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Yangian Algebras and Classical Riemann Problems

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Abstract

We investigate different Hopf algebras associated to Yang's solution of quantum Yang–Baxter equation. It is shown that for the precise definition of the algebra one needs the commutation relations for the deformed algebra of formal currents and the specialization of the Riemann problem for the currents. Two different Riemann problems are considered. They lead to the central extended Yangian double associated with $\mathfrak{sl}_2$ and to the degeneration of scaling limit of elliptic affine algebra. Unless the defining relations for the generating functions of the both algebras coincide their properties and the theory of infinite-dimensional representations are quite different. We discuss also the Riemann problem for twisted algebras and for scaled elliptic algebra.

1 Introduction

The Yangian $Y(\mathfrak{g})$, where $\mathfrak{g}$ is a simple Lie algebra was introduced by V. Drinfeld [1] as a Hopf algebra such that a quantization of the Yang rational solution of classical Yang-Baxter equation can be done in tensor category of finite-dimensional representations of $Y(\mathfrak{g})$. Later the algebraical structure of the quantum double $DY(\mathfrak{g})$ of the Yangian was studied in [2] and the corresponding universal R-matrix was calculated explicitly. The Yangian double admits a central extension $\widehat{DY}(\mathfrak{g})$ [3] (see also [4]) and the intertwining operators for its infinite-dimensional representations can be used for the calculation of form-factors of local operators in $SU(2)$ invariant Thirring model [5].

Recently, the detail analysis of the scaling limit $\mathcal{A}_{\hbar,\eta}(\widehat{\mathfrak{sl}_2})$ of the elliptic algebra was done in [6]. The corresponding intertwining operators can be used for the calculations of the form-factors in XXZ model in the gapless regime [7] and in the Sine-Gordon model [8]. It happens that rational degeneration $\mathcal{A}_{\hbar}(\widehat{\mathfrak{sl}_2})$ of the algebra $\mathcal{A}_{\hbar,\eta}(\widehat{\mathfrak{sl}_2})$ $(\eta \to 0)$ can be described on the level of generating functions (L-operators) by the same set of the relations as central extended Yangian double $\widehat{DY}(\mathfrak{sl}_2)$ while the structure of the algebras and their infinite-dimensional representations look very different.

For an investigation of this phenomena we go back to the original ideology of classical inverse scattering method, where the Riemann problem of factorizing the matrix valued function into

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the product of functions analytical in certain domains plays the crucial role [9]. We claim that the complete structure of the quantum algebra, including its coalgebraic structure, can be given by the following data: a formal algebra of (deformed) currents and a specific Riemann problem. For the algebras related to the $\mathfrak{sl}_2$ case which we study here it means that they are given by the formal relations for the total currents $e(u)$, $f(u)$ and $h^{\pm}(u)$ (here u is a spectral parameter) and the decompositions $e(u) = e^+(u) - e^-(u)$, $f(u) = f^+(u) - f^-(u)$. The operator valued generating functions of the spectral parameter $e^{\pm}(u)$, $f^{\pm}(u)$ are given as certain integral transforms of the total currents $e(u)$ and $f(u)$ which can be uniquely determined by the predicted analytical properties of $e^{\pm}(u)$ and $f^{\pm}(u)$ (see Section 3). The specification of the Riemann problem uniquely defines the precise expressions of the currents as generating functions of the elements of the algebra. It converts the relation between the currents into the relations between the generators of the algebra. Moreover it defines the comultiplication structure for the quantum algebra. We assume here that $e^{\pm}(u)$, $f^{\pm}(u)$ and $h^{\pm}(u)$ are Gauss coordinates of some L-operators and use the universal comultiplication formulas for Gauss coordinates (see Section 7).

For both algebras $\widehat{DY(\mathfrak{sl}_2)}$ and $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$ the commutation relations between the currents can be computed by a standard procedure called Ding-Frenkel isomorphism [10]. Actually the calculations are formal algebraical manipulations which use only the structure of the Yang (quantum) R-matrix $R(u) = 1 + \hbar P/u$ where P is a flip and thus coincide. This is done in Section 2.

The Riemann problem for $\widehat{DY(\mathfrak{sl}_2)}$ means a decomposition of a function with finite number of singularities into the sum of functions analytical at zero and at infinity. The solution is given by Cauchy integrals $e^{\pm}(u) = \oint \frac{e(v)dv}{2\pi i(u-v)}$, where the closed contour including zero goes from the left or from the right of the point u. The elements of the corresponding algebra are Taylor coefficients of $e^{\pm}(u)$, $f^{\pm}(u)$ and of $h^{\pm}(u)$. They generate precisely $\widehat{DY(\mathfrak{sl}_2)}$. The Riemann problem which corresponds to the algebra $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$ means a decomposition of a function vanishing at infinity into the sum of functions analytical in some half-planes. The solution is given by the Cauchy integral with the same kernel over the contour being the straight line parallel to the real axis going lower or upper u. The generators of the algebra, indexed by real numbers are the coefficients of inverse Laplace transforms of the currents $e^{\pm}(u)$, $f^{\pm}(u)$ and of $h^{\pm}(u)$ (Section 3). The analysis of the natural completions of the two algebras and of their properties show that they cannot be transformed one into another by means of projective transforms as well as corresponding Riemann problems are essentially different due to different asymptotical conditions for the involved functions (Section 4).

These two Riemann problems may be modified by imposing other asymptotical behaviors for $e^{\pm}(u)$, $f^{\pm}(u)$. Thus we get the twisted variants of these two algebras and as a limit the popular algebras with a simple comultiplication introduced by Drinfeld (new realization) (see Section 7). We demonstrate also in this Section that further trigonometric generalization of the Yangian current algebra essentially leads to the algebra defined in [6] and an application of the Riemann problem for a strip produces the scaled elliptic algebra $\mathcal{A}_{\hbar,\eta}(\widehat{\mathfrak{sl}_2})$.

It is worth to mention that the algebra $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$ fits much more for the applications in quantum integrable field theories. The main advantage is that the algebra $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$ is graded while $\widehat{DY(\mathfrak{sl}_2)}$ is a filtered algebra. Moreover the grading operator d can be diagonalized in infinite-dimensional representations of $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$ such that its trace has a sense contrary to the case of $\widehat{DY(\mathfrak{sl}_2)}$ where only the ratio of the traces is well defined (see Section 6). The application of the universal R-matrix to finite-dimensional representations gives integral forms of the corresponding R-matrices (Section 5). One may consider the universal R-matrix for $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$ as another quantization of classical rational solution of the Yang-Baxter equation.

2 Algebra of L-Operators

2.1. The aim of this section is to develop well known technique of the L–operators starting from the Yang's solution of the Yang-Baxter equation without referring to the precise meaning of the L-operators and to their analytical properties. The result can be thought of as a formal algebra which turns to be genuine Hopf algebra when one defines the L-operators to be explicit generating functions of its elements.

Let $\overline{R}(u)$ be a rational solution of the quantum Yang–Baxter equation:

$$\overline{R}(u) \;=\; \frac{u - i\hbar P}{u - i\hbar}\ , \tag{2.1}$$

where $P \in \mathrm{End}\,\mathbb{C}^2 \otimes \mathbb{C}^2$,

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is a permutation operator and $\hbar$ is the deformation parameter. Note that the algebras which we discuss below can be defined for arbitrary complex values of the deformation parameter. But the representation theory, especially the infinite-dimensional representation theory, depends on the specific values of this parameter. So we fix $\hbar \in \mathbb{R}$ and $\hbar > 0$. Moreover, this choice is in accordance with applications to massive field theory, where one should put $\hbar = \pi$ [8, 11].

Following the formalism of Faddeev–Reshetekhin–Takhtadjan [12] one can use $\overline{R}(u)$ for the construction of the bialgebra whose generators are gathered into the matrix elements of the quantum L-operator $L(u)$,

$$L(u) = \begin{pmatrix} L_{11}(u) & L_{12}(u) \\ L_{21}(u) & L_{22}(u) \end{pmatrix} \tag{2.2}$$

which satisfy the Yang-Baxter relation

$$\overline{R}(u-v)L_1(u)L_2(v) = L_2(v)L_1(u)\overline{R}(u-v)\ .$$

Here $L_1(u) = L(u) \otimes 1$ and $L_2(u) = 1 \otimes L(u)$.

More precisely, we would like to consider the Hopf algebras, which are quantum doubles and which have a family of $(2\ell+1)$-dimensional representations $\pi_z^{(\ell)}$ parametrized by the parameter $z \in \mathbb{C}$. Here ℓ is the spin of the representations. For the simplest nontrivial two-dimensional representation we require the L–operator to be proportional to the R-matrix (2.1):

$$\pi_z^{(1/2)}\left(L(u)\right) \sim \overline{R}(u-z)\ . \tag{2.3}$$

Due to [12] such an algebra can be defined via two generating matrix-valued functions $L^{\pm}(u)$ which satisfy the relations

$$\begin{aligned} \overline{R}(u_1-u_2)L_1^{+}(u_1)L_2^{-}(u_2) \;&=\; L_2^{-}(u_2)L_1^{+}(u_1)\overline{R}(u_1-u_2)\ , \\ \overline{R}(u_1-u_2)L_1^{\pm}(u_1)L_2^{\pm}(u_2) \;&=\; L_2^{\pm}(u_2)L_1^{\pm}(u_1)\overline{R}(u_1-u_2)\ , \qquad (2.4) \\ \text{q-det}\,L^{\pm}(u) = L_{11}^{\pm}(u-i\hbar)L_{22}^{\pm}(u) \;&-\; L_{12}^{\pm}(u-i\hbar)L_{21}^{\pm}(u) = 1\ . \qquad (2.5) \end{aligned}$$

As usual, the relation (2.5) on the q-determinant factorizes the algebra over primitive central elements which appear due to degeneracy of the R-matrix at critical points $u = \pm i\hbar$. This ensures the existence of antipode and turn the bialgebra into a Hopf algebra. Two L-operators $L^{\pm}(u)$ generate two dual Hopf subalgebras of the quantum double.

It was shown in [2] that the relations (2.4) (2.5) can be interpreted as the defining relations for the quantum double of the Yangian associated with $\mathfrak{sl}_2$ [13].

The two-dimensional representation (2.3) is

$$\pi_z^{(1/2)}\left(L^{\pm}(u)\right)=\rho^{\pm}(u-z)\overline{R}(u-z)\ , \tag{2.6}$$

where the functions $\rho^{\pm}(u)$ satisfy the equation

$$\rho^{\pm}(u)\rho^{\pm}(u-i\hbar)=\frac{u-i\hbar}{u}\ ,$$

which follows from q-determinant condition (2.5) and could be chosen, in particular, as follows:

$$\rho^{\pm}(u)=\left[\frac{\Gamma^2\left(\frac{1}{2}\mp\frac{u}{2i\hbar}\right)}{\Gamma\left(1\mp\frac{u}{2i\hbar}\right)\Gamma\left(\mp\frac{u}{2i\hbar}\right)}\right]^{\mp 1}\ . \tag{2.7}$$

Moreover, the solution (2.7) is unique for certain analyticity conditions on the L-operators $L^{\pm}(u)$ which we discuss later. Nevertheless the formal current algebras which we define further do not depend on the choice of $\rho^{\pm}(u)$.

The latter algebra of L-operators appears to have only finite-dimensional representations. In order to construct infinite-dimensional representations we need to perform the central extension of this formal algebra. It can be done as follows. The algebra of L-operators (2.4), (2.5) admits a family of shifting automorphisms $T_zL^{\pm}(u)=L^{\pm}(u-z)$. Then the extension of the quantum double by the infinitisemal shift operator d and the dual to it element c [14, 3] leads to the following commutation relations [15, 16]:

$$\begin{aligned}
[L(u),c] &= 0\ ,\\
\mathrm{e}^{ad}L^{\pm}(u) &= L^{\pm}(u+a)\mathrm{e}^{ad},\\
R^{+}(u_1-u_2+ic\hbar/2)L_1^{+}(u_1)L_2^{-}(u_2) &= L_2^{-}(u_2)L_1^{+}(u_1)R^{+}(u_1-u_2-ic\hbar/2),\\
R^{\pm}(u_1-u_2)L_1^{\pm}(u_1)L_2^{\pm}(u_2) &= L_2^{\pm}(u_2)L_1^{\pm}(u_1)R^{\pm}(u_1-u_2),\\
\text{q-det}\,L^{\pm}(u)=L_{11}^{\pm}(u-i\hbar)L_{22}^{\pm}(u) &- L_{12}^{\pm}(u-i\hbar)L_{21}^{\pm}(u)=1\ ,
\end{aligned} \tag{2.8}$$

where

$$R^{\pm}(u)=\rho^{\pm}(u)\overline{R}(u)\ .$$

We call this algebra the (central extended) algebra of L-operators. Let us stress that this is still a formal algebra since we did not specify the meaning of its generating functions $L^{\pm}(u)$.

We can go further to proceed in algebraic manipulations with formal algebra of L-operators. First, it is natural to factorize explicitly the quantum determinat and have three generating functions instead of four analogous to classical passage from $\mathfrak{gl}_2$ to $\mathfrak{sl}_2$. This can be done by means of Gauss decomposition of the L-operators.

Let

$$L^{\pm}(u)=\begin{pmatrix}1 & f^{\pm}(u)\\ 0 & 1\end{pmatrix}\begin{pmatrix}k_1^{\pm}(u) & 0\\ 0 & k_2^{\pm}(u)\end{pmatrix}\begin{pmatrix}1 & 0\\ e^{\pm}(u) & 1\end{pmatrix}\ , \tag{2.9}$$

be the Gauss decomposition of L-operators. The condition (2.5) implies that the product of the entries of diagonal matrix in r.h.s. of (2.9) is equal to one: $k_1^{\pm}(u)k_2^{\pm}(u+i\hbar)=1$, therefore they are invertible and

$$k_1^{\pm}(u)=\left(k_2^{\pm}(u+i\hbar)\right)^{-1}.$$

Let

$$h^{\pm}(u) = \left(k_2^{\pm}(u+i\hbar)\right)^{-1}\left(k_2^{\pm}(u)\right)^{-1} .$$

We call the operator valued generating functions $e^{\pm}(u)$, $f^{\pm}(u)$ and $h^{\pm}(u)$ the Gauss coordinates of the L-operators. We have the following

Proposition 1. *The Gauss coordinates $e^{\pm}(u)$, $f^{\pm}(u)$ and $h^{\pm}(u)$ satisfy the following commutation relations*

$$\begin{aligned}
h^{\pm}(u)h^{\pm}(v) &= h^{\pm}(v)h^{\pm}(u) , \\
[e^{\pm}(u), f^{\pm}(v)] &= i\hbar\frac{h^{\pm}(u)-h^{\pm}(v)}{u-v} , \\
[h^{\pm}(u), e^{\pm}(v)] &= i\hbar\frac{\{h^{\pm}(u), e^{\pm}(u)-e^{\pm}(v)\}}{u-v} , \\
[h^{\pm}(u), f^{\pm}(v)] &= -i\hbar\frac{\{h^{\pm}(u), f^{\pm}(u)-f^{\pm}(v)\}}{u-v} , \\
[e^{\pm}(u), e^{\pm}(v)] &= i\hbar\frac{(e^{\pm}(u)-e^{\pm}(v))^2}{u-v} , \\
[f^{\pm}(u), f^{\pm}(v)] &= -i\hbar\frac{(f^{\pm}(u)-f^{\pm}(v))^2}{u-v} , \\
h^{+}(u)h^{-}(v) &= \frac{(u-v-i\hbar(1+c/2))(u-v+i\hbar(1+c/2))}{(u-v+i\hbar(1-c/2))(u-v-i\hbar(1-c/2))}\; h^{-}(v)h^{+}(u) , \\
[e^{\pm}(u), f^{\mp}(v)] &= i\hbar\frac{h^{\pm}(u)}{u-v\pm ic\hbar/2} - i\hbar\frac{h^{\mp}(v)}{u-v\mp ic\hbar/2} , \\
[h^{\pm}(u), e^{\mp}(v)] &= i\hbar\frac{\{h^{\pm}(u), e^{\pm}(u)-e^{\mp}(v)\}}{u-v\mp ic\hbar/2} , \\
[h^{\pm}(u), f^{\mp}(v)] &= -i\hbar\frac{\{h^{\pm}(u), f^{\pm}(u)-f^{\mp}(v)\}}{u-v\pm ic\hbar/2} , \\
[e^{\pm}(u), e^{\mp}(v)] &= i\hbar\frac{(e^{\pm}(u)-e^{\mp}(v))^2}{u-v\mp ic\hbar/2} , \\
[f^{\pm}(u), f^{\mp}(v)] &= -i\hbar\frac{(f^{\pm}(u)-f^{\pm}(v))^2}{u-v\pm ic\hbar/2} .
\end{aligned}$$

Proof. The proof is a direct substitution of the Gauss decomposition of L-operators (2.9) into (2.8). The particular case of (2.8) at $u_1 = u_2 - i\hbar$:

$$\begin{aligned}
k_2^{\pm}(u)e^{\pm}(u) &= e^{\pm}(u-i\hbar)k_2^{\pm}(u) , \\
k_2^{\pm}(u)f^{\pm}(u-i\hbar) &= f^{\pm}(u)k_2^{\pm}(u) ,
\end{aligned}$$

is very useful for this algebraic exercise. For the analogous treatment of quantum affine algebras see [10].

One can see that the algebra of Gauss coordinates does not depend on a choice of the factor $\rho^{\pm}(u)$ in the definition of $R^{\pm}(u)$. It refers only to the original Yang R-matrix.

2.2. Hopf structure of the algebra of L-operators. The L-operator's language is convenient for the description of the coalgebraic structure. The comultiplication map for the algebra (2.8) of formal L-operators is given by the formulas

$$\begin{aligned} \Delta\, c &= c^{(1)} + c^{(2)} = c\otimes 1 + 1\otimes c\ , \\ \Delta\, d &= d\otimes 1 + 1\otimes d\ , \\ \Delta' L^{\pm}(u) &= L(u\pm i\hbar c^{(2)}/4)\dot{\otimes} L(u\mp i\hbar c^{(1)}/4) \end{aligned} \tag{2.10}$$

or in components

$$\Delta L^{\pm}_{ij}(u) = \sum_{k=1}^{2} L^{\pm}_{kj}(u\mp i\hbar c^{(2)}/4)\otimes L_{ik}(u\pm i\hbar c^{(1)}/4)\ . \tag{2.11}$$

The antipode and counit are:

$$S(L^{\pm}(u)) = (L^{\pm}(u))^{-1}\ , \tag{2.12}$$

$$\epsilon(L^{\pm}_{ij}(u)) = \delta_{ij}\ . \tag{2.13}$$

The comultiplications of the Gauss coordinates $e^{\pm}(u)$, $f^{\pm}(u)$ and $h^{\pm}(u)$ reads as follows:

$$\Delta e^{\pm}(u) = e^{\pm}(u')\otimes 1 + \sum_{p=0}^{\infty}(-1)^p\left(f^{\pm}(u'-i\hbar)\right)^p h^{\pm}(u')\otimes\left(e^{\pm}(u'')\right)^{p+1}, \tag{2.14}$$

$$\Delta f^{\pm}(u) = 1\otimes f^{\pm}(u'') + \sum_{p=0}^{\infty}(-1)^p\left(f^{\pm}(u')\right)^{p+1}\otimes h^{\pm}(u'')\left(e^{\pm}(u''-i\hbar)\right)^p, \tag{2.15}$$

$$\Delta h^{\pm}(u) = \sum_{p=0}^{\infty}(-1)^p(p+1)\left(f^{\pm}(u'-i\hbar)\right)^p h^{\pm}(u')\otimes h^{\pm}(u'')\left(e^{\pm}(u''-i\hbar)\right)^p, \tag{2.16}$$

where $u' = u\mp i\hbar c^{(2)}/4$ and $u'' = u\pm i\hbar c^{(1)}/4$. The proof of these formulas can be found in [6].

2.3. Current realization of the algebra of L-operators. To construct the infinite dimensional representation theory of the central extended L-operator algebra the Gauss coordinates are inconvenient and it is natural to introduce the total currents. Let

$$\begin{aligned} e(u) &= e^{+}\left(u+\frac{ic\hbar}{4}\right) - e^{-}\left(u-\frac{ic\hbar}{4}\right)\ , \\ f(u) &= f^{+}\left(u-\frac{ic\hbar}{4}\right) - f^{-}\left(u+\frac{ic\hbar}{4}\right)\ , \end{aligned} \tag{2.17}$$

be the combination of the Gauss coordinates of the L-operators which we call the total currents. One can verify that the commutation relations between the total currents and $h^{\pm}(u)$ close:

$$\begin{aligned} [d, e(u)] &= \frac{d}{du}e(u), \qquad [d, f(u)] = \frac{d}{du}f(u), \\ h^{\pm}(u)e(v) &= \frac{(u-v-i\hbar(1\pm c/4))}{(u-v+i\hbar(1\mp c/4))}\ e(v)h^{\pm}(u)\ , \\ h^{\pm}(u)f(v) &= \frac{(u-v+i\hbar(1\pm c/4))}{(u-v-i\hbar(1\mp c/4))}\ f(v)h^{\pm}(u)\ , \end{aligned}$$

$$
\begin{aligned}
e(u)e(v) &= \frac{(u-v-i\hbar)}{(u-v+i\hbar)}\, e(v)e(u)\ , \\
f(u)f(v) &= \frac{(u-v+i\hbar)}{(u-v-i\hbar)}\, f(v)f(u)\ ,
\end{aligned}
$$

$$
[e(u),f(v)] = -i\hbar\left[\delta\left(u-v+\frac{ic\hbar}{2}\right)h^{+}\left(u+\frac{ic\hbar}{4}\right) - \delta\left(u-v-\frac{ic\hbar}{2}\right)h^{-}\left(v+\frac{ic\hbar}{4}\right)\right], \qquad (2.18)
$$

where the δ-function is defined by the formal equality

$$
\delta(u-v) = \frac{1}{u-v} - \frac{1}{u-v}
$$

and satisfies the relation

$$
g(u)\delta(u-v) = g(v)\delta(u-v)\ .
$$

In the next section we develop a technique of extracting the correctly define (Hopf) algebra from the above current algebra provided certain analytical conditions are supposed.

3 Factorization and the Riemann problems

3.1. Until now we did not specify the analytical properties of L-operators. Let us fix them in the following manner:

$$
\begin{aligned}
&L^{+}(u) \quad \text{is analytical in some neighborhood of } u=\infty\ , \\
&L^{-}(u) \quad \text{is analytical in some neighborhood of } u=0\ .
\end{aligned}
$$

To simplify the consideration we set $c=0$ first and restore the central element only in the final formulas. We would like to invert the relation

$$
e(u) = e^{+}(u) - e^{-}(u)\ , \qquad (3.1)
$$

namely to express the generating functions $e^{\pm}(u)$ through the total current $e(u)$ by means of some integral transforms. More precisely, let us suppose that

- (i) the function $e(u)$ is analytical function which has only isolated singularities on compactified complex plane;
- (ii) the functions $e^{+}(u)$ is analytical at infinity and the function $e^{-}(u)$ is analytical at zero.

We call the solution of (3.1) with the conditions (i) and (ii) the classical Riemann problem for a circle. Let us first fix a closed counterclockwise oriented contour Γ around 0 (for instance, $\Gamma=\{|u|=1\}$). Then the Cauchy type integrals $\tilde{e}^{\pm}(u) = \oint_{\Gamma} \frac{dv}{2\pi i}\frac{e(v)}{u-v}$, where u is outside or inside Γ give two functions $\tilde{e}^{\pm}(u)$ analytical outside and inside of Γ such that the relation (3.1) is valid for the points $u\in\Gamma$. The limiting values of these functions when u tends to a contour are given by Sokhotsky-Plemely relations:

$$
\tilde{e}^{\pm}(u) = \frac{1}{2}\left[\text{V.P.}\oint_{\Gamma}\frac{dv}{\pi i}\frac{e(v)}{u-v} \pm e(u)\right], \quad u\in\Gamma\ . \qquad (3.2)
$$

Suppose also that the function $e(u)$ is analytical in the points of Γ so we have the identity

$$e(u) = \pm \int\limits_{C^{\pm}_{u,\epsilon}} \frac{dv}{\pi i} \frac{e(v)}{u-v} \ , \tag{3.3}$$

where $C^{\pm}_{u,\epsilon}$ are small semicircles around the point u of the radius ϵ drown in the different directions as shown on the Fig. 1.

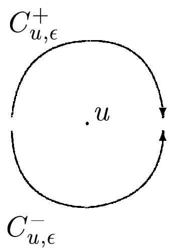

Fig. 1.

Using (3.3) we can obtain the solution of (3.1) in the domain of analyticity of $e(u)$ nearby the contour Γ:

$$e^{\pm}(u) = \oint\limits_{\Gamma_{\pm}} \frac{dv}{2\pi i} \frac{e(v)}{u-v} = \oint\limits_{|v| \lessgtr |u|} \frac{dv}{2\pi i} \frac{e(v)}{u-v} \ , \tag{3.4}$$

where the contours $\Gamma_{\pm} = \Gamma \oplus C^{\pm}_{u,\epsilon}$ are shown on the Fig. 2. The distinction between the generating functions $\tilde{e}^{\pm}(u)$ and $e^{\pm}(u)$ is that the former are related to the Riemann problem with fixed contour, while the latter with the problem where contours $\Gamma_{\pm}$ are not strictly fixed.

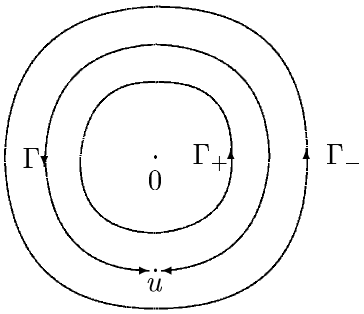

Fig. 2.

Now we forget the initially fixed contour Γ and use the integral transforms (3.4) as the solution of (3.1) satisfying the condition (ii), which is valid due to (i). The contours Γ_{+} (Γ_{-}) should be chosen close to the point u which means that they are located in the regipns $|u| - \epsilon < |v| < |u|$ $(|u| < |v| < |u| + \epsilon)$ of the analyticity of the functions $e(v)$.

Restoring the dependence on the central element in (2.17) we obtain analogously:

$$\begin{aligned}
e^{+}(u) &= \oint\limits_{|v|<|u-ic\hbar/4|} \frac{dv}{2\pi i} \frac{e(v)}{(u-v-ic\hbar/4)} \ , \\
f^{+}(u) &= \oint\limits_{|v|<|u+ic\hbar/4|} \frac{dv}{2\pi i} \frac{f(v)}{(u-v+ic\hbar/4)} \ , \\
e^{-}(u) &= \oint\limits_{|v|>|u+ic\hbar/4|} \frac{dv}{2\pi i} \frac{e(v)}{(u-v+ic\hbar/4)} \ , \\
f^{-}(u) &= \oint\limits_{|v|>|u-ic\hbar/4|} \frac{dv}{2\pi i} \frac{f(v)}{(u-v-ic\hbar/4)} \ ,
\end{aligned} \tag{3.5}$$

where all contours in the integrals are counterclockwise circles around the point $v = 0$ such that the corresponding points $u \pm i\hbar c/4$ are either outside of the contours for the generating functions $e^+(u)$, $f^+(u)$ or inside the contours for the generating functions $e^-(u)$, $f^-(u)$.

The integral transformations (3.5) are the solutions of the Riemann problems for the factorization of the entire function into sum of the functions analytical in the neighborhood of the given point ($u = 0$) and $u = \infty$ respectively.

The δ-function in the commutation relation of the total currents $e(u)$ and $f(v)$ should be also presented in terms of the same Riemann problem:

$$\delta(u - v) = \left.\frac{1}{u - v}\right|_{|u|>|v|} - \left.\frac{1}{u - v}\right|_{|u|<|v|} = \sum_{n+m=-1} u^n v^m \ . \tag{3.6}$$

The relations (3.5) dictate the precise sense of the operator valued functions $e(u)$, $f(u)$, $e^\pm(u)$ and $f^\pm(u)$ as the generating series of the elements of the algebra $\widehat{DY(\mathfrak{sl}_2)}$ (central extended Yangian double). If we decompose the currents $e^\pm(u)$ into Taylor series in the points of their regularity (∞ and 0):

$$e^\pm(u) = \mp i\hbar \sum_{\substack{k\geqslant 0\\ k<0}} e_k (u \mp ic\hbar/4)^{-k-1} \ , \quad f^\pm(u) = \mp i\hbar \sum_{\substack{k\geqslant 0\\ k<0}} f_k (u \pm ic\hbar/4)^{-k-1} \ ,$$

then, due to (3.1), we have the presentation

$$e(u) = -i\hbar \sum_{n\in\mathbb{Z}} e_n u^{-n-1} \ , \quad f(u) = -i\hbar \sum_{n\in\mathbb{Z}} f_n u^{-n-1} \ . \tag{3.7}$$

We see that the decompositions (2.17) modulo technical shifts are just the decompositions of a formal power series into the parts with positive and negative powers and the coefficients of the series are given as

$$e_n = \pm\frac{1}{2\pi\hbar} \oint_{C_\pm} v^n e^\pm(v \mp ic\hbar/4) dv, \qquad f_n = \pm\frac{1}{2\pi\hbar} \oint_{C_\pm} v^n f^\pm(v \pm ic\hbar/4) dv \ , \tag{3.8}$$

or, equivalently (this follows from the definition (3.4))

$$e_n = \frac{1}{2\pi\hbar} \oint_{C_\pm} v^n e(v) dv, \qquad f_n = \frac{1}{2\pi\hbar} \oint_{C_\pm} v^n f(v) dv \ , \tag{3.9}$$

where C_+ is a contour surrounding the infinity (clockwise, like Γ_+) and is taken for $n \geqslant 0$ while C_- is a contour surrounding zero (counterclockwise, like Γ_-) and is taken for $n < 0$. Note that in the following we use (3.8) as basic definitions since for the functions $e^\pm(u)$ and $f^\pm(u)$ of the distribution type, the integrals in (3.8) make precise sense contrary to the integrals in (3.9). It is natural also to assume that the currents $h^\pm(u)$ satisfy the same analyticity conditions as $e^\pm(u), f^\pm(u)$ and put

$$h^\pm(u) = 1 \mp i\hbar \sum_{\substack{k\geqslant 0\\ k<0}} h_k u^{-k-1},$$

where

$$h_n = \pm\frac{1}{2\pi\hbar} \int_{C_\pm} (h^\pm(u) - 1) u^n du \tag{3.10}$$

with the same rule for the signs. The unity in the above formula is introduced because of the group-like nature of the Cartan generating series.

3.2. There is another choice of the analytical data. Let us fix the following analytical behavior of the operators $L^{\pm}(u)$:

$$
\begin{aligned}
L^{+}(u) &\quad \text{is analytical in } \mathbb{C} \text{ for} \quad \operatorname{Im} u < -A \ , \\
L^{-}(u) &\quad \text{is analytical in } \mathbb{C} \text{ for} \quad \operatorname{Im} u > A \ ,
\end{aligned}
$$

for some positive A. In an analogous manner we obtain that for a meromorphic function $e(u)$ decreasing when $\operatorname{Re} u \to \pm\infty$ the Cauchy type integrals

$$
\tilde{e}^{\pm}(u) = \int_{-\infty}^{\infty} \frac{dv}{2\pi i} \frac{e(v)}{u-v}
$$

are analytical in closed half-planes $\operatorname{Im} u \leqslant 0$ and $\operatorname{Im} u \geqslant 0$; they satisfy the relation (3.1) for real u and can be used for the following presentations of the functions $e^{\pm}(u)$ satisfying (3.1) for all u and analytical in half-planes Im $u < -A$ and Im $u > A$ for some positive A:

$$
e^{\pm}(u) = \int_{\operatorname{Im} v \gtrless \operatorname{Im} u} \frac{dv}{2\pi i} \frac{e(v)}{u-v} \ . \tag{3.11}
$$

The contours of the integrations in (3.11) are close to the point u which means that there are no singularities of the function $e(z)$ in the strips $\operatorname{Im} u \lessgtr \operatorname{Im} z \lessgtr \operatorname{Im} v$.

Restoring the central elements we obtain:

$$
\begin{aligned}
e^{+}(u) &= \int_{\operatorname{Im}(u-v)<c\hbar/4} \frac{dv}{2\pi i} \ \frac{e(v)}{(u-v-ic\hbar/4)}, \\
f^{+}(u) &= \int_{\operatorname{Im}(u-v)<-c\hbar/4} \frac{dv}{2\pi i} \ \frac{f(v)}{(u-v+ic\hbar/4)} \ , \\
e^{-}(u) &= \int_{\operatorname{Im}(u-v)>-c\hbar/4} \frac{dv}{2\pi i} \ \frac{e(v)}{(u-v+ic\hbar/4)}, \\
f^{-}(u) &= \int_{\operatorname{Im}(u-v)>c\hbar/4} \frac{dv}{2\pi i} \ \frac{f(v)}{(u-v-ic\hbar/4)} \ .
\end{aligned} \tag{3.12}
$$

The corresponding integral transforms are related to the Riemann problem of factorizing the decreasing meromorphic function into a sum of functions analytical in certain upper and lower half-planes of the complex plane.

The δ-function in the commutation relation of the total currents $e(u)$ and $f(v)$ should be also the solution of the same Riemann problem and is given by the integral:

$$
\delta(u-v) = \lim_{\epsilon\to 0}\left[\frac{1}{u-v-i\epsilon} - \frac{1}{u-v+i\epsilon}\right] = i\int_{-\infty}^{\infty} d\lambda \ \mathrm{e}^{-i\lambda(u-v)} \ . \tag{3.13}
$$

Analogously to the case of Riemann problem for a circle, the integral relations (3.12) dictate the precise sense of the operator valued functions $e(u)$, $f(u)$, $e^{\pm}(u)$ and $f^{\pm}(u)$ as the generating integrals of the elements of the algebra $\mathcal{A}_{\hbar}(\widehat{\mathfrak{sl}_2})$ (degeneration of $\mathcal{A}_{\hbar,\eta}(\widehat{\mathfrak{sl}_2})$, $\eta \to 0$, [6]). Namely, the analytical in half-planes Im $u < -A$ and Im $u > A$ functions $e_+(u)$, $f_+(u)$ and $e_-(u)$, $f_-(u)$ can be presented via Laplace integrals:

$$e^{\pm}(u) = \hbar \int_0^{\pm\infty} d\lambda\ \mathrm{e}^{-i\lambda u}\ \hat{e}_\lambda \mathrm{e}^{-c\hbar|\lambda|/4}\ , \quad f^{\pm}(u) \ = \ \hbar \int_0^{\pm\infty} d\lambda\ \mathrm{e}^{-i\lambda u}\ \hat{f}_\lambda \mathrm{e}^{c\hbar|\lambda|/4}\ . \tag{3.14}$$

Then, due to (2.17), we have the presentation of the total currents

$$e(u) = \hbar \int_{-\infty}^{\infty} d\lambda\ \mathrm{e}^{-i\lambda u} \hat{e}_\lambda\ , \quad f(u) = \hbar \int_{-\infty}^{\infty} d\lambda\ \mathrm{e}^{-i\lambda u} \hat{f}_\lambda \tag{3.15}$$

and the relation (2.17) is the decomposition of formal Fourier integral into the sum of two Laplace transforms.

Due to the definition and the invertion formulas for the Laplace transforms, the Fourier modes $\hat{e}_\lambda$ and $\hat{f}_\lambda$ are the following integrals:

$$\hat{e}_\lambda \ = \ \pm\frac{\mathrm{e}^{c\hbar|\lambda|/4}}{2\pi\hbar} \int_{C_\pm} du\, \mathrm{e}^{i\lambda u} e^{\pm}(u), \quad \hat{e}_0 = \frac{1}{2\pi\hbar}\left(\int_{C_+} du\, e^+(u) - \int_{C_-} du\, e^-(u) \right), \tag{3.16}$$

$$\hat{f}_\lambda \ = \ \pm\frac{\mathrm{e}^{-c\hbar|\lambda|/4}}{2\pi\hbar} \int_{C_\pm} du\, \mathrm{e}^{i\lambda u} f^{\pm}(u), \quad \hat{f}_0 = \frac{1}{2\pi\hbar}\left(\int_{C_+} du\, f^+(u) - \int_{C_-} du\, f^-(u) \right), \tag{3.17}$$

where C_+ is the line parallel to the real axis and belonging to the half-plane Im $u < -A$ for $\lambda > 0$ and C_- is also the line parallel to the real axis but in another half-plane Im $u > A$ for $\lambda < 0$. All the integrals are principal value integrals. Again we can use total currents $e(u)$ and $f(u)$ in (3.17):

$$\begin{aligned} \hat{e}_\lambda \ &= \ \frac{1}{2\pi\hbar} \int_{C_\pm} du\, \mathrm{e}^{i\lambda u} e(u), \quad \hat{e}_0 = \frac{1}{2\pi\hbar}\left(\int_{C_+} du\, e(u) + \int_{C_-} du\, e(u) \right), \\ \hat{f}_\lambda \ &= \ \frac{1}{2\pi\hbar} \int_{C_\pm} du\, \mathrm{e}^{i\lambda u} f(u), \quad \hat{f}_0 = \frac{1}{2\pi\hbar}\left(\int_{C_+} du\, f(u) + \int_{C_-} du\, f(u) \right). \end{aligned}$$

It is natural to suppose also that

$$h^{\pm}(u) = 1 + \hbar \int_0^{\pm\infty} d\lambda\ \mathrm{e}^{-i\lambda u}\ \hat{h}_\lambda \mathrm{e}^{-c\hbar|\lambda|/4}\ , \tag{3.18}$$

where $\hat{h}_\lambda$ are given by principal value integrals

$$\hat{h}_\lambda \ = \ \pm\frac{\mathrm{e}^{c\hbar|\lambda|/4}}{2\pi\hbar} \int_{C_+} du\, \mathrm{e}^{i\lambda u} (h^+(u) - 1), \quad \text{for} \quad \lambda \neq 0\ , \tag{3.19}$$

$$\hat{h}_0 \ = \ \frac{1}{2\pi\hbar}\left(\int_{C_+} du\, (h^+(u) - 1) - \int_{C_-} du\, (h^-(u) - 1) \right). \tag{3.20}$$

We see that two different Riemann problems correspond to two different algebras. We show in the next sections that they are the central extended Yangian double and the rational degeneration of $\mathcal{A}_{\hbar,\eta}(\widehat{\mathfrak{sl}_2})$. Let us note once more that these Hopf algebras can be completely defined by the formal algebra of total currents $e(u)$, $f(u)$, $h^{\pm}(u)$ and by the type of the Riemann problem. We will return to this point in the last section.

4 Algebras $DY(\widehat{\mathfrak{sl}_2})$ and $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$

4.1. The algebra $DY(\widehat{\mathfrak{sl}_2})$. Let us consider the above algebra specialized by the Riemann problem (3.5) for a circle in more details. For the readers convenience we identify $-i\hbar = \nu$. As we have seen before, this algebra (we denote it for a short time by letter D) can be defined as an algebra generated by the elements $c, d, e_n, f_n, h_n, n \in \mathbb{Z}$ gathered into generating functions

$$e(u) = \nu \sum_{k\in\mathbb{Z}} e_k u^{-k-1}, \qquad f(u) = \nu \sum_{k\in\mathbb{Z}} f_k u^{-k-1},$$

$$h^{\pm}(u) = 1 \pm \nu \sum_{\substack{k\geqslant 0\\ k<0}} h_k u^{-k-1} \tag{4.1}$$

which satisfy the relations

$$\begin{aligned}
[d, e(u)] &= \frac{\mathrm{d}}{\mathrm{d}u} e(u), \qquad [d, f(u)] = \frac{\mathrm{d}}{\mathrm{d}u} f(u),\\
h^+(u)h^-(v) &= \frac{(u-v+\nu(1+c/2))(u-v-\nu(1+c/2))}{(u-v-\nu(1-c/2))(u-v+\nu(1-c/2))}\ h^-(v)h^+(u)\ ,\\
h^{\pm}(u)e(v) &= \frac{(u-v+\nu(1\pm c/4))}{(u-v-\nu(1\mp c/4))}\ e(v)h^{\pm}(u)\ ,\\
h^{\pm}(u)f(v) &= \frac{(u-v-\nu(1\pm c/4))}{(u-v+\nu(1\mp c/4))}\ f(v)h^{\pm}(u)\ ,\\
e(u)e(v) &= \frac{(u-v+\nu)}{(u-v-\nu)}\ e(v)e(u)\ ,\\
f(u)f(v) &= \frac{(u-v-\nu)}{(u-v+\nu)}\ f(v)f(u)\ ,
\end{aligned}$$

$$[e(u), f(v)] = \nu\left[\delta\left(u-v-\frac{c\nu}{2}\right)h^+\left(u-\frac{c\nu}{4}\right) - \delta\left(u-v+\frac{c\nu}{2}\right)h^-\left(v-\frac{c\nu}{4}\right)\right], \tag{4.2}$$

The total currents $e(u)$ and $f(u)$ decompose into the sums

$$e(u) = e^+(u-c\nu/4) - e^-(u+c\nu/4)\ , \quad f(u) = f^+(u+c\nu/4) - f^-(u-c\nu/4),$$

where

$$e^{\pm}(u) = \pm\nu \sum_{\substack{k\geqslant 0\\ k<0}} e_k (u\mp c\nu/4)^{-k-1}, \quad f^{\pm}(u) = \pm\nu \sum_{\substack{k\geqslant 0\\ k<0}} f_k (u\pm c\nu/4)^{-k-1},$$

due to the relations (3.5) and the generators of the algebra e_k, f_k, h_k are given by the inversion formulas (3.8), (3.10). The coalgebraic structure is given by the relations (2.14)–(2.16) for the generating functions $h^{\pm}(u)$ and $e^{\pm}(u)$, $f^{\pm}(u)$.

Let us translate these data into the language of generators e_k, f_k, h_k. Due to the definition of these generators, the translation should be done in two steps. In the first step, following the definitions (3.5) we get from the relations (4.2) the relations for the currents $e^{\pm}(u)$, $f^{\pm}(u)$, $h^{\pm}(u)$. These relations will be precise those given by Proposition 1. Then we put the spectral parameters into the domains of the analyticity of these currents and use the definitions (3.8) and (3.10) of the generators e_n, f_n and h_n. An example of such a calculation will be given for a Riemann problem on a line. Note that for a Riemann problem for a circle the result coincides with formal substitution of power series (4.1), (3.6) into relations (4.2).

At $c = 0$ the full set of the commutation relations is:

$$[d, e_n] = -ne_{n-1}\ , \quad [d, f_n] \ = \ -nf_{n-1}\ , \quad [d, h_n] \ = \ -nh_{n-1}\ , \tag{4.3}$$

$$[h_0, e_n] = 2e_n\ , \quad [h_0, f_n] \ = \ -2f_n\ , \tag{4.4}$$

$$[h_k, h_n] = 0\ , \quad [e_k, f_n] = h_{k+n}\ , \tag{4.5}$$

$$[h_{k+1}, e_n] - [h_k, e_{n+1}] = \nu\{h_k, e_n\}\ , \quad [h_{k+1}, f_n] - [h_k, f_{n+1}] = -\nu\{h_k, f_n\}\ , \tag{4.6}$$

$$[e_{k+1}, e_n] - [e_k, e_{n+1}] = \nu\{e_k, e_n\}\ , \quad [f_{k+1}, f_n] - [f_k, f_{n+1}] = -\nu\{f_k, f_n\}\ . \tag{4.7}$$

When $c \neq 0$ the commutation relations of the type $[h_k, e_n]$, $[h_k, f_n]$, $[h_k, h_n]$ and $[e_k, f_n]$ become more complicated while the rest are unchanged.

$$[h_{k+1}, e_n] - [h_k, e_{n+1}] + c\nu\vartheta(k)[h_k, e_n]/4 = \nu\{h_k, e_n\}\ , \tag{4.8}$$

$$[h_{k+1}, f_n] - [h_k, f_{n+1}] + c\nu\vartheta(k)[h_k, f_n]/4 = -\nu\{h_k, f_n\}\ , \tag{4.9}$$

$$[h_0, h_k] = 0, \quad k \in \mathbb{Z}$$

$$\begin{aligned}[][h^+_{k+2}, h^-_{-n-1}] - 2[h^+_{k+1}, h^-_{-n}] + [h^+_{k+2}, h^-_{-n-1}] &= \\ = \eta^2(1 + c^2/4)[h^+_k, h^-_{-n-1}] - 2c\eta^2\{h^+_k, h^-_{-n-1}\}, &\quad k, n \geqslant 0\ ,\end{aligned} \tag{4.10}$$

$$[e_n, f_p] = \sum_{k=0}^{n+p} h_{n+p-k}(-c\nu/4)^k B^k_{n,p}\ , \tag{4.11}$$

$$[e_{-n-1}, f_{-p-1}] = \sum_{k\geqslant 0} h_{-n-p-k-2}(-c\nu/4)^k D^k_{n,p}\ , \tag{4.12}$$

$$[e_n, f_{-p-1}] = \nu^{-1}\sum_{k\geqslant 0}[(-1)^k h^+_{n-p-1-k} - h^-_{n-p-1-k}](-c\nu/4)^k A^k_{n,p}\ , \tag{4.13}$$

$$[e_{-p-1}, f_n] = \nu^{-1}\sum_{k\geqslant 0}[h^+_{n-p-1-k} - (-1)^k h^-_{n-p-1-k}](-c\nu/4)^k A^k_{n,p}\ , \tag{4.14}$$

where in last four relations $n, p \geqslant 0$,

$$A^k_{n,p} = \sum_{k'=0}^{k} C^{k'}_n C^p_{k+p-k'}, \quad B^k_{n,p} = \sum_{k'=0}^{k}(-1)^{k'} C^{k'}_n C^{k-k'}_p, \quad D^k_{n,p} = \sum_{k'=0}^{k}(-1)^{k'} C^n_{n+k'} C^p_{p+k-k'},$$

$C^{k'}_k$ are binomial coefficients,

$$\vartheta(k) = \begin{cases} 1, & k \geqslant 0\ , \\ -1, & k < 0\ ,\end{cases}$$

and in order to write dow the commutation relations between Cartan generators h_k and the commutation relations (4.13) and (4.14) we introduced the short notations: $h^+_{-1} = 1$, $h^+_k = \nu h_k$, for $k > -1$ and $h^+_k = 0$ for $k < -1$. Analogously, $h^-_{-1} = 1 - \nu h_{-1}$, $h^-_k = -\nu h_k$ for $k < -1$ and $h^-_k = 0$ for $k > -1$.

One can see that the commutation relations (4.8), (4.14) coincide with the commutation relations for the generators of the central extended Yangian Double $\widehat{DY(\mathfrak{sl}_2)}$ (as well as comultiplication rules). More precisely, the algebra D defined by the relations (4.14) admit the filtration

$$\ldots \subset D_{-n} \subset \ldots \subset D_{-1} \subset D_0 \subset D_1 \ldots \subset D_n \ldots \subset D \tag{4.15}$$

defined by conditions $\deg e_k = \deg f_k = \deg h_k = k$; $\deg\{x \in D_m\} \leqslant m$. Then the formal completion of D over this filtration is a Hopf algebra which coincides with $\widehat{DY(\mathfrak{sl}_2)}$.

4.2. The algebra $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$. Let us turn our attention to the Riemann problem for half-planes (3.5). As we have seen in the previous section, the corresponding algebra $\mathcal{A}$ is generated by the elements $\hat{e}_\lambda$, $\hat{f}_\lambda$, $\hat{h}_\lambda$, $\lambda \in \mathbb{R}$, c and $\tilde{d}$ gathered into generating integrals

$$e(u) = \hbar \int_{-\infty}^{+\infty} d\lambda\ e^{-i\lambda u} \hat{e}_\lambda\ , \qquad f(u) = \hbar \int_{-\infty}^{+\infty} d\lambda\ e^{-i\lambda u} \hat{f}_\lambda\ ,$$

$$h^\pm(u) = 1 + \hbar \int_0^{\pm\infty} d\lambda\ \mathrm{e}^{-i\lambda u}\ \hat{h}_\lambda \mathrm{e}^{-c\hbar|\lambda|/4}\ .$$

They satisfy the commutation relations (2.18) and comultiplication rules (2.14)–(2.16) for the generating integrals $h^\pm(u)$ and $e^\pm(u)$, $f^\pm(u)$, which, due to (3.12), have the form

$$e^\pm(u) \;=\; \hbar \int_0^{\pm\infty} d\lambda\ \mathrm{e}^{-i\lambda u}\ \hat{e}_\lambda \mathrm{e}^{-c\hbar|\lambda|/4}\ , \quad f^\pm(u) \;=\; \hbar \int_0^{\pm\infty} d\lambda\ \mathrm{e}^{-i\lambda u}\ \hat{f}_\lambda \mathrm{e}^{c\hbar|\lambda|/4}\ , \tag{4.16}$$

The elements $\hat{e}_\lambda$, $\hat{f}_\lambda$, $\hat{h}_\lambda$ are given by the inversion formulas (3.16), (3.17), (3.19), (3.20). For the description of the algebraical structure of the algebra $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$ it is more convenient to use the generators $\hat{\kappa}$ being the Fourier modes of the logarithm of the currents $h^\pm(u)$:

$$\kappa^\pm(u) = \log h^\pm(u) = \hbar \int_0^{\pm\infty} d\lambda\ e^{-i\lambda u} \hat{\kappa}_\lambda\ .$$

The generators $\hat{\kappa}_\lambda$ and $\hat{h}_\lambda$ are connected as follows:

$$\hat{h}_\lambda = \mathrm{e}^{c\hbar|\lambda|/4} \left(\hat{\kappa}_\lambda + \sum_{n\geqslant 2} \frac{(\hbar)^{n-1}}{n!} \int_0^\lambda d\lambda_1 \ \cdots \int_0^{\lambda_{n-2}} d\lambda_{n-1}\ \hat{\kappa}_{\lambda_1} \ldots \hat{\kappa}_{\lambda_{n-1}} \hat{\kappa}_{\lambda - \sum_{i=1}^{n-1}\lambda_i} \right). \tag{4.17}$$

In particular $\hat{h}_0 = \hat{\kappa}_0$. The inverse to (4.17) relation is

$$\hat{\kappa}_\lambda = \mathrm{e}^{-c\hbar|\lambda|/4} \left(\hat{h}_\lambda + \sum_{n\geqslant 2} \frac{(-\hbar)^{n-1}}{n} \int_0^\lambda d\lambda_1 \ \cdots \int_0^{\lambda_{n-2}} d\lambda_{n-1}\ \hat{h}_{\lambda_1} \ldots \hat{h}_{\lambda_{n-1}} \hat{h}_{\lambda - \sum_{i=1}^{n-1}\lambda_i} \right).$$

The commutation relations given in Proposition 1 and written in terms of the generators $\hat{e}_\lambda$, $\hat{f}_\lambda$, $\hat{\kappa}_\lambda$, $\lambda \in \mathbb{R}$, c and $\tilde{d}$ take the form:

$$\begin{aligned} [c, \text{everything}] \;&=\; 0, \\ [\tilde{d}, \hat{e}_\lambda] \;=\; -\lambda \hat{e}_\lambda, \quad [\tilde{d}, \hat{f}_\lambda] \;&=\; -\lambda \hat{f}_\lambda, \quad [\tilde{d}, \hat{\kappa}_\lambda] \;=\; -\lambda \hat{\kappa}_\lambda\ , \end{aligned} \tag{4.18}$$

$$[\hat\kappa_\lambda,\hat e_\mu] = \frac{2\mathrm{sh}\,\hbar\lambda}{\hbar\lambda}\,\mathrm{e}^{-\hbar c|\lambda|/4}\hat e_{\lambda+\mu}\ ,\qquad [\hat\kappa_\lambda,\hat f_\mu] = -\frac{2\mathrm{sh}\,\hbar\lambda}{\hbar\lambda}\,\mathrm{e}^{\hbar c|\lambda|/4}\hat f_{\lambda+\mu}\ , \tag{4.19}$$

$$[\hat e_\lambda,\hat e_\mu] = \hbar\int_\mu^\lambda d\tau\ \left[\theta(\tau-\mu)-\theta(\tau-\lambda)\right]\ \hat e_\tau\hat e_{\lambda+\mu-\tau}\ , \tag{4.20}$$

$$[\hat f_\lambda,\hat f_\mu] = -\hbar\int_\mu^\lambda d\tau\ \left[\theta(\tau-\mu)-\theta(\tau-\lambda)\right]\ \hat f_\tau\hat f_{\lambda+\mu-\tau}\ , \tag{4.21}$$

$$[\hat\kappa_\lambda,\hat\kappa_\mu] = \frac{4}{\hbar^2\lambda}\,\mathrm{sh}(\hbar\lambda)\,\mathrm{sh}(\hbar\lambda c/2)\,\delta(\lambda+\mu)\ , \tag{4.22}$$

$$\begin{aligned}[\hat e_\lambda,\hat f_\mu] &= 2\hbar^{-1}\,\mathrm{sh}\,(\lambda\hbar c/2)\,\delta(\lambda+\mu)+\\ &+\left[\mathrm{e}^{(\lambda-\mu)\hbar c/4}\theta(\lambda+\mu)+\mathrm{e}^{(\mu-\lambda)\hbar c/4}\theta(-\lambda-\mu)\right]\hat h_{\lambda+\mu}\ ,\end{aligned} \tag{4.23}$$

where the step-function $\theta(\lambda)$ is defined as

$$\theta(\lambda)=\begin{cases} 1 & \text{for}\quad \lambda>0\ ,\\ 1/2 & \text{for}\quad \lambda=0\ ,\\ 0 & \text{for}\quad \lambda<0\ .\end{cases}$$

Note the similarity of the relations (4.19), (4.22) and (4.23) with some of the relations for the quantum affine algebra $U_q(\widehat{sl}_2)$ in new realization [14].

Let us obtain the formula (4.20) from the commutation relation in terms of the total current

$$[e(u),e(v)] = -i\hbar\frac{\{e(u),e(v)\}}{u-v}\ .$$

The first step is to obtain the commutation relations between the currents $e^\pm(u)$ using the Riemann problem (3.12). For simplicity we obtain the commutation relations between currents $e^+(u)$ and $e^+(v)$ and set in the calculations $c=0$. Others cases as well as the reconstraction of the central element can be easily considered. We have

$$\begin{aligned}[e^+(u),e^+(v)] &= -i\hbar\int_{C_+}\frac{d\tilde u}{2\pi i}\int_{C_+}\frac{d\tilde v}{2\pi i}\ \frac{\{e(\tilde u),e(\tilde v)\}}{(\tilde u-\tilde v)(u-\tilde u)(v-\tilde v)} =\\ &= -i\hbar\int_{C_+}\frac{d\tilde u}{2\pi i}\int_{C_+}\frac{d\tilde v}{2\pi i}\frac{e(\tilde u)e(\tilde v)}{\tilde u-\tilde v}\left[\frac{1}{(u-\tilde u)(v-\tilde v)}-\frac{1}{(u-\tilde v)(v-\tilde u)}\right] =\\ &= i\hbar\int_{C_+}\frac{d\tilde u}{2\pi i}\int_{C_+}\frac{d\tilde v}{2\pi i}\frac{e(\tilde u)e(\tilde v)}{u-v}\left[\frac{1}{u-\tilde u}-\frac{1}{v-\tilde u}\right]\left[\frac{1}{u-\tilde v}-\frac{1}{v-\tilde v}\right] =\\ &= i\hbar\frac{(e^+(u)-e^+(v))^2}{u-v}\ ,\end{aligned}$$

where the contour C_+ goes above the points u, v and in the second line we used the fact that $e^2(u)=0$ which follows from the commutation relation (2.18) in order to interchange the integration contours. Due to this condition the pole of the integrand when $\tilde u=\tilde v$ is superficial. Let us stress that the condition that square of the total current is vanishing is the special property of deformed infinite dimensional algebras (like central extended Yangian double, quantum affine algebra, etc.) and has no classical analogs.

The next step is to obtain from the commutation relation

$$[e^{+}(u),e^{+}(v)]=i\hbar\frac{(e^{+}(u)-e^{+}(v))^{2}}{u-v}$$

the relation (4.20) for $\lambda,\mu>0$. Others cases can be obtained from the commutation relations $[e^{\pm}(u),e^{\mp}(v)]$ and $[e^{-}(u),e^{-}(v)]$. This can be done using the convolution property of the inverse Laplace transform and the following Lemma (recall that we consider the case of $c=0$ for simplicity).

Lemma 2.

$$\hat{g}(\lambda,\mu)=i\int\limits_{C_+}\frac{du}{2\pi}\int\limits_{C_+}\frac{dv}{2\pi}\mathrm{e}^{i\lambda u}\mathrm{e}^{i\mu v}\,\frac{e^{+}(u)-e^{+}(v)}{u-v}=\begin{cases}\hat{e}_{\lambda+\mu} & \text{for}\ \ \lambda+\mu>0,\\ \hat{e}_{\lambda}/2 & \text{for}\ \ \lambda>0,\ \ \mu=0,\\ \hat{e}_{\mu}/2 & \text{for}\ \ \mu>0,\ \ \lambda=0,\\ 0 & \text{otherwise.}\end{cases}$$

Now we have

$$\begin{aligned}[][\hat{e}_{\lambda},\hat{e}_{\mu}] &= i\hbar\int\limits_{C_+}\frac{du}{2\pi}\int\limits_{C_+}\frac{dv}{2\pi}\mathrm{e}^{i\lambda u}\mathrm{e}^{i\mu v}\left[e^{+}(u)\frac{e^{+}(u)-e^{+}(v)}{u-v}-e^{+}(v)\frac{e^{+}(u)-e^{+}(v)}{u-v}\right]\\ &= \hbar\int\limits_{-\infty}^{\infty}d\tau\int\limits_{-\infty}^{\infty}d\nu\ [\hat{e}_{\tau}\theta(\tau)\delta(\nu)-\hat{e}_{\nu}\theta(\nu)\delta(\tau)]\ \hat{g}(\lambda-\tau,\mu-\nu)=\\ &= \hbar\int\limits_{-\infty}^{\infty}d\tau\ [\theta(\tau-\mu)-\theta(\tau-\lambda)]\ \hat{e}_{\tau}\hat{e}_{\lambda+\mu-\tau}\ .\end{aligned}\tag{4.24}$$

Note that the kernel of the integrand in (4.24) makes the integral to be supported on the finite domain of the real axis. One can verify that for $c\neq 0$ the calculation (4.24) will be modified a little bit but yields the same result (4.20).

In the terms of the generators $\hat{h}_{\lambda}$ the commutation relations (4.19) also can be written in a convolution form :

$$[\hat{h}_{\lambda},\hat{e}_{\mu}] = 2\hat{e}_{\lambda+\mu}+\hbar\int\limits_{0}^{\lambda}d\tau\ \ [\theta(\tau)-\theta(\tau-\lambda)]\ \ \{\hat{h}_{\tau},\hat{e}_{\lambda+\mu-\tau}\}\ , \tag{4.25}$$

$$[\tilde{\hat{h}}_{\lambda},\hat{f}_{\mu}] = -2\hat{f}_{\lambda+\mu}-\hbar\int\limits_{0}^{\lambda}d\tau\ \ [\theta(\tau)-\theta(\tau-\lambda)]\ \ \{\tilde{\hat{h}}_{\tau},\hat{f}_{\lambda+\mu-\tau}\}\ , \tag{4.26}$$

where $\tilde{\hat{h}}_{\lambda}=\hat{h}_{\lambda}\mathrm{e}^{-\hbar c|\lambda|/2}$.

The comultiplication rules (2.14)–(2.16) can be rewritten for the generators $\hat{e}_{\lambda}$, $\hat{f}_{\lambda}$ and $\hat{h}_{\lambda}$ in terms of multiple convolution integrals. For example, at $c=0$ we have:

$$\Delta\hat{e}_{\lambda}=\hat{e}_{\lambda}\otimes 1+1\otimes\hat{e}_{\lambda}+\hbar\int\limits_{0}^{\lambda}d\tau\ \hat{h}_{\lambda-\tau}\otimes\hat{e}_{\tau}+o(\hbar^{2}).$$

The precise definition of the algebra $\mathcal{A}_{\hbar}(\widehat{\mathfrak{sl}_2})$ means that one should consider the proper completion of the free tensor topological algebra generated by $\hat{e}_{\lambda}$, $\hat{f}_{\lambda}$, $\hat{h}_{\lambda}$, $\lambda\in\mathbb{R}$, c and $\tilde{d}$ over the ideal generated by the relations (4.21). It means in particular that the completed algebra is generated by formal integrals

$$\int_{-\infty}^{+\infty} \hat{e}_\lambda g(\lambda) d\lambda, \quad \int_{-\infty}^{+\infty} \hat{f}_\lambda g'(\lambda) d\lambda, \quad \int_{-\infty}^{+\infty} \hat{\kappa}_\lambda g''(\lambda) d\lambda, \tag{4.27}$$

where $g(\lambda)$, $g'(\lambda)$ and $g''(\lambda)$ are integrable functions decreasing faster then $\mathrm{e}^{-a|\lambda|}$ for some positive a (see details in [6]) . The completion $\widehat{\mathcal{A}}$ is a Hopf algebra which coincides with rational degeneration $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$ of the scaled elliptic algebra $\mathcal{A}_{\hbar,\eta}(\widehat{\mathfrak{sl}_2})$ [6]. As well as the Yangian double, it is the quantum double of its Hopf subalgebra, generated by the positive Fourier harmonics of the currents and one of the elements c and d.

4.3. The comparison. Let us compare the two Yangian type algebra obtained from the Yang R-matrix and the two Riemann problems. The commutation relations for their generators look very different while they were obtained from identical equations for the L-operators. In order to visualize their common nature one should look to the relations (4.4), (4.6)–(4.10) as to the difference equations (4.6)–(4.10) with initial conditions (4.4). One can solve these equations. For instance, the solution of (4.7) for the generators e_n has a form

$$[e_k, e_l] = \nu \sum_{p=l}^{k-1} e_p e_{k+l-p-1} \ , \qquad k > l \tag{4.28}$$

which is a discrete analog of the relation (4.20). Note that the relations (4.20), (4.21), (4.25) and (4.26) written in terms of convolutions allow to order the quadratic expressions over the generators in terms of iterated integrals (see details in [6]). On the other case, one can differentiate the relations (4.20), (4.21), (4.25) and (4.26) over the parameters λ and μ and get the continuous analogues of the relations (4.8)–(4.14):

$$[\hat{e}'_\lambda, \hat{e}_\mu] - [\hat{e}_\lambda, \hat{e}'_\mu] = \hbar\{\hat{e}_\lambda, \hat{e}_\mu\} \ . \tag{4.29}$$

Nevertheless only the integral relations (4.21), (4.25) and (4.26) have precise sense in completed algebra and admit further generalizations [6] to the contrary to their differential consequences (4.29).

The natural question then arises why the algebras $\widehat{DY(\mathfrak{sl}_2)}$ and $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$ are essentially different. The main difference concerns the properties of the elements d for $\widehat{DY(\mathfrak{sl}_2)}$ and of $\tilde{d}$ for the algebra $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$ and the different types of completions.

In the discrete case of the Yangian double the operator $[d, \ \cdot\ ] = \frac{d}{du}$ acts in $\widehat{DY(\mathfrak{sl}_2)}$ as degree -1 operator preserving the filtration (4.15) and has no eigenvalues in the completed algebra different from zero. It would encounter the divergence of the partition functions of infinite-dimensional representations (see Section 5).

To the contrary the operator $[\tilde{d}, \ \cdot\ ] = \frac{1}{i}\frac{d}{du}$ act as degree zero operator in the algebra $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$ preserving the grading. The elements $\hat{e}_\lambda, \hat{f}_\lambda, \hat{h}_\lambda$ are its eigenvectors with eigenvalue $\lambda, \lambda \in \mathbb{R}$. As a consequence, the characters of integrable finite-dimensional modules are well-defined (see Section 5).

The algebra $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$ possesses the Cartan antiinvolution θ (which can be treated as an involutive antiisomorphism of the algebra $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$ over the ring $\mathbb{C}[\hbar]$ or as involutive antiisomorphism from $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$ to $\mathcal{A}_{-\hbar}(\widehat{\mathfrak{sl}_2})$). On the level of generating functions it looks like

$$\theta e(u) = f(-u) \ , \qquad \theta f(u) = e(-u) \ , \qquad \theta h^{\pm}(u) = \pm h^{\mp}(-u) \ , \tag{4.30}$$

$$\theta d = d\ , \qquad \theta c = c\ , \qquad \theta \hbar = -\hbar\ ,$$

and for the generators

$$\theta \hat{e}_\lambda = -\hat{f}_{-\lambda}, \qquad \theta \hat{f}_\lambda = -\hat{e}_{-\lambda}, \qquad \theta \hat{h}_\lambda = \hat{h}_{-\lambda}\ .$$

The antiinvolution θ exchange the two subalgebras of $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$ generated by Fourier modes of $e^\pm(u), f^\pm(u), h^\pm(u)$; thus the algebra $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$ can be treated as contragredient algebra. To the contrary, there is no analog of Cartan antiinvolution for the Yangian double, since it reverses the filtration (4.15) and should exchange Laurent series from a subalgebra generated by negative Fourier harmonics with Laurent polynomials over positive Fourier harmonics. As a result we could not expect a good notion of restricted dual for highest weight modules of extended Yangian double.

The two algebras distinguish as well on the classical level. The classical limit of the central extended Yangian double $\widehat{DY(\mathfrak{sl}_2)}$ can be identify with semidirect sum of the central extension of meromorphic $\mathfrak{sl}_2$-valued functions and of one-dimensional Lie algebra generated by the derivative $\frac{d}{dz}$. The commutation relations for the generators are standard:

$$[h_n, e_m] = 2e_{n+m}, \qquad [h_n, f_m] = -2f_{n+m}, \qquad [e_n, e_m] = [f_n, f_m] = 0,$$
$$[e_n, f_m] = h_{n+m} + n\delta_{n,-m}c, \qquad [h_n, h_m] = 2n\delta_{n,-m}c$$
$$[d, e_n] = -ne_{n-1}, \quad [d, f_n] = -nf_{n-1}, \quad [d, h_n] = -nh_{n-1}\ .$$

The bialgebra structure can be defined via the decomposition of the algebra to the subalgebras of $\mathfrak{sl}_2$-functions regular at zero and at infinity:

$$(\mathfrak{s}l_2 \otimes \mathbb{C}[z] \oplus \mathbb{C}c) \oplus \left(\mathfrak{s}l_2 \otimes \mathbb{C}[[z^{-1}]] \oplus \mathbb{C}d\right)\ .$$

Again we have no symmetry between the two subalgebras and the (quasi)nilpotent action of the operator d.

The classical limit of the Hopf algebra $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$ can be identify with the completion of central extended algebra of meromorphic $\mathfrak{sl}_2$-valued functions vanishing at infinity (again with added element $\tilde{d} = \frac{1}{i}\frac{d}{dz}$). The commutation relations for the generators now are

$$[\hat{h}_\lambda, \hat{e}_\mu] = 2\hat{e}_{\lambda+\mu}, \qquad [\hat{h}_\lambda, \hat{f}_\mu] = -2\hat{f}_{\lambda+\mu}, \qquad [\hat{e}_\lambda, \hat{e}_\mu] = [\hat{f}_\lambda, \hat{f}_\mu] = 0,$$
$$[\hat{e}_\lambda, \hat{f}_\mu] = \hat{h}_{\lambda+\mu} + \lambda\delta(\lambda+\mu)c, \qquad [\hat{h}_\lambda, \hat{h}_\mu] = 2\lambda\delta(\lambda+\mu)c,$$
$$[\tilde{d}, \hat{e}_\lambda] = -\lambda\hat{e}_\lambda, \quad [\tilde{d}, \hat{h}_\lambda] = -\lambda\hat{h}_\lambda, \quad [\tilde{d}, \hat{f}_\lambda] = -\lambda\hat{f}_\lambda.$$

The bialgebra structure is given by the decomposition of the algebra into direct sum of subalgebras of $\mathfrak{sl}_2$-valued functions which are regular at upper or lower half-plane, which corresponds to taking positive or negative Fourier modes. Again we have the contragredient structure and the diagonalization of the operator $\tilde{d}$.

Note also that the different types of completions used for the definitions of the two algebras actually follow from different positions of the marked singular points in two Riemann problems. In the case of the Yangian double two separated points zero and infinity define Laurent series in infinity and Laurent polynomial in zero while in the case of $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$ the singular point near the contour (infinity) produces continuous family of Fourier harmonics which diagonalize the operator $\tilde{d}$. Thus these two algebras cannot be connected by a projective transform of the complex plane as well as the corresponding Riemann problems.

Let us note that the algebras $\widehat{DY(\mathfrak{sl}_2)}$ and $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$ given by the commutation relations (4.8)–(4.14) and (4.19)–(4.23) respectively and associated with simple Lie algebra $\mathfrak{sl}_2$ can be generalized for arbitrary simply-laced Lie algbera. See, for example, [3].

5 Finite-dimensional representations and R-matrices

5.1. Finite-dimensional representations. The finite-dimensional representations make sense for the subquotions of the algebras $DY(\widehat{\mathfrak{sl}_2})$ and $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$ defined by the condition $c=0$ with dropped elements d and $\tilde{d}$. Thus their structures are identical for both algebras. Due to classification theorem (see [17]), the irreducible finite-dimensional representations of the algebras $DY(\widehat{\mathfrak{sl}_2})$ and $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$ are certain tensor products of the evaluation representations. The evaluation representations can be defined via evaluation homomorphism $\mathcal{E}v_z : DY(\widehat{\mathfrak{sl}_2}) \to U(\mathfrak{sl}_2)$, $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2}) \to U(\mathfrak{sl}_2)$, $z\in\mathbb{C}$. For the Yangian double the homomorphism has a form

$$\mathcal{E}v_z\left(e(u)\right) = \nu\delta(u-z-\frac{h-1}{2}\nu)\cdot e\ , \quad \mathcal{E}v_z\left(f(u)\right) = \nu f\cdot\delta(u-z-\frac{h-1}{2}\nu)\ , \tag{5.1}$$

$$\mathcal{E}v_z\left(h^{\pm}(u)\right) = 1+\frac{\nu}{u-z-\nu(h-1)/2}\,ef-\frac{\nu}{u-z-\nu(h+1)/2}fe\ , \tag{5.2}$$

as well as for the algebra $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$ with the replacement $\nu=-i\hbar$. Here e,f and h are the generators of Lie algebra $\mathfrak{sl}_2$:

$$[h,e]=2e, \qquad [h,f]=-2f, \qquad [e,f]=h$$

and the right hand sides of (5.2) are taken for $\mathrm{Im}\,z \gtrless \mathrm{Im}\,u$ for $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$ and $|z|\gtrless|u|$ for $DY(\widehat{\mathfrak{sl}_2})$ respectively for "$\pm$" generating functions. For instance, for the simplest two-dimensional representation $\pi_z^{(1/2)}$ the action of the generators of the algebra $DY(\widehat{\mathfrak{sl}_2})$ and of the algebra $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$ has a form

$$e_k = z^k e_{1,2}, \quad f_k = z^k e_{2,1}, \quad h_k = z^k(e_{1,1}-e_{2,2})\ , \tag{5.3}$$

$$\begin{aligned}\hat{e}_\lambda &= \mathrm{e}^{i\lambda z}e_{1,2}, \quad \hat{f}_\lambda = \mathrm{e}^{i\lambda z}e_{2,1}, \quad \hat{h}_\lambda = \mathrm{e}^{i\lambda z}(e_{1,1}-e_{2,2})\ ,\\ \hat{\kappa}_\lambda &= \frac{\mathrm{e}^{i\lambda z}}{\hbar\lambda}\Big((1-\mathrm{e}^{-\lambda\hbar})e_{1,1}+(1-\mathrm{e}^{\lambda\hbar})e_{2,2}\Big),\end{aligned} \tag{5.4}$$

where $(e_{i,j})_{k,l}=\delta_{ik}\delta_{jl}$ are unit matrices in $\mathrm{End}\,\mathbb{C}^2\otimes\mathbb{C}^2$.

5.2. Universal $\mathcal{R}$-matrices. The universal $\mathcal{R}$-matrix for the algebra $DY(\widehat{\mathfrak{sl}_2})$ has been obtained in [2, 3] from the analysis of the canonical Hopf pairing of the two Hopf subalgebras $DY(\widehat{\mathfrak{sl}_2})^{\pm}$ of $DY(\widehat{\mathfrak{sl}_2})$, generated by Fourier coefficients of the currents $e^{\pm}(u), f^{\pm}(u),\ h^{\pm}(u)$. The pairing of these fields looks like

$$\langle e^+(u),f^-(v)\rangle = \langle f^+(u),e^-(v)\rangle = \frac{\nu}{u-v}\ , \tag{5.5}$$

$$\langle h^+(u),h^-(v)\rangle = \frac{u-v+\nu}{u-v-\nu} \tag{5.6}$$

or, in terms of the generators,

$$\langle c,d\rangle = -\nu^{-1}, \quad \langle e_k,f_{-l-1}\rangle = \langle f_k,e_{-l-1}\rangle = -\nu^{-1}\delta_{k,l}\ ,$$

$$\langle h_k,h_{-l-1}\rangle = \begin{cases} 2(\nu)^{k-l-1}\frac{k!}{l!(k-l)!}, & k\geqslant 0, \quad 0\leqslant l\leqslant k\ ,\\ 0, \quad \text{otherwise}\ .\end{cases}$$

The full pairing is described by the universal R-matrix which has a form

$$\mathcal{R} = \mathcal{R}_+ \cdot \mathcal{C} \cdot \mathcal{R}_0 \cdot \mathcal{C} \cdot \mathcal{R}_- \ , \tag{5.7}$$

where

$$\begin{aligned}
\mathcal{C} &= -\frac{\nu}{4}(c \otimes d + d \otimes c) \ , \\
\mathcal{R}_+ &= \prod_{k\geqslant 0}^{\rightarrow} \exp(-\nu e_k \otimes f_{-k-1}) = \exp(-\nu e_0 \otimes f_{-1}) \exp(-\nu e_1 \otimes f_{-2}) \cdots \ , && (5.8) \\
\mathcal{R}_- &= \prod_{k\geqslant 0}^{\leftarrow} \exp(-\nu f_k \otimes e_{-k-1}) = \cdots \exp(-\nu f_1 \otimes e_{-2}) \exp(-\nu f_0 \otimes e_{-1}) \ , && (5.9) \\
\mathcal{R}_0 &= \prod_{n\geqslant 0} \exp\left(- \operatorname*{Res}_{u=v} \left[\frac{d}{du} \ln\, h^+(u) \otimes \ln\, h^-(v-(2n+1)\nu) \right] \right) \ , && (5.10)
\end{aligned}$$

and a residue operation Res is defined as follows

$$\operatorname*{Res}_{u=v} \left(\sum_{i\geqslant 0} a_i u^{-i-1} \otimes \sum_{k\geqslant 0} b_k v^k \right) = \sum_{i\geqslant 0} a_i \otimes b_i \ .$$

As usual, the L-operators $L^{\pm}$ are given by the substitution to one tensor component of $\mathcal{R}$ the two-dimensional representation [16]

$$L^-(z) = (\pi_z^{(1/2)} \otimes \mathrm{id})\ \mathcal{C}^{-1} \cdot \mathcal{R} \cdot \mathcal{C}^{-1}, \quad L^+(z) = (\pi_z^{(1/2)} \otimes \mathrm{id})\ \mathcal{C} \cdot \left(\mathcal{R}^{21}\right)^{-1} \cdot \mathcal{C} \ . \tag{5.11}$$

The decomposition (5.7) produces the Gauss decomposition of the L-operators (2.9).

Analogously, the description of the universal R-matrix for the algebra $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$ can be done following the same scheme of [2]. Let us remind that the main arguments in [2] are: the triangular decomposition of the Yangian Double with respect to a Hopf pairing, the basic pairing (5.5), (5.6), and the expression of the tensor of the pairing for the subalgebras generated by Cartan currents as the exponent of the pairing between logarithms of Cartan fields. The last calculation uses the shift automorphisms of $\widehat{DY}(\mathfrak{sl}_2)$. All these arguments remain unchanged for the algebra $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$ (in basic pairing (5.5), (5.6) we use $\nu = -i\hbar$ as usual).

It means that the universal R-matrix for $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$ admits the decomposition (5.7) where the factor $\mathcal{C}$ is unchanged, the ordered products of exponents in the factors $\mathcal{R}_\pm$ turn to ordered exponential in integral form and the factor $\mathcal{R}_0$ is as before

$$\mathcal{R}_0 = \exp \Omega$$

where Ω is the tensor of the pairing of the fields $k^{\pm}(u) = \log h^{\pm}(u)$. The main distinction of the case of $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$ is that the pairing

$$\langle \kappa^+(u), \kappa^-(v) \rangle = \log \frac{u - v - i\hbar}{u - v + i\hbar}$$

can be explicitly and uniquely diagonalized in the generators $\hat{\kappa}_\lambda$:

$$\langle \hat{\kappa}_\lambda, \hat{\kappa}_\mu \rangle = -2 \frac{\mathrm{sh}\hbar\lambda}{\hbar^2 \lambda} \delta(\lambda + \mu) \ .$$

Summarizing the calculation we have.

The universal R-matrix for $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$ admits the decomposition (5.7) where $\mathcal{C}=\exp(-\hbar(\tilde d\otimes c+c\otimes\tilde d)/4)$,

$$\mathcal{R}_+ \;=\; \overrightarrow{P}\exp\left(-\hbar\int_0^{+\infty} d\lambda\,\hat e_\lambda\otimes\hat f_{-\lambda}\right),\quad \mathcal{R}_- \;=\; \overleftarrow{P}\exp\left(-\hbar\int_0^{+\infty} d\lambda\,\hat f_\lambda\otimes\hat e_{-\lambda}\right)\,. \tag{5.12}$$

and

$$\mathcal{R}_0=\exp\left(-\int_0^{+\infty} d\lambda\,\frac{\hbar^2\lambda}{2\,\mathrm{sh}\,\hbar\lambda}\hat\kappa_\lambda\otimes\hat\kappa_{-\lambda}\right)\,. \tag{5.13}$$

It is interesting to compare the evaluation of the two expressions of the universal R-matrix to tensor product $\pi^{(1/2)}_{z_1}\otimes\pi^{(1/2)}_{z_2}$ of two-dimensional representations. The formulas (5.8), (5.9) and (5.10) yield the following decomposition of four by four R-matrix $R(z)$:

$$R^-(z)=\begin{pmatrix}1&0&0&0\\0&1&\frac{\nu}{z}&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}\rho^-(z)&0&0&0\\0&\frac{z-\nu}{z}\rho^-(z)&0&0\\0&0&\frac{z}{z+\nu}\rho^-(z)&0\\0&0&0&\rho^-(z)\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&\frac{\nu}{z}&1&0\\0&0&0&1\end{pmatrix} \tag{5.14}$$

or $R^-(z)=\rho^-(z)\frac{z+\nu P}{z+\nu}$, where P is a permutation of tensor components and

$$\rho^-(z)=\prod_{n\geqslant 0}\frac{(z-2n\nu)(z-(2n+2)\nu)}{(z-(2n+1)\nu)^2}\,.$$

The infinite product converge for proper z and equals to the ratio of Γ–functions: $\rho^-(z)=\frac{\Gamma^2(\frac12-\frac{z}{2\nu})}{\Gamma(1-\frac{z}{2\nu})\Gamma(-\frac{z}{2\nu})}$. An application of (5.12) and (5.13) gives $R^-(z)$ in a form (5.14) but with a scalar factor $\rho^-(z)$ presented in an integral form

$$\rho^-(z)=\exp\left(-2\int_0^{+\infty} d\lambda\,\frac{\mathrm{sh}^2(\hbar\lambda/2)}{\lambda\,\mathrm{sh}\,\hbar\lambda}\mathrm{e}^{i\lambda z}\right),\quad \mathrm{Im}\,z>0\,.$$

We see that the two universal R-matrices presented in this section give two different quantization of the Yang rational solution of classical YB equation. The solution via algebra $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$ has an advantage of integral presentation and uniquely determined by the asymptotics of the scalar factor. To the contrary, there is no definite choice for such a solution for the double of the Yangian (the answer has no definite asymptotics when $|z|$ tends to infinity). Moreover, as we will see further analogous integral representation for divergent infinite products automatically appear in representation theory of $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$ in regularized form whereas the representation theory of the Yangian double has no instruments for such a regularization.

6 Basic infinite-dimensional representations

The goal of this section is to construct the examples of the infinite-dimensional representations of the algebras $\widehat{DY(\mathfrak{sl}_2)}$ and $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$. This construction demonstrate the distinctions of the considered algebras. We start from the discrete algebra $\widehat{DY(\mathfrak{sl}_2)}$ and for simplicity consider only the case of the central element $c=1$.

6.1. Basic representations of the algebra $\widehat{DY(\mathfrak{sl}_2)}$. Let $\mathcal{H}$ be the Heisenberg algebra generated by free bosons $a_{\pm n}$, $n = 1, 2, \dots$ with zero modes a_0, p and commutation relations

$$[a_n, a_m] = n\delta_{n+m,0} \ , \quad [p, a_0] = 2 \ .$$

Let

$$a_+(z) = \sum_{n\geqslant 1} \frac{a_n}{n} z^{-n} - p\log z \ , \quad a_-(z) = \sum_{n\geqslant 1} \frac{a_{-n}}{n} z^{n} + \frac{a_0}{2} \ , \quad \phi_\pm(z) = \exp a_\pm(z) \ ,$$

be the generating functions of the elements of the algebra $\mathcal{H}$. Let $\bar{e}(u)$, $\bar{f}(u)$, $\bar{h}^+(u)$ and $\bar{h}^-(u)$ be following generating series acting in the Fock space $\mathcal{H}$:

$$\begin{aligned}
\bar{e}(u) &= \nu\phi_-(u-\nu)\phi_-(u)\phi_+^{-1}(u) \ , \\
\bar{f}(u) &= \nu\phi_-^{-1}(u+\nu)\phi_-^{-1}(u)\phi_+(u) \ , \\
\bar{h}^+(u) &= \phi_+(u-\nu)\phi_+^{-1}(u) \ , \quad \bar{h}^-(u) = \phi_-(u-\nu)\phi_-^{-1}(u+\nu) \ .
\end{aligned}$$

We have the following

Proposition 3. [3] *The $\mathcal{H}$-valued generating functions (fields)*

$$e(u) = \bar{e}(u + c\nu/4), \quad f(u) = \bar{f}(u - c\nu/4), \quad h^+(u) = \bar{h}^+(u + c\nu/2), \quad h^-(u) = \bar{h}^-(u)$$

satisfy the commutation relations (4.2) *with* $c = 1$.

Let V_α be the formal power series extensions of the Fock spaces

$$V_i = \mathbb{C}[[a_{-1}, \dots, a_{-n}, \dots]] \otimes \left(\oplus_{n\in\mathbb{Z}} \mathbb{C}\mathrm{e}^{(n+\alpha)a_0}\right) \ , \quad 0 \leqslant \alpha < 1 \ , \tag{6.1}$$

with the action of bosons on these spaces

$$\begin{aligned}
a_n &= \text{the left multiplication by } a_n \otimes 1 \ \text{ for } n < 0 \ , \\
&= [a_n, \ \cdot\ ] \otimes 1 \quad \text{for } n > 0 \ , \\
\mathrm{e}^{n_1 a_0}(a_{-j_k} \cdots a_{-j_1} \otimes \mathrm{e}^{n_2 a_0}) &= a_{-j_k} \cdots a_{-j_1} \otimes \mathrm{e}^{(n_1+n_2)a_0} \ , \\
u^p(a_{-j_k} \cdots a_{-j_1} \otimes \mathrm{e}^{n a_0}) &= u^{2n} a_{-j_k} \cdots a_{-j_1} \otimes \mathrm{e}^{n a_0} \ .
\end{aligned} \tag{6.2}$$

It is clear from (6.2) that Fock spaces V_α becomes the irreducible representations of the algebra $\widehat{DY(\mathfrak{sl}_2)}$ at level 1 ($c = 1$) for $\alpha = 0$ or $1/2$ with the vacuum vectors $1 \otimes 1$ and $1 \otimes \mathrm{e}^{a_0/2}$. These representations are highest weight representations with respect to the Fourier components $\bar{e}_n$, $\bar{f}_n$ of the generating currents $\bar{e}(u) = \sum \bar{e}_n u^{-n-1}$ and $\bar{f}(u) = \sum \bar{f}_n u^{-n-1}$. The latter generators are related to e_n and f_n by the triangular transformations due to the relations given in Proposition 3

$$\begin{aligned}
\bar{e}_n(1 \otimes 1) &= \bar{f}_n(1 \otimes 1) = 0, \quad n < 0, \\
\bar{e}_n(1 \otimes \mathrm{e}^{a_0/2}) &= 0, \quad n < -1, \quad \bar{f}_n(1 \otimes \mathrm{e}^{a_0/2}) = 0, \quad n < 1.
\end{aligned}$$

Elements of the monomial basis (6.1) are *not* eigenvalues of the filtration operator d, since:

$$\begin{aligned}
[d, a_n] &= -n a_{n-1}, \quad n \leqslant -1, \quad n \geqslant 2 \ , \\
[d, a_1] &= -p, \quad [d, p] = 0, \quad [d, a_0/2] = a_{-1} \ .
\end{aligned}$$

They cannot be used for example for calculation the character of the Fock space V_α using the operator d. Usual expression $\mathrm{tr}_{V_\alpha}(\mathrm{e}^{pd})$ is divergent and only the ratio of such traces $\mathrm{tr}_{V_\alpha}(\mathrm{e}^{pd}O)\big/\,\mathrm{tr}_{V_\alpha}(\mathrm{e}^{pd})$ can be made finite for certain operators O in the Fock space $\mathcal{H}$ [18].

6.2. Realization of the algebra $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$ by the continuous fields. Let us define bosons a_λ, $\lambda \in \mathbb{R}$ which satisfy the commutation relations [8, 22]:

$$[a_\lambda, a_\mu] = \frac{4}{\hbar^2}\,\frac{\mathrm{sh}(\hbar\lambda)\mathrm{sh}(\hbar\lambda/2)}{\lambda}\;\delta(\lambda+\mu) = a(\lambda)\delta(\lambda+\mu)\ . \tag{6.3}$$

Consider the generating functions

$$e(u) \;=\; \hbar \mathrm{e}^{\gamma}\!:\exp\left(-\hbar\int_{-\infty}^{\infty} d\lambda\ \mathrm{e}^{-i\lambda u}\,\frac{a_\lambda \mathrm{e}^{-\hbar|\lambda|/4}}{2\mathrm{sh}(\hbar\lambda/2)}\right):\ , \tag{6.4}$$

$$f(u) \;=\; \hbar \mathrm{e}^{\gamma}\!:\exp\left(\hbar\int_{-\infty}^{\infty} d\lambda\ \mathrm{e}^{-i\lambda u}\,\frac{a_\lambda \mathrm{e}^{\hbar|\lambda|/4}}{2\mathrm{sh}(\hbar\lambda/2)}\right):\ , \tag{6.5}$$

$$\begin{aligned} h^{\pm}(u) &= (\hbar\mathrm{e}^{\gamma})^{-2} : e\left(u \mp \frac{i\hbar}{4}\right) f\left(u \pm \frac{i\hbar}{4}\right) : = \\ &= \exp\left(\hbar\int_{0}^{\pm\infty} d\lambda\ \mathrm{e}^{-i\lambda u} a_\lambda\right)\ , \end{aligned} \tag{6.6}$$

where γ is Euler constant. The notion of the normal ordered operator becomes more involved in case of continuous bosons. It requires some kind of ultraviolet regularization to be included in the definition of the normal ordered operators such that the product of these operators satisfy the rules [7]:

$$\begin{aligned} &:\exp\left(\int_{-\infty}^{\infty} d\lambda\ g_1(\lambda)\, a_\lambda\right): \;\cdot\; :\exp\left(\int_{-\infty}^{\infty} d\mu\ g_2(\mu)\, a_\mu\right): = \\ &= \exp\left(\int_{\tilde C} \frac{d\lambda\ \ln(-\lambda)}{2\pi i}\ a(\lambda)g_1(\lambda)g_2(-\lambda)\right) :\exp\left(\int_{-\infty}^{\infty} d\lambda\ (g_1(\lambda)+g_2(\lambda))\, a_\lambda\right):\ . \end{aligned} \tag{6.7}$$

The contour $\tilde C$ is shown in the Fig. 3.

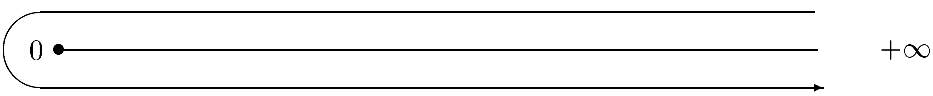

Fig. 3.

With this definition of the normal ordered exponents we can prove the following

Proposition 4. *The generating functions* (6.4)–(6.6) *satisfy the commutation relations* (2.18).

The main formula which should be used to prove the Proposition 4 is:

$$\exp\left(\int_{\tilde C} \frac{d\lambda\ \ln(-\lambda)}{2\pi i\lambda}\mathrm{e}^{-x\lambda}\right) = \mathrm{e}^{-\gamma} x^{-1}, \quad \mathrm{Re}\, x > 0\ .$$

For the description of the infinite-dimensional representations of the algebra $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$ in terms of continuous bosons we need a definition of a Fock space generated by the continuous family of free bosons. We borrow the construction below from [6].

Let $a(\lambda)$ be a meromorphic function, regular for $\lambda\in\mathbb{R}$ and satisfying the following conditions:

$$a(\lambda)=-a(-\lambda)\ ,$$

$$a(\lambda)\sim a_0\lambda,\quad \lambda\to 0,\qquad a(\lambda)\sim \mathrm{e}^{a'|\lambda|},\quad \lambda\to\pm\infty.$$

Let a_λ, $\lambda\in\mathbb{R}$, $\lambda\neq 0$ be free bosons which satisfy the commutation relations

$$[a_\lambda,a_\mu]=a(\lambda)\delta(\lambda+\mu)\ .$$

We define a (right) Fock space $\mathcal{H}_{a(\lambda)}$ as follows. $\mathcal{H}_{a(\lambda)}$ is generated as a vector space by the expressions

$$\int\limits_{-\infty}^{0} f_n(\lambda_n)a_{\lambda_n}d\lambda_n\ldots\int\limits_{-\infty}^{0} f_1(\lambda_1)a_{\lambda_1}d\lambda_1\ |\mathrm{vac}\rangle\ ,$$

where the functions $f_i(\lambda)$ satisfy the condition

$$f_i(\lambda)<C\,\mathrm{e}^{(a'/2+\epsilon)\lambda},\qquad \lambda\to-\infty\ ,$$

for some $\epsilon>0$ and $f_i(\lambda)$ are analytical functions in a neighborhood of $\mathbb{R}_-$ except $\lambda=0$, where they have a simple pole.

The left Fock space $\mathcal{H}^*_{a(\lambda)}$ is generated by the expressions

$$\langle\mathrm{vac}|\int\limits_{0}^{+\infty} g_1(\lambda_1)a_{\lambda_1}d\lambda_1\ldots\int\limits_{0}^{+\infty} g_n(\lambda_n)a_{\lambda_n}d\lambda_n\ ,$$

where the functions $g_i(\lambda)$ satisfy the conditions

$$g_i(\lambda)<C\,\mathrm{e}^{-(a'/2+\epsilon)\lambda},\qquad \lambda\to+\infty\ ,$$

for some $\epsilon>0$ and $g_i(\lambda)$ are analytical functions in a neighborhood of $\mathbb{R}_+$ except $\lambda=0$, where they also have a simple pole.

The pairing $(,):\mathcal{H}^*_{a(\lambda)}\otimes\mathcal{H}_{a(\lambda)}\to\mathbb{C}$ is uniquely defined by the following prescriptions:

(i) $\quad (\langle\mathrm{vac}|,|\mathrm{vac}\rangle)=1\ ,$

(ii) $$(\langle\mathrm{vac}|\int\limits_{0}^{+\infty}d\lambda\ g(\lambda)a_\lambda\ ,\int\limits_{-\infty}^{0}d\mu\ f(\mu)a_\mu\ |\mathrm{vac}\rangle)=\int\limits_{\tilde C}\frac{d\lambda\ \ln(-\lambda)}{2\pi i}g(\lambda)f(-\lambda)a(\lambda)\ ,$$

(iii) $\quad$ the Wick theorem.

Let the vacuums $\langle\mathrm{vac}|$ and $|\mathrm{vac}\rangle$ satisfy the conditions

$$a_\lambda|\mathrm{vac}\rangle=0,\quad \lambda>0,\qquad \langle\mathrm{vac}|a_\lambda=0,\quad \lambda<0\ ,$$

and $f(\lambda)$ be a function analytical in some neighborhood of the real line with possible simple pole at $\lambda=0$ and which has the following asymptotical behavior:

$$f(\lambda)<Ce^{-(a'/2+\epsilon)|\lambda|},\qquad \lambda\to\pm\infty$$

for some $\epsilon > 0$. Then, by definition, the operator

$$F = :\exp\left(\int_{-\infty}^{+\infty} d\lambda\ f(\lambda)a_\lambda\right):$$

acts on the right Fock space $\mathcal{H}_{a(\lambda)}$ as follows. $F = F_- F_+$, where

$$F_- = \exp\left(\int_{-\infty}^{0} d\lambda\ f(\lambda)a_\lambda\right) \quad \text{and} \quad F_+ = \lim_{\epsilon\to+0} \mathrm{e}^{\epsilon \ln \epsilon f(\epsilon) a_\epsilon} \exp\left(\int_{\epsilon}^{\infty} d\lambda\ f(\lambda)a_\lambda\right) .$$

The action of operator F on the left Fock space $\mathcal{H}^*_{a(\lambda)}$ is defined via another decomposition: $F = \tilde{F}_- \tilde{F}_+$, where

$$\tilde{F}_+ = \exp\left(\int_{0}^{+\infty} d\lambda\ f(\lambda)a_\lambda\right) \quad \text{and} \quad \tilde{F}_- = \lim_{\epsilon\to+0} \mathrm{e}^{\epsilon \ln \epsilon f(-\epsilon) a_{-\epsilon}} \exp\left(\int_{-\infty}^{-\epsilon} d\lambda\ f(\lambda)a_\lambda\right) .$$

These definitions imply the following statement:

Proposition 5. *(i) The defined above actions of the operator*

$$F =: \exp\left(\int_{-\infty}^{+\infty} d\lambda\ f(\lambda)a_\lambda\right):$$

on the Fock spaces $\mathcal{H}$ and $\mathcal{H}^*$ are adjoint;

(ii) The product of the normally ordered operators satisfy the property (6.7)*.*

Returning to level one representation of $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$ we choose $\mathcal{H} = \mathcal{H}_{a(\lambda)}$ for $a(\lambda)$ defined in (6.3):

$$a(\lambda) = \frac{4}{\hbar^2}\frac{\mathrm{sh}(\hbar\lambda)\mathrm{sh}(\hbar\lambda/2)}{\lambda} .$$

¿From the definition of the Fock space $\mathcal{H}$ and from the proposition 6 we have immediately the construction of a representation of $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$:

Proposition 6. *The relations* (6.4)–(6.6) *define a highest weight right representation of the algebra $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$ in the Fock space $\mathcal{H}$ and lowest weight left representation in the dual Fock space $\mathcal{H}^*$:*

$$\hat{e}_\lambda|\mathrm{vac}\rangle = 0, \quad \hat{f}_\lambda|\mathrm{vac}\rangle = 0, \quad \lambda \geqslant 0 \quad \text{and} \quad \langle\mathrm{vac}|\hat{e}_\lambda = 0, \quad \langle\mathrm{vac}|\hat{f}_\lambda = 0, \quad \lambda \leqslant 0 . \tag{6.8}$$

The highest weight property (6.8) means that all the matrix elements of the corresponding operators which do not vanishing identically satisfy this property. Let us demonstrate that $\langle v \mid \hat{e}_\lambda|\mathrm{vac}\rangle = 0$ for $\lambda > 0$ and certain $v \in \mathcal{H}^*$. Fix $\hbar > 0$. It is clear that any such matrix element has a form:

$$\langle\mathrm{vac}| \prod_{i=1}^{n} f(v_i) \prod_{j=1}^{n-1} e(u_j)\hat{e}_\lambda|\mathrm{vac}\rangle =$$

$$= \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} du_n\ \mathrm{e}^{i\lambda u_n} \langle \mathrm{vac}| \prod_{i=1}^{n} f(v_i) \prod_{j=1}^{n} e(u_j)|\mathrm{vac}\rangle =$$

$$= G(v;u) \int\limits_{-\infty}^{\infty} du_n\ \mathrm{e}^{i\lambda u_n} \frac{\prod\limits_{j=1}^{n-1} (u_n - u_j)(u_n - u_j + i\hbar)}{\prod\limits_{i=1}^{n} (u_n - v_i + i\hbar/2)(u_n - v_i - i\hbar/2)}\ , \tag{6.9}$$

where $G(v;u)$ is some factor which does not depend on the variable u_n and in order to calculate (6.9) using normal ordering relations for total currents (see [6] for details) we should impose $\mathrm{Im}\ v_i < -\hbar/2$, $i = 1,\ldots,n$ and $\mathrm{Im}\ u_j < 0$, $j = 1,\ldots,n-1$. Note that integrand is decreasing as u_n^{-2} function when $u_n \to \pm\infty$ so the integral is convergent. To calculate it we can close the contour of integration either along the big semicircle in upper half-plane of u_n for $\lambda > 0$ or along big semicircle in lower half-plane. But all the poles of the integrand are in lower half-plane so for $\lambda > 0$ the nonvanishing matrix elements of the operator $\hat{e}_\lambda$ equal to zero and the property $\hat{e}_\lambda|\mathrm{vac}\rangle = 0$ for $\lambda > 0$ is proved. When $\lambda = 0$ (6.8) follows from the continuity arguments.

In the contrary to the case of discrete algebra $DY(\widehat{\mathfrak{sl}_2})$ the trace function $\mathrm{tr}_{\mathcal{H}}(\mathrm{e}^{pd})$ is well defined now and can be calculated using the gradation property of the operator d:

$$[d, a_\lambda] = -\lambda a_\lambda\ .$$

By definition this trace is equal to:

$$\begin{aligned} \mathrm{tr}_{\mathcal{H}}(\mathrm{e}^{pd}) &= \sum_{n=0}^{\infty} \int\limits_{0\leqslant\lambda_1<\cdots<\lambda_n<\infty}\!\!\cdots\!\!\int d\lambda_1\cdots d\lambda_n \frac{\langle\mathrm{vac}|a_{\lambda_1}\cdots a_{\lambda_n}\mathrm{e}^{pd}a_{-\lambda_n}\cdots a_{-\lambda_1}|\mathrm{vac}\rangle}{\langle\mathrm{vac}|a_{\lambda_1}\cdots a_{\lambda_n}a_{-\lambda_n}\cdots a_{-\lambda_1}|\mathrm{vac}\rangle} \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\int\limits_0^{\infty} d\lambda\ \mathrm{e}^{p\lambda}\right)^n = \mathrm{e}^{1/p}, \quad \mathrm{Re}\, p < 0\ . \end{aligned} \tag{6.10}$$

For the interpretation the generalized form-factors in quantum integrable field theory as traces (6.10) one should put usually $\mathrm{Re}\, p = 0$, $(p = 2\pi i)$. In this case we should understand the result (6.10) as analytically continued from the domain, where $\mathrm{Re}\, p < 0$. This result can be compared with asymptotical expansion of the partition function $\prod_{n=1}^{\infty}(1-q^n)^{-1}$. Really, the trace of the operator e^{pd} can be presented as the integral over eigenvalues of this operator $\mathrm{e}^{p\lambda}$:

$$\mathrm{e}^{1/p} = 1 + \int\limits_0^{\infty} d\lambda\ \mathrm{e}^{p\lambda}\mathfrak{p}(\lambda)\ ,$$

where

$$\mathfrak{p}(\lambda) = \sum_{n\geqslant 0}\frac{1}{n!(n+1)!}\lambda^n\ . \tag{6.11}$$

On the other hand the coefficients $\mathfrak{p}_n$ of the partition function

$$\prod_{n>0}\frac{1}{(1-q^n)} = \sum_{n\geqslant 0}\mathfrak{p}_n q^n$$

have following asymptotical expansion

$$\mathfrak{p}_n \sim \sum_k \frac{1}{k!(k-1)!}n^k \tag{6.12}$$

in the region when $n \to \infty$. The explicit comparing (6.11) and (6.12) demonstrated their similarity.

7 Quantized Current Algebras

7.1. As we have seen in the previous section the total current algebra is apparently more suitable for the constructing the infinite-dimensional representations than standard RLL-formalism which used the Gauss coordinates of L-operators. On the other hand it is difficult to define the Hopf structure in terms of total currents while in the L-operator formalism the Hopf structure $\Delta' L = L \otimes L$ is quite natural due to RLL-relations and YB equation for R-matrix. We would like to discuss in this section the following point of view. It is possible to assign the Hopf algebra structure to the algebra of total currents adding to the commutation relations (2.18) the information on the Riemann problem and also some additional information which we will discuss below.

Let us start with formal total current algebra for the currents $e(u)$, $f(u)$ and $h^{\pm}(u)$ given for example by the commutation relations (2.18). This algebra is formal since in the commutation relations of total currents $[e(u), f(v)]$ the δ-function is defined formally as well as Cartan "half" currents $h^{\pm}(u)$.

The first step is to fix the Riemann problem which serves to divide the total currents $e(u)$ and $f(u)$ into "half"-currents $e^{\pm}(u)$ and $f^{\pm}(u)$. This fixes the notion of δ-function in the commutation relations $[e(u), f(v)]$, the contents of the generating functions $h^{\pm}(u)$ and also the full set of the commutation relations between all the generating functions $e^{\pm}(u)$, $f^{\pm}(u)$ and $h^{\pm}(u)$ (see the Proposition 1). On the other hand fixing first the analytical properties of δ-functions in this relation leaves the freedom for the factorization problem. This freedom is related to the possible twist in the Riemann problem and will be also discussed below.

Let us assume now that the generating functions $e^{\pm}(u)$, $f^{\pm}(u)$ and $h^{\pm}(u)$ are Gauss coordinates of some L-operator given by (2.9). It is natural to guess that this L-operators satisfy RLL-relations with some R-matrix.

We state that there exist the universal comultiplication rules for Gauss coordinates $e^{\pm}(u)$, $f^{\pm}(u)$, $h^{\pm}(u) = k_1^{\pm}(u)\left(k_2^{\pm}(u)\right)^{-1}$ and $\tilde{h}^{\pm}(u) = \left(k_2^{\pm}(u)\right)^{-1} k_1^{\pm}(u)$ ($h^{\pm}(u)$ may be not equal to $\tilde{h}^{\pm}(u)$ in the general case) which are based on the only assumption that the R-matrix in a critical point is given by the rank one operator. Let, for instance, $\overline{R}(i\hbar)$ is proportional to $1-P$, where P is a flip:

$$\overline{R}(i\hbar) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

It takes place for the Yang R-matrix, for Baxter's elliptic R-matrix and for Sine-Gordon R-matrix. Then the commutation relations for the L operators at the critical point of R-matrix basically reduces to the following:

$$\begin{aligned} k_1^{\pm}(u) f^{\pm}(u+i\hbar) &= f^{\pm}(u) k_1^{\pm}(u) , \\ k_2^{\pm}(u) f^{\pm}(u-i\hbar) &= f^{\pm}(u) k_2^{\pm}(u) , \\ k_1^{\pm}(u) e^{\pm}(u) &= e^{\pm}(u+i\hbar) k_1^{\pm}(u) , \\ k_2^{\pm}(u) e^{\pm}(u) &= e^{\pm}(u-i\hbar) k_2^{\pm}(u) . \end{aligned} \tag{7.1}$$

The natural map $\Delta' L = L\dot{\otimes}L$ implies the following universal comultiplication rules for the Gauss coordinates (see (2.10)):

$$\begin{aligned}
\Delta e^{\pm}(u) &= e^{\pm}(u')\otimes 1 + \sum_{p=0}^{\infty}(-1)^p \left(f^{\pm}(u'-i\hbar)\right)^p h^{\pm}(u')\otimes \left(e^{\pm}(u'')\right)^{p+1}, \\
\Delta f^{\pm}(u) &= 1\otimes f^{\pm}(u'') + \sum_{p=0}^{\infty}(-1)^p \left(f^{\pm}(u')\right)^{p+1}\otimes \tilde{h}^{\pm}(u'')\left(e^{\pm}(u''-i\hbar)\right)^p, \\
\Delta h^{\pm}(u) &= \sum_{p,p'=0}^{\infty}(-1)^{p+p'}\Big(\left(f^{\pm}(u')\right)^p\otimes\left(e^{\pm}(u''+i\hbar)\right)^p\Big)\times \\
&\times \quad \left(h^{\pm}(u')\otimes h^{\pm}(u'')\right)\Big(\left(f^{\pm}(u')\right)^{p'}\otimes\left(e^{\pm}(u''-i\hbar)\right)^{p'}\Big)\ . \qquad (7.2)
\end{aligned}$$

We see from (7.2) that the choice of the Riemann problem for the algebra of formal currents enables one to reconstruct also the comultiplication structure of the algebra. We will consider this ideology on some examples.

7.2. The twisting of the Yangian algebras. Besides the comultiplication for the Gauss coordinates given by the formulas (2.14)–(2.16) and related to the natural comultiplication in terms of L-operators there is exist another comultiplication introduced firstly in the paper [14] for the quantized affine algebras.

$$\begin{aligned}
\Delta e(u) &= e(u')\otimes 1 + h^{+}(u')\otimes e(u'')\ , \\
\Delta f(u) &= 1\otimes f(u'') + f(u')\otimes h^{-}(u'')\ , \\
\Delta h^{\pm}(u) &= h^{\pm}(u')\otimes h^{\pm}(u'')\ . \qquad (7.3)
\end{aligned}$$

This comultiplication describes the coproduct of the total currents $e(u)$ and $f(u)$ and at the first sight do not related to L-operator formulation of the corresponding deformed algebra. It was shown in the papers [2, 19] that the corresponding to (7.3) Hopf algebra can be obtained as twisted Hopf algebra, which is equivalent to infinite limit of the shifting automorphism in the quantum Weyl group. We would like to explain this twisting procedure on the example of the algebra $\mathcal{A}_{\hbar}(\widehat{\mathfrak{sl}_2})$. The twisting of the algebra $\widehat{DY(\mathfrak{sl}_2)}$ can be considered analogously. The essential part of the construction will be changing of the Riemann problems and comultiplication formulas.

Fix the parameter $a\in\mathbb{R}$. Consider the following automorphism

$$\omega_a\left(e(u)\right) = \mathrm{e}^{iau}e(u),\quad \omega_a\left(f(u)\right) = \mathrm{e}^{-iau}f(u),\quad \omega_a\left(h^{\pm}(u)\right) = \mathrm{e}^{\pm c\hbar a/2}h^{\pm}(u), \qquad (7.4)$$

which obviously conserve the commutation relations (2.18). The automorphism (7.4) being translated to the formal generators $\hat{e}_\lambda$, $\hat{f}_\lambda$ and $\hat{h}_\lambda$ takes the form:

$$\begin{aligned}
\omega_a\left(\hat{e}_\lambda\right) &= \hat{e}_{\lambda+a},\quad \omega_a\left(\hat{f}_\lambda\right) = \hat{f}_{\lambda-a}, \\
\omega_a\left(\hat{h}_\lambda\right) &= \hat{h}_\lambda\left[\mathrm{e}^{c\hbar a/2}\theta(\lambda) + \mathrm{e}^{-c\hbar a/2}\theta(-\lambda)\right] + 4\hbar^{-1}\mathrm{sh}(c\hbar a/2)\delta(\lambda)\ . \qquad (7.5)
\end{aligned}$$

Appearing of the δ-function in (7.5) is possible since we consider the elements of the algebra $\mathcal{A}_{\hbar}(\widehat{\mathfrak{sl}_2})$ as the integrals of the formal generators with exponentially decreasing weight functions.

The R-matrix will change and its value at the critical point yield the following modification of the commutation relations (7.1):

$$\begin{aligned} \mathrm{e}^{a\hbar}k_1^{\pm}(u)f^{\pm}(u+i\hbar) &= f^{\pm}(u)k_1^{\pm}(u)\ , \\ \mathrm{e}^{-a\hbar}k_2^{\pm}(u)f^{\pm}(u-i\hbar) &= f^{\pm}(u)k_2^{\pm}(u)\ , \\ k_1^{\pm}(u)e^{\pm}(u) &= \mathrm{e}^{-a\hbar}e^{\pm}(u+i\hbar)k_1^{\pm}(u)\ , \\ k_2^{\pm}(u)e^{\pm}(u) &= \mathrm{e}^{a\hbar}e^{\pm}(u-i\hbar)k_2^{\pm}(u)\ . \end{aligned} \tag{7.6}$$

The comultiplication in the twisted algebra due to (7.6) reads:

$$\begin{aligned} \Delta e^{\pm}(u) &= e^{\pm}(u')\otimes 1+\sum_{p=0}^{\infty}(-1)^p\left(\mathrm{e}^{-a\hbar}f^{\pm}(u'-i\hbar)\right)^p h^{\pm}(u')\otimes\left(e^{\pm}(u'')\right)^{p+1}, \\ \Delta f^{\pm}(u) &= 1\otimes f^{\pm}(u'')+\sum_{p=0}^{\infty}(-1)^p\left(f^{\pm}(u')\right)^{p+1}\otimes h^{\pm}(u'')\left(\mathrm{e}^{a\hbar}e^{\pm}(u''-i\hbar)\right)^p, \\ \Delta h^{\pm}(u) &= \sum_{p,p'=0}^{\infty}(-1)^{p+p'}\left(\left(f^{\pm}(u')\right)^p\otimes\left(\mathrm{e}^{-a\hbar}e^{\pm}(u''+i\hbar)\right)^p\right)\times \\ &\times\ \left(h^{\pm}(u')\otimes h^{\pm}(u'')\right)\left(\left(f^{\pm}(u')\right)^{p'}\otimes\left(\mathrm{e}^{a\hbar}e^{\pm}(u''-i\hbar)\right)^{p'}\right)\ . \end{aligned} \tag{7.7}$$

It is obvious that the automorphism (7.4) changes the asymptotical properties of the currents $e^{\pm}(u)$ and $f^{\pm}(u)$. These generating function defined by the Riemann problems (3.12) are analytical in the corresponding domains of the complex plane u and are decreasing as u^{-1} functions when $u\to\mp\infty$, respectively. The transformed currents $\omega_a\left(e^{\pm}(u)\right)$ and $\omega_a\left(f^{\pm}(u)\right)$ will have changed asymptotics

$$\omega_a\left(e^{\pm}(u)\right)\sim\mathrm{e}^{\pm a|\mathrm{Im}\,u|},\quad \omega_a\left(f^{\pm}(u)\right)\sim\mathrm{e}^{\mp a|\mathrm{Im}\,u|},\quad \mathrm{Im}\,u\to\mp\infty\ . \tag{7.8}$$

Consider now the limit of the twisted algebra when twisting parameter $a\to+\infty$. From (7.8) it is clear that

$$\lim_{a\to\infty}\omega_a\left(e^{-}(u)\right)=0\qquad \lim_{a\to\infty}\omega_a\left(f^{+}(u)\right)=0\ .$$

Because of the Ding-Frenkel relations (2.17) we have

$$\lim_{a\to\infty}e^{+}(u)=e(u-ic\hbar/4)\ ,\qquad \lim_{a\to\infty}f^{-}(u)=-f(u-ic\hbar/4)\ .$$

So we conclude that the limit of the twisted algebra is an algebra with the commutation relation given by (2.18) and the following Riemann problem:

$$e^{-}(u)=f^{+}(u)=0,\quad e(u)=e^{+}(u+ic\hbar/4),\quad f(u)=-f^{-}(u+ic\hbar/4). \tag{7.9}$$

¿From the general formulas (7.2) the comultiplication (7.3) follows. As result we claim that Drinfeld's new realization is the deformed current algebra with Riemann problem given in (7.9).

7.3. The algebra $\mathcal{A}_{\hbar,\eta}(\widehat{\mathfrak{sl}_2})$ [6]. Let us consider the following generalization of the total current algebra given by the commutation relations (2.18).

$$H^{+}(u)H^{-}(v)\ =\ \frac{\mathrm{sh}\,\pi\eta(u-v-i\hbar(1+c/2))}{\mathrm{sh}\,\pi\eta(u-v+i\hbar(1-c/2))}\times$$

$$\times \frac{\mathrm{sh}\,\pi\eta'(u-v+i\hbar(1+c/2))}{\mathrm{sh}\,\pi\eta'(u-v-i\hbar(1-c/2))}H^{-}(v)H^{+}(u)\ , \tag{7.10}$$

$$H^{\pm}(u)H^{\pm}(v) = \frac{\mathrm{sh}\,\pi\eta(u-v-i\hbar)\mathrm{sh}\,\pi\eta'(u-v+i\hbar)}{\mathrm{sh}\,\pi\eta(u-v+i\hbar)\mathrm{sh}\,\pi\eta'(u-v-i\hbar)}\ H^{\pm}(v)H^{\pm}(u)\ , \tag{7.11}$$

$$H^{\pm}(u)E(v) = \frac{\mathrm{sh}\,\pi\eta(u-v-i\hbar(1\pm c/4))}{\mathrm{sh}\,\pi\eta(u-v+i\hbar(1\mp c/4))}\ E(v)H^{\pm}(u)\ , \tag{7.12}$$

$$H^{\pm}(u)F(v) = \frac{\mathrm{sh}\,\pi\eta'(u-v+i\hbar(1\pm c/4))}{\mathrm{sh}\,\pi\eta'(u-v-i\hbar(1\mp c/4))}\ F(v)H^{\pm}(u)\ , \tag{7.13}$$

$$E(u)E(v) = \frac{\mathrm{sh}\,\pi\eta(u-v-i\hbar)}{\mathrm{sh}\,\pi\eta(u-v+i\hbar)}\ E(v)E(u)\ , \tag{7.14}$$

$$F(u)F(v) = \frac{\mathrm{sh}\,\pi\eta'(u-v+i\hbar)}{\mathrm{sh}\,\pi\eta'(u-v-i\hbar)}\ F(v)F(u)\ , \tag{7.15}$$

$$\begin{aligned}[E(u),F(v)] &= \hbar\left[\delta\left(u-v+\frac{ic\hbar}{2}\right)H^{+}\left(u+\frac{ic\hbar}{4}\right)-\right.\\ &- \left.\delta\left(u-v-\frac{ic\hbar}{2}\right)H^{-}\left(v+\frac{ic\hbar}{4}\right)\right],\end{aligned} \tag{7.16}$$

where the δ-function is defined by the formula (3.13) and the periods of the trigonometric functions η and η' are related as follows:

$$\frac{1}{\eta'}-\frac{1}{\eta} = -\hbar c\ . \tag{7.17}$$

Last relation is necessary in order to make the commutation relations given by (7.10)–(7.16) self consistent. The "rational" commutation relations (2.18) can be obtained from (7.10)–(7.16) by the degeneration $\eta\to 0$. Note that due to (7.17) also $\eta'\to 0$ in this limit. Note that the formulas (7.10)–(7.16) differ from the formulas given in [6] by reversing the sign of the central element. The algebra given by the commutation relations (7.10)–(7.16) is associated with simple Lie algebra $\mathfrak{sl}_2$ but can be formulated for arbitrary simply-laced Lie algebra, see for example [23].

One can see that the most unusual feature of these relations is the presence of two periods η and η' in trigonometric functions playing the role of structure constants. Let us make a comment on the appearance of such phenomena. If we put $c=0$ then the periods coincide and the relations are analogous to the ones for $c=0$ quantum affine algebra with $q=\mathrm{e}^{i\pi\eta\hbar}$ in variables $z=\mathrm{e}^{\pi\eta u}$, $w=\mathrm{e}^{\pi\eta v}$. The question is how to input the central charge to have nontrivial representation theory. Let us view first the classical picture. In the limit $\hbar\to 0$ $c=0$ the commutation relations (7.10)–(7.16) are the relations for Lie algebra of $\mathfrak{sl}_2$ valued (generalized) functions over z vanishing at $|\mathrm{Re}\ z|\to\pm\infty$ (the Cartan currents tend to ± 1) [20]

$$\begin{aligned}&[h^{\pm}(u),e(v)] = 2\pi i\eta\,\mathrm{cth}\pi\eta(u-v)e(v)\ ,\qquad [h^{\pm}(u),f(v)] = -2\pi i\eta\,\mathrm{cth}\pi\eta(u-v)e(v)\ ,\\ &[e(u),f(v)] = \delta(u-v)(h^{+}(u)-h^{-}(v))\ .\end{aligned} \tag{7.18}$$

The application of the Fourier transform to the generating functions $e(u)=e\otimes 2\pi\delta(u-z)$, $f(u)=f\otimes 2\pi\delta(u-z)$, $h^{+}(u)=h\otimes i\pi\eta\,\mathrm{cth}\pi\eta(u-z)$, $h^{-}(u)=h\otimes i\pi\eta\,\mathrm{cth}\pi\eta(u-z-i/\eta)$ which satisfy (7.18):

$$e(u)=\int_{-\infty}^{+\infty}d\lambda\ \mathrm{e}^{-i\lambda u}\hat{e}_\lambda\ ,\quad h^{\pm}(u)=\int_{-\infty}^{+\infty}d\lambda\ \mathrm{e}^{-i\lambda u}\frac{\hat{h}_\lambda}{1-\mathrm{e}^{\pm\lambda/\eta}}\ ,\quad f(u)=\int_{-\infty}^{+\infty}d\lambda\ \mathrm{e}^{-i\lambda u}\hat{f}_\lambda\ ,$$

turns these commutation relations into the standard form

$$[\hat{h}_\lambda, \hat{e}_\mu] = 2\hat{e}_{\lambda+\mu} \ , \quad [\hat{h}_\lambda, \hat{f}_\mu] = -2\hat{f}_{\lambda+\mu} \ , \quad [\hat{e}_\lambda, \hat{f}_\mu] = \hat{h}_{\lambda+\mu}$$

which admits the standard central extension

$$[\hat{e}_\lambda, \hat{f}_\mu] = \hat{h}_{\lambda+\mu} + \delta(\lambda+\mu)c \ , \quad [\hat{h}_\lambda, \hat{h}_\mu] = 2\delta(\lambda+\mu)c \ .$$

Let us look to the value of corresponding cocycle on the fields $h^+(u_i)$ [20]

$$B(h^+(u_1), h^+(u_2)) = 2i\pi\eta^2 \left(\frac{\pi\eta u}{\mathrm{sh}^2\, \pi\eta u} - \mathrm{cth}\, \pi\eta u \right) . \tag{7.19}$$

The first term in r.h.s. of (7.19) is no longer periodical function with a period $1/\eta$. Moreover, the cocycle can be written in term of the integral over the border of a strip $\Pi : 0 < \mathrm{Im}\, z < 1/\eta$ and derivatives over the period $1/\eta$:

$$B(x \otimes \varphi(z), y \otimes \psi(z)) = \frac{\eta^2}{4\pi} \int\limits_{\partial\Pi} dz \left(\frac{d\psi(z)}{d\eta}\varphi(z) - \psi(z)\frac{d\varphi(z)}{d\eta} \right) \langle x, y \rangle \ , \tag{7.20}$$

where $\langle \ , \ \rangle$ is Killing form, $x, y \in \mathfrak{sl}_2$. These arguments signify that the central extension for a quantum algebra should be achieved via the finite shift of the periods of the trigonometric functions in the defining relations (7.10)– (7.16).

We attach to the formal current algebra (7.10)–(7.16) the Riemann problem of the following type.

Given meromorphic function $g(u)$, we would like to find two functions $g^\pm(u)$ satisfying the following conditions:

- (i) $g(u) = g^+(u) - g^-(u)$;
- (ii) $g^\pm(u)$ are piecewise analytical functions. More precisely, $g^\pm(u)$ are analytical on the complement to some collection of the horizontal lines in a complex plane u;
- (iii) $g^+(u)$ $(g^+(u))$ has a boundary value on the lower (upper) boundaries of the corresponding strips;
- (iv) the following relations hold in any strip of analyticity of $g^\pm(u)$

  $$g^-(u) = -g^+(u - i\eta) \ , \quad g^+(u) = -g^-(u + i\eta) \ ,$$

  where $g^+(u - i\eta)$ and $g^-(u + i\eta)$ are the analytical continuations of $g^+(u)$ and $g^-(u)$.

The solution to this Riemann problem is given by the following integrals over the horizontal lines close to the point u:

$$g^\pm(u) = \pi\eta \int\limits_{\mathrm{Im}\, v \lessgtr \mathrm{Im}\, u} \frac{dv}{2\pi i} \, \frac{g(v)}{\mathrm{sh}\, \pi\eta(u - v)} \ .$$

More precisely, the Riemann problems for the currents $E(u)$ and $F(u)$ are chosen with the different periods $1/\eta$, $1/\eta'$ and certain shifts of the spectral parameter:

$$e^\pm(u) = \hbar^{-1} \sin \pi\eta\hbar \int\limits_{\mathrm{Im}\,(u-v) \lessgtr \pm c\hbar/4} \frac{dv}{2\pi i} \, \frac{E(v)}{\mathrm{sh}\, \pi\eta(u - v \mp ic\hbar/4)} \ , \tag{7.21}$$

$$f^\pm(u) = \hbar^{-1} \sin \pi\eta'\hbar \int\limits_{\mathrm{Im}\,(u-v) \lessgtr \pm c\hbar/4} \frac{dv}{2\pi i} \, \frac{F(v)}{\mathrm{sh}\, \pi\eta'(u - v \pm ic\hbar/4)} \ . \tag{7.22}$$

The coefficients in front of the integrals in (7.21) and (7.22) are chosen from the technical reasoning and the condition (i) have in this case the form:

$$\begin{aligned} e^{+}(u+ic\hbar/4)-e^{-}(u-ic\hbar/4) &= \frac{\sin\,\pi\eta\hbar}{\pi\eta\hbar}E(u)\ , \\ f^{+}(u-ic\hbar/4)-f^{-}(u+ic\hbar/4) &= \frac{\sin\,\pi\eta'\hbar}{\pi\eta'\hbar}F(u)\ . \end{aligned} \tag{7.23}$$

These Riemann problems are in accordance with the commutation relations (7.10)–(7.16) in the sense that they yields currents $e^{\pm}(u)$ and $f^{\pm}(u)$ which satisfy the commutation relations without the integral terms. As well as in the case of the Yangian algebras the Riemann problem is not formulated for the Cartan currents and we set

$$\tilde{h}^{\pm}(u)=\frac{\sin\,\pi\eta'\hbar}{\pi\eta'\hbar}H^{\pm}(u),\quad h^{\pm}(u)=\frac{\sin\,\pi\eta\hbar}{\pi\eta\hbar}H^{\pm}(u).$$

The commutation relations between currents $e^{+}(u)$, $f^{+}(u)$ and $h^{+}(u)$ are given by the relations (7.24)-(7.29).

$$e^{+}(u_1)f^{+}(u_2) - f^{+}(u_2)e(u_1)=\frac{\mathrm{sh}\,i\pi\eta'\hbar}{\mathrm{sh}\,\pi\eta' u}h^{+}(u_1)-\frac{\mathrm{sh}\,i\pi\eta\hbar}{\mathrm{sh}\,\pi\eta u}\tilde{h}^{+}(u_2), \tag{7.24}$$

$$\begin{aligned} \mathrm{sh}\,\pi\eta(u+i\hbar)h^{+}(u_1)e^{+}(u_2) &- \mathrm{sh}\,\pi\eta(u-i\hbar)e^{+}(u_2)h^{+}(u_1)= \\ &= \mathrm{sh}(i\pi\eta\hbar)\{h^{+}(u_1),e^{+}(u_1)\}, \end{aligned} \tag{7.25}$$

$$\begin{aligned} \mathrm{sh}\,\pi\eta'(u-i\hbar)h^{+}(u_1)f^{+}(u_2) &- \mathrm{sh}\,\pi\eta'(u+i\hbar)f^{+}(u_2)h^{+}(u_1)= \\ &= -\mathrm{sh}(i\pi\eta'\hbar)\{h^{+}(u_1),f^{+}(u_1)\}, \end{aligned} \tag{7.26}$$

$$\begin{aligned} \mathrm{sh}\,\pi\eta(u+i\hbar)e^{+}(u_1)e^{+}(u_2) &- \mathrm{sh}\,\pi\eta(u-i\hbar)e^{+}(u_2)e^{+}(u_1)= \\ &= \mathrm{sh}(i\pi\eta\hbar)\left(e^{+}(u_1)^2+e^{+}(u_2)^2\right), \end{aligned} \tag{7.27}$$

$$\begin{aligned} \mathrm{sh}\,\pi\eta'(u-i\hbar)f^{+}(u_1)f^{+}(u_2) &- \mathrm{sh}\,\pi\eta'(u+i\hbar)f^{+}(u_2)f^{+}(u_1)= \\ &= -\mathrm{sh}(i\pi\eta'\hbar)\left(f^{+}(u_1)^2+f^{+}(u_2)^2\right), \end{aligned} \tag{7.28}$$

$$h^{+}(u_1)h^{+}(u_2) = \frac{\mathrm{sh}\,\pi\eta'(u+i\hbar)\mathrm{sh}\,\pi\eta(u-i\hbar)}{\mathrm{sh}\,\pi\eta(u+i\hbar)\mathrm{sh}\,\pi\eta'(u-i\hbar)}h^{+}(u_2)h^{+}(u_1)\ . \tag{7.29}$$

The rest of the relations follows from the condition (iv) for the analytical continuations which in this case reads as:

$$e^{-}(u)=-e^{+}(u-i/\eta'')\ ,\quad f^{-}(u)=-f^{+}(u-i/\eta'')\ ,\quad \eta''=\frac{2\eta\eta'}{\eta+\eta'}\ .$$

We also impose analogous relation for the Cartan currents:

$$h^{-}(u)=h^{+}(u-i/\eta'')\ .$$

The integral formulas (7.21) and (7.22) dictate the following presentation of the currents $e^{\pm}(u)$ and $f^{\pm}(u)$:

$$e^{\pm}(u) = \pm\frac{\sin\,\pi\eta\hbar}{\pi\eta}\int\limits_{-\infty}^{\infty}d\lambda\ \mathrm{e}^{-i\lambda u}\ \frac{\hat{e}_\lambda \mathrm{e}^{\mp c\hbar\lambda/4}}{1+\mathrm{e}^{\mp\lambda/\eta}}\ , \tag{7.30}$$

$$f^{\pm}(u) \;=\; \pm\frac{\sin\pi\eta'\hbar}{\pi\eta'}\int_{-\infty}^{\infty} d\lambda\ \mathrm{e}^{-i\lambda u}\ \frac{\hat f_\lambda \mathrm{e}^{\pm c\hbar\lambda/4}}{1+\mathrm{e}^{\mp\lambda/\eta'}}\ , \tag{7.31}$$

and due to the formulas (7.23) the total currents are given by the Fourier transform:

$$E(u)=\hbar\int_{-\infty}^{\infty} d\lambda\ \mathrm{e}^{-i\lambda u}\hat e_\lambda\ ,\quad F(u)=\hbar\int_{-\infty}^{\infty} d\lambda\ \mathrm{e}^{-i\lambda u}\hat f_\lambda\ .$$

Note that in the limit $\eta\to 0$ formulas (7.30) and (7.31) go to (4.16). It is natural to guess that the Cartan generating functions are given also by the Fourier integrals of the generators $\hat t_\lambda$:

$$h^{\pm}(u)=\frac{\sin\pi\eta\hbar}{2\pi\eta}\int_{-\infty}^{\infty} d\lambda\ \mathrm{e}^{-i\lambda u}\ \hat t_\lambda \mathrm{e}^{\pm\lambda/2\eta''}\ .$$

The generators $\hat e_\lambda$ and $\hat f_\lambda$ can be defined by means of the inversion of the Fourier integrals in (7.30), (7.31) for the currents $e^+(u)$ and $f^+(u)$ respectively. This can be done in the domains of the analyticity of the corresponding currents. These domains depend on the choice of the representation of the algebra $\mathcal{A}_{\hbar,\eta}(\widehat{\mathfrak{sl}_2})$. For instance, the spectral parameter z of the evaluation representation shifts these domains. To obtain the commutation relations between the formal generators $\hat e_\lambda$, $\hat f_\lambda$ and $\hat t_\lambda$ one can use the strip $\{-1/\eta-\hbar c/4<\mathrm{Im}\,u<-\hbar c/4\}$ as it was done in [6]. As well as in the case of the algebra $\mathcal{A}_\hbar(\widehat{\mathfrak{sl}_2})$ the elements of the algebra $\mathcal{A}_{\hbar,\eta}(\widehat{\mathfrak{sl}_2})$ are the formal integrals similar to (4.27) (see details in [6]).

Considering the generating functions $e^{\pm}(u)$, $f^{\pm}(u)$, $h^{\pm}(u)=k_1^{\pm}(u)\left(k_2^{\pm}(u)\right)^{-1}$ as the Gauss coordinates of the L-operators and using (7.1) we can obtain from the natural comultiplication in terms of L-operator $\Delta' L(u)=L(u)\dot\otimes L(u)$ the comultiplication map for these generating functions. The verification of the accordance of these comultiplication rules with the commutation relations (7.24)–(7.29) bring us to the following phenomena. Let us demonstrate it on the first term in the comultiplication formulas for the Cartan generating functions $h^{\pm}(u)$ (7.2): $\Delta h^{\pm}(u)=h^{\pm}(u)\otimes h^{\pm}(u)$. Other terms can be considered analogously.

The commutation relation (7.29) can be rewritten in the form

$$g(u_1-u_2,\xi-\hbar c)h^{\pm}(u_1,\xi)h^{\pm}(u_2,\xi)=h^{\pm}(u_1,\xi)h^{\pm}(u_2,\xi)g(u_1-u_2,\xi)\ , \tag{7.32}$$

where

$$g(u,\xi)=\frac{\mathrm{sh}\,\pi\eta(u-i\hbar)}{\mathrm{sh}\,\pi\eta(u+i\hbar)},\quad \xi=\eta^{-1}.$$

Due to the fact that $\Delta c=c\otimes 1+1\otimes c=c^{(1)}+c^{(2)}$ we conclude that the commutation relation

$$g(u_1-u_2,\xi-\hbar(c^{(1)}+c^{(2)}))\Delta h^{\pm}(u_1,\xi)\Delta h^{\pm}(u_2,\xi)=\Delta h^{\pm}(u_1,\xi)\Delta h^{\pm}(u_2,\xi)g(u_1-u_2,\xi)$$

will follows from (7.32) if and only if the comultiplication for the generating function is defined as follows:

$$\Delta h^{\pm}(u,\xi)=h^{\pm}(u,\xi-\hbar c^{(2)})\otimes h^{\pm}(u,\xi)+\cdots \tag{7.33}$$

By the dots we denoted other terms in the series (7.2). Actually, analogous phenomena was observed in the paper [8] in the theory of quantum Sine-Gordon model and implicitly used for the definition of the intertwining operators in the elliptic algebra $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}_2})$ which was served as the dynamical symmetry algebra of the eight-vertex model [21]. The algebra $\mathcal{A}_{\hbar,\eta}(\widehat{\mathfrak{sl}_2})$ which is a scaling limit of $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}_2})$ was investigated in [6]. In the L-operator formalism the algebra $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}_2})$ as well as $\mathcal{A}_{\hbar,\eta}(\widehat{\mathfrak{sl}_2})$ is given by the commutation relation:

$$R^{+}(u_1-u_2,\xi-\hbar c)L(u_1,\eta)L_2(u_2,\eta)=L_2(u_2,\eta)L_1(u_1,\eta)R^{+}(u_1-u_2,\xi)\ .$$

Although the comultiplication map moves the algebra into the tensor product of the algebra with shifted by the central element parameter this unusual Hopf structure allows to construct the intertwining operators for the infinite-dimensional representations of the algebra $\mathcal{A}_{\hbar,\eta}(\widehat{\mathfrak{sl}_2})$ and interpret them as Zamolodchikov-Faddeev operators in the quantum Sine-Gordon model [6].

To conclude we would like to mention the papers [24] and [25] where the quasi-Hopf twisting of the quantum groups have been considered. These papers are the realization of the idea which belongs to C. Frønsdal [26] that the elliptic deformations of the quantum groups can be obtained by the twisting of the corresponding Hopf algebras and belong to the category of Drinfeld's quasi-Hopf algebras [27]. In particular, it means that the algebra $\mathcal{A}_{\hbar,\eta}(\widehat{\mathfrak{sl}_2})$ can be obtained by quasi-Hopf twisting of the algebra $\mathcal{A}_{\hbar}(\widehat{\mathfrak{sl}_2})$ and also explains the comultiplication formulas like (7.33) first considered in [6]. Also the result of these papers can be considered as a proof that all elliptic deformations of the Hopf algebras are actually isomorphic for the different values of the elliptic parameter, the fact which was observed in [20] on the level of the classical limit of the algebra $\mathcal{A}_{\hbar,\eta}(\widehat{\mathfrak{sl}_2})$.

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