SPECIAL VALUES OF HECKE L-FUNCTIONS AND ABELIAN INTEGRALS

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In this article we attempt to explain the formalism of Deligne's rationality conjecture for special values of motivic L-functions (see [D1]) in the particular case of L-functions attached to algebraic Hecke characters ("Größencharaktere of type A_0"). In this case the conjecture is now a theorem by virtue of two complementary results, due to D. Blasius and G. Harder, respectively: see §5 below.

For any "motive" over an algebraic number field, Deligne's conjecture relates certain special values of its L-function to certain periods of the motive. Most of the time when motives come up in a geometric situation, we tend to know very little about their L-functions. In the special case envisaged here, however, the situation is quite different: The L-functions of algebraic Hecke characters are among those for which Hecke proved analytic continuation to the whole complex plane and functional equation. But the "geometry" of the corresponding motives has emerged only fairly recently - see §3 below.

The relatively good command we now have of the motives attached to algebraic Hecke characters reveals that many non-trivial period relations are in fact but reflections of character-identities. This point of view is systematically perused in [Sch], and we shall illustrate it here by the so-called formula of Chowla and Selberg: see § 6.

This formula, in fact, goes back to the year 1897, as does the instance of Deligne's conjecture with which we start in § 1. Tying up these two relations in the motivic formalism, we hope to make it apparent that both results really should be viewed "comme les deux volets d'un même diptyque", as A. Weil has pointed out in [W III], p. 463.

Contents

§ 1. A formula of Hurwitz

In 1897, Hurwitz [Hu] proved that

$$(1) \qquad \sum_{a,b\in\mathbb{Z}}{}' \frac{1}{(a+bi)^{4\nu}} = \Omega^{4\nu} \times (\text{rational number}),$$

for all $\nu = 1,2,3,\ldots$, where

$$(2) \qquad \Omega = 2\int_0^1 \frac{dx}{\sqrt{1-x^4}} = 2.62205755\ldots = \frac{\Gamma(\frac{1}{4})^2}{2.\sqrt{2\pi}}$$

Notice the analogy of these identities with the well-known formula for the Riemann zeta-function at positive even integers:

$$(3) \qquad \sum_{a\in\mathbb{Z}}{}' \frac{1}{a^{2\nu}} = (2\pi i)^{2\nu} \times (\text{rational number}).$$

Both formulas are special cases of Deligne's conjecture. To understand this in Hurwitz' case, we look at the elliptic curve A given by the equation

$$A : y^2 = 4x^3 - 4x .$$

A is defined over $\mathbb{Q}$, but we often prefer to look at it as defined over the field $k = \mathbb{Q}(i) \subset \mathbb{C}$. Over this field of definition, we can see that A admits complex multiplication by the same field k :

$$k \xrightarrow{\sim} \mathrm{End}\,(A) \otimes \mathbb{Q}$$
$$i \longmapsto \begin{Bmatrix} x \mapsto -x \\ y \mapsto iy \end{Bmatrix}$$

Deligne's account of Hurwitz' formula would start from the observation that both sides of (1) express information about the homology

$$H_1(A)^{\otimes 4\nu} \subset H_{4\nu}(A^{4\nu}) .$$

The left hand side of (1) carries data collected at the finite places of k, as does the right hand side for the infinite places.

In fact, look at the different cohomology theories:

- <u>Etale cohomology:</u> Fix a rational prime number ℓ, and denote, for $n \geq 1$, by $A[\ell^n]$ the group of ℓ^n-torsion points in $A(\overline{\mathbb{Q}})$, $\overline{\mathbb{Q}}$ being the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Then

$$V_\ell(A) = \left(\varprojlim_n A[\ell^n]\right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

is the dual of the first ℓ-adic cohomology of $A \times_k \overline{\mathbb{Q}}$ with coefficients in $\mathbb{Q}_\ell$.

By functoriality, the isomorphism $k \cong \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{End}\, A$ makes $V_\ell(A)$ into a $k \otimes \mathbb{Q}_\ell$-module, free of rank 1. The natural continuous action of $\mathrm{Gal}\,(\overline{\mathbb{Q}}/k)$ on $V_\ell(A)$ is $k \otimes \mathbb{Q}_\ell$-linear, and therefore given by a continuous character

$$\Psi_\ell : \mathrm{Gal}\,(\overline{\mathbb{Q}}/k)_{ab} \longrightarrow GL_{k \otimes \mathbb{Q}_\ell}(V_\ell(A)) = (k \otimes \mathbb{Q}_\ell)^* .$$

This character was essentially determined - if from a rather different point of view - on July 6, 1814 by Gauss, [Ga]. The explicit analysis of the Galois-action on torsion-points of A was carried out (in a stunningly "modern" fashion) in 1850 by Eisenstein, [Ei]. - In any case, if π is a prime element of $\mathbb{Z}[i]$ not dividing 2ℓ, normalized so that $\pi \equiv 1 \pmod{(1+i)^3}$, and if $F_\pi \in \mathrm{Gal}\,(\overline{\mathbb{Q}}/k)_{ab}$ is a geometric Frobenius element at (π) (i.e., $F_\pi^{-1}(x) \equiv x^{\mathbb{N}\pi} \pmod{\mathfrak{P}}$ for any prime $\mathfrak{P}$ of k_{ab} dividing (π), any algebraic integer $x \in k_{ab}$), then one finds

$$\psi_\ell\,(F_\pi) = \pi^{-1} \in k^* \hookrightarrow (k \otimes \mathbb{Q}_\ell)^* .$$

The characters ψ_ℓ all fit together to give an "algebraic Hecke character" ψ defined on the group I_2 of ideals of k that are prime to 2:

$$\begin{array}{ccccc} I_2 & & \xrightarrow{\quad\psi\quad} & & k^* \\ \uparrow & & & & \searrow \\ I_{2\ell} & \xrightarrow{F} & \mathrm{Gal}\,(\overline{\mathbb{Q}}/k)_{ab} & \xrightarrow{\psi_\ell} & (k \otimes \mathbb{Q}_\ell)^* \end{array}$$

Then for all $\nu \geq 1$, the character $\psi^{4\nu}$ can be defined on all ideals of k by $(\alpha) \longmapsto \alpha^{-4\nu}$. Remember that k is embedded into $\mathbb{C}$, so that it makes sense to consider the L-functions

$$L(\psi^{4\nu}, s) = \prod_{\mathfrak{p}} \frac{1}{1 - \dfrac{\psi(\mathfrak{p})^{4\nu}}{\mathbb{N}\mathfrak{p}^{s}}} \qquad (\mathrm{Re}(s) > 1-2\nu)\ ,$$

where $\mathfrak{p}$ ranges over all prime ideals of $\mathbb{Z}[i]$. Then the left hand side of Hurwitz' formula (1) is simply $4.L(\psi^{4\nu},0)$. We have shown how this is a special value of the L-function afforded by the ℓ-adic cohomologies $V_\ell(A)^{\otimes_{k\otimes\mathbb{Q}_\ell} 4\nu}$.

- Betti and de Rham cohomology. Here we shall use the fact that the curve A (if not its complex multiplication) is already defined over $\mathbb{Q}$. Denote by $H_1^B(A) = H_1(A(\mathbb{C}),\mathbb{Q})$ the first rational singular homology of the Riemann surface $A(\mathbb{C})$, with the Hodge decomposition

$$H_1^B(A) \otimes_{\mathbb{Q}} \mathbb{C} = H^{-1,0} \oplus H^{0,-1} .$$

Complex conjugation on $A \times_{\mathbb{Q}} \mathbb{C}$ induces an endomorphism

$$F_\infty : H_1^B(A) \circlearrowleft \qquad \text{(the "Frobenius at } \infty\text{")}.$$

Call H_B^+ the fixed part of $H_1^B(A)$ under F_∞, and let η be a basis of this onedimensional $\mathbb{Q}$-vector space.

Let $H_1^{DR}(A) = H_{DR}^1(A)^\vee$ be the dual of the first algebraic de Rham cohomology of A over $\mathbb{Q}$, given with the Hodge filtration

$$H_1^{DR}(A) \supset F^+ \supset \{0\} ,$$

where $F^+ \otimes_{\mathbb{Q}} \mathbb{C} \cong H^{0,-1}$ under the GAGA isomorphism over $\mathbb{C}$:

$$I : H_1^B(A) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H_1^{DR}(A) \otimes_{\mathbb{Q}} \mathbb{C} .$$

I induces an isomorphism of onedimensional $\mathbb{C}$-vector spaces

$$I^+ : H_B^+(A) \otimes_{\mathbb{Q}} \mathbb{C} \longrightarrow (H_1^{DR}(A)/F^+) \otimes_{\mathbb{Q}} \mathbb{C} .$$

Then, $\frac{1}{\Omega} \cdot I^+(\eta) \in H_1^{DR}(A)/F^+$, for Ω defined by (2). In fact, $\Omega = \int_1^\infty \frac{dx}{\sqrt{x^3-x}}$ is a real fundamental period of our curve, and so, up to $\mathbb{Q}^*$, Ω is the determinant of the integration-pairing

$$(H_B^+(A) \otimes_{\mathbb{Q}} \mathbb{C}) \times (H^0(A,\Omega^1) \otimes_{\mathbb{Q}} \mathbb{C}) \xrightarrow{\int} \mathbb{C} ,$$

calculated in terms of $\mathbb{Q}$-rational bases of both spaces. This determinant equals that of the map I^+ since $H^0(A,\Omega^1) \subset H_{DR}^1(A)$ is the dual of $H_1^{DR}(A)/F^+$.

Passing to tensor powers of the onedimensional vector spaces above we find the periods $\Omega^{4\nu}$ occuring in (1).

In a sense, we have cheated a little in deriving the period Ω from the cohomological setup: In the ètale case we have used the action of k via complex multiplication to obtain a onedimensional situation (i.e., the k-valued character Ψ). In the calculation of the period, too, we should have considered $H_1^{DR}(A/k) = H_1^{DR}(A) \otimes_{\mathbb{Q}} k$, endowed with the further action of k via complex multiplication, and two copies of $H_1^B(A)$, indexed by the two possible embeddings of the base field k into $\mathbb{C}$.... But in the

resence of an elliptic curve over $\mathbb{Q}$, this would have seemed too arti-
icial, and the general procedure will be treated in § 4.

s a final remark about formula (1), it should be noted that it is proved airly easily. Any lattice $\Gamma = \lambda \cdot (\mathbb{Z} + \mathbb{Z} i)$ gives a Weierstraß $\wp$-function such that

$$\wp'(z,\Gamma) = 4\wp(z,\Gamma)^3 - g_2(\Gamma)\,\wp(z,\Gamma),$$

nd for $\lambda = \Omega$ we get $g_2(\Gamma) = 4$. The rational numbers left unspecified in (1) are then essentially the coefficients of the z-expansion of $\wp(z,\Gamma)$. t is these numbers that Hurwitz studied in his papers.

2. Algebraic Hecke Characters

et k and E be totally imaginary number fields (of finite degree over $\mathbb{Q}$), and write

$$\Sigma = \mathrm{Hom}\,(k,\overline{\mathbb{Q}}) \quad \text{and} \quad T = \mathrm{Hom}\,(E,\overline{\mathbb{Q}})$$

he sets of complex embeddings of k and E. The group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts n $\Sigma \times T$, transitively on each individual factor. An algebraic homomorphism

$$\beta : k^* \longrightarrow E^*$$

s a homomorphism induced by a rational character

$$\beta : R_{k/\mathbb{Q}}(\mathbb{G}_m) \longrightarrow R_{E/\mathbb{Q}}(\mathbb{G}_m).$$

This means that, for all $\tau \in T$, the composite

$$\tau\circ\beta : k^* \longrightarrow \overline{\mathbb{Q}}^*$$

s given by

(4) $$\tau\circ\beta\,(x) = \prod_{\sigma\in\Sigma} \sigma(x)^{n(\sigma,\tau)} ,$$

for certain *integers* $n(\sigma,\tau)$, such that $n(\rho\sigma,\rho\tau) = n(\sigma,\tau)$ for all $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Let $k^*_{\mathbb{A},f} \hookrightarrow k^*_{\mathbb{A}}$ be the topological group of finite idèles of k – *i.e.*, those idèles whose components at the infinite places are 1. For $x \in k^*$, let x also denote the corresponding principal idèle in $k^*_{\mathbb{A}}$,and x_f the finite idèle obtained by changing the infinite components of x to 1.

An *algebraic Hecke character* Ψ *of* k *with values in* E , *of (infinity-) type* β , is a continuous homomorphism

$$\Psi : k^*_{\mathbb{A},f} \longrightarrow E^*$$

such that, for all $x \in k^*$,

$$\Psi(x_f) = \beta(x) .$$

If β is the infinity-type of an algebraic Hecke character Ψ , then, by continuity, β has to kill a subgroup of finite index of the units of k. It follows that the integer

(5) $$w = n(\sigma,\tau) + n(c\sigma,\tau) = n(\sigma,\tau) + n(\sigma,c\tau)$$

(where c = complex conjugation on $\overline{\mathbb{Q}}$) is independent of σ,τ. It is called the *weight* of Ψ .

For any $\tau \in T$, we get a complex valued *Größencharakter* $\tau\circ\Psi$ which extends to a quasicharacter of the idèle-class-group:

$$\begin{array}{ccc}
k^*_{\mathbb{A},f} & \xrightarrow{\tau\circ\Psi} & \overline{\mathbb{Q}}^* \\
\downarrow & & \downarrow \\
k^*_{\mathbb{A}} & & \\
\downarrow & & \\
k^*_{\mathbb{A}}/k^* & \xrightarrow{\tau\circ\Psi} & \mathbb{C}^*
\end{array} .$$

Consider the array of L-functions, indexed by T:

$$L(\Psi,s) = (L(\tau\circ\Psi,s))_{\tau\in T} ,$$

where, for $\mathrm{Re}(s) > \frac{w}{2}+1$,

$$L(\tau\circ\Psi,s) = \prod_{\mathfrak{p}}\left(1-\frac{(\tau\circ\varphi)(\pi_{\mathfrak{p}})}{\mathbb{N}\mathfrak{p}^{s}}\right)^{-1} ,$$

the product being over all prime ideals $\mathfrak{p}$ of k for which the value $\Psi(\pi_{\mathfrak{p}})$ does not depend on the choice of uniformizing parameter $\pi_{\mathfrak{p}}$ of $k_{\mathfrak{p}}$.

The point $s = 0$ is called critical for Ψ , if for any τ , no Γ-factor on either side of the functional equation of $L(\tau\circ\Psi,s)$ has a pole at $s = 0$. This is really a property of the infinity-type β of Ψ , for it turns out that $s = 0$ is critical for Ψ if and only if there is a disjoint decomposition

$$\Sigma\times T = \{(\sigma,\tau) \mid n(\sigma,\tau) < 0\} \,\dot{\cup}\, \{(\sigma,\tau) \mid n(c\sigma,\tau) < 0\} .$$

In other words, for every $\tau\in T$, there is a "CM-type" $\Phi(\tau\circ\beta)\subset\Sigma$ such that

(6)
- $\Phi(\alpha\tau\circ\beta) = \Phi(\tau\circ\beta)^{\alpha}$, for $\alpha\in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$
- $\Phi(\tau\circ\beta) \,\dot{\cup}\, \Phi(c\tau\circ\beta) = \Sigma$
- $\sigma\in\Phi(\tau\circ\beta) \Leftrightarrow n(\sigma,\tau) < 0 \Leftrightarrow n(c\sigma,\tau) \geq 0$.

For Ψ such $s = 0$ is critical Deligne defined an array of periods $\Omega(\Psi) = (\Omega(\Psi,\tau))_{\tau\in T} \in (\mathbb{C}^*)^{T} = (E\otimes_{\mathbb{Q}}\mathbb{C})^*$, and conjectured that

$$\frac{L(\Psi,0)}{\Omega(\Psi)} \in E \hookrightarrow E\otimes\mathbb{C} . \tag{7}$$

In other words, he conjectured that there is $x\in E$ such that for all $\tau : E \hookrightarrow \mathbb{C}$,

$$L(\tau\circ\Psi,0) = \tau(x) . \Omega(\Psi,\tau) .$$

The definition of $\Omega(\Psi)$ is discussed in § 4. It requires attaching a motive to an algebraic Hecke character.

§ 3. Motives

3.1 In the example of § 1, we constructed a "motive" for our Hecke characters $\Psi^{4\nu}$ by taking tensor powers of $H_1(A)$, i.e., a certain direct factor of $H_{4\nu}(A^{4\nu})$, in the various cohomology theories. This illustrates fairly well the general idea of what a motive should be: Starting from an algebraic variety over a number field, we have the right to consistently choose certain parts of its cohomology. Just what "consistenly" means constitutes the difference between various notions of motive. Here we shall be concerned with a fairly weak and therefore half way manageable version: motives defined using "absolute Hodge cycles" - see [DMOS], I and II. In this theory motives can be shown to be isomorphic, very roughly speaking, whenever their L-functions and periods coincide. A little more precisely, giving a homomorphism between two such motives M and N amounts to giving a family of homomorphisms

$$\begin{cases} H_\sigma(M) \longrightarrow H_\sigma(N) & \text{(Betti cohomology depends on the choice of } \sigma : k \longrightarrow \mathbb{C} \text{ yielding } M \longmapsto M\times_\sigma\mathbb{C}\text{)} \\ H_{DR}(M) \longrightarrow H_{DR}(N) & \\ H_\ell(M) \longrightarrow H_\ell(N) & \text{(for all } \ell \text{)} \end{cases}$$

compatible with all the natural structures on these cohomology groups: Hodge decomposition, Hodge filtration, $\mathrm{Gal}(\bar{k}/k)$-action, as well as with the comparison isomorphisms between H_B and H_{DR}, H_B and the H_ℓ's .

3.2 Let us state more precisely what a <u>motive attached to an algebraic Hecke character</u> Ψ should be! - In the example of § 1, the curve $A/\mathbb{Q}$ defines the motive $H_1(A)$ <u>over</u> $\mathbb{Q}$ whose L-function is $L(\Psi,s)$. (<u>This</u> is really what Gauss observed in 1814; nowadays this follows from a result of Deuring, which has been further generalized by Shimura [Sh 1]...) But this is <u>not</u> what we are looking for. The complex multiplication of A and therefore the Hecke character Ψ are not visible over $\mathbb{Q}$. That is why we considered A <u>over</u> k in our treatment of the étale cohomology, and used the field of values of Ψ (which again happened to be k) to obtain onedimensional Galois-representations, and thus Ψ .

Given a general algebraic Hecke character Ψ like in § 2, a motive M for Ψ has to be a motive defined over the base field k such that the field E acts on all the realizations of M in the various cohomology theories, and such that for all ℓ, $H_\ell(M)$ is an $E\otimes\mathbb{Q}_\ell$-module of rank 1 with Gal $(\overline{\mathbb{Q}}/k)$ acting via Ψ . The action of E on the various realizations of M should of course be compatible with their extra structures and with the various comparison isomorphisms. In other words (see 3.1), E should embed into End M . Thus the rank-condition on $H_\ell(M)$ can also be stated by saying that Betti cohomology $H_\sigma(M)$ should form a onedimensional E-vector space.

<u>3.3</u> The typical example is $H_1(A)$, for an abelian variety A/k with $E \cong \mathbb{Q}\otimes_{\mathbb{Z}} \mathrm{End}_{/k}A$ and $2 \dim A = [E : \mathbb{Q}]$. The fact that these motives always give rise to an algebraic Hecke character was one of the main results of the theory of complex multiplication by Shimura and Taniyama. The Hecke characters occuring with abelian varieties of CM-type are precisely those of weight -1 such that $n(\sigma,\tau) \in \{-1,0\}$, for all $(\sigma,\tau) \in \Sigma\times T$.

In fact, given such an algebraic Hecke character Ψ of k with values in E , we can assume without loss of generality that E is the field generated by the values of Ψ on the finite idèles of k . Then E is a CM-field (i.e., quadratic over a totally real subfield), and a theorem of Casselman, [Sh 1], can be applied to get an abelian variety A defined over k such that:

- $2 \dim A = [E : \mathbb{Q}]$
- there is an isomorphism $E \xrightarrow{\sim} \mathbb{Q}\otimes_{\mathbb{Z}} \mathrm{End}_{/k}A$
- $H_1(A)$ is a motive for Ψ .

3.4. When Ψ has arbitray weight ($\neq 0$) the homogeneity condition (5) above still forces the infinity-type β to be of the form $\beta = \prod_i \beta_i$, with $|\text{weight}(\beta_i)| = 1$, $n_i(\sigma,\tau) \in \{\pm 1, 0\}$. Since twisting with finite order characters is easy to control motivically one would naively expect to be able to assemble a motive for any given algebraic Hecke character essentially as tensor product of constituents of the form $H_1(A)$ or $H^1(A)$ like in 3.3.

There is however the nasty problem of controlling the fields of values E. For example, if k is imaginary quadratic with class number $h > 1$, then a Hecke character of k with $\{n(\sigma,\tau)\} = \{-1,0\}$ or $\{n(\sigma,\tau)\} = \{1,0\}$ can never take all values in $E = k$, but its h-th power may.

Constructing a motive for the h-th power as an E-linear tensor power of a motive for the character of weight ± 1, one still has to show that the field of coefficients E can be "descended" to k in weight $\pm h$.

3.5 This "descent" of the field of coefficients can be dealt with directly. But we gain much more insight if we use a very elegant formalism due to Langlands, [La] § 5, and Deligne, [DMOS] IV. Langlands defined a group scheme over $\mathbb{Q}$, the "Taniyama group" T, of which Deligne was subsequently able to show that the category of its $\mathbb{Q}$-rational representations is equivalent to the category of those motives as can be obtained (eventually after twisting by a character of finite order) from abelian varieties over $\mathbb{Q}$ which admit complex multiplication over $\overline{\mathbb{Q}}$. Since the Taniyama group - along with many other beautiful properties - has, for every k, a certain subquotient S_k (isomorphic to a group scheme constructed by Serre in [Sℓ]) whose irreducible representations are given precisely by the algebraic Hecke characters Ψ of k, we "find" the motive attached to a given Ψ by lifting the corresponding representation of S_k back to the subgroup of T whose representations give the motives defined over k.

3.6 So, for every algebraic Hecke character Ψ of k with values in E, a motive over k equipped with an E-action can be constructed from CM-abelian varieties over k, whose ℓ-adic Galois representations are onedimensional $E \otimes \mathbb{Q}_\ell$-modules given by Ψ. Furthermore, the Tate-conjecture would imply that the ℓ-adic realizations determine a motive up to isomorphism - even in the strictest sense of "motives" (algebraic cycles) As we are dealing with motives for absolute Hodge cycles, it is perhaps not too surprising that one can actually prove: in the category of motives that can be obtained from all abelian varieties over k (not necessarily CM), any two motives attached to the same Hecke character are actually isomorphic - see [Sch], I. Still, this does not seem to be known in any larger category of motives. In fact, it hinges on Deligne's theorem that "every Hodge cycle on an abelian variety over an algebraically closed field is absolutely Hodge" - see [DMOS], I. Anyway, whenever we find two motives constructed from the cohomology of abelian varieties that belong to the same Hecke character they will have the same periods...

§ 4. Periods

As in the example of § 1, periods are going to arise from a comparison of the Betti and de Rham cohomology groups of our motive. So, let us first look at these cohomologies more closely in the case of a motive for an algebraic Hecke character. We are going to use some facts which are well-known for the cohomology of algebraic varieties, and which carry over to motives.

4.1 As in § 2, let k and E be totally imaginary number fields, and Ψ an algebraic Hecke character of k with values in E. Let M be a motive over k attached to Ψ (in the sense of 3.2 above). Then for any embedding $\sigma \in \Sigma$, the singular rational cohomology $H_\sigma(M)$ is an E-vector space of dimension 1. The E-action respects the Hodge-decomposition

$$H_\sigma(M) \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p,q} H^{p,q} \quad .$$

$H_\sigma(M) \otimes_{\mathbb{Q}} \mathbb{C}$ is an $E \otimes_{\mathbb{Q}} \mathbb{C} = \mathbb{C}^T$-module of rank 1. (Σ and T were defined at the beginning of § 2.)

Starting from the special case where $M = H_1(A)$ with an abelian variety A/k of CM-type, and using the uniqueness of the motive attached to a Hecke character (see 3.6), one finds that, for any embedding $\tau \in T$, the direct factor of $H_\sigma(M)\otimes_{\mathbb{Q}}\mathbb{C}$ on which E acts via τ lies in

$$H^{n(\sigma,\tau),\ w-n(\sigma,\tau)} .$$

(The $n(\sigma,\tau)$ are given by the infinity-type of Ψ : see § 2, formula (5).)

<u>4.2</u> Let us note in passing that, if $M(\Psi)$ and $M(\Psi')$ are motives for Hecke characters Ψ and Ψ' of k with values in E, then the following are equivalent:

- $M(\Psi) \cong M(\Psi')$ <u>over</u> $\overline{\mathbb{Q}}$.
- For some $\sigma\in\Sigma$, $H_\sigma(M(\Psi)) \cong H_\sigma(M(\Psi'))$, as rational Hodge-structures.
- Ψ and Ψ' have the same infinity-type β .

<u>4.3</u> Coming back to our motive M for Ψ, suppose now that $s = 0$ is critical for Ψ (see § 2, formula (6)), and consider the comparison isomorphism

$$I : \bigoplus_{\sigma\in\Sigma} H_\sigma(M)\otimes_{\mathbb{Q}}\mathbb{C} \xrightarrow{\ \sim\ } H_{DR}(M)\otimes_{\mathbb{Q}}\mathbb{C} .$$

Note that $H_{DR}(M)$ is by definition a k-vector space, and that $k\otimes_{\mathbb{Q}}\mathbb{C} \cong \mathbb{C}^\Sigma$. So, I is an isomorphism of $k\otimes E\otimes\mathbb{C}$ - modules of rank 1. For $\sigma\in\Sigma$, let e_σ be an E-basis of $H_\sigma(M)$, and put $e = (e_\sigma\otimes 1)_{\sigma\in\Sigma}$. On the right hand side, choose a basis ω of $H_{DR}(M)$ over $k\otimes_{\mathbb{Q}} E$, and decompose

$$\omega = \sum_{(\sigma,\tau)\in\Sigma\times T} \omega_{\sigma,\tau} ,$$

with $\omega_{\sigma,\tau} \in \tau$-eigenspace of $H_{DR}(M)\otimes_{k,\sigma}\mathbb{C}$. Writing $I(e_\sigma) = \sum_{\tau\in T} I(e_\sigma)_\tau$ for the corresponding decomposition of $I(e)$, we find for all $(\sigma,\tau)\in\Sigma\times T$ that

$$\omega_{\sigma,\tau} = p(\sigma,\tau) \cdot I(e_\sigma)_\tau \quad , \text{ for some } \quad p(\sigma,\tau) \in \mathbb{C}^* \; .$$

The unit

$$(p(\sigma,\tau))_{(\sigma,\tau)\in \Sigma\times T} \in (\mathbb{C}^*)^{\Sigma\times T} = (k \otimes E \otimes \mathbb{C})^*$$

gives the "matrix" of I and, up to multiplication by $(k \otimes E)^*$, depends only on Ψ.

<u>4.4</u> Modulo such a factor one has the relation

(8) $$p(\sigma,\tau) \cdot p(c\sigma,\tau) \sim (2\pi i)^w \; .$$

This amounts essentially to Legendre's period relation, and can be proved in our context (using uniqueness of motives for Hecke characters) from the identity $\Psi\overline{\Psi} = \mathbb{N}^w$. - The motive $\mathbb{Q}(-1)$ attached to the norm character is discussed in more detail, e.g., in [D1], § 3. For (8), it is enough to know that $\mathbb{Q}(-1)$ is a motive defined over $\mathbb{Q}$, with coefficients in $\mathbb{Q}$ such that

$$H_B(\mathbb{Q}(-1) = \frac{1}{2\pi i}\,\mathbb{Q} \quad \text{and} \quad H_{DR}(\mathbb{Q}(-1)) = \mathbb{Q} \; ,$$

with trivial comparison isomorphism. Incidentally, $\mathbb{Q}(-1)$ has no critical s, if considered over a totally imaginary field k.

With (8) and 3.4, calculating the $p(\sigma,\tau)$'s (or their inverses) usually reduces to integrating holomorphic differentials on which E acts via τ or $c\tau$.

<u>4.5</u> In terms of these $p(\sigma,\tau)$, Deligne's period $\Omega(\Psi) \in (E \otimes \mathbb{C})^*/E^*$ (see (7) above) can be defined componentwise by

(9) $$\Omega(\Psi,\tau) = D(\Psi)_\tau \cdot \prod_{\sigma\in\Phi(\tau\circ\beta)} p(\sigma,\tau)^{-1} \; .$$

For the definition of the "CM-types" $\Phi(\tau\circ\beta)$, see § 2, formula (6).

Note that the product in (9) is, in fact, well-defined up to a factor in $(E \otimes 1)^*$. - One definition of the "discriminant factor" $D(\Psi) = (D(\Psi)_\tau)_{\tau \in T}$ can be found in [D1], 8.15. This factor arises when one computes the cohomology of $R_{k/\mathbb{Q}} M$ by the Künneth formula: among other things, one has to choose an ordering of Σ. A definition of $D(\Psi)$ which was born out by these cohomological computations - cf. [Ha], esp. 2.4.1 and Cor. 5.7.2B - is as follows. Start with one $\tau \in T$, and let $K_\tau \subset E^\tau$ be the fixed field of

$$\{\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \mid \rho\, \Phi\,(\tau \circ \beta) = \Phi(\tau \circ \beta)\}\ .$$

$\mathrm{Gal}(\overline{\mathbb{Q}}/K_\tau)$ permutes the set $\Phi(\tau \circ \beta)$. Let $L_\tau \supset K_\tau$ be the fixed field of the kernel of the character

$$\mathrm{Gal}(\overline{\mathbb{Q}}/K_\tau) \longrightarrow \mathfrak{S}(\Phi(\tau \circ \beta)) \xrightarrow{\mathrm{sgn}} \{\pm 1\}\ .$$

Then $[L_\tau : K_\tau] \leq 2$, and $L_\tau = K_\tau(D(\Psi)_\tau)$ for some $D(\Psi)_\tau$ with $D(\Psi)_\tau^{\,2} \in K_\tau^*$.

Now, any $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ induces a permutation of the set of infinite places of k: both $\Phi(\tau \circ \beta)$ and $\Phi(\rho\tau \circ \beta)$ are in bijection with this set. Call $\varepsilon(\rho)$ the sign of this permutation. Then we set

$$D(\Psi)_{\rho\tau} = \varepsilon(\rho)\ (D(\Psi)_\tau)^\rho$$

The array $(D(\Psi)_\tau)_{\tau \in T}$ is independent, up to a factor in $(E \otimes 1)^*$, of the choices made in defining its components.

Let us list some properties of $D(\Psi)$ - cf. also [Sch].

<u>4.6</u> a) $D(\Psi)$ depends only on k, E, and the collection of "CM-types" $\{\Phi(\tau \circ \beta) \mid \tau \in T\}$.

b) $D(\Psi)^2 \in (E \otimes 1)^* \subset (E \otimes \mathbb{C})^*$.

c) If k is a CM-field, with maximal totally real subfield $k_\circ$, then $D(\Psi) \sim \sqrt{\mathrm{discr}(k_\circ)}$, up to a factor in $(E \otimes 1)^*$.

d) Let F/k be a finite extension of degree n. Then

$$\frac{D(\Psi\circ N_{F/k})}{D(\Psi)^n} \quad \left(\prod_{\sigma\in\Phi(\tau\circ\beta)} \sigma(\delta_{|\sigma|})\right)_{\tau\in T} ,$$

up to a factor in $(E\otimes 1)^*$. Here, the right hand side means the following:

Let $d(k^*)^2 \in k^*/(k^*)^2$ be the relative discriminant of F/k . For any infinite place v of k , choose a square root $\delta_v = \sqrt{d} \in k_v^*$. For $\sigma\in\Sigma$, let $|\sigma|$ be the infinite place of k determined by σ and $c\sigma$, and denote by $\sigma(\delta_{|\sigma|}) \in \mathbb{C}^*$ the well-defined image of $\delta_{|\sigma|} \in k_{|\sigma|}^*$ under the continuous isomorphism $k_{|\sigma|} \xrightarrow{\sim} \mathbb{C}$ given by σ . - Note that changing the representative of d or the signs of δ_v , at some places v , multiplies the right hand side of our formula only by a factor in $(E\otimes 1)^*$.

Assume the situation of 4.6,d). From the very definition of the $p(\sigma,\tau)$, and the properties 4.6,a) and d), one finds the following formula for the behaviour of the periods under extension of the base field:

(10)
$$\begin{aligned}\Delta(F/k,\beta) &:= \frac{\Omega(\Psi\circ N_{F/k})}{\Omega(\Psi^n)} = \frac{D(\Psi\circ N_{F/k})}{D(\Psi^n)} \\ &= \frac{D(\Psi\circ N_{F/k})}{D(\Psi)^n}\ \frac{D(\Psi)^n}{D(\Psi^n)} = \left(\prod_{\sigma\in\Phi(\tau\circ\beta)} \sigma(\delta_{|\sigma|})\right)_{\tau\in T} \cdot D(\Psi)^{n-1} .\end{aligned}$$

The array $\Delta(F/k,\beta) \in (E\otimes\mathbb{C})^*$ will reappear in the second theorem of § 5 below. Note that, if k is a CM-field, the second factor of $\Delta(F/k,\beta)$ can be evaluated by 4.6,c). Both factors of Δ are already present in [Ha], in the case $n = 2$, although the formalism there is still somewhat clumsier than the one employed here.

<u>4.7</u> Let us close this section with a few words on the behaviour of our periods under twisting. For the Tate twist, one finds

(11) $$\Omega(\Psi\cdot\mathbb{N}^m) \sim (2\pi i)^m\, \Omega(\Psi) .$$

If ω is a character <u>of finite order</u> on $K^*_{\mathbb{A},f}/k^*$ with values in E^*, one passes from $\Omega(\Psi)$ to $\Omega(\omega\Psi)$ by leaving $D(\Psi)$ unchanged, and multiplying the $p(\sigma,\tau)$ by certain algebraic numbers with eigen-properties under ω. The details can be found in [Sch]. All we need to know is the following invariance lemma:

If F is a finite extension of k, χ a character of finite order on $F^*_{\mathbb{A},f}/F^*$ with values in E^*, and ω the restriction of χ to $k^*_{\mathbb{A},f}$ (in other words, considering χ and ω on $\mathrm{Gal}(\bar{k}/F)$, $\mathrm{Gal}(\bar{k}/k)$, resp., via class field theory, $\omega = \chi\circ \mathrm{Ver}$, where $\mathrm{Ver} : \mathrm{Gal}(k^{ab}/k) \longrightarrow \mathrm{Gal}(F^{ab}/F)$ is the transfer map), then

$$(12) \qquad \frac{\Omega(\chi\cdot(\Psi\circ N_{F/k}))}{\Omega(\omega\cdot\Psi^n)} = \frac{\Omega(\Psi\circ N_{F/k})}{\Omega(\Psi^n)} = \Delta(F/k,\beta) .$$

Let us mention in passing that the proof of (12) also shows that the quotients

$$\frac{\Omega(\omega\Psi)}{\Omega(\Psi)}$$

may always be expressed by Gauss sums.

<u>§ 5. The rationality conjecture for Hecke L-functions</u>

The proof of Deligne's conjecture (see end of § 2) for the critical values of L-functions of algebraic Hecke characters falls into two parts The case where the base field k is a CM-field is treated first. From there one passes to the general case by a theorem about the behaviour of special values under extension of the base field.

(I) Let us briefly describe the CM-case:

Historically, the main idea for the CM-case goes back to Eisenstein. But it was Damerell who, in his thesis [Da], published the first comprehensive account of algebraicity results for critical values of Hecke L-functions of imaginary quadratic fields. He also announced finer rationality theorems in that case, but never published them. (The case of imaginary quadratic k was later settled completely in [GS] and [GS'].) In the Fall of 1974, André Weil gave an exposition of work of Eisenstein and Kronecker including, among other things, Damerell's theorem as an application. This course at the IAS - which was later on developed into the book [WEK] - inspired G. Shimura to generalize Damerell's algebraicity results to critical values of Hecke L-functions of arbitrary CM-fields: [Sh 3] . (At that point, he still needed a technical assumption on the infinity-type of the Hecke character.)

To explain the starting point of this method of proof, recall our example in § 1: the L-value there appeared (up to a factor of $\frac{1}{4}$) as an Eisenstein series :

$$\sum_{a,b\in\mathbb{Z}}' \frac{1}{(a+bi)^{4\nu}} ,$$

relative to the lattice $\mathbb{Z}+\mathbb{Z}i$. Now, sometimes the relation between L-value and Eisenstein series is not quite as straightforward - e.g., if, in § 1, we were to study the values $L(\Psi\mathbb{N}^a,0)$ for integers $a \neq 0$ such that $s = 0$ is critical for $\Psi\mathbb{N}^a$, then we would have to transform the Eisenstein series by certain (non holomorphic) differential operators. But except for such operators it remains true that, in any pair of critical values of an Hecke L-function of a CM-field k which are symmetric with respect to the functional equation, there is a value which can be written as a linear combination of Eisenstein series (viz., Hilbert modular forms with respect to the maximal totally real subfield of k), relative to lattices in k .

When k is imaginary quadratic, the algebraicity properties of the Eisenstein series can be derived directly from explicit polynomial relations among them (see, e.g., Weil's treatment of Damerell's theorem in [WEK]). But in general the proof of their algebraicity depends on a theory of canonical models for the Hilbert modular group (as in [Sh 3]) or, equivalently, on an algebraic theory of Hilbert modular forms.

This latter approach was used by Katz in [K1], [K2]. Just like Shimura, Katz did not stop to look at more precise rationality theorems about the special values he had determined up to an algebraic number. In fact, Katz' main concern was with integrality properties and p-adic interpolation.

When Deligne formulated his conjecture in 1977 he felt the need to check that, up to a factor in $\bar{\mathbb{Q}}^*$, it predicted Shimura's theorem. This turned out ot be a confusing problem, for the following reason. Shimura expresses the L-values in terms of periods of abelian varieties constructed from lattices in k, which therefore have <u>complex multiplication by</u> k, and are defined over some number field E'. On the other hand, the L-function in question is that of a Hecke character of the field k, with values in some number field E. The motive of such a character arises from abelian varieties <u>defined over</u> k, <u>with complex multiplication by</u> E (or some field closely related to E). This double role of k as field of definition and of coefficients was dealt with by Deligne - up to factors in $\bar{\mathbb{Q}}^*$ - by an ad hoc dualization, see [D1], 8.19. (Its refinement for more precise rationality statements remained the most serious obstacle in the attempt to prove Deligne's conjecture made in [Sch 1].)

Don Blasius managed to solve this problem by writing down an analogue of Deligne's dualization on the level of motives over k, resp. E: his "reflex motive". Thus he was able to prove

<u>Theorem 1</u>: <u>Let k be a CM-field, and Ψ a Hecke character of k, with values in some CM-field E. If $s = 0$ is critical for Ψ, then</u>

$$\frac{L(\Psi,0)}{\Omega(\Psi)} \in E \hookrightarrow E \otimes \mathbb{C} .$$

(Note that any algebraic Hecke character of any number field takes values in a CM-field.)

As Blasius' paper [B] is about to be available we shall not enter into describing the technique of his proof in detail. Suffice it to say that, apart from the "reflex motive" mentioned above, he needs, of course, a very careful analysis of the behaviour of the Eisenstein series under $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ (i.e., Shimura's reciprocity law in CM-points), and also the explicit description - due to Tate and Deligne - of the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on abelian varieties of CM-type: see [LCM], chapter 7.

(II) We shall now describe a little bit more in detail the second part of the proof of Deligne's conjecture for Hecke L-functions. It relies on a generalization of [Ha], § 3, from GL_2 to GL_n, and might not be published completely before some time.

Consider the following situation: Let k be a totally imaginary number field, and F/k a finite extension of degree $n \geq 2$. Let Ψ be an algebraic Hecke character with values in a number field E, of infinity-type β. Assume $s = 0$ is critical for Ψ. Let $\chi : F_{\mathbb{A}}^* / F^* \longrightarrow E^*$ be a character of finite order, and put $\omega = \chi|_{k_{\mathbb{A}}^*}$, like in § 4.7 above. Recall the array

$$\Delta(F/k,\beta) = (\Delta(F/k, \tau\circ\beta))_{\tau\in T}$$

defined in § 4.6, formula (10).

<u>Theorem 2</u>:

$$\Delta(F/k,\beta) \; \frac{L_F(\chi\cdot(\Psi\circ N_{F/k}),0)}{L_k(\omega\cdot\Psi^n,0)} \in E \hookrightarrow E\otimes\mathbb{C} .$$

<u>Remarks</u>: (i) As the Euler product for $L(\Psi,s)$ converges for $\mathrm{Re}(s) > \frac{w}{2} + 1$, and $s = 0$ is critical for Ψ, it is well-known that the denominator in the theorem is not zero.

(ii) Here is how theorems 1 and 2 imply Deligne's conjecture for all critical values of all Hecke L-functions: Given any totally imaginary number field F, and any Hecke character φ of F, with values in a number field E_o, of infinity-type β_o, the homogeneity condition (5) of § 2 forces β_o to factor through the <u>maximal CM-field</u> k <u>contained in</u> F :

$$\beta_o = \beta \circ N_{F/k} ,$$

for some algebraic homomorphism

$$\beta : k^* \longrightarrow E_o'^* .$$

Choose a Hecke character Ψ of k with infinity-type β, write $\varphi = \chi \cdot (\Psi \circ N_{F/k})$, for some finite order character χ of F, and choose $E \supset E_o$ big enough to contain the values of Ψ as well as those of χ. Define $\omega = \chi|_{k^*_{\mathbb{A}}}$. Put $n = [F : k]$. By theorem 1,

$$\frac{L(\omega\cdot\Psi^n,0)}{\Omega(\omega\cdot\Psi^n)} \in E .$$

But we know the behaviour of the periods Ω under twisting and base extension: see end of § 4. Theorem 2 therefore implies that

$$\frac{L(\varphi,0)}{\Omega(\varphi)} = \frac{L(\chi\cdot(\Psi\circ N_{F/k}))}{\Omega(\chi\cdot(\Psi\circ N_{F/k}))} \in E \longrightarrow E\otimes\mathbb{C} .$$

Finally, E may now be replaced by E_o because Deligne's conjecture is invariant under finite extension of the field of coefficients: [D1], 2.10.

This gives Deligne's conjecture for Hecke L-functions of totally imaginary number fields. These are the only fields with honest regard Hecke L-functions. But it should be said, for the sake of completeness, that Deligne's conjecture for Hecke (=Dirichlet) L-functions of <u>totally real fields</u> follows from results of Siegel's (cf. [D1], 6.7) and, in the case of number fields which are neither totally real nor totally imaginary, no Hecke (=Dirichlet) L-function has any critical value.

The remainder of this section is devoted to sketching the proof of theorem 2. Let us set up some notation.

We consider the following algebraic groups over k :

$$G_\circ/k = GL_n/k .$$

$$T_\circ/k = \text{standard maximal torus}$$

$$B_\circ/k = \text{standard Borel subgroup of upper triangular matrices,}$$

and the two maximal parabolic subgroups

$$P_\circ/k = \left\{ \begin{pmatrix} t & * \\ 0 & p \end{pmatrix} \;\middle|\; p \in GL_{n-1} , \; t \in GL_1 \right\}$$

$$Q_\circ/k = \left\{ \begin{pmatrix} q & * \\ 0 & t \end{pmatrix} \;\middle|\; q \in GL_{n-1} , \; t \in GL_1 \right\} .$$

Dropping the subscript zero will mean taking the restriction of scalars to $\mathbb{Q}$. So,

$$G/\mathbb{Q} = R_{k/\mathbb{Q}}(G_\circ/k) ,$$

and so on.

We introduce the two characters

$$\gamma_P : g = \begin{pmatrix} t & * \\ 0 & p \end{pmatrix} \longmapsto \frac{t^n}{\det(g)} ,$$

and

$$\gamma_Q : g = \begin{pmatrix} q & * \\ 0 & t \end{pmatrix} \longmapsto \frac{\det(g)}{t^n}$$

which we view as characters on the torus extending to $P_\circ$ (resp. $Q_\circ$). The representations of $G_\circ/k$ with highest weight $\nu\gamma_P$ (resp. $\mu\gamma_Q$) are the ν-th (resp. μ-th) symmetric power of the standard representation of $G_\circ/k$ on k^n (resp. its dual $(k^n)^\vee$).

Coming back to the situation of theorem 2, define a homomorphism

$$\Phi : P(\mathbb{Q}_{\mathbb{A},f}) = P_\circ(k_{\mathbb{A},f}) \longrightarrow E^*$$

by

$$\Phi : \begin{pmatrix} \underline{t}_f & * \\ 0 & \underline{p}_f \end{pmatrix} = \underline{g}_f \longmapsto \Psi_1(\underline{t}_f)\,\Psi(\det(\underline{g}_f)) .$$

We require that the central character of Φ be our ω. This means that Ψ_1 is determined by

$$\Phi \begin{pmatrix} \underline{t}_f & & \\ & \ddots & \\ & & \underline{t}_f \end{pmatrix} = \omega(\underline{t}_f) = \Psi_1(\underline{t}_f)\,\Psi(\underline{t}_f)^n .$$

We may view Φ as an "algebraic Hecke character" on $P/\mathbb{Q}$, and it has an infinity-type

$$\text{type}(\Phi) = \gamma \in \text{Hom}(P/\mathbb{Q}\,,\, R_{E/\mathbb{Q}}(\mathbb{G}_m)) .$$

Hence we get an array of types, indexed by $\tau \in T$, with components

$$\text{type}(\tau\circ\Phi) = \tau\circ\gamma \in \text{Hom}(P,\mathbb{G}_m) .$$

Recall that

$$\text{Hom}(P,\mathbb{G}_m) = \bigoplus_{\sigma\in\Sigma} \text{Hom}(P_\circ,\mathbb{G}_m) ,$$

and that the type β of Ψ is given by the integers $n(\sigma,\tau)$ - see § 2.

It is then easy to check that

$$\tau \circ \gamma = (n(\sigma,\tau) \cdot \gamma_P)_{\sigma \in \Sigma} \quad ,$$

for every $\tau \in T$.

Given τ , define an array of dominant weights

$$\Lambda(\tau) = (\lambda(\sigma,\tau))_{\sigma \in \Sigma}$$

by the rule

$$\lambda(\sigma,\tau) = \begin{cases} (-n(\sigma,\tau)-1)\gamma_Q & \text{if } n(\sigma,\tau) < 0 \\ n(\sigma,\tau)\gamma_P & \text{if } n(\sigma,\tau) \geq 0 \ . \end{cases}$$

This affords a representation

$$\rho : G \times_{\mathbb{Q}} \overline{\mathbb{Q}} = \prod_{\sigma \in \Sigma} (GL_n/k) \longrightarrow GL(M(\Lambda(\tau))) \quad ,$$

where $M(\Lambda(\tau)) = \bigotimes_{\sigma \in \Sigma} M(\lambda(\sigma,\tau))$, $M(\lambda(\sigma,\tau))$ being the representation with highest weight $\lambda(\sigma,\tau)$. The system $\{M(\Lambda(\tau))\}_{\tau \in T}$ is a $\mathbb{Q}$-rational system of representations in the sense of [Ha], 2.4 - i.e., the representations are conjugate under $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$.

As in [Ha], we study the cohomology of congruence subgroups of $GL_n(o)$ with coefficients in these modules: Form the quotients

$$S_K = G(\mathbb{Q}) \backslash G(\mathbb{Q}_{\mathbb{A}}) / K_\infty \cdot K_f \quad ,$$

where $K_\infty = \prod_{v|\infty} U(n) Z_\infty$ is a standard maximal compact subgroup, times the centre of $G(\mathbb{R}) = G_\infty$, and where K_f is open compact in $G(\mathbb{Q}_{\mathbb{A},f})$. The modules $M(\Lambda(\tau))$ provide coefficient systems $\widetilde{M(\Lambda(\tau))}$ on S_K , and we consider the $\overline{\mathbb{Q}}$ - $G(\mathbb{Q}_{\mathbb{A},f})$ - module

$$H^\bullet(\widetilde{S}, \widetilde{M(\Lambda(\tau))}) := \varinjlim_{K_f} H^\bullet(S_K, \widetilde{M(\Lambda(\tau))}) \ .$$

The embedding of S_K into its Borel-Serre compactification $\overline{S_K}$ is a homotopy equivalence. The boundary $\partial\overline{S_K}$ of this compactification has a stratification, with strata corresponding to the conjugacy classes of parabolic subgroups of $G/\mathbb{Q}$. The stratum of lowest dimension, $\partial_B\overline{S_K}$, corresponds to the conjugacy class of Borel-subgroups. The coefficient system can be extended to the boundary, and the limit

$$H^\bullet(\partial_B\widetilde{S}, \widetilde{M(\Lambda(\tau))}) = \varinjlim_{K_f} H^\bullet(\partial_B\overline{S_K}, \widetilde{M(\Lambda(\tau))})$$

is again a $G(\mathbb{Q}_{\mathbb{A},f})$ -module. The diagram

$$S_K \overset{i}{\hookrightarrow} \overline{S}_K \longleftarrow \partial_B\overline{S}_K$$

induces a $G(\mathbb{Q}_{\mathbb{A},f})$ -module homomorphism

$$r_B : H^\bullet(\widetilde{S}, \widetilde{M(\Lambda(\tau))}) \longrightarrow H^\bullet(\partial_B\widetilde{S}, \widetilde{M(\Lambda(\tau))}) \ .$$

Just as in [Ha] ,II, the right hand side turns out to be a direct sum of modules, induced from an algebraic Hecke character

$$\begin{array}{ccc} \eta : B(\mathbb{Q}_{\mathbb{A},f}) & \longrightarrow & \overline{\mathbb{Q}}^* \\ & \searrow \quad \nearrow & \\ & T(\mathbb{Q}_{\mathbb{A},f}) & \end{array}$$

on $B(\mathbb{Q}_{\mathbb{A},f})$, up to $G(\mathbb{Q}_{\mathbb{A},f})$. The types of these characters are determined by Kostant's theorem, [Ko] ; cf. [Ha], II, for $n = 2$. In particular, it is easily checked that the following induced module (for Φ as above, and $\tau \in T$) is contained in the cohomology of $\partial_B\widetilde{S}$:

$$V_{\tau\circ\Phi} = \operatorname{Ind}_{B(\mathbb{Q}_{\mathbb{A},f})}^{G(\mathbb{Q}_{\mathbb{A},f})} \tau\circ\Phi =$$

$$= \left\{ h : G(\mathbb{Q}_{\mathbb{A},f}) \longrightarrow \overline{\mathbb{Q}} \;\middle|\; \begin{array}{l} h \text{ is } C_\infty \text{, and} \\ h(\underline{b}_f \underline{g}_f) = (\tau\circ\Phi)(\underline{b}_f)\cdot h(\underline{g}_f) \,, \\ \text{for all } \underline{b}_f \in B(\mathbb{Q}_{\mathbb{A},f}) \text{ and} \\ \underline{g}_f \in G(\mathbb{Q}_{\mathbb{A},f}) \end{array} \right\}$$

(Here, "C_∞" means right invariance under a suitably small open compact subgroup in $G(\mathbb{Q}_{\mathbb{A},f})$.)

More precisely, we have

$$V_{\tau\circ\Phi} \xhookrightarrow{i_\tau} H^{(n-1)d_\circ}(\partial_B \tilde{S}, \widetilde{M(\Lambda(\tau))}) \quad ,$$

where $d_\circ = \frac{1}{2}[k:\mathbb{Q}]$, and the system of maps $\{i_\tau\}_{\tau\in T}$ is $\mathbb{Q}$-rational with respect to the two obvious $\mathbb{Q}$-structures on the systems on both sides.

Consider the non-trivial submodule

$$J_{\tau\circ\Phi} = \operatorname{Ind}_{P(\mathbb{Q}_{\mathbb{A},f})}^{G(\mathbb{Q}_{\mathbb{A},f})} \tau\circ\Phi \subset V_{\tau\circ\Phi} \quad .$$

Obviously, $\{J_{\tau\circ\Phi}\}_{\tau\in T}$ is a $\mathbb{Q}$-rational system of $G(\mathbb{Q}_{\mathbb{A},f})$-submodules of $H^{(n-1)d_\circ}(\partial_B \tilde{S}, \widetilde{M(\Lambda(\tau))})$. The first essential step of the proof is to construct a $\mathbb{Q}$-rational "section" of r_B ,

$$\mathrm{Eis}_\tau : J_{\tau\circ\Phi} \longrightarrow H^{(n-1)d_\circ}(\tilde{S}, \widetilde{M(\Lambda(\tau))}) \quad ,$$

for all $\tau\in T$. Thus, $r_B \circ \mathrm{Eis}_\tau = \mathrm{Id}$ on $J_{\tau\circ\Phi}$. This section is constructed first over $\mathbb{C}$ by means of residual Eisenstein series or, in other words, non cuspidal Eisenstein series attached to $P/\mathbb{Q}$. To prove that $\{\mathrm{Eis}_\tau\}_{\tau\in T}$ is defined over $\mathbb{Q}$ one has to use a multiplicity one argument, like in [Ha], III. But here this is more complicated. One has to use the spectral sequence which computes the cohomology of the boundary in terms of the cohomology of the strata. Then the cohomology has to be related to automorphic forms, and one has to appeal to results of Jacquet-Shalika on multiplicity one, and of Jacquet on the discrete non cuspidal spectrum.

Once we have the modules

$$\mathrm{Eis}_{\tau}\ (J_{\tau\circ\Phi}) \subset H^{(n-1)d_{\circ}}(\tilde{S}, \widetilde{M(\Lambda(\tau))})$$

we can proceed more or less in the same way as in [Ha], V: We construct an embedding

$$i_H : F^* \longrightarrow GL_n(k) \quad ,$$

H being the torus with $H(\mathbb{Q}) = i_H(F^*)$. Using this torus we can construct homology classes (compact modular symbols)

$$\mathfrak{Z}(i_H, \tau\circ\chi, \underline{g}) \in H_{(n-1)d_{\circ}}(\tilde{S}, \widetilde{M(\Lambda(\tau))})$$

depending on a point $\underline{g} \in G(\mathbb{Q}_{\mathbb{A}})$ and on a finite order character

$$\chi : F^*_{\mathbb{A}}/F^* \longrightarrow E^*$$

whose restriction to $k^*_{\mathbb{A}}$ should be ω .

As in [Ha], V, we get an intertwining operator

$$\mathrm{Int}(\mathfrak{Z}(i_H,\chi)) : J_{\tau\circ\Phi} \longrightarrow \mathrm{Ind}_{H(\mathbb{Q}_{\mathbb{A},f})}^{G(\mathbb{Q}_{\mathbb{A},f})}\ \tau\circ\chi$$

by evaluating $\mathrm{Eis}_{\tau}(J_{\tau\circ\Phi})$ on $\mathfrak{Z}(i_H,\chi,\underline{g})$. There is another intertwining operator

$$\mathrm{Int}^{\mathrm{loc}} : J_{\tau\circ\Phi} \longrightarrow \mathrm{Ind}_{H(\mathbb{Q}_{\mathbb{A},f})}^{G(\mathbb{Q}_{\mathbb{A},f})}\ \tau\circ\chi \quad ,$$

constructed as a product of local intertwining operators. Both operators are $\mathbb{Q}$-rational, and for some $x \in E^*$ we find that, for all $\tau \in T$,

$$\mathrm{Int}(\mathcal{Z}(i_H,\tau\circ\chi)) = \tau(x)\,\Delta(F/k,\tau\circ\beta)\,\frac{L_F(\tau\circ(\chi\cdot(\Psi\circ N_{F/k})),0)}{L_k(\tau\circ(\omega\cdot\Psi^n),0)}\,\mathrm{Int}^{\mathrm{loc}}\;.$$

This implies theorem 2. - The factor $x \in E^*$ can actually be given more explicitly.

§ 6. A formula of Lerch

The fact that a Hecke character determines its motive up to isomorphism produces a period relation whenever two different geometric constructions of a motive for the same character can be given. We have seen a first example of this principle in formula (8) of § 4. The periods $p(\sigma,\tau)$ occuring in this formula comprise those for which Shimura [Sh2] has proved various monomial period relations (up to an algebraic number). These monomial relations were reproven, by means of motives over $\overline{\mathbb{Q}}$, by Deligne, [D2]. They can be refined using the above principle. But we leave aside here this application, as well as some others, referring the reader to [Sch]. Instead, let us concentrate on a typical case involving G. Anderson's motives for Jacobi-sum Hecke characters.

Let $K = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field of discriminant $-D$. Assume for simplicity that $D > 4$. Recall the construction of the simplest Jacobi-sum Hecke character of K, in the sense of [W III], [1974 d]: K is contained in $\mathbb{Q}(\mu_D)$, the field of D-th roots of 1. Write $n = [\mathbb{Q}(\mu_D) : K] = \frac{\varphi(D)}{2}$. For a prime ideal $\mathfrak{P}$ of $\mathbb{Q}(\mu_D)$ not dividing D, put

$$G(\mathfrak{P}) = -\sum_{x\in\mathbb{Z}[\mu_D]/\mathfrak{P}} \chi_{D,\mathfrak{P}}(x)\cdot\lambda(x)\quad ,$$

with "$\chi_{D,\mathfrak{P}}(x) \equiv x^{(\mathbb{N}\mathfrak{P}-1)/D} \pmod{\mathfrak{P}}$" the $D^{\underline{th}}$-power residue symbol mod $\mathfrak{P}$, and $\lambda(x) = \exp(2\pi i\cdot \mathrm{tr}_{(\mathbb{Z}[\mu_D]/\mathfrak{P})/\mathbb{F}_p}(x))$.

Then extend the function of prime ideals $\mathfrak{p}$ of K with $\mathfrak{p} \nmid D$:

$$J(\mathfrak{p}) = \prod_{\mathfrak{P} | \mathfrak{p}} G(\mathfrak{P}) ,$$

multiplicatively to all ideals of K prime to D . Elementary properties of Gauss sums show that J takes values in K . By a theorem of Stickelberger and an explicit version of the analytic class number formula for K one finds that, if J is an algebraic Hecke character, then its infinity-type

$$\beta : K^{*} \longrightarrow K^{*}$$

is given by

$$x \longmapsto x^{(n+h)/2} \, \bar{x}^{(n-h)/2} ,$$

where h is the class number of K . (Note that n and h have the same parity, by genus theory.) In other words, if J is a Hecke character, then

(13) $$J . \mathbb{N}^{-(n+h)/2} = \mu \cdot \psi^{h} ,$$

for some character μ of $K^{*}_{\mathbb{A}}$ of finite order and some Hecke character ψ of K of weight -1 .

That J is in fact a Hecke character, i.e., is well-behaved at the places dividing D , if viewed on idèles, was proved by Weil (loc. cit). But it can also be deduced, e.g., from the following construction of a motive for J which was given by Greg Anderson, [A1],[A2].

Anderson finds a motive defined over K , with coefficients in K , whose ℓ-adic representations are given by J , in H^{n-2} of the zero-set Z of the function

$$\begin{aligned} \mathbb{Q}(\mu_D) &\longrightarrow K \\ x &\longmapsto tr_{\mathbb{Q}(\mu_D)/K}(x^D) \end{aligned} ,$$

viewed as a projective variety in $\mathbb{P}_K(\mathbb{Q}(\mu_D))$, the projective space of the K-vector space $\mathbb{Q}(\mu_D)$. Note that

$$Z \times_K \mathbb{Q}(\mu_D) = \left\{ x_1^D + \ldots + x_n^D = 0 \right\} \subset \mathbb{P}^{n-1} .$$

Anderson's construction is of course motivated by the well-known fact that Fermat-hypersurfaces contain motives (carved out by the action of their large automorphism groups) attached to Jacobi-sum Hecke characters of cyclotomic fields: see [DMOS], pp. 79 - 96. For details of Anderson's more general construction, we refer to his preprints, or to [Sch].

At any rate, thanks to Anderson's work, we have at our disposal a motive $M(J)$ for the character J , which lies in the category of motives obtained from abelian varieties. (This last fact is proved by Shioda-induction: [DMOS], p. 217). Thus, by (13), the periods of the motive

$$M(J\, \mathbb{N}^{-(n+h)/2}) = M(J) \otimes_K K(n+h)/2)$$

will be the same as those of any motive constructed for the character $\mu \cdot \psi^h$.

The period calculations on Fermat-hypersurfaces always reduce eventually to Beta-integrals. For $M(J)$ one essentially gets the product

$$\prod_{\chi(a)=-1} \Gamma\left(\left\langle \frac{a}{D} \right\rangle\right)^{-1} .$$

Here, $\chi(p) = (\frac{-D}{p})$ is the Dirichlet character of the quadratic field K , and the product is taken over those $a \in (\mathbb{Z}/D\mathbb{Z})^*$ for which $\chi(a) = -1$. $\langle \frac{a}{D} \rangle$ is the representative of the class $\frac{a}{D} \bmod \mathbb{Z}$ which lies between 0 and 1 .

A motive for $\mu\cdot\Psi^h$ can be built up from elliptic curves with complex multiplication by K. - Assume for simplicity that $\Psi\circ N_{H/K}$ takes values in K^*, for H the Hilbert class field of K. Choose any elliptic curve A/H such that $H_1(A)$ is a motive for $\Psi\circ N_{H/K}$, and call $B = R_{H/K}A$ its restriction of scalars to K. Calling E the field of values of Ψ, $H_1(B)^{\otimes_E h}$ can be shown to be a motive for Ψ^h (viewed as taking values in E): cf. [GS], § 4. Using formulas derived in [GS], § 9, the periods of this motive can be computed in terms of the periods Ω_σ of the conjugates A^σ/H of our elliptic curve A, for

$$\sigma \in \mathrm{Gal}(H/K) = C\ell(K) .$$

Straightening out the twists by the norm and the finite order character (cf. § 4.6 and 4.7), one finally obtains the following relation, up to a factor of K^*:

(14) $$y \cdot \prod_{\sigma \in C\ell(K)} \left(\sqrt{\frac{D}{2\pi}} \cdot \Omega_\sigma \right) \sim \prod_{\chi(a)=-1} \frac{\sqrt{\pi}}{\Gamma(<\frac{a}{D}>)} ,$$

where y generates the abelian extension of K belonging to μ. Multiplying (14) with its complex conjugate, we get

(15) $$z \cdot \prod_{\sigma \in C\ell(K)} \left(\frac{D}{2\pi} \ \Omega_\sigma \bar{\Omega}_\sigma \right) = \prod_{a \in (\mathbb{Z}/D\mathbb{Z})^*} \Gamma(<\frac{a}{D}>)^{\chi(a)} ,$$

for some z with $z^4 \in \mathbb{Q}^*$. Except for the different interpretation of z and the Ω_σ, this is the exponential of an identity proved analytically by Lerch in [Le], p.303. The first geometric proof of (15), up to a factor in $\bar{\mathbb{Q}}^*$, was given by Gross in [Gr], a paper which in turn inspired Deligne's proof of the theorem about absolute Hodge cycles on abelian varieties - which again is essential in proving uniqueness of the motive for an algebraic Hecke character.

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