DYNAMICS OF AUTOMORPHISMS ON PROJECTIVE COMPLEX MANIFOLDS

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ABSTRACT. We show that the dynamics of automorphisms on all projective complex manifolds X (of dimension 3, or of any dimension but assuming the Good Minimal Model Program or Mori's Program) are canonically built up from the dynamics on just three types of projective complex manifolds:

Complex Tori, weak Calabi-Yau, and Rationally Connected Manifolds.

As a by-product, we confirm the conjecture of Guedj [18] for automorphisms on 3-dimensional projective manifolds, and also determine $\pi_1(X)$.

1. INTRODUCTION

We work over the field $\mathbb{C}$ of complex numbers.

We show that the dynamics of automorphisms on all projective complex manifolds (of dimension 3, or of any dimension but assuming the Good Minimal Model Program or Mori's Program) are canonically built up from the dynamics on just three types of projective complex manifolds:

Complex Tori, weak Calabi-Yau, and Rationally Connected Manifolds

in the sense of [5] and [28] which are the high-dimensional analogues of rational surfaces. For a similar phenomenon in dimension 2, we refer to [6]. Here a smooth projective manifold X is *weak Calabi-Yau* or simply *wCY* if the Kodaira dimension $\kappa(X) = 0$ and the irregularity $q(X) = 0$.

For the recent development on complex dynamics, we refer to the survey article [12] and the references therein. See also [7], [9] and [34].

For algebro-geometric approach to dynamics of automorphisms due to Keiji Oguiso, see [40], [41] and [42].

We shall consider dynamics of automorophisms on projective complex manifolds of dimension ≥ 3. To focus on the dynamics of genuinely high dimension, we introduce the notions of *rigidly parabolic* pairs (X, g) and pairs (X, g) of *primitively positive entropy*, where X is a smooth projective manifold of $\dim X \geq 3$, and $g \in \text{Aut}(X)$. In other words, these are the pairs where the dynamics are not coming from the dynamics of lower dimension; see Convention 2.1, and also Lemma 2.20 for the classifications of the rigidly parabolic pairs in the case of surfaces. These notions might be the geometrical incarnations of McMullen's lattice-theoretical notion "essential lattice isometry" in [32] §4. By the way, all surface automorphisms of positive entropy, are of primitively positive entropy.

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In Theorems 1.1 and 3.1 below, it is shown that a pair (X, g) of rigidly parabolic or of primitively positive entropy exists only when the Kodaira dimension $\kappa(X) \le 0$ and irregularity $q(X) \le \dim X$. If (Y, g) is just of positive entropy, then one can only say that $\kappa(Y) \le \dim Y - 2$ which is optimal by Lemma 2.13.

In Theorems 3.2 and 3.3 of §3, we determine the g-structure of manifolds X of dimension ≥ 3, with $\kappa(X) = 0$ and $-\infty$, respectively. The difficult part in Theorem 3.2 is to show that the regular action of g on the initial manifold X is *equivariant* to a nearly regular action on another 'better' birational model X'; see Convention 2.1. Such difficulty occurs only in dimension ≥ 3 due to the high-dimensional new phenomenon of non-uniqueness and non-smoothness of minimal models of a birational class. Recall that a variety Z with mild 'terminal' singularities is *minimal* if the canonical divisor K_Z is nef (= numerically effective).

We now state the main results. The result below says that dynamics occur essentially only on those X with Kodaria dimension $\kappa(X) \le 0$.

Theorem 1.1. *Let X be a smooth projective complex manifold of $\dim X \ge 2$, and with $g \in \mathrm{Aut}(X)$. Then we have:*

1. *Suppose that (X, g) is either rigidly parabolic or of primitively positive entropy (see (2.1)). Then the Kodaira dimension $\kappa(X) \le 0$.*
2. *Suppose that $\dim X = 3$ and g is of positive entropy. Then $\kappa(X) \le 0$, unless $d_1(g^{-1}) = d_1(g) = d_2(g) = e^{h(g)}$ and it is a Salem number. Here $d_i(g)$ are dynamical degrees and $h(g)$ is the entropy (see (2.1)).*

In view of Theorem 1.1, we have only to treat the dynamics on those X with $\kappa(X) = 0$ or $-\infty$. This is done in Theorems 3.2 and 3.3. See the statements in §3 for details.

As sample results, we now give some applications to our results in §3 in the case of threefolds.

The result below says that 3-dimensional dynamics of positive entropy (not necessarily primitive) are just those of 3-tori, weak Calabi-Yau 3-folds and rational connected 3-folds, unless dynamical degrees are Salem numbers.

Theorem 1.2. *Let X' be a smooth projective complex threefold. Suppose that $g \in \mathrm{Aut}(X')$ is of positive entropy. Then there is a pair (X, g) birationally equivariant to (X', g), such that one of the cases below occurs.*

1. *There are a 3-torus $\tilde{X}$ and a g-equivariant étale Galois cover $\tilde{X} \to X$.*
2. *X is a weak Calabi-Yau threefold.*
3. *X is a rationally connected threefold in the sense of [5] and [27].*
4. *$d_1(g^{-1}|X) = d_1(g|X) = d_2(g|X) = e^{h(g|X)}$ and it is a Salem number.*

The following confirms the conjecture of Guedj [18] page 7 for automorphisms on 3-dimensional projective manifolds.

Theorem 1.3. *Let X be a smooth projective complex threefold admitting a cohomologically hyperbolic automorphism g in the sense of [18] p.3. Then we have:*

(1) *The Kodaira dimension* $\kappa(X) \le 0$.
(2) *More precisely, either* X *is a weak Calabi-Yau threefold, or* X *is rationally connected, or there is a* g*-equivariant birational morphism* $X \to T$ *onto a* Q*-torus.*

We can also determine the fundamental group below. For the case of $\kappa(X) = 0$ we refer to Namikawa-Steenbrink [39, Corollary (1.4)].

Theorem 1.4. *Let* X *be a smooth projective complex threefold with* $g \in \mathrm{Aut}(X)$ *of primitively positive entropy. Suppose that the Kodaira dimension* $\kappa(X) \neq 0$. *Then either* $\pi_1(X) = (1)$, *or* $\pi_1(X) = \mathbb{Z}^{\oplus 2}$.

For examples (X, g) of positive entropy with X a torus (well known case), or a rational manifold (take product of rational surfaces), or a Calabi-Yau manifold, we refer to [7], [33] and [34], and Mazur's example of multi-degree 2 hypersurfaces in $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ as in [9] Introduction.

See also Remark 1.7 below.

We define the following sets of dynamical degrees for automorphisms of positive entropy, where X is wCY = weak Calabi-Yau if $\kappa(X) = 0 = q(X)$, where rat.conn. = 'rationally connected' is in the sense of [5] and [28], where type (*) = type (t) (torus), or type (cy) (weak Calabi-Yau), or type (rc) (rational connected). Note that $D_1^*(n) \subset D_1^*(n)'$.

$$D_1^t(n) := \{\lambda \in \mathbb{R}_{>1} \mid \lambda = d_1(g) \text{ for } g \in \mathrm{Aut}(X) \text{ with } X \text{ an } n\text{-torus}\},$$
$$D_1^{cy}(n) := \{\lambda \in \mathbb{R}_{>1} \mid \lambda = d_1(g) \text{ for } g \in \mathrm{Aut}(X) \text{ with } X \text{ a wCY } n\text{-fold}\},$$
$$D_1^{rc}(n) := \{\lambda \in \mathbb{R}_{>1} \mid \lambda = d_1(g) \text{ for } g \in \mathrm{Aut}(X) \text{ with } X \text{ a rat.conn. } n\text{-fold}\},$$
$$Sa = \{\lambda \in \mathbb{R}_{>1} \mid \lambda \text{ is a Salem number }\},$$
$$D_1^*(n)' := \{\lambda \in \mathbb{R}_{>1} \mid X \text{ a type (*) } n\text{-fold}, H \text{ an ample Cartier divisor}, H \in L \subset \mathrm{NS}(X) \text{ a sublattice}, \sigma \in \mathrm{Hom}_{\mathbb{Z}}(L, L) \text{ is bijective and preserves the induced product form on } L,\ \sigma^* P = \lambda P \text{ for a nef } \mathbb{R}\text{-divisor } P \neq 0\}.$$

We conclude the introduction with the result / question below which suggest a connection between the existence of the dynamics and the theory of algebraic integers like the dynamical degrees $d_i(g)$.

See McMullen [32] for the realization of some Salem numbers as dynamical degrees of $K3$ automorphisms.

Theorem 1.5. *Let* X *be a smooth projective complex threefold. Suppose* $g \in \mathrm{Aut}(X)$ *is of positive entropy. Then the first dynamical degree satisfies*

$$d_1(g) \ \in \ Sa \cup D_1^t(3) \cup D_1^{cy}(3) \cup D_1^{rc}(3).$$

Further, for some $s > 0$,

$$d_1(g^s) \ \in \ D_1^{rc}(2)' \ \cup_{k=2}^{3} \ \{D_1^t(k) \cup D_1^{cy}(k) \cup D_1^{rc}(k)\}.$$

The question below has a positive answer in dimension 2; see [6].

Question 1.6. *Let X be a smooth projective complex manifold of dimension $n \ge 2$ and $g \in \mathrm{Aut}(X)$ of primitively positive entropy. Does the first dynamical degree $d_1(g)$ satisfy the following*

$$d_1(g) \ \in \ \cup_{k=2}^{n} \{D_1^t(k) \cup D_1^{cy}(k) \cup D_1^{rc}(k)\} \, ?$$

Remark 1.7. Let E_α be elliptic curves. Following Igusa's construction (see Oguiso - Sakurai [43, (2.17)], or Ueno [47]), one can construct free action of $\Gamma := (\mathbb{Z}/(2))^{\oplus 2}$ on the abelian variety $E_1 \times E_2 \times E_3$. We take $E_\alpha = E_i := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\sqrt{-1})$. Then $SL_3(\mathbb{Z})$ acts on $A := E_i \times E_i \times E_i$, and as observed in [8, (4.5)], $SL_3(\mathbb{Z})$ contains free abelian subgroups G of rank 2 so that the action on A by each $\mathrm{id} \ne g \in G$ is of positive entropy. Now $X := A/\Gamma$ is a smooth Calabi-Yau variety with $K_X \sim 0$. If we can find such a g normalizing Γ, then $g|A$ descends to a $\bar{g} \in \mathrm{Aut}(X)$ of positive entropy.

Remark 1.8. Like Oguiso [40] - [42], our approach is algebro-geometric in nature.

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2. Preliminary results

In this section we recall definitions and prove some lemmas.

Results like Lemmas 2.20 and 2.23 are not really / fully used by the paper, but are hopefully of independent interest.

2.1. Conventions and remarks

(1) For a linear transformation $T : V \to V$ of a vector space V, let $\rho(T)$ be the *spectral radius* of T, i.e., the largest modulus of eigenvalues of T.

(2) We shall use the terminology and notation in Kawamata-Matsuda-Matsuki [26] and Kollár - Mori [29]. Most of the divisors are $\mathbb{R}$-Cartier divisors: $\sum_{i=1}^{s} r_i D_i$ with $r_i \in \mathbb{R}$ and D_i Cartier prime divisor.

Let X be a smooth projective manifold. Set $H^*(X, \mathbb{C}) = \oplus_{i=0}^{2n} H^i(X, \mathbb{C})$. There is the Hodge decomposition:

$$H^k(X, \mathbb{C}) = \oplus_{i+j=k} H^{i,j}(X, \mathbb{C}).$$

Denote by $H_a^{i,i}(X, \mathbb{C})$ the subspace of $H^{i,i}(X, \mathbb{C})$ spanned by the algebraic subvarieties of complex codimension i.

(3) Let $\mathrm{Pic}(X)$ be the *Picard group*, and $\mathrm{NS}(X) = \mathrm{Pic}(X)/(\text{algebraic equivalence}) = H^1(X, \mathcal{O}_X^*)/\mathrm{Pic}^0(X) \subseteq H^2(X, \mathbb{Z})$ the *Neron-Severi group*. $\mathrm{NS}(X)$ is a finitely generated abelian group whose rank is the *Picard number*.

Set $\mathrm{NS}_{\mathbb{B}}(X) = \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{B} \subset H^2(X, \mathbb{B})$ for $\mathbb{B} = \mathbb{Q}$ and $\mathbb{R}$. Let $\mathrm{Nef}(X)$ be the closed cone of nef divisors in $\mathrm{NS}_{\mathbb{R}}(X)$. So $\mathrm{Nef}(X)$ is the closure of the ample cone. Also $\mathrm{Nef}(X) \subseteq \overline{\mathcal{K}(X)}$, the closure of the Kähler cone $\mathcal{K}(X)$.

Let $N_1(X)$ be the $\mathbb{R}$-space generated by algebraic 1-cycles modulo numerical equivalence; see [29] (1.16). When X is a surface, $N_1(X) = \mathrm{NS}_{\mathbb{R}}(X)$.

(4) Let $g \in \mathrm{Aut}(X)$. Denote by $\rho(g)$ the *spectral radius* of $g^*|H^*(X, \mathbb{C})$. It is known that either $\rho(g) > 1$, or $\rho(g) = 1$ and all eigenvalues of $g^*|H^*(X, \mathbb{C})$ are of modulus 1. When $\log \rho(g) > 0$ (resp. $\log \rho(g) = 0$) we say that g is of *positive entropy* (resp. *null entropy*).

We refer to Gromov [16], Yomdin [50], Friedland [11], and Dinh - Sibony [9] page 302, for the definition of the i-th *dynamical degree* $d_i(g)$ for $1 \le i \le n = \dim X$ (note that $d_n(g) = 1$ now and set $d_0(g) = 1$) and the actual definition of the *topological entropy* $h(g)$ which turns out to be $\log \rho(g)$ in the setting of our paper.

(5) Let Y be a projective variety and $g \in \mathrm{Aut}(Y)$. We say that g is of *positive entropy*, or *null entropy*, or *parabolic*, or *periodic*, or *rigidly parabolic*, or of *primitively positive entropy* (see the definitions below), if so is $g \in \mathrm{Aut}(\tilde{Y})$, where $\tilde{Y} \to Y$ is one (and hence all) g-equivariant resolutions guaranteed by Hironaka [22]. The definitions do not depend on the choice of $\tilde{Y}$ because every two g-equivariant resolutions are birationally dominated by a third one, by the work of Abramovich - Karu - Matsuki - Wlodarczyk; see Matsuki [31] (5-2-1); see also Lemma 2.6.

(6) We use $g|Y$ to signify that $g \in \mathrm{Aut}(Y)$.

(7) In this paper, by a *pair* (Y, g) we mean a projective variety Y and an automorphism $g \in \mathrm{Aut}(Y)$. Two pairs (Y', g) and (Y'', g) are *g-equivariantly birational*, if there is a birational map $\sigma : Y' \cdots \to Y''$ such that the birational action $\sigma(g|Y')\sigma^{-1} : Y'' \cdots \to Y''$ extends to the biregular action $g|Y''$.

(8) $g \in \mathrm{Aut}(Y)$ is *periodic* if the order $\mathrm{ord}(g)$ is finite. g is *parabolic* if $\mathrm{ord}(g) = \infty$ and if g is of null entropy.

(9) (Y', g) is *rigidly parabolic* if ($g|Y'$ is parabolic and) for every pair (Y, g) which is g-equivariantly birational to (Y', g) and for every g-equivariant surjective morphism $Y \to Z$ with $\dim Z > 0$, we have $g|Z$ parabolic.

(10) Let Y' be a projective variety and $g \in \mathrm{Aut}(Y')$ of positive entropy (so $\dim Y' \ge 2$). A pair (Y', g) is of *primitively positive entropy* if it is not of imprimitive positive entropy, while a pair (Y', g) is of *imprimitively positive entropy* if it is g-equivariantly birational to a pair (Y, g) and if there is a g-equivariant surjective morphism $f : Y \to Z$ such that either one of the two cases below occurs.

(10a) $0 < \dim Z < \dim Y$, and $g|Z$ is still of positive entropy.
(10b) $0 < \dim Z < \dim Y$, and $g|Z$ is periodic.

(11) **Remark.** We observe that in Case(10b), for some $s > 0$ we have $g^s|Z = \mathrm{id}$ and that g^s acts faithfully on the general fibre Y_z of $Y \to Z$, such that $g^s|Y_z$ is of positive entropy. To see it, we replace g^s by g for simplicity. In view of Lemma 2.6, we may assume that Y_z is connected by making use of the Stein factorization, and also that both Y and Z are smooth after g-equivariant resolutions as in Hironaka [22]. Let $0 \ne v_g \in \mathrm{Nef}(Y)$ be a nef divisor as in Lemma 2.4 such that $g^* v_g = d_1(g) v_g$ with $d_1(g) := d_1(g|Y) > 1$.

We claim that $v_g|Y_z \ne 0$. Indeed, take very ample divisors $H_1, \dots, H_k$ with $k = \dim Z$. Then $f^*H_1 \dots f^*H_k = cY_z$ homologously with $c = (H_1 \dots H_k) > 0$. Noting that $f^*H_1 \dots f^*H_i \ne 0$ and $g^*(f^*H_j) = f^*(g^*H_j) = f^*H_j$ and applying Lemma 2.3 repeatedly, we get $f^*H_1 \dots f^*H_i.v_g \ne 0$ for all $i \le k$. In particular, $0 \ne f^*H_1 \dots f^*H_k.v_g = cY_z.v_g = cv_g|Y_z$ homologously; see Fulton [13] (8.3) for the last equality. This claim is proved.

Next we claim that $d_{k+1}(g) := d_{k+1}(g|Y) \ge d_1(g|Y_z) \ge d_1(g)$ with $k = \dim Z$, so that $g|Y_z$ is of positive entropy. Indeed, $g^*(v_g|Y_z) = d_1(g)(v_g|Y_z)$ implies that $d_1(g|Y_z) \ge d_1(g)$ by Lemma 2.2. By Lemma 2.4, $g^*(v_{(g|Y_z)}) = (g|Y_z)^* v_{(g|Y_z)} = d_1(g|Y_z) v_{(g|Y_z)}$, so $d_{k+1}(g) \ge d_1(g|Y_z)$.

In the forthcoming paper [38, Appendix, Theorem A.10], we will show that $d_1(g^s|Y) = d_1(g^s|Y_z)$ for the $s > 0$ as given at the begining of this Remark.

(12) **Remark.** In view of the observation above and Lemma 2.2, if $\dim Y \le 2$ and if the pair (Y, g) is of positive entropy, then $\dim Y = 2$ and the pair (Y, g) is always of primitively positive entropy.

(13) We refer to Iitaka [23] for the definition of *D-dimension* $\kappa(X, D)$; the *Kodaira dimension* $\kappa(X) = \kappa(\tilde{X}) = \kappa(\tilde{X}, K_{\tilde{X}})$ with $\tilde{X} \to X$ a projective resolution; and the *Iitake fibring* (of X): $X' \to Y'$ with X' birational to X, both X' and Y' smooth projective, $\dim Y' = \kappa(X)$ $(= \kappa(X'))$ and $\kappa(X'_y) = 0$ for a general fibre X'_y over Y'. Note that $\kappa(X)$ attains one of the values: $-\infty, 0, 1, \dots, \dim X$. We say that X is of *general type* if $\kappa(X) = \dim X$.

(14) **Remark.** The Iitaka fibring is defined by the pluri-canonical system $|rK_X|$ for $r >> 0$ after g-equivariant blowup to resolve base points in the system; see Hironaka [22]. So we can always replace (X, g) by its g-equivariant blowup (X', g) such that there is a g-equivariant Iitaka fibring $X' \to Y'$ with smooth projective manifolds X' and Y' and with $\dim Y' = \kappa(X') = \kappa(X)$. Note that $\kappa(X)$ is a birational invariant.

(15) A smooth projective manifold X of dimension n is *uniruled* if there is a dominant rational map $\mathbb{P}^1 \times Y \cdots \to X$, where Y is a smooth projective manifold of dimension $n - 1$.

(16) A smooth projective manifold X is a *Q-torus* if there is a finite étale cover $T \to X$ from a torus T.

(17) A smooth projective manifold is a *weak Calabi-Yau manifold* (or *wCY* for short) if the Kodaira dimension $\kappa(X) = 0$ and if the irregularity $q(X) = 0$. A normal projective variety $\bar{X}$ with only terminal singularity is a *Calabi-Yau variety* if the canonical divisor $K_{\bar{X}}$ satisfies $sK_{\bar{X}} \sim 0$ for some integer $s > 0$ and if $q(\bar{X}) = 0$. So a proper resolution X of a Calabi-Yau variety $\bar{X}$ is a weak Calabi-Yau manifold. Conversely, assuming the minimal model program, every weak Calabi-Yau manifold is birational to a Calabi-Yau variety. We refer to [26] or [29] for the definition of singularity of type: *terminal, canonical, log terminal, or rational*.

(18) An algebraic integer $\lambda > 1$ of degree $2(r+1)$ over $\mathbb{Q}$ with $r \geq 0$, is a *Salem* number (see [45] or [33] §3) if all conjugates of λ over $\mathbb{Q}$ (including λ itself) are given as follows, where $|\alpha_i| = 1$:

$$\lambda,\ \lambda^{-1},\ \alpha_1,\ \bar{\alpha}_1,\ \ldots,\ \alpha_r,\ \bar{\alpha}_r.$$

The following result is fundamentally important in the study of complex dynamics. For the proof, we refer the readers to [16], [50], [11], [9] (2.5) and page 302, [10] before (1.4), [12] the Introduction, [17] (1.2) (1.5) (1.6).

Lemma 2.2. *Let X be a smooth projective manifold of dimension n, and $g \in \mathrm{Aut}(X)$. Then the following are true.*

(1) $d_i(g) = \rho(g^*|H^{i,i}(X,\mathbb{R})) = \rho(g^*|H_a^{i,i}(X,\mathbb{R}))$, $1 \leq i \leq n$.
(2) $d_1(g) = \rho(g^*|\mathrm{NS}_{\mathbb{R}}(X))$ (*see also* [9] (3.5)).
(3) $h(g) = \log \rho(g) = \max_{1 \leq i \leq n} \log d_i(g)$.
(4) *The entropy $h(g) > 0$ holds if and only if the dynamical degree $d_\ell(g) > 1$ for one (and hence for all) $1 \leq \ell \leq n-1$ (by (5)).*
(5) *The map $\ell \mapsto d_\ell(g)/d_{\ell+1}(g)$ with $0 \leq \ell \leq n-1$, is non-decreasing. So $d_\ell(g) \leq d_1(g)^\ell$ and $d_{n-\ell}(g) \leq d_{n-1}^\ell(g)$ for all $1 \leq \ell \leq n$. Also there are integers m, m' such that:*

$$1 = d_0(g) < d_1(g) < \cdots < d_m(g) = \cdots = d_{m'}(g) > \cdots > d_n(g) = 1.$$

The following very useful result is proved in Dinh-Sibony [8] (3.2) (4.4).

Lemma 2.3. *Let X be a smooth projective manifold of dimension n. Let $\mathrm{Nef}(X) \ni P,\ P',\ P_i$ $(1 \leq i \leq m;\ m \leq n-2)$ be nef divisors. Then we have:*

(1) *Suppose that $P_1.P_2 = 0$ in $H^{2,2}(X,\mathbb{R})$. Then one of P_1, P_2 is a multiple of the other.*
(2) *We have $P_1 \cdots P_m.P.P' \neq 0 \in H^{m+2,m+2}(X,\mathbb{R})$ if the two conditions below are satisfied.*
  (2a) *$P_1 \ldots P_m.P \neq 0$ and $P_1 \ldots P_m.P' \neq 0$.*
  (2b) *One has $g^*(P_1 \ldots P_m.P) = \lambda(P_1 \ldots P_m.P)$ and $g^*(P_1 \ldots P_m.P') = \lambda'(P_1 \ldots P_m.P)$, for some $g \in \mathrm{Aut}(X)$ and distinct (positive) real numbers λ and λ'.*

We refer to Dinh-Sibony [9] (3.5) for a result including the one below and with more analytical information.

Lemma 2.4. *(Generalized Perron-Frobenius Theorem) Let X be a smooth projective manifold and $g \in \mathrm{Aut}(X)$. Then there are non-zero nef divisors v_g and $v_{g^{-1}}$ in $\mathrm{Nef}(X)$ such that:*

$$g^* v_g = d_1(g) v_g, \quad (g^{-1})^* v_{g^{-1}} = d_1(g^{-1}) v_{g^{-1}}, \quad g^* v_{g^{-1}} = \frac{1}{d_1(g^{-1})} v_{g^{-1}}.$$

Proof. To get the first equality, we apply to the nef cone $\mathrm{Nef}(X)$ of the Perron - Fobenius Theorem for cones as in Schneider - Tam [46] page 4, item 5. The second or equivalently the third is the application of the first to g^{-1}. This proves the lemma. □

Here is the relation between dynamical degrees of automorphisms.

Lemma 2.5. *Let X be a smooth projective manifold of dimension n, and $g \in \mathrm{Aut}(X)$. Then we have:*

1. *Denote by $\Sigma(g)_i = \Sigma(g|X)_i$ the set of all eigenvalues of $g^*|H^{i,i}(X, \mathbb{C})$ (including multiplicities). Then $\Sigma(g)_1 = \Sigma(g^{-1})_{n-1}$.*
2. *The dynamical degrees satisfy $d_1(g) = d_{n-1}(g^{-1})$.*
3. *g is of positive entropy (resp. null entropy; periodic; parabolic) if and only if so is g^m for some (and hence for all) $m \ne 0$.*

Proof. There is a natural perfect pairing

$$H^{1,1}(X, \mathbb{C}) \times H^{n-1,n-1}(X, \mathbb{C}) \to \mathbb{C}$$

induced by the cup product / wedge product, via the Hodge decomposition. This pairing is preserved by the action of g^*; see Griffiths - Harris [15] page 59. So a simple linear algebraic calculation shows that if $g^*|H^{1,1}(X, \mathbb{C})$ is represented by a matrix A then $g^*|H^{n-1,n-1}(X, \mathbb{C})$ is represented by the matrix $(A^t)^{-1}$. Thus the lemma follows; see Lemma 2.2. □

The result below shows that the first dynamical degree of an automorphism is preserved even after lifting up or down by a generically finite and surjective morphism.

Lemma 2.6. *Let $f : X \to Y$ be a g-equivariant generically finite surjective morphism between smooth projective manifolds of dimension $n \ge 2$. Then we have:*

1. *$d_1(g|X) = d_1(g|Y)$.*
2. *$g|X$ is of positive entropy (resp. null entropy; periodic) if and only if so is $g|Y$.*
3. *$g|X$ is of parabolic if and only if so is $g|Y$.*
4. *If $g|X$ is rigidly parabolic then so is $g|Y$.*

Proof. (1) Set $\Sigma_X = \Sigma(g^*|\mathrm{NS}_{\mathbb{R}}(X)) = \{\lambda \in \mathbb{R} \mid g^* D = \lambda D$ for a divisor $0 \ne D \in \mathrm{NS}_{\mathbb{R}}(X)\}$. We show first that $\Sigma_Y \subseteq \Sigma_X$, whence $d_1(g|Y) \le d_1(g|X)$ by Lemma 2.2. Indeed, assume that $g^* \bar{L} = \lambda \bar{L}$ for some $0 \ne \bar{L} \in \mathrm{NS}_{\mathbb{R}}(Y)$ and $\lambda \in \Sigma_Y$. Set $L := f^* \bar{L}$. Then $g^* L = f^* g^* \bar{L} = \lambda L$. Note that $0 \ne L \in$

$\mathrm{NS}_{\mathbb{R}}(X) \subset H^2(X,\mathbb{R})$ because $f^* : H^*(Y,\mathbb{R}) \to H^*(X,\mathbb{R})$ is an injective ring homomorphism; see [1] I (1.3). Therefore, $\lambda \in \Sigma_X$.

Conversely, let $0 \neq L := v_g \in \mathrm{Nef}(X)$ such that $g^*L = d_1 L$ with $d_1 = d_1(g|X)$, as in Lemma 2.4. Set $\bar{L} := f_*L$. For any $H \in H^{2n-2}(Y,\mathbb{R})$. By the projection formula, we have $g^*\bar{L}.H = \bar{L}.g^{(-1)*}H = L.f^*g^{(-1)*}H = L.g^{(-1)*}f^*H = g^*L.f^*H = d_1L.f^*H = d_1\bar{L}.H$. So $(g^*\bar{L} - d_1\bar{L}).H = 0$ for *all* $H \in H^{2n-2}(Y,\mathbb{R})$. Hence $g^*\bar{L} = d_1\bar{L}$ in $H^2(Y,\mathbb{R})$.

We claim that $\bar{L} \neq 0$ in $\mathrm{NS}_{\mathbb{R}}(Y)$, whence $d_1 = d_1(g|X) \in \Sigma_Y$ and $d_1(g|X) \le d_1(g|Y)$ by Lemma 2.2 and we conclude the assertion (1). Assume the contrary that $\bar{L} = 0$. Take an ample divisor H_Y on Y. Then f^*H_Y is nef and big on X. So $f^*H_Y = A + D$ for an ample $\mathbb{Q}$-divisor A and an effective $\mathbb{Q}$-divisor on X, by Kodaira's lemma. By the projection formula and the nefness of L, one has $0 = \bar{L}.H_Y^{n-1} = L.f^*H_Y^{n-1} = L.f^*H_Y^{n-2}.(A + D) \ge L.f^*H_Y^{n-2}.A \ge \cdots \ge L.A^{n-1} \ge 0$. Applying the Lefschetz hyperplane section inductively to reduce to the Hodge index theorem for surfaces and using the nefness of L, we see that $L = 0 \in \mathrm{NS}_{\mathbb{R}}(X) \subseteq H^2(X,\mathbb{R})$, a contradiction. So the claim and hence the assertion (1) are proved.

With (1), the assertion (2) follows. Now (3) follows from (1) and (2).

(4) Assume that $g|X$ is rigidly parabolic. Let $Y \to Y_1$ be a g-equivariant surjective morphism with $\dim Y_1 > 0$. We have to show that $g|Y_1$ is parabolic. This follows from the assumption on $g|X$ and the g-equivariance of the composition $X \to Y \to Y_1$. This proves the lemma. □

We now describe the behavior of automorphisms dynamics in a fibration.

Lemma 2.7. *Let $X \to Y$ be a g-equivariant surjective morphism between smooth projective manifolds with* $\dim X > \dim Y > 0$. *Then we have:*

1. *If $g|X$ is of null entropy (resp. periodic), then so is $g|Y$.*
2. *Suppose that the pair (X, g) is either rigidly parabolic or of primitively positive entropy. Then $g|Y$ is rigidly parabolic.*

Proof. The proof of (1) is similar to that of Lemma 2.6. Suppose the contrary that (2) is false for some Y in (2). Then there is a g-equivariant surjective morphism $Y \to Z$ with $\dim Z > 0$ such that $g|Z$ is not parabolic. Thus, $g|Z$ is periodic or of positive entropy. This contradicts the rigidity or primitivity of (X, g) because $0 < \dim Z \le \dim Y < \dim X$. □

Here is the relation between Salem numbers and dynamical degrees in 2.1:

Lemma 2.8. *Let X be a smooth projective manifold and $g \in \mathrm{Aut}(X)$ of positive entropy. Then we have:*

1. *If* $\dim X = 2$, *then $d_1(g) = d_1(g^{-1}) = e^{h(g)}$ and it is a Salem number.*
2. *Suppose* $\dim X = 3$ *and there is a g-equivariant morphism $f : X \to Y$ onto a smooth projective curve Y with connected general fibre F. Then all $e^{h(g^{\pm})}$, $d_1(g^{\pm})$, $d_2(g^{\pm})$ are equal and it is a Salem number.*

Proof. The result in Case(1) is well known and follows from Lemmas 2.2 and 2.5 and the proof of McMullen [33] Theorem 3.2.

We consider Case(2). Set $L := (\mathrm{NS}(X)|F)/(\mathrm{torsion}) \subset \mathrm{NS}(F)/(\mathrm{torsion})$. We define the following intersection form $\langle , \rangle_L$ on the lattice L:

$$\langle D_1|F, D_2|F\rangle_L := D_1.D_2.F \in H^6(X, \mathbb{Z}) \cong \mathbb{Z}.$$

This $\langle , \rangle_L$ is compatible with the intersection form on $\mathrm{NS}(F)$ via the restriction $H^2(X, \mathbb{Z}) \to H^2(F, \mathbb{Z})$. This compatibility, the Hodge index theorem for the smooth projective surface F, and the fact that $H|F \in L$ with H an ample divisor of X, imply that the lattice L is non-degenerate and has signature $(1, r)$ with $1 + r \le \mathrm{rank}\,\mathrm{NS}(F)$. There is a natural action $g^*|L$ on L given by $g^*(D|F) = (g^*D)|F$. Since $g^*F = F$ in $\mathrm{NS}(X)$, this action is well defined and preserves the intersection form $\langle , \rangle_L$.

Since $g^*F = F$ and $g^*v_g = d_1 v_g$ with $d_1 = d_1(g) > 1$ in the notation of Lemma 2.4, our F and v_g are not proportional. So the Lefschetz hyperplane section theorem on cohomology and the Cauchy-Schwarz inequality or the Hodge index theorem for surfaces imply that $v_g.F.A = (v_g|A).(F|A) \ne 0$ for a very ample divisor A on X. So $v_g.F = v_g|F$ gives a non-zero $v := v_g|F \in L \otimes_{\mathbf{Z}} \mathbf{R}$. Further, $g^*v = (g^*v_g)|F = d_1 v$. Since L is an integral lattice of signature $(1, r)$ and $g^*|L$ is an isometry of L, by the proof of McMullen [33] Theorem 3.2, d_1 is a Salem number and all eigenvalues of $g^*|L$ are given as:

$$d_1,\ d_1^{-1},\ \alpha_1,\ \bar{\alpha}_1,\ \dots, \alpha_t,\ \bar{\alpha}_t$$

with $|\alpha_i| = 1$ and $2(t + 1) = r + 1$. Arguing with $g^{-1}|L$, we get $d_1(g|X) = d_1(g^{-1}|X)$ $(= d_2(g|X)$ by (2.5)). The result follows; see Lemma 2.2. □

The following result (though it will not be used in the sequel) is a generalization of a well-known result in the case of surfaces.

Lemma 2.9. *Let X be a smooth projective manifold of dimension $n \ge 2$ and $g \in \mathrm{Aut}(X)$ of positive entropy. Let $0 \ne v_i \in \mathrm{Nef}(X)$ $(1 \le i \le s)$ be nef divisors such that $g^*v_i = \lambda_i v_i$ for scalars λ_i with $\lambda_i > 1$ and that λ_i are pairwise distinct. Then we have:*

1. *$s \le n - 1$.*
2. *If $s = n - 1$, then $d_1(g) = \max_{1 \le i \le s}\{\lambda_i\}$.*

Proof. (1) Applying Lemma 2.3 repeatedly, we see that $u(s_1) := \prod_{i=1}^{s_1} v_i \ne 0$ if $s_1 \le n$. Note that $g^* = \mathrm{id}$ on $H^{n,n}(X, \mathbb{R}) \cong \mathbb{R}$. If $s \ge n$, then $u(n)$ is a non-zero scalar in $H^{n,n}(X, \mathbb{R})$, whence $u(n) = g^*u(n) = \lambda u(n)$ with $\lambda := \prod_{i=1}^{n} \lambda_i > 1$. This is a contradiction.

(2) Assume that $s = n - 1$. If $d_1 := d_1(g)$ is one of λ_i, then (2) is true by the maximality of $d_1(g)$ as in Lemma 2.2. Suppose that $d_1 \ne \lambda_i$ for all i. one gets a contradiction to (1) if one sets $v_n = v_g$ in the notation of Lemma 2.4. The lemma is proved. □

The result below shows that one may tell about the im/primitivity of threefold automorphisms by looking at the algebraic property of its dynamical degrees or entropy.

Lemma 2.10. *Let X be a smooth projective threefold and $g \in \mathrm{Aut}(X)$ such that the pair (X, g) is of imprimitively positive entropy. Then we have:*

(1) *All scalars $e^{h(g^{\pm})}$, $d_1(g^{\pm})$, $d_2(g^{\pm})$ are equal and it is a Salem number.*
(2) *For some $s > 0$, we have $d_1(g^s) \in D_1^t(2) \cup D_1^{cy}(2) \cup D_1^{rc}(2)$.*

We now prove Lemma 2.10. By the definition, there is a g-equivariant surjective morphism $f : X \to Y_1$ such that $\dim X > \dim Y_1 > 0$ and that either $g|Y_1$ is of positive entropy or $g|Y_1$ is periodic. Let $X \to Y \to Y_1$ be the Stein factorization. After g-equivariant blowups as in Hironaka [22], we may assume that X, Y and Y_1 are all smooth, $X \to Y$ has connected fibres F and $Y \to Y_1$ is generically finite and surjective. By Lemma 2.6, either $g|Y$ is of positive entropy or $g|Y$ is periodic. Since Question 1.6 has a positive answer in dimension 2 as in Cantat [6], our lemma follows from Lemma 2.8 and the claim below.

Claim 2.11. We have:

(1) Suppose that $g|Y$ is of positive entropy. Then the four scalars $d_1(g^{\pm}|X)$, $d_1(g^{\pm}|Y)$ coincide and we denote it by d_1.
(2) Suppose that $g|Y$ is periodic say $g^s|Y = \mathrm{id}$ for some $s > 0$. Then the four scalars $d_1(g^{\pm s}|X)$, $d_1(g^{\pm s}|F)$ coincide and we denote it by d_1^s or $d_1(g^s)$. So $d_1(g|X) = d_1(g^{-1}|X) = d_1$.
(3) For both cases in (1) and (2), if $0 \ne P \in \mathrm{Nef}(X)$ is a nef divisor such that $g^*P = \lambda P$ then $\lambda \in \{1, d_1^{\pm}\}$.
(4) For both cases in (1) and (2), $d_1 = d_1(g^{\mp}|X) = d_2(g^{\pm}|X) = e^{h(g^{\pm}|X)}$.

Let us prove the claim. (4) follows from (1) - (2) and Lemmas 2.5 and 2.2. Consider the case where $g|Y$ is of positive entropy. Thus $\dim Y \ge 2$, whence $\dim Y = 2$ and the fibre F is of dimension 1. The two scalars $d_1(g^{\pm}|Y)$ coincide and we denote it by d_1; see Lemma 2.5. Set $L^{\pm} := f^* v_{(g^{\pm}|Y)} \in \mathrm{Nef}(X)$; see Lemma 2.4 for the notation. Note that $L^{\pm} \ne 0$ in $\mathrm{NS}_{\mathbb{R}}(X)$; see [1] I (1.3). Further, $g^*L^{\pm} = f^*g^*v_{(g^{\pm}|Y)} = d_1^{\pm} L^{\pm}$. Thus, to prove the claim in the present case, we only have to show (3); see Lemma 2.4. If $\lambda \ne d_1^{\pm}$, then $u := L^+.L^-.P \in H^{3,3}(X, \mathbb{R}) \cong \mathbb{R}$ is a non-zero scalar by Lemma 2.3 and hence $u = g^*u = d_1 d_1^{-1} \lambda u = \lambda u$. So $\lambda = 1$. This proves the claim for the present case.

Consider the case where $g^s|Y = \mathrm{id}$. As remarked in (2.1), $g^s|F$ is of positive entropy, so $\dim F = 2$ and $\dim Y = 1$. Further, $d_1(g^{-s}|X) = d_2(g^s|X) \ge d_1(g^s|F) \ge d_1(g^s|X)$; see also Lemma 2.5. Arguing with g^{-1}, we get $d_1(g^s|X) \ge d_1(g^{-s}|F) \ge d_1(g^{-s}|X)$. Since the two scalars $d_1(g^{\pm s}|F)$ coincide for surface F by Lemma 2.5, the above two sequences of inequalities imply (2). Consider $u = v_{(g|X)}.v_{(g^{-1}|X)}.P$ as in the early case, one proves (3) for the present case. This proves the claim and also the lemma.

The following lemma is crucial, which is derived from a result of Nakamura - Ueno, and Deligne as in Ueno [47] Theorem 14.10.

Lemma 2.12. *Let $X \to Y$ a g-equivariant surjective morphism between smooth projective manifolds with $\dim Y > 0$. Suppose that the pair (X, g) is either rigidly parabolic or of primitively positive entropy. Then the Kodaira dimension $\kappa(Y) \le 0$. In particular, $\kappa(X) \le 0$.*

Proof. If $\kappa(Y) = \dim Y$, then $\mathrm{Aut}(X)$ is a finite group; see Iitaka [23] Theorem 11.12. Thus $g|Y$ is periodic, absurd. Suppose that $0 < \kappa(Y) < \dim Y$. After replacing by g-equivariant blowups of X and Y as in Hironaka [22], we may assume that $Y \to Z$ is a well-defined Iitaka fibring with Y and Z smooth projective and $\dim Z = \kappa(Y) > 0$. The natural homomorphism $\mathrm{Bir}(Y) \to \mathrm{Bir}(Z)$ between birational automorphism groups, has a finite group as its image; see [47] Theorem 14.10. In particular, $g|Z$ is periodic. This contradicts the assumption on g, noting that the composition $X \to Y \to Z$ is g-equivariant. This proves the lemma. □

For g of positive entropy (not necessarily being primitive), we have:

Lemma 2.13. *The following are true.*

1. *Let X be a smooth projective manifold of dimension n. Suppose that $g \in \mathrm{Aut}(X)$ is of positive entropy. Then the Kodaira dimension $\kappa(X) \le n - 2$.*
2. *Conversely, for every $n \ge 2$ and every $k \in \{-\infty, 0, 1, \ldots, n-2\}$, there are a smooth projective manifold X and $g \in \mathrm{Aut}(X)$ of positive entropy such that $\dim X = n$ and $\kappa(X) = k$.*

Proof. (1) Assume the contrary that $\kappa(X) \ge n - 1$. After g-equivariant blowup as in Hironaka [22], we may assume that for some $s > 0$, one has $|sK_X| = |M| + Fix$ with Fix the fixed part and with the movable part $|M|$ base point free, so that $f := \Phi_{|M|} : X \to Y \subset \mathbb{P}^N$ with $N = h^0(X, M)$, is the (g-equivariant) Iitaka fibring. Note that $M^{\kappa(X)}$ is homologous to a positive multiple of a fibre of f and hence $M^r \ne 0$ for every $r \le \kappa(X)$. Also $g^*M \sim M$ (linearly equivalent). With the notation of Lemma 2.4 and by Lemma 2.3, we have $v_g.M^r \ne 0$ in $H^{r+1,r+1}(X, \mathbb{R})$ for all $r \le n-1$. Since $g^* = \mathrm{id}$ on $H^{n,n}(X, \mathbb{R}) \cong \mathbb{R}$, for the scalar $u := v_g.M^{n-1} \in H^{n,n}(X, \mathbb{R})$, we have $u = g^*u = d_1 u$ with $d_1 = d_1(g) > 1$, so $u = 0$, a contradiction.

(2) Let S be a surface with $g \in \mathrm{Aut}(S)$ of positive entropy. Let Z be any $(n-2)$-fold. Set $X := S \times Z$ and $g|X := (g|S) \times (\mathrm{id}_Z)$. Then $g|X$ is of positive entropy by looking at the Kunneth formula for $H^2(X, \mathbb{C})$ as in Griffiths-Harris [15] page 58. Also $\kappa(X) = \kappa(S) + \kappa(Z)$. All values in $\{-\infty, 0\}$ (resp. $\{-\infty, 0, 1, \ldots, n-2\}$) are attainable with a suitable choice of S (resp. Z); see, for instance, Cantat [7] and McMullen [34]. Thus (2) follows. This proves the lemma. □

We need the following result on the eigenvalues of $g^*|H^*(X, \mathbb{C})$.

Lemma 2.14. *Let X be a smooth projective manifold of dimension n, and $g \in \mathrm{Aut}(X)$ of null entropy. Then there is an integer $s > 0$ such that $(g^s)^*|H^*(X, \mathbb{C})$ is unipotent, i.e., all eigenvalues are equal to 1.*

Proof. By Lemmas 2.2 and 2.5, every eigenvalue λ of $g^*|H^*(X,\mathbb{C})$ has modulus 1. Since g^* is defined over $\oplus_{i=0}^{2n} H^i(X,\mathbb{Z})/(\text{torsion})$ the monic characteristic polynomial of $g^*|H^*(X,\mathbb{C})$ has integer coefficients, whence all eigenvalues λ of g^* are algebraic integers. So every eigenvalue λ of g^* is an algebraic integer and all its conjugates (including itself) have modulus 1. Thus these λ are all units of 1. The lemma follows. □

The result below says that a rigidly parabolic action on an abelian variety is essentially the lifting of a translation.

Lemma 2.15. *Let $A \ne 0$ be an abelian variety and $g \in \mathrm{Aut}_{\mathrm{variety}}(A)$. Suppose that the pair (A, g) is rigidly parabolic. Then there are integers $s > 0$, $m \ge 0$ and a sequence of abelian subvarieties $B_1 \subset B_2 \cdots \subset B_m \subset A$ such that the following are true (setting $B_0 = 0$).*

(1) *The homomorphisms below are all g^s-equivariant:*

$$A \to A/B_1 \to \cdots \to A/B_m \ne 0.$$

(2) *$g^s|(A/B_m)$ is a translation of infinite order (so the pointwise fixed point set $(A/B_i)^{g^r} = \emptyset$ for all $r > 0$ and all $0 \le i \le m$).*

Proof. We may assume that $g^*|H^*(A,\mathbb{C})$ is already unipotent; see Lemma 2.14. Assume that the pointwise fixed locus $A^g \ne \emptyset$. Then we may assume that $g|A$ is a homomorphism after changing the origin. By BL [4] (13.1.2), A^g is a subgroup of positive dimension equal to that of the eigenspace of $g^*|H^{1,0}(A,\mathbb{C})$. Let B_1 be the identity component of A^g. Then $0 < \dim B_1 < \dim A$ by the parabolic rigidity of $g|A$. The homomorphism $A \to A/B_1$ is g-equivariant. But now $(A/B_1)^g$ contains the origin (= the image of B_1), and $g|(A/B_1)$ is again rigidly parabolic by the definition. The parabolic rigidity of $g|A$ helps us to continue this process forever. This contradicts the finiteness of $\dim A$.

Therefore, $A^g = \emptyset$. Write $g = t_0 h$ with a translation t_0 and a homomorphism h. Then $0 = |A^g| = |A^h|$ in the notation of BL [4] (13.1.1), whence A^h is of positive dimension equal to that of $\mathrm{Ker}(h^* - \mathrm{id}) = \mathrm{Ker}(g^* - \mathrm{id}) \subset H^{1,0}(A,\mathbb{C})$. Let B_1 be the identity component of A^h. Then $g(x + B_1) = g(x) + B_1$ and hence the homomorphism $A \to A/B_1$ is g-equivariant. If $B_1 = A$, then $h = \mathrm{id}$ and we are done. Otherwise, $g|(A/B_1)$ is again rigidly parabolic by the definition. Also $(A/B_1)^{g^r} = \emptyset$ for all $r > 0$ by the argument in the paragraph above for the early case. Continue this process and we see that $B_{m+1} = A$ for some m. So some positive power $g^s|(A/B_m)$ is a translation of infinte order. The lemma is proved. □

The density result (3) below shows that a rigidly parabolic action on an abelian variety is very ergodic.

Lemma 2.16. *Let $A \to Y$ be a g-equivariant generically finite surjective morphism from an abelian variety A onto a smooth projective manifold Y. Assume that $g|A$ is rigidly parabolic. Then we have:*

(1) *No proper subvariety of Y is stabilized by a positive power of g.*
(2) *g has no periodic points.*
(3) *For every $y_0 \in Y$, the Zariski closure $D := \overline{\{g^s(y_0) \mid s > 0\}}$ equals Y.*
(4) *Suppose that $f : X \to Y$ is a g-equivariant surjective morphism from a smooth projective manifold onto Y. Then f is a smooth morphism. In particular, if f is generically finite then it is étale.*

Proof. It suffices to show (1). Indeed, (2) is a special case of (1). If (3) is false, then some positive power g^s fixes an irreducible component of D, contradicting (1). If (4) is false, then the discriminant $D = D(X/Y)$ is stabilized by g and we get a contradiction as in (3).

For (1), since $A \to Y$ is g-equivariant, it is enough to show (1) for A; see Lemma 2.6. Suppose the contrary that a positive power g^r stabilizes a proper subvariety Z of A. To save the notation, rewrite g^r as g. If Z is a point, then $g^s|(A/B_m)$ fixes the image on A/B_m of Z in the notation of Lemma 2.15, absurd. Assume that $\dim Z > 0$. If the Kodaira dimension $\kappa(Z) = 0$, then Z is a translation of a subtorus by Ueno [47] Theorem 10.3, and we may assume that Z is already a torus after changing the origin; we let Z be the B_1 in Lemma 2.15 and then $g^s|(A/B_m)$ fixes the origin (= the image of Z), absurd.

Suppose that $\kappa(Z) \ge 1$. By Ueno [47] or Mori [35] (3.7), the identity component B_1 of $\{a \in A \mid a + Z = Z\}$ has positive dimension such that $Z \to Z/B_1 \subset A/B_1$ is birational to the Iitaka fibring with $\dim(Z/B_1) = \kappa(Z)$ and Z/B_1 of general type. We can check that $g(a + B_1) = g(a) + B_1$ (write g as the composition of a translation and a homomorphism and then argue), so the homomorphism $A \to A/B_1$ is g-equivariant and $g|(A/B_1)$ stabilizes a subvariety Z/B_1 of general type (having finite $\mathrm{Aut}(Z/B_1)$ by Iitaka [23] Theorem 11.12). Thus a positive power $g^r|(A/B_1)$ fixes every point in Z/B_1. So in Lemma 2.15, another positive power $g^s|(A/B_m)$ fixes every point in the image of Z/B_1, absurd. □

Here are two applications of Lemma 2.12 and Viehweg-Zuo [49] (0.2).

Lemma 2.17. *Let X be a smooth projective manifold of Kodaira dimension $\kappa(X) \ge 0$. Suppose that $f : X \to \mathbb{P}^1$ is a g-equivariant surjective morphism. Then $g|\mathbb{P}^1$ is periodic. In particular, (X, g) is neither rigidly parabolic nor of primitively positive entropy.*

Proof. By [49] Theorem 0.2, f has at least three singular fibres lying over a set of points of $\mathbb{P}^1$ on which $g|\mathbb{P}^1$ permutes. Thus a positive power $g^s|\mathbb{P}^1$ fixes every point in this set and hence is equal to the identity. □

Lemma 2.18. *Let $f : X \to Y$ be a g-equivariant surjective morphism from a smooth projective manifold onto a smooth projective curve. Suppose that the Kodaira dimension $\kappa(X) \ge 0$. Suppose further that the pair (X, g) is either rigidly parabolic, or of primitively positive entropy (so $\dim X \ge 2$).*

Then Y is an elliptic curve, $g^s|Y$ (for some $s|12$) is a translation of infinite order, and f is a smooth morphism.

Proof. By Lemma 2.12, the Kodaira dimension $\kappa(Y) \le 0$. So the arithmetic genus $p_a(Y) \le 1$. By Lemma 2.17, Y is an elliptic curve. So $g^s|Y$ is a translation for some $s|12$, which is of infinite order since $g|Y$ is rigidly parabolic by Lemma 2.7. In view of Lemma 2.16 the lemma follows. □

The following are sufficient conditions to have rational pencils on surfaces.

Lemma 2.19. *Let X be a smooth projective surface of Kodaira dimension $\kappa(X) \ge 0$, and let $g \in \mathrm{Aut}(X)$. Let $X \to X_m$ be the smooth blowdown to the minimal model. Then there is a g-equivariant surjective morphism $X \to \mathbb{P}^1$ such that $g|\mathbb{P}^1$ is periodic, if either Case* (1) *or* (2) *below occurs.*

1. *X_m is a hyperelliptic surface.*
2. *g is parabolic. X_m is $K3$ or Enriques.*

Proof. Since $\kappa(X) \ge 0$ in both Cases (1) and (2), the minimal model X_m of X is unique and hence $X \to X_m$ is g-equivariant. So we may assume that $X = X_m$.

There are exactly two elliptic fibrations on a hyperelliptic surface X (see Friedman-Morgan §1.1.4); one is onto an elliptic curve ($= \mathrm{Alb}(X)$) and the other is onto $\mathbb{P}^1$. Thus Lemma 2.17 implies the result in Case (1).

If X_m is $K3$, then the lemma follows from Cantat [7] (1.4).

An Enrques X can be reduced to the $K3$ case. Indeed, $g|X$ lifts to a parabolic $g|Y$ (see Lemma 2.6) on the universal $K3$ double cover Y of X so that a positive power $g^s|Y$ stabilizes every fibre of an elliptic fibration on Y. This fibration descends to one on X fibre-wise stabilized by $g^s|X$. This proves the lemma. □

Now we classify rigidly parabolic actions on surfaces.

Lemma 2.20. *Let X be a smooth projective surface and $g \in \mathrm{Aut}(X)$ such that the pair (X, g) is rigidly parabolic. Then there is a g^s-equivariant (for some $s > 0$) smooth blowdown $X \to X_m$ such that one of the following cases occurs (the description in* (3) *or* (4) *will not be used in the sequel).*

1. *X_m is an abelian surface (so $q(X) = 2$); see also* (2.16).
2. *$X_m \to E = \mathrm{Alb}(X_m)$ is an elliptic ruled surface (so $q(X) = 1$ and E an elliptic curve). $g^s|E$ is a translation of infinite order.*
3. *X_m is a rational surface such that $K_{X_m}^2 = 0$, the anti-canonical divisor $-K_{X_m}$ is nef and the anti-Kodaira dimension $\kappa(X_m, -K_{X_m}) = 0$. For a very general point $x_0 \in X_m$, the Zariski closure $D(x_0) := \overline{\{g^r(x_0) \,|\, r > 0\}}$ equals X_m.*
4. *X_m is a rational surface with $K_{X_m}^2 = 0$ and equipped with a* (*unique and relatively minimal*) *elliptic fibration $f : X_m \to \mathbb{P}^1$ such that f is g^s-equivariant.*
5. *One has $X_m = \mathbb{F}_e$ the Hirzebruch surface of degree $e \ge 0$ such that a/the ruling $\mathbb{F}_e \to \mathbb{P}^1$ is g^s-equivariant.*
6. *One has $X = X_m = \mathbb{P}^2$. So there are a g-equivariant blowup $\mathbb{F}_1 \to \mathbb{P}^2$ of a $g|\mathbb{P}^2$-fixed point and the g-equivariant ruling $\mathbb{F}_1 \to \mathbb{P}^1$.*

We now prove Lemma 2.20. By Lemma 2.12, the Kodaira dimension $\kappa(X) \le 0$. Consider first the case $\kappa(X) = 0$. Then X contains only finitely many (-1)-curves and has a unique smooth minimal model X_m. So $g|X$ descends to a biregular action $g|X_m$. Rewrite $X = X_m$. Then X is Abelian, Hyperelliptic, $K3$ or Enriques. By Lemma 2.19, X is an abelian surface.

Consider next the case where X is an irrational ruled surface. Then there is a $\mathbb{P}^1$-fibration $f : X \to E = \mathrm{Alb}(X)$ with genral fibre $X_e \cong \mathbb{P}^1$, so that $p_a(E) = q(X) \ge 1$. All rational curves (especially (-1)-curves) are contained in fibres. $g|X$ permutes finitely many such (-1)-curves. So we may assume that a positive power $g^s|X$ stabilizes every (-1)-curve and let $X \to X_m$ be the g^s-equivariant blowdown to a relatively minimal $\mathbb{P}^1$-fibration $f : X_m \to E$ where all fibres are $\mathbb{P}^1$. By the proof of Lemma 2.18 and replacing s, we may assume that E is an elliptic curve and $g^s|E$ is a translation of infinite order. So Case(2) occurs.

Consider the case where X is a rational surface. So $\mathrm{Pic}(X) = \mathrm{NS}(X)$. Assume that $g|\mathrm{Pic}(X)$ is finite, then $\mathrm{Ker}(\mathrm{Aut}(X) \to \mathrm{Aut}(\mathrm{Pic}(X)))$ is infinite. Hence X has only finitely many (-1)-curves by Harbourne [21] Proposition 1.3. As in the case above, let $X \to X_m$ be a g^s-equivariant smooth blowdown to a relatively minimal rational surface so that $X_m = \mathbb{P}^2$, or $\mathbb{F}_e$ the Hirzebruch surface of degree $e \ge 0$. Note that a/the ruling $\mathbb{F}_e \to \mathbb{P}^1$ is g^{2s}-equivariant (the $''2''$ is to take care of the case $e = 0$ where there are two rulings on $\mathbb{F}_e$). If $X_m = \mathbb{P}^2$ but $X \ne \mathbb{P}^2$, then we are reduced to the case $\mathbb{F}_1$. If $X = X_m = \mathbb{P}^2$, the last case in the lemma occurs (one trianglizes to see the fixed point).

Assume that X is rational and $g|\mathrm{Pic}(X)$ is infinite. By [51] Theorem 4.1 (or by Oguiso [**?**] Lemma 2.8 and the Riemann-Roch theorem applied to the v and the adjoint divisor $K_X + v$ there as well as Fujita's uniqueness of the Zariski-decomposition for pseudo-effective divisors like v and $K_X + v$ as formulated in Kawamata [26] Theorem 7-3-1), there is g-equivariant smooth blowdown $X \to X_m$ such that $K_{X_m}^2 = 0$, $-K_{X_m}$ is nef and $\kappa(X_m, -K_{X_m}) \ge 0$ (by the Riemann-Roch theorem). For simplicity, rewrite $X = X_m$.

If $\kappa(X, -K_X) \ge 1$, then Case(4) occurs by the claim (and the uniqueness of f) below.

Claim 2.21. Let X be a smooth projective rational surface such that $-K_X$ is nef, $K_X^2 = 0$ and $\kappa(X, -K_X) \ge 1$. Then X is equipped with a unique relatively minimal elliptic fibration $f : X \to \mathbb{P}^1$ such that $-K_X$ is a positive multiple of a fibre.

We now prove the claim. Write $|-tK_X| = |M| + Fix$ for some $t >> 0$, so that Fix is the fixed part. Note that $0 \le M^2 \le (-tK_X)^2 = 0$. Thus $M \sim rF$ (linearly equivalent) with $|F|$ a rational free pencil, noting that $q(X) = 0$. Since $K_X.F = \frac{-1}{t}(M + Fix).F \le 0$ and $F^2 = 0$, our F is a moving elliptic curve, rather than a (-2)-curve, by the genus formula. Thus $K_X.F = 0$ and $F.Fix = 0$. So $0 = (-tK_X)^2 = (M + Fix)^2 = (Fix)^2$. Hence Fix is a rational multiple of F; see Reid [44] page 36. Since $K_X^2 = 0$ and by going to a relative minimal model of the elliptic fibration and applying Kodaira's canonical divisor formula there, we see that $f := \Phi_{|F|} : X \to C$ ($\cong \mathbb{P}^1$) is already relatively minimal. The uniqueness of such f again follows from Kodaira's this formula. This proves the claim.

We return to the proof of Lemma 2.20. We still have to consider the case where X is rational, $g|\mathrm{Pic}(X)$ is infinite, $K_X^2 = 0$, $-K_X$ is nef and $\kappa(X, -K_X) = 0$. We shall show that Case(3) occurs. Take $x_0 \in X$ which does not lie on any negative curve or *the* anit-pluricanonical curve in some $|-tK_X|$ or the set $\cup_{r>0} X^{g^r}$ of g-periodic points. Suppose the contrary that the Zariski-closure $D(x_0)$ in Case(3) is not the entire X. Then $D(x_0)$ is 1-dimensional and we may assume that a positive power g^s stabilizes a curve $x_0 \in D_1$ in $D(x_0)$. By the choice of x_0, our $D_1^2 \ge 0$. If $-K_X$ is a rational multiple of D_1, then we have $\kappa(X, -K_X) \ge 1$, a contradiction. Otherwise, $H := D_1 - K_X$ is g^s-stable and $H^2 > 0$ by the Cauchy-Schwarz inequality (see the proof of [1] IV (7.2)) or the Hodge index theorem, whence g^s acts on $H^\perp := \{L \in \mathrm{Pic}(X) \,|\, L.H = 0\}$ which is a lattice with negative definite intersection form, so $g^s|H^\perp$ and hence $g|\mathrm{Pic}(X)$ is periodic, a contradiction. This proves Lemma 2.20.

The key for the 'splitting' of action below is from Lieberman [30].

Lemma 2.22. *Let X and Y be smooth projective manifolds. Suppose that the second projection $f : V = X \times Y \to Y$ is g-equivariant. Then there is an action $g|X$ such that we can write $g(x, y) = (g.x, g.y)$, with $x \in X$, $y \in Y$, if either Case* (1) *or* (2) *below occurs.*

(1) *The irregularity $q(X) = 0$, and X is non-uniruled (or non-ruled).*
(2) $\dim X = \dim Y = 1$ *and* $\mathrm{rank}\, \mathrm{NS}_{\mathbb{Q}}(V) = 2$. *(e.g. when one of X, Y is $\mathbb{P}^1$, or when X, Y are non-isogenius elliptic curves).*

Proof. As in Hanamura [20] the proof of Theorem 2.3 there, we express $g(x, y) = (\rho_g(y).x,\ g.y)$ where $\rho_g : Y \to \mathrm{Aut}(X)$ is a morphism. We consider Case (1). By [30] Theorem 3.12 and the proof of Theorem 4.9 there, the identity connected component $\mathrm{Aut}_0(X)$ of $\mathrm{Aut}(X)$ is trivial, so $\mathrm{Aut}(X)$ is discrete. Thus $\mathrm{Im}(\rho_g)$ is a single point, denoted as $g|X \in \mathrm{Aut}(X)$. The lemma is proved in this case.

For Case(2), let F be a fibre of f. Then $g^*F = F'$ (another fibre). Let L be a fibre of the projection $f_X : V \to X$. Since $\mathrm{rank}\, \mathrm{NS}_{\mathbb{Q}}(V) = 2$, we have $\mathrm{NS}_{\mathbb{Q}}(V) = \mathbb{Q}[F] + \mathbb{Q}[L]$. Write $g^*L = aL + bF$. Then $1 = F.L = g^*F.g^*L = F.g^*L = a$ and $0 = (g^*L)^2 = 2ab$ implies that $g^*L = L$ in $\mathrm{NS}_{\mathbb{Q}}(V)$. Thus $g(L)$ is a curve with $g(L).L = L^2 = 0$, whence $g(L)$ is another fibre of f_X. The result follows. This proves the lemma. □

We use Lieberman [30] Proposition 2.2 and Kodaira's lemma to deduce the result below, though it is not needed in this paper (see also Dinh-Sibony [8] the proof of Theorem 4.6 there for a certain case).

Lemma 2.23. *Let X be a smooth projective manifold of dimension n, and $H \in \mathrm{Nef}(X)$ a nef and big $\mathbb{R}$-Cartier divisor (i.e. H is nef and $H^n > 0$). Then $\mathrm{Aut}_H(X)/\mathrm{Aut}_0(X)$ is a finite group. Here $\mathrm{Aut}_0(X)$ is the identity component of $\mathrm{Aut}(X)$, $\mathrm{Aut}_H(X) := \{\sigma \in \mathrm{Aut}(X) \mid \sigma^* H = H$ in $\mathrm{NS}_{\mathbb{R}}(X)\}$.*

Proof. By Nakayama [37] II (3.16) and V (2.1), one may write $H = A + D$ in $\mathrm{NS}_{\mathbb{R}}(X)$ with A a $\mathbb{Q}$-ample divisor and D an effective $\mathbb{R}$-divisor. We follow the proof of Lieberman [30] Proposition 2.2. For $\sigma \in \mathrm{Aut}_H(X)$, the volume of the graph Γ_σ is given by:

$$\mathrm{vol}(\Gamma_\sigma) = (A+\sigma^* A)^n \le (A+\sigma^* A)^{n-1}(H+\sigma^* H) \le \cdots \le (H+\sigma^* H)^n = 2^n H^n.$$

The rest of the proof is the same as [30]. This proves the lemma. □

The two results below will be used in the proofs in the next section.

Lemma 2.24. *The following are true.*

1. *A Q-torus Y does not contain any rational curve.*
2. *Let $f : X \cdots \to Y$ be a rational map from a normal variety X with only log terminal singularities to a Q-torus Y. Then f is a well-defined morphism.*

Proof. (1) Let $T \to Y$ be a finite étale cover from a torus T. Suppose the contrary that $\mathbf{P}^1 \to Y$ is a non-constant morphism. Then $P := T \times_Y \mathbf{P}^1 \to \mathbf{P}^1$ is étale and hence P is a disjoint union of $\mathbf{P}^1$ by the simply connectedness of $\mathbf{P}^1$. So the image in T of P is a union of rational curves, contradicting the fact that a torus does not contain any rational curve.

(2) By Hironaka's resolution theorem, there is a birational proper morphism $Z \to X$ such that the composition $Z \to X \cdots \to Y$ is a well defined morphism. By Hacon-McKernan's solution to the Shokurov conjecture [19] Corollary 1.6, every fibre of $Z \to X$ is rationally chain connected and is hence mapped to a point in Y, by (1). By the normality of X and considering the Stein factorization of $Z \to Y$ or taking the normalization of Y in the function field of Z, (2) follows. □

Lemma 2.25. *Let $f : X \to Y$ be a g-equivariant surjective morphism between smooth projective manifolds and with connected general fibre F. Assume the following conditions.*

1. *All of X, Y and F are of positive dimension.*
2. *Y is a Q-torus.*
3. *The Kodaira dimension $\kappa(X) = 0$ and X has a good (terminal) minimal model $\bar{X}$, i.e., $\bar{X}$ has only terminal singularities and $sK_{\bar{X}} \sim 0$ for some $s > 0$.*

Then there is a g-equivariant finite étale Galois extension $\tilde{Y} \to Y$ from a torus $\tilde{Y}$ such that the following are true.

1. *The composition $\bar{X} \cdots \to X \to Y$ is a well defined morphism with a general fibre $\bar{F}$. One has $sK_{\bar{F}} \sim 0$, so $\bar{F}$ is a good terminal minimal model of F.*

(2) $X_1 := X \times_Y \tilde{Y}$ *is birational to* $\bar{X}_1 := \bar{F} \times \tilde{Y}$ *over* $\tilde{Y}$ *with* $sK_{\bar{X}_1} \sim 0$.

(3) *Denote by the same* G *the group* $\mathrm{Gal}(\tilde{Y}/Y)$ *and the group* $\mathrm{id}_X \times_Y \mathrm{Gal}(\tilde{Y}/Y) \le \mathrm{Aut}(X_1)$, *and by the same* g *the automorphism* $(g|X)\times_Y (g|\tilde{Y}) \in \mathrm{Aut}(X_1)$. *Then* $g = g|X_1$ *normalizes* $G = G|X_1$.
In the assertions $(4) - (7)$ *below, suppose further that* $q(F) = 0$.

(4) $g|X_1 \in \mathrm{Aut}(X_1)$ *induces a birational action* g *on* $\bar{X}_1$ *such that* $g|\bar{X}_1 = (g|\bar{F}) \times (g|\tilde{Y})$, *where* $g|\bar{F} \in \mathrm{Bir}(\bar{F})$ *and* $g|\tilde{Y} \in \mathrm{Aut}(\tilde{Y})$.

(5) *Suppose that* $1 \le \dim F \le 2$. *Then* $\dim F = 2$ *and* F *is birational to a smooth minimal surface* $\hat{F}$ *which is either a* $K3$ *or an Enriques. Further, the induced birational action of* $g = (g|\hat{F}) \times (g|\tilde{Y})$ *on* $\hat{X}_1 := \hat{F} \times \tilde{Y}$ *is regular, i.e.,* $g|\hat{F} \in \mathrm{Aut}(\hat{F})$.

(6) *In* (5), $G = G|X_1 \le \mathrm{Aut}(X_1)$ *induces a biregular action by* $G = G|\hat{X}_1 \le G_{\hat{F}} \times G_{\tilde{Y}}$ *on* $\hat{X}_1$ *with* $G_{\hat{F}} \le \mathrm{Aut}(\hat{F})$ *and* $G_{\tilde{Y}} = \mathrm{Gal}(\tilde{Y}/Y)$.

(7) $g|X$ *is neither rigidly parabolic nor of primitively positive entropy.*

Proof. (1) follows from Lemma 2.24 and the fact that $K_{\bar{F}} = K_{\bar{X}}|\bar{F}$. (2) is proved in Nakayama [36] Theorem at page 427. Indeed, for the g-equivalence of $\tilde{Y} \to Y$, by [36], (2) is true with an étale extension $Y' \to Y$. Let $T \to Y$ be an étale cover of a torus T of minimal degree. Then $g|Y$ lifts to $g|T$ as in Beauville [2] §3. Now the projection $T' := T \times_Y Y' \to T$ is étale. So there is another étale cover $T'' \to T'$ such that the composition $T'' \to T' \to T$ is just the multiplicative map $m_{T''}$ for some $m > 0$. In particular, $T = m_{T''}(T'')$ is isomorphic to T''. Clearly, the natural action $g|T''$ is compactible with the action $g|T$ via the map $m_{T''}$. Now the composition $\tilde{Y} := T'' \to Y$ is g-equivariant and factors through $Y' \to Y$, so that (2) is satisfied. (3) is true because $g|X_1$ is the lifting of the action g on $X = X_1/G$.

We now assume $q(F) = 0$. Assume that a group $\langle h \rangle$ acts on both X_1 and $\tilde{Y}$ compactibly with the cartesian projection $X_1 \to \tilde{Y}$. For instance, we may take $\langle h \rangle$ to be a subgroup of $G|X_1$ or $\langle g|X_1 \rangle$. This h acts birationally on $\bar{X}_1$. To be precise, for $(x, y) \in \bar{X}_1$, we have $h.(x, y) = (\rho_h(y).x,\ h.y)$, where $\rho_h : \tilde{Y} \cdots \to \mathrm{Bir}(\bar{F})$ is a rational map. By Hanamura [20] (3.3), (3.10) and page 135, $\mathrm{Bir}(\bar{F})$ is a disjoint union of abelian varieties of dimension equal to $q(\bar{F}) = q(F) = 0$ (the first equality is true because the singularities of $\bar{F}$ are canonical and hence rational). Thus $\mathrm{Im}(\rho_h)$ is a single element and denoted as $h|\bar{F} \in \mathrm{Bir}(\bar{F})$. So $h|\bar{X}_1 = (h|\bar{F}) \times (h|\tilde{Y})$.

(4) follows by applying the arguments above to $h = g$. For (5), suppose $\dim F = 1, 2$. Note that $\kappa(F) = 0 = q(F)$. So F is birational to $\hat{F}$, a $K3$ or an Enriques, by the classification theory of surfaces. Now (5) follows from the fact that $\mathrm{Bir}(S) = \mathrm{Aut}(S)$ for smooth minimal surface S, by the uniqueness of surface minimal model. The argument in the preceding paragraph also shows $G|\hat{X}_1 \le G_{\hat{F}} \times G_{\tilde{Y}}$ with $G_{\hat{F}} \le \mathrm{Bir}(\hat{F}) = \mathrm{Aut}(\hat{F})$ and $G_{\tilde{Y}} = \mathrm{Gal}(\tilde{Y}/Y) \le \mathrm{Aut}(\tilde{Y})$. This proves (6).

(7) Set $X' := \hat{X}_1/G$. Then g acts on X' biregularly such that the pairs (X', g) and (X, g) are birationally equivalent, $g|X_1$ being the lifting of $g|X$

and X_1 being birational to $\hat{X}_1$. The projections $X' = \hat{X}_1/G \to \hat{F}/G$ and $X' \to \tilde{Y}/G = Y$ are g-equivariant, since g normalizes G.

Suppose the contrary that $g|X$ is either rigidly parabolic, or of primitively positive entropy. Then both $g|(\hat{F}/G)$ and $g|Y$ are rigidly parabolic by Lemma 2.7 (applied to g-equivariant resolutions of both the source and targets of the projections). In particular, $g|\hat{F}$ is parabolic by Lemma 2.6 (applied to g-equivariant resolutions of the source and target of $\hat{F} \to \hat{F}/G$).

By Lemma 2.19, there is a g-equivariant surjective morphism $\hat{F} \to \mathbb{P}^1$ (with fibre $\hat{F}_p$) such that a positive power $g^s|\mathbb{P}^1 = \mathrm{id}$. Rewrite g^s as g. Pushing down the rational free pencil $\{F_p \times \tilde{Y} \mid p \in \mathbb{P}^1\}$ by the finite map $\pi : \hat{X}_1 \to X'$, we get a rational pencil $\Lambda := \{\pi_*(F_p \times \tilde{Y}) \mid p \in \mathbb{P}^1\}$ on X'; see Fulton [13] Theorem 1.4. Resolving the base locus of Λ and replacing X' by a g-equivariant blowup X'' as in Hironaka [22], we get a g-equivariant surjective morphism $X'' \to C \cong \mathbb{P}^1$ with $g|C = \mathrm{id}$. So (X'', g) (and (X, g)) are neither rigidly parabolic nor of primitively positive entropy. This is a contradiction. This proves (7) and also the lemma. □

3. Results in arbitrary dimension; the proofs

The results in Introduction follow from Theorem 1.1 and three general results below in dimension ≥ 3.

In the case of dimension ≤ 3, the good (terminal) minimal model program (as in Kawamata [25], or Mori [35] §7) has been completed. So in view of Theorems 1.1, 3.2 and 3.3, we are able to describe the dynamics of (X, g) in (3.5) $\sim$ (3.6). See also Remark 3.4.

The result below is parallel to the conjecture (resp. theorem) of Demailly - Peternell - Schneider (resp. Qi Zhang) to the effect that the Albanese map $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$ is surjective whenever X is a compact Kähler (resp. projective) manifold with $-K_X$ nef (and hence $\kappa(X) = -\infty$).

Theorem 3.1. *Let X be a smooth projective complex manifold of $\dim X \geq 1$, and $g \in \mathrm{Aut}(X)$. Suppose that the pair (X, g) is either rigidly parabolic or of primitively positive entropy (see (2.1)). Then we have:*

1. *The albanese map $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$ is a g-equivariant surjective morphism with connected fibres.*
2. *The irregularity $q(X)$ satisfies $q(X) \leq \dim X$.*
3. *$q(X) = \dim X$ holds if and only if X is g-equivariantly birational to an abelian variety.*
4. *$\mathrm{alb}_X : X \to \mathrm{Alb}(X)$ is a smooth morphism if $q(X) < \dim X$; see also (2.16), (2.7).*

Theorem 3.2. *Let X be a smooth projective complex manifold of dimension $n \geq 3$, with $g \in \mathrm{Aut}(X)$. Assume the following conditions.*

1. *The Kodaira dimension $\kappa(X) = 0$ and the irregularity $q(X) > 0$.*
2. *The pair (X, g) is either rigidly parabolic or of primitively positive entropy (see (2.1)).*

(3) *X has a good terminal minimal model* (*so* (3) *is automatic if* $n \le 3$)*; see* (2.25)*,* (3.4)*,* [25] *p.4,* [35] §7.

Then Case (1) *or* (2) *below occurs.*

(1) *There are a g-equivariantly birational morphism $X \to X'$, a pair $(\tilde{X}', g)$ of a torus $\tilde{X}'$ and $g \in \mathrm{Aut}(\tilde{X}')$, and a g-equivariant étale Galois cover $\tilde{X}' \to X'$. In particular, X' is a Q-torus.*

(2) *There are a g-equivariant étale Galois cover $\tilde{X} \to X$, a Calabi-Yau variety F with $\dim F \ge 3$* (*see* (2.1)) *and a birational map $\tilde{X} \cdots \to F \times \tilde{A}$ over $\tilde{A} := \mathrm{Alb}(\tilde{X})$. Further, the biregular action $g|\tilde{X}$ is conjugate to a birational action $(g|F)\times(g|\tilde{A})$ on $F\times\tilde{A}$, where $g|F \in \mathrm{Bir}(F)$ with the first dynamical degree $d_1(g|F) = d_1(g|X)$, where $g|\tilde{A} \in \mathrm{Aut}(\tilde{A})$ is parabolic. In particular, $\dim X \ge \dim F + q(X) \ge 4$.*

Theorem 3.3. *Let X' be a smooth projective complex manifold of dimension $n \ge 3$, with $g \in \mathrm{Aut}(X')$. Assume the following conditions* (*see* (3.4))*.*

(1) *The Kodaira dimension $\kappa(X') = -\infty$.*

(2) *The pair (X', g) is either rigidly parabolic or of primitively positive entropy* (*see* (2.1))*.*

(3) *The good terminal minimal model program is completed for varieties of dimension $\le n$* (*so* (3) *is automatic if* $n \le 3$)*; see* [25] *p.4,* [35] §7.

Then there is a g-equivariant birational morphism $X \to X'$ from a smooth projective manifold X such that one of the cases below occurs.

(1) *X is a rationally connected manifold in the sense of* [5] *and* [28]*.*

(2) *$q(X) = 0$. The maximal rational connected fibration $\mathrm{MRC}_X : X \to Z$ in the sense of* [*ibid*] *is a well defined g-equivariant surjective morphism. Z is a weak Calabi-Yau manifold with $\dim X > \dim Z \ge 3$.*

(3) *$q(X) > 0$. There is a g-equivariant étale cover $\tilde{X} \to X$ such that the surjective g-equivariant albanese map $\mathrm{alb}_{\tilde{X}} : \tilde{X} \to \mathrm{Alb}(\tilde{X})$ coincides with the maximal rationally connected fibration $\mathrm{MRC}_{\tilde{X}}$.*

(4) *$q(X) > 0$. There is a g-equivariant étale Galois cover $\tilde{X} \to X$ such that the surjective albanese map $\mathrm{alb}_{\tilde{X}} : \tilde{X} \to \tilde{A} := \mathrm{Alb}(\tilde{X})$ factors as the g-equivariant $\mathrm{MRC}_{\tilde{X}} : \tilde{X} \to \tilde{Z}$ and $\mathrm{alb}_{\tilde{Z}} : \tilde{Z} \to \mathrm{Alb}(\tilde{Z}) = \tilde{A}$. Further, there are a Calabi-Yau variety F with $\dim F \ge 3$* (*see* (2.1))*, and a birational morphism $\tilde{Z} \to F \times \tilde{A}$ over $\tilde{A}$, such that the biregular action $g|\tilde{Z}$ is conjugate to a birational action $(g|F) \times (g|\tilde{A})$ on $F \times \tilde{A}$, where $g|F \in \mathrm{Bir}(F)$, where $g|\tilde{A} \in \mathrm{Aut}(\tilde{A})$ is parabolic. Also $\dim X > \dim F + q(X) \ge 4$.*

Remark 3.4.

(a) By the proof, the condition (3) in Theorem 3.3 can be weakened to:

(3)' X' is uniruled. For every projective variety Z dominated by a proper subvariety ($\ne X'$) of X', if the Kodaira dimension $\kappa(Z) = -\infty$ then Z is uniruled, and if $\kappa(Z) = 0$ then Z has a good terminal minimal model Z_m (i.e., Z_m has only terminal singularities and $sK_{Z_m} \sim 0$ for some $s > 0$); see [25] p.4, and [35] §7.

In dimension three, the good terminal minimal model program has been completed; see [26] and [29]. For the recent break through on the minimal model program in arbitrary dimension, we refer to Birkar - Cascini - Hacon - McKernan [3].
(b) The good minimal model program also implies the equivalence of the Kodaira dimension $\kappa(X) = -\infty$ and the uniruledness of X. It is known that the uniruledness of X always implies $\kappa(X) = -\infty$ in any dimension.
(c) The birational automorphisms $g|F$ in Theorems 3.2 and 3.3 are indeed isomorphisms in codimenion 1; see [20] (3.4).

As consequences of Theorems 3.2 and 3.3 for all dimension ≥ 3 and as illustrations, we have the simple 3-dimensional formulations of them as in (3.5) $\sim$ (3.6) below.

The result below says that the dynamics on an irregular threefold of Kodaira dimension 0, are essentially the dynamics of a torus.

Corollary 3.5. *Let X' be a smooth projective complex threefold, with $g \in \mathrm{Aut}(X')$. Assume that the Kodaira dimension $\kappa(X') = 0$, irregularity $q(X') > 0$, and the pair (X', g) is either rigidly parabolic or of primitively positive entropy; see* (2.1).

Then there are a g-equivariant birational morphism $X' \to X$, a pair $(\tilde{X}, g)$ of a torus $\tilde{X}$ and $g \in \mathrm{Aut}(\tilde{X})$, and a g-equivariant étale Galois cover $\tilde{X} \to X$. In particular, X is a Q-torus.

The result below shows that the dynamics on a threefold of Kodaira dimension $-\infty$ are (or are built up from) the dynamics on a rationally connected threefold (or on a rational surface and that on a 1-torus).

Theorem 3.6. *Let X be a smooth projective complex threefold, with $g \in \mathrm{Aut}(X)$. Assume that $\kappa(X) = -\infty$, and the pair (X, g) is either rigidly parabolic or of primitively positive entropy* (*see* (2.1)). *Then we have:*

(1) *If $q(X) = 0$ then X is rationally connected in the sense of* [5] *or* [28].
(2) *Suppose that $q(X) \geq 1$ and the pair (X, g) is of primitively positive entropy. Then $q(X) = 1$ and the albanese map $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$ is a smooth morphism with every fibre F a smooth projective rational surface of Picard number* $\mathrm{rank}\,\mathrm{Pic}(F) \geq 11$.

3.7. Proof of Theorem 1.1.

The assertion (1) follows from Lemma 2.12. For (2), in view of (1), we may assume that (X, g) is of imprimitively positive entropy. Then the assertion (2) follows from Lemma 2.10. This proves Theorem 1.1.

3.8. Proof of Theorem 3.1.

We may assume that $q(X) > 0$. By the universal property of $A := \mathrm{Alb}(X)$, the albanese map $\mathrm{alb}_X : X \to A$ is always equivariant with respect to the actions of g as automorphisms of varieties. By Lemma 2.12, $\kappa(\mathrm{alb}_X(X)) \le 0$. Thus, by Ueno [47] Lemma 10.1, $\kappa(\mathrm{alb}_X(X)) = 0$ and $\mathrm{alb}_X(X) = A = \mathrm{Alb}(X)$, i.e., alb_X is surjective. Let $X \to X_0 \to A$ be the Stein factorization with $f : X_0 \to A$ a finite surjective morphism from a normal variety X_0, and $X \to X_0$ having connected fibres. Note that $\kappa(X_0) \ge 0$ by the ramification divisor formula for (the resolution of the domain of) $X_0 \to A$ as in Iitaka [23] Theorem 5.5. So by Lemma 2.12, $\kappa(X_0) = 0$. By the result of Kawamata-Viehweg as in Kawamata [24] Theorem 4, $X_0 \to A$ is étale, so X_0 is an abelian variety too. By the universal property of $A = \mathrm{Alb}(X)$, we have $X_0 = A$. Thus, $f : X \to A = X_0$ has connected fibres. Theorem 3.1 (1) is proved. Now Theorem 3.1 (2) and (3) follow from (1). If $q(X) < \dim X$ then $g|A$ is rigidly parabolic by Lemma 2.7; so Theorem 3.1 (4) follows from Lemma 2.16. This proves Theorem 3.1.

3.9. Albanese variety.

For a projective variety Z, we denote by $A(Z)$ or $\mathrm{Alb}(Z)$ the albanese variety $\mathrm{Alb}(Z')$ with $Z' \to Z$ a proper resolution. By Lemma 2.24, this definition is independent of the choice of Z', and $A(Z)$ depends only on the birational equivalence class of Z. Further, if Z is log terminal, then the composition $Z \cdots \to Z' \to A(Z)$ is a well defined morphism.

3.10. Proof of Theorem 3.2.

By Theorem 3.1, we may assume that $q(X) < \dim X$, so $g|A(X)$ is rigidly parabolic by Lemma 2.7. The albanese map $\mathrm{alb}_X : X \to A(X)$ has connected fibre F_1 and is smooth and surjective; see Theorem 3.1 and also [24] Theorem 1.

By the assumption, X has a good terminal minimal model $\bar{X}$ with $sK_{\bar{X}} \sim 0$ for some $s > 0$. We apply Lemma 2.25 to $\mathrm{alb}_X : X \to Y_1 := A(X)$. Then there is a g-equivariant étale Galois extension $\tilde{Y}_1 \to Y_1$ from a torus $\tilde{Y}_1$, such that $X_1 := X \times_{Y_1} \tilde{Y}_1$ is birational to $\bar{X}_1 := \bar{F}_1 \times \tilde{Y}_1$ with $\bar{F}_1$ a good terminal minimal model of F_1 and $sK_{\bar{F}_1} \sim 0$. Also $g|X_1$ normalizes $G_1|X_1$ ($\cong \mathrm{Gal}(\tilde{Y}_1/Y_1)$).

Assume that $0 < q(F_1) < \dim F_1$. By [24] Theorem 1, $\mathrm{alb}_{X_1} : X_1 \to A(X_1)$ is a surjective morphism with connected smooth general fibre F_2. By the universal property of the albanese map, $\mathrm{alb}_{X_1} : X_1 \to A(X_1)$ is $\langle g, G_1 \rangle$-equivariant. Both of the natural morphisms $X = X_1/G_1 \to Y_2 := A(X_1)/G_1$ and $A(X_1) \to Y_2$ are g-equivariant and surjective. Since G_1 acts freely on $\tilde{Y}_1$ and $A(X_1) = A(\bar{X}_1) = A(F_1) \times \tilde{Y}_1$, the latter map is étale. By the same reason, every fibre of $X \to Y_2$ can be identified with a fibre F_2, so it is connected.

We apply Lemma 2.25 to $X \to Y_2$. Then there is a g-equivariant étale Galois extension $\tilde{Y}_2 \to Y_2$ from a torus $\tilde{Y}_2$, such that $X_2 := X \times_{Y_2} \tilde{Y}_2$ is birational to $\bar{X}_2 := \bar{F}_2 \times \tilde{Y}_2$ with $\bar{F}_2$ a good terminal minimal model of F_2 and $sK_{\bar{F}_2} \sim 0$. Also $g|X_2$ normalizes $G_2|X_2$ ($\cong \mathrm{Gal}(\tilde{Y}_2/Y_2)$).

If $0 < q(F_2) < \dim F_2$, we can consider $X \to Y_3 := A(X_2)/G_2$. Continue this process, we can define $X \to Y_{i+1} := A(X_i)/G_i$ with $G_i \cong \mathrm{Gal}(\tilde{Y}_i/Y_i)$ the Galois group of the étale Galois extension $\tilde{Y}_i \to Y_i$ from a torus $\tilde{Y}_i$, such that $X_i := X \times_{Y_i} \tilde{Y}_i$ is birational to $\bar{X}_i := \bar{F}_i \times \tilde{Y}_i$ with $sK_{\bar{X}_i} \sim 0$, where $\bar{F}_i$ a good terminal model of a general fibre F_i of $X \to Y_i$ (and also of $X_i \to \tilde{Y}_i$ and $X_{i-1} \to A(X_{i-1})$).

Note that $q(F_i) \le \dim F_i$ because $\kappa(F_i) = \kappa(\bar{F}_i) = 0$ (see [24] Theorem 1). Also $\dim X \ge \dim Y_{i+1} = \dim Y_i + q(F_i)$. So there is an $m \ge 1$ such that $q(F_m)$ equals either 0 or $\dim F_m$.

Consider the case where $q(F_m) = 0$ and $\dim F_m > 0$. Then $\mathrm{Alb}(X_m) = \mathrm{Alb}(\bar{X}_m) = \tilde{Y}_m$, and by Lemma 2.25, Theorem 3.2 Case(2) occurs with $\tilde{X} = X_m$, $F = \bar{F}_m$ and $T = \tilde{Y}_m$. Indeed, for the second part of Theorem 3.2 (2), since $X \to Y_m$ is g-equivariant, $g|Y_m$ is rigidly parabolic by Lemma 2.7 and hence $g|T$ is parabolic by Lemma 2.6 (with $d_1(g|T) = 1$). The first dynamical degrees satisfy $d_1(g|X) = d_1(g|\tilde{X}) = d_1((g|F) \times (g|T)) = d_1(g|F)$ by Lemma 2.6, Guedj [17] Proposition 1.2 and the Kunneth formula for H^2 as in Griffiths-Harris [15] page 58. Also $\dim X = \dim F + \dim \tilde{Y}_m \ge \dim F + \dim Y_1 = \dim F + q(X)$.

Consider the case $(q(\bar{F}_m) =)$ $q(F_m) = \dim F_m$. Then $q(X_m) = \dim X_m$. By Kawamata [24] Theorem 1, the albanese map $X_m \to \tilde{X}' := A(X_m) = A(F_m) \times \tilde{Y}_m$ is a $\langle g, G_m \rangle$-equivariant birational surjective morphism. It induces a g-equivariant birational morphism $X = X_m/G_m \to X' := \tilde{X}'/G_m$, since g normalizes G_m as in Lemma 2.25. Also G_m acts freely on $\tilde{Y}_m$, and hence the quotient map $\tilde{X}' \to X'$ is étale. Note that $X' \times_{Y_m} \tilde{Y}_m \cong \tilde{X}'$ over $\tilde{Y}_m$, since both sides are finite (étale) over X' and birational to each other. Thus Case(1) of Theorem 3.2 occurs with the étale Galois cover $\tilde{X}' \to X'$. This proves Theorem 3.2.

3.11. Proof of Theorem 3.3.

Let $\mathrm{MRC}_{X'} : X' \cdots \to Z$ be a maximal rationally connected fibration; see [5], or [27] IV Theorem 5.2. The construction there, is in terms of an equivalence relation, which is preserved by $g|X'$. So we can replace (X', g) by a g-equivariant blowup (X, g) such that $\mathrm{MRC}_X : X \to Z$ is a well defined surjective morphism with general fibre rationally connected, $g|X \in \mathrm{Aut}(X)$, and X, Z smooth projective manifolds; see Hironaka [22]. Further, Z is non-uniruled by Graber-Harris-Starr [14] (1.4). The natural homomorphism $\pi_1(X) \to \pi_1(Z)$ is an isomorphism; see Campana [5] or Kollár [27]. So $q(X) = q(Z)$. If $\dim Z = 0$, then Case(1) of the theorem occurs.

Consider the case $\dim Z > 0$. Since X' is uniruled by the assumptions of the theorem (see Remark 3.4), $\dim Z < \dim X'$. Since Z is non-uniruled, we have $\kappa(Z) \ge 0$ by the assumption. So $\kappa(Z) = 0$ by Lemma 2.12.

Now $g|Z$ is rigidly parabolic by Lemma 2.7. If $q(Z) = 0$ then $\dim Z \ge 3$ because $\kappa(Z) = 0$ and by Lemma 2.20. So Case(2) of the theorem occurs.

Suppose $q(Z) > 0$. Since an abelian variety contains no rational curves, $\mathrm{alb}_X : X \to A := \mathrm{Alb}(X)$ factors as $\mathrm{MRC}_X : X \to Z$ and $\mathrm{alb}_Z : Z \to \mathrm{Alb}(Z) = A$; see Lemma 2.24 and [24] Lemma 14. By Lemma 2.7, $g|A$ is rigidly parabolic. Also alb_X and alb_Z are smooth and surjective with connected fibres by Theorem 3.1 and Lemma 2.16.

We apply Theorem 3.2 to (Z, g), so two cases there occur; in the first case there, we may assume that K_Z is torsion after replacing Z by its g-equivariant blowdown. Let $\tilde{Z} \to Z$ be the g-equivariant étale Galois extension as there. So either $\tilde{Z}$ is a torus, or $\tilde{Z} \to F \times \tilde{A}$ is a well defined birational morphism over $\tilde{A} := \mathrm{Alb}(\tilde{Z})$ (after replacing Z and X by their g-equivariant blowups), with $q(F) = 0$ etc as described there. Set $\tilde{X} := X \times_Z \tilde{Z}$. Then the projection $\tilde{X} \to \tilde{Z}$ coincides with $\mathrm{MRC}_{\tilde{X}}$. So $\mathrm{alb}_{\tilde{X}} : \tilde{X} \to \mathrm{Alb}(\tilde{X}) = \tilde{A}$ factors as $\tilde{X} \to \tilde{Z}$ and $\mathrm{alb}_{\tilde{Z}} : \tilde{Z} \to \tilde{A}$. If $\tilde{Z}$ is a torus, then Case(3) of Theorem 3.3 occurs. In the situation $\tilde{Z} \to F \times \tilde{A}$, Case(4) of Theorem 3.3 occurs in view of Theorem 3.2. This proves Theorem 3.3.

3.12. Proofs of Theorems 1.2 $\sim$ 1.5 and 3.6, and Corollary 3.5.

Corollary 3.5 and Theorem 3.6 (1) follow respectively from Theorems 3.2 and 3.3, while Theorem 1.2 follows from Lemma 2.10, Corollary 3.5, Theorem 3.6, and Lemma 2.8 applied to alb_X. Theorem 1.3 follows from Theorem 1.2 (and its proof). Theorem 1.5 follows from Lemma 2.10, Corollary 3.5, Theorem 3.6, the proof of Lemma 2.8, and Lemma 2.6. See Remark 3.4.

For Theorem 1.4, by Theorems 1.1 and 3.6, we have only to consider the case in Theorem 3.6 (2). But then $\pi_1(X) = \pi_1(\mathrm{Alb}(X)) = \mathbb{Z}^{\oplus 2}$ since general (indeed all) fibres of $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$ are smooth projective rational surfaces (see [5] or [28]).

We now prove Theorem 3.6 (2) directly. We follow the proof of Theorem 3.3. Let $\mathrm{MRC}_X : X \cdots \to Y$ be a maximal rationally connected fibration, where $\kappa(Y) \ge 0$. Replacing Y by a g-equivariant modification, we may assume that Y is smooth and minimal. Since $\kappa(X) = -\infty$, our X is uniruled (see Remark 3.4). So $\dim Y < \dim X$. Our $\mathrm{alb}_X : X \to A := \mathrm{Alb}(X)$ is smooth and surjective (with connected fibre) and factors as $\mathrm{MRC}_X : X \cdots \to Y$ and $\mathrm{alb}_Y : Y \to \mathrm{Alb}(Y) = A$; also $3 = \dim X > \dim Y \ge q(Y) = q(X) > 0$ (by the assumption of Theorem 3.6 (2)), $\kappa(Y) = 0$ and $g|Y$ and $g|A$ are rigidly parabolic; see the proof of Theorem 3.3. Theorem 3.6 (2) follows from the two claims below.

Claim 3.13. In Theorem 3.6 (2), $q(X) = 1$.

Proof. Suppose the contrary that $q(X) \ge 2$. Then $\dim Y = q(Y) = q(X) = 2$, so $Y = A$ and $\mathrm{alb}_X = \mathrm{MRC}_X$, by Theorem 3.1, Lemma 2.16, and the proof of Theorem 3.3. By Theorem 3.1, $\mathrm{alb}_X : X \to A$ is surjective with every fibre F a smooth projective curve. F is a rational curve by the definition of $\mathrm{MRC}_X : X \to Y = A$. Take a nef $L = v_{(g|X)} \in \mathrm{Nef}(X)$ as in Lemma 2.4

such that $g^*L = d_1L$ with $d_1 = d_1(g|X) > 1$. By Lemma 2.15, either $g^s|A$ is a translation and we let C be a very ample divisor on A, or $g^s(C) = C + t_0$ for an elliptic curve C and $t_0 \in A$. Rewrite g^s as g and we always have $g^*C = C$ in $N_1(A) = \mathrm{NS}_{\mathbb{R}}(A)$.

Let $X_C \subset X$ be the inverse of C. Then the restriction $\mathrm{alb}_X|X_C : X_C \to C$ is a ruling. Note that $g^*F = F$ in $N_1(X)$. Also $g^* = \mathrm{id}$ on $H^6(X, \mathbb{R}) \cong \mathbb{R}$. So $L.F = g^*L.g^*F = d_1L.F$, whence $L.F = 0$. By the adjunction formula, $K_{X_C} = (K_X + X_C)|X_C = K_X|X_C + eF$ with the scalar $e = C^2 \geq 0$. Note that $\mathbb{R} \cong H^6(X, \mathbb{R}) \ni L.K_X.X_C = g^*L.g^*K_X.g^*X_C = d_1L.K_X.X_C$, whence $0 = L.K_X.X_C = (L|X_C).(K_X|X_C) = (L|X_C).K_{X_C}$ since $L.F = 0$. Now $(L|X_C).F = L.F = 0$ and $(L|X_C).K_{X_C} = 0$ imply that $L.X_C = L|X_C = 0$ in the lattice $\mathrm{NS}_{\mathbb{R}}(X_C)$ because the ruling F and K_{X_C} span this lattice. Hence X_C and L are proportional in $\mathrm{NS}_{\mathbb{R}}(X)$ by Lemma 2.3 (1), noting that any curve like C in the abelian surface A is nef and hence X_C is nef. But $g^*X_C = X_C$ while $g^*L = d_1L$ with $d_1 > 1$, so X_C and L are not proportional in $\mathrm{NS}_{\mathbb{R}}(X)$. So the claim is true. □

Claim 3.14. In Theorem 3.6 (2), every fibre X_a of $\mathrm{alb}_X : X \to A = \mathrm{Alb}(X)$ is a smooth projective rational surface (so $\mathrm{NS}(X) = \mathrm{Pic}(X)$) with non-big $-K_{X_a}$. Further, $\operatorname{rank}\mathrm{Pic}(X_a) \geq 11$.

Proof. If $\dim Y > \dim A = q(X) = 1$, then Y is a surface with $\kappa(Y) = 0$, $q(Y) = 1$ and $g|Y$ rigidly parabolic, which contradicts Lemma 2.20. Thus $\dim Y = 1 = q(Y)$, so $Y = \mathrm{Alb}(Y) = A$ and $\mathrm{alb}_X = \mathrm{MRC}_X$ by Theorem 3.1 and Lemma 2.16. Therefore, every fibre $F = X_a$ of $X \to A$ is a smooth projective rational surface; see Kollár [27] IV Theorem 3.11.

Assume the contrary that $-K_F$ is big or $\operatorname{rank}\mathrm{Pic}(F) \leq 10$ (i.e., $K_F^2 \geq 0$) and we shall derive a contradiction. If $K_F^2 \geq 1$, then $-K_F$ is big by the Riemann-Roch theorem applied to $-nK_F$. Thus we may assume that either $-K_F$ is big or $K_F^2 = 0$ and shall get a contradiction.

As in the proof of the previous claim, for $K_F = K_X|F$ and $L := v_{(g|X)}$, we have $0 = L.K_X.F = (L|F).K_F$, so $L|F \equiv cK_F = cK_X|F$ for some scalar c by the Hodge index theory; see the proof of [1] IV (7.2). If $c \neq 0$, applying g^*, we get $d_1(g) = 1$, absurd. Hence $c = 0$ and $L.F = 0$. Then $L \equiv eF$ for some scalar $e > 0$ by Lemma 2.3. Applying g^*, we get the same contradiction. This proves the claim and also Theorem 3.6. □

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