A proof of the uniqueness of static stellar model with small $\frac{d\rho}{dp}$.

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1. Introduction

In earlier works (Bunting et al. (1987), Masood-ul-Alam (1987)) positive-mass theorem has been used to prove uniqueness theorems for certain asymptotically Euclidean static space-times. In this paper we modify the technique to prove the spherical symmetry of a static stellar model whenever there exists a spherically symmetric model with the same equation of state $\rho = \rho(p)$ and surface potential V_s, provided $\frac{d\rho}{dp}$ is "small" in some appropriate sense. (For precise statements see theorem 2 and corollaries 1 and 2 where a measure for $\frac{d\rho}{dp}$ is given. $\frac{d\rho}{dp}$ can be allowed to be large near the surface of the star; see theorem 3). Roughly the argument is as follows. If g be the Riemannian metric on a $t = \text{constant}$ hypersurface of a static stellar model then we can find a function $\Omega = \Omega(V)$ such that the metric $\Omega^2 g$ is asymptotically Euclidean with zero mass and it has scalar curvature $P(V)(W_o(V) - |\nabla V|^2)$. Now for "small" $\frac{d\rho}{dp}$ we can have $P(V) \geq 0$ and $W_o(V) \geq |\nabla V|^2$. Hence in this case the positive-mass theorem gives $\Omega^2 g$ to be an Euclidean metric and the result follows. We now state the corollary of the positive-mass theorem we are going to use (notation: greek letters $\alpha, \beta, \ldots,$ run from 1 to 3 and italic letters $i, j, \ldots,$ run from 0 to 3).

Theorem 1. Let (N, γ) be a complete oriented three-dimensional Riemannian manifold which is asymptotically Euclidean in the sense that N is topologically Euclidean outside a compact set and in the standard coordinate system (x^α) in $\mathbb{R}^3$ the metric satisfies the decay condition

$$\gamma_{\alpha\beta} = (1 + 2m/|x|)\,\delta_{\alpha\beta} + a_{\alpha\beta}$$

where $a_{\alpha\beta} = O(|x|^{-2})$, $\partial a_{\alpha\beta} = O(|x|^{-3})$, $\partial\partial a_{\alpha\beta} = O(|x|^{-4})$ as

$$|x|^2 = \sum_{\tau=1,2,3} (x^\tau)^2 \longrightarrow \infty \ .$$

If the scalar curvature of γ is non-negative and the mass $m = 0$, then (N,γ) is isometric to $\mathbb{R}^3$ with the standard Euclidean metric.

Theorem 1 was proved by Schoen and Yau (1979), (an alternative proof was given by Witten (1981); see also Parker et al. (1982). For a proof suitable to our purpose as regards asymptotic decay of γ and regularity of γ (we shall assume γ to be $C^{1,1}$) we refer to Bartnik, (1986).

We give some preliminaries about static stellar model in § 2. In § 3 we consider spherically symmetric models and list some results which are needed in later sections. In § 4 and § 5 we give the main results and complete their proofs. The strongest result obtained is the proposition 1 which however is too technical. The weaker results involve much simpler conditions on the equations of state and are given in § 5. In the conclusion we comment on the possibility of generalizing the method to remove the conditions on $\frac{d\rho}{dp}$.

2. Static stellar model

By a (nonsingular) static stellar model we shall mean a geodesically complete spacetime $(M, {}^4g)$ such that

(i) M is a C^∞ manifold diffeomorphic to $N \times \mathbb{R}$ where, for each $t \in \mathbb{R}$, $N_t = N \times \{t\}$ is an oriented spacelike 3-manifold.

(ii) The Lorentzian metric ${}^4g$ can be written as

$$ {}^4g = -V^2(dt \otimes dt) + g \tag{2.1} $$

where V is a positive function (called the potential), g is a tensor such that g restricted to N is a complete Riemannian metric on N , and V and g are independent of t .

(iii) $(M, {}^4g)$ satisfies Einstein's equation

$$ Ric({}^4g)_{ij} - \frac{1}{2}\, Scalar({}^4g)\, {}^4g_{ij} = 8\pi[(\rho + p)u_i u_j + p\, {}^4g_{ij}] \tag{2.2} $$

where u_i is a unit timelike vector field, and ρ and

p are bounded measurable functions. These functions are, respectively, called the density and pressure. It is clear that they are independent of t.

(iv) ρ and p are related by an "equation of state" of the form $\rho = \rho(p)$, with $\frac{d\rho}{dp} \geq 0$.

(v) There exists a bounded open set $Q \subset N$ such that $\rho = p = 0$ on $N \sim Q$ and in Q $\rho > 0$, $p > 0$. Q and $N \sim Q$ are called the fluid (or star) and the vacuum respectively.

(vi) $(M, {}^4g)$ is asymptotically Euclidean in the following sense :

There exists a compact set $N_1 \subset N$ such that $N \sim N_1$ is diffeomorphic to $\mathbb{R}^3 \sim \bar{B}_1(0)$ where $\bar{B}_1(0)$ is the closed unit ball centred at the origin and with respect to the standard coordinate system $\{x^\alpha\}$ in $\mathbb{R}^3$ we have on $N \sim N_1$

$$g_{\alpha\beta} = (1 + 2m/|x|)\delta_{\alpha\beta} + h_{\alpha\beta} \tag{2.3}$$

$$V = 1 - m/|x| + v \tag{2.4}$$

where $h_{\alpha\beta} = O(|x|^{-2})$, $\partial h_{\alpha\beta} = O(|x|^{-3})$, $\partial\partial h_{\alpha\beta} = O(|x|^{-4})$, $v = O(|x|^{-2})$, $\partial v = O(|x|^{-3})$ and $\partial\partial v = O(|x|^{-4})$ as $|x| \to \infty$. The constant m is called the mass of the stellar model.

Remark 1. For the sake of simplicity, we shall make following regularity assumptions. V and g are globally $C^{1,1}$ and C^3 on Q and $N \sim \bar{Q}$. (It is well known that V and g will then be autometically analytic in $N \sim \bar{Q}$. See Müller zum Hagen (1970)). ∂Q is assumed to be smooth. Furthermore we shall assume that ρ is a (locally) $C^{1,1}$ function of p on $(0,\infty)$ and $\int_0^p (\rho(s) + s)^{-1} ds < \infty$ for finite p. In particular the last condition allows the "polytropic" equations of state of the form $\rho = Ap^q$, $q < 1$ and A = constant. It is wellknown (Künzle et al. (1980), Masood-ul-Alam (1987)) that if ρ is a Lipschitz function of p then ρ has a discontinuity on ∂Q. See also Remark 4 below.

Remark 2. (2,2) decomposes into

$$Ric\ (g)_{\alpha\beta} = V^{-1} V_{;\alpha\beta} + 4\pi(\rho - p) g_{\alpha\beta} \tag{2.5}$$

$$\Delta V = 4\pi V(\rho + 3p) \tag{2.6}$$

where ; denotes the covariant derivative with respect to g and Δ denotes the Laplacian with respect to g.

Remark 3. (2.3) and (2.4) follow naturally from (2.5) and (2.6) under mild asymptotic decay conditions (see Beig (1980), Bunting et al. (1987) for details).

Remark 4. The contracted Bianchi identity (in a suitable weak sense) for g implies that p is a Lipschitz function on N and

$$p_{;\alpha} = - V^{-1}(\rho + p) V_{;\alpha} \ .$$

It follows that (see Künzle and Savage (1980)) ρ and p are functions of V and ∂Q is a level set of V. The above equation becomes

$$\frac{dp}{dV} = - V^{-1}(\rho + p) \ . \tag{2.7}$$

Since we are assuming that $\int_0^p (\rho(s) + s)^{-1} ds < \infty$ for $p < \infty$, we can integrate (2.7) to get

$$V(p) = V_s \exp\left(- \int_0^p (\rho(\tau) + \tau)^{-1} d\tau \right) \tag{2.7'}$$

where $V_s = V(p)$ at $p = 0$.

It is conjectured that the stellar model defined above is spherically symmetric. This seems to be physically obvious. Also there are partial results due to Avez, Künzle (1971), Müller zum Hagen (1970), Künzle et al. (1980), and Lindblom (1980, 1981). However a rigorous analytical proof of the conjecture in full generality is still lacking.

One quantity which proves to be very important (see Lindblom

(1980) is the tensor $R_{\alpha\beta\delta}$ defined by

$$R_{\alpha\beta\delta} = Ric(g)_{\alpha\beta;\delta} - Ric(g)_{\alpha\delta;\beta} + \frac{1}{4}\,(g_{\alpha\delta}R_{;\beta} - g_{\alpha\beta}R_{;\delta}) \tag{2.9}$$

R is the scalar curvature of g . For a 3-dimensional Riemannian metric, $R_{\alpha\beta\delta} = 0$ if and only if the metric is (locally) conformally flat. A straightforward but tedious calculation using (2.5) and (2.6) yields (Lindblom),

$$\begin{aligned} \frac{1}{4}\, V^4 W^{-1} R_{\alpha\beta\delta} R^{\alpha\beta\delta} = {} & \Delta W - V^{-1} W_{;\alpha} V^{;\alpha} + 8\pi W(\rho + p) - 8\pi V \rho_{;\alpha} V^{;\alpha} \\ & - \frac{3}{4}\, W^{-1} |\nabla W|^2 - 16\pi^2 V^2 (\rho + 3p)^2 \\ & + 4\pi V(\rho + 3p) W^{-1} V_{;\alpha} W^{;\alpha} \end{aligned} \tag{2.10}$$

where $W \equiv |\nabla V|^2 \equiv g^{\alpha\beta} V_{;\alpha} V_{;\beta}$ and indices are raised with respect to g .

3. Spherically symmetric stellar model

In this section we list some results related to the spherically symmetric stellar model. These are required for later reference. Most of them are wellknown while the remaining are not difficult to obtain. The metric (for details see page 608 in Misner et al.) has the form

$${}^4\tilde{g} = -\tilde{V}^2 dt^2 + \tilde{g}_{rr} dr^2 + r^2 d\Sigma^2 \tag{3.1}$$

where $\tilde{V}$ and $\tilde{g}_{rr}$ are functions of r only and where

$$d\Sigma^2 = d\theta^2 + \sin^2\theta d\varphi^2 \; . \tag{3.2}$$

The pressure $\tilde{p}$ and density $\tilde{\rho} = \tilde{\rho}(\tilde{p})$ must satisfy the relation (for $r \leq r_s$, the value of r at ∂Q)

$$8\pi\tilde{p} = 2\tilde{g}^{rr}\tilde{V}^{-1}r^{-1}\frac{d\tilde{V}}{dr} + r^{-2}(\tilde{g}^{rr} - 1) \tag{3.3}$$

$$8\pi\tilde{\rho} = - r^{-1}\frac{d\tilde{g}^{rr}}{dr} + r^{-2}(1 - \tilde{g}^{rr}) \tag{3.4}$$

$$\frac{d\tilde{p}}{dr} = - (\tilde{\rho} + \tilde{p})\tilde{V}^{-1}\frac{d\tilde{V}}{dr} \tag{3.5}$$

Defining

$$\hat{m} = \hat{m}(r) = \int_0^r 4\pi\rho s^2 ds \tag{3.6}$$

we can write the metric as

$$4g = \begin{cases} -\tilde{V}^2 dt^2 + (1 - 2\hat{m}r^{-1})^{-1}dr^2 + r^2 d\Sigma^2 & r \leq r_s \\ -(1 - 2\tilde{m}r^{-1})dt^2 + (1 - 2\tilde{m}r^{-1})^{-1}dr^2 + r^2 d\Sigma^2 & r > r_s \end{cases} \tag{3.7}$$

where $\tilde{m} = \hat{m}(r_s)$ is the mass of the stellar model. The desired regularity of $\tilde{g}$ is obtained in the harmonic coordinates relative to $\tilde{V}^2\tilde{g}$. It follows from (3.3), (3.4), (3.5) and the regularity assumption in Remark 1 that near the "centre" (i.e. $r = 0$) we have

$$\tilde{p}(r) = \tilde{p}_c - (2\pi/3)(\tilde{\rho}_c + \tilde{p}_c)(\tilde{\rho}_c + 3\tilde{p}_c)r^2 + O(r^3) \qquad (3.8)$$

$$\tilde{\rho}(r) = \tilde{\rho}_c - (2\pi/3)(\tilde{\rho}_c + \tilde{p}_c)(\tilde{\rho}_c + 3\tilde{p}_c)\left(\frac{d\tilde{\rho}}{d\tilde{p}}\right)_c r^2 + O\ (r^3) \qquad (3.9)$$

where subscript c is used to denote the value of any function at the centre.

It is wellknown (Buchdahl (1959), Bondi (1964)) that for a non-singular spherically symmetric stellar model with $\frac{d\rho}{dp} \geq 0$ we have a lower bound for $\tilde{V}_s \equiv \tilde{V}(r_s)$ and $\tilde{g}^{rr}$. We have (see Buchdahl (1981) : Lecture 12)

$$\tilde{V}_s > \frac{1}{3} \ . \qquad (3.10)$$

We shall denote the upper bound for $\tilde{g}_{rr}$ by a constant a :

$$\tilde{g}_{rr} < a \ . \qquad (3.11)$$

It was shown by Bondi that (for $\rho > 0$) $a = (17 - 12\sqrt{2})^{-1} \approx 34.$ If $\rho \geq kp$ with $k > 0$, and $\frac{d\rho}{dp} \geq 0$, this value can be reduced further. (For example, when $k = 3$ we have $a < 6$. For details see figure 1 and the subsequent discussion in Bondi (1964)).

An important quantity is the function $\Omega = \Omega(\tilde{V})$ which makes $\Omega^2\tilde{g}$ Euclidean. Writing $\tilde{g} = \Omega^{-2}(d\hat{r}^2 + \hat{r}^2 d\Sigma^2)$ where $\hat{r} = \hat{r}(r)$ we find that $\Omega(\tilde{V})$ satisfies the following equation

$$\Omega^{-1} \frac{d\Omega}{d\tilde{V}} = \left(\sqrt{\tilde{g}_{rr}} - 1\right) r^{-1} \frac{dr}{d\tilde{V}} \tag{3.12}$$

Differentiating the above equation relative to $\tilde{V}$ and using

$$W_o := \tilde{g}^{rr}\left(\frac{d\tilde{V}}{dr}\right)^2 \tag{3.13}$$

as well as equations (2.6),(3.3) and (3.4) we get the crucial indentity (which expresses the fact that $Scalar(\Omega^2\tilde{g})=0$)

$$2\Omega^{-1} \frac{d^2\Omega}{d\tilde{V}^2} - \Omega^{-2}\left(\frac{d\Omega}{d\tilde{V}}\right)^2 = \left\{8\pi\tilde{\rho} - 8\pi\tilde{V}(\tilde{\rho} + 3\tilde{p})\Omega^{-1} \frac{d\Omega}{d\tilde{V}}\right\} W_o^{-1} \tag{3.14}$$

In vacuum region we take

$$\Omega(\tilde{V}) = \frac{1}{4}(1 + \tilde{V})^2 \qquad \tilde{V} > \tilde{V}_s \tag{3.15}$$

so that $\Omega \longrightarrow 1$ towards infinity. Ω is $C^{1,1}$ in the neighbourhood of the surface ∂Q . This is clear from (3.12) and (3.14) since $\frac{dr}{d\tilde{V}}$ is continuous across ∂Q . Since we are assuming $\tilde{g}$ to be smooth in the interior of the fluid, Ω will also be $C^{1,1}$ in the neighbourhood of the centre of the fluid. In fact calculations up to order r^2 near the centre yields (one can use (3.8) and (3.9)).

$$8\pi\tilde{\rho} - 8\pi\mathcal{V}(\tilde{\rho} + 3\tilde{p})\Omega^{-1}\frac{d\Omega}{d\mathcal{V}} = (16\pi^2/15)\left[5\tilde{\rho}_c^2 - 6(\tilde{\rho}_c + \tilde{p}_c)\tilde{p}_c\left(\frac{d\tilde{\rho}}{d\tilde{p}}\right)_c\right]r^2 + O(r^3) \quad . \tag{3.16}$$

The method we shall demonstrate in this paper, will work only if the expression $2\Omega^{-1}\frac{d^2\Omega}{d\mathcal{V}^2} - \Omega^{-2}\left(\frac{d\Omega}{d\mathcal{V}}\right)^2$ is nonnegative. This expression vanishes in the vacuum. Since $\tilde{p}$ vanishes at the boundary of the fluid, this quantity is nonnegative in Q near the boundary whenever $\tilde{\rho}_s \equiv \tilde{\rho}(r_s) > 0$. This follows from (3.14) and the fact that $\mathcal{V}\Omega^{-1}\frac{d\Omega}{d\mathcal{V}} = 2\hat{m}r^{-1}[(\hat{m}r^{-1} + 4\pi\tilde{p}r^2)(1 + \sqrt{\tilde{g}_{rr}})]^{-1}$ is strictly less than unity for $r \in (0, r_s]$. It follows from (3.16) that for some equations of state this expression would not be nonnegative near the centre. On the other hand we have the following lemma.

Lemma 1. As usual we assume $\tilde{\rho} \geq 0$, $\tilde{p} \geq 0$, $\frac{d\tilde{\rho}}{d\tilde{p}} \geq 0$, suppose that

$$5\tilde{\rho}^2 \geq 6(\tilde{\rho} + \tilde{p})\tilde{p}\frac{d\tilde{\rho}}{d\tilde{p}} \tag{3.17}$$

Then $8\pi\tilde{\rho} - 8\pi\mathcal{V}(\tilde{\rho} + 3\tilde{p})\Omega^{-1}\frac{d\Omega}{d\mathcal{V}} \geq 0$.

Proof: Let $f = 8\pi\tilde{\rho} - 8\pi\mathcal{V}(\tilde{\rho} + 3\tilde{p})\Omega^{-1}\frac{d\Omega}{d\mathcal{V}}$.

Then differentiating f and using (3.14) we get

$$16\pi\mathcal{V}(\tilde{\rho}+3\tilde{p})\ \frac{ds}{d\mathcal{V}} = s^2 - \left[4\pi\tilde{\rho} + 16\pi\ \frac{d\tilde{\rho}}{d\tilde{p}}\ (\tilde{\rho}+\tilde{p}) + 64\pi^2\mathcal{V}^2(\tilde{\rho}+3\tilde{p})^2 w_o^{-1}\right] s$$

$$+ 64\pi^2\left[5\tilde{\rho}^2 - 6p(\tilde{\rho}+\tilde{p})\ \frac{d\tilde{\rho}}{d\tilde{p}}\right]$$

Hence for $s < 0$, $\frac{ds}{dV} > 0$. But it follows from (3.16) that s tends to zero as $r \to 0$. Hence the lemma follows. □

Now we shall describe some of the properties of the function $w_o(\tilde{\mathcal{V}})$. For the spherical stellar model the tensor $R_{\alpha\beta\delta}$ vanishes so that we have from (2.10)

$$w_o\ \frac{d^2 w_o}{d\mathcal{V}^2} - \frac{3}{4}\left(\frac{dw_o}{d\mathcal{V}}\right)^2 + \{8\pi\mathcal{V}(\tilde{\rho}+3\tilde{p}) - \mathcal{V}^{-1}w_o\}\ \frac{dw_o}{d\mathcal{V}}$$

$$- 8\pi w_o\mathcal{V}\ \frac{d}{d\mathcal{V}}\ (\tilde{\rho}+\tilde{p}) - 16\pi^2\mathcal{V}^2(\tilde{\rho}+3\tilde{p})^2 = 0\ . \tag{3.18}$$

The value of $\frac{dw_o}{d\tilde{\mathcal{V}}}$ at the centre will be important. At $\tilde{\mathcal{V}} = \tilde{\mathcal{V}}_c$ we have $Ric(\tilde{g})_{\alpha\beta} = (16\pi/3)\tilde{\rho}\tilde{g}_{\alpha\beta}$. Hence using (2.5) we get

$$\frac{dw_o}{d\mathcal{V}}\ (\tilde{\mathcal{V}}_c) = (8\pi/3)\ \tilde{\mathcal{V}}_c\ (\tilde{\rho}_c + 3\tilde{p}_c)\ . \tag{3.19}$$

For future reference we also need the formulae computed in the following lemma.

Lemma 2.

$$\frac{d}{d\mathcal{V}}\left\{\tilde{\mathcal{V}}^{-1}\ \frac{dw_o}{d\tilde{\mathcal{V}}} - 8\pi(\tilde{\rho}+\tilde{p})\right\} = 4(3\hat{m}r^{-3} - 4\pi\tilde{\rho})\,r^{-1}\ \frac{dr}{d\tilde{\mathcal{V}}} \tag{3.20}$$

$$8\pi(\tilde{\rho}+3\tilde{p}) - \frac{3}{2}\tilde{V}^{-1}\frac{dW_o}{d\tilde{V}} = 4\pi(\tilde{\rho}+3\tilde{p}) + 2(3\hat{m}r^{-3} - 4\pi\tilde{\rho}) \tag{3.21}$$

Remark 5. Note that since $\tilde{\rho}$ is a nonincreasing function of $\tilde{V}$ we have $3\hat{m}r^{-3} - 4\pi\tilde{\rho} \geq 0$.

Proof: We show that

$$\tilde{V}^{-1}\frac{dW_o}{d\tilde{V}} - 8\pi(\tilde{\rho}+\tilde{p}) = -4\hat{m}r^{-3} \tag{3.22}$$

An easy way to do this is to decompose $\tilde{\Delta}\tilde{V}$ along the level sets of $\tilde{V}$ and in normal direction. We have

$$\tilde{\Delta}\tilde{V} = \frac{1}{2}\frac{dW_o}{d\tilde{V}} + H_o\sqrt{W_o} \quad .$$

Now the mean curvature H_o of the level set is given by $H_o = 2r^{-1}\sqrt{\tilde{g}^{rr}}$. Hence using (3.3) and (3.6) we get (3.22), from which the lemma follows easily. □

4. The conformal metric $\Omega^2(V)g$

Our aim in this section is to prove the proposition 1 below. Let $(M, {}^4g)$ be a stellar model with equation of state $\rho = \rho(p)$ and surface potential (i.e. the value of V at ∂Q) V_s . Suppose there exists a (nonsingular) spherically symmetric stellar model $(\tilde{M}, {}^4\tilde{g})$ with the same equation of state and surface potential (i.e. $V_s = \tilde{V}_s$) . For $V \geq \tilde{V}_c$ we define the functions $\Omega^2(V)$

and $W_o(V)$ as those obtained from $\Omega^2(\tilde{V})$ and $W_o(\tilde{V})$ respectively by replacing $\tilde{V}$ by V .

Remark 6. Also for $V \geq \tilde{V}_c$ we have $p(V) = \tilde{p}(\tilde{V})$ and $\rho(V) = \tilde{\rho}(\tilde{V})$. This follows from (2.7') and the fact that corresponding to $p = 0$ we have $V_s = \tilde{V}_s$.

Now let $\varphi = \varphi(V)$. Then by virtue of (2.5) and (2.6) we find that the scalar curvature $\hat{R}(\varphi^2 g)$ of the metric $\varphi^2 g$ is given by

$$\hat{R}(\varphi^2 g) = 2\varphi^{-2}\Big[8\pi\rho - 8\pi V(\rho + 3p)\varphi^{-1}\frac{d\varphi}{dV} - \Big\{2\varphi^{-1}\frac{d^2\varphi}{dV^2} - \varphi^{-2}\Big(\frac{d\varphi}{dV}\Big)^2\Big\}|\nabla V|^2\Big] \tag{4.1}$$

For $V \geq \tilde{V}_c$ we take $\varphi(V) = \Omega(V)$. Then using the definition of $\Omega(V)$ and $W_o(V)$, Remark 6 and (3.14) we get

$$\hat{R}(\Omega^2 g) = 2\Omega^{-2}\Big\{2\Omega^{-1}\frac{d^2\Omega}{dV^2} - \Omega^{-2}\Big(\frac{d\Omega}{dV}\Big)^2\Big\}(W_o - W) \quad , \quad V \geq \tilde{V}_c \ . \tag{4.2}$$

W is as in (2.10) . $\hat{R}(\Omega^2 g)$ vanishes in vacuum.

The next step is to show that $V_{min} \equiv \inf_N V$ is not less than $\tilde{V}_c$ and $W_o(V) \geq W$. In lemma 3 below we show this in case the spherically symmetric model satisfies certain condition. We deduce a Robinson-type identity to which we can apply the maximum principle. One problem arises because a priori it is not known that $V_{min} \geq \tilde{V}_c$. Hence we extend $W_o(V)$ beyond $\tilde{V}_c$ by taking it to be the solution of the ODE

$$\frac{dW_o}{dV} = (8\pi/3)V(\rho + 3p) \quad , \quad V \leq \tilde{V}_c \tag{4.3}$$

with $W_o(\tilde{V}_c) = 0$. $W_o(V)$ so extended is $C^{1,1}$ (see equations (3.19) and (3.20)). It is now defined all over N . We shall now prove the following lemma.

Lemma 3 . Suppose

$$16\pi V(\rho + 3p) - 3\frac{dW_o}{dV} \geq (b - V^2)\frac{d}{dV}\left\{V^{-1}\frac{dW_o}{dV} - 8\pi(\rho + p)\right\} \tag{4.4}$$

where

$$b \equiv \frac{1}{3}(1 + 2V_s^2) > V_s^2 \tag{4.5}$$

Then $V_{min} \geq \tilde{V}_c$ and $W_o \geq W$ in Q .

Proof: For $V \leq V_s$ we have

$$\left[V^{-1}\left\{(W - W_o)(b - V^2)^{-1}\right\}_{\alpha}\right]^{\alpha} + \left[-4(b - V^2)^{-1} + W^{-1}\left\{4\pi(\rho + 3p) - \frac{3}{2}V^{-1}\frac{dW_o}{dV}\right\}\right]$$

$$\{(W - W_o)(b - V^2)^{-1}\}^{\alpha}V_{\alpha}$$

$$+ \left[\left\{3V^{-1}\frac{dW_o}{dV} - 16\pi(\rho + 3p)\right\}V(b - V^2)^{-1}\right.$$

$$+\frac{d}{dV}\left\{V^{-1}\frac{dW_o}{dV}-8\pi(\rho+p)\right\}\Big](W-W_o)(b-V^2)^{-1} \tag{4.6}$$

$$= (b-V^2)^{-1}\left[\frac{1}{4}V^3W^{-1}R_{\alpha\beta\delta}R^{\alpha\beta\delta}+\frac{3}{4}V^{-1}|\nabla(W-W_o)|^2-I\right]$$

with

$$I = \begin{cases} 0 \quad \text{for} \quad V \geq \tilde{V}_c \\ \\ (16\pi/3)\ W_o\ (\rho+p)\ \frac{d\rho}{dp} \quad \text{for} \quad V_{min} \leq V < \tilde{V}_c\ . \end{cases} \tag{4.7}$$

To obtain (4.6) we first use (3.18), (2.6) and the definition of I above, to deduce that,

$$W_o\frac{d^2W_o}{dV^2}+\frac{dW_o}{dV}\Delta V-V^{-1}\frac{dW_o}{dV}W_o+8\pi(\rho+p)W_o-8\pi V\frac{d\rho}{dV}W_o$$

$$-\frac{3}{4}\left(\frac{dW_o}{dV}\right)^2-16\pi^2V^2(\rho+3p)^2+4\pi V(\rho+3p)\frac{dW_o}{dV}=I\ .$$

Straightforward calculation using (2.10) now yields (4.6). For $V > V_s$ we use the following form of the Robinson's (1977) identity

$$[V^{-1}\{(1-V^2)^{-3}(W-W_o)\}_\alpha]^\alpha = [V^4R_{\alpha\beta\delta}R^{\alpha\beta\delta}+3|\nabla W+8(1-V^2)^{-2}VW\nabla V|^2]$$

$$[4VW(1-V^2)^3]^{-1} \tag{4.8}$$

Now $(W-W_o)(1-V^2)^{-3}$ goes to zero as V tends to 1. Hence by applying maximum principle to (4.8) in the domain $N \sim Q$ we get that either $W - W_o \leq 0$ at ∂Q or $W - W_o$ attains a positive maximum at a point $T_1 \in \partial Q$ and at T_1

$$(1 - V_s^2)^{-1} (W - W_o)_{;\alpha} n^\alpha + 6 V_s (1 - V_s^2)^{-2} (W - W_o) V_{;\alpha} n^\alpha < 0 \qquad (4.9)$$

where n^α is the normal to ∂Q at T_1 (pointing inward to $N \sim Q$) . We now show that either $W \leq W_o$ in Q or inequality (4.9) is contradicted. We shall apply maximum principle to (4.6) in Q . Note that by virtue of (4.4) the coefficient of $(W - W_o)(b - V^2)^{-1}$ in (4.6) is nonpositive. Furthermore, by virtue of the definition of W_o (see the statement following (4.3)) and Theorem 7.8 in Gilberg et. al. (1984) $W - W_o$ belongs to the Sobolev space $W^{2,3}_{loc}(Q)$. Finally we note that for $V \geq \tilde{V}_c$ we can stay away from the critical points of V because at these point we have $W - W_o \leq 0$ and hence maximum of $(W - W_o)(b - V^2)^{-1}$ is not reached near these critical points unless $W - W_o \leq 0$ in Q . (For $V \leq \tilde{V}_c$ the term involving W^{-1} in (4.6) vanishes by virtue of (4.3)). Now applying the weak maximum principle of A.D. Aleksandrov (Theorem 9.1 in Gilberg et al.) we find that either $W - W_o \leq 0$ in Q or positive maximum of $(W - W_o)(b - V^2)^{-1}$ is reached at a point $T_2 \in \partial Q$. If the second case occur we can take $T_2 = T_1$ since W has the maximum value over ∂Q at the point T_1 . But in a neighbourhood of T_1 in Q $W - W_o$ is C^2 and we can use the strong maximum principle (Theorem 3.5 in Gilberg et al.) and the boundary point lemma (Lemma 3.4 in Gilberg et al.) to deduce that at T_1 ,

$$(b - V_s^2)^{-1} (W - W_o)_{;\alpha} n^\alpha + 2(b - V_s^2)^{-2} (W - W_o) V_s V_{s;\alpha} n^\alpha \geq 0 \tag{4.10}$$

Since $b - V_s^2 = \frac{1}{3}(1 - V_s^2)$ and $(W - W_o)_\alpha n^\alpha$ is continuous across ∂Q , (4.10) contradicts (4.9). Hence we have $W - W_o \leq 0$ in Q . In particular $\tilde{V}_c \leq V_{min}$. □

We are now in a position to prove the following proposition.

Proposition 1: Suppose ${}^4g$ and ${}^4\tilde{g}$ are such that (4.4) holds (in particular this implies $V_{min} \geq \tilde{V}_c$) . If in addition $\Omega(\tilde{V})$ satisfies

$$2\Omega^{-1} \frac{d^2\Omega}{d\tilde{V}^2} - \Omega^{-2}\left(\frac{d\Omega}{d\tilde{V}}\right)^2 \geq 0 , \quad V \geq \tilde{V}_c \tag{4.11}$$

then (N,g) and V are spherically symmetric. In fact $(M,{}^4g)$ is isometric to $(\tilde{M},{}^4\tilde{g})$.

Proof: From (4.1) and Lemma 3 we get that the scalar curvature of the metric $\Omega^2 g$ (which is now defined over all N since $V_{min} \geq \tilde{V}_c$) is nonnegative. But $\Omega^2 g$ is asymptotically Euclidean with mass zero. Thus by the corollary of positive-mass theorem (stated above as theorem 1) g is conformally Euclidean, Ω^2 being the conformal factor. Hence the result follows by standard argument (Lindblom (1980) or by straight forward integration of (2.5) and (2.6). □

By virtue of the existence of spherically symmetric stellar model, most probably the two conditions appearing in Proposition 1 (viz. (4.4) and (4.11)) are conditions on the equation of state and the surface potential V_s . Equivalent conditions involving only the equation of state and V_s may be difficult to obtain. However, in the next section, we give some sufficient conditions involving only these two objects.

5. Main results

We now ask for which equations of state, spherically symmetric stellar models satisfy inequality (4.4). Although this inequality is always satisfied near the surface $\partial\Omega$ (see later), in general towards the interior it my be violated. It follows from lemma 2 that for a physically reasonable stellar model the expression on the left hand side of (4.4) is strictly positive. But, unfortunately, the expression $\frac{d}{dV}\left\{V^{-1}\frac{dW_o}{dV} - 8\pi(\rho + p)\right\}$ is nonnegative. However this quantity vanishes for a uniform density star, and provided $\frac{d\rho}{dp}$ is "small enough" we can expect (4.4) to hold everywhere. We shall give some sufficient conditions (both dependent and independent of the value of $\tilde{V}_s$) such that the equations of state satisfy inequaltity (4.4).

Using lemma 2 we find that for $V \geq \tilde{V}_c$, (4.4) is equivalent to

$$2\pi V(\rho + 3p) \geq \left(b - V^2 - r V \frac{d\tilde{V}}{dr}\right) (3\hat{m}r^{-3} - 4\pi\rho) r^{-1} \frac{dr}{d\tilde{V}} \tag{5.1}$$

We now estimate $(3\hat{m}r^{-3} - 4\pi\rho) r^{-1} \frac{dr}{d\tilde{V}}$.

Lemma 4. As usual let $\tilde{\rho} \geq 0$, $\tilde{p} \geq 0$, $\frac{d\tilde{\rho}}{d\tilde{p}} \geq 0$. Then

$$(3\hat{m}r^{-3} - 4\pi\tilde{\rho}) r^{-1} \frac{dr}{d\tilde{V}} \leq (8\pi/5) \sup_{(0,r)} \left\{\frac{d\tilde{\rho}}{d\tilde{p}} (\tilde{\rho} + \tilde{p}) (\tilde{\rho} + 3\tilde{p}) \tilde{g}_{rr}\right\} \tilde{V}^{-1} (\tilde{\rho} + 3\tilde{p})^{-1} \tilde{g}^{rr} \tag{5.2}$$

Proof: Using (3.3) - (3.6) we get

$$3\hat{m}r^{-3} - 4\pi\tilde{\rho} = 16\pi^2 r^{-3} \int_0^r \left[\int_s^r \frac{d\tilde{\rho}}{d\tilde{p}} (\tilde{\rho} + \tilde{p}) (\tilde{\rho} + 3\tilde{p}) \tilde{g}_{rr} \tau d\tau\right] s^2 ds$$

$$+ 4\pi r^{-3} \int_0^r \left[\int_s^r \frac{d\tilde{\rho}}{d\tilde{p}} (\tilde{\rho} + \tilde{p}) (3\hat{m}(\tau)\tau^{-3} - 4\pi\tilde{\rho}) \tilde{g}_{rr} \tau d\tau\right] s^2 ds .$$

This implies

$$3\hat{m}r^{-3} - 4\pi\tilde{\rho} \leq (16/15)\pi^2 r^2 \sup_{(0,r)} \left\{\frac{d\tilde{\rho}}{d\tilde{p}} (\tilde{\rho} + \tilde{p}) (\tilde{\rho} + 3\tilde{p}) \tilde{g}_{rr}\right\}$$

$$+ (4\pi/15) r^2 \sup_{(0,r)} \left\{\frac{d\tilde{\rho}}{d\tilde{p}} (\tilde{\rho} + \tilde{p}) \tilde{g}_{rr}\right\} \sup_{(0,r)} (3\hat{m}(s)s^{-3} - 4\pi\tilde{\rho}(s)) .$$

Thus for

$$r^2 < \left[(8\pi/15) \sup_{(0,r)} \left\{\frac{d\tilde{\rho}}{d\tilde{p}} (\tilde{\rho} + \tilde{p}) \tilde{g}_{rr}\right\}\right]^{-1} , \tag{5.3}$$

$$3\hat{m}r^{-3} - 4\pi\tilde{\rho} \leq (32/15)\pi^2 \sup_{(0,r)} \left\{\frac{d\tilde{\rho}}{d\tilde{p}}(\tilde{\rho} + \tilde{p})(\tilde{\rho} + 3\tilde{p})\tilde{g}_{rr}\right\} r^2 \tag{5.4}$$

Now using (3.3) and $\hat{m}r^{-3} \geq (4/3)\pi\tilde{\rho}r^{-3}$ we find that (5.2) holds for r satisfying (5.3). On the other hand for r not satisfying (5.3) we have

$$r^{-1}\frac{dr}{d\tilde{V}} \leq (8/15)\pi \sup_{(0,r)} \left\{\frac{d\tilde{\rho}}{d\tilde{p}} (\tilde{\rho} + \tilde{p})\tilde{g}_{rr}\right\} V^{-1}(\hat{m}r^{-3} + 4\pi\tilde{p})^{-1}\tilde{g}^{rr} .$$

Hence the result follows because of the fact that $\tilde{\rho}$ and $\tilde{p}$ decrease as r increases. □

From lemma 4 and (5.1) we find that if for $V \geq \tilde{V}_c$,

$$V^2(\rho + 3p)^2 \geq (4/5)(b - V^2) \sup_{(\tilde{V}_c, V)} \left\{\frac{d\rho}{dp}(\rho + p)(\rho + 3p)\right\} \sup_{(0,r)} \{\tilde{g}_{rr}\}\tilde{g}^{rr} \tag{5.5}$$

then (4.4) is satisfied for $V \geq \tilde{V}_c$. For $V < \tilde{V}_c$ it is easy to show that (4.4) is equivalent to

$$V^2(\rho + 3p) \geq (2/3)(b - V^2) \frac{d\rho}{dp} (\rho + p) . \tag{5.6}$$

Now $\tilde{g}_{rr} \geq 1$ and by virtue of (3.11) we have an absolute upper bound a for $\tilde{g}_{rr}$. Hence a sufficient condition on the equation of state such that (4.4) is satisfied is given by

$$V^2(\rho + 3p)^2 \geq (4a/5)(b - V^2) \sup_{(0,V)} \left\{\frac{d\rho}{dp} (\rho + p)(\rho + 3p)\right\} \tag{5.7}$$

Clearly (5.7) is a condition on the equation of state and V_s only; by virtue of (2.7') ρ and p are uniquely determined as functions of V, once the equation of state and V_s are given. Thus we get the following theorem.

Theorem 2. Let $(M, {}^4g)$ be a (nonsingular) static stellar model (asymptotically Euclidean in the sense of assumption (vi)) with bounded fluid region Q such that ∂Q is a smooth level set $V = V_s$ of the "potential" V. Suppose in addition to satisfying the usual conditions (viz, $p \geq 0$, $\rho \geqq 0$, $\frac{d\rho}{dp} \geqq 0$ and $\int_0^p (\rho(s) + s)^{-1} ds < \infty$ for finite p) $\rho = \rho(p)$ satisfies the following two conditions

$$5\rho^2 \geqq 6p(\rho + p) \frac{d\rho}{dp} \tag{5.8}$$

and

$$x^2(\rho + 3p)^2 \geqq (4a/5)(b - x^2) \sup_{(0,x)} \left\{\frac{d\rho}{dp} (\rho + p)(\rho + 3p)\right\} \tag{5.9}$$

where $x(p) = V_s exp\left(-\int_0^p (\rho(\tau) + \tau)^{-1} d\tau\right)$.

If there exists a spherically symmetric (nonsingular) static stellar model $(\tilde{M}, {}^4\tilde{g})$ with the same equation of state $\rho = \rho(p)$ and surface potential V_s, then $(M, {}^4g)$ is also spherically symmetric. In fact $(M, {}^4g)$ and $(\tilde{M}, {}^4\tilde{g})$ are isometric.

Proof: Because of (5.10), (5.7) and hence (4.4) is satisfied. By virtue of lemma 1, 4.11 is also satisfied. Hence we can use proposition 1. □

We can get a condition independent of V_s as follows. Equation (2.7) is invariant under the transformation $V \longrightarrow \text{constant} \times V$ and also we have $V_s > \frac{1}{3}$. Hence we can prove the following lemma.

Lemma 5. Let $y(p) = \frac{1}{3} \exp\left(-\int_0^p (\rho(s)+s)^{-1} ds\right)$. Then

$$y^2(\rho + 3p)^2 \geq (4a/5)((11/27) - y^2) \sup_{(0,y)} \left\{\frac{d\rho}{dp}(\rho+p)(\rho+3p)\right\} \qquad (5.11)$$

implies (5.9).

Proof: We have $x = \tau y$ where $\tau = 3V_s > 1$. Multiplying (5.11) by τ^2 and noting that

$$11/27 = \frac{1}{3}(1 + 2V_s^2\tau^{-2}) > \frac{1}{3}\tau^{-2}(1 + 2V_s^2) = \tau^{-2}b$$

we get (5.9). □

Hence theorem 2 implies the following corollary.

Corollary 1. Conclusion of theorem 2 remains valid if (5.9) is replaced by (5.11).

Remark 7. Inequality (4.4) allows only small values of $\frac{d\rho}{dp}$ towards the interior. For example, even in the case of a spherical star it gives that at the centre

$$\left(\frac{d\rho}{dp}\right)_c \leq (5/2)\tilde{V}_c^2(b-\tilde{V}_c^2)^{-1}(\tilde{\rho}_c+3\tilde{p}_c)(\tilde{\rho}_c+\tilde{p}_c)^{-1} \tag{5.12}$$

Hence (5.11) allows still smaller values of $\left(\frac{d\rho}{dp}\right)_c$. Of course the actual numerical value of $\left(\frac{d\rho}{dp}\right)_c$ in 5.12 would depend on the ratio $\tilde{V}_c{}^2/(b-\tilde{V}_c^2)$.

Remark 8. Suppose $\rho_s \neq 0$ where $\rho_s \equiv \rho(p)$ at $p = 0$. Then it is possible to estimate V from below in terms of V_s, ρ and ρ_s. Hence we can replace the second condition in theorem 2 with a simpler condition. Furthermore by virtue of (3.10) even the dependence on V_s can be removed. Although this condition restricts $\frac{d\rho}{dp}$ to smaller values, it is much simpler and gives a better feeling about the nontriviality of theorem 2 at first sight.

For example, since $\frac{d}{dV}\{V(\rho+3p)\} \leq 0$ we have $V(\rho+3p) \geq V_s\rho_s$. Thus we have the following corollary to theorem 2:

Corollary 2. The conclusion of theorem 2 remains valid with condition (5.9) replaced by the following condition

$$5\rho_s^2 \geq 4a\,\frac{d\rho}{dp}\,(\rho+p)(\rho+3p) \tag{5.13}$$

For nontrivial condition we need $\rho_s \neq 0$ in theorem 2 and its corollaries 1 and 2. However the inequality (4.4) is automatically satisfied near the surface ∂Q (see below) and we can allow ρ to decrease rapidly to zero in this narrow range. Thus $\frac{d\rho}{dp}$ can be allowed to become large near ∂Q. The inequality in (5.1) is automatically satisfied if

$$b - \mathcal{V}^2 - r\mathcal{V}\,\frac{d\mathcal{V}}{dr} \leqq 0 \tag{5.14}$$

This is the case near the surface of the fluid because at $\mathcal{V} = V_s$ we have

$$r\mathcal{V}\,\frac{d\mathcal{V}}{dr} = \frac{1}{2}\,(1 - V_s^2) = \frac{3}{2}\,(b - V_s^2) \tag{5.15}$$

Thus as $\mathcal{V}$ decreases from V_s, the left hand side of inequality (5.14) remains negative until (say) $\mathcal{V} = \mathcal{V}_1(r_1) \in (\mathcal{V}_c, V_s)$ where it becomes zero. The following lemma gives an estimate of $\mathcal{V}_1$ from above in terms of V_s.

Lemma 6. There exists a solution $\mathcal{V} = \mathcal{V}_2$ of

$$V_s^2 + \frac{9}{2}\,(b - V_s^2)\,\log\{(b - \mathcal{V}^2)/(b - V_s^2)\} - \frac{3}{2}\,\mathcal{V}^2\{(b - V_s^2)/(b - \mathcal{V}^2)\}^{5/2} = 0 \tag{5.16}$$

in the range $[\mathcal{V}_1, V_s)$. In particular on $[\mathcal{V}_2, V_s]$ (5.14), and hence (4.4) holds.

Proof. By virtue of (3.22) and (3.20) $\hat{m}r^{-3}$ is nonincreasing in r. Hence using (3.3), (3.4) and (3.6) we get

$$\tilde{V} r \frac{d\tilde{V}}{dr} \geq \tilde{V}^2 \tilde{g}_{rr} r^2 r_s^{-3} \tilde{m} .$$

Now on $[\tilde{V}_1, V_s]$ we have $-d(\log(b - \tilde{V}^2)) \geq 2r^{-1} dr$. This implies that on $[\tilde{V}_1, V_s]$

$$r \geq r_s \{(b - V_s^2)/(b - \tilde{V}^2)\}^{1/2}.$$

Hence on $[\tilde{V}_1, V_s]$ we have

$$\tilde{V} r \frac{d\tilde{V}}{dr} \geq \frac{3}{2} \tilde{V}^2 \tilde{g}_{rr} (b - \tilde{V}^2)^{-1} (b - V_s^2)^2$$

Finally using

$$\tilde{g}_{rr}(V) \geq \{(b - V_s^2)/(b - \tilde{V}^2)\}^{1/2} [V_s^2 + \frac{9}{2} (b - V_s^2) \log\{(b - \tilde{V}^2)/(b - V_s^2)\}]^{-1} \tag{5.17}$$

we get that on $[\tilde{V}_1, V_s]$,

$$\tilde{V} r \frac{d\tilde{V}}{dr} \geq \frac{3}{2} \tilde{V}^2 (b - \tilde{V}^2) \{(b - V_s^2)/(b - \tilde{V}^2)\}^{5/2} [V_s^2 + \frac{9}{2} (b - V_s^2) \log\{(b - \tilde{V}^2)/(b - V_s^2)\}]^{-1} .$$

where at $\tilde{V} = V_s$ equality holds. Thus the expression

$$1-\frac{3}{2}\,\tilde{V}^2\{(b-V_s^2)/(b-\tilde{V}^2)\}^{5/2}[V_s^2+\frac{9}{2}\,(b-V_s^2)\,\ell og\,\{(b-\tilde{V}^2)/(b-V_s^2)\}]^{-1}$$

is negative at V_s and nonnegative at $\tilde{V}_1$. Hence this expression is nonpositive on $[\tilde{V}_2, V_s] \subset [\tilde{V}_1, V_s]$ where $\tilde{V}_2$ is as in the statement of the lemma. Since $\tilde{V}_1 < \tilde{V}_2$ (5.14) holds on $[\tilde{V}_2, V_s]$. □

Remark 9. Note that $\tilde{V}_2$ is the unique solution of (5.16) in the range (V_{min}, V_s).

Thus we can improve theorem 2 to obtain the following result.

Theorem 3. The conclusion of theorem 2 remains valid if the condition (5.9) holds only for $x \in (0, \tilde{V}_2)$ where $\tilde{V}_2 = \tilde{V}_2(V_s)$ is as in lemma 6.

6. Conclusion

Given a stellar model we try to find a conformal transformation such that the resulting metric of the t = constant hypersurface is asymptotically Euclidean with mass zero and has nonnegative scalar curvature. If there exists a spherical star with the same equation of state and the value of V_s, then we can find such a conformal transformation provided the equation of state is restricted to satisfy two conditions implying small $\frac{d\rho}{dp}$. In this case the scalar curvature of the conformal metric on the t = constant hypersurface is of the form $P(V)(W_o - W)$ where $P(V) \geq 0$ and $W_o \geq W$. We use the first condition (for example, (4.11) or (5.8)) to have $P(V) \geq 0$. If we compare (5.8) with the Harrison-Wheeler equation of state we find that (5.8) is satisfied for central density up to $\rho = 2.39 \times 10^{-17}$ ($\equiv 3.22 \times 10^{11}$ gm/cm^3) at which stage the "compressibility index" ($\Gamma = (\rho + p)p^{-1} \frac{dp}{d\rho}$) drops discontinuously below 1.2 due to "neutron drip", (see page 624 in Misner et al. See also chapter 10 in Harrison et al. (1965)). As density increases, Γ increases rapidly above 1.2 again. But without further analysis it is not clear how badly $P(V)$ becomes negative in case it does. The second condition (viz, 4.4 or (5.9)) is used to show that $W_o \geq W$. Such a condition is artificial as we expect $W_o \geq W$ to hold in any case. As expected this condition is very badly violated by the Harrison-Wheeler equation of state. For example, rough calculation shows that even (5.12) can be violated by a factor of $50 \sim 500$ at density $\rho = 7.42 \times 10^{-23}$ ($\equiv 1 \times 10^6$ gm/cm^3. $\tilde{V}_c/V_s$ can be easily calculated from the column for "n" in Table 13 on page 109 in Harrison et al. Then V_s can be estimated from 5.16). In fact, the function $(b - V^2)^{-1}$ which has been used to multiply $W - W_o$ in the Robinson-type identity viz. (4.6), is an artificial choice. This choice is not bad towards ∂Q but in the interior one would not expect such a simple function to work nicely. Perhaps following the methods due to Lindblom (1981) one can get better identities implying $W_o \geq W$. In general, however, we like to show that

$\hat{R}(\Omega^2 g) = P(V)\ (W_o - W)$ is nonpositive. One can try to find a second order quasilinear elliptic PDE for $\hat{R}(\Omega^2 g)$ with non-negative right hand side !

It may be possible to remove the assumption of the existence of spherically symmetric stellar model having the same surface potential V_s by comparing the given nonsingular stellar model with the one obtained by integrating the "spherically symmetric field equations" from ∂Q inward with the same V_s and mass m . (Q need not to be connected and W_o need not to be globally a single-valued function of V). A result of the form $W \leq W_o < \infty$ would possibly then allow one to find a suitable conformal function Ω, globally defined on N .

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