathcal{M}))|$ and $|\beta^*(nm(K_Y + \lambda\mathcal{B}))|$ have the same dimension, and

$$\phi_{|nm(K_W+\lambda\mathcal{D})|} = \phi_{|\alpha^*(nm(K_X+\lambda\mathcal{M}))|} = \phi_{|\beta^*(nm(K_Y+\lambda\mathcal{B}))|},$$

which concludes the claim. □

Remark A.11. Let $(X, \lambda\mathcal{M})$ be a mobile log pair, where $K_X + \lambda\mathcal{M} \sim_{\mathbb{Q}} 0$. Then $\kappa(X, \lambda\mathcal{M}) \leqslant 0$, and the equality $\kappa(X, \lambda\mathcal{M}) = 0$ holds if and only if the singularities of $(X, \lambda\mathcal{M})$ are canonical.

Example A.12. Let X be a smooth quartic hypersurface in $\mathbb{P}^4$, and $\psi : W \to X$ be a blow up of a line in X. Then the linear system $|-K_W|$ does not have base points and induces the elliptic fibration $\phi : W \to \mathbb{P}^2$. Let $\mathcal{M}$ be a proper transform of the linear system $|-K_W|$ on the hypersurface X, and λ be a rational number. Then

$$\kappa(X, \lambda\mathcal{M}) = \begin{cases} -\infty \text{ for } \lambda < 1, \\ 0 \text{ for } \lambda = 1, \\ 2 \text{ for } \lambda > 1. \end{cases}$$

Definition A.13. Let $(X, \lambda\mathcal{M})$ and $(V, \lambda\mathcal{D})$ be mobile log pairs. Then $(V, \lambda\mathcal{D})$ is a canonical model[8] of the mobile log pair $(X, \lambda\mathcal{M})$ if the log pairs $(X, \lambda\mathcal{M})$ and $(V, \lambda\mathcal{D})$ are birationally equivalent, the divisor $K_V + \lambda\mathcal{D}$ is ample, and the singularities of $(V, \lambda\mathcal{D})$ are canonical.

The proof of lemma A.10 implies the following result.

Theorem A.14. *A canonical model is unique if it exists.*

The following result is due to [9].

Theorem A.15. *Let $(X, \lambda\mathcal{M})$ be a mobile log pair such that $\dim(X) = 3$, and O be a singular point of the variety X such that O is a quotient singularity of type $\frac{1}{r}(1, a, r-a)$, where a and r are coprime natural numbers such that $r > a$. Then*

$$\mathcal{D} \sim_{\mathbb{Q}} \pi^*(\mathcal{M}) - \mathrm{mult}_O(\mathcal{M})G,$$

where $\pi : U \to X$ is a weighted blow up of the point O with weights $(1, a, r-a)$, G is the exceptional divisor of the weighted blow up π, $\mathcal{D}$ is a proper transform of the linear system $\mathcal{M}$ on the variety U, and $\mathrm{mult}_O(\mathcal{M})$ is a rational number. Suppose that the set $\mathbb{CS}(X, \lambda\mathcal{M})$ contains either the point O, or a curve passing through the point O. Then $\mathrm{mult}_O(\mathcal{M}) \geqslant 1/(r\lambda)$.

Proof. We consider only the case $r = 2$. Then $\mathcal{D}|_G \sim_{\mathbb{Q}} -\mathrm{mult}_O(\mathcal{M})G|_G$ and $G|_G \sim_{\mathbb{Q}} \mathcal{O}_{\mathbb{P}^2}(-2)$, where $G \cong \mathbb{P}^2$. Suppose that $\mathrm{mult}_O(\mathcal{M}) < 1/(2\lambda)$. Let P be a point of G. Intersecting the divisor $\mathcal{D}|_G$ with a general line on G that passes through P, we see that the inequalities

$$\mathrm{mult}_P(\mathcal{D}) \leqslant 2\mathrm{mult}_O(\mathcal{M}) < \frac{1}{\lambda}$$

hold. Suppose that $\mathbb{CS}(X, \lambda\mathcal{M})$ contains a curve C that passes through O. Then

$$\frac{1}{\lambda} > \mathrm{mult}_Q(\mathcal{D}) \geqslant \mathrm{mult}_Z(\mathcal{D}) \geqslant \mathrm{mult}_C(\mathcal{M}) \geqslant \frac{1}{\lambda},$$

where Z is a proper transform of the curve C on the variety U, and Q is a point of the intersection of the curve Z and the exceptional divisor G. Therefore, the set $\mathbb{CS}(X, \lambda\mathcal{M})$ does not contain a curve passing through the point O. Hence, the set $\mathbb{CS}(X, \lambda\mathcal{M})$ contains the point O. Then

$$K_U + \lambda\mathcal{D} \sim_{\mathbb{Q}} \pi^*(K_X + \lambda\mathcal{M}) + \Big(\frac{1}{r} - \lambda\mathrm{mult}_O(\mathcal{M})\Big)G,$$

which implies that the set $\mathbb{CS}(U, \lambda\mathcal{D})$ contains a proper subvariety $\Delta \subsetneq G$. In particular, the inequality $\mathrm{mult}_\Delta(\mathcal{D}) \geqslant 1/\lambda$ holds, which is a contradiction. $\square$

The claim of Theorem A.15 implies the following result.

Lemma A.16. *Under the assumptions and notations of Theorem A.15, suppose that the singularities of $(X, \lambda\mathcal{M})$ are canonical and $\mathbb{CS}(X, \lambda\mathcal{M}) = \{O\}$, but $\mathbb{CS}(U, \lambda\mathcal{D}) \neq \varnothing$. Then*

- *the set $\mathbb{CS}(U, \lambda\mathcal{D})$ does not contain smooth points of the surface $G \cong \mathbb{P}(1, a, r-a)$,*
- *if the set $\mathbb{CS}(U, \lambda\mathcal{D})$ contains a curve L, then $L \in |\mathcal{O}_{\mathbb{P}(1,\,a,\,r-a)}(1)|$, and every singular point of the surface G is contained in the set $\mathbb{CS}(U, \lambda\mathcal{D})$.*

Proof. We consider only the case when $r = 5$ and $a = 2$, because the proof in general case is very similar. Thus, we have $G \cong \mathbb{P}(1, 2, 3)$.

Let P and Q be different singular points of the surfaces G, and L be the unique curve of the linear system $|\mathcal{O}_{\mathbb{P}(1,\,2,\,3)}(1)|$. Then the curve L passes through the points P and Q, but it follow from the claim of Theorem A.15 that $\mathrm{mult}_O(\mathcal{M}) = 1/(5\lambda)$, which implies $\mathcal{D}|_G \sim_{\mathbb{Q}} \lambda L$.

Suppose that the set $\mathbb{CS}(U, \lambda\mathcal{D})$ contains a subvariety Z of subvariety U that is different from the curve L and the points P and Q. Then $Z \subset G$.

Suppose that Z is a point. Then the point Z is smooth on the variety U, which implies the inequality $\mathrm{mult}_Z(\mathcal{D}) > 1/\lambda$. Let C be a general curve in the linear system $|\mathcal{O}_{\mathbb{P}(1,\,2,\,3)}(6)|$ that passes through the point Z. Then the curve C is not contained in the base locus of the linear system $\mathcal{D}$. Hence, we have

$$\frac{1}{\lambda} = C \cdot \mathcal{D} \geqslant \mathrm{mult}_Z(C)\mathrm{mult}_Z(\mathcal{D}) > \frac{1}{\lambda},$$

[8] The given definition coincides with the classical definition of a canonical model for $\lambda = 0$.

which is a contradiction.

Therefore, the subvariety Z is a curve. Then $\mathrm{mult}_Z(\mathcal{D}) \geqslant 1/\lambda$. Let C be a sufficiently general curve in the linear system $|\mathcal{O}_{\mathbb{P}(1,2,3)}(6)|$. Then the curve C is not contained in the base locus of the linear system $\mathcal{D}$. Hence, we have

$$\frac{1}{\lambda} = C \cdot \mathcal{D} \geqslant \mathrm{mult}_Z(\mathcal{D}) C \cdot Z \geqslant \frac{C \cdot Z}{\lambda},$$

which implies that $C \cdot Z = 1$. The equality $C \cdot Z = 1$ implies that the curve Z is contained in the linear system $|\mathcal{O}_{\mathbb{P}(1,2,3)}(1)|$, which is impossible due to our assumption. $\square$

Consider the following simple result.

Lemma A.17. *Let X be a variety, and $\mathcal{M}$ be a linear system on the variety X such that the linear system $\mathcal{M}$ does not have fixed components and is not composed from a pencil. Then there is no proper Zariski closed subvariety $\Sigma \subsetneq X$ such that*

$$\mathrm{Supp}(S_1) \cap \mathrm{Supp}(S_2) \subset \Sigma \subsetneq X,$$

where S_1 and S_2 are sufficiently general divisors in the linear system $\mathcal{M}$.

Proof. Suppose that there is a proper Zariski closed subvariety $\Sigma \subset X$ such that the set-theoretic intersection of the sufficiently general divisors S_1 and S_2 of the linear system $\mathcal{M}$ is contained in Σ. Let $\rho : X \dashrightarrow \mathbb{P}^n$ be a rational map induced by the linear system $\mathcal{M}$, where n is the dimension of the linear system $\mathcal{M}$. Then there is a commutative diagram

$$\begin{tikzcd} & W \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X \arrow[rr, dashed, "\rho"'] & & \mathbb{P}^n, \end{tikzcd}$$

where W is a smooth variety, α is a birational morphism, and β is a morphism. Let Y be the image of the morphism β. Then $\dim(Y) \geqslant 2$, because $\mathcal{M}$ is not composed from a pencil.

Let Λ be a Zariski closed subvariety of the variety W such that the morphism

$$\alpha|_{W \setminus \Lambda} : W \setminus \Lambda \longrightarrow X \setminus \alpha(\Lambda)$$

is an isomorphism, and Δ be a union of the subset $\Lambda \subset W$ and the closure of the proper transform of the set $\Sigma \setminus \alpha(\Lambda)$ on the variety W. Then Δ is a Zariski closed proper subset of W.

Let B_1 and B_2 be general hyperplane sections of the variety Y, and D_1 and D_2 be proper transforms of the divisors B_1 and B_2 on the variety W respectively. Then $\alpha(D_1)$ and $\alpha(D_2)$ are general divisors of the linear system $\mathcal{M}$. Hence, in the set-theoretic sense we have

$$\varnothing \neq \beta^{-1}\Big(\mathrm{Supp}(B_1) \cap \mathrm{Supp}(B_2)\Big) = \mathrm{Supp}(D_1) \cap \mathrm{Supp}(D_2) \subset \Delta \subsetneq W, \tag{A.18}$$

because $\dim(Y) \geqslant 2$. However, the set-theoretic identity A.18 is an absurd. $\square$

The following result is implied by Lemma 0.3.3 in [10] and Lemma A.20.

Corollary A.19. *Let X be a three-dimensional variety with canonical singularities, D be divisor on the variety X that is big and nef, $\mathcal{M}$ be a linear system on the variety X that does not have fixed components and is not composed from a pencil, and S_1 and S_2 be sufficiently general surfaces of the linear system $\mathcal{D}$. Then the inequality $D \cdot S_1 \cdot S_2 > 0$ holds.*

The proof of Lemma A.17 implies the following result.

Lemma A.20. *Let X be a variety, $\mathcal{M}$ be a linear system on the variety X that does not have fixed components and is not composed from a pencil, and $\mathcal{D}$ be a linear system on X that does not have fixed components. Then there is no Zariski closed subvariety $\Sigma \subsetneq X$ such that*

$$\mathrm{Supp}(S) \cap \mathrm{Supp}(D) \subset \Sigma \subsetneq X,$$

where S and D are sufficiently general divisors of the linear system $\mathcal{M}$ and $\mathcal{D}$ respectively.

Many applications of Lemma A.20 are related to the following simple result.

Lemma A.21. *Let S be a projective normal surface, and D be an effective divisor on S such that the equivalence $D \equiv \sum_{i=1}^{r} a_i C_i$ holds, where a_i is a rational number, and $C_1, \ldots, C_r$ are irreducible curves on the surface S such that the intersection form of the curves $C_1, \ldots, C_r$ on the surface S is negatively defined. Then $D = \sum_{i=1}^{r} a_i C_i$.*

Proof. Let $D = \sum_{i=1}^{k} c_i B_i$, where B_i is an irreducible curve on S, and c_i is a nonnegative rational number. Suppose that

$$\sum_{i=1}^{k} c_i B_i \neq \sum_{i=1}^{r} a_i C_i,$$

and the curve B_i is not one of the curves among $C_1, \ldots, C_r$ for every i. We have

$$0 \geqslant \Big(\sum_{a_i>0} a_i C_i\Big) \cdot \Big(\sum_{a_i>0} a_i C_i\Big) = \Big(\sum_{i=1}^{k} c_i B_i\Big) \cdot \Big(\sum_{a_i>0} a_i C_i\Big) - \Big(\sum_{a_i\leqslant 0} a_i C_i\Big) \cdot \Big(\sum_{a_i>0} a_i C_i\Big) \geqslant 0,$$

which implies the numerical equivalence

$$\sum_{c_i\geqslant 0} c_i B_i \equiv \sum_{a_i\leqslant 0} a_i C_i,$$

which implies that $c_i = 0$ and $a_i = 0$ for every i, which is a contradiction. $\square$

Let X be a Fano variety such that the singularities of the variety X are terminal and $\mathbb{Q}$-factorial, and the equality $\mathrm{rk}\,\mathrm{Pic}(X) = 1$ holds. The following result is due to [4].

Theorem B.1. *Suppose that the singularities of every mobile log pair $(X, \lambda\mathcal{M})$ are canonical, where λ is a rational number such that $K_X + \lambda\mathcal{M} \sim_{\mathbb{Q}} 0$. Then X is birationally superrigid.*

Proof. Let $\rho : X \dashrightarrow Y$ be a birational map such that either Y is a Fano variety having terminal and $\mathbb{Q}$-factorial singularities such that the equality $\mathrm{rk}\,\mathrm{Pic}(Y) = 1$ holds, or there is a dominant morphism $\tau : Y \to Z$ such that a general fiber of the morphism τ is a variety of negative Kodaira dimension, where $\dim(Y) \neq \dim(Z) \neq 0$. To conclude the proof we must show that the latter case is impossible, but in the former case the map ρ is an isomorphism.

Suppose that there is a morphism $\tau : Y \to Z$ such that a general fiber of τ is a variety of negative Kodaira dimension. Let $\mathcal{D}$ be a linear system $|\tau^*(H)|$, where H is a very ample divisor on the variety Z, and μ be a positive rational number. Then

$$\kappa(X, \mu\mathcal{M}) = \kappa(Y, \mu\mathcal{D}) = -\infty,$$

where $\mathcal{M}$ is the proper transform of the linear system $\mathcal{D}$ on the variety X. Now choose μ such that $K_X + \mu\mathcal{M} \sim_{\mathbb{Q}} 0$. Then $\kappa(X, \mu\mathcal{M}) = 0$, because the singularities of $(X, \mu\mathcal{M})$ are canonical.

Therefore, the variety Y is a Fano variety with terminal $\mathbb{Q}$-factorial singularities such that the equality $\mathrm{rk}\,\mathrm{Pic}(Y) = 1$ holds. Let $\mathcal{D}$ be a linear system $| - nK_Y|$, and $\mathcal{M}$ be the proper transform of the linear system $\mathcal{D}$ on the variety X. Now choose rational number μ such that the equivalence $K_X + \mu\mathcal{M} \sim_{\mathbb{Q}} 0$ holds. Then the singularities of the log pair $(X, \mu\mathcal{M})$ are canonical, which implies that the equality $\kappa(Y, \mu\mathcal{D}) = 0$ holds. Therefore, we have $\mu = 1/n$.

Consider commutative diagram

$$\begin{tikzcd} & W \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X \arrow[rr, dashed, "\rho"'] & & Y, \end{tikzcd}$$

where W is a smooth variety, and α and β are birational morphisms. Then

$$\sum_{j=1}^{k} a(X, \mu\mathcal{M}, F_j)F_j \sim_{\mathbb{Q}} \sum_{i=1}^{l} a(Y, \mu\mathcal{D}, G_i)G_i,$$

where G_i and F_j are exceptional divisors of the morphism β and α respectively. On the other hand, the singularities of the log pairs $(X, \mu\mathcal{M})$ and $(Y, \mu\mathcal{D})$ are canonical and terminal respectively. Therefore, the equality $a(X, \mu\mathcal{M}, E) = a(Y, \mu\mathcal{D}, E)$ holds for every divisor E on the variety W (see chapter 1.1 in [15]). In particular, we have

$$\sum_{j=1}^{k} a(X, \mu\mathcal{M}, F_j)F_j = \sum_{i=1}^{l} a(Y, \mu\mathcal{D}, G_i)G_i,$$

where $k = l$, because $\mathrm{rk}\,\mathrm{Cl}(Y) = \mathrm{rk}\,\mathrm{Cl}(X) = 1$. Thus, the singularities of $(X, \mu\mathcal{M})$ are terminal.

Let ζ be a positive rational number such that $\zeta > \mu$, but the log pair $(X, \zeta\mathcal{M})$ has terminal singularities (see Remark A.5). Then $(X, \zeta\mathcal{M})$ and $(Y, \zeta\mathcal{D})$ are canonical models, which implies that ρ is biregular by Theorem A.14. □

It is easy to see that the claim of Theorem B.1 is a criterion of the birational superrigidity of the variety X up to the existence of extremal blow ups (see Proposition 2.10 in [4]).

Corollary B.2. *Suppose that* $\dim(X) = 3$. *Then X is birationally superrigid if and only if the singularities of every mobile log pair $(X, \lambda\mathcal{M})$ has canonical singularities, where λ is a positive rational number such that $K_X + \lambda\mathcal{M} \sim_{\mathbb{Q}} 0$.*

The proof of Theorem B.1 implies the following result.

Corollary B.3. *Suppose that* $\dim(X) = 3$. *Then* X *is birationally rigid if and only if for every linear system* $\mathcal{M}$ *on the variety* X *that does not have fixed components there is a birational automorphism* $\chi \in \mathrm{Bir}(X)$ *such that the singularity of* $(X, \lambda\chi(\mathcal{M}))$ *are canonical, where* λ *is a positive rational number such that* $K_X + \lambda\chi(\mathcal{M}) \sim_{\mathbb{Q}} 0$.

The following generalization of Theorem B.1 is due to [1].

Theorem B.4. *Let* $\zeta : X \dashrightarrow Z$ *be a rational map such that the normalization of a general fiber of the rational map* ζ *is an elliptic curve, and* $\rho : V \to X$ *be a birational morphism such that the variety* V *is a smooth, and there is a commutative diagram*

$$\begin{tikzcd} & V \arrow[dl, "\rho"'] \arrow[dr, "\tau"] & \\ X \arrow[rr, dashed, "\zeta"'] & & Z, \end{tikzcd}$$

where τ *is a morphism. Take a very ample divisor* D *on the variety* Z. *Let* $\mathcal{M}$ *be a proper transform of the linear system* $|\tau^*(D)|$ *on the variety* X. *Then the set* $\mathbb{CS}(X, \gamma\mathcal{M})$ *is not empty, where* γ *is a positive rational number such that* $K_X + \gamma\mathcal{M} \sim_{\mathbb{Q}} 0$.

Proof. Suppose that $\mathbb{CS}(X, \gamma\mathcal{M}) = \varnothing$. Then the log pair $(X, \epsilon\mathcal{M})$ is a canonical model for some rational number $\epsilon > \gamma$, which implies $\kappa(V, \epsilon\mathcal{D}) = \dim(X)$, but $\kappa(V, \epsilon\mathcal{D}) \leqslant \dim(X) - 1$. $\square$

The proof of Theorem B.4 implies the following result.

Lemma B.5. *Let* Y *be a variety with terminal* $\mathbb{Q}$*-factorial singularities such that* $-K_Y$ *is nef and big. Suppose that there is a rational map* $\zeta : Y \dashrightarrow Z$ *such that the normalization of a general fiber of the rational map* ζ *is an elliptic curve. Let* $\rho : V \to Y$ *be a birational morphism such that the variety* V *is a smooth, and there is a commutative diagram*

$$\begin{tikzcd} & V \arrow[dl, "\rho"'] \arrow[dr, "\tau"] & \\ Y \arrow[rr, dashed, "\zeta"'] & & Z, \end{tikzcd}$$

where τ *is a morphism. Take a very ample divisor* D *on the variety* Z. *Let* $\mathcal{M}$ *be a proper transform of the linear system* $|\tau^*(D)|$ *on the variety* Y. *Suppose that there is a positive rational number* γ *such that* $K_Y + \gamma\mathcal{M} \sim_{\mathbb{Q}} 0$. *Then* $\mathbb{CS}(Y, \gamma\mathcal{M}) \neq \varnothing$.

The proof of Theorem B.4 implies the following generalization of Theorem B.4.

Theorem B.6. *Let* $\rho : X \dashrightarrow V$ *be a birational map such that* V *is a smooth variety, and there is a dominant morphism* $\tau : V \to Z$, *whose general fiber is a variety of Kodaira dimension zero, where* $\dim(X) \neq \dim(Z) \neq 0$. *Take a very ample divisor* D *on the variety* Z. *Let* $\mathcal{M}$ *be a proper transform of the linear system* $|\tau^*(D)|$ *on the variety* X, *and* γ *be a positive rational number such that* $K_X + \gamma\mathcal{M} \sim_{\mathbb{Q}} 0$. *Then the set* $\mathbb{CS}(X, \gamma\mathcal{M})$ *is not empty.*

Theorem B.7. *Let* $\rho : X \dashrightarrow V$ *be a birational map such that* V *is a Fano variety with canonical singularities, and* ρ *is not biregular. Take a sufficiently big natural number* n. *Let* $\mathcal{M}$ *be a proper transform of the linear system* $|-nK_V|$ *on the variety* X, *and* γ *be a positive rational number such that* $K_X + \gamma\mathcal{M} \sim_{\mathbb{Q}} 0$. *Then the set* $\mathbb{CS}(X, \gamma\mathcal{M})$ *is not empty.*

Proof. Suppose that $\mathbb{CS}(X, \gamma\mathcal{M}) = \varnothing$. Then the singularities of $(X, \gamma\mathcal{M})$ are terminal, which implies that $\kappa(X, \gamma\mathcal{M}) = 0$. Therefore, the equality $\gamma = 1/n$ holds. Thus, for some rational number $\epsilon > \gamma$ the mobile log pairs $(X, \epsilon\mathcal{M})$ and $(V, \epsilon|-nK_V|)$ are canonical models, which implies that the rational map ρ is biregular by Theorem A.14. $\square$

The claim of Theorem B.1 is called Noether–Fano–Iskovskikh inequality. Therefore, the claims of Theorems B.4, B.6 and B.7 can be considered as strengthened Noether–Fano–Iskovskikh inequalities.

Properties of mobile log pairs (see Definition A.1) reflects birational geometry of algebraic varieties (see Theorem A.14). Canonical and terminal singularities are the most appropriate classes of singularities for mobile log pairs. Many geometrical problems can be easily translated into the language of mobile log pairs (see Theorem B.1).

We can consider log pairs with both mobile and fixed components, which is similar to the fact that linear systems can have both mobile and fixed parts. Moreover, we can consider log pairs with negative coefficients. Such generalizations are important because of the following reasons:

- log pull backs of mobile log pairs with respect to birational morphisms reflect the property of original log pairs (see Definition C.1), but log pull backs can have fixed components and negative boundary coefficients;
- centers of canonical singularities (see Definition A.6) do not have good biregular properties in contrast to centers of log canonical singularities (see Definition C.2, [15], [11]).

In the given chapter we do not assume any restrictions on the boundary coefficients, namely, boundaries may not be effective. We assume that log canonical divisors are $\mathbb{Q}$-Cartier divisors.

Definition C.1. A log pair (V, B^V) is called a log pull back of a log pair (X, B_X) with respect to a birational morphism $f : V \to X$ if we have

$$B^V = f^{-1}(B_X) - \sum_{i=1}^{n} a(X, B_X, E_i)E_i$$

such that the equivalence $K_V + B^V \sim_{\mathbb{Q}} f^*(K_X + B_X)$ holds, where E_i is an f-exceptional divisor and $a(X, B_X, E_i)$ is a rational number.

Definition C.2. A proper irreducible subvariety $Y \subset X$ is called a center of log canonical singularities of the log pair (X, B_X) if there are a birational morphism $f : V \to X$ and a not necessary f-exceptional divisor $E \subset V$ such that E is contained in the effective part of the support of the divisor $\lfloor B^V \rfloor$ and $f(E) = Y$. The set of all centers of log canonical singularities of the log pair (X, B_X) is denoted as $\mathbb{LCS}(X, B_X)$. The set-theoretic union of all elements of the set $\mathbb{LCS}(X, B_X)$ considered as a subset of the variety X is called a locus of log canonical singularities of the log pair (X, B_X) and is denoted as $LCS(X, B_X)$.

Remark C.3. Let (X, B_X) be a log pair, H be a general hyperplane section of X, and Z be an element of the set (X, B_X). Then every component of $Z \cap H$ is contained in $\mathbb{LCS}(H, B_X|_H)$.

Let X be a variety and $B_X = \sum_{i=1}^{n} a_i B_i$ be a boundary on X, where a_i is a rational number, and B_i is either a prime divisor on the variety X, or a linear system on the variety X that does not have fixed components. We say that the boundary B_X is effective if $a_i \geqslant 0$ for every possible value of i. We say that the boundary B_X is mobile if B_i is a linear system on X that does not have fixed components for every index i.

Example C.4. Let O be a smooth point of the variety X. Suppose that the point O is contained in the set $\mathbb{LCS}(X, B_X)$. Let $f : V \to X$ be a blow up of the point O, and E be an exceptional divisor of the birational morphism f. Then either $E \in \mathbb{LCS}(V, B^V)$, or there is an irreducible proper subvariety $Z \subsetneq E$ such that $Z \in \mathbb{LCS}(V, B^V)$. Moreover, the exceptional divisor E is contained in the set $\mathbb{LCS}(V, B^V)$ if and only if the inequality $\mathrm{mult}_O(B_X) \geqslant \dim(X)$ holds.

Let $f : Y \to X$ be a birational morphism such that the variety Y is smooth, and the union of all f-exceptional divisors and $\cup_{i=1}^{n} \bar{B}_i$ is a divisor with simple normal crossing, where $\bar{B}_i$ is the proper transform of B_i on Y if B_i is a divisor, or a proper transform of a general element of the linear system B_i otherwise. Then f is called a log resolution of the log pair (X, B_X), and

$$K_Y + B^Y \sim_{\mathbb{Q}} f^*(K_X + B_X),$$

where (Y, B^Y) is a log pull back of the log pair (X, B_X).

Definition C.5. Let $\mathcal{I}(X, B_X) = f_*(\mathcal{O}_Y(\lceil -B^V \rceil))$. Then the subscheme $\mathcal{L}(X, B_X)$ associated to the ideal sheaf $\mathcal{I}(X, B_X)$ is called a log canonical singularity subscheme of (X, B_X).

It should be pointed out, that $\mathrm{Supp}(\mathcal{L}(X, B_X)) = LCS(X, B_X) \subset X$. The following result is the Shokurov vanishing theorem (see [15]).

Theorem C.6. *Suppose that B_X is effective. Let H be a nef and big divisor on X such that the divisor $D = K_X + B_X + H$ is a Cartier divisor. Then $H^i(X, \mathcal{I}(X, B_X) \otimes D) = 0$ for $i > 0$.*

Proof. It follows from the the Kawamata-Viehweg vanishing that

$$R^i f_*\big(f^*(K_X + B_X + H) + \lceil -B^W \rceil\big) = 0$$

for all $i > 0$ (see [10]). The degeneration of the local–to–global spectral sequence and

$$R^0 f_*\big(f^*(K_X + B_X + H) + \lceil -B^W \rceil\big) = \mathcal{I}(X, B_X) \otimes D$$

imply that

$$H^i\big(X, \mathcal{I}(X, B_X) \otimes D\big) = H^i\big(W, f^*(K_X + B_X + H) + \lceil -B^W \rceil\big)$$

for $i \geqslant 0$, but

$$H^i\big(W, f^*(K_X + B_X + H) + \lceil -B^W \rceil\big) = 0$$

for $i > 0$ by the Kawamata-Viehweg vanishing. $\square$

Let us consider the following elementary application of Theorem C.6.

Lemma C.7. *Let $V = \mathbb{P}^1 \times \mathbb{P}^1$, and B_V be an effective boundary on V of bi-degree (a, b) such that a and $b \in \mathbb{Q} \cap [0, 1)$. Then $\mathbb{LCS}(V, B_V) = \emptyset$.*

Proof. Let $B_V = \sum_{i=1}^k a_i B_i$, where a_i is a positive rational number, and B_i is an irreducible reduced curve on the surface V. Intersecting the boundary B_V with the rulings of V we get the inequality $a_i < 1$. Thus the set $\mathbb{LCS}(V, B_V)$ does not contains curves on V.

Suppose that the set $\mathbb{LCS}(V, B_V)$ contains a point O. Take a divisor $H \in \mathrm{Pic}(V) \otimes \mathbb{Q}$ having bi-degree $(1 - a, 1 - b)$. Then the divisor H is ample. Moreover, there is a divisor

$$D \sim_{\mathbb{Q}} K_V + B_V + H$$

such that D is a Cartier divisor and $H^0(\mathcal{O}_V(D)) = 0$. On the other hand, the map

$$H^0(\mathcal{O}_V(D)) \to H^0(\mathcal{O}_{\mathcal{L}(V, B_V)}(D))$$

is surjective by Theorem C.6, which is a contradiction. $\square$

The idea of the proof of Lemma C.7 can be used to get a more general result. Namely, for an arbitrary Cartier divisor D on the variety X consider the exact sequence of sheafs

$$0 \to \mathcal{I}(X, B_X) \otimes D \to \mathcal{O}_X(D) \to \mathcal{O}_{\mathcal{L}(X,\, B_X)}(D) \to 0$$

and the corresponding exact sequence

$$H^0(\mathcal{O}_X(D)) \to H^0(\mathcal{O}_{\mathcal{L}(X,\, B_X)}(D)) \to H^1(\mathcal{I}(X, B_X) \otimes D)$$

of cohomology groups. Then Theorem C.6 implies the following result (see [15]).

Theorem C.8. *Suppose that the boundary B_X is effective, but the divisor $-(K_X + B_X)$ is nef and big. Then the set $LCS(X, B_X)$ is connected.*

Similar arguments implies the following result (see Lemma 5.7 in [15].

Theorem C.9. *Let $g : X \to Z$ be a surjective morphism such that the fibers of the morphism g are connected, the divisor $-(K_X + B_X)$ is g-nef and g-big, and $\mathrm{codim}(g(B_i) \subset Z) \geqslant 2$ whenever the inequality $b_i < 0$ holds. Then the set $LCS(Y, B^Y)$ is connected in the neighborhood of every fiber of the morphism $g \circ f : Y \to Z$.*

The main application of Theorem C.9 is the following inductive result[9].

Theorem C.10. *Suppose that B_X is effective. Let Z be an element of $\mathbb{CS}(X, B_X)$, and H be an effective and reduced Cartier divisor on the variety X such that H is not a component of the boundary B_X, the subvariety Z is contained in the divisor H, but the divisor H is smooth in the general point of the subvariety Z. Then $\mathbb{LCS}(H, B_X|_H) \neq \varnothing$.*

[9]We define centers of canonical singularities and set of centers of canonical singularities only for mobile log pairs (see Definition A.6), but the same definition is valid for any log pair.

Proof. Consider the log pair $(X, B_X + H)$. Then
$$\{Z, H\} \subset \mathbb{LCS}(X, B_X + H).$$
Let $f : W \to X$ be a log resolution of the log pair $(X, B_X + H)$. Then
$$K_W + \hat{H} \sim_{\mathbb{Q}} f^*(K_X + B_X + H) + \sum_{E \neq \hat{H}} a(X, B_X + H, E)E,$$
where where $\hat{H}$ is the proper transform of H on W. Applying Theorem C.9 to the log pull back of the log pair $(X, B_X + H)$ on the variety W, we see that $\hat{H} \cap E \neq \varnothing$ for some f-exceptional divisor E on the variety W such that $f(E) = Z$ and $a(X, B_X, E) \leqslant -1$. Now the equivalence
$$K_{\hat{H}} \sim_{\mathbb{Q}} (K_W + \hat{H})|_{\hat{H}} \sim_{\mathbb{Q}} f|_{\hat{H}}^*(K_H + B_X|_H) + \sum_{E \neq \hat{H}} a(X, B_X + H, E)E|_{\hat{H}}$$
concludes the proof. □

The proof of Theorem C.10 implies the following result.

Corollary C.11. *Let O be a smooth point of the variety X such that the singularities of the log pair (X, B_X) are log terminal in a punctured neighborhood of O, the point O is contained in the set $\mathbb{CS}(X, B_X)$, where B_X is effective. Then $O \in \mathbb{LCS}(H, B_H)$, where H is a reduced irreducible effective Cartier divisor on the variety X that passes through the point O, and $B_H = B_X|_H$.*

The natural generalization of Theorem C.10 is the following result (see Theorem 7.5 in [11]).

Theorem C.12. *Suppose that the boundary B_X is effective and $\lfloor B_X \rfloor = \varnothing$. Let S be an effective reduced and irreducible Cartier divisor on the variety X such that $K_X + S + B_X$ is a $\mathbb{Q}$-Cartier divisor. Then the singularities of the log pair $(X, S + B_X)$ are purely log terminal if and only if the singularities of the log pair $(S, \mathrm{Diff}_S(B_X))$ are Kawamata log terminal*[10].

The following result is Theorem 3.1 in [5].

Theorem C.13. *Let H be a surface, O be a smooth point of the surface H, and M_H be an effective mobile boundary on the surface H such that*
$$O \in \mathbb{LCS}(H, (1 - a_1)\Delta_1 + (1 - a_2)\Delta_2 + M_H),$$
where Δ_1 and Δ_2 are irreducible and reduced curves on the surface H that intersects normally at point O, and a_1 and a_2 are positive rational numbers. Then
$$\mathrm{mult}_O(M_H^2) \geqslant \begin{cases} 4a_1a_2 & a_1 \leqslant 1 \quad a_2 \leqslant 1, \\ 4(a_1 + a_2 - 1) & a_1 > 1 \quad a_2 > 1. \end{cases}$$

Proof. Let $D = (1 - a_1)\Delta_1 + (1 - a_2)\Delta_2 + M_H$, and $f : S \to H$ be a birational morphism such that the surface S is smooth and
$$K_S + f^{-1}(D) \sim_{\mathbb{Q}} f^*(K_H + D) + \sum_{i=1}^{k} a(H, D, E_i)E_i,$$
where E_i is an exceptional divisor of the birational morphism f, $a(H, D, E_i)$ is a rational number, the inequality $a(H, D, E_1) \leqslant -1$ holds, and f is a composition of k blow ups of smooth points.

In the case $k = 1$ the proof of the claim is obvious. Suppose that the claim is already proved for all cases when either $a_1 \leqslant 1$, or $a_2 \leqslant 1$. Thus, we may assume that $a_1 > 1$ and $a_2 > 1$. Then
$$O \in \mathbb{LCS}(H, (2 - a_1 - a_2)\Delta_2 + M_H),$$
which implies that $\mathrm{mult}_O(M_H^2) \geqslant 4(a_1 + a_2 - 1)$. Hence, to conclude the proof we may assume that $a_1 \leqslant 1$. Now we can obtain the required claim by induction on k. □

Corollary C.14. *Let O be a smooth point of X such that $O \in \mathbb{LCS}(X, B_X)$, where $\dim(X) = 2$, and the boundary B_X is effective and mobile. Then the inequality $\mathrm{mult}_O(B_X^2) \geqslant 4$ holds, and the inequality $\mathrm{mult}_O(B_X) \neq 2$ implies that the inequality $\mathrm{mult}_O(B_X^2) > 4$ holds.*

[10] The definition of purely log terminal and Kawamata log terminal singularities can be found in [11].

The claims of Corollaries C.11 and C.14 imply the following result (see Proposition 1 in [12].

Theorem C.15. *Let O be a smooth point of X such that $O \in \mathbb{CS}(X, B_X)$, where $\dim(X) = 3$, and the boundary B_X is effective and mobile. Then the inequality $\mathrm{mult}_O(B_X^2) \geqslant 4$ holds, and the equality $\mathrm{mult}_O(B_X^2) = 4$ implies the equality $\mathrm{mult}_O(B_X) = 2$.*

The following result is Theorem 3.10 in [5].

Theorem C.16. *Let O be an isolated ordinary double point of the variety X such that O is contained in the set $\mathbb{CS}(X, B_X)$, where $\dim(X) = 3$, and B_X is effective. Then $\mathrm{mult}_O(B_X) \geqslant 1$.*

Proof. Let $f : W \to X$ be a blow up of the point O. Then

$$B_W \sim_{\mathbb{Q}} f^*(B_X) - \mathrm{mult}_O(B_X)E,$$

where B_W is a proper transform of the boundary B_X on the variety W, and E is an exceptional divisor of the morphism f. Suppose that $\mathrm{mult}_O(B_X) < 1$. Then the equivalence

$$K_W + B_W \sim_{\mathbb{Q}} f^*(K_X + B_X) + (1 - \mathrm{mult}_O(B_X))E$$

implies the existence of a proper subvariety $Z \subset E$ such that Z is a center of canonical singularities of the log pair (W, B_W). Hence, the set $\mathbb{LCS}(E, B_W|_E)$ is not empty by Theorem C.10, which contradicts Lemma C.7, because $E \cong \mathbb{P}^1 \times \mathbb{P}^1$. $\square$

The geometrical meaning of Theorems C.15 and C.16 can be explained in the framework of extremal birational contractions to smooth point of three-dimensional variety and isolated ordinary double point of three-dimensional variety respectively.

Let X be a general quasismooth hypersurface in $\mathbb{P}(1, a_1, a_2, a_3, a_4)$ of degree $d = \sum_{i=1}^{4} a_i$, where $a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$ such that X has terminal singularities. Then in the given paper we explicitly constructed the set Ξ consisting of dominant rational maps $\xi : X \dashrightarrow S$ such that the normalization of a general fiber of the rational map ξ is an elliptic curve.

n	d	a_1	a_2	a_3	a_4	$-K_X^3$	$\mathrm{Sing}(X)$	$\lvert\Xi\rvert$
1	4	1	1	1	1	4	$\varnothing$	∞
2	5	1	1	1	2	5/2	$\frac{1}{2}(1,1,1)$	15
3	6	1	1	1	3	2	$\varnothing$	0
4	6	1	1	2	2	3/2	$3\times\frac{1}{2}(1,1,2)$	1
5	7	1	1	2	3	7/6	$\frac{1}{2}(1,1,1)$, $\frac{1}{3}(1,1,2)$	1
6	8	1	1	2	4	1	$2\times\frac{1}{2}(1,1,1)$	1
7	8	1	2	2	3	2/3	$4\times\frac{1}{2}(1,1,1)$, $\frac{1}{3}(1,1,2)$	5
8	9	1	1	3	4	3/4	$\frac{1}{4}(1,1,3)$	1
9	9	1	2	3	3	1/2	$\frac{1}{2}(1,1,1), 3\times\frac{1}{3}(1,1,2)$	2
10	10	1	1	3	5	2/3	$\frac{1}{3}(1,1,2)$	1
11	10	1	2	2	5	1/2	$5\times\frac{1}{2}(1,1,1)$	5
12	10	1	2	3	4	5/12	$2\times\frac{1}{2}(1,1,1)$, $\frac{1}{3}(1,1,2)$, $\frac{1}{4}(1,1,3)$	1
13	11	1	2	3	5	11/30	$\frac{1}{2}(1,1,1)$, $\frac{1}{3}(1,1,2)$, $\frac{1}{5}(1,2,3)$	1
14	12	1	1	4	6	1/2	$\frac{1}{2}(1,1,1)$	1
15	12	1	2	3	6	1/3	$2\times\frac{1}{2}(1,1,1), 2\times\frac{1}{3}(1,1,2)$	1
16	12	1	2	4	5	3/10	$3\times\frac{1}{2}(1,1,1)$, $\frac{1}{5}(1,1,4)$	1
17	12	1	3	4	4	1/4	$3\times\frac{1}{4}(1,1,3)$	4
18	12	2	2	3	5	1/5	$6\times\frac{1}{2}(1,1,1)$, $\frac{1}{5}(1,2,3)$	1
19	12	2	3	3	4	1/6	$3\times\frac{1}{2}(1,1,1), 4\times\frac{1}{3}(1,1,2)$	4
20	13	1	3	4	5	13/60	$\frac{1}{3}(1,1,2)$, $\frac{1}{4}(1,1,3)$, $\frac{1}{5}(1,1,4)$	2
21	14	1	2	4	7	1/4	$3\times\frac{1}{2}(1,1,1)$, $\frac{1}{4}(1,1,3)$	1
22	14	2	2	3	7	1/6	$7\times\frac{1}{2}(1,1,1)$, $\frac{1}{3}(1,1,2)$	1
23	14	2	3	4	5	7/60	$3\times\frac{1}{2}(1,1,1)$, $\frac{1}{3}(1,1,2)$, $\frac{1}{4}(1,1,3)$, $\frac{1}{5}(1,2,3)$	2
24	15	1	2	5	7	3/14	$\frac{1}{2}(1,1,1)$, $\frac{1}{7}(1,2,5)$	1
25	15	1	3	4	7	5/28	$\frac{1}{4}(1,1,3)$, $\frac{1}{7}(1,3,4)$	2
26	15	1	3	5	6	1/6	$2\times\frac{1}{3}(1,1,2)$, $\frac{1}{6}(1,1,5)$	3
27	15	2	3	5	5	1/10	$\frac{1}{2}(1,1,1), 3\times\frac{1}{5}(1,2,3)$	1
28	15	3	3	4	5	1/12	$5\times\frac{1}{3}(1,1,2)$, $\frac{1}{4}(1,1,3)$	1
29	16	1	2	5	8	1/5	$2\times\frac{1}{2}(1,1,1)$, $\frac{1}{5}(1,2,3)$	1

n	d	a_1	a_2	a_3	a_4	$-K_X^3$	$\mathrm{Sing}(X)$	$\|\Xi\|$
30	16	1	3	4	8	1/6	$\frac{1}{3}(1,1,2), 2\times\frac{1}{4}(1,1,3)$	2
31	16	1	4	5	6	2/15	$\frac{1}{2}(1,1,1), \frac{1}{5}(1,1,4), \frac{1}{6}(1,1,5)$	1
32	16	2	3	4	7	2/21	$4\times\frac{1}{2}(1,1,1), \frac{1}{3}(1,1,2), \frac{1}{7}(1,3,4)$	1
33	17	2	3	5	7	17/210	$\frac{1}{2}(1,1,1), \frac{1}{3}(1,1,2), \frac{1}{5}(1,2,3), \frac{1}{7}(1,2,5)$	1
34	18	1	2	6	9	1/6	$3\times\frac{1}{2}(1,1,1), \frac{1}{3}(1,1,2)$	1
35	18	1	3	5	9	2/15	$2\times\frac{1}{3}(1,1,2), \frac{1}{5}(1,1,4)$	1
36	18	1	4	6	7	3/28	$\frac{1}{4}(1,1,3), \frac{1}{2}(1,1,1), \frac{1}{7}(1,1,6)$	2
37	18	2	3	4	9	1/12	$4\times\frac{1}{2}(1,1,1), 2\times\frac{1}{3}(1,1,2), \frac{1}{4}(1,1,3)$	1
38	18	2	3	5	8	3/40	$2\times\frac{1}{2}(1,1,1), \frac{1}{5}(1,2,3), \frac{1}{8}(1,3,5)$	1
39	18	3	4	5	6	1/20	$3\times\frac{1}{3}(1,1,2), \frac{1}{4}(1,1,3), \frac{1}{2}(1,1,1), \frac{1}{5}(1,1,4)$	1
40	19	3	4	5	7	19/420	$\frac{1}{3}(1,1,2), \frac{1}{4}(1,1,3), \frac{1}{5}(1,2,3), \frac{1}{7}(1,3,4)$	1
41	20	1	4	5	10	1/10	$\frac{1}{2}(1,1,1), 2\times\frac{1}{5}(1,1,4)$	1
42	20	2	3	5	10	1/15	$2\times\frac{1}{2}(1,1,1), \frac{1}{3}(1,1,2), 2\times\frac{1}{5}(1,2,3)$	1
43	20	2	4	5	9	1/18	$5\times\frac{1}{2}(1,1,1), \frac{1}{9}(1,4,5)$	1
44	20	2	5	6	7	1/21	$3\times\frac{1}{2}(1,1,1), \frac{1}{6}(1,1,5), \frac{1}{7}(1,2,5)$	2
45	20	3	4	5	8	1/24	$\frac{1}{3}(1,1,2), 2\times\frac{1}{4}(1,1,3), \frac{1}{8}(1,3,5)$	1
46	21	1	3	7	10	1/10	$\frac{1}{10}(1,3,7)$	1
47	21	1	5	7	8	3/40	$\frac{1}{5}(1,2,3), \frac{1}{8}(1,1,7)$	1
48	21	2	3	7	9	1/18	$\frac{1}{2}(1,1,1), 2\times\frac{1}{3}(1,1,2), \frac{1}{9}(1,2,7)$	1
49	21	3	5	6	7	1/30	$3\times\frac{1}{3}(1,1,2), \frac{1}{5}(1,2,3), \frac{1}{6}(1,1,5)$	2
50	22	1	3	7	11	2/21	$\frac{1}{3}(1,1,2), \frac{1}{7}(1,3,4)$	1
51	22	1	4	6	11	1/12	$\frac{1}{4}(1,1,3), \frac{1}{2}(1,1,1), \frac{1}{6}(1,1,5)$	2
52	22	2	4	5	11	1/20	$5\times\frac{1}{2}(1,1,1), \frac{1}{4}(1,1,3), \frac{1}{5}(1,1,4)$	1
53	24	1	3	8	12	1/12	$2\times\frac{1}{3}(1,1,2), \frac{1}{4}(1,1,3)$	1
54	24	1	6	8	9	1/18	$\frac{1}{2}(1,1,1), \frac{1}{3}(1,1,2), \frac{1}{9}(1,1,8)$	1
55	24	2	3	7	12	1/21	$2\times\frac{1}{2}(1,1,1), 2\times\frac{1}{3}(1,1,2), \frac{1}{7}(1,2,5)$	1
56	24	2	3	8	11	1/22	$3\times\frac{1}{2}(1,1,1), \frac{1}{11}(1,3,8)$	1
57	24	3	4	5	12	1/30	$2\times\frac{1}{3}(1,1,2), 2\times\frac{1}{4}(1,1,3), \frac{1}{5}(1,2,3)$	1
58	24	3	4	7	10	1/35	$\frac{1}{2}(1,1,1), \frac{1}{7}(1,3,4), \frac{1}{10}(1,3,7)$	1
59	24	3	6	7	8	1/42	$4\times\frac{1}{3}(1,1,2), \frac{1}{2}(1,1,1), \frac{1}{7}(1,1,6)$	1
60	24	4	5	6	9	1/45	$2\times\frac{1}{2}(1,1,1), \frac{1}{5}(1,1,4), \frac{1}{3}(1,1,2), \frac{1}{9}(1,4,5)$	0
61	25	4	5	7	9	5/252	$\frac{1}{4}(1,1,3), \frac{1}{7}(1,2,5), \frac{1}{9}(1,4,5)$	1
62	26	1	5	7	13	2/35	$\frac{1}{5}(1,2,3), \frac{1}{7}(1,1,6)$	1
63	26	2	3	8	13	1/24	$3\times\frac{1}{2}(1,1,1), \frac{1}{3}(1,1,2), \frac{1}{8}(1,3,5)$	1

n	d	a_1	a_2	a_3	a_4	$-K_X^3$	$\mathrm{Sing}(X)$	$\lvert\Xi\rvert$
64	26	2	5	6	13	1/30	$4\times\frac{1}{2}(1,1,1), \frac{1}{5}(1,2,3), \frac{1}{6}(1,1,5)$	2
65	27	2	5	9	11	3/110	$\frac{1}{2}(1,1,1), \frac{1}{5}(1,1,4), \frac{1}{11}(1,2,9)$	1
66	27	5	6	7	9	1/70	$\frac{1}{5}(1,1,4), \frac{1}{6}(1,1,5), \frac{1}{3}(1,1,2), \frac{1}{7}(1,2,5)$	1
67	28	1	4	9	14	1/18	$\frac{1}{2}(1,1,1), \frac{1}{9}(1,4,5)$	1
68	28	3	4	7	14	1/42	$\frac{1}{3}(1,1,2), \frac{1}{2}(1,1,1), 2\times\frac{1}{7}(1,3,4)$	1
69	28	4	6	7	11	1/66	$2\times\frac{1}{2}(1,1,1), \frac{1}{6}(1,1,5), \frac{1}{11}(1,4,7)$	1
70	30	1	4	10	15	1/20	$\frac{1}{4}(1,1,3), \frac{1}{2}(1,1,1), \frac{1}{5}(1,1,4)$	1
71	30	1	6	8	15	1/24	$\frac{1}{2}(1,1,1), \frac{1}{3}(1,1,2), \frac{1}{8}(1,1,7)$	1
72	30	2	3	10	15	1/30	$3\times\frac{1}{2}(1,1,1), 2\times\frac{1}{3}(1,1,2), \frac{1}{5}(1,2,3)$	1
73	30	2	6	7	15	1/42	$5\times\frac{1}{2}(1,1,1), \frac{1}{3}(1,1,2), \frac{1}{7}(1,1,6)$	1
74	30	3	4	10	13	1/52	$\frac{1}{4}(1,1,3), \frac{1}{2}(1,1,1), \frac{1}{13}(1,3,10)$	1
75	30	4	5	6	15	1/60	$\frac{1}{4}(1,1,3), 2\times\frac{1}{2}(1,1,1), 2\times\frac{1}{5}(1,1,4), \frac{1}{3}(1,1,2)$	0
76	30	5	6	8	11	1/88	$\frac{1}{2}(1,1,1), \frac{1}{8}(1,3,5), \frac{1}{11}(1,5,6)$	1
77	32	2	5	9	16	1/45	$2\times\frac{1}{2}(1,1,1), \frac{1}{5}(1,1,4), \frac{1}{9}(1,2,7)$	1
78	32	4	5	7	16	1/70	$2\times\frac{1}{4}(1,1,3), \frac{1}{5}(1,1,4), \frac{1}{7}(1,5,2)$	1
79	33	3	5	11	14	1/70	$\frac{1}{5}(1,1,4), \frac{1}{14}(1,3,11)$	1
80	34	3	4	10	17	1/60	$\frac{1}{3}(1,1,2), \frac{1}{4}(1,1,3), \frac{1}{2}(1,1,1), \frac{1}{10}(1,3,7)$	1
81	34	4	6	7	17	1/84	$\frac{1}{4}(1,1,3), 2\times\frac{1}{2}(1,1,1), \frac{1}{6}(1,1,5), \frac{1}{7}(1,4,3)$	1
82	36	1	5	12	18	1/30	$\frac{1}{5}(1,2,3), \frac{1}{6}(1,1,5)$	1
83	36	3	4	11	18	1/66	$2\times\frac{1}{3}(1,1,2), \frac{1}{2}(1,1,1), \frac{1}{11}(1,4,7)$	1
84	36	7	8	9	12	1/168	$\frac{1}{7}(1,2,5), \frac{1}{8}(1,1,7), \frac{1}{4}(1,1,3), \frac{1}{3}(1,1,2)$	0
85	38	3	5	11	19	2/165	$\frac{1}{3}(1,1,2), \frac{1}{5}(1,1,4), \frac{1}{11}(1,3,8)$	1
86	38	5	6	8	19	1/120	$\frac{1}{5}(1,1,4), \frac{1}{6}(1,1,5), \frac{1}{2}(1,1,1), \frac{1}{8}(1,3,5)$	1
87	40	5	7	8	20	1/140	$2\times\frac{1}{5}(1,2,3), \frac{1}{7}(1,1,6), \frac{1}{4}(1,1,3)$	0
88	42	1	6	14	21	1/42	$\frac{1}{2}(1,1,1), \frac{1}{3}(1,1,2), \frac{1}{7}(1,1,6)$	1
89	42	2	5	14	21	1/70	$3\times\frac{1}{2}(1,1,1), \frac{1}{5}(1,1,4), \frac{1}{7}(1,2,5)$	1
90	42	3	4	14	21	1/84	$2\times\frac{1}{3}(1,1,2), \frac{1}{4}(1,1,3), \frac{1}{2}(1,1,1), \frac{1}{7}(1,3,4)$	1
91	44	4	5	13	22	1/130	$\frac{1}{2}(1,1,1), \frac{1}{5}(1,2,3), \frac{1}{13}(1,4,9)$	1
92	48	3	5	16	24	1/120	$2\times\frac{1}{3}(1,1,2), \frac{1}{5}(1,1,4), \frac{1}{8}(1,3,5)$	1
93	50	7	8	10	25	1/280	$\frac{1}{7}(1,3,4), \frac{1}{8}(1,1,7), \frac{1}{2}(1,1,1), \frac{1}{5}(1,2,3)$	0
94	54	4	5	18	27	1/180	$\frac{1}{4}(1,1,3), \frac{1}{2}(1,1,1), \frac{1}{5}(1,2,3), \frac{1}{9}(1,4,5)$	1
95	66	5	6	22	33	1/330	$\frac{1}{5}(1,2,3), \frac{1}{2}(1,1,1), \frac{1}{3}(1,1,2), \frac{1}{11}(1,5,6)$	1

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