RECENT PROGRESS IN TRANSCENDENCE THEORY

G.Wüstholz
Max-Planck-Institut für
Mathematik
Gottfried-Claren-Str.26
53 Bonn 3

MPI/SFB 17-83

1. Introduction

In the last few years, methods from commutative algebra, algebrai geometry, complex analysis of several variables, and even cohomology theory have been used to solve problems in transcendence theory which had long been regarded as inaccessible. One of the central objects is the so-called zero-estimates, or more generally multiplicity-estimate on certain algebraic objects. Using these estimates and the technique developed to obtain them, many open problems in transcendence theory have been solved. On the other hand, many new problems have arisen, a it seems that transcendence theory has finally become a theory. In th article, we would like to describe this developement, and for this we have to begin with a short description of the theory of multiplicity-estimates. Let us begin with two very elementary examples to desribe what we mean by multiplicity-estimates.

Example 1. Let $a_1,\dots,a_l$ be pairwise different complex numbers and $n_1,\dots,n_l$ positive integers. Then it is of course always possible to construct a polynomial $F(X)$ in one variable of degree

$$n \geq n_1+\dots+n_l$$

such that F vanishes at the points $a_1,\dots,a_l$ to order at least $n_1,\dots,n_l$. On the other hand it is well-known that a polynomial $F(X)$ of degree n that has $n+1$ zeroes (counted with multiplicities) i necessarily the zero-polynomial.

This is a trivial example of what we mean by multiplicity estimat It generalizes to several variables in the following way.

Example 2. Let $P_1,\dots,P_l$ be given points in $\mathbb{C}^d$ for $d \geq 1$ which pairwise different and again let $n_1,\dots,n_l$ be positive integers. We want to construct a polynomial $F(X_1,\dots,X_d)$ of a given degree D t vanishes in $P_1,\dots,P_l$ with multiplicity $n_1,\dots,n_l$ respectively. It turns out that this is possible as soon as

$$N = \binom{n_1+d-1}{d} + \dots + \binom{n_l+d-1}{d} < \binom{D+d}{d}.$$

On the other hand it is not clear at all that the second implication Example 1 should hold. It is in general not true in the case $d \geq 2$ that a polynomial $F(X_1,\dots,X_d)$ in d variables and of degree D t vanishes at all these points with multiplicity $n_1, \ldots, n_l$ is necessarily the zero polynomial if only

$$N \geq \binom{D+d}{d}$$

holds.

These two examples illustrate that the situation in several variables is much more complicated than in one variable. Nevertheless situations like this have been studied quite extensively by many authors (Waldschmidt [Wa1], Chudnovsky [Ch1], Demailly [D], Masser [Ma1], [Ma2], Wüstholz [Wü1], Philippon [Ph1], Bombieri [Bo], Esnault-Vieweg [E-V1], [E-V2]) under very different points of view. It has turned out that the answer for such types of questions depends very much on the distribution of the points in question. Fortunately in most cases arrising in transcendence theory we are concerned with a much more special situation which we want to describe now.

2. Zero estimates on group varieties

We consider a commutative algebraic group (group variety) G defined over the field of complex numbers, for example $\mathbb{G}_a$, the additive group, $\mathbb{G}_m$, the multiplicative group, E, an elliptic curve, or an arbitrary abelian variety. An arbitrary algebraic group G is then obtained by forming products or so-called extensions. Such an algebraic group G is a quasi-projective variety and can be embedded in some projective space in a nice way (see [Se] and [Fa-Wü])

$$G \hookrightarrow \mathbb{P}^N$$

for some N. For this one first compactifies G to $\overline{G}$ in such a way that G operates on $\overline{G}$ and after this embeds $\overline{G}$ into $\mathbb{P}^N$ by means of a certain very ample divisor. We denote the dimension of G by d. Then G replaces the space $\mathbb{C}^d$ of Example 2.

Next we take a finite set $X \subset G$ of elements of G that contains the neutral element of G. Then for integers $r \geq 1$ we define the sets rX as

$$X + \ldots + X \quad (\text{r-times}).$$

Then dX replaces the set of points $\{P_1, \ldots, P_l\}$ in Example 2. Now we define for integers r with $1 \leq r \leq d$ the numbers $Q_r(X)$ as

$$Q_r(X) = \begin{cases} |X| & \text{if } G \text{ has no algebraic subgroup of codimension } r \\ \min\limits_{\substack{H \subset G \\ H \text{ algebraic} \\ \text{cod } H = r \\ H \text{ connected}}} |X+H/H| & \text{otherwise.} \end{cases}$$

The numbers $Q_r(X)$ measure the distribution of the set X in G. The Example 1 and Example 2 generalize in the following way.

<u>Theorem 1</u> ([Ma-Wü1], [Ma-Wü2]). Let $P(X_0,\ldots,X_N)$ be a homogeneous polynomial of degree at most D that vanishes on dX. Then if

$$Q_r(X) > \deg G \cdot D^r \qquad (1 \leq r \leq d)$$

the polynomial P vanishes on all of G.

Here $\deg G$ is the degree of G in $\mathbb{P}^N$.

<u>Remarks</u>. 1. A geometric proof of a weaker version of Theorem 1 was given by Moreau ([Mo1] and [Mo2]). Unfortunately this proof does not seem to generalize to zeroes counted with multiplicities.
2. Various extensions and modifications of this result have been prove now. So for example there exists a version which also takes into accou different degrees ([Ma-Wü2], see also [Wü4]). Another modification are the so-called "zero estimates with knobs on" used to prove resul on large transcendence degree (see section 5 and [Ma-Wü3]). They also appear very useful for obtaining transcendence measures and measures for algebraic independence.

3. Multiplicity estimates on group varieties

As in the two examples at the beginning we now allow the zeroes t have multiplicities. We can do this even more generally than in these examples. Instead of taking into account multiplicities in all directions we take multiplicities only with respect to a certain subset of all directions. First results in this direction were obtained by Neste renko [Ne1] and by Brownawell and Masser [Br-Ma], [Br1], [Br2]. But we are considering the following general situation.

Let $A \subseteq G$ be an analytic subgroup of G and let $a = \dim A$ be the dimension of A. This is the image of an analytic homomorphism

$$\Phi: \mathbb{C}^a \to G(\mathbb{C})$$

from the complex a-space to the complex Lie-group $G(\mathbb{C})$. This is the set of complex valued points of G. (It would be more precise but not more illuminating to work here with the notion of group schemes.) Such an analytic subgroup A provides us via its Lie-algebra with derivations $\Delta_1,\ldots,\Delta_a$ on G. This enables us to define the order of vanishing in the following way. We say that a homogeneous polynomial $P(X_0,\ldots,X_N)$ vanishes to order at least T along A (or Φ) at a point g in G with $X_0(g)\neq 0$ if

$$\Delta_1^{t_1}\ldots\Delta_a^{t_a} P(1,x_1/x_0,\ldots,x_N/x_0)(g) = 0$$

for all non-negative integers $t_1,\ldots,t_d$ with $t_1+\ldots+t_a < T$. Here $x_0,\ldots,x_N$ are the restrictions of the coordinate functions $X_0,\ldots,X_N$ on $\mathbb{P}^N$ to G.

Obvious examples force us in the same way as with the points to measure the "distribution" of the derivations correspondig to A. For this we define integers $\tau(V)$ in the following way. For algebraic subvarieties V of G with $V \cap A \neq \emptyset$ we define

$$\tau(V) = \mathrm{cod}_A V \cap A$$

where cod_A denotes the codimension in A. Then for integers r with $1 \leq r \leq d$ we put

$$\tau_r = \min_{\mathrm{cod}\, V = r} \tau(V)$$

and we can state the following fundamental result.

<u>Theorem 2</u> ([Wü2]). There exists a positive constant c with the following property. Let $P(X_0,\ldots,X_N)$ be a homogeneous polynomial of degree at most equal to D that vanishes in dX along A with multiplicity at least T. Then if

$$(T/n)^{\tau_r}\cdot Q_r(X) > (cD)^r \qquad (1 \leq r \leq n)$$

the polynomial P vanishes on all of G.

Remark. It can be shown that Theorem 3 is best possible up to a numerical constant. It is very likely that Teorem 2 has the same property. It would be interesting to verify this.

we now illustrate our result with an explicit example.

Example 3. Let E be an elliptic curve and $K = (\mathrm{End}\, E)\otimes\mathbb{Q}$ the field of complex multiplications, where End E is the ring of endomorphisms of E. Associated with E is a Weierstraß elliptic function $\wp(z)$. Then let $x_1,\ldots,x_n$ be complex numbers which generate over K a vector space of dimension n. This implies, as one verifies without any difficulties, that the functions

$$\wp(z_1),\ \ldots\ ,\ \wp(z_n),\ \wp(x_1z_1+\ldots+x_nz_n)$$

are algebraically independent over $\mathbb{C}$ or even over $\mathbb{C}(z_1,\ldots,z_n)$. This also follows from a general result of Brownawell and Kubota [Br-Ku] . Now we take a polynomial $P(X_1,\ldots,X_{n+1})$ of degree D and define the function $\Phi(z_1,\ldots,z_n)$ by

$$\Phi(z_1,\ldots,z_n) = P(\wp(z_1),\ldots,\wp(z_n),\wp(x_1z_1+\ldots+x_nz_n)).$$

This function is identically zero only if the polynomial P is the zero polynomial. Let now $w = (w_1,\ldots,w_n)$ be a point in $\mathbb{C}^n$ such that Φ is defined for all non-zero integer multiples of w and suppose that for some integers $S \leq 1$, $T \leq 1$ we have

$$\left(\frac{\partial}{\partial z_1}\right)^{t_1}\cdots\left(\frac{\partial}{\partial z_n}\right)^{t_n}\Phi(sw) = 0 \qquad (1 \leq s \leq S,\ 0 \leq t_1+\ldots+t_n < T).$$

Then we have the following corollary to Theorem 2.

Corollary. If $(T/n)^n(S/n) > (cD)^{n+1}$ then the polynomial P is the zero polynomial.

Here the coefficients τ_r, as well as the n other inequalities, do not appear anymore. The reason for this is that the conditions on the numbers $x_1,\ldots,x_n$ imply that $\tau_r \geq r$ for $1 \leq r \leq n$ and $\tau_{n+1} = n$. It then follows easily that the missing inequalities are consequences of the given one. What this example also shows is that the conditions of

Theorem 2, which seem to be very abstract, can be verified in all given situations without much difficulty. It reduces to a problem in linear algebra.

4. Small transcendence degree

In 1949 Gel'fond developed a new method to prove algebraic independence of two numbers out of a certain given set of numbers. The central point in his method was a general criterion for algebraic independence of two complex numbers, called "Gel'fond's criterion". In practice this criterion was until recently only applicable to numbers connected with the exponential function. The most striking result here was Gel'fond's result on the algebraic independence of the numbers

$$\alpha^{\beta},\ \alpha^{\beta^2}$$

where $\alpha \neq 0,1$ is algebraic and β cubic over the rationals.

One of tthe motivations for developing the theory of zero and multiplicity estimates was to prove the elliptic analogues of these types of results. In the same way as in the exponential case (for a complete account of this see [Wa2]) it is possible to embed this kind of results in a more general context. Here we will content ourselves with giving the elliptic analogue of Gel'fond's result. The general result can be found in [Ma-Wü4].

For this let $\wp(z)$ be a Weierstraß elliptic function with algebraic invariants g_2 and g_3. Denote as before by K its field of complex multiplications and denote by d its degree over $\mathbb{Q}$. Then it is known that $d = 1,2$. Let then β be an algebraic number of degree $l=\frac{2(d+1)}{d}$ over K and u a complex number such that $\wp(u)$ is algebraic. Then we have the following result.

<u>Theorem 3</u> ([Ma-Wü4]). At least two of the numbers

$$\wp(\beta u), \ldots, \wp(\beta^{l-1}u)$$

are algebraically independent (over $\mathbb{Q}$).

As an immediate consequence of this theorem we obtain in the case $d=2$ the elliptic analogue of Gel'fond's result.

The proof of this and related results will be given in [Ma-Wü5].

5. Large transcendence degree

The method initiated by Gel'fond was considerably extended by sev ral authors to obtain algebraic independence of more than two numbers connected with the exponential function (Smelev [Sm], Waldschmidt [Wa2]) but these were still small transcendence degree results. It was G.V. Chudnovsky who then developed a method for proving the firs result on large transcendence degree in this context ([Ch2]). He pr that the field generated over $\mathbb{Q}$ by numbers of the form

$$u_1,\ldots,u_n,v_1,\ldots,v_m,e^{u_1v_1},\ldots,e^{u_nv_m}$$

has transcendence degree which tends to infinity if m and n tend infinity. Here one has to impose certain measures of linear independe on the $u_1,\ldots,u_n$ and $v_1,\ldots,v_m$. A number of authors have developed Chudnovsky's ideas (Reyssat [R], Philippon [Ph2], Endell [E], Neste renko [N2]) and this area of research is at the moment very active (Waldschmidt [Wa3], Zhu Yao Chen [Wa-Z] and Endell). A very natura and interesting question of course is to extend this sort of results arbitrary algebraic groups. The main obstacle for doing this was the lack of the lemma of Tijdeman [T] in the general situation but sinc we have the zero estimates and the multiplicity estimates and both al "with knobs on" it is now possible to consider the general situation The first step in this direction is due to D.W. Masser and the author [Ma-Wü3] where one elliptic curve is considered and an elliptic analo of Chudnovsky's result is obtained.

In order to state it let E be an elliptic curve defined over th field of algebraic numbers and $\wp(z)$ an associated Weierstraß ellipt function with algebraic invariants g_2 and g_3. We assume that E h no complex multiplication. Then let $u_1,\ldots,u_n$ and $v_1,\ldots,v_m$ be com numbers linear independent over the rationals respectively such that there exists a positive real number κ with the property that

$$|s_1u_1+\ldots+s_nu_n| > \exp(-S^{\kappa})$$

and

$$|t_1v_1+\ldots+t_mv_m| > \exp(-T^{\kappa})$$

for all integers $s_1,\ldots,s_n,t_1,\ldots,t_m$ with $S = |s_1|+\ldots+|s_n|$ and T $|t_1|+\ldots+|t_m|$ sufficiently large. Then we have the following result.

Theorem 4 ([Ma-Wü3]). Suppose that the integers m and n satisfy

$$mn \geq \{2^{k+1}(k+7)+4\kappa\}(m+2n)$$

for some integer $k \geq 0$. Then the transcendence degree of the field generated over the rationals by the numbers

$$\wp(u_i v_j) \qquad (1 \leq i \leq n;\ 1 \leq j \leq m)$$

is at least equal to k.

This result is of course only relatively weak and there might be some chance to improve it slightly using the result of Waldschmidt and Zhu Yao Chen already mentioned. Of course on can also ask for measures of algebraic independence. The general problem in this direction is the following one.

Problem. Let G be a commutative algebraic group defined over a finitely generated subfield K of the field of complex numbers and let Γ be a finitely generated subgroup of the tangent space $T(G)$ of G in the neutral element of G. Determine the transcendence degree of the smallest field L contained in the field of complex numbers such that

$$\exp_G(\Gamma) \subseteq G(L).$$

Of course in many special cases the answer is trivial. But in general this problem seems to be very difficult. For example it contains Schanuel's conjecture with $G = \mathbb{G}_a{}^n \times \mathbb{G}_m{}^n$ and $\Gamma = \mathbb{Z}\cdot(x_1,\ldots,x_n,x_1,\ldots,x_n)$.

6. Results of Lindemann's type

In 1882 Lindemann [Li] proved his famous theorem which is sill one of the most beautiful results in transcendence theory. This result solved the old greek problem of squaring the circle. The theorem says that if $\alpha_1,\ldots,\alpha_n$ are pairwise different algebraic numbers then the numbers

$$e^{\alpha_1},\ldots,e^{\alpha_n}$$

are linearly independent over the field of algebraic numbers. It is an immediate consequence of this result that if $\beta_1,\ldots,\beta_m$ are Q-linearly independent algebraic numbers then the numbers

$$e^{\beta_1},\ldots,e^{\beta_m}$$

are algebraically independent over $\mathbb{Q}$. This result created a big theor developed by C.L. Siegel [Si1] in 1929 and later by Shidlovsky [Sh] and his school, namely the theory of so-called E- and G-functions. He one is able to prove the algebraic independence of certain values of th E- and G-functions.

It seemed hopeless to obtain results of this sharpness for other classes of functions. But recently the newly developed theory of zero and multiplicity estimates together with effective algebraic constructions made it possible to prove in the complex multiplication case the abelian analogue of Lindemann's result on algebraic independence. The general result is very complicated to state. Therefore we restrict our selves for simplicity to the elliptic case. For this let $\wp(z)$ be as usual a Weierstraß elliptic function with algebraic invariants g_2 an g_3. Furthermore we assume that the associated elliptic curve has compl multiplication. As usual we denote by K the associated imaginary qua dratic field.

Theorem 5 ([Wü3], [Wü4]). Let $\beta_1,\ldots,\beta_m$ be algebraic numbers linea ly independent over K. Then the numbers

$$\wp(\beta_1),\ldots,\wp(\beta_m)$$

are algebraically independent over $\mathbb{Q}$.

A slightly different proof of this result was given by Philippon [Ph4] using some ideas of the author. We should perhaps remark that ou proof also gives the best possible (up to a ε) measure for algebraic independence. Let ε be a positive number.

Theorem 6 (see [Wü4]). Under the same hypothesis as before let $0 \neq P(X_1,\ldots,X_m)$ be a polynomial with integer coefficients of degree d(P and height H(P). Then

$$\log |P(\wp(\beta_1),\ldots,\wp(\beta_m))| > -c\, d(P)^{n+\varepsilon}\log H(P)$$

where c is an effectively computable positive constant.

These results were proved for $m = 2,3$ by G.V.Chudnovsky [Ch3].

A somewhat related problem was posed by Gel'fond and by Schneider. Suppose that $\alpha \neq 0,1$ is an algebraic number and $\beta_1,\dots,\beta_m$ are algebraic numbers that are linearly independent over $\mathbb{Q}$. Then the problem is to show that the numbers

$$\alpha^{\beta_1},\dots,\alpha^{\beta_m}$$

are algebraically independent.

We do not know much on this problem. We know only in special situations from the result of Chudnovsky that the transcendence degree of the field generated by numbers of this type tends to infinity with m. And at present we do not know how to improve this in a significant way. Another special case which we know is the already mentioned result of Gel'fond.

Surprisingly the situation is quite different in the case of abelian functions. Here we know very sharp (but not yet best possible) results. They were proved by Philippon [Ph3] again relying on multiplicity estimates on group varieties. Here we will state only the simplest case of a more general result, and only the case of complex multiplication, which yields the best result. For this let $\wp(z)$ again be a Weiersraß elliptic function with algebraic g_2 and g_3. Let K have its usual meaning. Furthermore let F be an algebraic extension of K of degree m and $\beta_1,\dots,\beta_m$ be a K-basis of F. Finally let u be a complex number such that $\wp(\beta_i u)$ is finite for $1 \leq i \leq m$. Then the following holds.

<u>Theorem 7</u> ([Ph3]). At least $\frac{m}{2} - 1$ of the numbers

$$\wp(\beta_1 u),\dots,\wp(\beta_m u)$$

are algebraically independent.

Both, Theorem 5 and Theorem 7, give a partial answer to the problem stated at the end of the last section. We end this section with the following

<u>Conjecture</u>. Let $E_1,\dots,E_n$ be pairwise non-isogeneous elliptic curves defined over the field of algebraic numbers and let $G = \mathbb{G}_m \times E_1 \times \dots \times E_n$. Let $u \in T(G)(\mathbb{Q})$ be a rational tangent vector. Then the dimension of

the Zariski closure X of $\exp_G(u)$ with respect to the $\overline{\mathbb{Q}}$-topolgy is equal to n+1. In other words $e, \wp_1(1), \ldots, \wp_n(1)$ are algebraically in dependent .

7. Analytic subgroups of algebraic groups

Since A.Baker had proved his famous theorem on linear forms in logarithms (see for this [Ba1]) a great number of authors have tried to prove analoguous results for elliptic and more generally abelian logarithms. These studies were initiated by Masser [Ma3] and further developed by Masser, Lang [L] and Coates and Lang [C-L]. It turne out that the crucial point here was a general multiplicity estimate which was not available. Therefore the authors had to use adhoc argume to overcome these difficulties in very special situations. In addition the results that could be derived were rather weak except in the ellip tic case with complex multiplication (Masser and Anderson). More recently Bertrand and Masser [B-M] succeeded to treat the elliptic cas completely using a quite different method. Very surprisingly but in so sense quite naturally they could apply the criterion of Schneider and Lang in this situation. Nevertheless their approach is not very satisfactory for different reasons.

Since the necessary multiplicity estimates are now available (The rem 2) we obtain a completely general and universal result that cov all problems in this field.

In order to state it let G be as usual a commutative algebraic group defined over $\overline{\mathbb{Q}}$ and $A \subset G$ an analytic subgroup defined over By this we mean that A is defined in the tangent space T(G) of G in the neutral element, which is in a natural way a $\overline{\mathbb{Q}}$-vector space, by $\overline{\mathbb{Q}}$-subspace T(A). Then the exponential map $\exp_G\colon T(G) \to G$ induces a local diffeomorphism between T(A) and A (or more precisely between their complex valued points $T(A)(\mathbb{C})$ and $A(\mathbb{C})$). Then we have the fol lowing result.

<u>Theorem 8</u> ([Wü5]). Suppose that A contains a non-trivial algebraic point, i.e. $A(\overline{\mathbb{Q}}) \neq 0$. Then there exists an algebraic subgroup H of G with the following properties:

(i) H is defined over $\overline{\mathbb{Q}}$,

(ii) $\dim H > 0$,

(iii) $H \subseteq A$.

Remark. 1. It is obvious that the existence of such a subgroup H of G implies that $A(\overline{\mathbb{Q}}) \neq 0$. Hence the condition is also necessary. ____

2. As was pointed out by Gabber the theorem can also be stated as $\overline{\overline{A(\overline{\mathbb{Q}})}} \subseteq A$. Here the second bar denotes the Zariski closure with respect to the $\overline{\mathbb{Q}}$-topology. In other words the Zariski closure (with respect to $\overline{\mathbb{Q}}$) of $A(\overline{\mathbb{Q}})$ is contained in A.

It is not very difficult to deduce Baker's theorem (in its qualitative version) from this general result. But we prefer to give the following more general result conjectured by Waldschmidt.

For this let $n,m \geq 1$ be integers and $\alpha_1,\ldots,\alpha_n$ be algebraic numbers such that $\log\alpha_1,\ldots,\log\alpha_n$ are $\mathbb{Q}$-linearly independent (for any fixed choice of the logarithms). Let further $\wp(z)$ be a Weierstraß elliptic function with algebraic g_2 and g_3 and $u_1,\ldots,u_m$ be complex numbers that are K-linearly independent ($K = (\mathrm{End}\ E)\otimes\mathbb{Q}$) and satisfy $\wp(u_1),\ldots,\wp(u_m) \in \overline{\mathbb{Q}}$. Finally let $\beta_1,\ldots,\beta_n$ and $\gamma_1,\ldots,\gamma_m$ be algebraic numbers not all zero.

Theorem 9 ([Wü6]). $\beta_1\log\alpha_1+\ldots+\beta_n\log\alpha_n+\gamma_1u_1+\ldots+\gamma_mu_m \neq 0$.

Remark. The case $\gamma_1= \ldots =\gamma_m = 0$ is Baker's theorem and for $\beta_1= \ldots =\beta_m = 0$ we obtain the theorem of Bertrand and Masser.

A nice corollary of this result is the following result which was pointed out by Masser. Here we do not assume that $\log\alpha_1,\ldots,\log\alpha_n$ are $\mathbb{Q}$-linearly independent but that the numbers 1 and the non-zero ones among $\beta_1,\ldots,\beta_n$ are $\mathbb{Q}$-linearly independent.

Corollary. The complex number

$$\alpha_1^{\beta_1}\cdots\alpha_n^{\beta_n}e^{\gamma_1u_1+\ldots+\gamma_mu_m}$$

is transcendental.

Of course it is not difficult to extend this result to abelian logarithms. Furthermore it is also possible to obtain quantitative versions of Theorem 8 nearly as precise as the bounds given by Baker for linear forms in classical logarithms. This applies for example to Siegel's theorem on integer points on curves and can be used to eliminate Roth's theorem, which is non-effective, in the proof of this theorem.

8. Periods of rational integrals.

Transcendence properties of periods of certain differential forms, especially elliptic and abelian, have been studied for a long time. Th first result in this field was proved by Lindemann who obtained the transcendency of the non-zero periods of the differential form dx/x. Then Siegel [Si2] proved the transcendency of the periods of the ell tic differential $dx/\sqrt{4x^3-4x}$. Since then many authors (Schneider [Sch1], [Sch2], [Sch3], Baker [Ba2], [Ba3], Coates [Co1] , [Co2], [Co3 Masser [Ma3], [Ma4], [Ma5], Laurent [La1], [La2] and Bertrand [Be] have obtained partial results. With the help of Theorem 8 it is now possible to prove a general result on arbitrary periods of rational li integrals. This result covers all these results just refered to.

Let X be a smooth quasiprojective variety defined over $\overline{\mathbb{Q}}$ and a closed holomorphic 1-form on X defined over $\overline{\mathbb{Q}}$. Suppose that γ i a closed path in X (in other words: γ represents an element in the first homology group $H_1(X,\mathbb{Z})$). Then we have the following result.

<u>Theorem 10</u> ([Wü5]). The rational integral

$$\int_\gamma \xi$$

is either zero or transcendental.

One obtains immediately the following Corollary.

<u>Corollary</u>. The periods of an abelian integral (first, second and thir kind) are either zero or transcendental.

At the present we can only deal with periods of 1-forms. It would be extremely interesting to extend this result as far as possible to a bitrary r-forms with $r \geq 2$. Of course it is not too hard to decide when the periods are zero. This can happen without the differential be ing exact or the path being homotopic to zero. This is done in [Wü7] in the case of arbitrary elliptic integrals where the group generated the polar divisor of the part of the third kind plays the essential ro Furthermore this is worked out in the case of abelian integrals of the second kind at most for products of elliptic curves ([Wü8]). This solv a problem of Baker.

Of course it is possible to extend Theorem 10 to non-closed pathe going from one algebraic point on X to another. One obtains then a r sult of the following type. Either the integral is transcendental or it is algebraic and one can determine why this is the case. But the details have yet to be worked out.

9. References

[Ba1] A.Baker, Transcendental number theory, Cambridge Universi Press, Cambridge (1975, 1979).

[Ba2] A.Baker, On the periods of the Weierstrass ℘-function, S posia Math. IV, INDAM Rome, 1968 (Academic Press, London 1970), 155-174.

[Ba3] A.Baker, On the quasi-periods of the Weierstrass ζ-funct Göttinger Nachr. (1969), No.16, 145-157.

[Be] D.Bertrand, Endomorphismes de groupes algébriques; applic tions arithmétiques, Progr.Math.31, 1-46 (1983).

[Be-Ma] D.Bertrand, D.W.Masser, Formes linéaires d'integrales abé liennes, CRAS Paris, 290 (1980), 725-727.

[Bo] E.Bombieri, On the Thue-Siegel-Dyson theorem, Acta Math.1 255-296 (1982).

[Br1] W.D.Brownawell, On the orders of zero of certain function Bull.Soc.Math. France, Mémoire 2 (1980), 5-20.

[Br2] W.D.Brownawell, Zero estimates for solutions of different equations, Progr.Math.31, 67-94 (1983).

[Br-K] W.D.Brownawell, K.K.Kubota, The algebraic independence of Weierstrass functions and some related numbers, Acta Arit 33 (1977), 111-149.

[Br-Ma] W.D.Brownawell, D.W.Masser, Multiplicity estimates for an lytic functions, Duke Math.J.47 (1980), 273-295.

[Ch1] G.V.Chudnovsky, Singular points on complex hypersurfaces multidimensional Schwarz Lemma, Progr.Math.12, 29-69 (198

[Ch2] G.V.Chudnovsky, Some analytic methods in the theory of tr cendental numbers, Inst.of Math.,Ukr. SSSR Acad.Sci., Pre print IM 74-8 and 74-9, Kiev (1974).

[Ch3] G.V.Choodnovsky, Algebraic independence of the values of elliptic function at algebraic points, Inv.math.61 (1980) 267-290.

[Co1] J.Coates, Linear forms in the periods of the exponential and elliptic functions, Inv.math.12 (1971), 290-299.

[Co2] J.Coates, The transcendence of linear forms in $\omega_1, \omega_2, \eta_1,$ $2\pi i$, Amer.J.Math.93 (1971), 385-397.

[Co3] J.Coates, Linear relations between $2\pi i$ and the periods two elliptic curves, Diophantine approximation and its ap cations (Acad.Press, London, 1973), 77-99.

[Co-L] J.Coates, S.Lang, Diophantine approximation on Abelian va rieties with complex multiplications, Inv.math.34 (1976), 129-133.

[D] J.P.Demailly, Formules de Jensen en plusieurs variables et applications arithmétiques, Bull.Soc.Math. France 110 (1982), 75-102.

[E] R.Endell, Zur algebraischen Unabhängigkeit gewisser Werte der Exponentialfunktion, this volume.

[Es-V1] H.Esnault, E.Vieweg, Sur une minoration du degré d'hypersurfaces s'annulant en certains points, Math.Ann.263, 75-86 (1983).

[Es-V2] H.Esnault, E.Vieweg, Dyson's lemma for polynomials in several variables, preprint (1983).

[F-Wü] G.Faltings, G.Wüstholz, Einbettungen kommutativer algebraischer Gruppen und einige ihrer Eigenschaften, Manuskript.

[L] S.Lang, Diophantine approximation on abelian varieties with complex multiplication, Adv. in Math.17 (1975), 281-336.

[La1] M.Laurent, Transcendance de périodes d'integrales elliptiques, J. reine u. angew. Math. 316, 122-139 (1980).

[La2] M.Laurent, Transcendance de périodes d'integrales elliptiques II, J. reine u. angew. Math. 333, 144-161 (1982).

[Li] F.Lindemann, Über die Zahl π, Math.Ann. 20, 213-223 (1882).

[Ma1] D.W.Masser, A note on multiplicities of polynomials, Publ. Math.Univ.Paris VI (1981).

[Ma2] D.W.Masser, Interpolation on group varieties, Progr.Math.31, 151-171 (1983).

[Ma3] D.W.Masser, Elliptic functions and transcendence, SLN 437 (1975).

[Ma4] D.W.Masser, Some vector spaces associated with two elliptic functions, Transcendence theory: advances and applications, Acad.Press, London, 101-119 (1977).

[Ma5] D.W.Masser, The transcendence of certain quasi-periods associated with Abelian functions, Comp.Math.35 (1977), 239-258.

[Ma-Wü1] D.W.Masser, G.Wüstholz, Zero estimates on group varieties I, Inv.math.64 (1981), 489-516.

[Ma-Wü2] D.W.Masser, G.Wüstholz, Zero estimates on group varieties II, preprint (1983).

[Ma-Wü3] D.W.Masser, G.Wüstholz, Fields of large transcendence degree generated by values of elliptic functions, Inv.math.72 (1983), 407-464.

[Ma-Wü4] D.W.Masser, G.Wüstholz, Algebraic independence properties of values of elliptic functions, London Math.Soc.Lecture

Note 56 (1982), 360-363, Cambridge Univ. Press.

[Ma-Wü5] D.W.Masser, G.Wüstholz, Algebraic independence of values o elliptic functions, preprint (1983).

[Mo1] J.-C.Moreau, Démonstration géométrique de lemmes de zéros, I, Séminaire de Theorie des Nombres, Paris 1981-1982, to appear in Progr.Math.

[Mo2] J.-C. Moreau, Démonstration géométrique de lemmes de zéros II, Progr.Math.31 (1983), 191-197.

[Ne1] Ju.V.Nesterenko, Bounds for the orders of zeros of functio in a special class and application to the theory of transcendental numbers, Izvestia Ser.math. 41 (1977), n°2, 253-284.

[Ne2] Ju.V.Nesterenko, On the algebraic independence of algebrai numbers to algebraic powers, Progr.Math.31 (1983), 199-220

[Ph1] P.Philippon, Interpolation dans les espaces affines, Progr Math.22 (1982), 221-236.

[Ph2] P.Philippon, Indépendance algébrique de valeurs de fonctio exponentielles p-adiques, J.reine u. angew. Math. 329 (198 42-51.

[Ph3] P.Philippon, Variétés abéliennes et indépendance algébriqu I, Inv.math. 70 (1983), 259-318.

[Ph4] P.Philippon, Variétés abéliennes et indépendence algébriqu II: un analogue abélien du théorème de Lindemann-Weierstra Inv.math. 72 (1983), 389-405.

[R] E.Reyssat, Un critère d'indépendance algébrique, J.Reine u angew. Math. 329 (1981), 66-81.

[Sch1] Th.Schneider, Transzendenzuntersuchungen periodischer Funktionen I,II, J.reine u. angew. Math. 172 (1934), 65-74.

[Sch2] Th.Schneider, Arithmetische Untersuchungen elliptischer Integrale, Math.Ann. 113 (1937), 1-13.

[Sch3] Th.Schneider, Zur Theorie der Abelschen Funktionen und Integrale, J.reine u. angew. Math. 183 (1941), 110-128.

[Se] J.P.Serre, Quelques propriétés des groupes algébriques commutatifs, Astérisque 69-70 (1979), 191-202.

[Sh] A.V.Shidlovsky, On a criterion of algebraic independence, Izvestia Akad. Nauk SSSR 23 (1959), 35-66.

[Si1 C.L.Siegel, Über einige Anwendungen diophantischer Approximationen, Abh. Preuss. Akad. Wiss. 1 (1929).

[Si2] C.L.Siegel, Über die Perioden elliptischer Funktionen, J. reine u. angew. Math. 167 (1932), 62-69.

[Sm] A.A.Shmelev, On the question of the algebraic independence of algebraic powers of algebraic numbers, Mat.Zam.11 (1972), 635-644.

T] R.Tijdeman, An auxiliary result in the theory of transcendental numbers, J. Number Theory 5 (1973), 80-94.

Wa1] M.Waldschmidt, Nombres transcendants et groupes algébriques, Astérisque 69-70 (1979).

Wa2] M.Waldschmidt, Nombres transcendants, SLN 402 (1974).

Wa3] M.Waldschmidt, Indépendance algébrique et exponentielles en plusieurs variables, to appear in this volume.

Wa-Z] M.Waldschmidt, Zhu Yao Chen, Une généralisation en plusieurs variables d'un critère de transcendance de Gel'fond, to appear in CRAS Paris.

Wü1] G.Wüstholz, Nullstellenabschätzungen auf Varietäten, Progr. Math. 22 (1982), 359-362.

Wü2] G.Wüstholz, Multiplicity estimates on group varieties, to appear.

Wü3] G.Wüstholz, Sur l'analogue abélien du théorème de Lindemann, CRAS Paris,Ser.I, 295 (1982), 35-37.

Wü4] G.Wüstholz, Über das abelsche Analogon des Lindemannschen Satzes I, Inv.math.72 (1983), 363-388.

[Wü5] G.Wüstholz, Algebraic values of analytic homomorphisms of algebraic groups, to appear.

[Wü6] G.Wüstholz, Some remarks on a conjecture of Waldschmidt, Progr.Math. 31 (1983), 329-336.

[Wü7] G.Wüstholz, Zum Periodenproblem, to appear in Inv.math.

[Wü8] G.Wüstholz, Transzendenzeigenschaften von Perioden elliptischer Integrale, to appear in J. reine u. angew. Math.