CHARACTERISTIC CLASSES AND SPECTRAL SEQUENCES OF ATIYAH-HIRZEBRUCH TYPE

(Preliminary version)

by

Ch.B.Thomas

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Straße 26
5300 Bonn 3

West Germany

MPI/86-52

CHARACTERISTIC CLASSES AND SPECTRAL SEQUENCES OF ATIYAH-HIRZEBRUCH TYPE

(Preliminary version)

by

Ch.B.Thomas

This paper develops some of the ideas in my survey article for the First Singapore Topology Conference, June 1985, which has been available for some time as a preprint (ETH-Zürich). The aim is to rephrase Atiyah's famous result on the topological K-theory of the classifying space of a finite group G in the language of algebraic K-theory, and to prove it for as general a family of coefficient rings as possible. The role of characteristic classes is to suggest the correct form of the spectral sequence linking group cohomology and the K-theory of the classifying space in general, and to help with calculations in the few cases where enough is known about $K_*(A)$ to make these worthwhile. The best result which we obtain is Theorem 13 below which states that there is a fourth quadrant spectral sequence

$$E_2^{r,-s} = H^r_{\text{conts}}(\text{Spec } \mathcal{O}[\tfrac{1}{\ell}],\ G_\ell\ ;\ \mathbf{Z}_\ell^{\otimes(\frac{s}{2})}) \Rightarrow \hat{R}\mathcal{O}[\tfrac{1}{\ell}](G_\ell) \quad ,$$

where G_ℓ is a group of ℓ-primary order and $\mathcal{O}$ is the ring of integers in the ℓ th. cyclotomic extension of $\mathbb{Q}$ (ℓ = odd and regular). Completion is to be understand in the ℓ-adic sense, but is equivalent to completion with respect to powers of the augmentation ideal, if we restrict attention to groups of prime power order. We wish to emphasise that the results we obtain are certainly not the best possible, and that in particular the proofs in section three below on extensions of rings of algebraic integers will need to be expanded in a later version. There is also considerable overlap with the work of R.Thomason (see numerous preprints), although we have chosen to work with êtale K-theory rather than with algebraic K-theory with the Bott element inverted.

1. General Framework

Let A be a commutative ring with 1, satisfying the following conditions:

(i) $K_0(A) \cong \mathbb{Z}$,

(ii) The algebraic K-groups $K_s(A)$ are finitely generated groups for all $s > 0$. Examples of such rings are the finite fields $\mathbb{F}_q (q=p^t)$ and rings of algebraic integers $\mathcal{O}$ in a number field F . For technical reasons it will be necessary to enlarge $\mathcal{O}$ to a ring of S-integers by inverting some prime ℓ - this does not affect the finite generation of the K-groups, see [D-F1]. Let $BGL(A)^+$ be the classifying space for algebraic K-theory obtained by adding 2- and 3-cells to some classifying space (unique up to homotopy) for the discrete group $GL(A) = \bigcup_{n=1}^{\infty} GL(n,A)$. Consider the negatively graded cohomology theory given on the category of countable CW-complexes with finite skeleta in each dimension by

$$KA^s(X) = [S^{-s}X, \mathbb{Z}\times BGL(A)^+], \; s \leq 0 .$$

The coefficients of this theory are given by

$$KA^s(\text{point}) = \begin{cases} K_{-s}(A), & s < 0 \\ \mathbb{Z}, & s = 0 \\ 0, & s > 0 . \end{cases}$$

If $X = \bigcup_{k=0}^{\infty} X^k$, where X^k equals the finite k-skeleton of X , we wish to restrict attention to those spaces X such that the

Atiyah-Hirzebruch spectral sequence with

$$E_2^{r,s} = H^r(X, KA^s(\text{pt.})) \quad \text{converges}$$

to a graded group associated to $\varprojlim_k KA^{r+s}(X^k)$, and the short exact sequence

$$0 \to R^1 \varprojlim_k K^{s-1}(X^k) \to K^s(X) \to \varprojlim_k K^s(X^k) \to 0$$

has vanishing left hand term. For example let $X = BG$, when G is a finite group. Since $K_s(A)$ is a finitely generated abelian group $H^r(BG, K_s(A))$ is finite for all values of r and s , so that the inverse systems $\{H^r(BG^k, K_s(A))\}_k$ and $\{K^s(BG^k)\}_k$ both satisfy condition (F) in [A] page 33, which implies the Mittag-Leffler condition, op.cit. Lemma 4.6. Again because the cohomology groups of a finite group with finitely generated coefficients are finite, the derived functor R^1 vanishes, so that

THEOREM 1. <u>If the ring</u> A <u>satisfies the conditions</u> (i) <u>and</u> (ii) <u>and</u> G <u>is a finite group, there is a fourth quadrant spectral sequence</u>

$$E_2^{r,s} = H^r(G, K^s(\text{pt})) \ ,$$

<u>which is strongly convergent to a graded group associated to</u> $KA^{r+s}(BG)$.

Before listing the conditions which we would like our characteristic classes to satisfy, it is convenient to define the flat bundle homomorphism

$$\alpha : RA(G) \to KA^{C}(BG) = [BG, \mathbb{Z} \times BGL(A)^{+}] .$$

Here $RA(G)$ equals the Grothendieck group of finitely generated $A(G)$-modules which are (stably) free over A, modulo the equivalence relation of short exact sequences, i.e. $[E] = [E'] + [E"]$ if there is a short exact sequence $0 \to E' \to E \to E" \to 0$ of G-representation modules over A. If W equals the disjoint union of the objects $\bar{W}GL(n,A)$ where $\bar{W}$ is the simplicial classifying complex associated to $GL(n,A)$, then the operation of taking the direct sum of two matrices induces a free simplicial semigroup structure on W. If UW denotes its enveloping group there is a homotopy equivalence

$$\mathbb{Z} \times BGL(A)^{+} \simeq |UW| \text{ , and}$$

using this model, it is clear how to construct a flat bundle map α. Under very general assumptions on A, which hold for example if A is a field or $A = O[\frac{1}{\ell}]$, a ring of algebraic integers with a single prime inverted, α is well-defined on equivalence classes of representations and

$$\alpha(E') + \alpha(E") = \alpha(E) \text{ , see [Q3, Theorem 2'] .}$$

The characteristic classes $\theta(E)$ of the representation module

E are the characteristic classes of the "flat $GL(n,A)$-bundle" $\alpha(E)$ over BG .

If the ring A satisfies condition (i) and (ii) we would like to construct a family of characteristic classes

$$\Theta_j(\rho) \in H^j(G,K_j(A)) ,$$

where $\rho : G \to GL(n,A)$ is the homomorphism defining a G-structure on the A-module E (possibly stabilised). These classes should have the following properties:

1. $\Theta_0(\rho) = 1$, $_j(\rho) = 0$ for $j > n = \dim_A E$.

2. Given a homomorphism $f : G' \to G$, then

$$\Theta c_j(f^! \rho) = f^*(\Theta_j \rho) ,$$

where $f^!$ and f^* denote the restriction maps in representation and cohomology theory.

3. Assuming that there is a natural, associative, (skew) commutative product

$$K_s(A) \otimes K_t(A) \to K_{s+t}(A) ,$$

which induces a product on $H^*(G,K_*(A))$, the total characteristic class

$$\Theta_{\cdot}(\rho) = 1 + \sum_{j=1}^{n} \Theta_j(\rho) ,$$

satisfies the Whitney formula

$$\theta.(p_1+p_2) = \theta.(p_1)\theta.(p_2) \ .$$

Note that if 1 is the trivial representation, and $\theta.(1) = 1$, then the class θ_j are <u>stable</u>, i.e.

$$\theta.(\rho+1) = \theta.(\rho) \ .$$

In addition we ask for formulae corresponding to the classical ones for the components of $\theta.(\rho_1 \otimes_A \rho_2)$ and $\theta.(\Lambda^i\rho)$.

4. If θ_j is indexed by the natural numbers, then $\theta_1(\rho)$ is defined by the homomorphism $G \to K_1(A)$, which associates to each element $g \in G$ the class of the representing matrix $\rho(g)$. However if the suffix j belongs to the <u>even</u> natural numbers, as is the case for the classical Chern classes and the variations on them discussed below, we need some rule for normalizing the definition of $\theta_2(\rho) \in H^2(G,K_2(A))$. In the most general case the following procedure seems natural: above let $\bar{\rho} : G \to K_1(A)$ be induced by ρ . Then since $K_1(A)$ is abelian, $\bar{\rho}$ factors through the commutator subgroup $[G,G]$. There is an extension of the group of elementary matrices $E(A)$ by G , induced from the extension

$$1 \to E(A) \to GL(A) \to K_1(A) \to 1 \ ,$$

and since the relations in the Steinberg group $St(A)$ are modelled on the familiar relations satisfied by elementary matrices, the lift of the map $G \to \mathrm{Out}(E(A))$ to $\mathrm{Aut}(E(A))$ needed to define the extension can be copied in the Steinberg group to give

$$1 \to \mathrm{St}(A) \to * \to G \to 1 .$$

Since $K_2(A)$ is isomorphic to the centre of $\mathrm{St}(A)$, the extension (*) is classified by $\Theta_2(\rho)$ in $H^2(G,K_2(A))$. This putative definition of Θ_2 generalises that of the second Stiefel-Whitney class $w_2(\rho)$ for real representations, which equals the (mod 2) reduction of $c_1(\rho)$, whenever ρ has an underlying complex structure.

5. If the ring A has a natural automorphism $\#$, for example conjugation if $A \subseteq \mathbb{C}$, or the Frobenius automorphism if $A = \mathbb{F}_q$, then the characteristic classes of $\rho^{\#}$ should be simply related to those of ρ .

6. In order to use characteristic classes in conjunction with the spectral sequence we need three filtrations on the representation ring $RA(G)$, each of which also exists on $KA^0(BG)$.

   (i) Since $K_0(A) \cong \mathbb{Z}$ we can define the augmentation map $\varepsilon : RA(G) \to \mathbb{Z}$; let $I = \mathrm{Ker}(\varepsilon)$ the ideal of representations with virtual dimension equal to 0 . Then the I-adic filtration equals

$$RA(G) \supseteq I \supseteq I^2 \supseteq \ldots\ldots \supseteq I^k \supseteq \ldots ,$$

and topologically defines a nested sequence of neighbourhoods of zero.

(ii) The topological filtration is defined by

$$RA(G)_k^{top} = \mathrm{Ker}\{RA(G) \underset{\alpha}{\to} KA^\circ(BG) \to KA^\circ(BG^{k-1})\} .$$

(iii) If we introduce exterior powers (λ^i) and Grothendieck operations (γ^i) into the graded ring $K_*(A)$ as in [B, Chapter 13], then

$$RA(G)_k^\gamma = \text{the subgroup generated by monomials of}$$

the form $\gamma^{i_1}(x_1)\ldots\ldots\gamma^{i_r}(x_r)$, where $i_1+i_2+\ldots\ldots+i_r \geq k$.

Note that since we suppose in general that the classes θ_j may be defined for odd as well as for even suffixes j , it is necessary to define R_k^{top} for all suffixes k . However, since BG is connected, and we may suppose that the 0-skeleton consists of a single point,

$$I(G) \subseteq RA(G)_1^{top} .$$

Both filtrations are multiplicative, so $I^k \subseteq RA(G)_k^{top}$, so that the topological filtration on $RA(G)$ is weaker than the I-adic. It follows that if we filter

$$A(G) = 1 + \prod_{j=1}^{\infty} H^j(G,K_j(A))$$

by

$$A(G)_k = \{a = \prod_{j=0}^{\infty} a_j : a_j = 0 \quad \text{for} \quad 1 \le j \le k-1\} ,$$

then the total characteristic class $\Theta.$ defines a homomorphism of filtered objects

$$\Theta. : RA(G) \to A(G) ,$$

where the domain is filtered as in either (i) or (ii). Completing I-adically we have

$$\Theta.^{\wedge} : \hat{R}A(G)^{I} \to A(G) ,$$

which, even when $A = \mathbb{C}$, need not be a monomorphism for an arbitrary group G .

Turning to the spectral sequence let $H'(G,KA^*(pt.))$ denote the subgroup of universal cycles on the line of total degree 0 , i.e. those elements of $\bigoplus_{j=0}^{\infty} H^j(G,K_j(A))$ which are killed by each differential. Dividing out by the universal boundaries we obtain a family of epimorphisms

$$\psi_j : H'^{j}(G,K_j(A)) \twoheadrightarrow \frac{KA(BG)_j^{top}}{KA(BG)_{j+1}^{top}} .$$

Then

7. For all $\rho \in RA(G)$ and all values of j , $\Theta_j(\rho)$ is a universal cycle, and if $\rho \in RA(G)_j^{top}$ then $\psi_j\Theta_j(\rho)$ equals

, modulo elements of filtration degree $(j+1)$; up to some simple numerical factor.

Hence wherever they can be defined the family of characteristic classes $\{\Theta_1, \Theta_2, \ldots\}$ provide a means of detecting universal cycles in the spectral sequence. Properties 1. to 5. are not enough to prove the existence and uniqueness of such classes, except for representation modules having composition series, such that successive factors are either one or two dimensional. In order to include more general representations one needs a good formula for the classes

$$\Theta_j(f_! \rho) \ ,$$

where ρ is a representation of some subgroup G' of G . Using the methods of [E2] or [F-M] it ought to be possible to find such a formula, which under suitable restrictions or for components of sufficiently low degree reduces to

$$\Theta^{\cdot}(f_! \rho) = N(\Theta \cdot \rho)(f_! 1)^r, \ r = \deg_A \rho \ ,$$

where $N(.)$ is Evens' multiplicative transfer, see [E1] . Such a formula is particularly useful if there is an analogue of Blichfeldt's theorem for representations over the ring A , i.e. for a suitable class of groups (containing groups of prime power order) an irreducible representation of augmentation greater than 2 is induced up from a 1 or 2-dimensional representation of a proper subgroup.

In addition to characteristic classes we wish to study the relation between $RA(G)$ and $KA(BG)$ in the hope of showing that after I-adic completion the homomorphism α induces an isomorphism. For various technical reasons we replace algebraic K-theory with the étale K-theories $\hat{K}A(\ )$ and $KA(\ ,\mathbf{Z}/_{\ell^\nu})$, defined for example in [D-F1, §4]. Since there is a natural transformation (which is compatible with transfer)

$$\varphi_* : KA^*(\ ) \to \hat{K}A^*(\ ),$$

with a parallel definition for finite coefficients, by composition with α we obtain a flat bundle homomorphism into $\hat{K}A^0(BG)$, which by definition is ℓ-adically complete. If A is a suitable Dedekind domain with quotient field F, consider the diagram below, the construction of which is motivated by [S, Prop.2.1]. Here p denotes a prime, and hence maximal, ideal in A.

$$\begin{tikzcd}
RF^{-1}(G) \arrow[r, "d"'] \arrow[d, "\alpha_F^{-1}"] & \coprod_p RA/p(G) \arrow[r, "\beta"'] \arrow[d, "\coprod \alpha_p"] & RA(G) \arrow[r, two heads, "\gamma"'] \arrow[d, "\alpha_A"] & RF(G) \arrow[d, "\alpha_F"] \\
\hat{K}F^{-1}(BG) \arrow[r, "d'"'] & \coprod_p \hat{K}A/p(BG) \arrow[r, "\beta'"'] & \hat{K}A(BG) \arrow[r, two heads, "\gamma'"'] & \hat{K}F(BG)
\end{tikzcd}$$

The upper row is Swan's exact sequence extended to the left by Quillen's higher Grothendieck's groups (which are modules over the representation ring, see [R]), β is obtained by transfer along the projections $A \to A/p$ and γ by extension of scalars from A to F. The lower row is localisation in étale K-theory, justified formally by first carrying out the known localisation argument in the Dwyer-Friedlander model for algebraic K-theory, and then replacing geometric by ℓ-adic maps. Working from the right α_F and α_p are always defined, α_A if A contains an invertible prime. As usual we assume this, e.g. $A = \mathcal{O}[\frac{1}{\ell}]$. The map α_F^{-1} is then defined as in [R] . The right hand square commutes by naturality, the middle square because the original flat bundle map commutes with transfer (compare [A, 26-27]) , as does φ_* above, see [D-F, §6] . The commutativity of the left hand square is formal, depending on the definitions of d and d' .

In order to make sense of completion we have to know that the representation rings concerned are finitely generated and free as abelian groups, i.e. that each representation module is equivalent modulo short exact sequences to a unique finite direct sum of indecomposable modules. For F and A/p this is immediate, for $A = \mathcal{O}[\frac{1}{\ell}]$ we have:

LEMMA 2. _If_ $A = \mathcal{O}[\frac{1}{\ell}]$ _and_ $G = G_\ell$ _is a finite group of order a power of_ ℓ , _then_ $RA(G_\ell)$ _is finitely generated_.

Proof. The usual proof of Maschke's Theorem holds, since $|G|$ is invertible in A , i.e. we can replace an A-splitting by an $A(G_\ell)$ splitting. Hence the ring $A(G_\ell)$ is semisimple, and the classical theory applies.

LEMMA 3. _If_ $RF^{-1}(G_\ell)$ _is a finitely generated free abelian group, and_ $A = \mathcal{O}[\frac{1}{\ell}]$, _the top row of the commutative exact diagram above remains exact when tensored with_ $\mathbb{Z}_\ell$, i.e. _after_ ℓ_-adic completion_.

Proof. This is immediate, since we are considering finitely generated free abelian groups throughout. Note that since ℓ is invertible in A, $A/p(G_\ell)$ is semisimple for all prime ideals p.

LEMMA 4. Let $A = \mathcal{O}[\frac{1}{\ell}]$, $G = G_\ell$. For each of the coefficient rings A/p, A and F the ℓ-adic and I-adic topologies coincide on the representation ring.

Proof. Recall the argument for this lemma over the complex numbers $\mathbb{C}$. Let $|G| = \ell^e$, then (i) $\ell^e \cdot I^n \hookrightarrow I^{n+1}$ for $n > 0$. To see this consider the pair of maps

$$R(G) \underset{i_*}{\overset{i^*}{\rightleftarrows}} R(1)$$

related by Frobenius reciprocity. Then $i_* i^*(x) = x i_*(1) = 0$, because $i^*(x) = 0$. Therefore $(\ell^e - i_*(1)) \in I$, and

$$\ell^e x = (\ell^e - i_*(1))x \text{ belongs to } I^{n+1} \text{ if } x \text{ belongs to } I^n .$$

In other direction (ii) $(x - \varepsilon(x))^{\ell^e} \in \ell \cdot I$.

We have
$$\left.\begin{aligned} (x - \varepsilon(x))^{\ell^e} &= x^{\ell^e} - \varepsilon(x)^{\ell^e} \\ &= \psi^{\ell^e}(x) - \varepsilon(x)^{\ell^e} \\ &= 0 \end{aligned}\right\} \text{ all modulo } \ell R(G) .$$

Part (i) holds for more general coefficients, because Frobenius reciprocity does. Part (ii) uses a property of the Adams operations which holds in any finite extension of $\mathbb{Q}$ and in the finite field A/p, hence also in A, since $RA(G_\ell)$ is trapped between representation rings over fields.

As a purely formal consequence of Lemmas 2-4 we have

LEMMA 5. *Let* $A = \mathcal{O}[\frac{1}{\ell}]$, $G = G_\ell$ *and* $RF^{-1}(G_\ell)$ *be a finitely generated free abelian group. Then if* $\alpha^\wedge$ *denotes completion with respect to the* ℓ*-adic topology, and* $\hat{\alpha}_F^{-1}$, $\hat{\alpha}_p$ *and* $\hat{\alpha}'_F$ *are all isomorphisms, then so is* $\hat{\alpha}_A$.

Proof. This is a simple diagram chase. In order to prove that $\hat{\alpha}_A$ is a monomorphism, one needs $\hat{\alpha}_p$, $\hat{\alpha}_F$ to be monic, and $\hat{\alpha}_F^{-1}$ to be epic. For the epimorphism one needs the weaker conditions that $\hat{\alpha}_F$ and $\hat{\alpha}_p$ be epic. It is this half of the lemma, which needs no assumption on α_F^{-1}, which we shall actually use in the final section below.

Given the framework which we have now set up, we can ask two questions:

(I) Do the classes $\theta_j(\rho)$ generate a subring of $H^*(G,K_*(A))$ which projects onto the image of ψ_* ?

(II) Does the completion of α in the I-adic topology define an isomorphism onto either $KA^{0,alg}(BG)$ or $\hat{KA}^0(BG)$?

In the three sections which follow we discuss both problems for $A = \mathbb{F}_q$ (obtaining complete answers), and for $A = \mathcal{O}[\frac{1}{\ell}]$ (where the situation is much less satisfactory).

2. Modular Representations

In the case when $A = \mathbb{F}_q (q=p^t)$, a finite field, the answer to the test question (II) is the best possible.

THEOREM 6. *If G is a finite group the I-adic completion*

$$\alpha^{\wedge} : R\,\mathbb{F}_q(G)^{\wedge} \xrightarrow{\cong} K\,\mathbb{F}_q^{\,0}(BG)$$

of the flat bundle homomorphism is a continuous isomorphism, which is natural with respect to group homomorphisms and field extensions.

Proof. This depends on the homotopy equivalence between the spaces $F\psi^q$ and $BGL(\mathbb{F}_q)^+$, see [Q2] and [R] . Because of [D-F, Cor.8.6] there is no need to distinguish between algebraic and étale K-theory.

The theory of charactersitic classes is equally satisfactory, since universal classes are provided by the isomorphism, see [F-P]

$$H^*(BGL(\mathbb{F}_q);\mathbb{Z}) = \mathbb{Z}[c_1,c_2,\ldots]/_{\{(q^i-1)c_i = 0\}}\ ,$$

where the degree of c_i equals $2i$. We obtain these classes by pulling back the Chern classes in characteristic zero along the Brauer map $BGL(\mathbb{F}_q) \to BGL(\mathbb{C})$, see [Q2] .

DEFINITION. If $\rho : G \to GL(n, \mathbb{F}_q) \hookrightarrow GL(\mathbb{F}_q)$ is a representation of the finite group G, the jth modular characteristic class equals

$$c_j(\rho) = \rho^*(c_j) \in H^{2j}(G, \mathbb{Z}) \ , \ j = 1,2,\dots n \ .$$

We explain the shift in dimension and the change in coefficients from the rather experimental earlier definition as follows. We have

$$K_{2i}(\mathbb{F}_q) = 0 \ , \ K_{2i-1}(\mathbb{F}_q) \cong \mathbb{Z}/_{q^i-1} \ .$$

The Bockstein homomorphism β associated with the coefficient sequence

$$\mathbb{Z} \underset{\text{multn.}}{\rightarrowtail} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/_{q^i-1}$$

induces

$$\to \underset{\substack{\shortparallel \\ 0}}{H^{2i-1}(GL(\mathbb{F}_q),\mathbb{Z})} \to H^{2i-1}(GL(\mathbb{F}_q),K_{2i-1}(\mathbb{F}_q)) \overset{\beta}{\rightarrowtail} H^{2i}(GL(\mathbb{F}_q),\mathbb{Z}) \xrightarrow{\times(q^i-1)} \ .$$

Since the order of the universal class c_i is (q^i-1), c_i belongs to the image of β, which is a monomorphism. Hence there exists a well-defined element

$$\Theta_i \in H^{2i-1}(GL(\mathbb{F}_q), K_{2i-1}(\mathbb{F}_q)) \ ,$$

mapping to c_i under β . Define $\Theta(p) = \rho^*\Theta_i'$.

Working with the evenly indexed classes c_i or with the odd classes $_i$ properties 1. and 2. are satisfied. There is an isomorphism from $K_1\,\mathbb{F}_q$ to $\mathbb{F}_q{}^x \cong \mathbb{Z}/_{q-1}$ given by taking determinants, hence as over $\mathbb{C}$, $c_1 = \beta O_1$ is obtained by applying β to the class in H^1 given by $\det(\rho)$. The only possible ambiguity arises from the identification of $\mathbb{F}_q{}^x$ with a subgroup of $\mathbb{C}^x$ used in the construction of the Brauer map. Hence 4. is satisfied.

The Whitney sum formula (3) holds both for O_i and c_i . For the latter this follows as over $\mathbb{C}$, for the former from the structure of $K_*(\mathbb{F}_q)$ as a ring. This can be read off from the corresponding result for complex K-theory (Bott periodicity) - again we identify $K_*(\mathbb{F}_q)$ with the kernel of ψ^q-1 .

A natural automorphism of $\mathbb{F}_q$ is the Frobenius map, $Fr : \mathbb{F}_q \to \mathbb{F}_q$ given by $x \longmapsto x^p$, the effect of which on representations is measured by powers of the Adams operation ψ^p . Furthermore

LEMMA 7. $\quad c_i(\psi^k\rho) = k^i c_i(\rho)$.

<u>Proof</u>. See [E-M] .

As a corollary of this lemma we obtain a test for the field of definition of the modular representation ρ . Suppose that ρ takes values in $GL(n,\mathbb{F}_q)$, where $q = p^t$, so that $\mathbb{F}_q$ is the unique extension of the prime field $\mathbb{F}_q$ of degree t . The elements of $\mathbb{F}_q$ are invariant under ψ^q, so that $\psi^q\rho = \rho$. Hence by Lemma 7 $(q^i-1)c_i(\rho) = 0$. This numerical bound corresponds to the familiar result in characteristic zero that the odd indexed Chern classes of a real representation have order dividing 2.

In discussing the relation with the spectral sequence we start with the evenly indexed classes $c_i(\rho)$, and hence slightly modify the definition of the γ-filtration on $R\,\mathbb{F}_q(G)$ to make it even. For the topological filtration

$$\begin{aligned} R\,\mathbb{F}^{top}_{q,2k}(G) &= \operatorname{Ker}\{R\,\mathbb{F}_q(G) \underset{\alpha}{\longrightarrow} K\,\mathbb{F}_q(BG) \longrightarrow K\,\mathbb{F}_q(BG^{2k-1})\} \\ &= R\,\mathbb{F}^{top}_{q,2k-1}(G) \text{ , since } K\,\mathbb{F}^{even}_q \text{ (point) is trivial.} \end{aligned}$$

(Note the shift in index in comparison with the complex case.)

In this case the augmentation ideal $I(G) = R\,\mathbb{F}^{top}_{q,2}(G) = R\,\mathbb{F}^{top}_{q,1}(G)$, so that $I(G)^k \subseteq R\,\mathbb{F}^{top}_{q,2k}(G)$. Theorem 6 implies that the two filtrations actually define the same topology on the completed representation ring.

Arguing as in the previous section we see that the total modular Chern class defines a homomorphism

$$c_{\cdot} : R\,\mathbb{F}_q(G) \to A(G) = 1 + \prod_{j=1}^{\infty} H^{2j}(G,\mathbb{Z}) \, ,$$

which may be extended to a continous homomorphism of completions. As a simple illustration let $G = C_n$, a cyclic group of order n with $(n,p) = 1$. Assume that q is sufficiently large so that $\mathbb{F}_q$ contains a primitive nth. root of unity ζ . Then if ρ is the one-dimensional representation which maps some generator of C_n to ζ , write $x = c_1(\rho)$, so that

$$A(G) = 1 + \sum_{j=1}^{\infty} a_k x^k ,$$

a ring of formal power series in one variable over $\mathbb{Z}/_n$. Since q is sufficiently large, the irreducible representations are $1, \rho, \ldots, \rho^{n-1}$, and

$$c.(\sum_{r=0}^{n-1} m_r \rho^r) = \prod_{r=0}^{n-1} (1 + rx)^{m_r} .$$

If $n = \ell$ a prime number distinct from p , the characteristic of $\mathbb{F}_q$, the right hand product vanishes if and only if $m_r = 0$ for all $r \geq 1$, so that in this case $c.$ is a monomorphism. As Atiyah shows in the appendix to [A] this is not the case more generally, even when $n = \ell^2$ - consider the equation

$$c\ (\ell_{\rho}{}^{\ell} - \ell) = (1+\ell x)^{\ell} \equiv 1 (\mathrm{mod}\ \ell^2) .$$

Another simple example is provided by the identity map $\iota_p : GL(n, \mathbb{F}_p) \to GL(n, \mathbb{F}_p)$. Lifted to the complex numbers this is a virtual representation, rather than a homomorphism, the character of which is trivial at all elements of p-power order. Again let ℓ be a prime different from p . If $h_\ell(p)$ is the order of the residue class of p in $\mathbb{Z}/\ell^x$, then the cohomology of $GL(n, \mathbb{F}_p)$ is ℓ-periodic with period $2h_\ell(p) = 2h$, provided that $\frac{n}{2} < h \leq n$, see [B-E, 4.2] . Localising the coefficients from $\mathbb{Z}$ to $\mathbb{Z}_{(\ell)}$ it follows that the modular characteristic class $c_h(\iota_p)_{(\ell)}$ generates the polynomial algebra $H^*(GL(n,\mathbb{F}_p),\mathbb{Z}_{(\ell)})$.

Next consider the role of the characteristic classses in the fourth quadrant spectral sequence

$$\left.\begin{aligned} E_2^{r,-2s} &= 0 \\ E_2^{r,-2s+1} &= H^r(G,\mathbb{Z}/_{q^s-1}) \end{aligned}\right\} 0 \leq r,\ 1 \leq s;\ E_2^{r,0} = H^r(G,\mathbb{Z}) .$$

Since the terms of negative even fibre degree are all zero, there are no non-zero even differentials d_{2j} entering or leaving terms of total degree zero. On this line the universal cycles in degree $2s-1$ from a subgroup of

$$H^{2s-1}(G,K_{2s-1}(\mathbb{F}_q)) \cong H^{2s-1}(G,\mathbb{Z}/q^s-1) ,$$

which contains the elements $\Theta_s(\rho) = \beta^{-1}c_s(\rho)$, as ρ runs through the modular representations of G over $\mathbb{F}_q$. As in [T2] the Grothendieck operation γ^s is compatible with the Brauer lifting of representations, and hence $\Theta_s(\rho)$ projects to $\gamma^s(\rho-\varepsilon(\rho))$, modulo terms of higher topological filtration. However note that in accordance with our numerical conventions $\gamma^s(\rho-\varepsilon(\rho)) \in R_{2s-1}^{top}$.

EXAMPLE 8. $G = SL(2,\mathbb{F}_{q'})$, $q' = p^{t'}$, $p \geq 5$.

The group G has order $(q'-1)q'(q'+1)$, G is perfect, and has cohomological ℓ-period equal to 4 for all primes $\ell \neq p$. Since a p-Sylow subgroup is elementary abelian of rank t' , a p-period exists only when $q' = p$, but in the context of modular representations this case does not concern us. With coefficients in $\mathbb{Z}/_{q^s-1}$ (note the different roles of q and q' !), the cohomology groups are as follows:

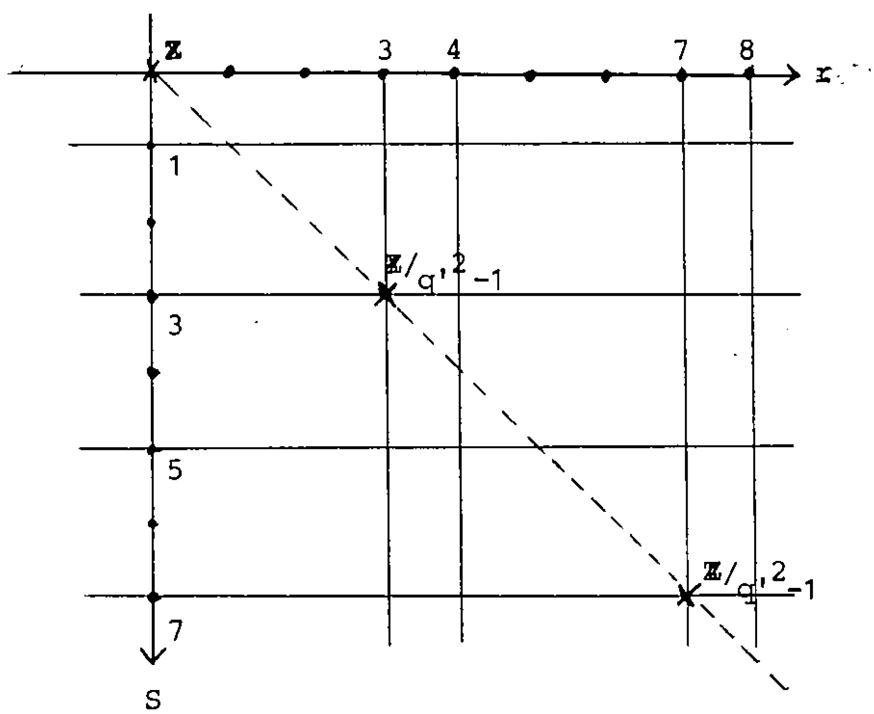
$$H^{4r}(G, \mathbb{Z}/_{q^s-1}) \cong \hat{H}^0(G, \mathbb{Z}/_{q^s-1}) = \operatorname{Coker}(\mathbb{Z}/_{q^s-1} \xrightarrow[x|G|]{} \mathbb{Z}/_{q^s-1}) .$$

$$H^{4r+1}(G, \mathbb{Z}/_{q^s-1}) = H^{4r+2}(G, \mathbb{Z}/_{q^s-1}) = 0, \text{ because } G \text{ is perfect.}$$

$$H^{4r+3}(G, \mathbb{Z}/_{q^s-1}) \cong \hat{H}^{-1}(G, \mathbb{Z}/_{q^s-1}) = \operatorname{Ker}(\mathbb{Z}/_{q^s-1} \xrightarrow[x|G|]{} \mathbb{Z}/_{q^s-1}) .$$

Here q and q' are powers of the same prime p ; given q' choose the field of coefficients $\mathbb{F}_q$ such that $t = nt'$. We have

$$E_2^{4r-1,-4r+1} = H^{4r-1}(G, \mathbb{Z}/_{q^{2r}-1}) \cong \mathbb{Z}/_{q'^2-1}$$; indeed if n is even all non-zero terms at the E_2-level of the spectral sequence away from the origin have the same order.



q = even power of q', all non-zero terms except $E_2^{0,0}$ are isomorphic, and all differentials are trivial.

On the line of total degree zero consider the term

$$H^{4r-1}(G,\ K_{4r-1}(\mathbb{F}_q)) \cong \mathbb{Z}/_{q'^2-1} \ .$$

The Brauer character of the lift of $\iota_{q'}$ to the complex numbers depends on the eigenvalues of the matrices $A \in SL(2, \mathbb{F}_{q'})$ in some extension field. If $A = \begin{pmatrix} \zeta & 0 \\ 0 & \bar{\zeta} \end{pmatrix}$, where ζ generates $\mathbb{F}_{q'}^{\times}$, since ζ lifts to a primitive root of unity, $c_2(\iota_{q'}|\langle A\rangle)$ generates $H^4(G, \mathbb{Z}_{(q'-1)})$. Similarly if B is an element of order $(q'+1)$, which diagonalises over $\mathbb{F}_{q'^2}$, then $c_2(\iota_{q'}|\langle B\rangle)$ generates $H^4(G, \mathbb{Z}_{(q'+1)})$. Pulling back this Chern class along the monomorphism

$$\beta : H^3(G, \mathbb{Z}/_{q^2-1}) \rightarrowtail H^4(G, \mathbb{Z}) \cong \mathbb{Z}/_{q'(q'^2-1)}$$

we see that $E_2^{3,-3}$ is generated by $\Theta_3(\iota_{q'})$. Furthermore as a ring $H^{3*}(G, K_{3*}(\mathbb{F}_q))$ is polynomial on this class. Passing to the E_∞-level of the spectral sequence we see that $R_{4r-1}^{top} = R_{4r}^{top}$ is generated by $\gamma^r(\iota_{q'}-2)$, modulo $R_{4r+1}^{top} = R_{4r+2}^{top}$. Therefore these Grothendieck classes generate $R\,\mathbb{F}_q(G)$, and hence as a λ-ring the modular representation ring of $SL(2, \mathbb{F}_{q'})$ is generated by the single class $\iota_{q'}$.

Remark. If we wish to study modular representations over small fields, i.e. with $q < q'$, then the spectral sequence is a little more complicated. However for large values of s $\mathbb{Z}/_{q^{2s}-1}$ will be divisible by q'^2-1, and hence the behaviour of modular representations of "sufficiently high filtration" is independent of the ground field.

EXAMPLE 9. $G = A_4 = PSL(2, \mathbb{F}_3)$, the alternating group on four symbols. A_4 has four conjugacy classes, of which $\{1\}$ and {elements of order 2} are 3-regular. Hence there are two irreducible representations over $\mathbb{F}_3$, the trivial representation and σ of dimension 3. Furthermore

$$H^*(A_4, \mathbb{Z}/_3 s_{-1}) \hookrightarrow H^*(C_2{}^A \times C_2{}^B, \mathbb{F}_2) ,$$

which is polynomial on two 1-dimensional generators. Write α and β for the reductions of the Chern classes of the complex representations which map A (resepectively B) to -1 . Then

$$c.(\sigma|A_{4,2}) = (1+\alpha)(1+\beta)(1+\alpha+\beta) ,$$

and in particular $c_3(\sigma|A_{4,2}) = \alpha^2\beta+\alpha\beta^2 \in H^6(C_2 \times C_2, \mathbb{Z})$. However an easy calculation shows that $\alpha^3+\beta^3+\beta\alpha^2$ and $\alpha^3+\beta^3+\alpha\beta^2$ are also invariant under the action of the cyclic group $A_4/A_{4,2}$ of order 3, and thus belong to $H^6(A_4, \mathbb{Z})$. Comparison of the spectral sequences for A_4 and $C_2 \times C_2$ shows that all invariant elements survive to infinity, and hence that $R_6^{top}(A_4)$ contains $R_6^{\gamma}(A_4)$ as a subgroup of index 2. Put another way there are insufficient Chern classes to generate the universal cycles.

Remarks. (i) We work with the class c_3 rather than with Θ_5 to emphasise the similarity to ordinary complex representations.

(ii) A similar argument applies to $PSL(2,\mathbb{F}_p)$ for all $p \equiv \pm 3 \pmod 8$, since a 2-Sylow subgroup has order 4 and is isomorphic to $C_2 \times C_2$, compare [T1] .

3. Extensions of rings of algebraic integers

Let $\mathcal{O}$ be the ring of integers in a global number field F, and $A = \mathcal{O}[\frac{1}{\ell}]$, with ℓ a rational prime, which will usually be odd. In contrast to the previous section it is not immediately obvious that there are characteristic classes in $H^j(G,K_j(A))$, although Grothendieck's theory of "mixed Chern classes" does at least provide a means of detecting universal cycles in the Atiyah-Hirzebruch spectral sequence. Indeed, working with the ring A rather than with some extension of its field of fractions does allow us to avoid problems raised by the huge Galois cohomology of F. However we shall find it more convenient to work with étale K-theory rather than with algebraic. If the (mod ℓ) cohomological dimension of A is sufficiently small, the coefficients $\hat{KA}^*$ (point) can be calculated using the étale cohomology of Spec A, which are known, at least in principle, see [D-F1] or [D-F2]. In particular we have

LEMMA 10. *The groups* $K_j(A)$ *are finitely generated for all* j. *If* ℓ *is odd, the groups* $\hat{KA}^j$ (point) *are finitely generated* $\mathbb{Z}$ *-modules*.

Proof. Quillen has proved that $K_j(\mathcal{O})$ is finitely generated, [Q1]. The same holds for $K_j(A)$ by the localisation exact sequence in algebraic K-theory (compare the construction of the commutative exact diagram in section one). The second statement follows from the existence of the natural surjection

$$\varphi_j : K_j(A) \otimes \mathbb{Z} \to \hat{K}_j(\mathrm{Spec}\ A),\ j \geq 2\ ,$$

see Theorem 8.7 in [D-F1] , or alternatively from knowledge of the étale cohomology groups and the spectral sequence below. When $j = 1$ or 2 , use Proposition 8.2 in the same paper. When $\ell = 2$, the same conclusion holds, provided that $\sqrt{-1} \in \mathcal{O}$.

Dwyer and Friedlander establish the existence of a spectral sequence linking cohomology and étale K-theory, op.cit. Proposition 5.2 and 5.2. Thus let X be a connected simplicial scheme over A of finite (mod ℓ) cohomological dimension. Then there is a natural, strongly convergent fourth quadrant spectral sequence

$$E_2^{r,-s} = H^r(X_{et}, \mathbb{Z}/_{\ell}\nu^{\otimes (\frac{s}{2})}) \Rightarrow KA^{s-r}(X, \mathbb{Z}/_{\ell}\nu)\ .$$

Replacing $\mathbb{Z}/_{\ell}\nu$ by $\mathbb{Z}_{\ell}$ and étale by continuous cohomology we obtain $\hat{K}A^{s-r}(X)$ in the limit. We need the assumption of finite dimension to ensure strong convergence in both cases, and by convention $\mathbb{Z}/_{\ell}\nu^{\otimes (\frac{s}{2})}$ vanishes, unless s is a non-negative even integer. Let Y be another finite dimensional scheme over A admitting G as a finite group of operators and write

$$X^{(k)} = EG^k \times_G Y\ ,$$

with G acting on the right on EG and on the left on Y . Here EG^k is a finite dimensional approximation to an (algebraic) model for the classifying space of G . Such certainly exists - for example allow G to act on sufficiently high dimensional Stiefel variety via an embedding in some unitary group. In particular let $Y = \mathrm{Spec}\, A$ with trivial G-action, and if ℓ is not odd, again suppose that $\sqrt{-1} \in \mathcal{O}$.

PROPOSITION 11. Let $A = \mathcal{O}[\frac{1}{\ell}]$, ℓ = odd or $\sqrt{-1} \in \mathcal{O}$. There is a strongly convergent fourth quadrant spectral sequence, natural with respect to ring extensions and group homomorphisms

$$E_2^{r,-s} = H^r(\mathrm{Spec}\, A, G;\ \mathbf{Z}/_{\ell}\nu^{\otimes(\frac{s}{2})}) \Rightarrow KA^{s-r}(BG;\ \mathbf{Z}/_{\ell}\nu) .$$

Proof. This is the limit spectral sequence obtained from $\lim\limits_{k\to\infty} X^{(k)} = EG \times_G Y$, i.e. we define the equivariant cohomology groups of Y to be the Borel groups. The Künneth formula gives a short exact sequence

$$0 \to \bigoplus_{r=s+t} xt^1(H_{s-1}(G, \mathbf{Z}),\ H^t(\mathrm{Spec}\, A,\ \mathbf{Z}/_{\ell}\nu^{\otimes(\frac{s}{2})})) \to E_2^{r,-s}$$
$$\to \bigoplus_{r=s+t} \mathrm{Hom}(H_s(G, \mathbf{Z}),\ H^t(\mathrm{Spec}\, A,\ \mathbf{Z}/_{\ell}\nu^{\otimes(\frac{s}{2})})) \to 0 .$$

Since the cohomology groups of G with coefficients in $\mathbf{Z}$ are finite torsion groups, the same holds for $E_2^{r,-s}$, and hence, as in the first section, the inverse limit $\varprojlim_k E(k)_2^{r,-s}$ satisfies the Mittag-Leffler condition. Similar considerations show that the spectral sequence converges to $KA^*(BG, \mathbf{Z}/_{\ell}\nu)$ rather than a homomorphic image, i.e. $R^1 \varprojlim_k$ is trivial.

Passing to the limit with ν we can replace $\mathbf{Z}/_{\ell}\nu$ by the ℓ-adic integers $\mathbf{Z}_{\ell}$, so long as we work with continuous cohomology, and take a little care with the edge terms of the spectral sequence, see [D-F1] Proposition 5.1 and the remark following its statement.

Given the spectral sequence in Proposition 11 we detect universal cycles by means of the Grothendieck characteristic classes

$$c_j(\rho) \in H^{2j}(\mathrm{Spec}\ A, G; \boldsymbol{\mu}_{\ell^\nu}^{\otimes j}) \in E_2^{2j,-2j} .$$

Recall from [G] that these satisfy properties 1. to 4. in the first section. In particular normalisation (4) takes the form: if $\rho : G \to A^* = \text{units}\ (\mathcal{O}[\frac{1}{\ell}])$ defines a 1-dimensional representation module L, then L is determined by an element in $H^1(\mathrm{Spec}\ A, G; A^*)$, and using the Kummer exact sequence (ℓ is invertible in A !)

$$0 \to \boldsymbol{\mu}_{\ell^\nu} \to A^* \to A^* \to 0 ,$$
$$a \longmapsto a^{\ell^\nu}$$

we define $c_1(\rho) = \beta[L] \in H^2(\mathrm{Spec}\ A, G; \boldsymbol{\mu}_{\ell^\nu})$. (At this point we interpret the Kummer exact sequence as an exact sequence of abelian sheaves with operators.) The discussion of the total Chern class as a homomorphism into $\mathbb{A}(G)$, defined in the obvious way, and the definition of the topological and γ-filtrations on $RA(G)$, are similar to the case of finite fields. In order to see that $c_j(\rho)$ is a universal cycle it is necessary to look a little more closely at the definition of

γ^j in étale K-theory. One way to do this is via the projective A-scheme $\mathrm{Grass}_{m,n}$ representing locally free coherent sheaves of rank n generated by m global sections. In the limit over $\mathbb{C}$ this gives a model for the classifying space $BGL(\mathbb{C})$, which admits a self-map γ^j corresponding to the classical operation on vector bundles. Working backwards mimic the construction of $\gamma^j_{\mathbb{C}}$ and obtain an algebraic map $\gamma^j_A : \mathrm{Grass}_{m,n} \to \mathrm{Grass}_{m,n}$ (m and n large). Composition with γ^j_A then defines $\gamma^j_A(x) \in \hat{K}A(X)$ with a similar construction for finite coefficients.

As in the classical case a 1-dimensional representation of a finite group G is determined by its first Chern class. Furthermore it is clear from the spectral sequence that $E_2^{2,-2} = E_\infty^{2,-2}$ and that if ρ is 1-dimensional, $c_1(\rho)$ coincides with $\gamma^1(\rho-1) = \rho-1$, modulo elements of filtration level 4. More generally the class $c_j(\rho) \in E_2^{2j,-2j}$ survives to infinity as $\gamma^j(\rho)$, modulo elements of level $2j+2$. We show this first for sums of 1-dimensional elements, and then use a splitting principle [A-T, Thm. 6.1], which holds, since using the Grassmann scheme again $\hat{K}A(BG)$ is a special λ-ring.

Although the Grothendieck classes are naturally associated with the Dwyer-Friedlander spectral sequence, in line with the framework of the first section we wish to make a first tentative approach to classes Θ_j associated with the spectral sequence with coefficients in $K_s^{alg}(A)$. Consider the projection map given by the Künneth formula

$$H^{2j}(\mathrm{Spec}\ A, G, \mu_{\ell^\nu}^{\otimes j}) \to \bigoplus_{k=0}^{2j} \mathrm{Hom}(H_{2j-k}(G, \mathbf{Z}), H^k(\mathrm{Spec}\ A, \mu_{\ell^\nu}^{\otimes j}))$$

$$c_j(\rho) \longmapsto \{c_{j,k}(\rho)\} .$$

Since $\mathrm{cd}\ (\mathrm{Spec}\ A) \leq 2$, we need only consider the components $c_{j,0}$, $c_{j,1}$ and $c_{j,2}$. $c_{j,0}$ is the $(\mathrm{mod}\ \ell^\nu)$ reduction of the usual Chern class, obtained by embedding A in the complex numbers; the other two components are more interesting. Put $X = \mathrm{Spec}\ A$ in the non-equivariant version of the Dwyer-Friedlander spectral sequence, and obtain isomorphisms

$$H^k(\mathrm{Spec}\ A, \mu_{\ell^\nu}^{\otimes j}) \cong K_{2j-k}^{\acute{e}t}(A, \mathbf{Z}/\ell^\nu), \ k = 1,2 .$$

It follows that $c_{j,k}(\rho)$ belongs to a summand of $H^{2j-k}(G, K_{2j-k}^{et}(A, \mathbf{Z}/\ell^\nu))$, and for any ring for which the Quillen conjecture that the isomorphism above still holds with algebraic K-theory replacing K-theory, i.e. such that φ_k is an isomorphism see [D-F1], we will obtain a sequence of classes Θ_{2j-k}. These will have at least some of the properties of section one, although it is clear that the Whitney sum formula (3) must be replaced by something more complicated.

EXAMPLE 12. $c_1(\rho)$ determines a component of mixed type

$$\Theta_1(\rho) \in H^1(G, H^1(\mathrm{Spec}\ A, \mu_{\ell^\nu})) .$$

The coefficients are in turn determined by a short exact sequence

$$0 \to A^*/_{(A^*)^{\ell^\nu}} \to H^1(\mathrm{Spec}\ A, \mu_{\ell^\nu}) \to \mathrm{Pic}(A)_{(\ell^\nu)} \to 0 ,$$

where the term on the right equals the kernel of multiplication by ℓ^ν, and hence vanishes if this group has no ℓ-torsion. Consider the special case when $G = C_\ell$, a cyclic group of order ℓ, $A = [\frac{1}{\ell}]$ with $\mathcal{O} = \mathbf{Z}(\zeta)$, ζ a primitive ℓth-root of unity and $\nu = 1$. Then (compare the claculation in Lemma 4.4 of [D-F2]) H^1 is an extension of the direct sum of $(\ell+1)/_2$ copies of $\mathbf{Z}/_\ell$ by a finite group which vanishes if ℓ is regular. Note that the number of summands in A^* is determined by the Dirichlet unit theorem together with the fact that ℓ^{-1} generates a further free summand. Let $\bar{\Theta}_1(\rho)$ denote the projection of $\Theta_1(p)$ in $H^1(C_\ell, (\mathrm{Pic}\ A)_{(\ell)}) \cong \mathrm{Pic}(A)_{(\ell)}$, since this group is elementary and therefore killed by multiplication by ℓ. From the point of view of representations an ideal $\mathbb{A}$ representing a class in $\mathrm{Pic}(\mathcal{O})$ can be thought of as a one-dimensional representation $_{\mathbb{A}}$ of C_ℓ over A (extend the scalars from $\mathbf{Z}$ to $\mathbf{Z}[\frac{1}{\ell}]$). The discussion shows that $\bar{\Theta}_1(\rho_{\mathbb{A}})$ is non-zero if and only if the order of $\mathbb{A}$ in the Picard group equals ℓ. In particular, if ℓ is regular $\bar{\Theta}_1(\rho_{\mathbb{A}}) = 0$.

4. Completion of the representation ring

Let us continue with the ring of coefficients from Example 12. If ℓ is odd and regular, the ℓ-adic homotopy type of the classifying space $\underset{\sim}{\hat{K}A}$ is determined by the fibre square

$$\begin{array}{ccc} \underset{\sim}{\hat{K}A} & \longrightarrow & \prod_{\frac{\ell-1}{2}} U \\ \big\downarrow & & \big\downarrow \\ \underset{\sim}{\hat{K}\mathbb{F}_q} \underset{\ell}{\simeq} F\psi^q & \longrightarrow & * \end{array}$$

, see [D-F2] .

Here is a rational prime number whose image in the ℓ-adic units $\mathbb{Z}_\ell^*$ is a topological generator of $\mathrm{Ker}(\mathbb{Z}_\ell^* \to \mathbb{Z}/_{\ell-1})$. In order to understand this homotopy equivalence, first claculate the (mod ℓ) ètale homology, thus as in Example 12,

$$H_i(\mathrm{Spec}\ A\ \ \mathbb{Z}/\ ) = \begin{cases} \mathbb{Z}/_\ell, & i = 0 \\ \frac{(\ell+1)}{2}\,\mathbb{Z}/_\ell, & i = 1,\ (\ell \text{ is regular}) \\ 0, & i \geq 2,\ (\ell \text{ is odd}) . \end{cases}$$

It follows that there is a (mod ℓ) homology equivalence between $(\mathrm{Spec}\ A)_{\acute{e}t}$ and $\bigvee_{\frac{\ell+1}{2}} S^1$. Interpret the ètale fundamental group as the Galois group of a maximal unramified extension of A , and allow $\pi_1(\mathrm{Spec}\ \mathbb{Z}[\frac{1}{\ell}])$ to act on $\mathbb{Z}_\ell^*$ via the action of the Galois group on roots of unity. The homology equivalence above then implies, that apart from a wedge of circles (corresponding to the copies of U in the upper right hand corner of the fibre space) the ℓ-adic homotopy type of Spec A is determined by the last circle S^1 with fundamental group having a generator which maps to a topological generator of $\mathrm{Ker}(\mathbf{Z}_\ell^* \to \mathbf{Z}/_{\ell-1})$. (Taking the kernel allows for the replacement of $\mathbf{Z}[\frac{1}{\ell}]$ by A .) Replace the circle S^1 by a copy of $(\mathrm{Spec}\,\mathbb{F}_q)_{\text{ét}}$ with q as above, obtaining f : $\mathrm{Spec}(\mathbb{F}_q)_{\text{ét}} \vee V_{\ell-1/2}S^1 \to \mathrm{Spec}\,\mathcal{O}[\frac{1}{\ell}]_{\text{ét}}$. The relation between étale K-theory and cohomology discussed in [D-F1, § 5] is strong enough to give the ℓ-adic homotopy equivalence claimed in the diagram.

Now let $G = G_\ell$ be a finite group of order equal to a power of ℓ . Since $[BG_\ell, U] = 0$, see [A] ,

$$[BG_\ell, \underset{\sim}{\hat{K}A}] \cong [BG_\ell, \underset{\sim}{\hat{K}\mathbb{F}_q}] ,$$

with a similar statement for K-theory with coefficients in the finite ring $\mathbf{Z}/_{\ell^\nu}$. Lemma 2 implies that $RA(G_\ell)$ is finitely generated, and the choice of q as topological generator of $\mathbf{Z}$ is equivalent under the Brauer lifting from $\mathbb{F}_q$ to $\mathbb{C}$ to an identification of $R\,\mathbb{F}_q(G_\ell)$ with those representations invariant under the action of $\pi_1(\mathrm{Spec}\,A_{\text{ét}}) = G$. As before G acts on the ℓ-primary roots of unity contained in a sufficiently large finite extension of $\mathbb{Q}$. This means that

$$R\,\mathbb{F}_q(G_\ell) \cong RU(G_\ell)^G \supseteq RA(G_\ell) .$$

The right hand inclusion is clear, and at least if ℓ is an odd prime, we may identify $RA(G_\ell)$ with the Galois invariant subgroup.

THEOREM 13. *If G_ℓ is a finite group of order ℓ^e (ℓ = odd and regular), then the ℓ-adic completion of the flat bundle homomorphism*

$$\hat{\alpha} : \hat{R}(G_\ell)^G \xrightarrow{\ \simeq\ } \hat{K}A(BG_\ell)$$

is a continuous isomorphism.

Proof. This follows from the dicussion above, Theorem 6 and Lemma 4.

For the prime 2 the situation is not quite so simple. When $A = \mathbf{Z}[\frac{1}{2}]$ one obtains the 2-adic homotopy type of $\hat{\mathbf{K}}\mathbf{Z}[\frac{1}{2}]$ from the fibre sqare

$$\begin{array}{ccc} \hat{\mathbf{K}}\mathbf{Z}[\frac{1}{2}] & \longrightarrow & BO \\ \downarrow & & \downarrow \\ \hat{\mathbf{K}}\mathbb{F}_3 \underset{2}{\simeq} F\psi^3 & \longrightarrow & BU \end{array} .$$

The prime 3 plays the role of q in the odd case, that is 3 maps to a topological generator of $Z_2^*/_{\{\pm 1\}}$. Turning to representation rings define B_2 by the pull-back diagram

$$\begin{array}{ccc} B_2 & \dashrightarrow & RO(G_2) \\ \vdots & & \downarrow \\ R\,\mathbb{F}_3(G_2) & \xrightarrow[\mathrm{Br}]{} & RU(G_2) \end{array} .$$

Since the complexification (c) and the Brauer map (Br) are both injective, we can consider B_2 as a subgroup of $RU(G_2)$, which can be identified with the subset of elements invariant under the action of $\pi_1(\mathrm{Spec}\,\mathbb{Z}[\frac{1}{2}]_{\acute{e}t})$ on the 2-primary roots of unity. (The space BO enters into the determination of the 2-adic homotopy type, since for a ring $\mathcal{O}$ of algebraic integers $H^{2i+1}(\mathrm{Spec}\,\mathcal{O}, \mathbb{Z}/_2) \cong (\mathbb{Z}/_2)^{r_1}$, where r_1 equals the number of real embeddings of the quotient field.) The elements of $R\,\mathbb{Z}[\frac{1}{2}](G_2)$ are certainly Galois invariant in this sense, and hence contained in B_2. The familiar example of representations of quaternion type shows that for an arbitrary 2-group this inclusion is proper.

Arguing as for an odd prime ℓ one sees that $B_2\hat{}$ (2-adic completion) is isomorphic to $\hat{K}\,\mathbb{Z}[\frac{1}{2}](BG_2)$. The proof needs the fact that $\alpha_F\hat{}$ is an isomorphism for $F = \mathbb{F}_3$, $\mathbb{R}$ and $\mathbb{C}$; the middle case depends on [A-S, Thm. 7.1]. In all three cases 2-adic completion is equivalent to I-adic completion (Lemma 4 again), and is given by tensoring a finitely generated free abelian group with $\mathbb{Z}_2$. This preserves exactness, hence $\hat{\alpha}|B_2\hat{}$ is an isomorphism onto $\hat{K}\,\mathbb{Z}[\frac{1}{2}](BG_2)$.

Remarks. There are versions of the theorem just proved with finite coefficients both for $\ell = 2$ and ℓ = odd and regular. If one uses algebraic rather than étale K-theory, then since the Dwyer-Friedlander map φ_* is split, $\hat{\alpha}$ maps the completed representation ring onto a direct summand of $KO[\frac{1}{\ell}]^{alg}(BG_\ell) \otimes \mathbb{Z}_\ell$.

Although Theorem 13 depends on the description of the ℓ-adic homotopy type of $\hat{K}A$ given in [D-F2] some further information can be extracted from the commutative exact ladder at the end of the first section. Let A be the ring of integers in a global number field F with the prime ℓ inverted, and let $G = G_\ell$.

$$\begin{array}{ccccccc}
RF^{-1}(G_\ell) & \xrightarrow[d]{} & \coprod_p RA/p(G_\ell) & \xrightarrow[\beta]{} & RA(G_\ell) & \xrightarrow[\gamma]{} & RF(G_\ell) \\
\downarrow{\scriptstyle \alpha_F^{-1}} & & \downarrow{\scriptstyle \coprod \alpha_p} & & \downarrow{\scriptstyle \alpha_A} & & \downarrow{\scriptstyle \alpha_F} \\
\hat{K}F^{-1}(BG_\ell) & \xrightarrow[d']{} & \coprod_p \hat{K}A/p(BG_\ell) & \xrightarrow[\beta']{} & \hat{K}A(BG_\ell) & \xrightarrow[\gamma']{} & \hat{K}F(BG_\ell)
\end{array}$$

By Theorem 6 each completed map $\hat{\alpha}_p$ is an isomorphism, since $A/_p$ is a finite field. In certain cases we have just shown that $\hat{\alpha}_A$ is also an isomorphism, where in both cases completion is to be understood in the ℓ-adic sense. Take the tensor product of the upper row with $\mathbb{Z}_\ell$ - except perhaps at the left exactness is preserved. An easy diagram chase, which uses no assumptions on α_F^{-1} , shows that if $\hat{\alpha}_A$ and $\hat{\alpha}_p$ are isomorphisms, then so is $\hat{\alpha}_F$. This proves part of

THEOREM 14. Let ℓ be an odd regular prime and G_ℓ a finite group of order a power of . Then

$$\hat{\alpha_{\mathbb{Q}}} : R\mathbb{Q}(G_\ell)\hat{} \xrightarrow{\simeq} \hat{K\mathbb{Q}}(BG_\ell)$$

is a continuous isomorphism.

Proof. The diagram above with $A = \mathbb{Z}[\zeta,\frac{1}{\ell}]$ shows that $\alpha\hat{}$ is an isomorphism over the ℓth. cyclotomic field $\mathbb{Q}(\zeta)$. Finite Galois descent holds for étale K-theory [D-F1, § 7] , hence $\alpha\hat{}$ restricts to an isomorphism between $\hat{R\mathbb{Q}}(\zeta)(G_\ell)^{\mathrm{inv}}$ and $\hat{K\mathbb{Q}}(BG_\ell)$. Since ℓ is odd the Galois invariant representations coincide with $\hat{R\mathbb{Q}}(G\ )$.

It would be interesting to have a more direct proof of the last theorem, which would apply to fields more general than those lying between $\mathbb{Q}$ and $\mathbb{Q}(\zeta)$, and without restricting ℓ to be regular. The evidence that such exists as follows: as a consequence of A.Suslin's description of the algebraic K-theory of algebraically closed subfields of $\mathbb{C}$, see [T2] ,

$$\hat{\alpha_{\bar{\mathbb{Q}}}} : \hat{R\bar{\mathbb{Q}}}(BG_\ell) \xrightarrow{\simeq} K\bar{\mathbb{Q}}(BG_\ell) ,$$

where completion is again ℓ-adic, and the cohomology theory on the right can be taken to be either étale or algebraic. This follows because the coefficients differ only by uniquely divisible groups, which make no contribution with domain space BG_ℓ . Choosing étale K-theory the problem is to show that

$$\hat{K}\bar{\mathbb{Q}}(BG_{\ell}) \cong \hat{K}\mathbb{Q}(\zeta)(BG_{\ell}) ,$$

where ζ is an ℓ-primary root of unity of sufficiently high order, since such an isomorphism certainly exists for the cmmpleted representation rings. A priori such an isomorphism looks improbable, given that $\hat{K}_*(\mathbb{Q}(\zeta))$ has a huge component coming from the Brauer group of $\mathbb{Q}(\zeta)$. However it is possible that it makes no contribution to be K-theory of BG_{ℓ} , since it corresponds to mapping into copies of U - compare the calculation of the ℓ-adic homotopy type of the spectrum $\hat{K}O[\frac{1}{\ell}]$ above.

REFERENCES

[A] M.F.Atiyah, Characters & cohomology of finite gorups, Publ. IHES 9 (1961), 23-64.

[A-S] M.F.Atiyah, G.B.Segal, Equivariant K-theory and completion, J.Diff.Geom.3 (1969), 1-18.

[A-T] M.F.Atiyah, D.O.Tall, Group representations, λ-rings and the J-homomorphism, Topology 8 (1969) 253-297.

[B] A.J.Berrick, An approach to algebraic K-theory, Pitman Research Notes in Mathematics, 56 (1982).

[B-E] B.Bürgisser - B.Eckmann, The p-periodicity of the groups $GL(n,\mathcal{O}_S(K))$ & $SL(n,\mathcal{O}_S(K))$, Mathematika 31 (1984) 89-97.

[D-F1] W.G.Dwyer - E.M.Friedlander, Algebraic and Étale K-theory, Trans.Amer.Math.Soc., 292 (1985), 247-280.

[D-F2] W.G.Dwyer - E.M.Friedlander, Conjectural calculations of general linear group homology (preprint).

[E1] L.Evens, A generalisation of the transfer map in the cohomology of groups, Trans.Amer.Math.Soc., 108 (1963) 54-65.

[E2] L.Evens, On the Chern classes of representations of finite groups, Trans.Amer.Math.Soc., 115 (1965) 180-193.

[E-M] B.Eckmann, G.Mislin, Chern classes of group representations, Compositio Math. 44 (1981).

[F-M] W.Fulton, R.MacPherson, Characteristic classes of direct image bundles for covering maps, CR.Acad.Sci. Paris, ser I 299 (1984) 379-382.

[F-P] Z.Fiedorowicz , S.Priddy, Homology of classical groups over a finite field, Springer LN 551 (1976) 269-277.

[G] A.Grothendieck, Classes de Chern et représentation linéaires des groupes discrets, 10 exposés sur la cohomologie des schémas (1968), 41-65 North-Holland publishing Co., 215-305.

[Q1] D.Quillen, Finite generation of the groups K_i of rings of algebraic integers, Springer LN 341 (1973) 179-198.

[Q2] D.Quillen, On the cohomology & K-theory of the general linear group over a finite field, Ann. of Math. 96 (1972), 552-586.

[Q3] D.Quillen, Characteristic classes of representations, Springer LN 551 (1976), 189-216.

[R] D.L.Rector, Modular characters & K-theory with coefficients in a finite field, J.Pure and Applied Alg. 4 (1974), 137-158.

[S] R.G.Swan, Induced representations and projective modules, Ann. of Math. 71 (1960), 552-578.

[T1] C.B.Thomas, Filtrations on the representation ring of a finite group, Contemporary Math. 19 (1983), 397-405.

[T2] C.B.Thomas, Modular representations & the cohomology of finite groups, ETH preprint (February 1986).