SPECTRAL EXTENSION OF THE QUANTUM GROUP COTANGENT BUNDLE

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Abstract. The structure of a cotangent bundle is investigated for quantum linear groups $GL_q(n)$ and $SL_q(n)$. Using a q-version of the Cayley-Hamilton theorem we construct an extension of the algebra of differential operators on $SL_q(n)$ (otherwise called the Heisenberg double) by spectral values of the matrix of right invariant vector fields. We consider two applications for the spectral extension. First, we describe the extended Heisenberg double in terms of a new set of generators — the Weyl partners of the spectral variables. Calculating defining relations in terms of these generators allows us to derive $SL_q(n)$ type dynamical R-matrices in a surprisingly simple way. Second, we calculate an evolution operator for the model of q-deformed isotropic top introduced by A.Alekseev and L.Faddeev. The evolution operator is not uniquely defined and we present two possible expressions for it. The first one is a Riemann theta function in the spectral variables. The second one is an almost free motion evolution operator in terms of logarithms of the spectral variables. Relation between the two operators is given by a modular functional equation for Riemann theta function.

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Contents

1. Introduction

A notion of a Heisenberg double over quantum group has been formulated and attracted substantial researcher's interest in the early 90-s [AF.91, S, SWZ.92, SWZ.93]. From the algebraic point of view it is a smash product algebra (see [M]) of the quantum group (or, the quantized universal enveloping algebra) and its dual Hopf algebra (see [D.86, FRT]). In the differential geometric interpretation it may be viewed as an algebra of quantized differential operators over group or, equivalently, as an algebra of quantized functions over cotangent bundle of the group. Since the group's cotangent bundle serve a typical phase space for integrable classical dynamics, it is natural to attach the same role to the Heisenberg double over quantum group for quantum physical models. As a test example, a model of q-deformed isotropic top was suggested in [AF.91, AF.92]. A discrete time evolution in this model is given by a series of automorphisms of the Heisenberg double. It turns out however that finding an explicit expression for the model's evolution operator is not just a technical problem.[1] The automorphisms defining the evolution by no means can be treated as inner ones in the original algebra and so, for a proper realization of the q-top one needs an appropriate extension of the Heisenberg double.

Also stimulated by the invention of quantum groups were general studies of the algebras whose generators satisfy quadratic relations (see [PP] and references therein) and investigations of minor identities for matrices over noncommutative rings [GR.91, GR.92, KL]. These two lines of research are meeting together in the theory of the so-called quantum matrix algebras [H, IOP.99] whose structure theory can be developed in a full analogy with the usual matrix analysis. In particular, one can define quantum versions of the matrix trace and determinant [FRT], introduce notions of a spectrum and a power of quantum matrix, and formulate the Cayley-Hamilton theorem (see [GPS.97, IOP.99, OP.05] and references therein).

A remarkable fact about quantum matrix algebras is that their most known examples — the RTT algebra [FRT] and the reflection equation algebra [KS] — serve the building blocks for a construction of the quantum group differential geometry in general [SWZ.92] and so, also for the Heisenberg double. It is the aim of the present paper to apply the structure results on the quantum matrix algebras for investigation of the dynamics of the isotropic q-top. Following [AF.91, S, SWZ.92] we begin with a definition of the Heisenberg double as a smash product algebra of a pair of quantum matrix algebras. These are the RTT algebra, playing the role of quantized functions over group, and the reflection equation algebra, interpreted as quantized right invariant differential operators over group. We then consider a central extension of the reflection equation algebra by the spectrum of it's generating matrix of quantized right invariant vector fields, and define a proper (non central) extension of the whole Heisenberg double by these spectral variables. Finally, after the spectral extension is made, the evolution of the isotropic q-top becomes an inner automorphism of the Heisenberg double.[2] Constructing the evolution operator is then straightforward.

The paper is organized as follows. In the next section we recall some facts about universal R-matrix and the R-matrix techniques. We are mainly discussing the case of (numeric) R-matrices of a type $GL_q(n)$. These type R-matrices are later on used for description of the cotangent bundles (or, the Heisenberg doubles) over quantum linear groups.

In section 3 we introduce the RTT algebra, the reflection equation algebra and define their smash product algebra — the Heisenberg double. We are describing the algebras of the two

[1] This problem was suggested to authors by L.D. Faddeev in summer 1996, during Alushta conference "Nonlocal, nonrenormalizable field theories".

[2] Strictly speaking, one has to extend the algebra by a formal power series in the spectral variables.

linear types — $GL_q(n)$ and $SL_q(n)$. For the reflection equation algebra we formulate in these cases the Cayley-Hamilton theorem and use it for the spectral extension of the Heisenberg double. This is the first main result of the paper (see theorem 3.27).

The Heisenberg double is initially defined in terms of the quantized right invariant vector fields. In order to demonstrate the left-right symmetry of the Heisenberg double, in subsection 3.4 we describe it using quantized left invariant vector fields. We also derive explicit relations between the spectra of the matrices of left and right generators, see corollaries 3.17 and 3.34. To keep clearness of a presentation some technical lemmas are moved from this subsection to appendix B.

The spectral extension suggests yet another distinguished generating set for the Heisenberg double, namely, the one which satisfy the simplest possible – Weyl algebraic – relations with the spectral variables. In the subsection 3.5 we derive defining relations for this set, see theorem 3.36. Quite expectantly, the relations involve dynamical R-matrices whose dynamical arguments are the spectral variables (see corollary 3.37). Surprising facts are that the dynamical R-matrices are coming in pairs, and that they are derived by solving a simple system of (at most three) linear equations.

Section 4 is devoted to solving a dynamical problem for the isotropic q-top. This is our second main result. Noticing that an evolution operator of the model is not uniquely defined, we derive two different expressions for it. The first one is given in terms of the Riemann theta function whose matrix of periods is proportional to Gram matrix of the lattice A^*_{n-1}, see relations (4.3.4), (4.3.5). This solution converges for $|q| < 1$, or for q a rational root of 1. The second solution converging for arbitrary values of q is given in terms of logarithms of the spectral variables, see (4.4.2), (4.4.3). The idea for the logarithmic substitution (that means passing from Weyl type to Heisenberg type commutation relations) was suggested to authors by L.D. Faddeev (for argumentation see [F.94, F.95]). The evolution in the logarithmic variables reduces to an almost free motion. A relation between the two solutions is given then by a modular functional equation for Riemann theta function (4.4.4).

Concluding the introduction we would like to mention a number of open problems which, in our opinion, deserve further investigation. First of all, it is straightforward to formulate a problem of spectral extension for the Heisenberg doubles over orthogonal and symplectic quantum groups and over quantum linear supergroups. Technical prerequisites for this were developed, respectively, in [OP.05] and [GPS.05, GPS.06].

Another interesting problem is an extension of a modular double construction [F.95, F.99] (see also [KLS, GKL]) for the case of Heisenberg double over quantum group. A starting point for investigation here would be a modular functional relation (4.4.4) between the two evolution operators constructed in section 4. Riemann theta function standing in the denominator in this relation could be considered as an evolution operator for the modular dual Heisenberg double.

At last, an observation that a ribbon element serves a q-top evolution operator on the smash product algebra of a ribbon Hopf algebra with it's dual Hopf algebra (see example 4.3) could open a way for the spectral extension of a quasi-triangular Hopf algebra. A partial step in this direction is made in appendix A, where pairing of the quasi-triangular Hopf algebra with its dual Hopf algebra is extended for the set of spectral variables, see corollary A.2.

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2. R-matrices

In this introductory section we collect some necessary information about R-matrices and an R-matrix technique.

2.1. **Universal R-matrix.** First, we recall few basic notions from the theory of quasi-triangular Hopf algebras [D.86, D.89] and ribbon Hopf algebras [RT] (for review see [ChP, KSch]).

Let $\mathfrak{A}$ be a Hopf $\mathbb{C}$-algebra supplied with a unit $1 : \mathbb{C} \to \mathfrak{A}$, a counit $\epsilon : \mathfrak{A} \to \mathbb{C}$, a product $\mathbf{m} : \mathfrak{A} \otimes \mathfrak{A} \to \mathfrak{A}$, a coproduct $\Delta : \mathfrak{A} \to \mathfrak{A} \otimes \mathfrak{A}$, and an antipode $S : \mathfrak{A} \to \mathfrak{A}$ mappings subject to standard axioms.

A Hopf algebra $\mathfrak{A}$ is called *almost cocommutative* if there exists an invertible element $\mathcal{R} \in \mathfrak{A} \otimes \mathfrak{A}$ that intertwines the coproduct Δ and the opposite coproduct Δ^{op} (in Sweedler's notation: $\Delta^{op}(x) = x_{(2)} \otimes x_{(1)}$ if $\Delta(x) = x_{(1)} \otimes x_{(2)}$)

$$\mathcal{R}\Delta(x) = \Delta^{op}(x)\mathcal{R} \qquad \forall\, x \in \mathfrak{A}_{\mathcal{R}}. \tag{2.1.1}$$

In this case the element $\mathcal{R}$ is called a *universal R-matrix*, and the corresponding almost cocommutative Hopf algebra is denoted as $\mathfrak{A}_{\mathcal{R}}$. The algebra $\mathfrak{A}_{\mathcal{R}}$ is called *quasi-triangular* if additionally $\mathcal{R}$ satisfies relations

$$(\Delta \otimes id)\mathcal{R} \,=\, \mathcal{R}_{13}\mathcal{R}_{23}\,, \quad (id \otimes \Delta)\mathcal{R} \,=\, \mathcal{R}_{13}\mathcal{R}_{12}\,, \tag{2.1.2}$$

where $\mathcal{R}_{12} = \mathcal{R} \otimes 1$, $\mathcal{R}_{23} = 1 \otimes \mathcal{R}$, and $\mathcal{R}_{13} = \sum_i a_i \otimes 1 \otimes b_i$ for $\mathcal{R} = \sum_i a_i \otimes b_i$. Relations (2.1.1), (2.1.2) together imply an equality

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}\,, \tag{2.1.3}$$

which is called the *Yang-Baxter equation*.

In the almost cocommutative case an element $u := \mathbf{m}(S \otimes id)(\mathcal{R}_{21}) \in \mathfrak{A}_{\mathcal{R}}$ is invertible. In terms of u the square of antipode is expressed as

$$S^2(x) \,=\, u\, x\, u^{-1} \qquad \forall\, x \in \mathfrak{A}_{\mathcal{R}}\,. \tag{2.1.4}$$

In the quasi-triangular case one has following formulas

$$\begin{aligned} S(u) \,&=\, \mathbf{m}(id \otimes S)(\mathcal{R}_{12})\,, \qquad u^{-1} \,=\, \mathbf{m}(id \otimes S^2)(\mathcal{R}_{21})\,, \\ \Delta(u) \,&=\, (\mathcal{R}_{21}\mathcal{R}_{12})^{-1}\, u \otimes u\,, \qquad \mathcal{R}\,(u \otimes u) \,=\, (u \otimes u)\,\mathcal{R}\,. \end{aligned} \tag{2.1.5}$$

An element $uS(u) = S(u)u$ (in the almost cocommutative case) belongs to the center of $\mathfrak{A}_{\mathcal{R}}$. A *central* extension of the quasi-triangular Hopf algebra $\mathfrak{A}_{\mathcal{R}}$ by a so-called *ribbon element* v such that

$$v^2 \,=\, uS(u)\,, \qquad \Delta(v) \,=\, (\mathcal{R}_{21}\mathcal{R}_{12})^{-1}\, v \otimes v \tag{2.1.6}$$

is called a *ribbon Hopf algebra*. The ribbon element also fulfills relations

$$\epsilon(v) \,=\, 1\,, \qquad S(v) \,=\, v\,, \qquad \mathcal{R}\,(v \otimes v) \,=\, (v \otimes v)\,\mathcal{R}\,.$$

Throughout this paper our basic reference example of the quasi-triangular Hopf algebra $\mathfrak{A}_{\mathcal{R}}$ is the *quantized universal enveloping algebra* $U_q(\mathfrak{g})$ of a complex Lie algebra $\mathfrak{g} = \mathfrak{sl}(n)$ [D.86, J.85, J.86].

2.2. **Braid groups and their R-matrix representations.** In the rest of the section we introduce standard notation and recall basic results on R-matrix representations of the braid groups.

The braid group $\mathcal{B}_k$ in Artin's presentation is given by a set of generators $\{\sigma_i\}_{i=1}^{k-1}$ and relations

$$\sigma_i\sigma_{i+1}\sigma_i \;=\; \sigma_{i+1}\sigma_i\sigma_{i+1} \qquad \forall\, i=1,2,\dots,k-1, \tag{2.2.1}$$

$$\sigma_i\sigma_j \;=\; \sigma_j\sigma_i \qquad \forall\, i,j: |i-j|>1\,. \tag{2.2.2}$$

Let V be a finite dimensional $\mathbb{C}$-linear space. For any operator $X\in \mathrm{End}(V^{\otimes 2})$ and for all integer $i>0$, $j>0$ denote

$$X_i \;:=I^{\otimes(i-1)}\otimes X\otimes I^{\otimes(j-1)}\in \mathrm{End}(V^{\otimes(i+j)})\,, \tag{2.2.3}$$

where $I\in \mathrm{Aut}(V)$ is the identity operator.[3] We also use notation X_{ij} for an operator in $\mathrm{End}(V^{\otimes k})$, $1\le i\ne j\le k$, acting as X in *component spaces* V with labels i and j and as identity in the rest. In these notation $X_{i\,i+1}\equiv X_i$.

An operator $R\in \mathrm{Aut}(V\otimes V)$ satisfying equality

$$R_1R_2R_1 \;=\; R_2R_1R_2\,, \tag{2.2.4}$$

is called an R-*matrix*. Any R-matrix generates representations ρ_R of the braid groups $\mathcal{B}_k$, $k=2,3,\dots$

$$\rho_R:\quad \mathcal{B}_k\to \mathrm{Aut}(V^{\otimes k}),\quad \rho_R(\sigma_i)=R_i,\quad 1\le i\le k-1.$$

By a slight abuse of notation we assign the same symbol ρ_R to the R-matrix representations of the braid groups $\mathcal{B}_k$ for different values of index k. This should not cause problems as the braid groups admit a series of monomorphisms commuting with ρ_R

$$\mathcal{B}_k\hookrightarrow \mathcal{B}_{k+1}:\quad \sigma_i\mapsto\sigma_i\quad \forall i=1,\dots k-1. \tag{2.2.5}$$

Definition 2.1. *An* R-*matrix* R *is called* skew invertible *if there exists an operator* $\Psi_R\in$ *End*$(V^{\otimes 2})$ *such that*

$$\mathrm{Tr}_{(2)}R_{12}\Psi_{R23}=\mathrm{Tr}_{(2)}\Psi_{R12}R_{23}=P_{13}\,. \tag{2.2.6}$$

Here by $\mathrm{Tr}_{(i)}$ *we denote trace operation in* i*-th space, and by* P *— the permutation operator:* $P(u\otimes v)=v\otimes u\ \ \forall\, u,v\in V$.

With any skew invertible R-matrix R we associate a pair of operators $D_R, C_R\in \mathrm{End}(V)$

$$D_{R1}=\mathrm{Tr}_{(2)}\Psi_{R12}\,,\quad C_{R2}=\mathrm{Tr}_{(1)}\Psi_{R12}\,, \tag{2.2.7}$$

which, by (2.2.6), satisfy equalities

$$\mathrm{Tr}_{(2)}R_{12}D_{R2}=I_1\,,\quad \mathrm{Tr}_{(1)}C_{R1}R_{12}=I_2\,. \tag{2.2.8}$$

Further properties of the operators D_R and C_R are summarized below.

Proposition 2.2. [Is.04, O] *Let* R *be a skew-invertible* R-*matrix. The operators* D_R *and* C_R *(2.2.7) satisfy equalities*

$$D_{R1}\,I_2=\mathrm{Tr}_{(3)}D_{R3}R_2^{\pm1}\,P_{12}\,R_2^{\mp1}\,,\qquad C_{R3}\,I_2=\mathrm{Tr}_{(1)}C_{R1}R_1^{\pm1}\,P_{23}\,R_1^{\mp1}\,,$$

$$R_{12}\,D_{R1}\,D_{R2}=D_{R1}\,D_{R2}\,R_{12}\,,\qquad R_{12}\,C_{R1}\,C_{R2}=C_{R1}\,C_{R2}\,R_{12}\,. \tag{2.2.9}$$

[3] Strictly speaking a proper notation for the l.h.s. of (2.2.3) would be, say, $X_i^{(i+j)}$. We use the shortened notation X_i since a dependence on j is not critical for our considerations. All formulas below have sense if the index j is large enough. A minimal possible value for j in each case is obvious from the context.

Let W be a $\mathbb{C}$-linear space. For any skew invertible R-matrix R we define an R-*trace* map[4] $\mathrm{Tr}_{R}: \mathrm{End}_W(V) \to W$

$$Y \mapsto \mathrm{Tr}_{R}(Y) := \mathrm{Tr}\,(D_{R} Y)\,, \quad Y \in \mathrm{End}_W(V)\,.$$

Following properties of the R-trace are simple consequences of the relations given in the proposition 2.2.

Corollary 2.3. *Let R be skew invertible* R*-matrix. For any operator $Y \in \mathrm{End}_W(V)$ the* R*-trace associated with R satisfies relations*

$$\mathrm{Tr}_{R\,(2)}(R_{12}^{\varepsilon}\, Y_1\, R_{12}^{-\varepsilon}) = I_1\, \mathrm{Tr}_{R}(Y)\,. \tag{2.2.10}$$

where $\varepsilon = \pm 1$ and the symbol $\mathrm{Tr}_{R\,(i)}$ denotes taking the R*-trace in i-th space.*

For an element $x^{(k)} \in \mathbb{C}[\mathcal{B}_k]$ denote $X_{R}^{(k)} := \rho_R(x^{(k)}) \in \mathrm{End}(V^{\otimes k})$. Following cyclic property

$$\mathrm{Tr}_{R\,(1,\dots,k)} \left(X_{R}^{(k)}\; Y^{(k)} \right) = \mathrm{Tr}_{R\,(1,\dots,k)} \left(Y^{(k)}\; X_{R}^{(k)} \right)\,.$$

is fulfilled for any $k \ge 1$ and $Y^{(k)} \in \mathrm{End}_W(V^{\otimes k})$, and for all $x^{(k)} \in \mathbb{C}[\mathcal{B}_k]$.

Example 2.4. Permutation P: $P(u \otimes v) := v \otimes u \;\; \forall\, u, v \in V$, is the skew invertible R-matrix. Identity operator $I^{\otimes 2}$ is the R-matrix which is not skew invertible.

Example 2.5. Assume that the quasi-triangular Hopf algebra $\mathfrak{A}_{\mathcal{R}}$ admits a representation

$$\rho_V : \mathfrak{A}_{\mathcal{R}} \to \mathrm{End}(V).$$

As follows from the Yang-Baxter equation (2.1.3) an operator

$$R := \eta P\, (\rho_V \otimes \rho_V)(\mathcal{R}), \tag{2.2.11}$$

satisfies relation (2.2.4). Here scaling factor $\eta \in \{\mathbb{C} \setminus 0\}$ is introduced for sake of future convenience.

The R-matrix (2.2.11) is skew invertible, its skew inverse matrix is given by formula (see, e.g., [O], section 4.1.2)

$$\Psi_{R} \,=\, \eta^{-1} P\, (\rho_V \otimes \rho_V)((id \otimes S)\mathcal{R})$$

The matrices D_{R} and C_{R} associated with the R-matrix (2.2.11) are:

$$D_{R} \,=\, \eta^{-1}\, \rho_V(u)\,, \qquad C_{R} \,=\, \eta^{-1}\, \rho_V(S(u))\,. \tag{2.2.12}$$

Both, they are invertible and their properties (2.2.9) are descending from (2.1.5).

2.3. **Hecke algebras and Hecke type R-matrix.** An A-type *Hecke algebra* $\mathcal{H}_k(q)$ is a quotient algebra of the group algebra $\mathbb{C}[\mathcal{B}_k]$ (2.2.1), (2.2.2) by relations

$$(\sigma_i - q1)(\sigma_i + q^{-1}1) = 0 \qquad \forall\, 1 \le i \le k-1.$$

Under following conditions on the parameter q

$$\textbf{[k]} \qquad i_q := (q^i - q^{-i})/(q - q^{-1}) \neq 0 \quad \forall i = 2, 3, \dots, k, \tag{2.3.1}$$

the algebra $\mathcal{H}_k(q)$ is isomorphic to the group algebra of the symmetric group $\mathbb{C}[S_k]$ and, hence, semisimple. It's irreducible representations as well as its central idempotents are labeled by a set of partitions $\lambda \vdash k$. We are particularly interested in a series of idempotents corresponding to the one dimensional representations $\lambda = (1^k)$, $k = 1, 2, \dots$. These idempotents – we denote

[4] This map is often called a quantum trace or, shortly, a q-trace. In our opinion, the name R-trace is better appropriate to it.

them as $a^{(k)}$ – admit a recursive construction (see, e.g., [HIOPT], section 1, or [GPS.97], section 2.3, or [TW], lemma 7.2)

$$a^{(1)} = 1, \qquad a^{(k)} = \frac{(k-1)_q}{k_q}\, a^{(k-1)} \Big(\frac{q^{k-1}}{(k-1)_q}\, 1 \,-\, \sigma_{k-1} \Big) a^{(k-1)} \tag{2.3.2}$$

$$= \frac{(k-1)_q}{k_q}\, a^{(k-1)\uparrow 1} \Big(\frac{q^{k-1}}{(k-1)_q}\, 1 \,-\, \sigma_1 \Big) a^{(k-1)\uparrow 1} \qquad \forall\, k = 2, 3, \dots , \tag{2.3.3}$$

where we use symbol $x^{(k)\uparrow 1} \in \mathcal{H}_{k+1}(q)$ for an image of element $x^{(k)} \in \mathcal{H}_k(q)$ under following algebra monomorphism (c.f. with (2.2.5)):

$$\mathcal{H}_k \hookrightarrow \mathcal{H}_{k+1} : \quad \sigma_i \mapsto \sigma_{i+1} \quad \forall i = 1, \dots k-1 \,.$$

The idempotents $a^{(k)}$ obey relations

$$a^{(k)}\sigma_i = \sigma_i a^{(k)} = -q^{-1} a^{(k)} \qquad \forall\, i = 1, 2, \dots , k-1 \,, \tag{2.3.4}$$

$$a^{(k)} a^{(i)\uparrow j} = a^{(i)\uparrow j} a^{(k)} = a^{(k)} \,, \qquad \text{if } \; i + j \le k \,. \tag{2.3.5}$$

An R-matrix R satisfying quadratic minimal characteristic identity is called a *Hecke type* R-matrix. By an appropriate rescaling of R one always can turn its characteristic identity to a form

$$(R - qI)(R + q^{-1} I) = 0 \,. \tag{2.3.6}$$

In this case the corresponding representations ρ_R become representations of the Hecke algebras $\mathcal{H}_k(q)$

$$\rho_R : \quad \mathcal{H}_k(q) \to \mathrm{Aut}(V^{\otimes k}), \quad \rho_R(\sigma_i) = R_i, \quad 1 \le i \le k-1. \tag{2.3.7}$$

We reserve special notation for the R-matrix images of idempotents $a^{(k)}$:

$$A^{(k)} := \rho_R(a^{(k)}) \,, \quad A^{(k)\uparrow 1} := \rho_R(a^{(k)\uparrow 1}) \quad \forall\, k \ge 1 \,. \tag{2.3.8}$$

We also put $A^{(0)} := 1$. The elements $A^{(k)}$ will be further referred as *k-antisymmetrizers*.

Remark 2.6. The R-matrix analogues of relations (2.3.2)–(2.3.5) have been described in literature (see [J.86], [G]) even earlier then their algebraic prototypes.

2.4. $GL_q(n)$ **type R-matrix.**

Definition 2.7. *Consider a Hecke type* R*-matrix* R*. Assume that parameter* q *in its characteristic identity (2.3.6) satisfies conditions* **[n]** *(2.3.1), so that antisymmetrizers* $A^{(2)}, \dots , A^{(n)}$ *are well defined.* R *is called* $GL_q(n)$ type R-matrix *if two conditions*

$$A^{(n)} \Big(\frac{q^n}{n_q}\, I \,-\, R_n \Big) A^{(n)} \;=\; 0 \tag{2.4.1}$$

and

$$\mathrm{rk} A^{(n)} \;=\; 1 \tag{2.4.2}$$

are fulfilled.

Remark 2.8. Assuming $(n+1)_q \ne 0$, the condition (2.4.1) is equivalent to $A^{(n+1)} = 0$. For generic values of q, assuming validity of (2.4.1), the condition (2.4.2) is equivalent to demanding skew invertibility of R (see [G], propositions 3.6 and 3.10).

Proposition 2.9. [G, Is.04] *Let R be a skew invertible* R*-matrix of the type $GL_q(n)$. Then $C_{\rm R}$ and $D_{\rm R}$ are invertible and following relations are fulfilled*

$$D_{\rm R} C_{\rm R} \;=\; C_{\rm R} D_{\rm R} \,=\, q^{-2n}\, I, \tag{2.4.3}$$

$$\mathrm{Tr}_{{\rm R}(k)} A^{(k)} \;=\; q^{-n}\, \frac{(n+1-k)_q}{k_q} A^{(k-1)} \quad \forall\, k=1,2,\dots,n, \tag{2.4.4}$$

$$A^{(n)} \prod_{i=1}^{n} (D_{\rm R})_i \;=\; \prod_{i=1}^{n} (D_{\rm R})_i\, A^{(n)} \,=\, q^{-n^2} A^{(n)}. \tag{2.4.5}$$

Example 2.10. Consider the case $\mathfrak{A}_{\mathcal{R}}$ is the quantized universal enveloping algebra $U_q\mathfrak{sl}(n)$. Let V be a vector representation of $U_q\mathfrak{sl}(n)$, $\dim V = n$. In this case formula (2.2.11) with the scaling factor chosen as $\eta = q^{1/n}$ gives a standard *Drinfeld-Jimbo's* R-matrix R° of the $GL_q(n)$ type (see [KSch], section 8.4.2):

$$R^\circ \;=\; \sum_{i,j=1}^{n} q^{\delta_{ij}} E_{ij} \otimes E_{ji} \,+\, (q-q^{-1}) \sum_{i<j} E_{ii} \otimes E_{jj}\, . \tag{2.4.6}$$

Here $(E_{ij})_{kl} := \delta_{ik}\delta_{jl},\;\; i,j=1,\dots,n,$ is a standard basis of $n\times n$ matrix units. Via the so-called twist procedure (for details see [R.90]) R° gives rise to a multiparametric family of $GL_q(n)$ type R-matrices

$$R^f \;:=\; F R^\circ F^{-1} \;=\; \sum_{i,j=1}^{n} q^{\delta_{ij}} \frac{f_{ij}}{f_{ji}}\, E_{ij} \otimes E_{ji} + (q-q^{-1}) \sum_{i<j} E_{ii} \otimes E_{jj}\, , \quad \forall\, f_{ij} \in \{\mathbb{C}\setminus 0\}. \tag{2.4.7}$$

Here $F := \sum_{i,j=1}^{n} f_{ij}\, E_{ii} \otimes E_{jj}$ is a *twisting R-matrix*. In what follows we use these particular R-matrices for illustration purposes. Their corresponding matrices D_{R° and D_{R^f} are

$$D_{R^\circ} \;=\; D_{R^f} \;=\; \sum_{i=1}^{n} q^{2(i-n)-1} E_{ii}\, .$$

Remark 2.11. Generally speaking, $GL_q(n)$ type R-matrix can be realized in a tensor square of space V whose dimension is different from n. Examples of the R-matrices for any $\dim V \geq n$ are given in [G], in section 4. In what follows we do not assume any relation between the parameter n in the definition 2.7 and the dimension of the space V, unless it is stated explicitly.

3. Quantized functions on a cotangent bundle over matrix group

In this section we recall definition of a quantum group cotangent bundle and develop in linear cases – $GL_q(n)$ and $SL_q(n)$ – a basic techniques for it's structure investigation.

3.1. Quantized functions over matrix group (RTT algebra).

Definition 3.1. [D.86, FRT] *Let R be a skew invertible* R*-matrix. An associative unital algebra generated by a set of matrix components $\|T^i_j\|_{i,j=1}^{\dim V}$ satisfying relations*

$$R_{12}\, T_1\, T_2 = T_1\, T_2\, R_{12} \tag{3.1.1}$$

is denoted as $\mathfrak{F}[R]$ and called an RTT algebra*. The RTT algebra is endowed in a standard way with the coproduct and the counit*

$$\Delta(T^i_j) = \sum_k T^i_k \otimes T^k_j\, , \qquad \epsilon(T^i_j) = \delta^i_j\, . \tag{3.1.2}$$

Let further extend the RTT algebra by a set of inverse matrix components $\|(T^{-1})^i_j\|_{i,j=1}^{\dim V}$*:*

$$\sum_k T^i_k\,(T^{-1})^k_j \;=\; \sum_k (T^{-1})^i_k\,T^k_j \;=\; \delta^i_j\,1\,. \qquad (3.1.3)$$

The extended algebra can be endowed with the antipode mapping

$$S(T^i_j) \;=\; (T^{-1})^i_j\,, \qquad \textit{so that (see [R.89]):} \qquad S^2(T)\,D_R \;=\; D_R\,T\,. \qquad (3.1.4)$$

The resulting Hopf algebra is further denoted as $\mathfrak{FG}[R]$.

Example 3.2. Consider the quasi-triangular Hopf algebra $\mathfrak{A}_{\mathcal{R}}$ together with its representation ρ_V (see example 2.5). For any $x \in \mathfrak{A}_{\mathcal{R}}$ denote $\|\rho_V(x)^i_j\|$ a matrix of the operator $\rho_V(x)$ in a certain basis in the space V.

Let $\mathfrak{A}^*_{\mathcal{R}}$ be the dual Hopf algebra and let $\langle\cdot,\cdot\rangle$ denote a non degenerate pairing between $\mathfrak{A}_{\mathcal{R}}$ and $\mathfrak{A}^*_{\mathcal{R}}$. Consider two matrices of linear functionals on $\mathfrak{A}_{\mathcal{R}}$ — T^i_j and $(T^{-1})^i_j$ — such that

$$\langle T^i_j, x\rangle = \rho_V(x)^i_j\,, \quad \langle (T^{-1})^i_j\,, x\rangle = \rho_V(S(x))^i_j \quad \forall\, x \in \mathfrak{A}_{\mathcal{R}}\,. \qquad (3.1.5)$$

It is easy to see that these functionals satisfy conditions of the definition 3.1 (for details see, e.g., [B]), the numeric R-matrix R in (3.1.1) in this case is given by (2.2.11), relation (3.1.4) for the square of antipode descends from (2.1.4). The functionals T^i_j and $(T^{-1})^i_j$ generate a Hopf subalgebra in $\mathfrak{A}^*_{\mathcal{R}}$.

In case if $\mathfrak{A}_{\mathcal{R}}$ is a universal enveloping algebra $U\mathfrak{g}$ of some Lie algebra $\mathfrak{g}$, the dual Hopf algebra $(U\mathfrak{g})^*$ can be treated as $\mathrm{Fun}(\mathfrak{G}) \equiv \mathfrak{FG}$, where $\mathfrak{G}$ is a formal group corresponding to $\mathfrak{g}$. Therefore, heuristically we can treat the RTT algebras $\mathfrak{FG}[R]$ and $\mathfrak{F}[R]$ as algebras of *quantized functions over matrix group and matrix semigroup*, respectively. Here the term *matrix* refers to a matrix form of the coproduct (3.1.2); the term *quantized* means that relations (3.1.1) in general define a noncommutative product.

In the rest of the subsection we describe a construction of the inverse matrix T^{-1} for the RTT algebra associated with the $GL_q(n)$ type R-matrix.

Consider an element

$$\mathrm{det}_R T := \mathrm{Tr}_{(1,\dots,n)}\big(A^{(n)} T_1 T_2 \dots T_n\big). \qquad (3.1.6)$$

By the definition of the coproduct (3.1.2) and due to the rank 1 condition (2.4.2) the element $\mathrm{det}_R T$ is group-like

$$\Delta(\mathrm{det}_R T) = \mathrm{det}_R T \otimes \mathrm{det}_R T\,,$$

and it satisfies relations

$$A^{(n)}\, T_1 T_2 \dots T_n = T_1 T_2 \dots T_n\, A^{(n)} = A^{(n)}\, \mathrm{det}_R T\,.$$

Therefore, it is natural to call $\mathrm{det}_R T$ a *determinant of the matrix* T.

Proposition 3.3. [G] *Let R be a skew invertible $GL_q(n)$ type* R*-matrix. The following relation is satisfied in the corresponding RTT algebra* $\mathfrak{F}[R]$

$$(\mathrm{det}_R T)\, T \;=\; (O_R T O_R^{-1})\, \mathrm{det}_R T\,,$$

where O_R, $O_R^{-1} \in \mathrm{Aut}(V)$ *are mutually inverse matrices:*

$$O_{R1} \;=\; n_q\, \mathrm{Tr}_{(2,\dots,n+1)}\Big(P_1 P_2 \dots P_n\, A^{(n)}\Big)\,, \qquad (3.1.7)$$

$$(O_R^{-1})_1 \;=\; n_q\, \mathrm{Tr}_{(2,\dots,n+1)}\Big(A^{(n)} P_n \dots P_2 P_1\Big)\,,$$

(recall that P_i are permutation operators acting in components spaces $V_i \otimes V_{i+1}$).

Corollary 3.4. *In the assumptions of proposition 3.3 consider an extension of the RTT algebra* $\mathfrak{F}[R]$ *by an element* $(\det_R T)^{-1}$ *subject to relations*

$$(\det_R T)^{-1}\, T = (O_R^{-1} T O_R)(\det_R T)^{-1}\,, \quad \det_R T\, (\det_R T)^{-1} = (\det_R T)^{-1} \det_R T = 1\,,$$

In the extended algebra the inverse matrix T^{-1} *satisfying relations (3.1.3) is given by formula*

$$(T^{-1})_1 \;=\; q^{n(n-1)} n_q\, \mathrm{Tr}_{R\,(2,\dots,n)} \Big(T_2 \dots T_n A^{(n)}\Big)\, (\det_R T)^{-1}\,,$$

The resulting Hopf algebra is called $GL_q(n)$ type RTT algebra *and denoted as* $\mathfrak{F}_{GL_q(n)}[R]$.

Assume additionally that for the R*-matrix* R *the corresponding matrix* O_R *(3.1.7) is scalar:* $O_R \propto I$. *In this case* R *is called the R-matrix of* $SL_q(n)$ type. *In the corresponding RTT algebra* $\mathfrak{F}_{GL_q(n)R}$ *the element* $\det_R T$ *is central. A quotient of this algebra by relation* $\det_R T = 1$ *is called* $SL_q(n)$ type RTT algebra *and denoted as* $\mathfrak{F}_{SL_q(n)}[R]$.

Remark 3.5. For a skew invertible $GL_q(n)$ type R-matrix R consider a system of equations

$$R_{12}\, N_1\, N_2 \;=\; N_1\, N_2\, R_{12}\,, \qquad N^n \,\propto\, O_R \qquad \text{for some } N \in \mathrm{Aut}(V).$$

Note that a consistency condition for these equations — $R_{12} O_{R1} O_{R2} = O_{R1} O_{R2} R_{12}$ — is satisfied (see [OP.05]). By any solution N of these equations one can construct the $SL_q(n)$ type R-matrix

$$\widetilde{R}_{12} \;:=\; N_1 R_{12} N_1^{-1} \;=\; N_2^{-1} R_{12} N_2\,. \tag{3.1.8}$$

Example 3.6. For the R-matrices described in the example 2.10 one has

$$O_{R^\circ} \;=\; -I\,, \qquad O_{R^f} \;=\; -\textstyle\sum_{i=1}^n \Big(\prod_{j\neq i} f_{ji}/f_{ij}\Big)\, E_{ii}\,.$$

So, R° is $SL_q(n)$ type, while R^f is $SL_q(n)$ type only if $\forall\, i = 1,\dots,n:\; \prod_{j\neq i}(f_{ji}/f_{ij}) = \sqrt[n]{1}$. Taking a diagonal n-th root $O_{R^f}^{1/n}$ of the diagonal matrix O_{R^f} one finds the $SL_q(n)$ type R-matrix associated with R^f:

$$\widetilde{R}^f \;=\; R^{\tilde f}\,, \quad \text{where} \quad \tilde f_{ij} \;:=\; \textstyle\prod_{k\neq i,j} (f_{ij} f_{jk} f_{ki})^{1/n}\,, \quad \text{so that} \quad O_{R^{\tilde f}} = -I\,.$$

3.2. **Quantized right invariant vector fields (reflection equation algebra).**

Definition 3.7. [KS] *Let* R *be a skew invertible* R*-matrix. An associative unital algebra* $\mathfrak{LG}[R]$ *generated by a set of matrix components* $\|L^i_j\|_{i,j=1}^{\dim V}$ *satisfying relations*

$$L_1 R_{12} L_1 R_{12} \;=\; R_{12} L_1 R_{12} L_1 \tag{3.2.1}$$

is called a reflection equation algebra *or, shortly,* RE algebra*. The RE algebra* $\mathfrak{LG}[R]$ *is naturally endowed with a structure of left coadjoint* $\mathfrak{FG}[R]$*-comodule algebra*

$$\delta_\ell(L^i_j) = \sum_{k,m} T^i_k (T^{-1})^m_j \otimes L^k_m\,. \tag{3.2.2}$$

Example 3.8. [FRT] In notations of the examples 2.5, 3.2 consider following $\mathfrak{A}_{\mathcal{R}}$-valued matrices

$$\begin{aligned} L^{(+)}{}^i_j \;&=\; \langle id \otimes T^i_j,\, \mathcal{R}\rangle\,, \qquad L^{(-)}{}^i_j \;=\; \langle S(T^i_j) \otimes id\,, \mathcal{R}\rangle \;=\; \langle T^i_j \otimes id\,, \mathcal{R}^{-1}\rangle\,, \\ ((L^{(+)})^{-1})^i_j &= \langle id \otimes T^i_j,\, \mathcal{R}^{-1}\rangle, \qquad ((L^{(-)})^{-1})^i_j = \langle T^i_j \otimes id\,,\, \mathcal{R}\rangle. \end{aligned} \tag{3.2.3}$$

As a consequence of the Yang-Baxter equation (2.1.3) components of these matrices satisfy relations

$$R_{12}\, L_2^{(\pm)}\, L_1^{(\pm)} = L_2^{(\pm)}\, L_1^{(\pm)}\, R_{12}\,, \quad R_{12}\, L_2^{(+)}\, L_1^{(-)} = L_2^{(-)}\, L_1^{(+)}\, R_{12}\,, \tag{3.2.4}$$

where R is given by (2.2.11). By (2.1.2), the elements $\big((L^{(\pm)})^{\pm 1}\big)^i_j$ generate a Hopf $\mathfrak{A}_{\mathcal{R}}$-subalgebra

$$\Delta(L^{(\pm)}{}^i_j) = \sum_k L^{(\pm)}{}^i_k \otimes L^{(\pm)}{}^k_j\,, \quad \epsilon(L^{(\pm)}{}^i_j) = \delta^i_j\,, \quad S(L^{(\pm)}{}^i_j) = ((L^{(\pm)})^{-1})^i_j\,.$$

Consider a composite matrix L with components

$$L^i_j \,:=\, q^{(n-\frac{1}{n})}\sum_k ((L^{(-)})^{-1})^i_k L^{(+)}{}^k_j \,=\, q^{(n-\frac{1}{n})}\langle id\otimes T^i_j\,, \mathcal{R}_{21}\mathcal{R}_{12}\rangle\,, \tag{3.2.5}$$

where our choice of a numeric factor $q^{n-\frac{1}{n}}$ is argued in appendix A. By (3.2.4), components of L satisfy reflection equation (3.2.1), where R is given by (2.2.11). Note that an $\mathfrak{A}_{\mathcal{R}}$-subalgebra generated by the elements L^i_j (3.2.5) does not carry a natural Hopf algebra structure. Instead, it obeys a coadjoint comodule algebra structure (3.2.2) with respect to the Hopf $\mathfrak{A}^*_{\mathcal{R}}$-subalgebra generated by the components of the matrices T and T^{-1} (3.1.5).

Let us comment on a geometric interpretation of the RE algebra. In [FRT] the matrices $L^{(\pm)}$ were used to develop an RTT type description for the quantized universal enveloping algebra $U_q\mathfrak{g}$. Consider the case $\mathfrak{g} = \mathfrak{sl}(n)$ and let V be its vector representation. The corresponding $GL_q(n)$ type R-matrix R is given in the example 2.10. Making a linear change of generators $L^i_j \to \ell^i_j$:

$$L^i_j \,=\, \delta^i_j + (q-q^{-1})\ell^i_j\,. \tag{3.2.6}$$

and using the Hecke condition (2.3.6) the reflection equation (3.2.1), for $q^2 \neq 1$, can be equivalently rewritten as

$$\ell_1 R_{12}\ell_1 R_{12} \,-\, R_{12}\ell_1 R_{12}\ell_1 \,=\, R_{12}\ell_1 \,-\, \ell_1 R_{12}\,. \tag{3.2.7}$$

In a "classical" limit $q \to 1$ the R-matrix (2.4.6) tends to the permutation and the equations (3.2.7) go into commutation relations for the basis of generators of the Lie algebra $\mathfrak{gl}(n)$

$$[\ell_1, \ell_2] \,=\, P_{12}(\ell_1 - \ell_2)\,. \tag{3.2.8}$$

Classically we can treat $\|\ell^i_j\|^p_{i,j=1}$ as a basis of right invariant vector fields on $GL(n)$. Transformation of these basic fields under the left transition by a group element $t \in GL(n)$ is given by formula (c.f. with (3.2.2))

$$\delta_\ell(t)\,:\;\; \ell^i_j \mapsto \sum_{k,m=1}^n t^i_k\,\ell^k_m\,(t^{-1})^m_j\,, \qquad \text{where}\quad t^i_j := \rho_V(t)^i_j\,.$$

Extrapolating this interpretation to a "quantum" case $q \neq 1$ we call $\|L^i_j\|^n_{i,j=1}$ *a basis of quantized right invariant vector fields over matrix group.*

It is technically convenient to introduce notation

$$L_{\overline{1}} := L_1\,, \qquad L_{\overline{k+1}} := R_k\, L_{\overline{k}}\, R_k^{-1}\,, \tag{3.2.9}$$
$$L_{\underline{1}} := L_1\,, \qquad L_{\underline{k+1}} := R_k^{-1}\, L_{\underline{k}}\, R_k \qquad \forall\, k \geq 1.$$

In terms of these R-*copies* $L_{\overline{k}}$, $L_{\underline{k}}$ of the matrix L the reflection equation (3.2.1) can be equivalently written in any of the following forms

$$R_k\, L_{\overline{k}}\, L_{\overline{k+1}} \,=\, L_{\overline{k}}\, L_{\overline{k+1}}\, R_k\,, \qquad R_k\, L_{\underline{k+1}}\, L_{\underline{k}} \,=\, L_{\underline{k+1}}\, L_{\underline{k}}\, R_k \qquad \forall\, k \geq 1\,. \tag{3.2.10}$$

Taking into account commutativity relations

$$R_i\, L_{\overline{k}} \,=\, L_{\overline{k}}\, R_i\,, \qquad R_i\, L_{\underline{k}} \,=\, L_{\underline{k}}\, R_i \qquad \forall\, i, k:\; k \neq i, i+1\,, \tag{3.2.11}$$

one sees that the R-copies $L_{\overline{k}}$ ($L_{\underline{k}}$) of the matrix L in the RE algebra $\mathfrak{LG}[R]$ formally satisfy the same relations as the usual copies T_k (T_k^{-1}) of the matrix T (T^{-1}) in the RTT algebra $\mathfrak{FG}[R]$.

Matrix monomials in two different series of the R-copies satisfy relations

$$L_{\overline{1}}\, L_{\overline{2}} \dots L_{\overline{k}} = L_{\underline{k}} \dots L_{\underline{2}}\, L_{\underline{1}} \qquad \forall\, k \geq 1\,, \tag{3.2.12}$$

For $k = 2$ the equality (3.2.12) is identical to the reflection equation (3.2.1). For $k > 2$ this equality follows by induction on k. Note that monomials (3.2.12) transform covariantly under the left coadjoint coaction (3.2.2)

$$\delta_\ell\left(L_{\overline{1}} \dots L_{\overline{k}}\right) \;=\; \left(T_1 \dots T_k \otimes 1\right)(1 \otimes L_{\overline{1}} \dots L_{\overline{k}})\left(S(T_1 \dots T_k) \otimes 1\right). \tag{3.2.13}$$

Following proposition goes back to theorem 14 from [FRT] (see also [Is.04], proposition 5).

Proposition 3.9. *Let R be a skew invertible* R*-matrix. For an element $x^{(k)} \in \mathbb{C}[\mathcal{B}_k]$ denote*

$$ch(x^{(k)}) := \mathrm{Tr}_{R\,(1\dots k)}\left(X_R^{(k)}\, L_{\overline{1}}\, L_{\overline{2}} \dots L_{\overline{k}}\right), \tag{3.2.14}$$

where $X_R^{(k)} := \rho_R(x^{(k)}) \in \mathrm{End}(V^{\otimes k})$. Consider a linear subspace $\mathfrak{Ch}[R] \subset \mathfrak{LG}[R]$ spanned by the unity and by elements $ch(x^{(k)})$ $\;\forall k \geq 1$ and $\forall x^{(k)} \in \mathbb{C}[\mathcal{B}_k]$. The space $\mathfrak{Ch}[R]$ is a subalgebra of the center of the RE algebra $\mathfrak{LG}[R]$. It is called a characteristic subalgebra *of the RE algebra $\mathfrak{LG}[R]$. The characteristic subalgebra is invariant with respect to the left $\mathfrak{FG}[R]$ coadjoint coaction (3.2.2).*

Proof. In a setting of the quasi-triangular Hopf algebras these statements were proved in [D.89, R.89] (see there sec.3 and sec.4, respectively). Below we prove the proposition in the RE algebra setting.

Consider an arbitrary element $ch(x^{(k)})$ of the characteristic subalgebra. We first prove a following version of the formula (3.2.14)

$$\begin{aligned} ch(x^{(k)})\, I_1 \;&=\; \mathrm{Tr}_{R\,(2,\dots,k+1)}\left(X_R^{(k)\uparrow 1}\, L_{\overline{2}}\, L_{\overline{3}} \dots L_{\overline{k+1}}\right) = \\ &=\; \mathrm{Tr}_{R\,(2,\dots,k+1)}\left(X_R^{(k)\uparrow 1}\, L_{\underline{k+1}} \dots L_{\underline{3}}\, L_{\underline{2}}\right). \end{aligned} \tag{3.2.15}$$

Here the first equality results from a calculation

$$\begin{aligned} &\mathrm{Tr}_{R\,(2,\dots,k+1)}\left(X_R^{(k)\uparrow 1}\, L_{\overline{2}} \dots L_{\overline{k+1}}\right) \\ &\quad=\; \mathrm{Tr}_{R\,(2,\dots,k+1)}\left(X_R^{(k)\uparrow 1}\, R_1 \cdots R_k\, L_{\overline{1}} \dots L_{\overline{k}}\, R_k^{-1} \cdots R_1^{-1}\right) \\ &\quad=\; \mathrm{Tr}_{R\,(2,\dots,k+1)}\left(R_1 \cdots R_k\, (X_R^{(k)}\, L_{\overline{1}} \dots L_{\overline{k}})\, R_k^{-1} \cdots R_1^{-1}\right) \\ &\quad=\; \dots \;=\; \mathrm{Tr}_{R\,(1,\dots,k)}\left(X_R^{(k)}\, L_{\overline{1}}\, L_{\overline{2}} \dots L_{\overline{k}}\right), \end{aligned}$$

where in the last line we applied (2.2.10) k times. To prove the second equality in (3.2.15) we first use the relation (3.2.12) and then perform similar transformations.

With the use of (3.2.15) and (3.2.12) checking centrality of $ch(x^{(k)})$ is straightforward

$$\begin{aligned} L_1\, ch(x^{(k)}) \;&=\; \mathrm{Tr}_{R\,(2,\dots,k+1)}\left(X_R^{(k)\uparrow 1}\, L_{\overline{1}} L_{\overline{2}}\, L_{\overline{3}} \dots L_{\overline{k+1}}\right) \\ &=\; \mathrm{Tr}_{R\,(2,\dots,k+1)}\left(X_R^{(k)\uparrow 1}\, L_{\overline{k+1}} \dots L_{\overline{2}}\, L_{\overline{1}}\right) = ch(x^{(k)})\, L_1\,. \end{aligned}$$

The invariance of $ch(x^{(k)})$ under the left $\mathfrak{F}\mathfrak{G}[R]$ coadjoint coaction follows immediately from (3.2.13) together with the relation (3.1.4) for the square of antipode. ■

Consider a series of elements of the RE algebra $\mathfrak{L}\mathfrak{G}[R]$

$$p_i := \mathrm{Tr}_R(L^i), \quad i = 1, 2, \ldots . \tag{3.2.16}$$

Further on they are called *power sums*. Following calculation

$$L_1 p_i = \mathrm{Tr}_{R(2)} L_1 R_{12} L_1^i R_{12}^{-1} = \mathrm{Tr}_{R(2)} R_{12}^{-1} L_1^i R_{12} L_1 = p_i L_1 ,$$

proves centrality of the power sums. Here in the first and the last equalities we use formula (2.2.10), and the second equality is a consequence of (3.2.1). Actually, the power sums belong to the characteristic subalgebra $\mathfrak{Ch}[R]$:

$$p_i = ch(\sigma_{i-1} \ldots \sigma_2 \sigma_1) ,$$

which is verified by a following transformation

$$\begin{aligned}
ch(\sigma_{i-1} \ldots \sigma_2 \sigma_1) &= \mathrm{Tr}_{R(1,\ldots,i)} (L_{\overline{1}} \ldots L_{\overline{i}} (R_{i-1} \ldots R_1)) \\
&= \mathrm{Tr}_{R(1,\ldots,i)} \big(L_{\overline{1}} \ldots L_{\overline{i-1}} (R_{i-1} \ldots R_1) L_1 (R_1^{-1} \ldots R_{i-1}^{-1}) (R_{i-1} \ldots R_1)\big) \\
&= \mathrm{Tr}_{R(1,\ldots,i-1)} \Big(L_{\overline{1}} \ldots L_{\overline{i-1}} \Big(\mathrm{Tr}_{R(i)} R_{i-1}\Big) (R_{i-2} \ldots R_1) L_1\Big) \\
&= \mathrm{Tr}_{R(1,\ldots,i-2)} \Big(L_{\overline{1}} \ldots L_{\overline{i-2}} \Big(\mathrm{Tr}_{R(i-1)} R_{i-2}\Big) (R_{i-3} \ldots R_1) L_1^2\Big) \\
&= \ldots = \mathrm{Tr}_R(L^i) = p_i .
\end{aligned}$$

Here we repeatedly expand the notation $L_{\overline{j}} = (R_{j-1} \ldots R_1) L_1 (R_1^{-1} \ldots R_{j-1}^{-1})$ for $j = i, \ldots, 2$, and use (2.2.8).

Let R be a skew invertible R-matrix of the Hecke type. Assuming that conditions **[k]** (2.3.1) are fulfilled consider a series of elements $a_i \in \mathfrak{Ch}[R]$, $i = 0, 1, \ldots k$, in the corresponding *Hecke type* RE algebra $\mathfrak{L}\mathfrak{G}[R]$

$$a_0 := 1 , \qquad a_i := ch(a^{(i)}) = \mathrm{Tr}_{R(1,\ldots,i)} \Big(A^{(i)} L_{\overline{1}} \ldots L_{\overline{i}}\Big) \qquad \forall\, 1 \le i \le k , \tag{3.2.17}$$

where notations $a^{(i)}$, $A^{(i)}$ were explained in (2.3.2), (2.3.8). The elements a_i are called *elementary symmetric functions*.

Definition 3.10. *Let R be a skew invertible $GL_q(n)$ type* R*-matrix. A central extension of the corresponding RE algebra $\mathfrak{L}\mathfrak{G}[R]$ by an element a_n^{-1} : $a_n a_n^{-1} = 1$ is called $GL_q(n)$* type RE algebra *and denoted as $\mathfrak{L}_{GL_q(n)}[R]$. A quotient of this algebra by a relation*

$$a_n = q^{-1} 1 \tag{3.2.18}$$

is called $SL_q(n)$ type RE algebra *and denoted as $\mathfrak{L}_{SL_q(n)}[R]$.*

Remark 3.11. An actual value of a numeric factor in the right hand side of (3.2.18) is not relevant for the definition. Our choice allows avoiding numeric factors later in formula (4.1.1) (see proof of proposition 4.1).

Consider realization of the RE algebra $\mathfrak{L}_{SL_q(n)}[R]$ as a subalgebra in the quasi-triangular Hopf algebra $\mathfrak{A}_{\mathcal{R}}$ (see example 3.8). In this case the condition (3.2.18) is consistent with the pairing $\langle\cdot,\cdot\rangle$ of the dual Hopf algebras $\mathfrak{A}_{\mathcal{R}}$ and $\mathfrak{A}^*_{\mathcal{R}}$ only for the chosen normalizations (3.2.5) for L and $\eta = q^{1/n}$ for R (2.2.11). This point is explained in appendix A, see (A.3).

Remark 3.12. The $GL_q(n)$ type R-matrix R and its $SL_q(n)$ partner R-matrix $\widetilde{R}$ (3.1.8) define identical RE algebras.

In the theorem below we describe Cayley-Hamilton and Newton identities specific to the $GL_q(n)$ type and Hecke type RE algebras.

Theorem 3.13. *Let R be a skew invertible* R*-matrix of the Hecke type. Assume that the conditions* $[\mathbf{k}]$ *(2.3.1) are fulfilled. Then in the corresponding RE algebra $\mathfrak{LG}[R]$ following* Cayley-Hamilton-Newton identities [IOP.98, IOP.99]

$$i_q\,\mathrm{Tr}_{R\,(2,\dots,i)}(A^{(i)}L_{\overline{2}}L_{\overline{3}}\dots L_{\overline{i}}) = (-1)^{i+1}\sum_{j=0}^{i-1}(-q)^j\,a_j\,L_1^{i-j-1}\quad \forall\, 2\le i\le k \tag{3.2.19}$$

take place. Multiplying by L_1 from the left and taking the R*-trace* $\mathrm{Tr}_{R\,(1)}$ *of these identities one obtains* Newton relations *for the sets of power sums $\{p_i\}_{i\ge 1}$ and the set of elementary symmetric functions $\{a_i\}_{i\ge 0}$* [GPS.97]

$$i_q\,a_i + (-1)^i\sum_{j=0}^{i-1}(-q)^j a_j\,p_{i-j} = 0\quad \forall\, 1\le i\le k\,. \tag{3.2.20}$$

Both sets $\{1,p_j\}_{j\ge 1}$ and $\{a_j\}_{j\ge 0}$ in this case generate the characteristic subalgebra $\mathfrak{Ch}[R]$.

Assume additionally that R is an R*-matrix of the $GL_q(n)$ type. Then the finite set $\{a_i\}_{i=0}^n$ generates the characteristic subalgebra of the RE algebra $\mathfrak{L}_{GL_q(n)}[R]$, and following* Cayley-Hamilton identity *is fulfilled* [GPS.97]*:*

$$\sum_{i=0}^{n}(-q)^i\,a_i\,L^{n-i} \;=\; 0\,. \tag{3.2.21}$$

This identity leads, in particular, to an invertibility of the matrix L:

$$L^{-1} \;=\; q^{-1}\,a_n^{-1}\sum_{i=0}^{n-1}(-q)^{-i}\,a_{n-i-1}\,L^i\,.$$

Remark 3.14. One can introduce generating functions $a(x)$, $p(x)$ for the elementary symmetric functions and for the power sums

$$a(x) := \sum_{i\ge 0} a_i\,x^i\,,\qquad p(x) := \sum_{i\ge 1} p_i\,x^i\,.$$

The Newton relations (3.2.20) can be written as a finite difference equation for the generating functions

$$a(qx)\,p(-x) \;=\; \frac{a(q^{-1}x) - a(qx)}{q-q^{-1}}\,.$$

For the $GL_q(n)$ type RE algebra we now construct its central extension by roots of the characteristic polynomial (3.2.21).

Definition 3.15. *Denote $\mathfrak{S}_n$ a $\mathbb{C}$-algebra of polynomials in n pairwise commuting invertible indeterminates $\mu_\alpha^{\pm 1}$ and their differences $(\mu_\alpha-\mu_\beta)^{\pm 1}$, $\alpha,\beta = 1,\dots,n$, $\alpha\ne\beta$.*

Let R be a skew invertible R*-matrix of the $GL_q(n)$ type, $\mathfrak{L}_{GL_q(n)}[R]$ be the corresponding RE algebra, and $\mathfrak{Ch}[R]$ be it's characteristic subalgebra. Consider a monomorphism $\mathfrak{Ch}[R]\hookrightarrow\mathfrak{S}_n$ defined on generators as* [5]

$$a_i \mapsto e_i(\mu_1,\dots,\mu_n) := \sum_{1\le j_1<\dots<j_i\le n}\mu_{j_1}\mu_{j_2}\dots\mu_{j_i}\qquad \forall\, i = 0,1,\dots,n\,, \tag{3.2.22}$$

[5] When defining the map (3.2.22) we implicitly assume an algebraic independence of the elements a_i, $i = 1,\dots,n$. Otherwise, we should impose the same algebraic conditions on functions $e_i(\mu_1,\dots,\mu_n)$.

where e_i are the elementary symmetric functions of their arguments. The map (3.2.22) defines naturally a structure of, say, left $\mathfrak{Ch}[R]$-module on $\mathfrak{S}_n$. A central extension of the algebra $\mathfrak{L}_{GL_q(n)}[R]$

$$\overline{\mathfrak{L}}_{GL_q(n)}[R] \,:=\, \mathfrak{L}_{GL_q(n)}[R] \underset{\mathfrak{Ch}_R}{\otimes} \mathfrak{S}_n \,:$$

$$a_\alpha \,=\, e_\alpha(\mu_1,\dots,\mu_n)\,,\quad L^i_j\,\mu_\alpha \,=\, \mu_\alpha\, L^i_j \qquad \forall\, i,j=1,\dots,\dim V,\ \ \forall\,\alpha=1,\dots,n\,, \tag{3.2.23}$$

is called a (semisimple) spectral completion *of $\mathfrak{L}_{GL_q(n)}[R]$. A quotient of this algebra by relations*

$$a_n \,=\, \prod_{\alpha=1}^n \mu_\alpha \,=\, q^{-1}\,.$$

is called a (semisimple) spectral completion *of $\mathfrak{L}_{SL_q(n)}[R]$ and denoted as $\overline{\mathfrak{L}}_{SL_q(n)}[R]$. Variables μ_α are called* spectral variables.

Remark 3.16. Assuming that the spectral variables μ_α are invariants of the coadjoint coaction, the algebra $\overline{\mathfrak{L}}_{GL_q(n)}[R]$ ($\overline{\mathfrak{L}}_{SL_q(n)}[R]$) inherits the structure of left coadjoint $\mathfrak{F}_{GL_q(n)}[R]$- ($\mathfrak{F}_{SL_q(n)}[R]$-) comodule algebra.

Corollary 3.17. *In the spectrally completed algebra $\overline{\mathfrak{L}}_{GL_q(n)}[R]$ the characteristic identity (3.2.21) assumes a factorized form*

$$\prod_{\alpha=1}^n \bigl(L - q\mu_\alpha I\bigr) \,=\, 0\,. \tag{3.2.24}$$

One can construct a resolution of the matrix unity

$$P^\alpha \,:=\, \prod_{\substack{\beta=1\\ \beta\neq\alpha}}^n \frac{\bigl(L-q\mu_\beta I\bigr)}{q(\mu_\alpha-\mu_\beta)} \ :\quad P^\alpha P^\beta \,=\, \delta_{\alpha\beta}\, P^\alpha\,,\quad \sum_{\alpha=1}^n P^\alpha \,=\, I\,, \tag{3.2.25}$$

so that

$$L\,P^\alpha \,=\, P^\alpha L \,=\, q\mu_\alpha P^\alpha\,. \tag{3.2.26}$$

Remark 3.18. In papers [GS.99, DM.01, DM.02, GS.04] the factorized form of the Cayley-Hamilton identity and the projectors P^α where used to construct explicitly quantized semisimple coadjoint orbits of $GL(n)$ and line bundles over them.

3.3. **Quantized differential operators over matrix group (Heisenberg double).**

Definition 3.19. [AF.91, S] *Let R, T and L be as described in the definitions 3.1 and 3.7. A* Heisenberg double (HD) algebra *$\mathfrak{D}\mathfrak{G}[R,\gamma]$ of the two algebras $\mathfrak{F}\mathfrak{G}[R]$ and $\mathfrak{L}\mathfrak{G}[R]$ is an associative unital algebra generated by the components of the matrices T and L subject additionally to a permutation relation*

$$\gamma^2\, T_1 L_2 \,=\, R_{12} L_1 R_{12} T_1\,,\quad \text{where } \gamma\in\{\mathbb{C}\backslash 0\}\,. \tag{3.3.1}$$

The HD algebra carries structures of left and right $\mathfrak{F}\mathfrak{G}[R]$-comodule algebra, respectively,

$$\delta_\ell(T^i_j) \,=\, \sum_k T^i_k\otimes T^k_j\,,\qquad \delta_\ell(L^i_j) \,=\, \sum_{k,m} T^i_k\,(T^{-1})^m_j\otimes L^k_m\,; \tag{3.3.2}$$

$$\delta_r(T^i_j) \,=\, \sum_k T^i_k\otimes T^k_j\,,\qquad \delta_r(L^i_j) = L^i_j\otimes 1\,. \tag{3.3.3}$$

Example 3.20. The Heisenberg double is closely related to a *smash product* of two mutually dual Hopf algebras (see, e.g., [M]). Namely, given a pair $\mathfrak{A}_\mathcal{R}$ and $\mathfrak{A}^*_\mathcal{R}$ their smash product algebra $\mathfrak{A}_\mathcal{R}\sharp\mathfrak{A}^*_\mathcal{R}$ is a linear space $\mathfrak{A}_\mathcal{R}\otimes\mathfrak{A}^*_\mathcal{R}$ supplied with a multiplication

$$(x\,\sharp\, u)(y\,\sharp\, v) \,:=\, \langle u_{(1)}, y_{(2)}\rangle\,(xy_{(1)}\,\sharp\, u_{(2)}v)\,, \tag{3.3.4}$$

where $x, y \in \mathfrak{A}_{\mathcal{R}}$, $u, v \in \mathfrak{A}^*_{\mathcal{R}}$, and symbols $(x \,\sharp\, u)$, $(y \,\sharp\, v)$ denote elements of $\mathfrak{A}_{\mathcal{R}} \sharp \mathfrak{A}^*_{\mathcal{R}}$.

Let us calculate in the settings of the examples 3.2, 3.8 the smash product of the elements $(T^i_j \,\sharp 1) \,=\, T^i_j$ and $(1 \,\sharp\, L^{(\pm)}{}^i_j) \,=\, L^{(\pm)}{}^i_j$

$$\begin{aligned} T_1\, L_2^{(+)} \;&=\; L_2^{(+)} \langle T_1,\, L_2^{(+)} \rangle\, T_1 \;=\; \eta^{-1}\, L_2^{(+)}\, P_{12} R_{12}\, T_1\,, \\ T_1\, L_2^{(-)} \;&=\; L_2^{(-)} \langle T_1,\, L_2^{(-)} \rangle\, T_1 \;=\; \eta\, L_2^{(-)}\, P_{12} R_{12}^{-1}\, T_1\,, \end{aligned}$$

wherefrom it follows that the smash product of T_1 and $L_2 \propto (L^{(-)}{}^{-1} L^{(+)})_2$ is given by (3.3.1) with $\gamma = \eta$. However, we stress that in general one can keep γ independent of the normalization η of the R-matrix at a price of loosing universality of formulas. Indeed, the multiplication in the smash product algebra is given by (3.3.4) universally for any pair of its elements, while the relation (3.3.1) in the HD algebra is written for the generators L and T only.

Now we consider a geometric interpretation of the HD algebra. Applying the substitution $L^i_j \rightarrow \ell^i_j$ (3.2.6) and taking the "classical" limit $q \rightarrow 1$ in relation (3.3.1) we find

$$\bigl[T_1, \ell_2\bigr] \;=\; \bigl(P_{12} - \gamma'\, I_{12}\bigr)\, T_1\,, \qquad (3.3.5)$$

where we used the Hecke condition (2.3.6) in a form $R^2 = I + (q - q^{-1})R$ and assumed additionally $R \stackrel{q\rightarrow 1}{\longrightarrow} P$ (which is true for the Drinfeld-Jimbo R-matrix (2.4.6)) and $\gamma \equiv \gamma(q) = 1 + (q - q^{-1})\frac{\gamma'}{2} + o(q - q^{-1})$. The commutation relations (3.3.5), (3.2.8) are realized by operators

$$\ell^i_j \;=\; \sum_{k=1}^{n} g^i_k \frac{\partial}{\partial g^j_k}\,, \quad T^i_j \;=\; |g|^{-\gamma'} g^i_j\,, \quad \text{where} \quad g^i_j := \rho_V(g)^i_j\,,\; g \in GL(n)\,,\; |g| := \det \|g^i_j\|\,.$$

These are, respectively, right invariant vector fields and properly normalized coordinate functions on $GL(n)$. Together they generate an algebra of differential operators over $GL(n)$[6].

Extrapolating the classical picture we can treat $\mathfrak{D}\mathfrak{G}[R, \gamma]$ as an algebra of *quantized differential operators over matrix group* or, equivalently, as *quantized functions over cotangent bundle of a matrix group* (see [AF.91, AF.92, SWZ.92, IP]). The form of the substitution (3.2.6) suggests that the quantized vector fields L^i_j possess properties of finite difference operators rather then of the differential operators. In particular, they do not satisfy classical Leibniz rule when acting on functions (see (3.3.1)).

The next proposition describes an action of the characteristic subalgebra on quantized functions in the Hecke case.

Proposition 3.21. *Let R be a skew invertible Hecke type* R*-matrix. Assume that the conditions* $[\mathbf{k}]$ *(2.3.1) are satisfied, so that the elementary symmetric functions $a_i \in \mathfrak{Ch}[R] \subset \mathfrak{D}\mathfrak{G}[R, \gamma]$, $0 \le i \le k$, (3.2.17) are well defined. Then relations*

$$\gamma^{2i} T\, a_i \;=\; a_i\, T \;-\; (q^2 - 1) \sum_{j=1}^{i} (-q)^{-j} a_{i-j}\, (L^j T) \qquad \forall\, 0 \le i \le k \qquad (3.3.6)$$

are fulfilled for the Hecke type HD algebra $\mathfrak{D}\mathfrak{G}[R, \gamma]$.

Proof: For any operator $Y \in \mathrm{End}_W(V^{\otimes i})$, where W is an arbitrary $\mathbb{C}$-linear space, we denote

$$Y^{\uparrow 1} \;:=\; (P_1 P_2 \dots P_i) Y (P_1 P_2 \dots P_i)^{-1}\,. \qquad (3.3.7)$$

[6] Imposing conditions $\gamma' = 1/n$, $\det T = 1$, $\mathrm{Tr}\, \ell = 0$ one can make a reduction to a subalgebra of differential operators over $SL(n)$.

For any R-matrix R we define series of operators J_i, Z_i

$$J_1 := I\,, \qquad J_{i+1} := R_i\, J_i\, R_i \qquad \forall\, i \ge 1\,, \qquad Z_i := \prod_{j=1}^{i} J_j\,. \tag{3.3.8}$$

Remark 3.22. Elements J_j, $1 \le j \le i$, are R-matrix realizations of a remarkable set of *Jucys-Murphy elements* in the braid group $\mathcal{B}_i$:

$$j_1 := 1\,, \qquad j_{j+1} := \sigma_j\, j_j\, \sigma_j \qquad \forall\, j = 1, \dots, i-1\,.$$

These elements generate a commutative subgroup in $\mathcal{B}_i$ and their product $z_i := \prod_{j=1}^{i} j_j$ is a central element in $\mathcal{B}_i$. For their applications and for historical references see, e.g., [OP.01].

With these notations permutation relations (3.2.1), (3.3.1) can be suitably written for arbitrary R-copies of the matrix L:

$$\begin{aligned} (L_{\bar i} J_i)(L_{\bar j} J_j) \;&=\; (L_{\bar j} J_j)(L_{\bar i} J_i), &\quad (3.3.9)\\ \gamma^2\, T_1 (L_{\bar i} J_i)^{\uparrow 1} \;&=\; (L_{\overline{i+1}} J_{i+1})\, T_1 \;=\; R_i\, (L_{\bar i} J_i)\, R_i\, T_1 \qquad \forall\, i, j \ge 1\,. &\quad (3.3.10) \end{aligned}$$

Here the second equality follows from the recursive definitions of $L_{\overline{i+1}}$ and J_{i+1}, while the first equality can be easily proved by induction on i.

Next, we prepare a suitable expression for a_i (3.2.17):

$$\begin{aligned} a_i \;&=\; \mathrm{Tr}_{R(1,\dots,i)}\Big(L_{\bar 1} \dots L_{\bar i}\, A^{(i)}\Big) \;=\; q^{i(i-1)}\, \mathrm{Tr}_{R(1,\dots,i)}\Big(L_{\bar 1} \dots L_{\bar i}\, Z_i\, A^{(i)}\Big) \\ &=\; q^{i(i-1)}\, \mathrm{Tr}_{R(1,\dots,i)}\Big((L_{\bar 1} J_1) \dots (L_{\bar i} J_i)\, A^{(i)}\Big) \end{aligned} \tag{3.3.11}$$

Here we substituted $Z_i A^{(i)} = q^{-i(i-1)} A^{(i)}$ in the first line and used a commutativity relation

$$L_{\bar i}\, J_j \;=\; J_j\, L_{\bar i} \qquad \forall\, i, j \,:\, i > j \tag{3.3.12}$$

in the second line. By relabelling the subscript indices of the R-traces we then recast (3.3.11) in a following form[7]

$$I_1\, a_i \;=\; q^{i(i-1)}\, \mathrm{Tr}_{R(2,\dots,i+1)}\Big((L_{\bar 1} J_1) \dots (L_{\bar i} J_i)\, A^{(i)}\Big)^{\uparrow 1}\,. \tag{3.3.13}$$

Now we are ready to permute T_1 and a_i. Substituting expression (3.3.13) for a_i and using relations (3.3.10) and (3.3.12) we calculate

$$\begin{aligned} \gamma^{2i} T_1\, a_i \;&=\; \gamma^{2i} T_1\, (I_1\, a_i) \;=\; q^{i(i-1)} \gamma^{2i}\, \mathrm{Tr}_{R(2,\dots,i+1)} T_1 \Big((L_{\bar 1} J_1) \dots (L_{\bar i} J_i)\, A^{(i)}\Big)^{\uparrow 1} \\ &=\; q^{i(i-1)} \mathrm{Tr}_{R(2,\dots,i+1)}\Big((L_{\bar 2} J_2) \dots (L_{\overline{i+1}} J_{i+1})\, A^{(i)\uparrow 1}\Big)\, T_1 \\ &=\; q^{i(i-1)} \mathrm{Tr}_{R(2,\dots,i+1)}\Big((L_{\bar 2} \dots L_{\overline{i+1}})\, Z_{i+1}\, A^{(i)\uparrow 1}\Big)\, T_1\,. \end{aligned}$$

To continue the calculation we need following formula

$$Z_{i+1}\, A^{(i)\uparrow 1} \;=\; A^{(i)\uparrow 1}\, Z_{i+1}\, A^{(i)\uparrow 1} \;=\; q^{-i(i-1)}\, \Big(q^2 A^{(i)\uparrow 1} \;-\; q^{-i}(q^2-1)(i+1)_q\, A^{(i+1)}\Big)\,,$$

---

[7] Notice a similarity of the formula (3.3.13) with the relation (2.2.10). The role of the R-matrices $R^{\pm\varepsilon}$ is now played by the permutation matrix P (see (3.3.7)).

which follows by a combination of the definitions (2.3.3), (2.3.8), (3.3.8), and relations (2.3.4), (2.3.5), (2.3.6). So we finish the calculation

$$\begin{aligned}\gamma^{2i}T_1\, a_i &= \mathrm{Tr}_{R\,(2,\dots,i+1)}\Big[(L_{\overline{2}}\dots L_{\overline{i+1}})\,\Big(q^2A^{(i)\uparrow 1} - q^{-i}(q^2-1)(i+1)_q\,A^{(i+1)}\Big)\Big]\,T_1\\ &= q^2 a_i\,T_1 + (-q)^{-i}(q^2-1)\sum_{j=0}^{i}(-q)^{-j}\,a_j\big(L^{i-j}\,T\big)_1\\ &= a_i\,T_1 - (q^2-1)\sum_{j=1}^{i}(-q)^{-j}a_{i-j}(L^jT)_1\,. \end{aligned} \tag{3.3.14}$$

Here we calculate the first summand in the second line taking into account equality

$$(L_{\overline{2}}\dots L_{\overline{i+1}})\,A^{(i)\uparrow 1} = (R_1\dots R_i)(L_{\overline{1}}\dots L_{\overline{i}})\,A^{(i)}(R_1\dots R_i)^{-1}$$

and using i times formula (2.2.10). For calculation of the second summand we use the Cayley-Hamilton-Newton identity (3.2.19). Thus (3.3.6) is proved. ■

Remark 3.23. For the set of power sums (3.2.16) the permutation relations with T^i_j in the Hecke case read

$$\gamma^{2i}T\,p_i = p_i\,T + (q-q^{-1})^2\sum_{j=1}^{i-1}\frac{(2j)_q}{2_q}p_{i-j}\,(L^jT) + (q-q^{-1})\frac{(2i)_q}{2_q}(L^iT)\,.$$

One can derive this formula applying the R-trace $\mathrm{Tr}_{R\,(2)}$ to an equality $\gamma^{2i}T_1(L_2)^i = (RL_1R)^iT_1$ and taking into account relations

$$\begin{aligned}(RL_1R)^i &= R(L_1)^iR + (q-q^{-1})\sum_{j=1}^{i-1}R^{2j}(L_1)^{i-j}R(L_1)^j\,,\\ R^{2j} &= 2_q^{-1}\left((q^{2j-1}+q^{-2j+1})I + (q^{2j}-q^{-2j})\,R\right)\,,\end{aligned}$$

These relations, in turn, follow inductively from the Hecke condition (2.3.6) and the reflection equation (3.2.1). Note that in this case there is no need to impose restrictions (2.3.1) on q.

Proposition 3.24. *Let R be a skew invertible $GL_q(n)$ type* R*-matrix. An extension of the corresponding HD algebra $\mathfrak{DG}[R,\gamma]$ by the elements $(\det_R T)^{-1}$ and $(a_n)^{-1}$, satisfying relations*

$$\gamma^{2n}\,L\,(\det_R T)^{-1} = q^2(\det_R T)^{-1}\,(O_R L O_R^{-1})\,, \tag{3.3.15}$$

$$\gamma^{2n}\,(a_n)^{-1}\,T = q^2 T\,(a_n)^{-1}\,, \tag{3.3.16}$$

in addition to those, given in definitions 3.4 and 3.10, is called $GL_q(n)$ type HD algebra *and denoted as $\mathfrak{D}_{GL_q(n)}[R,\gamma]$.*

Let R be a skew invertible $SL_q(n)$ type R*-matrix. In the corresponding HD algebra $\mathfrak{DG}[R,\gamma]$ let us restrict the parameters by condition $\gamma^n = q$ and take a quotient by relations $\det_R T = 1$ and $a_n = q^{-1}1$. The quotient algebra is called $SL_q(n)$* type HD algebra *and denoted as $\mathfrak{D}_{SL_q(n)}[R]$.*

Remark 3.25. Notice consistency of the $SL_q(n)$ reduction condition $\gamma^n = q$ with the parameter restrictions $\eta = q^{1/n}$ in the example 2.10 and $\gamma = \eta$ in the example 3.20.

Proof: Relations (3.3.15) and (3.3.16) should be consistent with permutation relations for $\det_R T$ and a_n in the algebra $\mathfrak{DG}[R,\gamma]$. Permutation relation for a_n with T were in fact derived in the first line of the calculation (3.3.14) (put $i = n$ and take into account that $A^{(n+1)} = 0$ in the $GL_q(n)$ case). Permutation relation for $\det_R T$ with L can be derived by the same method as for $\det_R T$ with T (see [G], sec.5, or [Is.04], calculation (3.5.39)). Given these results the consistency is obvious.

In the $SL_q(n)$ case ($O_R \propto I$, $\gamma^n = q$) the elements $\det_R T$ and a_n are central. Hence, $\mathfrak{D}_{SL_q(n)}[R]$ is consistently defined. ∎

Corollary 3.26. *In the $GL_q(n)$ type HD algebra elements of the characteristic subalgebra satisfy following commutation relations with* $\det_R T$

$$\gamma^{2nk} \det_R T\, ch(x^{(k)}) \,=\, q^{2k}\, ch(x^{(k)}) \det_R T \quad \forall\, x^{(k)} \in \mathcal{H}_k(q),\; k = 1, 2, \dots .$$

Proof: A proof is a direct calculation of permutation of $ch(x^{(k)})$ (3.2.14) and $\det_R T$ exploiting relations (3.3.15) and properties of the matrix O_R (3.1.7)

$$R_{12}\, O_{R1}\, O_{R2} \,=\, O_{R1}\, O_{R2}\, R_{12}\,, \qquad O_R\, D_R \,=\, D_R\, O_R$$

The latter relations are proved in [OP.05], section 5.3. ∎

Theorem 3.27. *Let R be a skew invertible $GL_q(n)$ ($SL_q(n)$) type* R*-matrix. An extension of the corresponding HD algebra $\mathfrak{D}_{GL_q(n)}[R,\gamma]$ ($\mathfrak{D}_{SL_q(n)}[R]$) by the algebra $\mathfrak{S}_n$ of polynomials in mutually commuting indeterminates $\mu_\alpha^{\pm 1}$, $(\mu_\alpha - \mu_\beta)^{\pm 1}$ satisfying relations (3.2.23) together with*

$$\gamma^2\, (P^\beta T)\, \mu_\alpha \,=\, q^{2\delta_{\alpha\beta}} \mu_\alpha\, (P^\beta T) \qquad \forall\, \alpha, \beta = 1, \dots, n\,, \tag{3.3.17}$$

or, equivalently,

$$\gamma^2\, T\, \mu_\alpha \,=\, \mu_\alpha\, T \,+\, (q^2 - 1)\mu_\alpha\, (P^\alpha T)\,,$$

is called a (semisimple) spectral completion *of the $GL_q(n)$ ($SL_q(n)$) type HD algebra and denoted as $\overline{\mathfrak{D}}_{GL_q(n)}[R,\gamma]$ ($\overline{\mathfrak{D}}_{SL_q(n)}[R]$).*

Remark 3.28. To avoid problems with permutations of $(\mu_\alpha - \mu_\beta)^{-1}$ with $P^\sigma T$ one could assume invertibility of all elements $(\mu_\alpha - q^{2k}\mu_\beta)$ $\forall\, \alpha \neq \beta$, $k \in \mathbb{Z}$. Further on we will not make such permutations and so we don't impose the corresponding restrictions.

Remark 3.29. Assuming that the spectral variables μ_α are invariants of both left and right coactions, the algebra $\overline{\mathfrak{L}}_{GL_q(n)}[R,\gamma]$ ($\overline{\mathfrak{L}}_{SL_q(n)}[R]$) inherits the structures of left and right $\mathfrak{F}_{GL_q(n)}[R]$- ($\mathfrak{F}_{SL_q(n)}[R]$-) comodule algebra (see definition 3.19).

Remark 3.30. Note that relation (3.3.17) is typical for Weyl algebra generators. In fact there are many ways to combine from the elements $(P^\beta T)_{ij}$ a set of n generators satisfying Weyl relations with the spectral variables μ_α. One such possibility is used later in section 4.4.

Proof: We have to check consistency of relations (3.3.17), (3.3.6) with the conditions $a_i = e_i(\mu_1, \dots, \mu_n) \equiv e_i(\mu)$ for $1 \le i \le n$. Denote $e_i(\mu^{/\alpha}) := e_i(\mu)|_{\mu_\alpha = 0}$. We have

$$e_i(\mu) \,=\, e_i(\mu^{/\alpha}) \,+\, \mu_\alpha\, e_{i-1}(\mu^{/\alpha}) \quad \Rightarrow \quad e_i(\mu^{/\alpha}) \,=\, \sum_{j=0}^{i} (-\mu_\alpha)^j\, e_{i-j}(\mu)\,. \tag{3.3.18}$$

Using relations (3.3.17), (3.3.18), (3.2.25) and (3.2.26) we calculate

$$
\begin{aligned}
\gamma^{2i}\, T\, e_i(\mu) &= \gamma^{2i} \sum_{\alpha=1}^{n} (P^\alpha T) \Big(e_i(\mu^{/\alpha}) \,+\, \mu_\alpha\, e_{i-1}(\mu^{/\alpha}) \Big) \\
&= \sum_{\alpha=1}^{n} \Big(e_i(\mu^{/\alpha}) \,+\, q^2 \mu_\alpha\, e_{i-1}(\mu^{/\alpha}) \Big) (P^\alpha T) \\
&= \sum_{\alpha=1}^{n} \Big(e_i(\mu) \,+\, (q^2-1)\, \mu_\alpha \sum_{j=0}^{i-1} (-\mu_\alpha)^j e_{i-j-1}(\mu) \Big) (P^\alpha T) \\
&= \Big(e_i(\mu) \,-\, (q^2-1) \sum_{j=1}^{i} (-L/q)^j e_{i-j}(\mu) \Big) \sum_{\alpha=1}^{n} (P^\alpha T) \\
&= e_i(\mu)\, T \,-\, (q^2-1) \sum_{j=1}^{i} (-q)^{-j} e_{i-j}(\mu)\, (L^j T)\,,
\end{aligned}
$$

which coincides with (3.3.6) under identification $e_i(\mu) = a_i$. ■

Corollary 3.31. *In the completed $GL_q(n)$ type HD algebra $\overline{\mathfrak{D}}_{GL_q(n)}[R,\gamma]$ following permutation relations hold*

$$
\gamma^{2n} \det_R T\, \mu_\alpha \,=\, q^2\, \mu_\alpha \det_R T \quad \forall\ \alpha = 1, 2, \ldots, n\,. \tag{3.3.19}
$$

Proof: Using formulas (3.1.6), (3.2.25), (3.3.17) we can permute $\det_R T$ and μ_α:

$$
\begin{aligned}
\gamma^{2n} \det_R T \mu_\alpha &= \gamma^{2n} \sum_{\beta_1,\ldots,\beta_n=1}^{n} \mathrm{Tr}_{(1,\ldots,n)} \big(A^{(n)} (P^{\beta_1} T)_1 \ldots (P^{\beta_n} T)_n \big) \mu_\alpha \\
&= \mu_\alpha \sum_{\beta_1,\ldots,\beta_n=1}^{n} q^{2\sum_{j=1}^{n} \delta_{\alpha\beta_j}}\, \mathrm{Tr}_{(1,\ldots,n)} \big(A^{(n)} (P^{\beta_1} T)_1 \ldots (P^{\beta_n} T)_n \big).
\end{aligned} \tag{3.3.20}
$$

Assuming that

$$
\mathrm{Tr}_{(1,\ldots,n)} \big(A^{(n)} (P^{\beta_1} T)_1 \ldots (P^{\beta_n} T)_n \big) \,=\, 0\,, \quad \text{if there exists a pair } i,j :\ \beta_i = \beta_j\,, \tag{3.3.21}
$$

we conclude that for any nonzero summand in (3.3.20) the coefficient $q^{2\sum_{j=1}^{n} \delta_{\alpha\beta_j}}$ equals q^2, and therefore we can complete the calculation

$$
\gamma^{2n} \det_R T\, \mu_\alpha \,=\, q^2 \mu_\alpha \sum_{\beta_1,\ldots,\beta_n=1}^{n} \mathrm{Tr}_{(1,\ldots,n)} \big(A^{(n)} (P^{\beta_1} T)_1 \ldots (P^{\beta_n} T)_n \big) \,=\, q^2\, \mu_\alpha \det_R T\,.
$$

It lasts to prove the assumption. First, we note that conditions on β_i in (3.3.21) stand that there exists integer σ: $1 \leq \sigma \leq n$, and $\sigma \neq \beta_i\ \forall\, i$. Therefore, any projector P^{β_i} in (3.3.21) contains factor $(L - q\mu_\sigma I)$. Using relations (3.3.10), (3.3.17) we can move all such factors to the left side of the expression. Thus we obtain

$$
\text{left hand side of (3.3.21)} \,\propto\, \mathrm{Tr}_{(1,\ldots,n)} \Big(A^{(n)} \Big\{ \textstyle\prod_{j=1}^{n} \big(L_{\bar{j}} J_j - q\mu_\sigma I \big) \Big\} \ldots \Big)\,. \tag{3.3.22}
$$

Next, we note that the expression in braces is a symmetric function in a commuting set of matrices $L_{\bar{j}} J_j$ (see (3.3.9)) which by relations

$$
R_i (L_{\bar{i}} J_i)(L_{\overline{i+1}} J_{i+1}) = (L_{\bar{i}} J_i)(L_{\overline{i+1}} J_{i+1}) R_i, \quad R_i (L_{\bar{i}} J_i + L_{\overline{i+1}} J_{i+1}) = (L_{\bar{i}} J_i + L_{\overline{i+1}} J_{i+1}) R_i,
$$

and by (3.2.11) together with the same formulas for J_k commutes with R_i, $i = 1, \ldots, n-1$, and so with $A^{(n)}$. Hence, using relations $A^{(n)} = (A^{(n)})^2$ and $\mathrm{rk} A^{(n)} = 1$ we can separate a left factor $\kappa := \mathrm{Tr}_{(1,\dots,n)}\big(A^{(n)}\prod_{j=1}^{n}\big(L_{\overline{j}}J_j - q\mu_\sigma I\big)\big)$ in (3.3.22). This factor we now calculate explicitly.

Taking into account relations (2.4.5), (3.3.9), (3.3.12) and $A^{(n)}J_i = q^{-2(i-1)}A^{(n)}$ we transform the expression for κ:

$$\kappa \;=\; q^n\,\mathrm{Tr}_{R\,(1,\dots,n)}\Big(A^{(n)}\textstyle\prod_{j=1}^{n}\big(L_{\overline{j}}\,-\,q^{2j-1}\mu_\sigma I\big)\Big)\,. \tag{3.3.23}$$

Expanding this expression in powers of L and noticing that (2.4.4) assumes $\mathrm{Tr}_{R\,(k+1,\dots,n)}A^{(n)} = q^{n(k-n)}\frac{(k)_q!(n-k)_q!}{(n)_q!} := q^{n(k-n)}\binom{n}{k}^{-1}$ we find that k-th order monomials

$$\mathrm{Tr}_{R\,(1,\dots,n)}\Big(A^{(n)}L_{\overline{i_1}}\dots L_{\overline{i_k}}\Big) = \mathrm{Tr}_{R\,(1,\dots,n)}\Big(A^{(n)}L_{\overline{1}}\dots L_{\overline{k}}\Big) \;=\; q^{n(k-n)}\binom{n}{k}_q^{-1} a_k$$

are equal to each other for any choice of indices $1 \le i_1 < \dots i_k \le n$. Their corresponding coefficients in (3.3.23) sum up to

$$(-q^{-1}\mu_\sigma)^{n-k}\sum_{1\le i_1<\dots i_{n-k}\le n} q^{2\sum_{r=1}^{n-k} i_r} \;=\; q^{n(n-k)}\binom{n}{k}_q(-\mu_\sigma)^{n-k}\,,$$

and so we obtain

$$\kappa \;=\; q^n\sum_{k=0}^{n} a_k(-\mu_\sigma)^{n-k} \;=\; q^n\prod_{\alpha=1}^{n}(\mu_\alpha - \mu_\sigma) \;=\; 0\,,$$

where we took (3.2.23) into account. ■

3.4. **Quantized left invariant vector fields.** In a classical differential geometry of the Lie groups one uses two global bases on tangent bundles – the bases of right and left invariant vector fields. In previous sections we discussed quantization of the right invariant vector fields only and defined the HD algebra $\mathfrak{DG}[R,\gamma]$ in their terms. To demonstrate a left-right symmetry of the whole construction we now describe the HD algebra using a set of left invariant generators. We also find explicit relations between the spectra of left and right invariant vector fields in both, the $GL_q(n)$ and the $SL_q(n)$ cases.

In the assumptions of the definition 3.19 consider a matrix M whose components belong to $\mathfrak{DG}[R,\gamma]$:

$$M^i_j \;:=\; \sum_{k,m}(T^{-1})^i_k\,L^k_m\,T^m_j\,. \tag{3.4.1}$$

Taking into account transformation properties of the matrix elements M^i_j with respect to the left and right $\mathfrak{FG}[R]$-coactions (3.3.2) and (3.3.3)

$$\delta_\ell(M^i_j) \;=\; 1\otimes M^i_j\,,\qquad \delta_r(M^i_j) \;=\; \sum_{k,m}(T^{-1})^i_k\,T^m_j\otimes M^k_m\,, \tag{3.4.2}$$

we shall call them a *basis of quantized left invariant vector fields over matrix group*.

One can give presentation of the HD algebra $\mathfrak{DG}[R,\gamma]$ in terms of generators T^i_j and M^i_j, and relations

$$\begin{aligned}
\overset{*}{R}_{12}\,T_2\,T_1 \;&=\; T_2\,T_1\overset{*}{R}_{12}\,,\\
\overset{*}{R}_{12}\,M_1\,\overset{*}{R}_{12}\,M_1 \;&=\; M_1\,\overset{*}{R}_{12}\,M_1\,\overset{*}{R}_{12}\,, &(3.4.3)\\
\gamma^{-2}\,M_2\,T_1 \;&=\; T_1\,\overset{*}{R}_{12}\,M_1\,\overset{*}{R}_{12}\,, &(3.4.4)
\end{aligned}$$

where we denote

$$\overset{*}{R}_{12} \;:=\; (PR^{-1}P)_{12} \;=\; (R_{21})^{-1}\,. \tag{3.4.5}$$

Necessary technical data about $\overset{*}{R}$ are collected in lemma B.1 in appendix B.

By (3.4.3), the entries of matrix M generate yet another RE subalgebra $\mathfrak{LG}[\overset{*}{R}]$ in the HD algebra $\mathfrak{DG}[R,\gamma]$. By (3.4.2), the subalgebra $\mathfrak{LG}[\overset{*}{R}]$ is a right coadjoint $\mathfrak{FG}[R]$-comodule algebra. We also notice a nontrivial but quite expected property of the quantized left and right invariant vector fields — their mutual commutativity

$$M_1\, L_2 \;=\; L_2\, M_1\, .$$

In the rest of this section we investigate the characteristic subalgebra $\mathfrak{Ch}[\overset{*}{R}] \subset \mathfrak{LG}[\overset{*}{R}]$. In particular, we shall see that $\mathfrak{Ch}[\overset{*}{R}] = \mathfrak{Ch}[R]$ for the $\mathfrak{DG}[R,\gamma]$-subalgebras $\mathfrak{LG}[\overset{*}{R}]$ and $\mathfrak{LG}[R]$.

It is suitable to introduce $\overset{*}{R}$-*copies* of the matrix M (c.f. with (3.2.9))

$$M_{\overline{\overset{*}{1}}} \;:=\; M_1\, , \qquad M_{\overline{\overset{*}{k+1}}} \;:=\; \overset{*}{R}_k\, M_{\overline{\overset{*}{k}}}\, (\overset{*}{R}_k)^{-1}\, , \tag{3.4.6}$$

and $\overset{*}{R}$-matrix realizations of the Jucys-Murphy elements (c.f. with (3.3.8) and remark 3.22)

$$\overset{*}{J}_1 \;:=\; I\, , \qquad \overset{*}{J}_{k+1} \;:=\; \overset{*}{R}_k \overset{*}{J}_k \overset{*}{R}_k \qquad \forall\, k \geq 1\, . \tag{3.4.7}$$

In their terms the relations (3.4.3), (3.4.4) can be written as (c.f. with (3.2.10), (3.3.10))

$$\begin{aligned} \overset{*}{R}\, M_{\overline{\overset{*}{k}}}\, M_{\overline{\overset{*}{k+1}}} \;&=\; M_{\overline{\overset{*}{k}}}\, M_{\overline{\overset{*}{k+1}}}\, \overset{*}{R}\, , \\ \gamma^{-2}\Big(M_{\overline{\overset{*}{k}}}\, \overset{*}{J}_k\Big)^{\uparrow 1} T_1 \;&=\; T_1\Big(M_{\overline{\overset{*}{k+1}}}\, \overset{*}{J}_{k+1}\Big)\, . \end{aligned} \tag{3.4.8}$$

We now assume that the R-matrix $\overset{*}{R}$ is skew invertible[8] and introduce two generating sets in the characteristic subalgebra $\mathfrak{Ch}[\overset{*}{R}] \subset \mathfrak{LG}[\overset{*}{R}]$: the power sums $\overset{*}{p}_k$

$$\overset{*}{p}_k := \,\mathrm{Tr}_{\overset{*}{R}}(M^k)\, , \qquad k = 1, 2, \dots\, ,$$

and, assuming additionally that conditions **[k]** (2.3.1) are fulfilled, the elementary symmetric functions $\overset{*}{a}_i$

$$\overset{*}{a}_0 := \,1\, , \qquad \overset{*}{a}_i := \,\mathrm{Tr}_{\overset{*}{R}(1,\dots,i)}\Big(\overset{*}{A}{}^{(i)}\; M_{\overline{\overset{*}{1}}} M_{\overline{\overset{*}{2}}} \dots M_{\overline{\overset{*}{i}}}\Big) \qquad \forall\, 1 \leq i \leq k\, . \tag{3.4.9}$$

Proposition 3.32. *Let R be a skew invertible Hecke type* R*-matrix and D_{R} be invertible. Assume that conditions* **[k]** *(2.3.1) are satisfied. Then for two sets of elements in $\mathfrak{Ch}[R] \subset \mathfrak{DG}[R,\gamma]$ — a_i (3.2.17) and $\overset{*}{a}_i$ (3.4.9) — following relations are satisfied*

$$\overset{*}{a}_i = \,\gamma^{2i}\, a_i \qquad \forall\, 0 \leq i \leq k\, .$$

Proof: We transform the expression (3.4.9) for $\overset{*}{a}_i$ in a following way

$$\overset{*}{a}_i = \,\mathrm{Tr}_{\overset{*}{R}(1,\dots,i)}\Big(M_{\overline{\overset{*}{1}}} \dots M_{\overline{\overset{*}{i}}}\; \overset{*}{A}{}^{(i)}\Big) \;=\; q^{-i(i-1)}\, \mathrm{Tr}_{\overset{*}{R}(1,\dots,i)}\Big((M_{\overline{\overset{*}{1}}} \overset{*}{J}_1) \dots (M_{\overline{\overset{*}{i}}} \overset{*}{J}_i)\; \overset{*}{A}{}^{(i)}\Big)$$

Here we used formulas

$$\overset{*}{J}_k \overset{*}{A}{}^{(i)} = q^{2(k-1)} \overset{*}{A}{}^{(i)} \;\; \forall\, 1 \leq k \leq i\, , \quad \text{and} \quad M_{\overline{\overset{*}{i}}} \overset{*}{J}_k = \overset{*}{J}_k M_{\overline{\overset{*}{i}}} \; \forall\, 1 \leq k < i\, , \tag{3.4.10}$$

which, in turn, follow from (B.3), (3.4.6), (3.4.7).

[8] This is indeed the case if R is skew invertible and D_{R} is invertible (see lemma B.1 in appendix B).

Then we apply lemma B.2 from the appendix B and use the relations (B.6) to move $\overset{*}{A}^{(i)}$ leftwards

$$\overset{*}{a}_i = q^{-i(i-1)}\,\gamma^{i(i+1)}\,\mathrm{Tr}_{\overset{*}{R}(1,\dots,i)}\mathrm{Tr}_{R(i+1,\dots,2i)}\Big(\Upsilon_P^{(i)}\Upsilon_P^{(2i)}(LT)_i\dots(LT)_1\Upsilon_{\overset{*}{R}}^{(2i)}\overset{*}{A}^{(i)\uparrow i}\times$$
$$\times(T_i\dots T_1)^{-1}\Upsilon_P^{(2i)}\Upsilon_P^{(i)}\Big),$$

Here matrices $\Upsilon_*^{(*)}$ are defined in (B.4).

Next, we permute $\overset{*}{A}^{(i)\uparrow i}$ with $\Upsilon_{\overset{*}{R}}^{(2i)}$ and cancel terms $\Upsilon_P^{(i)}\Upsilon_P^{(2i)}$ on the left and $\Upsilon_P^{(2i)}\Upsilon_P^{(i)}$ on the right. The latter cancellation exchange the R-traces Tr_R and $\mathrm{Tr}_{\overset{*}{R}}$:

$$\overset{*}{a}_i = q^{-i(i-1)}\,\gamma^{i(i+1)}\,\mathrm{Tr}_{R(1,\dots,i)}\mathrm{Tr}_{\overset{*}{R}(i+1,\dots,2i)}\Big((LT)_i\dots(LT)_1\,\overset{*}{A}^{(i)}\,\Upsilon_{\overset{*}{R}}^{(2i)}(T_i\dots T_1)^{-1}\Big).$$

In the resulting expression all the R-traces $\mathrm{Tr}_{\overset{*}{R}(i+1,\dots,2i)}$ can be evaluated with the help of lemma B.3. So, we continue

$$\overset{*}{a}_i = q^{i(i-1)}\,\gamma^{i(i+1)}\,\mathrm{Tr}_{R(1,\dots,i)}\Big((LT)_i\dots(LT)_1\,\overset{*}{A}^{(i)}\,(T_i\dots T_1)^{-1}\Big)$$
$$= q^{i(i-1)}\,\gamma^{i(i+1)}\,\mathrm{Tr}_{R(1,\dots,i)}\Big((LT)_1\dots(LT)_i(T_1\dots T_i)^{-1}A^{(i)}\Big).$$

Here in the first line we used formula (B.14) and (3.4.10); in the second line we applied formula (B.2) and then, moved the two terms $\Upsilon_P^{(i)}$, respectively, to the left and to the right and cancelled them under the R-traces $\mathrm{Tr}_{R(1,\dots,i)}$. Finally, using repeatedly permutation relations (3.3.10) and then formula (3.3.11) we complete the transformation

$$\overset{*}{a}_i = q^{i(i-1)}\,\gamma^{i(i+1)}\,\mathrm{Tr}_{R(1,\dots,i)}\Big((LT)_1\dots(LT)_{i-1}\Big((L_{\bar 1}J_1)^{\uparrow 1}T_1^{-1}\Big)^{\uparrow(i-2)}(T_1\dots T_{i-2})^{-1}A^{(i)}\Big)$$
$$\dots = q^{i(i-1)}\,\gamma^{2i}\,\mathrm{Tr}_{R(1,\dots,i)}\Big((L_{\bar 1}J_1)\dots(L_{\bar i}J_i)A^{(i)}\Big) \;=\; \gamma^{2i}\,a_i\,. \qquad \blacksquare$$

Remark 3.33. For the sets of power sums p_i and $\overset{*}{p}_i$ one can prove following recurrent relations

$$\overset{*}{p}_i \;=\; \gamma^{2i}p_i \;-\; (q-q^{-1})\sum_{k=1}^{i-1}\gamma^{2k}p_k\overset{*}{p}_{i-k}\,.$$

Corollary 3.34. *Let R be a skew invertible* R*-matrix of the $GL_q(n)$ type (in which case D_R is invertible, see proposition 2.9). Then for the matrix M (3.4.1) generating the RE algebra $\mathfrak{L}_{GL_q(n)}[\overset{*}{R}]\subset\mathfrak{D}_{GL_q(n)}[R,\gamma]$ following Cayley-Hamilton identity is valid:*

$$\sum_{i=0}^{n}(-1/q)^i\;\overset{*}{a}_i\;M^{n-i} \;=\; \sum_{i=0}^{n}(-\gamma^2/q)^i\,a_i\,M^{n-i} \;=\; 0\,.$$

In the spectrally completed algebra $\overline{\mathfrak{L}}_{GL_q(n)}[\overset{*}{R}]\subset\overline{\mathfrak{D}}_{GL_q(n)}[R,\gamma]$ this identity assumes a completely factorized form

$$\prod_{\alpha=1}^{n}\Big(M-\frac{\gamma^2\mu_\alpha}{q}I\Big) \;=\; 0\,. \tag{3.4.11}$$

With the factorized characteristic identity (3.4.11) one can construct yet another resolution of matrix unity (c.f. with (3.2.25))

$$S^\alpha \;:=\; \prod_{\substack{\beta=1\\ \beta\neq\alpha}}^{n}\frac{\big(M-\gamma^2q^{-1}\mu_\beta I\big)}{\gamma^2q^{-1}(\mu_\alpha-\mu_\beta)} \;\;:\;\; S^\alpha S^\beta \;=\; \delta_{\alpha\beta}\,S^\alpha\,, \quad \sum_{\alpha=1}^{n}S^\alpha \;=\; I\,, \tag{3.4.12}$$

so that

$$M\,S^\alpha \,=\, S^\alpha M \,=\, \gamma^2 q^{-1}\mu_\alpha S^\alpha\,.$$

Relation between the two sets of projectors P^α and S^α is explained in the following proposition.

Proposition 3.35. *In the spectrally completed algebra $\overline{\mathfrak{D}}GL_q(n)[R,\gamma]$ ($\overline{\mathfrak{D}}SL_q(n)[R]$) one has*

$$P^\alpha T S^\beta \,=\, \delta_{\alpha\beta}\, P^\alpha T \qquad \text{or, equivalently,} \qquad P^\alpha T \,=\, T S^\alpha\,. \tag{3.4.13}$$

Proof: Taking into account relations $TM = LT$, (3.2.26) and (3.3.17) one finds

$$P^\alpha T M \,=\, P^\alpha L T \,=\, q\mu_\alpha (P^\alpha T) \,=\, (P^\alpha T)\gamma^2 q^{-1}\mu_\alpha\,.$$

Hence, in view of (3.4.12)

$$P^\alpha T S^\beta \,=\, P^\alpha T \prod_{\sigma\neq\beta} \frac{\gamma^2 q^{-1}(\mu_\alpha-\mu_\sigma)}{\gamma^2 q^{-1}(\mu_\beta-\mu_\sigma)}\,.$$

In case $\alpha\neq\beta$ the factor with $\sigma=\alpha$ in the product vanishes. In case $\alpha=\beta$ (and so, $\sigma\neq\alpha$) all the terms in the product are equal to 1. So, the relation above reduces to the first equality in (3.4.13). ■

3.5. **Derivation of dynamical R-matrix.** In [AF.91] A.Alekseev and L.Faddeev used dynamical R-matrix in their construction of the Heisenberg double algebra. Namely, they observed an appearance of the classical dynamical r-matrix in the Poisson relations for certain classical variables and then, by postulating a quantum counterpart of those relations, they derived defining formulas (as in the definition 3.19) for the algebra $\mathfrak{D}\mathfrak{G}[R,\gamma]$.

In this section we are aimed to explain an origin of the dynamical R-matrix in the context of the HD algebras. We show that the dynamical R-matrix – $R(\mu)_{\alpha\beta}$ – appears in the permutation relations for matrix components of the matrices

$$W^\alpha \,:=\, P^\alpha T \,=\, T S^\alpha\,, \tag{3.5.1}$$

and the arguments of the dynamical R-matrix are just the spectral variables μ_α. In a sense, we solve an inverse problem to that considered in [AF.91].

Recall the definition of two projectors associated with the Hecke type R-matrix (see (2.3.6))

$$A^{(2)} \,=\, \frac{qI - R_1}{q+q^{-1}}\,, \qquad S^{(2)} \,=\, \frac{q^{-1}I + R_1}{q+q^{-1}}\,. \tag{3.5.2}$$

These projectors, called the *antisymmetrizer* and the *symmetrizer*, serve for suitable separation of the different eigenspaces of R.

Theorem 3.36. *In the completed HD algebra $\overline{\mathfrak{D}}GL_q(n)[R,\gamma]$ ($\overline{\mathfrak{D}}SL_q(n)[R]$) the matrices W^α (3.5.1) satisfy relations*

$$S^{(2)}\Big\{W_1^\alpha W_2^\beta + W_1^\beta W_2^\alpha\Big\}A^{(2)} = A^{(2)}\Big\{W_1^\alpha W_2^\beta + W_1^\beta W_2^\alpha\Big\}S^{(2)} \;=\; 0 \quad \forall\,\alpha,\beta, \tag{3.5.3}$$

$$S^{(2)}\Big\{(\mu_\beta - q^2\mu_\alpha)W_1^\alpha W_2^\beta \,+\, (\mu_\alpha - q^2\mu_\beta)W_1^\beta W_2^\alpha\Big\}S^{(2)} \;=\; 0 \quad \forall\,\alpha\neq\beta, \tag{3.5.4}$$

$$\begin{aligned} & A^{(2)}\Big\{(\mu_\alpha - q^2\mu_\beta)W_1^\alpha W_2^\beta \,+\, (\mu_\beta - q^2\mu_\alpha)W_1^\beta W_2^\alpha \\ & -(q^4-1)\,\mu_\alpha\,\varphi_{\alpha\beta}\,W_1^\alpha W_2^\alpha \,-\, (q^4-1)\,\mu_\beta\,\varphi_{\beta\alpha}\,W_1^\beta W_2^\beta\Big\}A^{(2)} \;=\; 0 \quad \forall\,\alpha\neq\beta, \end{aligned} \tag{3.5.5}$$

where $\varphi_{\alpha\beta} := \prod_{\sigma\neq\alpha,\beta}\frac{\mu_\sigma - q^2\mu_\alpha}{\mu_\sigma-\mu_\beta}$. Relations (3.5.3)–(3.5.4) and (3.3.17) (together with the appropriate conditions on the spectral variables μ_α) define the algebra $\overline{\mathfrak{D}}GL_q(n)[R,\gamma]$ ($\overline{\mathfrak{D}}SL_q(n)[R]$) in terms of generators W^α, μ_α, $\alpha=1,\dots,n$.

Proof: Consider the product $W_1^\alpha W_2^\beta$, where $\alpha \neq \beta$. With the help of (3.3.17) and (3.3.10) we can reorder terms of the product in a following way

$$W_1^\alpha W_2^\beta \;=\; \frac{(L_1 - q\mu_\beta)(L_{\overline{2}}J_2 - q^3\mu_\alpha)}{q^2(\mu_\alpha-\mu_\beta)(\mu_\beta - q^2\mu_\alpha)}\, W_{12}^{\alpha\beta}\,, \qquad W_{12}^{\alpha\beta} := \prod_{\sigma\neq\alpha,\beta} \frac{(L_1 - q\mu_\sigma)(L_{\overline{2}}J_2 - q\mu_\sigma)}{q^2(\mu_\alpha-\mu_\sigma)(\mu_\beta-\mu_\sigma)}\, T_1T_2 .$$

Here factor $W_{12}^{\alpha\beta}$ commutes with the R-matrix R_{12}, which follows by the same arguments as in the proof of corollary 3.31, see below (3.3.22). We now extract symmetric and antisymmetric parts of the product using projectors (3.5.2)

$$S^{(2)}W_1^\alpha W_2^\beta \;=\; S^{(2)}\, \frac{L_1L_{\overline{2}}J_2 + q^4\mu_\alpha\mu_\beta I - \frac{q^2(\mu_\beta+\mu_\alpha)}{q+q^{-1}}(L_1 + L_{\overline{2}}J_2) + \frac{\mu_\beta - q^2\mu_\alpha}{q+q^{-1}}(L_1 - L_{\overline{2}})}{q^2(\mu_\alpha-\mu_\beta)(\mu_\beta-q^2\mu_\alpha)}\, W_{12}^{\alpha\beta}, \tag{3.5.6}$$

$$A^{(2)}W_1^\alpha W_2^\beta \;=\; A^{(2)}\, \frac{L_1L_{\overline{2}}J_2 + q^4\mu_\alpha\mu_\beta I - \frac{\mu_\beta+q^4\mu_\alpha}{q+q^{-1}}(L_1 + L_{\overline{2}}J_2) + \frac{q^2(\mu_\beta - q^2\mu_\alpha)}{q+q^{-1}}(L_1 - L_{\overline{2}})}{q^2(\mu_\alpha-\mu_\beta)(\mu_\beta-q^2\mu_\alpha)}\, W_{12}^{\alpha\beta}. \tag{3.5.7}$$

Here we separated linear in L terms with the opposite symmetry properties

$$R_1(L_1 + L_{\overline{2}}J_2) = (L_1 + L_{\overline{2}}J_2)R_1\,, \qquad R_1(L_1 - L_{\overline{2}}) = -(L_1 - L_{\overline{2}})R_1^{-1}\,, \tag{3.5.8}$$

which was done by the use of relation

$$q^3\mu_\alpha L_1 + q\mu_\beta L_{\overline{2}}J_2 = \frac{q(I+R_1^{-2})}{(q+q^{-1})^2}\Big\{(\mu_\beta R_1^2 + q^2\mu_\alpha I)(L_1 + L_{\overline{2}}J_2) + (q^2\mu_\alpha - \mu_\beta)(L_1 - L_{\overline{2}})\Big\}\,.$$

The symmetry properties (3.5.8) imply, in particular, that the only term contributing to expressions $A^{(2)}W_1^\alpha W_2^\beta S^{(2)}$ and $S^{(2)}W_1^\alpha W_2^\beta A^{(2)}$ is the one proportional to $(L_1 - L_{\overline{2}})$, while the terms $(L_1L_{\overline{2}}J_2)$, I and $(L_1 + L_{\overline{2}}J_2)$ contribute to $S^{(2)}W_1^\alpha W_2^\beta S^{(2)}$ and $A^{(2)}W_1^\alpha W_2^\beta A^{(2)}$.

It is now straightforward to check that formulas (3.5.3) and (3.5.4) follow from relations (3.5.6), (3.5.7). To check formula (3.5.5) one needs also similar expression for the product $W_1^\alpha W_2^\alpha$:

$$W_1^\alpha W_2^\alpha \;=\; \frac{L_1L_{\overline{2}}J_2 + q^2\mu_\beta^2 I - q\mu_\beta(L_1 + L_{\overline{2}}J_2)}{q^2\varphi_{\alpha\beta}(\mu_\alpha-\mu_\beta)(q^2\mu_\alpha-\mu_\beta)}\, W_{12}^{\alpha\beta}\,,$$

and analogous formula for $W_1^\beta W_2^\beta$. Here factor $\varphi_{\alpha\beta}$ was defined in the proposition.

It lasts checking that defining relations for the algebras $\overline{\mathfrak{D}}_{GL_q(n)}[R,\gamma]$ and $\overline{\mathfrak{D}}_{SL_q(n)}[R]$ can be derived from (3.5.3)–(3.5.5) and (3.3.17). It is convenient to check relations for the matrices T and LT:

$$R_1T_1T_2 = T_1T_2R_1, \quad R_1(LT)_1(LT)_2 = (LT)_1(LT)_2R_1, \quad \gamma^2 T_1(LT)_2 = R_1(LT)_1T_2R_1. \tag{3.5.9}$$

For $GL_q(n)$ and $SL_q(n)$ types HD algebras, where T is invertible, these formulas imply (3.2.1) and (3.3.1). Substituting expressions

$$T = \textstyle\sum_{\alpha=1}^n W^\alpha, \qquad LT = q\sum_{\alpha=1}^n \mu_\alpha W^\alpha$$

one can easily prove that the first two relations (3.5.9) follow from (3.5.3) and (3.3.17). Checking the last formula in (3.5.9) is also straightforward, although more lengthy. To this end one has to use the whole set of relations for W's and to take into account the identity $\sum_{\alpha\neq\beta}\varphi_{\alpha\beta} = 1$. ■

Corollary 3.37. *Relations (3.5.3)–(3.5.5) can be equivalently written as*

$$S^{(2)}\Big[W_1^\alpha W_2^\beta R_1 \,-\, \sum_{\alpha',\beta'=1}^{n} R^S(q;\mu)_{\alpha'\beta'}^{\alpha\;\beta}\, W_1^{\alpha'} W_2^{\beta'}\Big] \;=\; 0\,, \qquad (3.5.10)$$

$$A^{(2)}\Big[W_1^\alpha W_2^\beta R_1 \,-\, \sum_{\alpha',\beta'=1}^{n} R^A(q;\mu)_{\alpha'\beta'}^{\alpha\;\beta}\, W_1^{\alpha'} W_2^{\beta'}\Big] \;=\; 0\,, \qquad (3.5.11)$$

where $n^2 \times n^2$ matrix $R^S(q;\mu)$ has following nonzero components

$$R^{S\,\alpha\alpha}_{\;\;\alpha\alpha} \,=\, q, \qquad R^{S\,\alpha\beta}_{\;\;\alpha\beta} \,=\, -\frac{(q-q^{-1})\mu_\beta}{\mu_\alpha-\mu_\beta}, \qquad R^{S\,\alpha\beta}_{\;\;\beta\alpha} \,=\, \frac{q^{-1}\mu_\alpha - q\mu_\beta}{\mu_\alpha-\mu_\beta} \quad \forall\, \alpha\neq\beta,$$

and $n^2 \times n^2$ matrix $R^A(q;\mu)$ has nonzero components at the same places as R^S with values $R^A(q;\mu) = R^S(-q^{-1};\mu)$, and the additional nonzero components

$$R^{A\,\alpha\alpha}_{\;\;\beta\alpha} \,=\, -R^{A\,\alpha\alpha}_{\;\;\alpha\beta} \,=\, \frac{(q^4-1)\,\mu_\alpha\,\varphi_{\alpha\beta}}{q(\mu_\alpha-\mu_\beta)} \quad \forall\, \alpha\neq\beta.$$

Both matrices $R^{S/A}(q;\mu) \equiv R^{S/A}(\mu)$ satisfy dynamical Yang-Baxter equation*:*

$$R(\mu)^{12} R\big(\nabla^1(\mu)\big)^{23} R(\mu)^{12} \,=\, R\big(\nabla^1(\mu)\big)^{23} R(\mu)^{12} R\big(\nabla^1(\mu)\big)^{23}\,. \qquad (3.5.12)$$

Here superscript labels denote endomorphism spaces for the spectral indices, e.g., $R(\mu)^{\alpha_1\alpha_2}_{\beta_1\beta_2} \equiv R(\mu)^{12}$, and ∇^1 is a diagonal finite shift operator

$$\nabla^1 \,=\, \mathrm{diag}\{\nabla^\alpha\}_{\alpha=1}^{n}\,: \quad \nabla^\alpha(\mu_\beta) \,:=\, q^{2\delta_{\alpha\beta}}\gamma^{-2}\mu_\beta\,. \qquad (3.5.13)$$

Proof: Apply the symmetrizer $S^{(2)}$ and the antisymmetrizer $A^{(2)}$ from the right to both sides of the equalities (3.5.10), (3.5.11). The resulting projections is easy to compare with (3.5.3)–(3.5.5).

To prove the dynamical Yang-Baxter equation for the matrices $R^A(q;\mu)$ and $R^S(q;\mu)$ we consider, respectively, the following cubic terms:

$$A^{(3)} W_1^\alpha W_2^\beta W_3^\sigma \qquad \text{and} \qquad S^{(3)} W_1^\alpha W_2^\beta W_3^\sigma\,.$$

Here 3-antisymmetrizer $A^{(3)} = \rho_R(a^{(3)})$ is the R-matrix realization of the idempotent $a^{(3)}$ (see (2.3.2), (2.3.3)), and 3-symmetrizer $S^{(3)}$ is a similar projector which differs from $A^{(3)}$ by substitution $q \leftrightarrow -q^{-1}$ in the formulas (2.3.2), (2.3.3). Now applying two equal operators $R_1R_2R_1$ and $R_2R_1R_2$ from the right side to these terms and using relations (3.5.10), (3.5.11) and (3.3.17) one eventually proves (3.5.12) for $R^{A/S}(q;\mu)$. ■

Remark 3.38. The dynamical R-matrix $R^S(q;\mu)$ was constructed in [F.90, AF.91, Is.95]. A review on the dynamical Yang-Baxter equation and the dynamical R-matrices is given in [ES]. It is surprising that in our approach the dynamical R-matrices $R^{A/S}(q;\mu)$, being the solutions of the nonlinear finite difference equation (3.5.12), are calculated by solving a system of (at most three) linear equations.

In conclusion of the section we comment how relations (3.5.10) can be reduced to dynamical quadratic relations considered in [F.90, AF.91]. Recall that a (Hecke type) *quantum plane* $\mathcal{V}[R]$ is an algebra generated by components of vector $\{x_i\}_{i=1}^{\dim V}$ subject to relations

$$x_{\langle 1|}x_{\langle 2|}\, A^{(2)} \,=\, 0 \qquad \Leftrightarrow \qquad x_{\langle 1|}x_{\langle 2|}\, S^{(2)} \,=\, x_{\langle 1|}x_{\langle 2|}\,. \qquad (3.5.14)$$

In the tensor product algebra $\mathcal{V}[R]\otimes \overline{\mathfrak{D}}_{GL_q(n)}[R,\gamma]$ consider a rectangular matrix

$$\Lambda_i^\alpha \,:=\, \textstyle\sum_{j=1}^{\dim V} x_j \otimes W_{ji}^\alpha\,, \qquad \alpha = 1,\dots n\,,\;\; i = 1,\dots,\dim V\,.$$

As a consequence of (3.5.10), (3.5.14) the matrix components of Λ fulfill relations

$$\Lambda^{|1\rangle}_{\langle 1|}\,\Lambda^{|2\rangle}_{\langle 2|}\,R_{12} \;=\; R^S(q;\mu)^{12}\,\Lambda^{|1\rangle}_{\langle 1|}\,\Lambda^{|2\rangle}_{\langle 2|}\,. \tag{3.5.15}$$

Assume additionally that *i)* $\dim V = n$, and *ii)* the quantum plane admits a one dimensional representation $\chi:\ \mathcal{V}[R]\to\mathbb{C}$ (note that both these conditions are satisfied for the R-matrices from the example 2.10).

It is the square matrix $\chi(\Lambda)\in\overline{\mathfrak{D}}_{GL_q(n)}[R,\gamma]$ whose dynamical quadratic relations (3.5.15) were introduced in [F.90, AF.91] and also investigated in [HIOPT, FHIOPT].

4. Discrete time evolution on quantum group cotangent bundle

4.1. **Automorphisms of the Heisenberg double algebra.** In this section we investigate a sequence of automorphisms on the HD algebra $\mathfrak{D}\mathfrak{G}[R,\gamma]$. These automorphisms were introduced by A. Alekseev and L. Faddeev [AF.91, AF.92], who interpreted them as a discrete time evolution of a q-deformed quantum isotropic Euler top. The automorphisms $\theta^k:\mathfrak{D}\mathfrak{G}[R,\gamma]\to\mathfrak{D}\mathfrak{G}[R,\gamma]$ are given on generators

$$\begin{gathered}\{T,\ L\}\stackrel{\theta^k}{\longrightarrow}\{T(k),\ L(k)\}\,,\quad \forall\, k=0,1,2,\dots\,,\\ T(0):=T\,,\quad T(k+1):=LT(k)=L^{k+1}\,T\,,\quad L(k):=L\,.\end{gathered}\tag{4.1.1}$$

It is easy to see (c.f. (3.5.9)) that the map θ agrees with the defining relations (3.1.1), (3.2.1), (3.3.1) of the algebra $\mathfrak{D}\mathfrak{G}[R,\gamma]$. Less obvious is its consistency with the $SL_q(n)$ type reduction conditions.

Proposition 4.1. *The map θ (4.1.1) defines an automorphism of the algebra $\mathfrak{D}_{SL_q(n)}[R]$.*

Proof: It is necessary to check that $\det_R(LT)=1$ in the $SL_q(n)$ case. To this end, we use formula

$$(LT)_1(LT)_2\dots(LT)_k \;=\; \gamma^{-k(k-1)}Z_k\big(L_{\overline{1}}L_{\overline{2}}\dots L_{\overline{k}}\big)\big(T_1T_2\dots T_k\big)$$

to separate matrices L and T in the expression for $\det_R(LT)$. This formula follows from (3.3.1), (3.2.10), (3.2.11) and (3.3.8) by induction on k.

The calculation of $\det_R(LT)$ proceeds as follows

$$\begin{aligned}\det{}_R LT :=\ & \mathrm{Tr}_{(1,\dots,n)}\Big(A^{(n)}(LT)_1\dots(LT)_n\Big)=\gamma^{n(n-1)}\mathrm{Tr}_{(1,\dots,n)}\Big(A^{(n)}Z^{(n)}L_{\overline{1}}\dots L_{\overline{n}}T_1\dots T_n\Big)\\ &=(\gamma q)^{-n(n-1)}\,\mathrm{Tr}_{(1,\dots,n)}\big(A^{(n)}L_{\overline{1}}\dots L_{\overline{n}}\big)\,\mathrm{Tr}_{(1,\dots,n)}\big(A^{(n)}T_1\dots T_n\big)\\ &=q^n\gamma^{-n(n-1)}\,\mathrm{Tr}_{R\,(1,\dots,n)}\big(A^{(n)}L_{\overline{1}}\dots L_{\overline{n}}\big)\det{}_R T \;=\; \big(q\gamma^{-n}\big)^{n-1}q\,a_n\det{}_R T\,,\end{aligned}\tag{4.1.2}$$

and so, under conditions $\det_R T=1$, $a_n=q^{-1}1$, $\gamma^n=q$ we have $\det_R(LT)=1$. Here in the second line we substituted $A^{(n)}Z_n=q^{-n(n-1)}A^{(n)}$ and used the condition $\mathrm{rk}A^{(n)}=1$; in the last line we applied (2.4.5) and the definitions of $\det_R T$ and a_n. ■

In what follows we will investigate the automorphisms (4.1.1) for HD algebras of the types $\mathfrak{D}_{GL_q(n)}[R,\gamma]$ and $\mathfrak{D}_{SL_q(n)}[R]$. A key point for their dynamical interpretation is a possibility to write down following ansatz

$$T(k+1) \;=\; L\,T(k) \;=\; \big(qa_n\big)^{1/n}\,\Theta\,T(k)\,\Theta^{-1}\,,\quad \text{where }\ \Theta\in\overline{\mathfrak{Ch}}[R]\,. \tag{4.1.3}$$

Here the dynamical process – evolution – is thought as an inner HD algebra automorphism, and Θ plays a role of the *evolution operator*. As the evolution keeps L unchanged, it is natural to assume that Θ belongs to the center of the RE algebra generated by the matrix L. More specifically, we will look for Θ as a formal power series in spectral variables μ_α, $\alpha=1,\dots,n$, which we denote as $\overline{\mathfrak{Ch}}[R]$. We also note that the condition $\Theta \in \overline{\mathfrak{Ch}}[R]$ makes the ansatz manifestly covariant with respect to both left and right coactions (3.3.2), (3.3.3).

Factor $(qa_n)^{1/n}$ in the ansatz (4.1.3) becomes trivial for the $SL_q(n)$ type HD algebra. In the $GL_q(n)$ case one adds this scaling factor to make the ansatz consistent with the evolution of $\det_R T$, see (4.1.2). One assumes following relation for the newly introduced element $a_n^{1/n}$ (c.f. with (3.3.16))

$$T\, a_n^{1/n} \;=\; \left(q\gamma^{-n}\right)^{2/n} a_n^{1/n}\, T\,. \tag{4.1.4}$$

Then, consistency of (4.1.3) and (4.1.2) results in a commutativity of $\det_R T$ with Θ:

$$\det_R T\,\Theta \;=\; \Theta \det_R T\,, \tag{4.1.5}$$

which again trivializes in the $SL_q(n)$ case.

Remark 4.2. The action of the automorphisms θ^k on T can be equally treated as write multiplications by powers of the left invariant matrix M:

$$T(k+1) \;=\; T(k)\,M \;=\; T\,M^k\,, \quad M(k) \;=\; M\,.$$

The relation (3.4.1) between L and M would no more be valid if one would treat them as *quantized right and left invariant Lie derivatives* acting on *quantized external algebra of differential forms over matrix group.* In this case one would have a two-parametric series of automorphisms:

$$\{T,\ L,\ M\} \stackrel{\theta^{(k,m)}}{\longrightarrow} \{L^kTM^m,\ L,\ M\}\,, \quad \forall\, k\ge 0,\ m\ge 0\,.$$

Example 4.3. Let us show that in the ribbon Hopf algebra setting the ribbon element $\upsilon \in \mathfrak{A}_{\mathcal{R}}$ generates the evolution (4.1.3) in the smash product algebra $\mathfrak{A}_{\mathcal{R}}\sharp\,\mathfrak{A}^*_{\mathcal{R}}$. For this we first have to specify pairing for the ribbon element. Using the definition (2.1.6) and relations (2.2.12), (2.4.3) and setting $\eta = q^{1/n}$ as in the example 2.10 we calculate

$$\langle T,\, \upsilon^2\rangle \;=\; \langle T,\, uS(u)\rangle \;=\; \rho_V(u)\,\rho_V(S(u)) \;=\; \eta^2 D_R\, C_R \;=\; q^{2(\frac{1}{n}-n)}\, I\,.$$

Therefore, taking into account centrality of the ribbon element υ in $\mathfrak{A}_{\mathcal{R}}$, it is natural to define

$$\langle T,\, \upsilon\rangle \;=\; q^{(\frac{1}{n}-n)}\, I\,.$$

Using this formula and relations (2.1.6), (3.2.5), (3.3.4) we now calculate conjugation of matrix T with the ribbon element

$$\begin{aligned}\upsilon\, T\, \upsilon^{-1} \;&=\; (\upsilon\otimes id)\,\langle id\otimes T,\, \Delta(\upsilon^{-1})\rangle\, T \;=\; (\upsilon\otimes id)\,\langle id\otimes T,\, (\upsilon^{-1}\otimes\upsilon^{-1})\mathcal{R}_{21}\mathcal{R}_{12}\rangle\, T\\ &=\; \langle T,\, \upsilon^{-1}\rangle\,\langle id\otimes T,\, \mathcal{R}_{21}\mathcal{R}_{12}\rangle\, T \;=\; LT\,.\end{aligned}$$

Note that defining relations for the evolution operator Θ (as a function of the spectral variables μ_α) and for the ribbon element υ, both admit multiple solutions. [9] Therefore, a problem of finding explicit expression of the ribbon element υ in terms of spectral variables μ_α demands further investigations.

4.2. **Equations for the evolution operator Θ.** Using the results of section 3 it is straightforward to derive equations for Θ. We consider in details, the evolution in the $SL_q(n)$ type HD algebra. In this case we assume

$$\Theta = \Theta(\mu_1,\dots,\mu_n)\,, \qquad \text{where} \quad a_n = \textstyle\prod_{\alpha=1}^n \mu_\alpha = q^{-1} \quad \text{and} \quad \gamma = q^{-1/n}\,. \tag{4.2.1}$$

Applying from the left the projector P^α to both sides of (4.1.3) we obtain

$$q\mu_\alpha\,(P^\alpha T) \;=\; \Theta(P^\alpha T)\Theta^{-1}\,, \quad \forall\, \alpha = 1,\dots,n\,.$$

[9] The ribbon element is defined modulo central factor $z\in\mathfrak{A}_{\mathcal{R}}$: $z^2=1\,,\ \ S(z)=z\,,\ \ \epsilon(z)=1\,,\ \ \Delta(z)=z\otimes z\,.$ For the evolution operator $\Theta(\mu)$ different solutions are constructed in the next sections.

Multiplying this equality by Θ from the right and permuting Θ with $P^\alpha T$ in the left hand side with the help of (3.3.17) we finally get

$$q\mu_\alpha\,\Theta(q^{-2/n}\mu_1,\dots,q^{2-2/n}\mu_\alpha,\dots,q^{-2/n}\mu_n)\,(P^\alpha T) \;=\; \Theta(\mu_1,\dots,\mu_n)\,(P^\alpha T)\,.$$

We state the result in a following proposition

Proposition 4.4. *For the Heisenberg double algebra $\mathfrak{D}_{SL_q(n)}[R]$ the evolution operator $\Theta(\mu_\alpha)$ in (4.1.3), (4.2.1) is a solution of equations*

$$q\mu_\alpha\,\Theta\big(\nabla^\alpha(\mu_\beta)\big) \;=\; \Theta(\mu_\beta) \quad \forall\,\alpha=1,\dots,n\,, \tag{4.2.2}$$

where ∇^α are finite shift operators introduced in (3.5.13). In the $SL_q(n)$ case their actions are

$$\nabla^\alpha(\mu_\beta) \;:=\; q^{2X_{\alpha\beta}}\,\mu_\beta\,, \qquad X_{\alpha\beta} \;:=\; \delta_{\alpha\beta} - \frac{1}{n} \quad \forall\,\alpha,\beta=1,\dots,n\,. \tag{4.2.3}$$

The $n\times n$ matrix X is a Gram matrix for the set of vectors $\vec{e}^{\,*}_\alpha\in\mathbb{Q}^n$, $\alpha=1,\dots,n$:

$$\vec{e}^{\,*}_\alpha := \frac{1}{n}\,(\underbrace{-1,\dots,-1}_{(\alpha-1)\ times},n-1,-1,\dots,-1\,)\,, \qquad X_{\alpha\beta} \;=\; \langle\vec{e}^{\,*}_\alpha,\vec{e}^{\,*}_\beta\rangle\,.$$

X is positive semi-definite of the rank $n-1$ ($\sum_{\alpha=1}^n\vec{e}^{\,*}_\alpha=0$).

For the Heisenberg double algebra $\mathfrak{D}_{GL_q(n)}[R,\gamma]$ the evolution operator Θ is suitably parameterized by variables $z:=(qa_n)^{1/n}$ and ν_α

$$\nu_\alpha := \mu_\alpha(qa_n)^{-1/n}\,, \qquad \text{such that} \qquad \textstyle\prod_{\alpha=1}^n\nu_\alpha \;=\; q^{-1}\,, \quad \nu_\alpha\,\det_R T \;=\; \det_R T\,\nu_\alpha \;\;\forall\,\alpha\,.$$

The evolution equations for $\Theta(\nu_1,\dots,\nu_n;z)$ read

$$q\nu_\alpha\,\Theta(\nabla^\alpha(\nu_\beta);\,(q\gamma^{-n})^{2/n}\,z) \;=\; \Theta(\nu_\alpha;z) \quad \forall\,\alpha=1,\dots,n\,, \tag{4.2.4}$$

where shift operators ∇^α are defined as in (4.2.3). Since $\prod_{\alpha=1}^n\nabla^\alpha = 1$, this system is consistent provided that $\Theta(\nu_\beta;\,q^2\gamma^{-2n}\,z) = \Theta(\nu_\beta;z)$ (c.f. with (4.1.5)). Demanding that Θ does not actually depend on z one reduces (4.2.4) to (4.2.2).

Proof: The $SL_q(n)$ case is already considered. Taking into account relations (4.1.4) and (3.3.19) a derivation of the evolution equations in the $GL_q(n)$ case is the same. ■

In the next two subsections we will construct particular solutions of the SL-type evolution equations (4.2.2).

4.3. **Solution in case $|q|<1$.** Let us look for solution of (4.2.2) as a series in μ_α. Taking into account condition (4.2.1) we exclude one dependent variable, say μ_n, from the expansion

$$\Theta(\mu_\alpha) = \sum_{\vec{k}\in\mathbb{Z}^{n-1}} c(\vec{k})\,\mu_1^{k_1}\,\mu_2^{k_2}\dots\mu_{n-1}^{k_{n-1}}\,. \tag{4.3.1}$$

where $\mathbb{Z}^{n-1}=\{(k_1,\dots,k_{n-1}):\ k_i\in\mathbb{Z}\}$, and the coefficients $c(\vec{k})$ are $\mathbb{C}$-valued functions on $\mathbb{Z}^{n-1}$. Substitution of (4.3.1) into (4.2.2) gives conditions on the coefficients:

$$c(\vec{k}+\vec{\epsilon}_\alpha) \;=\; q^{\left(1+\frac{2}{n}\sum_{\beta=1}^{n-1}A^*_{\alpha\beta}k_\beta\right)}\,c(\vec{k}) \quad \forall\,\alpha=1,\dots,n-1. \tag{4.3.2}$$

Here $\vec{\epsilon}_\alpha := (\underbrace{0,\dots,0}_{(\alpha-1)\text{ times}},1,0,\dots,0)$, and $A^*_{\alpha\beta} := n\,X_{\alpha\beta}$ is a $(n-1)\times(n-1)$ positive-definite matrix.

The general solution to (4.3.2) is

$$c(\vec{k}) \;=\; q^{\frac{1}{n}\left\{(\vec{k},A^*\vec{k})+(\vec{1}\,\vec{k})\right\}}\,, \tag{4.3.3}$$

where we choose normalization $c(\vec{0}) = 1$ and use notation $(\vec{k},\, A^*\,\vec{k}) = \sum_{\alpha,\beta=1}^{n-1} k_\alpha A^*_{\alpha\beta} k_\beta$, and $\vec{1} = (1,\dots,1)$, so that $(\vec{1},\, \vec{k}) = \sum_{\alpha=1}^{n-1} k_\alpha$.

Remark 4.5. The matrix $A^*_{\alpha\beta}$ is a Gram matrix of a lattice A^*_{n-1} dual to the root lattice A_{n-1} (see [CS], chapter 4, section 6.6). The corresponding quadratic form $(\vec{k},\, A^*\,\vec{k})$ is often referred to as Voronoi's principal form of the first type.

The ansatz (4.3.1) gives a particular solution (4.3.3) of the evolution equations, we denote it $\Theta^{(1)}$. Introducing a parameterization

$$q = \exp(2\pi\mathrm{i}\,\tau), \quad q^{1/n}\mu_\alpha = \exp(2\pi\mathrm{i}\,z_\alpha) : \sum_{\alpha=1}^{n-1} z_\alpha = 0, \quad \Omega_{\alpha\beta} = \frac{2\tau}{n} A^*_{\alpha\beta} \;=\; 2\tau\,(\delta_{\alpha\beta} - \frac{1}{n}), \tag{4.3.4}$$

we can write $\Theta^{(1)}$ as a Riemann theta function $\theta(\vec{z},\Omega)$ (see [Mum, I])

$$\Theta^{(1)}(\mu_\alpha) \;=\; \theta(\vec{z},\Omega) \;=\; \sum_{\vec{k}\in\mathbb{Z}^{n-1}} \exp\Big\{\pi\mathrm{i}\,(\vec{k},\, \Omega\,\vec{k}) + 2\pi\mathrm{i}\,(\vec{k}\,,\, \vec{z})\Big\}. \tag{4.3.5}$$

Here τ is a *modular parameter* and Ω is a *matrix of periods.* Expression (4.3.5) converges either if $|q| < 1$, or if q is a rational root of unity, in which case the series can be truncated.

Remark 4.6. One can present formula (4.3.5) in a manifestly covariant form:

$$\Theta^{(1)} \;\equiv\; \Theta^{(1)}(\vec{z}, A^*_{n-1}, \tau) \;=\; \sum_{\vec{\mathrm{k}}\in A^*_{n-1}} \exp\Big\{2\pi\mathrm{i}\,\frac{\tau}{n}\,\langle \vec{\mathrm{k}}, \vec{\mathrm{k}}\rangle + 2\pi\mathrm{i}\,\langle\vec{\mathrm{k}}\,,\, \vec{\mathrm{z}}\rangle\Big\}.$$

Here vectors $\vec{\mathrm{k}} = \sum_{\alpha=1}^{n-1} k_\alpha e^*_\alpha$ label vertices of the lattice A^*_{n-1}, and $\vec{\mathrm{z}} = \sum_{\alpha=1}^{n-1} z_\alpha e_\alpha$, where $e_\alpha = \epsilon_\alpha - \epsilon_n$, $\alpha = 1,\dots,n-1$, (see line below (4.3.2)) are basic vectors of the root lattice A_{n-1}: $\langle e^*_\alpha,\, e_\beta\rangle = \delta_{\alpha\beta}$.

In the simplest $SL_q(2)$ case the evolution operator $\Theta^{(1)}$ becomes the Jacobi theta function:

$$\Theta^{(1)}(\mu_1) \;=\; \sum_{k\in\mathbb{Z}} q^{\frac{1}{2}k(k+1)}\mu_1^k \;=\; \sum_{k\in\mathbb{Z}} \exp(\pi\mathrm{i}\,k^2\tau + 2\pi\mathrm{i}\,kz_1) \;=\; \theta_3(z_1;\, q)\,,$$

or, in a multiplicative form

$$\Theta^{(1)}(\mu_1) \;=\; \prod_{n=1}^{\infty}(1-q^n)(1+q^n\mu_1)(1+q^{n-1}/\mu_1)\,.$$

4.4. **Solution for arbitrary** q**.** In this section we derive yet another particular solution of the evolution equations (4.2.2), the one which is well defined for arbitrary values of q. The idea of such solution was proposed in L.D. Faddeev's lectures on two-dimensional integrable quantum field theory [F.94] (see also [F.95]). We use heuristic arguments inspired by considerations in [AF.91]. For the moment we assume $\dim V = n$, so that the range of the indices α and i, j in the projectors P^α_{ij}, S^α_{ij} is the same. Consider following $n\times n$ matrices

$$U_{ij} \;:=\; \sum_{k=1}^{n} u_{ik}\, P^{\alpha=i}_{kj}\,, \qquad V_{ij} \;:=\; \sum_{k=1}^{n} S^{\alpha=j}_{ik}\, v_{kj}\,,$$

where the only restriction for the auxiliary parameters u_{ij} and v_{ij} is their commutativity with the spectral variables μ_α

$$[u_{ij},\, \mu_\alpha] \;=\; [v_{ij},\, \mu_\alpha] \;=\; 0 \quad \forall\, i, j, \alpha\,.$$

As a result of the Cayley-Hamilton identities (3.2.24), (3.4.11) we have matrix relations

$$UL \;=\; qDU\,, \qquad MV \;=\; \gamma^2 q^{-1}\, VD\,, \qquad \text{where} \quad D := \mathrm{diag}\{\mu_1,\dots,\mu_n\}\,.$$

Moreover, by (3.4.13), matrix $Q := UTV$ is diagonal

$$Q = \mathrm{diag}\{w_1, \dots, w_n\}\,, \qquad \text{where } \ w_i \;:=\; \Big(uP^iTv\Big)_{ii}\,,$$

and, by (3.3.17), w_i satisfy following permutation relations with μ_j and z_j:

$$w_i\,\mu_j \;=\; q^{2\delta_{ij}}\gamma^{-2}\,\mu_j\,w_i \quad \Leftrightarrow \quad w_i\,z_j \;=\; \Big(z_j + 2\tau(\delta_{ij} - 1/n)\Big)w_i\,, \tag{4.4.1}$$

where in the latter formula we used the $SL_q(n)$ type condition $\gamma = q^{1/n}$.

Assuming invertibility of the matrices U and V we can write diagonal decompositions for the matrices L, M and T

$$L \;=\; q\,UDU^{-1}\,, \qquad M \;=\; \gamma^2 q^{-1}\,VDV^{-1}\,, \qquad T \;=\; U^{-1}QV^{-1}\,,$$

which after substitution into the ansatz (4.1.3) reduce the evolution equations to a following form

$$qDQ \;=\; \Theta\,Q\,\Theta^{-1} \qquad \Leftrightarrow \qquad q\mu_i w_i \;=\; \Theta\,w_i\,\Theta^{-1}\,.$$

Taking into account (4.4.1) these equations clearly have following solution

$$\Theta^{(2)}(z_\alpha) \;:=\; \exp\Big(-\frac{\pi\mathrm{i}}{2\tau}\textstyle\sum_{\beta=1}^{n} z_\beta^2\Big)\,. \tag{4.4.2}$$

Now, it is easy to check that the function $\Theta^{(2)}$ fulfills the evolution equations (4.2.2) without additional assumptions we made for the derivation. Written in the independent variables $\vec{z} = \{z_1, \dots, z_{p-1}\}$ it reads

$$\Theta^{(2)}(\vec{z}) \;=\; \exp\Big(-\frac{\pi\mathrm{i}}{\tau}\sum_{1\le\alpha\le\beta\le n-1} z_\alpha z_\beta\Big) \;=\; \exp\Big\{-\pi\mathrm{i}\,(\vec{z},\,\Omega^{-1}\vec{z})\Big\}\,, \tag{4.4.3}$$

where the inverse matrix of periods is

$$\Omega^{-1}_{\alpha\beta} = \frac{1}{2\tau}\,(\delta_{\alpha\beta} + 1) = \frac{1}{2\tau}\,A_{\alpha\beta}\;,$$

and $A_{\alpha\beta} = \langle e_\alpha, e_\beta\rangle$ is the Gram matrix for the root lattice A_{n-1} (see remark 4.6). Let us stress that the logarithmic change of variables $\mu_\alpha \mapsto z_\alpha$ (4.3.4) which was rather superficial in case of $\Theta^{(1)}$, is inevitable for the derivation of $\Theta^{(2)}$.

Finally, we comment on relation between the two evolution operators $\Theta^{(1)} = \theta(\vec{z},\,\Omega)$ and $\Theta^{(2)}$. The relation is based on a functional equation for Riemann theta function

$$\theta(\Omega^{-1}\vec{z},\,-\Omega^{-1}) \;=\; \Big(\det\big(\Omega/\mathrm{i}\big)\Big)^{\frac{1}{2}}\exp\Big\{\pi\mathrm{i}(\vec{z},\,\Omega^{-1}\vec{z})\Big\}\,\theta(\vec{z},\,\Omega)\,,$$

which is the special case of a more general modular functional equation (for derivation and generalization see [Mum], chapter 2, section 5). With our particular matrix of periods Ω (4.3.5) we find

$$\Theta^{(2)}(\vec{z}) \;=\; \frac{1}{\sqrt{n}}\Big(\frac{2\tau}{\mathrm{i}}\Big)^{\frac{n-1}{2}}\frac{\theta(\vec{z},\,\Omega)}{\theta(\Omega^{-1}\vec{z},\,-\Omega^{-1})}\,. \tag{4.4.4}$$

Note that theta function in the denominator – $\theta(\Omega^{-1}\vec{z},\,-\Omega^{-1})$ – commutes with the elements of $\mathfrak{D}_{SL_q(n)}[R]$ and can be thought as an evolution operator on a 'modular dual' quantum cotangent bundle [F.99].

Appendix A. Pairing between spectral variables and quantized functions

Here we calculate pairing of the elementary symmetric functions a_i (3.2.17) with the generators of quantized functions T^i_j. We assume that T and a_i are realized respectively, as elements of dual quasi-triangular Hopf algebras $\mathfrak{A}^*_{\mathcal{R}}$ and $\mathfrak{A}_{\mathcal{R}}$. We further extend this pairing also for the spectral variables μ_α.

For the calculation we use formula

$$\langle T_1,\, L_2\rangle \;=\; \eta^{-2} q^{(n-\frac{1}{n})} R^2_{12} \tag{A.1}$$

which follows from the definitions (2.2.11), (3.1.5), (3.2.3), (3.2.5).

Proposition A.1. *Let a_i (3.2.3), (3.2.5) and T (3.1.5) be elements of the dual quasi-triangular Hopf algebras, respectively, $\mathfrak{A}_{\mathcal{R}}$ and $\mathfrak{A}^*_{\mathcal{R}}$ Assume that the R-matrix R (2.2.11) is $GL_q(n)$ type with scaling parameter $\eta = q^{1/n}$ as in the example 2.10. Then*

$$\langle T,\, a_i\rangle \;=\; q^{-3i/n}\, n_q^{-1} \binom{n}{i}_q \Big\{ n_q \,+\, q^{n+1} \,-\, q^{n-2i+1} \Big\} I. \tag{A.2}$$

Proof: The calculation proceeds as follows

$$\begin{aligned}
\langle T_1,\, a_i\rangle \;&=\; \langle T_1,\, \mathrm{Tr}_{R(2,\dots,i+1)}\big(A^{(i)} L_1 \dots L_{\bar{i}}\big)^{\uparrow 1}\rangle \\
&=\; q^{i(n-\frac{3}{n})}\, \mathrm{Tr}_{R(2,\dots,i+1)}\big(A^{(i)\uparrow 1}(J_2 \dots J_{i+1})(J_1^{-1} \dots J_i^{-1})^{\uparrow 1}\big) \\
&=\; q^{i(n-\frac{3}{n})}\, \mathrm{Tr}_{R(2,\dots,i+1)}\big(R_1 \dots R_i\, A^{(i)} R_i \dots R_1\big) \\
&=\; q^{i(n-\frac{3}{n})} \Big\{ q^{-n}\, (n+1-i)_q\, i_q^{-1}\; \mathrm{Tr}_{R(2,\dots,i)}\big(R_1 \dots R_{i-1}\, A^{(i-1)} R_{i-1} \dots R_1\big) \\
&\qquad + (q-q^{-1})\, q^{-(n+2)(i-1)}\, n_q^{-1} \tbinom{n}{i}_q I_1 \Big\} \\
\dots \;&=\; q^{i(n-\frac{3}{n})} \tbinom{n}{i}_q \Big\{ q^{-in} + (q-q^{-1})\, q^{-(n+2)(i-1)}\, n_q^{-1} \big(1+q^2+\dots+q^{2(i-1)}\big) \Big\} I_1.
\end{aligned}$$

Here in the first line we substituted expression for a_i similar to (3.3.13). In the second line we evaluated pairing using formulas $\langle T_1,\, (L_{\bar{i}})^{\uparrow 1}\rangle = \eta^{-2} q^{(n-\frac{1}{n})} J_{i+1} (J_i^{-1})^{\uparrow 1}$ following from (A.1). In the third line we first, used the cyclic property of the R-trace to evaluate the term $(J_1^{-1} \dots J_i^{-1})^{\uparrow 1}$ on $(A^{(i)})^{\uparrow 1}$ and then, rearranged the product $J_2 \dots J_{i+1} = Z_{i+1} = J_2^\dagger \dots J_{i+1}^\dagger$, where $J_1^\dagger := I$, $J_{k+1}^\dagger := R_{i-k+1} J_k^\dagger R_{i-k+1}$, and evaluated the term $J_2^\dagger \dots J_i^\dagger$ on $(A^{(i)})^{\uparrow 1}$. After that we recollected terms in the product: $(A^{(i)})^{\uparrow 1} J_{i+1}^\dagger = R_1 \dots R_i\, A^{(i)} R_i \dots R_1$. In the forth line we substituted $R_i = R_i^{-1} + (q-q^{-1})I$ for one of R_is and used formulas (2.2.8) and (2.2.10) to evaluate $\mathrm{Tr}_{R(i+1)}$. Then, in the summand which is proportional to $(q-q^{-1})$ all the R-traces can be evaluated with the help of (2.4.4). Omission points in the fifth line stand for similar evaluations of $\mathrm{Tr}_{R(i)} \dots \mathrm{Tr}_{R(2)}$, the resulting expression coincides with (A.2). ■

For a_n relation (A.2) simplifies to

$$\langle T,\, a_n\rangle \;=\; q^{-1} I\,, \tag{A.3}$$

which obviously agrees with (3.2.18). So, we checked a consistency of the normalizations $q^{n-\frac{1}{n}}$ in (3.2.5) and $\eta = q^{1/n}$ for the Drinfeld-Jimbo R-matrices with the $SL_q(n)$ reduction condition (3.2.18).

Corollary A.2. *In the conditions of proposition A.1 the pairing $\langle\cdot,\cdot\rangle$ can be extended for the spectral variables (3.2.22):*

$$\langle T,\, \mu_\alpha\rangle \,=\, q^{(2\alpha+2\delta_{\alpha n}-n-\frac{3}{n}-1)} I, \qquad \alpha=1,\dots,n. \tag{A.4}$$

Proof: Let us rescale spectral variables $\tilde\mu_\alpha := q^{\frac{3}{n}-2\delta_{\alpha n}}\mu_\alpha$. For rescaled variables (A.4) reads

$$\langle T,\, \tilde\mu_\alpha\rangle = q^{(2\alpha-n-1)} I. \tag{A.5}$$

Using q-binomial identity $q^i\binom{n-1}{i}_q + q^{i-n}\binom{n-1}{i-1}_q = \binom{n}{i}_q$ it is straightforward to derive from (A.5) pairings of the elementary symmetric functions $e_i(\tilde\mu)$ by induction on n: $\langle T,\, e_i(\tilde\mu)\rangle = \binom{n}{i}_q$. Using (3.3.18) it is then straightforward to derive pairings for elementary symmetric functions in original spectral variables — $\langle T,\, e_i(\mu)\rangle$ — which under identification $a_i \mapsto e_i(\mu)$ coincide with (A.2). ■

Appendix B. Three lemmas for subsection 3.4

Here we collect some technical results which are used for establishing relation between the spectra of the left and right invariant vector fields.

Lemma B.1. a) *If the* R*-matrix R is skew invertible then the following four statements are equivalent:* i) *the matrix D_R is invertible;* ii) *the matrix C_R is invertible;* iii) *the* R*-matrix R^{-1} is skew invertible;* iv) *the R-matrix $\overset{*}{R} := PR^{-1}P$ is skew invertible. One has*

$$D_{\overset{*}{R}} \,=\, C_{R^{-1}} \,=\, (D_R)^{-1}, \qquad C_{\overset{*}{R}} \,=\, D_{R^{-1}} \,=\, (C_R)^{-1}. \tag{B.1}$$

b) *Let R be the Hecke type R-matrix generating representations ρ_R (2.3.7) of the algebras $\mathcal{H}_k(q)$. Then the* R*-matrix $\overset{*}{R}$ is Hecke type as well, and $\rho_{\overset{*}{R}}$ are representations of the algebras $\mathcal{H}_k(q^{-1})$. If additionally the parameter q satisfies conditions* **[k]** *(2.3.1) so that the idempotent $a^{(k)}|_{q\leftrightarrow q^{-1}} \in \mathcal{H}_k(q^{-1})$, (see (2.3.2)) is well defined, then*

$$\overset{*}{A}{}^{(k)} \;:=\; \rho_{\overset{*}{R}}(a^{(k)}|_{q\leftrightarrow q^{-1}}) \,=\, \Upsilon_P^{(k)}\, A^{(k)}\, \Upsilon_P^{(k)}\,, \tag{B.2}$$

$$\overset{*}{R}_i\, \overset{*}{A}{}^{(k)} \;=\; -q\, \overset{*}{A}{}^{(k)}\,, \quad \forall\, i=1,\dots,k-1\,. \tag{B.3}$$

Here $\Upsilon_P^{(k)} = (\Upsilon_P^{(k)})^{-1}$ is a particular $R=P$ case of an operator $\Upsilon_R^{(k)} \in \mathrm{Aut}(V^{\otimes k})$, defined inductively for any R*-matrix R*

$$\Upsilon_R^{(1)} := 1\,, \quad \Upsilon_R^{(k+1)} := \bigl(R_1R_2\dots R_k\bigr)\,\Upsilon_R^{(k)} \,=\, \Upsilon_R^{(k)}\bigl(R_k\dots R_2R_1\bigr) \quad \forall\, k=2,3,\dots\,. \tag{B.4}$$

This operator performs reflection of the indices of the R*-matrices*

$$R_i\,\Upsilon_R^{(k)} \,=\, \Upsilon_R^{(k)}\, R_{k-i} \qquad \forall\, i,k:\; 1\le i<k\,. \tag{B.5}$$

The particular element $\Upsilon_P^{(k)}$ enjoys also relations

$$R_i\,\Upsilon_P^{(k)} \;=\; \Upsilon_P^{(k)}\,(PRP)_{k-i} \quad \forall\, i,k:\; 1\le i<k\,, \quad \forall\ \text{R-matrix } R, \tag{B.6}$$

$$M_i\,\Upsilon_P^{(k)} \;=\; \Upsilon_P^{(k)}\, M_{k-i+1} \qquad \forall\, i,k:\; 1\le i\le k\,, \quad \forall\, M\in \mathrm{End}_W(V)\,, \tag{B.7}$$

where W is an arbitrary $\mathbb{C}$-linear space.

Proof: The first equality in both formulas (B.1) is proved in a more general setting in [OP.05], lemma 3.6 c). The second equality is proved in [Is.04], section 3.1, proposition 2. Relations (B.5) and (B.6) for matrices $\Upsilon_R^{(k)}$, $\Upsilon_P^{(k)}$ follow directly from (2.2.4) and from equalities

$$R_1P_2P_1 \,=\, P_2P_1R_2\,, \quad R_2P_1P_2 \,=\, P_1P_2R_1\,. \tag{B.8}$$

Equalities (B.7) are obvious. Relations (B.3) are byproducts of (B.2) and (2.3.4). The second equality in (B.2) follows from (2.3.2), (2.3.3), (B.6), and the Hecke relation for $\overset{*}{R}$: $\overset{*}{R} = PRP - (q - q^{-1})I$. ■

Lemma B.2. *Let M be a matrix of left invariant vector fields for the Hecke type HD algebra $\mathfrak{D}\mathfrak{G}[R,\gamma]$. For any $i \geq 1$ and $j \geq 0$ one has*

$$\Bigl(M_{\bar{\overset{*}{1}}}\overset{*}{J}_1\Bigr)\Bigl(M_{\bar{\overset{*}{2}}}\overset{*}{J}_2\Bigr)\dots\Bigl(M_{\bar{\overset{*}{i}}}\overset{*}{J}_i\Bigr)\, I_{i+1,\dots,i+j} \tag{B.9}$$

$$= \gamma^{i(i+1)}\,\mathrm{Tr}_{R\,(i+j+1,\dots,2i+j)}\Bigl(\Upsilon_P^{(i+j)}\Upsilon_P^{(2i+j)}(LT)_i\dots(LT)_1\Upsilon_{\overset{*}{R}}^{(2i)}(T_i\dots T_1)^{-1}\Upsilon_P^{(2i+j)}\Upsilon_P^{(i+j)}\Bigr),$$

where $I_{i+1,\dots,i+j}$ is the identity operator acting in the component spaces V with labels $i+1,\dots,i+j$.

Proof: Consider following sequence of transformations

$$\begin{aligned} M_{\bar{\overset{*}{1}}}\overset{*}{J}_1 &= M_1 = (T^{-1}LT)_1 = \mathrm{Tr}_{R\,(2)}\Bigl(\underline{T_1^{-1}R_1L_1}T_1\Bigr) = \gamma^2\,\mathrm{Tr}_{R\,(2)}\Bigl(L_2\underline{T_1^{-1}R^{-1}T_1}\Bigr)\\ &= \gamma^2\,\mathrm{Tr}_{R\,(2)}\Bigl((LT)_2\,R_1^{-1}\,T_2^{-1}\Bigr) = \gamma^2\,\mathrm{Tr}_{R\,(2)}\Bigl(P_1(LT)_1\,\overset{*}{R}_1\,T_1^{-1}P_1\Bigr). \end{aligned} \tag{B.10}$$

Here in the first line we transform underlined expressions using (3.3.1) and (3.1.1), and in the last line we apply the definition (3.4.5). Relation (B.10) reproduces formula (B.9) for $i = 1$ and $j = 0$. By a repeated application of formula (c.f. with (2.2.10))

$$\mathrm{Tr}_{R\,(j+1)}(P_jXP_j) = I_j\ \mathrm{Tr}_{R\,(j)}(X) \qquad \forall\, X \in \mathrm{End}_W(V^{\otimes j}). \tag{B.11}$$

we can rewrite it as (B.9) with $i = 1$ and arbitrary $j > 0$

$$\Bigl(M_{\bar{\overset{*}{1}}}\overset{*}{J}_1\Bigr)\, I_{2,\dots j+1} = \gamma^2\,\mathrm{Tr}_{R\,(j+2)}\Bigl((P_{j+1}\dots P_1)(LT)_1\,\overset{*}{R}_1\,T_1^{-1}(P_1\dots P_{j+1})\Bigr). \tag{B.12}$$

In a similar way, for any value of i relations (B.9) with $j > 0$ follow from that with $j = 0$ by a repeated application of (B.11). Therefore, it is enough considering the case $j = 0$.

Using relations (B.12) and (B.8) we can rewrite an expression $M_{\bar{\overset{*}{i}}}\overset{*}{J}_i$ in a following way

$$\begin{aligned} M_{\bar{\overset{*}{i}}}\overset{*}{J}_i &= (\overset{*}{R}_{i-1}\dots\overset{*}{R}_1)\,M_1\,(\overset{*}{R}_1\dots\overset{*}{R}_{i-1})\\ &= \gamma^2\,\mathrm{Tr}_{R\,(i+1)}\Bigl((P_i\dots P_2P_1)(LT)_1\,(\overset{*}{R}_i\dots\overset{*}{R}_2\overset{*}{R}_1\overset{*}{R}_2\dots\overset{*}{R}_i)\,(T_1)^{-1}(P_1P_2\dots P_i)\Bigr). \end{aligned} \tag{B.13}$$

Now we are ready to prove formula (B.9) by induction on i. Assuming that (B.9) with $j = 1$ is valid for the product of $(i-1)$ factors we transform the product of i factors

$$\begin{aligned} \Bigl(M_{\bar{\overset{*}{1}}}\overset{*}{J}_1\Bigr)\Bigl(M_{\bar{\overset{*}{2}}}\overset{*}{J}_2\Bigr)\dots\Bigl(M_{\bar{\overset{*}{i}}}\overset{*}{J}_i\Bigr) &= \gamma^{(i-1)i}\,\mathrm{Tr}_{R\,(i+1,\dots 2i-1)}\Bigl(\Upsilon_P^{(i)}\Upsilon_P^{(2i-1)}(LT)_{i-1}\dots(LT)_1\,\times\\ &\times\Upsilon_{\overset{*}{R}}^{(2i-2)}(T_{i-1}\dots T_1)^{-1}\Upsilon_P^{(2i-1)}\Upsilon_P^{(i)}\Bigr)\Bigl(M_{\bar{\overset{*}{i}}}\overset{*}{J}_i\Bigr) \end{aligned}$$

Next, we apply formulas (B.6), (B.7) to move the last factor $(M_{\bar{\overset{*}{i}}}\overset{*}{J}_i)$ in this expression leftwards. The result is

$$\begin{aligned} &= \gamma^{(i-1)i}\,\mathrm{Tr}_{R\,(i+1,\dots 2i-1)}\Bigl(\Upsilon_P^{(i)}\Upsilon_P^{(2i-1)}(LT)_{i-1}\dots(LT)_1\Upsilon_{\overset{*}{R}}^{(2i-2)}(T_{i-2}\dots T_1)^{-1}\,\times\\ &\times\Bigl(T_1^{-1}(M_{\bar{\overset{*}{i}}}\overset{*}{J}_i)^{\uparrow 1}\Bigr)^{\uparrow(i-2)}\Upsilon_P^{(2i-1)}\Upsilon_P^{(i)}\Bigr), \end{aligned}$$

where we have used identities $(T_{i-1})^{-1} = (T_1^{-1})^{\uparrow(i-2)}$ and $(M_{\overline{\overset{*}{i}}}\overset{*}{J}_i)^{\uparrow(i-1)} = ((M_{\overline{\overset{*}{i}}}\overset{*}{J}_i)^{\uparrow 1})^{\uparrow(i-2)}$ to arrange the terms $(T_{i-1})^{-1}$ and $(M_{\overline{\overset{*}{i}}}\overset{*}{J}_i)^{\uparrow(i-1)}$ in a suitable way. Next, we use formula (3.4.8) for their permutation and then, in a similar way we move term $(M_*\overset{*}{J})$ to the left of all the terms $(T_*)^{-1}$:

$$\cdots = \gamma^{(i-1)i+2(i-1)}\,\mathrm{Tr}_{R\,(i+1,\dots 2i-1)}\Big(\Upsilon_P^{(i)}\Upsilon_P^{(2i-1)}(LT)_{i-1}\dots(LT)_1\Upsilon_{\overset{*}{R}}^{(2i-2)}\times$$
$$\times(M_{\overline{\overset{*}{2i-1}}}\overset{*}{J}_{2i-1})(T_{i-1}\dots T_1)^{-1}\Upsilon_P^{(2i-1)}\Upsilon_P^{(i)}\Big).$$

Now we substitute the expression (B.13) for $(M_{\overline{\overset{*}{2i-1}}}\overset{*}{J}_{2i-1})$

$$= \gamma^{i(i+1)}\,\mathrm{Tr}_{R\,(i+1,\dots 2i)}\Big(\Upsilon_P^{(i)}\Upsilon_P^{(2i-1)}(LT)_{i-1}\dots(LT)_1\Upsilon_{\overset{*}{R}}^{(2i-2)}(P_{2i-1}\dots P_1)(LT)_1\times$$
$$\times(\overset{*}{R}_{2i-1}\dots\overset{*}{R}_2\overset{*}{R}_1\overset{*}{R}_2\dots\overset{*}{R}_{2i-1})(T_1)^{-1}(P_1\dots P_{2i-1})(T_{i-1}\dots T_1)^{-1}\Upsilon_P^{(2i-1)}\Upsilon_P^{(i)}\Big)$$

and move the term $(P_{2i-1}\dots P_1)$ leftwards and the term $(P_1\dots P_{2i-1})$ rightwards close to the terms $\Upsilon_P^{(2i-1)}$. Finally, using (B.4) we complete the calculation

$$= \gamma^{i(i+1)}\,\mathrm{Tr}_{R\,(i+1,\dots 2i)}\Big(\Upsilon_P^{(i)}\Upsilon_P^{(2i)}(LT)_i\dots(LT)_1\Upsilon_{\overset{*}{R}}^{(2i)}(T_i\dots T_1)^{-1}\Upsilon_P^{(2i)}\Upsilon_P^{(i)}\Big),$$

Here we transformed terms containing $\overset{*}{R}$ in a following way

$$\Upsilon_{\overset{*}{R}}^{(2i-2)\uparrow 1}(\overset{*}{R}_{2i-1}\dots\overset{*}{R}_2\overset{*}{R}_1\overset{*}{R}_2\dots\overset{*}{R}_{2i-1}) = \Upsilon_{\overset{*}{R}}^{(2i-2)\uparrow 1}(\overset{*}{R}_1\dots\overset{*}{R}_{2i-2}\overset{*}{R}_{2i-1}\overset{*}{R}_{2i-2}\dots\overset{*}{R}_1)$$
$$= (\overset{*}{R}_1\dots\overset{*}{R}_{2i-1})\Upsilon_{\overset{*}{R}}^{(2i-2)}(\overset{*}{R}_{2i-2}\dots\overset{*}{R}_1) = (\overset{*}{R}_1\dots\overset{*}{R}_{2i-1})\Upsilon_{\overset{*}{R}}^{(2i-1)} = \Upsilon_{\overset{*}{R}}^{(2i)}. \qquad \blacksquare$$

Lemma B.3. *The operators J_k (3.3.8) and $\Upsilon^{(k)}$ (B.4) associated with a skew invertible* R-*matrix R satisfy relations*

$$\mathrm{Tr}_{R\,(i+1,\dots,2i)}\Upsilon_R^{(2i)} = \Big(\Upsilon_R^{(i)}\Big)^4 = \Big(J_1J_2\dots J_i\Big)^2. \tag{B.14}$$

Proof: Calculation proceeds as follows:

$$\begin{aligned}
\mathrm{Tr}_{R\,(i+1,\dots,2i)}\Upsilon_R^{(2i)} &= \mathrm{Tr}_{R\,(i+1,\dots,2i-1)}\Big(\Upsilon_R^{(2i-1)}\,(\mathrm{Tr}_{R\,(2i)}R_{2i-1})\,(R_{2i-2}\dots R_1)\Big)\\
&= \mathrm{Tr}_{R\,(i+1,\dots,2i-1)}\Big((R_1\dots R_{i-1})\,\Upsilon_R^{(2i-1)}\,(R_{i-1}\dots R_1)\Big)\\
\dots &= (R_1\dots R_{i-1})^i\,\Upsilon_R^{(i)}\,(R_{i-1}\dots R_1)(R_{i-2}\dots R_1)\dots(R_2R_1)R_1\\
&= \Big(J_1J_2\dots J_i\Big)\Big(\Upsilon_R^{(i)}\Big)^2 = \Big(\Upsilon_R^{(i)}\Big)^4.
\end{aligned}$$

Here in passing to the second line we calculated the R-trace $\mathrm{Tr}_{R\,(2i)}$ with the help of (2.2.8) and then used (B.5) to move $(i-1)$ R-matrices to the left of the term $\Upsilon_R^{(2i-1)}$. Expression in the third line results from a similar calculations of the R-traces $\mathrm{Tr}_{R\,(2i-1)}, \dots, \mathrm{Tr}_{R\,(i+1)}$, consecutively. Equalities in the last line result from rearranging factors of the product $(R_1\dots R_{i-1})^i$.
$\blacksquare$

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