On Some Global Aspects of the Theory of Partial Differential Equations on Manifolds with Singularities

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Abstract

In this paper we investigate the connection between asymptotic expansions of solutions to elliptic equations near different points of singularities of the underlying manifold. We propose the procedure of computation of the asymptotic expansion of solution at any point of singularity via the asymptotics given at one (fixed) of these points.

Introduction

One of the main problems of the theory of differential equations on manifolds with singularities is the investigation of the behavior of solutions near singular points of the manifold. In fact, this problem is strongly connected with the other main problem of the theory — the solvability problem. Actually, the knowledge of the asymptotics of solutions in a neighborhood of the singularity set allows one to give the adequate functional spaces in which the finiteness theorem for the corresponding operator can be proved.

It is well-known (see, for example, [1]) that a solution to homogeneous elliptic equation near singular (say, conical) point has the so-called *conormal asymptotic* *expansion*, that is, the expansion of the form

$$u(r,x) \simeq \sum_{k} r^{S_k} \sum_{j=0}^{m_k} a_{kj}(x) \ln^j r,$$

where (r,x) are special coordinates in a neighborhood of the conical point corresponding to the representation

$$M = ([0,1] \times X) / (\{0\} \times X)$$

of the manifold M near this point; $r \in [0,1]$, $x \in X$. Far less known is the fact that the conormal asymptotics of solutions near points of singularity are strongly connected with one another, namely, that the asymptotics near one of these points uniquely determines the asymptotics near all of them (see [2]).

Hence, the problem arises to find out the method allowing to compute the asymptotics near any point of singularity provided that such an asymptotics is known near some (fixed) point. We remark that the solution of this problem, being of interest by itself, allows one to find out the *exact functional spaces* in which the considered equation is uniquely solvable (or, at least, possesses a Fredholm property).

The outline of the paper is as follows:

In the first section, we present simple examples which are aimed at the *motivation* of the general statement of the problem. Namely, in the first subsection, the connection between asymptotic expansion of solutions at singular points and correct statements of problems for the corresponding operators is illustrated. The second subsection illustrates (on the example) the *mechanism of transportation* of asymptotic expansions from one singular point to the another throug the complex domain.

The second section contains (as this follows from its title) the formulation of the *main problem* of investigation in this paper.

Later on, the two last sections contain the investigation of the above stated problem for the two-dimensional and multidimensional cases, respectively. The reason of the distinct consideration of the two-dimensional case is that in this case the main ideas of the computational algorythm are not darkened by thechnical difficulties. Both these sections are divided, in turn, into the two subsections, the first aimed at the consideration of the propagation of singularities along the regular component of the singularity set of the solution, and the second on the consideration of "jumps" of the asymptotic expansion from one of such components to the another.

1 Examples

1.1 Singularities of solutions to a homogeneous equation and correct statements of the problem

As it was already told in the introduction, the aim of this paper is to investigate global aspects of asymptotic theory of partial differential equations on manifolds with singularities. Namely, if the manifold M has two or more connected components of its singularity set and $\widehat{a}$ is an elliptic partial differential operator on M, then the problem is to compute the asymptotic expansion of solution to the homogeneous equation

$$\widehat{a}u = 0 \tag{1}$$

on the whole singularity set of M provided that this expansion is known on one of the connected components of this set.

The stated problem has important applications to investigation of properties of operator $\widehat{a}$ in the weighted Sobolev spaces $H^s_\gamma(M)$ (see, e.g. [2]). Let us illustrate the connection between asymptotic behavior of solutions to (1) and the investigation of the operator

$$\widehat{a} \,:\, H^s_\gamma(M) \to H^{s-m}_\gamma(M)$$

on a simple example (here m is the order of the operator $\widehat{a}$).

Consider the operator

$$\widehat{a} = \frac{1}{(1-x^2)^2}\left[\left(\left(1-x^2\right)\frac{\partial}{\partial x}\right)^2 + \frac{\partial^2}{\partial \varphi^2}\right] \tag{2}$$

given on the surface of the spindle S (see Figure 1). Here x is a coordinate along the axis of the spindle, $-1 < x < 1$ and φ is the coordinate corresponding to the rotation around this axis. It is easy to check that the full system of solutions to equation (1) for such an operator is

$$\begin{aligned} u_k^{\pm}(x,\varphi) &= \left(\frac{1+x}{1-x}\right)^k e^{\pm ik\varphi},\ k \neq 0, \\ u_0(x,\varphi) &= A + B\ln\frac{1+x}{1-x}, \end{aligned} \tag{3}$$

where k is an integer. So, one can see that if a solution to equation (1) behaves as $(1+x)^k$ at one of the vertexes $x = -1$ of the spindle, then this solution necessarily behaves as $(1-x)^{-k}$ at the other its vertex $x = 1$. This fact allows

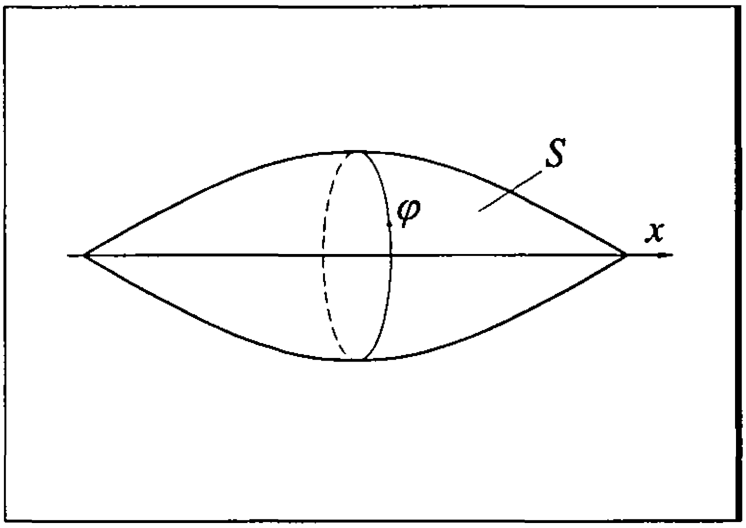

Figure 1. The spindle.

one, in particular, to investigate the correct statements of the problem for the operator

$$\widehat{a}_1 = \left(\left(1-x^2\right)\frac{\partial}{\partial x}\right)^2 + \frac{\partial^2}{\partial \varphi^2} \ : \ H^s_\gamma(S) \to H^s_\gamma(S) \tag{4}$$

(to be short, we had omitted the factor $(1-x^2)^{-2}$ in the expression (2) for the operator $\widehat{a}_1$) considered in the weighted Sobolev spaces $H^s_\gamma(S)$, $\gamma = (\gamma_0, \gamma_1)$. We recall that the latter spaces are defined with the help of the norm

$$\|u\|^2_{s,\gamma} = \int_{-1}^{1}\int_{0}^{2\pi} (1+x)^{-2\gamma_0}(1-x)^{-2\gamma_1}\left|(1-\widehat{a}_1)^{s/2}u(x,\varphi)\right|^2 d\varphi \frac{dx}{(1-x^2)}.$$

Here γ_0 and γ_1 are weights at points $x=-1$ and $x=1$, respectively.

One of the requirements for operator (4) to be an isomorphism is that the kernel of this operator must vanish. From the other hand, the functions $u^\pm_k(x,\varphi)$ given by (3) belong to spaces $H^s_\gamma(S)$ for $\gamma_0 < k$, $\gamma_1 < -k$ (and arbitrary values of s). From this fact it follows that for operator (4) to have zero kernel it is necessary to require that the interval $(\gamma_0, -\gamma_1)$ does not contain any integer k. Actually, if we fix the value γ_0, then all elements $u^\pm_k(x,\varphi)$ of the kernel with $k > \gamma_0$ will belong to the space $H^s_\gamma(S)$ at the left vertex $x=-1$ of the spindle S. Later on, as it was already mentioned, the behavior of the solution $u^\pm_k(x,\varphi)$ at the left vertex prescribes the behavior of this solution at the right vertex. Namely,

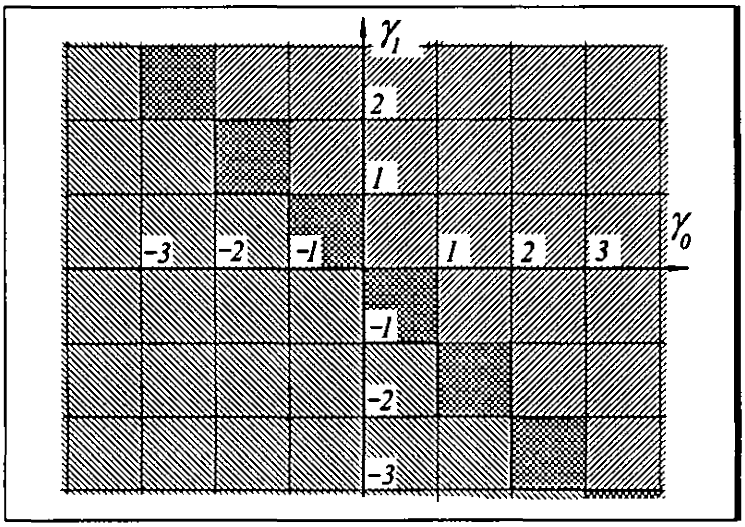

Figure 2. Kernel, cokernel, and the isomorphism region of the operator $\widehat{a}$.

if the solution is of order $(1+x)^k$ at $x=-1$, then it is of order $(1-x)^{-k}$ at $x=1$. Hence, for operator (4) to have the zero kernel, it is necessary to require that all elements of its kernel $u_k^{\pm}(x,\varphi)$ which belong to the space $H_\gamma^s(S)$ at the left vertex $x=-1$ do not belong to this space at the right vertex $x=1$ of the spindle S. This means that any value of k subject to the inequality $k>\gamma_0$ must satisfy the condition $k\geq-\gamma_1$, because the latter inequality is equivalent to the fact that the function $u_k^{\pm}(x,\varphi)$ does not belong to the space $H_\gamma^s(S)$ at $x=1$. So, the domain on the plane (γ_0,γ_1) where the operator $\widehat{a}$ has zero kernel is such as it is shown on Figure 2 (the upper dashed region).

To investigate the cokernel of this operator we remark that the adjoint operator[1]

$$\widehat{a}^* : H_{-\gamma}^{-s+m} \to H_{-\gamma}^{-s}$$

is given by the same expression. So, the domain on the plane (γ_0,γ_1), where operator (4) has zero cokernel is such as it is drawn on Figure 2 (the lower dashed region).

[1]with respect to the paring

$$(u,v)=\int_{-1}^{1}\int_{0}^{2\pi} u(x,\varphi)\,v(x,\varphi)\,d\varphi\frac{dx}{(1-x^2)}.$$

Now, combining the two obtained results one can construct the *isomorphism* region for the considered operator which is dashed twice on Figure 2.

The above analysis shows that the dependence between the asymptotic behavior of one and the same solution at different points of singularity of the underlying manifold affects the correct statements of the problems of the corresponding differential operator. This paper is aimed at the investigation of this dependence.

Further, we are mostly interested in the investigation of partial differential equations on manifolds with singularities of the wedge and edge types (see [1]). Since all these singularities can be obtained with the help of the operation of constructing a cone over a manifold and taking direct product with a smooth manifold, it suffices to investigate the above stated problem on manifolds with conical singularities. For simplicity, we shall consider here singularities of the type of the circular cones in different dimensions though methods developed in this paper seem to be applicable to the conical singularities of an arbitrary type.

1.2 Propagation of singularities. Metamorphosis

Here, on simple examples, we illustrate that the investigation of the above stated problem requires the *analytic continuation* of the differential equation in question into the complex space.

1. In this example, we shall show that the singularities of solutions, being located at points of singularities of the underlying manifold in the real domain, propagate to the complex domain along the degeneration set of the analytic continuation of the considered differential operator. To do this, let us consider the Laplace equation on the surface of the two-dimensional cone C:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{c^2}{r^2}\frac{\partial^2 u}{\partial\varphi^2}=0. \tag{5}$$

Here (r,φ) are polar coordinates on the surface of the cone, and c is a constant determined by the opening of the cone.

It is easy to see that the full system of solutions to equation (5) is

$$u_k^{\pm}(r,\varphi)=r^{\pm ck}e^{ik\varphi},\ k\in\mathbf{Z}. \tag{6}$$

The latter formula shows that these solutions have singularities on the set

$$r^2=x^2+y^2=0$$

in the complexification of the cone C (we use here the coordinates

$$(x,y)=(r\cos\varphi,r\sin\varphi)\in\mathbf{C}^2$$

on the above mentioned complexification).

Certainly, this observation is based on the fact that for our simple example we have written down the solutions to the homogeneous equation in the explicit form. Since for general equations on manifolds with singularities it is not possible, one has to understand the reason of appearence of the singularities in terms of the considered operators. To do this, we notice that, due to the relations

$$r\frac{\partial}{\partial r}=z\frac{\partial}{\partial z}+\zeta\frac{\partial}{\partial \zeta},\ \frac{\partial}{\partial \varphi}=i\left(z\frac{\partial}{\partial z}-\zeta\frac{\partial}{\partial \zeta}\right), \tag{7}$$

$z=x+iy$, $\zeta=x-iy$, equation (5) can be rewritten in the form

$$\frac{1}{z\zeta}\left\{\left(1-c^2\right)\left(z\frac{\partial}{\partial z}\right)^2+2\left(1+c^2\right)z\zeta\frac{\partial^2}{\partial z\partial\zeta}+\left(1-c^2\right)\left(\zeta\frac{\partial}{\partial \zeta}\right)^2\right\}u=0. \tag{8}$$

We remark that the set of singularities $x^2+y^2=0$ of solutions to equation (5) exactly coinsides with the degeneration set of this equation. Clearly, this fact is not an occasional one, and we shall see below that the *singularities of solution propagate from singular points of the manifold into the complex domain along the degeneration set of the considered equation.* Hence, one can imagine that if the equation has more than one point of singularity, then the asymptotic expansion comes from one of these point to another *through the complex domain along the degeneration set of the corresponding equation.* This guess can be confirmed with the help of the following example which is a slight modification of the above considered one.

2. Consider the equation

$$\frac{1+r^2}{r^2}\left[\left(r\frac{\partial}{\partial r}\right)^2+\frac{\partial^2}{\partial \varphi^2}\right]u=0 \tag{9}$$

as an equation on the surface of the spindle. In this case one of the vertexes of the spindle corresponds to the origin in the plane (x,y), and the other corresponds to infinity. We remark that the variable change $\rho=1/r$ does not change the form of equation (9). In this example, the singular points $r=0$ and $r=\infty$ are not connected by the set of singularities of the solution on the (real) spindle, but are lying on one and the same connected set $\{z=0\}\cup\{\zeta=0\}$ of singularities of solution in the complexification of this spindle. Such a situation is drawn schematically on Figure 3. So, the form of the asymptotic expansion at one of (real) points of singularities can be found out by that at the other if the propagation of a singularity along the degeneration set (more exactly, along the

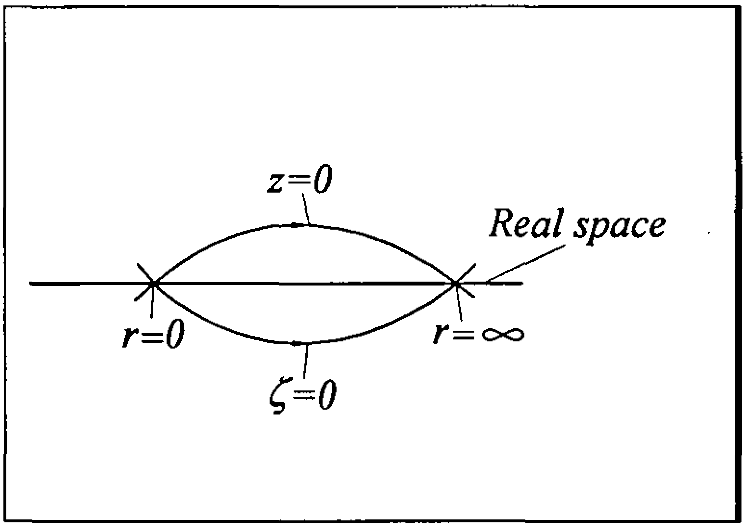

Figure 3. Degeneration set (simple configuration).

regular part of the degeneration set) of the equation under consideration can be computed. The method of this computation will be described below.

3. However, for more general equations the situation can be quite different. Namely, in the latter example the degeneration set of solutions to homogeneous equation was a *regular* complex manifold apart from the real singular points of M. So, to compute the asymptotic expansion at one of singular points via that at the other it was sufficient to examine the propagation of singularities along this regular part. In general, however, the regular part of the degeneration set of the equation (and, hence, the regular part of the set of singularities of solutions) can split into different connected components, and points of singularity of the manifold M can lye on different connected components. To illustrate that this situation can really take place, let us turn our mind once more to the consideration of equation (2) on the surface of the spindle. The degeneration set for this equation can be decomposed into two irreducible components $x = 1$ and $x = -1$. Consequently, the regular part of this degeneration set splits into the (disjoint) union of two connected components (these two components intersect each other at infinity; this fact is clearly quite occasional). It is clear that one of the two points of singularity of the spindle lyes on one of these components, and the other point of singularity lyes on the other of them; this situation is drawn schematically on Figure 4.

Hence, if one knows the asymptotic expansion at one of the vertexes, say,

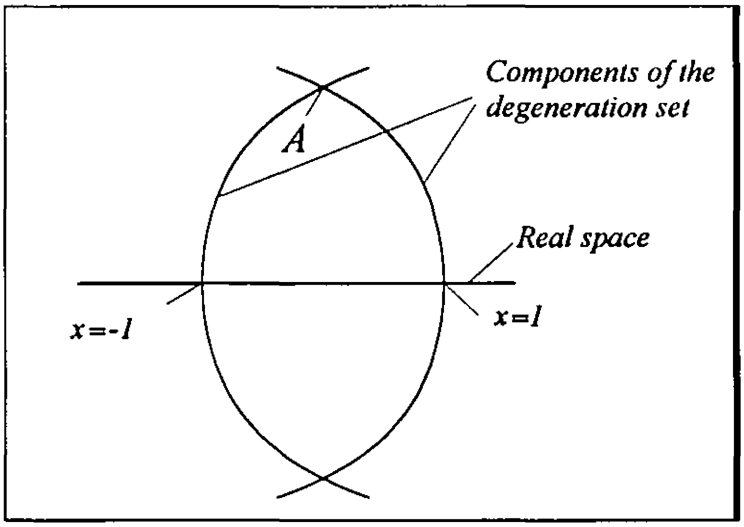

Figure 4. Degeneration set (general case).

$x = 1$, then to compute the corresponding asymptotic expansion at the other vertex one has:

1) to investigate the propagation of singularity along the component $x = 1$ of the degeneration set;

2) to examine the "jump" of the singularity from one of the connected components to the other. Clearly, this jump takes place at the point A of intersection of the two irreducible components $x = 1$ and $x = -1$ of the degeneration set (see Figure 4. So, to examine the mentioned jump one has to investigate the asymptotic expansion of the solution near the point A and then to find out the asymptotic expansion of the solution at points of the component $x = -1$ near the point A;

3) to investigate the propagation of singularity along the component $x = -1$ from A to the vertex $x = -1$.

Thus, in general, the following two problems arise:

First,to investigate the propagation of singularities along regular parts of degeneration sets of the equation in question.

Second, to find out the asymptotic expansion of a singularity of solution on one of the components of its singularity set provided that the corresponding asymptotic expansion is known on the other component. This problem can be solved by investigation of asymptotic expansion of solution in a neighborhood of the *intersection* of the two components in question.

In the rest part of the paper we shall consider both these problems. To make the presentation more clear, we shall consider first the two-dimensional case.

2 General Statement

So, the general statement of the problem is as follows:

Let M be an n-dimensional real-analytic manifold which is a real part of a complex-analytic manifold $M_{\mathbf{C}}$. Suppose that the manifold M has a finite number of singular points $m_1,\ldots,m_k$ each being a point of conical type. This means that in a neighborhood U_j of each point m_j, $j=1,\ldots,k$ the manifold M is diffeomorphic to the cone

$$(0,1)\times S^{n-1}/\{0\}\times S^{n-1}, \tag{10}$$

where S^{n-1} is an $n-1$-dimensional unit sphere in the Cartesian space $\mathbf{R}^n$. So, near each point m_j there exists a "coordinate system" (r,ω), $r\in(0,1)$, $\omega\in S^{n-1}$. We suppose that this coordinate system is compatible with the complex extension $M_{\mathbf{C}}$ of the manifold M, that is, that (r,ω), $r\in D_1$, $\omega\in Q^{n-1}$ forms a coordinate system in a neighborhood of the point $m_j\in M_{\mathbf{C}}$. Here by Q^{n-1} we denote the complex quadrics

$$Q^{n-1}=\left\{x\in\mathbf{C}^n|\sum_{j=1}^{n}\left(x^j\right)^2=1\right\},$$

and by D_1 we denote the unit disk

$$D_1=\{r\in\mathbf{C}|\ |r|<1\}$$

in the complex plane $\mathbf{C}$. So, we suppose that the complex manifold $M_{\mathbf{C}}$ is biholomorphic to the complex cone

$$D_1\times Q^{n-1}/\{0\}\times Q^{n-1}.$$

Consider an elliptic differential operator $\widehat{a}$ of order m on the manifold M. This means, in particular, that this operator has the form

$$\widehat{a}=\sum_{|\alpha|\leq m}a_\alpha(x)\left(\frac{\partial}{\partial x}\right)^\alpha \tag{11}$$

near each regular point of M and the form

$$\widehat{a}=\sum_{j=1}^{m}\widehat{b}_j\left(r\frac{\partial}{\partial r}\right)^j \tag{12}$$

near each singular point m_j, where $\widehat{b}_j$ are differential operators on the complex quadrics Q^{n-1} (we use here the above mentioned representation (10) of the manifold M). Suppose that the operator $\widehat{a}$ can be analytically continued on the manifold $M_{\mathbf{C}}$; the continuation is given by the same formulas (11) and (12), where $a_\alpha(x)$ are now holomorphic functions of the variable $x \in \mathbf{C}^n$, and $\widehat{b}_j$ are differential operators on the complex quadrics Q^{n-1} with holomorphic coefficients.

Now the problem is *to investigate the asymptotic expansion of analytic continuations of solutions to the homogeneous equation*

$$\widehat{a}u = 0 \tag{13}$$

to the complex manifold $M_{\mathbf{C}}$ near the degeneration set of the operator $\widehat{a}$.

In particular, we must investigate the propagation of singularities along the degeneration set of the operator and the "metamorphosis" of the singularity which takes place at points of intersection of different components of the degeneration set.

3 Two-Dimensional Case

3.1 Propagation of singularities

It is well-known (see [1], [3]) that, under the above conditions, the asymptotics of solutions to (13) near each singular point of the manifold M has the form

$$u(r,\varphi) \simeq \sum_k r^{S_k} \sum_{j=0}^{m_k} a_{kj}(\varphi) \ln^j r$$

(conormal asymptotics), where r is a radial variable and φ is the angular variable along the directrix of the cone and the outer sum is taken over the set of values of $s = S_k$ having a finite intersection with any half-plane $\operatorname{Re} s < A$ for any real value of A.

In this subsection, we shall investigate how asymptotics of the conormal type propagate along the degeneration set of equation (13). Since in the two-dimensional case the sphere S^{n-1} is simply a circle, we shall use the notation φ instead of ω for the coordinate on this circle. Hence, equation (13) can be rewritten in the form

$$\widehat{a}u = a\left(r, \varphi, r\frac{\partial}{\partial r}, \frac{\partial}{\partial \varphi}\right) u = 0.$$

This equation can be considered as an equation on the real two-dimensional plane $\mathbf{R}^2$ with the coordinates x, y:

$$x = r\cos\varphi,\ y = r\sin\varphi$$

(sure, this equation degenerates at the origin). As above, we suppose that the coefficients of the obtained equation can be analytically continued to the complex domain.

To describe the degeneration set of the considered equation, it is convenient to introduce the complex variables $z = x + iy$, $\zeta = x - iy$. Using the relations (7) one can rewrite the equation in the form

$$a_1\left(z, \zeta, z\frac{\partial}{\partial z}, \zeta\frac{\partial}{\partial \zeta}\right) u = 0 \tag{14}$$

with some symbol $a_1(z, \zeta, p, q)$. The reader can notice that the latter equation is an equation of the Fuchs type in the variables (z, ζ) with the degeneration set $\{z = 0\} \cup \{\zeta = 0\}$. Hence, the singularities of solutions to this equation are posited on its degeneration set (see [4], [5]).

To be definite, let us consider the equation near the set $z = 0$; in this case z is a small complex variable and ζ is separated from zero. Thus, it is useful to rewrite the equation as[2]

$$a\left(z, \zeta, z\frac{\partial}{\partial z}, \frac{\partial}{\partial \zeta}\right) u = 0. \tag{15}$$

For example, equation (5) have the following form in the variables (x, y):

$$\left(x^2 + c^2y^2\right)\frac{\partial^2 u}{\partial x^2} + 2\left(1 - c^2\right)\frac{\partial^2 u}{\partial x \partial y} + \left(c^2x^2 + y^2\right)\frac{\partial^2 u}{\partial y^2} + \left(1 - c^2\right)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 0.$$

In the variables (z, ζ) we obtain

$$\left\{\left(1 - c^2\right)\left(z\frac{\partial}{\partial z}\right)^2 + 2\left(1 + c^2\right)\left(z\frac{\partial}{\partial z}\right)\left(\zeta\frac{\partial}{\partial \zeta}\right) + \left(1 - c^2\right)\left(\zeta\frac{\partial}{\partial \zeta}\right)^2\right\} u = 0.$$

If one examines this equation near the set $z = 0$ (under the assumption that ζ is not small), the equation can be rewritten as

$$\left\{\left(1 - c^2\right)\left(z\frac{\partial}{\partial z}\right)^2 + 2\zeta\left(1 + c^2\right)\left(z\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial \zeta}\right) + \zeta^2\left(1 - c^2\right)\left(\frac{\partial}{\partial \zeta}\right)^2 \right.$$
$$\left. + \zeta\left(1 - c^2\right)\left(\frac{\partial}{\partial \zeta}\right)\right\} u = 0.$$

[2]In what follows we omit subscripts of symbols. So, one and the same letter a can denote different symbols.

Let us search for solutions to equation (15) in the form

$$u(z,\zeta) = z^S \sum_{j=0}^{k} u_j(z,\zeta) \ln^j z, \tag{16}$$

where $u_j(z,\zeta)$ are regular functions of z near the origin:

$$u_j(z,\zeta) = \sum_{l=0}^{\infty} u_{jl}(\zeta) z^l. \tag{17}$$

Taking into account the relations

$$\begin{aligned} z\frac{\partial}{\partial z} u(z,\zeta) &= Sz^S \sum_{j=0}^{k} u_j(z,\zeta) \ln^j z + z^S \sum_{j=0}^{k} \left(z\frac{\partial}{\partial z} u_j(z,\zeta) \right) \ln^j z \\ &+ \sum_{j=1}^{k} j u_j(z,\zeta) \ln^{j-1} z, \end{aligned}$$

one can write down the result of the substitution of (16) into (15) in the form

$$\begin{aligned} a\left(z,\zeta,S+z\frac{\partial}{\partial z},\frac{\partial}{\partial \zeta}\right) u_k(z,\zeta) &= 0, \\ a\left(z,\zeta,S+z\frac{\partial}{\partial z},\frac{\partial}{\partial \zeta}\right) u_j(z,\zeta) &= \sum_{i=j+1}^{k} a_i\left(z,\zeta,z\frac{\partial}{\partial z},\frac{\partial}{\partial \zeta}\right) u_i(z,\zeta), \\ & \quad j = k-1,\dots,0. \end{aligned} \tag{18}$$

Here $a_i(z,\zeta,z\partial/\partial z,\partial/\partial\zeta)$ are differential operators of order $m+j-k$ with holomorphic in (z,ζ) coefficients.

Let us consider the first of equations (18). Expanding the coefficients of the operator $a(z,\zeta,S+z\partial/\partial z,\partial/\partial\zeta)$ into the Taylor series in z and substituting (17) for $u_k(z,\zeta)$, we arrive at the recurrent system of equations for $u_{kl}(\zeta)$:

$$\begin{aligned} a_0'\left(\zeta,S,\frac{\partial}{\partial \zeta}\right) u_{k0}(\zeta) &= 0, \\ a_0'\left(\zeta,S+l,\frac{\partial}{\partial \zeta}\right) u_{kl}(\zeta) &= -\sum_{j=0}^{l-1} a_{l-j}'\left(\zeta,S+j,\frac{\partial}{\partial \zeta}\right) u_{kj}(\zeta), \\ & \quad l = 1,2,\dots. \end{aligned} \tag{19}$$

Here $a'_j(\zeta, p, q)$ are Taylor coefficients of the symbol $a(z, \zeta, p, q)$ of the operator $a(z, \zeta, z\partial/\partial z, \zeta\partial/\partial\zeta)$:

$$a(z, \zeta, p, q) = \sum_{j=0}^{\infty} z^j a'_j(\zeta, p, q).$$

In particular,

$$a'_0(\zeta, p, q) = a(0, \zeta, p, q).$$

The recurrent system of equations for Taylor coefficients of functions $u_j(z, \zeta)$ can be obtained quite similar to system (19). Thus, one can see that to construct the asymptotic expansion of the form (15) to equation (16) it is sufficient to solve an ordinary differential equation for each coefficient of this expansion.

3.2 Asymptotics near the intersection

In this subsection we investigate the asymptotic expansion of solutions to equation (13) near a point $m_0 \in M_{\mathbb{C}}$ of intersection of different components of degeneration set. We require that the following condition is valid:

Condition 1 The considered point m_0 is a (proper) point of transversal intersection of two regular components of the degeneration set of the equation in question.

Under this condition, it is clear that the equation can be rewritten in the form (14) near the point m_0. However, unlike the previous subsection, one cannot neglect the factor ζ in the operator $\zeta\partial/\partial\zeta$ since at the point m_0 one has $z = \zeta = 0$.

Let us show that the asymptotics of a solution to equation (14) has a specific form in a neighborhood of the point m_0.

Lemma 1 *Let $u(z, \zeta)$ be a solution to equation (14) near $z = \zeta = 0$ having the form (16) of conormal asymptotics apart from the origin. Then this solution has an asymptotic expansion of the form*

$$u(z, \zeta) = z^{S_1} \zeta^{S_2} \sum_{j=0}^{k_1} \sum_{l=0}^{k_2} u_{jl}(z, \zeta) \ln^j z \, \ln^l \zeta \tag{20}$$

with $u_{jl}(z, \zeta)$ regular near m_0.

Proof. For simplicity, we shall consider the case when both multiplicities are equal to 1. So, for the solution $u(z,\zeta)$ we have the following asymptotic expansions:

$$\begin{aligned} u(z,\zeta) &= z^{S_1}\left[u_1(z,\zeta)\ln z + u_0(z,\zeta)\right] \text{ for } z \to 0, |\zeta| > \varepsilon, \\ u(z,\zeta) &= \zeta^{S_2}\left[u_1'(z,\zeta)\ln \zeta + u_0'(z,\zeta)\right] \text{ for } \zeta \to 0, |z| > \varepsilon \end{aligned} \tag{21}$$

(for any positive ε) with regular functions $u_j(z,\zeta)$, $u_j'(z,\zeta)$, $j = 1,2$. Denote

$$v(z,\zeta) = z^{-S_1}\zeta^{-S_2}u(z,\zeta).$$

This function have the form

$$\begin{aligned} v(z,\zeta) &= u_1(z,\zeta)\ln z + u_0(z,\zeta) \text{ for } z \to 0, |\zeta| > \varepsilon, \\ v(z,\zeta) &= u_1'(z,\zeta)\ln \zeta + u_0'(z,\zeta) \text{ for } \zeta \to 0, |z| > \varepsilon. \end{aligned}$$

Hence, the function

$$v_1(z,\zeta) = v(z,\zeta) - u_1(z,\zeta)\ln z - u_1'(z,\zeta)\ln \zeta$$

possesses the following properties:

1) It is univalent both around the manifold $z = 0$ and $\zeta = 0$.

2) It has at most logarithmic growth as $z \to 0$ and $\zeta \to 0$.

Hence, $v_1(z,\zeta)$ is a holomorphic function in a deleted neighborhood of the point m_0. Using the theorem on removable singularity, we obtain that this function is regular at the point m_0 as well. This proves the Lemma.

Now let us search for solutions to equation (14) in the form (20). Taking into account the relations

$$\begin{aligned} z\frac{\partial u}{\partial z} &= z^{S_1}\zeta^{S_2}\left\{\sum_{j=0}^{k_1}\sum_{l=0}^{k_2}\left[\left(S_1 + z\frac{\partial}{\partial z}\right)u_{jl}(z,\zeta)\right]\ln^j z\,\ln^l \zeta\right. \\ &+ \left.\sum_{j=1}^{k_1}\sum_{l=0}^{k_2} j u_{jl}(z,\zeta)\ln^{j-1} z\,\ln^l \zeta\right\}, \\ \zeta\frac{\partial u}{\partial \zeta} &= z^{S_1}\zeta^{S_2}\left\{\sum_{j=0}^{k_1}\sum_{l=0}^{k_2}\left[\left(S_2 + \zeta\frac{\partial}{\partial \zeta}\right)u_{jl}(z,\zeta)\right]\ln^j z\,\ln^l \zeta\right. \\ &+ \left.\sum_{j=0}^{k_1}\sum_{l=1}^{k_2} l u_{jl}(z,\zeta)\ln^j z\,\ln^{l-1} \zeta\right\}, \end{aligned}$$

we arrive at the following recurrent system of equations for the unknown coefficients $u_{jl}(z,\zeta)$ of asymptotic expansion (20):

$$a\left(z,\zeta,S_1+z\frac{\partial}{\partial z},S_2+\zeta\frac{\partial}{\partial\zeta}\right)u_{k_1k_2}(z,\zeta)=0,$$
$$a\left(z,\zeta,S_1+z\frac{\partial}{\partial z},S_2+\zeta\frac{\partial}{\partial\zeta}\right)u_{jl}(z,\zeta)= \tag{22}$$
$$\sum a\left(z,\zeta,z\frac{\partial}{\partial z},\zeta\frac{\partial}{\partial\zeta}\right)u_{j'l'}(z,\zeta)$$

where $a_{j'l'}(z,\zeta,z\partial/\partial z,\zeta\partial/\partial\zeta)$ are differential operators of order $m+j+l-j'-l'$ and the summation is taken over all indices j',l' such that $j'\geq j$, $l'\geq l$, $(j',l')\neq(j,l)$. Let us consider the first of equations (22). We shall search for a solution to this equation in the form of the Taylor series

$$u_{k_1k_2}(z,\zeta)=\sum_{\alpha,\beta=0}^{\infty}u_{k_1k_2}^{\alpha\beta}z^{\alpha}\zeta^{\beta}.$$

Substituting this relation into the first from equations (22) and expanding the coefficients of the operator $a(z,\zeta,S_1+z\partial/\partial z,S_2+\zeta\partial/\partial\zeta)$ in powers of z and ζ, we obtain

$$a'_{00}(S_1,S_2)\,u_{k_1k_2}^{00}=0,$$
$$a'_{00}(S_1+\alpha,S_2+\beta)\,u_{k_1k_2}^{\alpha\beta}= \tag{23}$$
$$-\sum_{(\gamma,\delta)<(\alpha,\beta)}a'_{\alpha-\gamma,\beta-\delta}(S_1+\gamma,S_2+\delta)\,u_{k_1k_2}^{\gamma\delta}.$$

Here $a'_{\gamma\delta}(p,q)$ are the Taylor coefficients of the symbol $H(z,\zeta,p,q)$ in powers of z and ζ:

$$a(z,\zeta,p,q)=\sum_{|(\gamma,\delta)|\leq m}a'_{\gamma\delta}(p,q)\,z^{\gamma}\zeta^{\delta}.$$

So, there exists a nontrivial solution to equation (14) of the form (20) only in the case when the numbers S_1 and S_2 satisfy the following equation

$$a'_{00}(S_1,S_2)=0. \tag{24}$$

The last equation is a homogeneous algebraic equation with respect to (S_1,S_2) of order m. Using this equation one can find the possible values of the ratio S_1/S_2 necessary for equation (14) to possess a solution of the form (20). These values allow to compute the power of ζ in asymptotic expansions (21) if the corresponding power of z is known.

The rest equation in (23) can be solved in a quite similar manner if the following nondegeneracy condition is valid:

Condition 2 The pair $(S_1 + \alpha, S_2 + \beta)$ is not a solution to equation (24) except for $(\alpha, \beta) = 0$.

We remark that the latter condition takes place in the generic position.

Similar considerations can be also used for computing asymptotic expansions of solutions to differential equations near singular points of the manifold. We shall illustrate this on the example of the equation (8):

$$\frac{1}{z\zeta}\left\{\left(1-c^2\right)\left(z\frac{\partial}{\partial z}\right)^2 + 2\left(1+c^2\right) z\zeta\frac{\partial^2}{\partial z \partial \zeta} + \left(1-c^2\right)\left(\zeta\frac{\partial}{\partial \zeta}\right)^2\right\} u = 0.$$

Searching for solutions to this equation in the form

$$u = z^\alpha \zeta^\beta$$

we are lead to the following equation for (α, β):

$$\left(1-c^2\right)\alpha^2 + 2\left(1+c^2\right)\alpha\beta + \left(1-c^2\right)\beta^2 = 0.$$

The ratio α/β can be computed from this equation:

$$\frac{\alpha}{\beta} = -\frac{1+c}{1-c} \quad \text{or} \quad \frac{\alpha}{\beta} = -\frac{1-c}{1+c}.$$

Taking into account that we are searching for solutions to (8) which are univalent on the real space $z = \overline{\zeta}$, we obtain the second equation for determining the numbers α and β:

$$\alpha - \beta = k \in \mathbf{Z}.$$

The last two equations give us possible values of α and β:

$$\begin{cases} \alpha = \dfrac{(c+1)k}{2}, \\ \beta = \dfrac{(c-1)k}{2}, \end{cases}$$

or

$$\begin{cases} \alpha = -\dfrac{(c-1)k}{2}, \\ \beta = -\dfrac{(c+1)k}{2}. \end{cases}$$

The reader can easily verify that all the obtained values of (α, β) really determine univalent solutions to equation (8). Actually, the corresponding solutions to the homogeneous equation are

$$z^{\frac{(c+1)k}{2}}\zeta^{\frac{(c-1)k}{2}} = r^{ck}e^{ik\varphi}$$

and

$$z^{-\frac{(c-1)k}{2}}\zeta^{-\frac{(c+1)k}{2}} = r^{-ck}e^{ik\varphi}.$$

The latter expressions correspond to formula (6) above.

4 Multidimensional case

In this section, we consider asymptotic expansions of solutions to the homogeneous equation (1) for some operator of the corner type (12) on n-dimensional manifold M with conical singularities. Similar to the previous section, we divide the presentation into two parts: investigation of the propagation of singularities along regular parts of the degeneration set of the considered equation and investigation of "metamorphosis" of singularity on the intersection points of these regular parts.

4.1 Propagation of singularities

Considerations similar to those of Subsection 3.1 show that the complexification of the equation of the corner type in the complex domain in a neighborhood of any point from regular part of its degeneration set reads

$$a\left(z, \zeta, z\frac{\partial}{\partial z}, \frac{\partial}{\partial \zeta}\right) u = 0, \tag{25}$$

where $z \in \mathbf{C}$ is an one-dimensional complex variable transversal to the degeneration set at the considered point and $\zeta = (\zeta_1, \ldots, \zeta_{n-1}) \in \mathbf{C}^{n-1}$ are coordinates on the degeneration set itself.

Let us search for solutions to this equation in the form

$$u(z,\zeta) = z^S \sum_{j=0}^{k} u_j(z,\zeta) \ln^j z \tag{26}$$

with some regular functions $u_j(z,\zeta)$. As above, we search the functions $u_j(z,\zeta)$ in the form of the Taylor series in the variable z:

$$u_j(z,\zeta) = \sum_{l=0}^{\infty} z^l u_{jl}(\zeta). \tag{27}$$

Substituting relations (26) and (27) into equation (25) and equating the coefficients of powers of z, we arrive at the following recurrent system of equations for the Taylor coefficients $u_{kj}(\zeta)$ of the main term $u_k(z,\zeta)$ of expansion (26):

$$\begin{aligned} a_0'\left(\zeta, S, \frac{\partial}{\partial \zeta}\right) u_{k0}(\zeta) &= 0, \\ a_0'\left(\zeta, S+l, \frac{\partial}{\partial \zeta}\right) u_{kl}(\zeta) &= -\sum_{j=0}^{l-1} a_{l-j}'\left(\zeta, S+j, \frac{\partial}{\partial \zeta}\right) u_{kj}(\zeta), \\ & \qquad l = 1, 2, \ldots, \end{aligned}$$

and similar recurrent systems for Taylor coefficients of the rest terms $u_j(z,\zeta)$, $j=0,\ldots,k-1$. Here, as above, $a'_j(\zeta,p,q)$ are the Taylor coefficients of the full symbol $a(z,\zeta,p,q)$ in the variable z:

$$a(z,\zeta,p,q)=\sum_{j=0}^{\infty} z^j a'_j(\zeta,p,q).$$

We remark that in the multidimensional case the Taylor coefficients $u_{ij}(\zeta)$ are to be determined from a partial differential equation along the components of the degeneration set of equation (25).

4.2 Asymptotics near the intersection

Here we investigate the asymptotics of solutions *near the intersection* of different components of the degeneration set of the equation considered. The geometrical situation in this case is quite different from that in the two-dimensional one. The matter is that in the multidimensional case the intersection between different components of the degeneration set is a complex-analytic manifold of non-zero dimension whereas in the two-dimensional case this intersection is simply a discrete set of points.

So, let us consider two submanifolds X_1 and X_2 in the complexification $M_{\mathbf{C}}$ which are two irreducible components of the degeneration set of the operator $\widehat{a}$ and suppose that these two manifolds intersect each other transversely at points of their intersection $X_1\cap X_2$. Then, similar to the results of the previous section, one can rewrite the operator $\widehat{a}$ in the form

$$\widehat{a}=a\left(z,\zeta,\eta,z\frac{\partial}{\partial z},\zeta\frac{\partial}{\partial\zeta},\frac{\partial}{\partial\eta}\right),$$

where the coordinates (z,ζ,η) are chosen in such a way that $z=0$ and $\zeta=0$ are equations of the manifolds X_1 and X_2, correspondingly.

Suppose that $u(z,\zeta,\eta)$ is a solution to the homogeneous equation

$$a\left(z,\zeta,\eta,z\frac{\partial}{\partial z},\zeta\frac{\partial}{\partial\zeta},\frac{\partial}{\partial\eta}\right)u(z,\zeta,\eta)=0 \tag{28}$$

such that $u(z,\zeta,\eta)$ has the asymptotic expansion of the conormal type near both X_1 and X_2. This means that

$$u(z,\zeta,\eta)=z^{S_1}\sum_{j=0}^{k_1} u_j(z,\zeta,\eta)\ln^j z \tag{29}$$

with regular functions $u_j(z,\zeta,\eta)$ near X_1 apart from X_2 (that is, for $|z| < \varepsilon_1$ and $|\zeta| > \varepsilon_2$ for some positive ε_1 and ε_2) and

$$u(z,\zeta,\eta) = \zeta^{S_2} \sum_{j=0}^{k_2} v_j(z,\zeta,\eta) \ln^j \zeta \tag{30}$$

with regular functions $v_j(z,\zeta,\eta)$ near X_2 apart from X_1. Then, similar to the result of Subsection 3.2, the following affirmation takes place:

Lemma 2 *Let $u(z,\zeta,\eta)$ be a solution to equation* (28) *having asymptotic expansions* (29) *and* (30) *at points of X_1 and X_2, correspondingly. Then this solution has the asymptotic expansion of the form*

$$u(z,\zeta,\eta) = z^{S_1} \zeta^{S_2} \sum_{j=0}^{k_1} \sum_{l=0}^{k_2} u_{jl}(z,\zeta,\eta) \ln^j z \ln^j \zeta \tag{31}$$

with regular functions $u_{jl}(z,\zeta,\eta)$ near the intersection $X_1 \cap X_2$, that is, at the points with $z = \zeta = 0$.

The proof of this lemma is quite similar to that of Lemma 1 in Subsection 3.2.

To establish the connection between numbers S_1 and S_2 let us search for a solution to equation (28) in the form (31). The substitution of expansion (31) into equation (28) goes quite similar to that in Subsection 3.2. The result is

$$a\left(z,\zeta,\eta,S_1 + z\frac{\partial}{\partial z}, S_2 + \zeta\frac{\partial}{\partial \zeta}, \frac{\partial}{\partial \eta}\right) u_{k_1k_2}(z,\zeta,\eta) = 0,$$

$$a\left(z,\zeta,\eta,S_1 + z\frac{\partial}{\partial z}, S_2 + \zeta\frac{\partial}{\partial \zeta}, \frac{\partial}{\partial \eta}\right) u_{jl}(z,\zeta,\eta) =$$

$$\sum_{j' \geq j, l' \geq l, (j',l') \neq (j,l)} a_{j'l'}\left(z,\zeta,\eta, z\frac{\partial}{\partial z}, \zeta\frac{\partial}{\partial \zeta}, \frac{\partial}{\partial \eta}\right) u_{j'l'}(z,\zeta,\eta)$$

where $a_{j'l'}(z,\zeta,\eta,z\partial/\partial z,\zeta\partial/\partial\zeta,\partial/\partial\eta)$ are differential operators of order $m+j+l-j'-l'$. Similar to the two-dimensional case we can construct the recurrent system of equations for the Taylor coefficients $u_{jl}^{\alpha\beta}(\eta)$ of the function $u_{jl}(z,\zeta,\eta)$ in (z,ζ):

$$u_{jl}(z,\zeta,\eta) = \sum_{\alpha,\beta=0}^{\infty} u_{jl}^{\alpha\beta}(\eta) z^\alpha \zeta^\beta.$$

For the main term $u_{k_1k_2}(z,\zeta,\eta)$ this system reads

$$a'_{00}\left(\eta,S_1,S_2,\frac{\partial}{\partial\eta}\right)u^{00}_{k_1k_2}(\eta)=0,$$

$$a'_{00}\left(\eta,S_1+\alpha,S_2+\beta,\frac{\partial}{\partial\eta}\right)u^{\alpha\beta}_{k_1k_2}= \tag{32}$$

$$-\sum_{(\gamma,\delta)<(\alpha,\beta)}a'_{\alpha-\gamma,\beta-\delta}\left(\eta,S_1+\gamma,S_2+\delta,\frac{\partial}{\partial\eta}\right)u^{\gamma\delta}_{k_1k_2}(\eta),$$

where, as above, $a'_{\gamma\delta}(\eta,p_z,p_\zeta,p_\eta)$ are the Taylor coefficients of the symbol a in powers of z and ζ:

$$a(z,\zeta,\eta,p_z,p_\zeta,p_\eta)=\sum_{|(\gamma,\delta)|\le m}a'_{\gamma\delta}(\eta,p_z,p_\zeta,p_\eta)z^\gamma\zeta^\delta.$$

Clearly, similar systems can be obtained for all coefficients $u^{\alpha\beta}_{jl}(\eta)$.

Thus, for coefficients $u^{\alpha\beta}_{jl}(\eta)$ we have obtained a recurrent system of differential equations of the form (32). Since the first equation in this system (see (32)) is a homogeneous one (its right-hand part vanishes), for this system to admit nontrivial solutions it is necessary that the homogeneous equation

$$a'_{00}\left(\eta,S_1,S_2,\frac{\partial}{\partial\eta}\right)u=0$$

on the manifold $X_1\cap X_2$ has a non-zero univalent solution for the given values of S_1, S_2. This is exactly the condition for determining the connection between values of S_1 and S_2 on the two components X_1 and X_2 of the degeneration set of the considered equation.

Remark 1 This condition can be formulated in the more explicit terms if there exists a real-type compact submanifold X of the intersection $X_1\cap X_2$ such that the operator expression $a'_{00}(\eta,S_1,S_2,\partial/\partial\eta)$ determines an elliptic analytic operator family on X with parameters S_1 and S_2. Denote by $\sum$ the set in the plane $\mathbf{C}_2$ with coordinates (S_1,S_2) such that the operator family $a'_{00}(\eta,S_1,S_2,\partial/\partial\eta)$ is invertible outside $\sum$. Then the connection between S_1 and S_2 is described by the inclusion $(S_1,S_2)\in\sum$.

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