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LOCALIZATION IN EQUIVARIANT OPERATIONAL K-THEORY AND THE CHANG-SKJELBRED PROPERTY

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Abstract. We establish a localization theorem of Borel-Atiyah-Segal type for the equivariant operational K-theory of Anderson and Payne [AP]. Inspired by the work of Chang-Skjelbred and Goresky-Kottwitz-MacPherson, we establish a general form of GKM theory in this setting, applicable to singular schemes with torus action. Our results are deduced from those in the smooth case via Gillet-Kimura's technique of cohomological descent for equivariant envelopes. As an application, we extend Uma's description of the equivariant K-theory of smooth compactifications of reductive groups to the equivariant operational K-theory of all, possibly singular, projective group embeddings.

1. Introduction and Motivation

Goresky, Kottwitz and MacPherson, in their seminal paper [GKM], developed a theory, nowadays called GKM theory, that makes it possible to describe the equivariant cohomology of certain T-*skeletal* varieties: complete algebraic varieties upon which a complex algebraic torus T acts with a finite number of fixed points and invariant curves. Let X be a T-skeletal variety and denote by X^T the fixed point set. The main purpose of GKM theory is to identify the image of the functorial map $i^* : H^*_T(X) \to H^*_T(X^T)$, assuming X is *equivariantly formal*. GKM theory has been mostly applied to smooth projective T-skeletal varieties, because of the Bialynicki-Birula decomposition [B1]. Additionally, the GKM data issued from the fixed points and invariant curves has been explicitly obtained for some interesting cases: flag varieties [C], and regular embeddings of reductive groups [Br1, Br2]. In contrast, regarding singular varieties, GKM theory has been applied to Schubert varieties [C] and to *rationally smooth* projective group embeddings, due to the author's work [G1, G2].

Because of its power as a computational tool, GKM theory has been implemented in other equivariant cohomology theories on schemes with torus actions. For instance, Brion established GKM theory for equivariant Chow groups [Br1], Vezzosi-Vistoli did it for equivariant algebraic K-theory [VV], and Krishna provided the tool in equivariant algebraic cobordism [Kr]. Nevertheless, in all of these generalizations, a crucial assumption on *smoothness* of the ambient space needs to be made.

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This paper is concerned with the equivariant K-theory of possibly *singular* schemes equipped with an action of an algebraic torus T (i.e. T-schemes). Our main goal is to increase the applicability of GKM theory as a tool for understanding the geometry of singular T-schemes in this setting. For convenience of the reader, we briefly review some of the basic underlying notions, as well as the previous progress made on this problem. Equivariant K-theory was developed by Thomason [Th1]. Let X be a T-scheme. Let $K_T(X)$ denote the Grothendieck group of T-equivariant vector bundles on X. This is a ring, with the product given by the tensor product of equivariant vector bundles. Let $K^T(X)$ denote the Grothendieck group of T-equivariant coherent sheaves on X. This is a module for the ring $K_T(X)$. If we identify the representation ring $R(T)$ with $K_T(pt)$, then pullback by the projection $X \to pt$ gives a natural map $R(T) \to K_T(X)$. In this way, $K_T(X)$ becomes an $R(T)$-algebra and $K^T(X)$ an $R(T)$-module. The functor $K_T(-)$ is contravariant with respect to arbitrary equivariant maps. In contrast, $K^T(-)$ is covariant for equivariant proper morphisms and contravariant for equivariant flat maps. If X is smooth, then every T-equivariant coherent sheaf has a finite resolution by T-equivariant locally free sheaves, and thus $K_T(X) \simeq K^T(X)$. When X is complete, the equivariant Euler characteristic

$$\mathcal{F} \mapsto \chi(X, \mathcal{F}) = \sum_i (-1)^i [H^i(X, \mathcal{F})]$$

yields the pushforward map $\chi : K^T(X) \longrightarrow K^T(pt) \simeq R(T)$. By work of Merkurjev [M1], one recovers the usual K-theory from the equivariant one via the identity $K^T(X) \otimes_{R(T)} \mathbb{Z} \simeq K(X)$.

In general, the K-theory groups are difficult to compute. In the case of singular varieties, they can be quite large [AP, Introduction, p. 2]. In the smooth case, however, there are three powerful theorems that allow many computations and important comparison theorems of Riemann-Roch type. The first one is the localization theorem of Borel-Atiyah-Segal type.

Localization theorem of Borel-Atiyah-Segal type ([Th2, Théorème 2.1])**.** *Let X be a smooth complete scheme with an action of T. Let X^T be the subscheme of fixed points and let $i_T : X^T \to X$ be the natural inclusion. Then the pullback $i_T^* : K_T(X) \to K_T(X^T)$ is injective, and it becomes surjective over the quotient field of $R(T)$.*

Let X be a smooth complete T-scheme. The second fundamental theorem in this context identifies the image of i_T^* inside $K_T(X^T) \simeq K(X^T) \otimes R(T)$. To state it, we introduce some notation. Let $H \subset T$ be a subtorus of codimension one. Observe that i_T factors as $i_{T,H} : X^T \to X^H$ followed by $i_H : X^H \to X$. Thus, the image of i_T^* is contained in the image of $i_{T,H}^*$. In symbols,

$$\mathrm{Im}[i_T^* : K_T^0(X) \to K_T^0(X^T)] \subseteq \bigcap_{H \subset T} \mathrm{Im}[i_{T,H}^* : K_T^0(X^H) \to K_T^0(X^T)],$$

where the intersection runs over all codimension-one subtori H of T. This criteria, which dates back to the work of Chang-Skjelbred [CS] in equivariant cohomology, yields a complete description of $K_T(X)$ as a subring of $K_T(X^T) \simeq K(X^T) \otimes R(T)$.

CS property ([VV, Theorem 2]). *Let X be a smooth complete T-scheme. Then the image of the injective map $i_T^* : K_T(X) \to K_T(X^T)$ equals the intersection of the images of $i_{T,H}^* : K_T(X^H) \to K_T(X^T)$, where H runs over all subtori of codimension one in T.*

Now let X be a (complete) T-skeletal variety. Assume, for simplicity, that each T-invariant irreducible curve has exactly two fixed points (e.g. X is equivariantly embedded in a normal T-variety). In this setting, it is possible to define a ring $PE_T(X)$ of *piecewise exponential* functions. Indeed, let $K_T(X^T) = \oplus_{x\in X^T} R_x$, where R_x is a copy of the representation ring $R(T)$. We then define $PE_T(X)$ as the subalgebra of $K_T(X^T)$ given by

$$PE_T(X) = \{(f_1, \ldots, f_m) \in \oplus_{x\in X^T} R_x \,|\, f_i \equiv f_j \mod 1 - e^{-\chi_{i,j}}\}$$

where x_i and x_j are the two distinct fixed points in the closure of the one-dimensional T-orbit $C_{i,j}$, and $\chi_{i,j}$ is the character of T associated with $C_{i,j}$. This character is uniquely determined up to sign (permuting the two fixed points changes $\chi_{i,j}$ to its opposite). In light of the CS property, one obtains:

GKM theorem ([VV, Corollary 5.12], [U, Theorem 1.3]). *Let X be a smooth T-skeletal variety. Then $i_T^* : K_T(X) \to K_T(X^T)$ induces an isomorphism between $K_T(X)$ and $PE_T(X)$. If X is also projective, then $K_T(X)$ is a free $R(T)$-module of rank $|X^T|$.*

Thus far, it is clear that to any complete T-skeletal variety X we can associate the ring $PE_T(X)$, regardless of whether X is smooth or not. (In fact, if X is a projective compactification of a reductive group G with maximal torus T, then X is $T \times T$-skeletal, and $PE_{T\times T}(X)$ has been explicitly identified in [G2].) Nonetheless, as it stems from the previous facts, $PE_T(X)$ does not always describe $K_T(X)$. This phenomena yields some natural *questions:* Let X be a T-skeletal variety. What kind of information does $PE_T(X)$ encode? If not equivariant K-theory, is it still reasonable to expect that $PE_T(X)$ encodes certain topological/geometric information that is *common* to all possible T-equivariant resolution of singularities of X? The work of Payne [P] and Anderson-Payne [AP], inspired in turn by the works of Fulton-MacPherson-Sottile-Sturmfels [FMSS] and Totaro [To], gives a positive answer to these questions *when X is a toric variety.* Namely, the GKM data (i.e. $PE_T(X)$) of a toric variety encodes all the information needed to reconstruct Bott-Chern operators defined on the structure sheaves $\mathcal{O}_{\overline{Tx}}$ of the T-orbit closures $\overline{Tx} \subseteq X$ (and their equivariant resolutions). This positive result is our motivation. In the pages to follow we will show that Anderson-Payne's assertion on toric varieties holds more generally for *all* T-skeletal varieties. But first, and in order to put these statements in a much clearer form, we recall some of the main aspects of Anderson and Payne's equivariant operational K-theory.

Fulton and MacPherson [FM] devised a machinery that produces a cohomology theory out of a homology theory. This cohomology has all the formal properties one could hope for, and it is well suited for the study of singular schemes. Taking as input the homology functor $K^T(-)$, Anderson-Payne [AP] obtained a theory that is very well suited for computations. Moreover, it agrees with Thomason's equivariant K-theory when X is smooth (Properties (a) and (b) below). We outline here the main notions of [AP]. Let X be a T-scheme. The *T-equivariant operational K-theory* ring of X, denoted $\mathrm{op}K_T(X)$, is defined as follows: an element $c \in \mathrm{op}K_T(X)$ is a collection of homomorphisms $c_f : K^T(Y) \to K^T(Y)$ for every T-map $f : Y \to X$. (Recall that $K^T(Y)$ denotes the Grothendieck group of T-equivariant coherent sheaves on Y.) These homomorphisms must be compatible with (T-equivariant) proper pushforward, flat pullback and Gysin morphisms [AP]. For any X, the ring structure on $\mathrm{op}K_T(X)$ is given by composition of such homomorphisms. With this product, $\mathrm{op}K_T(X)$ becomes an associative commutative ring with unit. Moreover, $\mathrm{op}K_T(X)$ is contravariantly functorial in X. Other salient functorial properties of $\mathrm{op}K_T(-)$ are:

(a) For any X, there is a canonical homomorphism $K_T(X) \to \mathrm{op}K_T(X)$ of $R(T)$-algebras, sending a class γ to the operator $[\gamma]$ which acts via $[\gamma]_g = g^*\gamma \cdot \xi$, for any T-map $g : Y \to X$ and $\xi \in K^T(Y)$. There is also a canonical map $\mathrm{op}K_T(X) \to K^T(X)$ defined by $c \mapsto c_{\mathrm{id}_X}[\mathcal{O}_X]$, where $\mathcal{O}_X$ is the structure sheaf of X. Put together, they provide a factorization of the canonical homomorphism $K_T(X) \to K^T(X)$ [AP, Theorem 5.6].

(b) When X is smooth, the homomorphisms

$$K_T(X) \to \mathrm{op}K_T(X) \to K^T(X),$$

defined in (a), are all isomorphisms of $R(T)$-modules [AP, Corollary 4.5 and Theorem 5.6].

(c) *$\mathbb{A}^1$-homotopy invariance* [AP, Corollary 4.7]*:* For any scheme X, the natural pull back map from $\mathrm{op}K_T(X)$ to $\mathrm{op}K_T(X \times \mathbb{A}^1)$ is an isomorphism.

(d) *Gillet-Kimura's cohomological descent for equivariant envelopes* [AP, Theorem 5.3]*:* If $\pi : \tilde{X} \to X$ is an equivariant envelope (that is, any T-invariant subvariety of X is the birational image of a T-invariant subvariety of $\tilde{X}$) and π_1, π_2 are the projections $\tilde{X} \times_X \tilde{X} \to \tilde{X}$, then the following sequence is exact

$$0 \longrightarrow \mathrm{op}K_T(X) \xrightarrow{\pi^*} \mathrm{op}K_T(\tilde{X}) \xrightarrow{\pi_1^* - \pi_2^*} \mathrm{op}K_T(\tilde{X} \times_X \tilde{X}).$$

(e) Let X be a complete T-variety. If X is a toric variety (i.e. X is normal and has a dense orbit isomorphic to T), then $\mathrm{op}K_T(X) \simeq PE_T(X)$ [AP, Theorem 1.6]. Similar results hold for non-complete toric varieties [AP].

(f) *Equivariant Kronecker duality for spherical varieties* [AP, Theorem 6.1]*:* Let B be a connected solvable linear algebraic group with maximal torus T. Let X be a scheme with an action of B. If B acts on X with finitely many orbits, then the natural equivariant Kronecker map

$$\mathcal{K}_T : \mathrm{op}K_T(X) \to \mathrm{Hom}_{R(T)}(K^T(X), R(T)),$$

induced by pushforward to a point, namely,

$$\mathcal{K}_T : \; c \longmapsto \{\xi \mapsto \chi(X, c_{\mathrm{id}_X}(\xi))\},$$

is an isomorphism. This holds e.g. for Schubert varieties and spherical varieties. There is a more general version of equivariant Kronecker duality, valid for T-linear schemes ([AP, Section 6]). This class encompasses all the B-schemes mentioned above (see e.g. [G3, Theorem 2.5]). For equivariant Kronecker duality in the context of equivariant operational Chow groups, see [G3, Theorem 3.6].

In this paper we use the functorial properties listed above, together with resolution of singularities, to establish:

(I) The localization theorem of Borel-Atiyah-Segal type for $\mathrm{op}K_T(X)$, whenever X is a complete T-scheme (Theorem 4.1).

(II) The CS property for $\mathrm{op}K_T(X)$, where X is any complete T-scheme (Theorem 4.4).

(III) GKM theory for possibly singular complete T-varieties: if X is a T-skeletal variety, then $\mathrm{op}K_T(X) \simeq PE_T(X)$ (Theorem 5.4).

Together with the combinatorial results of [G2], this extends Anderson's and Payne's work on toric varieties to all projective group embeddings (Theorems 6.2 and 6.4). See Section 7 (as well as [G3]) for the corresponding statements in operational Chow groups with rational coefficients.

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2. Conventions and Notation

Conventions. Throughout this paper, we fix an algebraically closed field k of characteristic zero. All schemes and algebraic groups are assumed to be defined over k. By a scheme we mean a separated scheme of finite type. A variety is a reduced scheme. Observe that varieties need not be irreducible. A subvariety is a closed subscheme which is a variety. A curve on a scheme is an irreducible one-dimensional subscheme. Unless explicit mention is made to the contrary, we will assume all schemes are equidimensional. A point on a scheme will always be a closed point.

Notation. We denote by T an algebraic torus. A scheme X provided with an algebraic action of T is called a *T-scheme.* If X is a T-scheme, the class in $K^T(X)$ of a T-equivariant coherent sheaf $\mathcal{F}$ will be denoted by $[\mathcal{F}]$. In particular, if $Y \subset X$ is a T-stable closed subscheme, then the structure sheaf of Y defines a class $[\mathcal{O}_Y]$ in $K^T(X)$. For a T-scheme X, we denote by X^T the fixed point subscheme and by $i_T : X^T \to X$ the natural inclusion. If H is a closed subgroup of T, we similarly denote by $i_H : X^H \to X$ the inclusion of the fixed point subscheme. When comparing X^T and X^H we write $i_{T,H} : X^T \to X^H$ for the natural (T-equivariant) inclusion. If $g : Y \to X$ is a T-equivariant morphism of T-schemes, then we write $g_T : Y^T \to X^T$, or simply $g : Y^T \to X^T$, for the associated morphism of fixed point subschemes. Likewise, we write $g^* : \mathrm{op}K_T(X) \to \mathrm{op}K_T(Y)$ for the pullback in equivariant operational K-theory.

We denote by Δ the character group of T, and by $\mathbb{Z}[\Delta]$ the group ring over $\mathbb{Z}$ of Δ. We let e^χ denote the element of $\mathbb{Z}[\Delta]$ corresponding to $\chi \in \Delta$. Then $\{e^\chi\}_{\chi\in\Delta}$ is a basis of the $\mathbb{Z}$-module $\mathbb{Z}[\Delta]$. For a k-linear representation V of T, we put

$$\mathrm{tr}(V) = \sum_{\chi\in\Delta} (\mathrm{rank}_k V_\chi) e^\chi,$$

where V_χ is the subspace of invariants of T of weight χ in V. It is well-known that tr induces an isomorphism from the representation ring of T, denoted $R(T)$, to $\mathbb{Z}[\Delta]$.

3. Equivariant envelopes and computability of equivariant operational K-theory

Recall that an envelope $p : \tilde{X} \to X$ is a proper map such that for any subvariety $W \subset X$ there is a subvariety $\tilde{W}$ mapping birationally to W via p ([F, Definition 18.3]). In the case of T-actions, we say that $p : \tilde{X} \to X$ is an *equivariant envelope* if p is T-equivariant, and if we can take $\tilde{W}$ to be T-invariant for T-invariant W. If there is an open set $U \subset X$ over which p is an isomorphism, then we say that $p : \tilde{X} \to X$ is a *birational* envelope. The following is recorded in [EG-2, Proposition 7.5].

Lemma 3.1. *Let X be a T-scheme. Then there exists a T-equivariant birational envelope $p : \tilde{X} \to X$, where $\tilde{X}$ is a smooth quasi-projective T-scheme.* □

Anderson and Payne's version of Gillet and Kimura's notion of cohomological descent (Property (d), Introduction) implies that $\mathrm{op}K_T(X)$ of a singular scheme X injects into $\mathrm{op}K_T(\tilde{X})$ of a smooth equivariant envelope (which is the usual equivariant K-theory ring of a smooth scheme) with an explicit cokernel. More precisely, suppose that $p : \tilde{X} \to X$ is a T-equivariant birational envelope which is an isomorphism over an open set $U \subset X$. Let $\{Z_i\}$ be the irreducible components of $Z = X - U$, and let $E_i = p^{-1}(Z_i)$, with $p_i : E_i \to Z_i$ denoting the restriction of p. The next theorem is Kimura's fundamental result [Ki, Theorem 3.1]) adapted to equivariant operational K-theory.

Theorem 3.2 ([AP, Theorem 5.4])**.** *Let $p : \tilde{X} \to X$ be a T-equivariant envelope. Then the induced map $p^* : \mathrm{op}K_T(X) \to \mathrm{op}K_T(\tilde{X})$ is injective. Furthermore, if p is birational (and notation is as above), then the image of p^* is described inductively as follows: a class $\tilde{c} \in \mathrm{op}K_T(\tilde{X})$ equals $p^*(c)$, for some $c \in \mathrm{op}K_T(X)$ if and only if, for all i, we have $\tilde{c}|_{E_i} = p_i^*(c_i)$ for some $c_i \in \mathrm{op}K_T(Z_i)$.* □

Since E_i and Z_i may be arranged to have smaller dimension than that of X, we can use this result to compute $\mathrm{op}K_T(X)$ using a resolution of singularities (Lemma 3.1) and induction on dimension. This is one of the reasons why cohomological descent (Property (d), Introduction) makes equivariant operational K-theory more computable than the Grothendieck ring of equivariant vector bundles, when it comes to singular T-schemes.

Corollary 3.3. *Notation being as above, the sequence*

$$0 \longrightarrow \mathrm{op}K_T(X) \longrightarrow \mathrm{op}K_T(\tilde{X}) \oplus \mathrm{op}K_T(Z) \longrightarrow \mathrm{op}K_T(E)$$

is exact, where $E = p^{-1}(Z)$. □

Corollary 3.4. *Let Y be a T-scheme, and let $Y = \cup_{i=1}^n Y_i$ be the decomposition of Y into irreducible components. Let $Y_{ij} = Y_i \cap Y_j$. Then the sequence*

$$0 \to \mathrm{op}K_T(Y) \to \bigoplus_i \mathrm{op}K_T(Y_i) \to \bigoplus_{i,j} \mathrm{op}K_T(Y_{ij}).$$

is exact.

Proof. First recall that $\bigsqcup_i Y_i \to Y$ is an equivariant envelope. Now use cohomological descent to get the result. □

The following was first observed in [EG-2, Lemma 7.2].

Lemma 3.5. *Let X be a T-scheme, and let $\pi : \tilde{X} \to X$ be an equivariant envelope. If H is a closed subgroup of T, then the induced map $\tilde{X}^H \to X^H$ is also a T-equivariant envelope.*

Proof. The argument here is basically that of [EG-2, Lemma 7.2]. First, notice that the map $\pi_H : \tilde{X}^H \to X^H$ is T-equivariant, because T is an abelian group. Now let $W \subset X^H$ be a T-invariant irreducible subvariety and let $\tilde{W}$ be an irreducible subvariety of $\tilde{X}$ mapping birationally to W via π. To prove that $\tilde{X}^H$ is an equivariant envelope, it suffices to prove that we can take $\tilde{W} \subset \tilde{X}^H$. The restricted map $\pi : \tilde{W} \to W$ is a T-equivariant isomorphism over a dense open subspace U of W. Replace $\tilde{W}$ with the closure of $\pi^{-1}(U)$. Because H acts trivially on $\pi^{-1}(U)$ (for $U \subset W \subset X^H$), and $\tilde{W}^H$ is closed, we get $\tilde{W} \subset \tilde{X}^H$, as desired. □

An important technical result is stated next.

Corollary 3.6. *Let $p : \tilde{X} \to X$ be an equivariant envelope. If H is a closed subgroup of T, then the diagram of exact sequences*

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{op}K_T(X) & \longrightarrow & \mathrm{op}K_T(\tilde{X}) & \longrightarrow & \mathrm{op}K_T(\tilde{X} \times_X \tilde{X}) \\
& \downarrow i_H^* & & \downarrow i_H^* & & \downarrow i_H^* \\
0 \longrightarrow & \mathrm{op}K_T(X^H) & \longrightarrow & \mathrm{op}K_T(\tilde{X}^H) & \longrightarrow & \mathrm{op}K_T(\tilde{X}^H \times_{X^H} \tilde{X}^H).
\end{array}$$

commutes. Moreover, if p is birational, and notation being as in Theorem 3.2, then the diagram of exact sequences

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{op}K_T(X) & \longrightarrow & \mathrm{op}K_T(\tilde{X}) \oplus \mathrm{op}K_T(Z) & \longrightarrow & \mathrm{op}K_T(E) \\
& \downarrow i_H^* & & \downarrow i_H^* & & \downarrow i_H^* \\
0 \longrightarrow & \mathrm{op}K_T(X^H) & \longrightarrow & \mathrm{op}K_T(\tilde{X}^H) \oplus \mathrm{op}K_T(Z^H) & \longrightarrow & \mathrm{op}K_T(E^H).
\end{array}$$

commutes.

Proof. First, apply cohomological descent to $p : \tilde{X} \to X$. Due to Lemma 3.5, we can also apply this tool to the T-equivariant envelope $p_H : \tilde{X}^H \to X^H$, noticing that $(\tilde{X} \times_X \tilde{X})^H = \tilde{X}^H \times_{X^H} \tilde{X}^H$. Now write the associated short exact sequences as the rows of the first square diagram displayed above. An straightforward check shows that the diagram is commutative. A similar argument yields the second assertion, in view of Corollary 3.3. □

In the upcoming proposition we state another crucial consequence of Kimura's work. Put in perspective, it asserts that the equivariant operational K-theory ring $\mathrm{op}K_T(X)$ of *any* complete T-scheme X is a subring of $\mathrm{op}K_T(X^T)$. Moreover, there is a natural isomorphism

$$\mathrm{op}K_T(X^T) \simeq \mathrm{op}K_T(X^T) \otimes_{\mathbb{Z}} R(T),$$

by [AP, Corollary 5.5]. In many cases of interest, X^T is finite (e.g. for spherical varieties) and so one has $\mathrm{op}K_T(X) \subseteq \bigoplus_1^\ell \mathrm{op}K_T(pt) = R(T)^\ell$, where $\ell = |X^T|$. This motivates our introduction of localization techniques, and ultimately GKM theory, into the study of the functor $\mathrm{op}K_T(-)$.

Proposition 3.7. *Let X be a complete T-scheme and let $i_T : X^T \to X$ be the inclusion of the fixed point subscheme. Then the pull-back map*

$$i_T^* : \mathrm{op}K_T(X) \to \mathrm{op}K_T(X^T)$$

is injective.

Proof. First, choose a T-equivariant envelope $p : \tilde{X} \to X$, with $\tilde{X}$ projective and smooth (Lemma 3.1). Thus $p^* : \mathrm{op}K_T(X) \to \mathrm{op}K_T(\tilde{X})$ is injective (Theorem 3.2). Since $\tilde{X}$ is smooth and projective, the pull-back $i_T^* : \mathrm{op}K_T(\tilde{X}) \to \mathrm{op}K_T(\tilde{X}^T)$, is injective (by Property (b) of the Introduction together with the CS property for smooth T-schemes [VV, Theorem 2]). Besides, the chain of inclusions $\tilde{X}^T \subset p^{-1}(X^T) \subset \tilde{X}$ indicate that $\tilde{i}_T^*$ factors through $\iota^* : \mathrm{op}K_T(\tilde{X}) \to \mathrm{op}K_T(p^{-1}(X^T))$, where $\iota : p^{-1}(X^T) \hookrightarrow \tilde{X}$ is the natural inclusion. Thus, ι^* is injective as well. Finally, adding this information to the commutative diagram

$$\begin{array}{ccc} \mathrm{op}K_T(X) & \xrightarrow{p^*} & \mathrm{op}K_T(\tilde{X}) \\ \downarrow{\scriptstyle i_T^*} & & \downarrow{\scriptstyle \iota^*} \\ \mathrm{op}K_T(X^T) & \xrightarrow{p^*} & \mathrm{op}K_T(p^{-1}(X^T)). \end{array}$$

renders $i_T^* : \mathrm{op}K_T(X) \to \mathrm{op}K_T(X^T)$ injective. ☐

Corollary 3.8. *Let X be a complete T-scheme. Let Y be a T-invariant closed subscheme containing X^T. Denote by $\iota : Y \to X$ the natural inclusion. Then the $R(T)$-algebra map $\iota^* : \mathrm{op}K_T(X) \to \mathrm{op}K_T(Y)$ is injective. In particular, if H is a closed subgroup of T, then $i_H^* : \mathrm{op}K_T(X) \to \mathrm{op}K_T(X^H)$ is injective.*

Proof. Simply notice that $\iota : Y \to X$ fits into the commutative triangle

$$\begin{array}{ccccc} & & Y & & \\ & {\scriptstyle i_{T,Y}}\nearrow & & \searrow{\scriptstyle \iota} & \\ X^T & & \xrightarrow[i_T]{} & & X. \end{array}$$

In other words, the functorial map $i_T^* : \mathrm{op}K_T(X) \to \mathrm{op}K_T(X^T)$ factors as $\iota^* : \mathrm{op}K_T(X) \to \mathrm{op}K_T(Y)$ followed by $i_{T,Y}^* : \mathrm{op}K_T(Y) \to \mathrm{op}K_T(X^T)$. By Proposition 3.7, i_T^* is injective, hence so is ι^*. As for the second assertion, just note that X^H is T-invariant and $X^T \subset X^H$. ☐

Remark 3.9. Of particular interest is the case $Y = \cup_{i=1}^n Y_i$, where Y_i are the irreducible components of Y. Let $Y_{ij} = Y_i \cap Y_j$. By Corollary 3.4 the following sequence is exact

$$0 \to \mathrm{op}K_T(Y) \to \bigoplus_i \mathrm{op}K_T(Y_i) \to \bigoplus_{i,j} \mathrm{op}K_T(Y_{ij}).$$

When Y^T is finite, the sequence above yields the commutative diagram (Corollary 3.6):

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{op}K_T(Y) & \longrightarrow & \bigoplus_i \mathrm{op}K_T(Y_i) & \longrightarrow & \bigoplus_{i,j} \mathrm{op}K_T(Y_{ij}) \\ & & \downarrow{\scriptstyle i_T^*} & & \downarrow{\scriptstyle i_T^*} & & \downarrow{\scriptstyle i_T^*} \\ 0 & \longrightarrow & \mathrm{op}K_T(Y^T) & \xrightarrow{p} & \bigoplus_i \mathrm{op}K_T(Y_i^T) & \xrightarrow{q} & \bigoplus_{i,j} \mathrm{op}K_T(Y_{ij}^T) \end{array}$$

Since all vertical maps are injective (Proposition 3.7), it is important to observe that we can describe the image of the first vertical map in terms of the image of the second vertical map and the kernel of q. In other words, the map

$$p : im(i^*_{T,Y}) \to \{w \in \bigoplus_i \mathrm{op}K_T(Y_i) \mid w \in im(\sum_i i^*_T) \,\mathrm{and}\, q(w) = 0\}$$

sending $u \to p(u)$ is an isomorphism. Now, since Y^T is finite, the kernel of the map q consists of all families $(f_i)_i$ such that $f_i(x_k) = f_j(x_k)$ (equality of k-components), whenever x_k is in the intersection of Y_i and Y_j.

Back to the general case, let X be a complete T-scheme. We wish to describe the image of the injective map

$$i^*_T : \mathrm{op}K_T(X) \to \mathrm{op}K_T(X^T).$$

For this, let $T' \subset T$ be a subtorus of codimension one. Observe that $i_T : X^T \to X$ factors as $i_{T,T'} : X^T \to X^{T'}$ followed by $i_{T'} : X^{T'} \to X^T$. Thus, the image of i^*_T is contained in the image of $i^*_{T,T'}$. In symbols,

$$\mathrm{Im}[i^*_T : \mathrm{op}K_T(X) \to \mathrm{op}K_T(X^T)] \subseteq \bigcap_{T' \subset T} \mathrm{Im}[i^*_{T,T'} : \mathrm{op}K_T(X^{T'}) \to \mathrm{op}K_T(X^T)],$$

where the intersection runs over all codimension-one subtori of T. This observation will lead, as in the smooth case, to an explicit description of the image of i^*_T. Such is the central theme of the subsequent sections.

4. The Localization theorem of Borel-Atiyah-Segal type and the Chang-Skjelbred property

Let T be an algebraic torus. We recall a construction of Thomason [Th2, Lemma 1.1, Proposition 1.2]. Let $\mathfrak{p} \subset R(T)$ be a prime ideal. Set $K_\mathfrak{p} = \{n \in \Delta \mid 1 - n \in \mathfrak{p}\}$, where Δ is the character group of T. It is well-known, see e.g. [Bo2], that the quotient $\Delta/K_\mathfrak{p}$ determines a unique subgroup $T_\mathfrak{p} \subset T$ with the property that $R(T_\mathfrak{p}) = \mathbb{Z}[\Delta/K_\mathfrak{p}]$. Following [Th2], we call $T_\mathfrak{p}$ the *support* of $\mathfrak{p}$. When $\mathfrak{p}$ is maximal, $K_\mathfrak{p}$ has finite index and $T_\mathfrak{p}$ is a finite group.

Theorem 4.1. *Let X be a T-scheme. Let $\mathfrak{p} \subset R(T)$ be a prime ideal and $T_\mathfrak{p}$ be its support. Then the $R(T)$-algebra map $i^*_{T_\mathfrak{p}} : \mathrm{op}K_T(X) \to \mathrm{op}K_T(X^{T_\mathfrak{p}})$ becomes an isomorphism after localizing at $\mathfrak{p}$:*

$$i^*_{T_\mathfrak{p}} : \mathrm{op}K_T(X)_\mathfrak{p} \xrightarrow{\sim} \mathrm{op}K_T(X^{T_\mathfrak{p}})_\mathfrak{p}\ .$$

Proof. Choose a T-equivariant birational envelope $p : \tilde{X} \to X$, with $\tilde{X}$ quasi-projective and smooth. Then p is an isomorphism outside some T-invariant closed subscheme Z. Let $E = p^{-1}(Z)$. Notice that p can be chosen so that

Z and E have dimension smaller than that of X. Now, in light of Corollary 3.6, form the commutative diagram of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{op}K_T(X) & \longrightarrow & \mathrm{op}K_T(\tilde{X}) \oplus \mathrm{op}K_T(Z) & \longrightarrow & \mathrm{op}K_T(E) \\ & & \downarrow i^*_{T_\mathfrak{p}} & & \downarrow i^*_{T_\mathfrak{p}} & & \downarrow i^*_{T_\mathfrak{p}} \\ 0 & \longrightarrow & \mathrm{op}K_T(X^{T_\mathfrak{p}}) & \longrightarrow & \mathrm{op}K_T(\tilde{X}^{T_\mathfrak{p}}) \oplus \mathrm{op}K_T(Z^{T_\mathfrak{p}}) & \longrightarrow & \mathrm{op}K_T(E^{T_\mathfrak{p}}). \end{array}$$

It follows from Noetherian induction and Thomason's concentration theorem [Th2, Théorème 2.1] that the last two vertical maps become isomorphisms after localizing at $\mathfrak{p}$; hence so does the first one. □

Corollary 4.2. *Let X be a complete T-scheme. Notation being as above, the $R(T)$-algebra map $i^*_{T_\mathfrak{p}} : \mathrm{op}K_T(X) \to \mathrm{op}K_T(X^{T_\mathfrak{p}})$ is injective and it becomes surjective after localizing at $\mathfrak{p}$.*

Proof. Use Proposition 3.7 and Theorem 4.1. □

Definition 4.3. Let X be a complete T-scheme. We say that X has the **Chang-Skjelbred property** (or **CS property**, for short) if the image of

$$i^*_T : \mathrm{op}K_T(X) \to \mathrm{op}K_T(X^T)$$

is exactly the intersection of the images of

$$i^*_{T,H} : \mathrm{op}K_T(X^H) \to \mathrm{op}K_T(X^T),$$

where H runs over all subtori of codimension one in T.

By [VV, Theorem 2], every nonsingular complete T-scheme has the CS property. Remarkably, it holds over $\mathbb{Z}$. We extend this result to include all, possibly singular, complete schemes with an action of T.

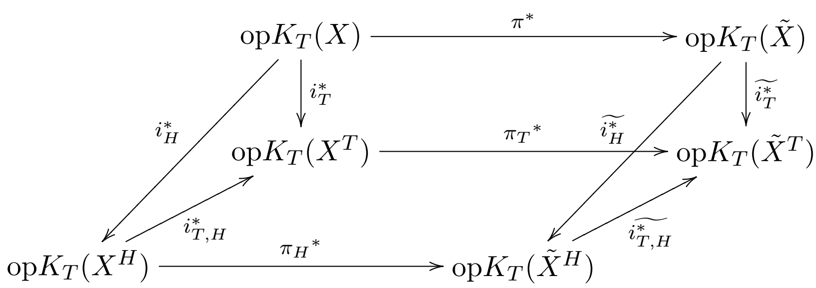
Theorem 4.4. *Let X be a complete T-scheme. Then X has the CS property.*

Proof. Let $\pi : \tilde{X} \to X$ be a T-equivariant envelope with $\tilde{X}$ projective and smooth (Lemma 3.1). Because of Corollary 3.6 we get the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{op}K_T(X) & \xrightarrow{\pi^*} & \mathrm{op}K_T(\tilde{X}) & \xrightarrow{\pi_1^* - \pi_2^*} & \mathrm{op}K_T(X') \\ & & i^*_T \downarrow & & \widetilde{i^*_T} \downarrow & & {i^*_T}' \downarrow \\ 0 & \longrightarrow & \mathrm{op}K_T(X^T) & \xrightarrow{{\pi_T}^*} & \mathrm{op}K_T(\tilde{X}^T) & \xrightarrow{\pi^*_{T,1} - \pi^*_{T,2}} & \mathrm{op}K_T({X'}^T). \end{array}$$

A simple diagram chasing shows that $u \in \mathrm{op}K_T(X^T)$ is in the image of i^*_T if and only if ${\pi_T}^*(u)$ is in the image of $\widetilde{i^*_T}$. Indeed, this follows from the fact that all vertical maps in the diagram are injective (Proposition 3.7).

On the other hand we have the commutative diagram



obtained by combining and comparing the sequences that Corollary 3.6 assigns to the envelopes $\pi : \tilde{X} \to X$, $\pi_H : \tilde{X}^H \to X^H$ and $p_T : \tilde{X}^T \to X^T$. From the diagram it follows that if $u \in \mathrm{op}K_T(X^T)$ is in the image of $i^*_{T,H}$, then ${\pi_T}^*(u)$ is in the image of $\widetilde{i^*_{T,H}}$. Hence, if u is in the intersection of the images of all $i^*_{T,H}$, then ${\pi_T}^*(u)$ is in the intersection of the images of all $\widetilde{i^*_{T,H}}$, where H runs over all codimension-one subtori of T. Since $\tilde{X}$ satisfies the CS condition, then ${\pi_T}^*(u)$ is in the image of $\widetilde{i^*_T}$. Finally, from the observation made at the end of the previous paragraph, we conclude that u is in the image of i^*_T. □

5. GKM theory

Vistoli and Vezzosi established a version of GKM theory applicable to nonsingular complete T-schemes [VV, Theorem 2]. Based on Theorem 4.4, we establish here a version of GKM-theory valid for the equivariant operational K-theory of *singular* complete T-schemes (Theorem 5.4). As a consequence, we extend [AP, Theorem 1.6] to the larger class of T-skeletal varieties, a family of objects that includes all equivariant projective embeddings of reductive groups (Theorem 6.2). We start by recalling a few definitions from [GKM] and [G1].

Definition 5.1. Let X be a complete T-variety. Let $\mu : T \times X \to X$ be the action map. We say that μ is a ***T*-skeletal action** if

(1) X^T is finite, and
(2) The number of one-dimensional orbits of T on X is finite.

In this context, X is called a ***T*-skeletal variety**. The associated graph of fixed points and invariant curves is called the **GKM graph** of X. We shall denote this graph by $\Gamma(X)$.

Example 5.2. Smooth T-skeletal varieties include regular compactifications of reductive groups ([BCP], [LP]) and, more generally, regular compactifications of symmetric varieties of minimal rank. The Chow rings of these varieties are described in [BJ] by means of GKM theory. In constrast, Schubert varieties and projective group embeddings of reductive groups are examples of *singular* T-skeletal varieties. The former have a paving by affine spaces and their equivariant cohomology is well-known [C]. The latter are spherical varieties and, when rationally smooth, their equivariant cohomology has been described by the author in [G2]. Our version of GKM theory (Theorem 5.4) will generalize these topological descriptions to the corresponding equivariant operational K-theory rings (Example 5.5 and Section 6).

Let X be a complete T-variety and let C be a T-invariant irreducible curve of X, which is not fixed pointwise by T. Let $\pi : \tilde{C} \to C$ be the (T-equivariant) normalization. Then $\tilde{C}$ is isomorphic to $\mathbb{P}^1$. Denote by $0, \infty$ the two fixed points of T in $\tilde{C}$, and denote by x_0, x_∞ their corresponding images via π. Then $\tilde{C} \setminus \{0, \infty\} = C \setminus \{x_0, x_\infty\}$ identifies to k^*, where T acts on $\tilde{C} \setminus \{0, \infty\}$ via a unique character χ (when interchanging 0 and ∞, one replaces χ by $-\chi$). Clearly, T has either one or two fixed points in C.

Notice that, in principle, Definition 5.1 allows for T-invariant irreducible curves with exactly one fixed point (i.e. the GKM graph $\Gamma(X)$ may have simple loops). We shall see that the functor $\mathrm{op}K_T(-)$ "contracts" such loops to a point.

Proposition 5.3. *Let X be a complete T-variety and let C be a T-invariant irreducible curve of X which is not fixed pointwise by T. Then the image of the injective map $i_T^* : \mathrm{op}K_T(C) \to \mathrm{op}K_T(C^T)$ is described as follows:*

(i) If C has only one fixed point, say x, then $i_T^* : \mathrm{op}K_T(C) \to \mathrm{op}K_T(x)$ is an isomorphism; that is, $\mathrm{op}K_T(C) \simeq R(T)$.

(ii) If C has two fixed points, then

$$\mathrm{op}K_T(C) \simeq \{(f_0, f_\infty) \in R(T) \oplus R(T) \mid f_0 \cong f_\infty \mod 1 - e^{-\chi}\},$$

where T acts on C via the character χ.

Proof. Let $\pi : \mathbb{P}^1 \to C$ be the normalization map. By [VV, Theorem 2] (see also [U, Theorem 1.3])

$$K_T(\mathbb{P}^1) = \{(f_0, f_\infty) \in R(T) \oplus R(T) \mid f_0 \cong f_\infty \mod 1 - e^{-\chi}\},$$

where χ is the character of the T-action on C. Moreover, given that $\mathbb{P}^1$ is smooth, we get $\mathrm{op}K_T(\mathbb{P}^1) = K_T(\mathbb{P}^1)$. In view of this, and Gillet-Kimura criterion (Theorem 3.2), it suffices to find the image of the injective map $\pi^* : \mathrm{op}K_T(C) \to \mathrm{op}K_T(\mathbb{P}^1)$ explicitly. First, assume that C has only one fixed point, say $x = \pi(0) = \pi(\infty)$. Then an element $f \in \mathrm{op}K_T(\mathbb{P}^1)$ is in the image of π^* if and only if the restriction $(f_0, f_\infty) \in \mathrm{op}K_T(\{0, \infty\})$ is in the image of the induced map $\pi^* : \mathrm{op}K_T(x) \to \mathrm{op}K_T(\{0, \infty\})$. But the latter morphism is simply the diagonal inclusion, so we get that $f \in \mathrm{op}K_T(\mathbb{P}^1)$ is in the image of π^* if and only if $f_0 = f_\infty$. Therefore, $\mathrm{op}K_T(C) = R(T)$ and $i_T^* : \mathrm{op}K_T(C) \to \mathrm{op}K_T(x)$ is an isomorphism. Finally, if $\pi(0) \neq \pi(\infty)$, a similar analysis yields assertion (ii). □

Let X be a T-skeletal variety. Now, as done in the Introduction, it is possible to define a ring $PE_T^*(X)$ of **piecewise exponential functions**. We recall the construction here, taking into account Proposition 5.3. Let

$K_T(X^T) = \oplus_{x \in X^T} R_x$, where R_x is a copy of the representation ring $R(T)$. We then define $PE_T(X)$ as the subalgebra of $K_T^0(X^T)$ given by

$$PE_T(X) = \{(f_1, \ldots, f_m) \in \oplus_{x \in X^T} R_x \mid f_i \equiv f_j \mod 1 - e^{-\chi_{i,j}}\}$$

where x_i and x_j are the two (perhaps equal) fixed points in the closure of the one-dimensional T-orbit $C_{i,j}$, and $\chi_{i,j}$ is the character of T associated with $C_{i,j}$. This character is uniquely determined up to sign (permuting the two fixed points changes $\chi_{i,j}$ to its opposite). Invariant curves with only one fixed point do not impose any relation (this is compatible with Proposition 5.3).

Theorem 5.4. *Let X be a complete T-skeletal variety. Then the pullback $i_T^* : \mathrm{op}K_T(X) \to \mathrm{op}K_T(X^T)$ induces an isomorphism between $\mathrm{op}K_T(X)$ and $PE_T(X)$.*

Proof. Observe that a codimension one subtorus of T is the kernel of a primitive (i.e. indivisible) character of T. Such character is uniquely defined up to sign.

Let π be a primitive character of T. Let $X^{\ker \pi} = \bigcup_j X_j$ be the decomposition into irreducible components. Notice that each X_j is either a fixed point, or a T-invariant irreducible curve. Now, with the notation of Corollary 3.4, we have the commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{op}K_T(X^{\ker \pi}) & \longrightarrow & \bigoplus_i \mathrm{op}K_T(X_j) & \longrightarrow & \bigoplus_{i,j} \mathrm{op}K_T(X_{i,j}) \\ & \downarrow{\scriptstyle i^*_{T,\ker \pi}} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ 0 \longrightarrow & \mathrm{op}K_T(X^T) & \longrightarrow & \bigoplus_i \mathrm{op}K_T(X_j^T) & \longrightarrow & \bigoplus_{i,j} \mathrm{op}K_T(X_{i,j}), \end{array}$$

where each $X_{i,j}$ is just a fixed point, and T acts on those X_j's that are curves via a character χ_j, a multiple of π. The image of the middle vertical map is completely characterized by Proposition 5.3, and so is the image of $i^*_{T,\ker \pi}$, as it follows from Remark 3.9. In short, $\mathrm{Im}(i^*_{T,\ker \pi}) \simeq PE_T(X^{\ker \pi})$. Now apply Theorem 4.4 to conclude the proof. □

Let X be a T-skeletal variety. Notice that $\Gamma(X)$ is a singular projective T-variety with the same equivariant operational K-theory as that of X. In symbols, $\mathrm{op}K_T(\Gamma(X)) = \mathrm{op}K_T(X)$. This is simply a rephrasing of Theorems 4.4 and 5.4.

Example 5.5. (Bruhat graph.) Let G be a connected reductive group with Borel subgroup B and maximal torus $T \subset B$. Let W be the Weyl group of (G, T). It is a finite group generated by reflections $\{s_\alpha\}_{\alpha \in \Phi}$, where Φ stands for the set of roots of (G, T). Now let $X(w) = \overline{BwB/B} \subset G/B$ be the Schubert variety associated to $w \in W$. In what follows we extend the usual picture of $K_T(G/B)$ to $\mathrm{op}K_T(X(w))$. Denote by I_w the Bruhat interval

$$[1, w] = \{x \in W \mid x \leq w\}.$$

Notice that $X(w)^T = I_w$. As a shorthand, set $R = R(T)$. Then, by Theorem 5.4, op$K_T(X(w))$ is the subring of $\oplus_{x\in I_w} R$ consisting of all $\sum_{x\in I_w} f_x x$ such that $f_x \cong f_{s_\alpha x} \mod 1 - e^{-\alpha}$, whenever (i) s_α is a reflection of W and (ii) $x, s_\alpha x \in I_w$. Finally, op$K_T(X(w))$ is a free $R(T)$-module of rank $|I_w|$. This is a consequence of equivariant Kronecker duality (Property (f), Introduction) together with the fact that $K^T(X(w))$ is a free $R(T)$-module of rank $|I_w|$ (for $X(w)$ has a paving by affine spaces, cf. [U, Lemma 1.6]).

Remark 5.6. Let X be a T-skeletal variety. By Theorem 5.4, the $R(T)$-algebra op$K_T(X)$ identifies to $PE_T(X) \subset R(T)^m$. Moreover, $PE_T(X)$ and $R(T)^m$ have the same quotient field (by the localization theorem). It follows that $PE_T(X)$ is a reduced, finitely generated $\mathbb{Z}$-algebra. The same holds for the natural extension $PE_T(X)_k := PE_T(X) \otimes k$, a k-algebra of dimension $d = dim(T)$. Let $V(X)$ be the corresponding affine k-variety (defined over a finite algebraic extension of $\mathbb{Q}$). It is worth noting that the associated map $i_T^* : \text{op}K_T(X)_k \to k[T]^m$ is the normalization (cf. [Br3, Proposition 2]). For this, first observe that $k[T]$ is a subring of op$K_T(X)_k$, as a choice of fixed point yields a section of the structural map $\text{op}K_T(X)_k \to \text{op}K_T(pt)_k \simeq k[T]$. Secondly, $k[T]^m$ is a finite module over $k[T]$, so it is also a finite module over op$K_T(X)$. Finally, since $k[T]^m$ is integrally closed in its quotient field and op$K_T(X)_k$ and $k[T]^m$ have the same quotient field (by the localization theorem), we conclude that i^* is the normalization. Hence, the normalization of the affine variety $V(X)$ is the union of $|X^T|$ disjoint copies of T. Moreover, the set $V(X)$ is obtained as follows: for any character χ associated to a T-invariant curve with fixed points x and y we identify the toric hyperplanes $\{1 - e^\chi = 0\}$ in T_x and T_y, provided $x \neq y$. If the aforementioned x and y are the same, then we set $T_x = T_y$, in accordance with Proposition 5.3.

6. Equivariant operational K-theory rings of projective group embeddings

Throughout this section we denote by G a connected reductive linear algebraic group (over k) with Borel subgroup B and maximal torus $T \subset B$. We denote by W the Weyl group of (G, T). Observe that W is generated by reflections $\{s_\alpha\}_{\alpha\in\Phi}$, where Φ stands for the set of roots of (G, T). We write U_α for the unipotent subgroup of G associated to $\alpha \in \Phi$. Since W acts on Δ, the character group of T, there is a natural action of W on $\mathbb{Z}[\Delta]$ given by $w(e^\lambda) = e^{w(\lambda)}$, for each $w \in W$ and $\lambda \in \Delta$. Recall that we can identify $R(G)$ with $R(T)^W$ via restriction to T, where $R(T)^W$ denotes the subring of $R(T)$ invariant under the action of W.

An affine algebraic monoid M is called *reductive* it is irreducible, normal, and its unit group is a reductive algebraic group. See [R1] for many details. Let M be a reductive monoid with zero and unit group G. We denote by $E(\overline{T})$ the idempotent set of the associated affine torus embedding $\overline{T}$, that is, $E(\overline{T}) = \{e \in \overline{T} \,|\, e^2 = e\}$. One defines a partial order on $E(\overline{T})$ by declaring $f \leq e$ if and only if $fe = f$. Denote by $\Lambda \subset E(\overline{T})$, the cross section lattice of

M. The Renner monoid $\mathcal{R} \subset M$ is a finite monoid whose group of units is W and contains $E(\overline{T})$ as idempotent set. In fact, any $x \in \mathcal{R}$ can be written as $x = fu$, where $f \in E(\overline{T})$ and $u \in W$. Given $e \in E(\overline{T})$, we write $C_W(e)$ for the centralizer of e in W. Denote by $\mathcal{R}_k$ the set of elements of rank k in $\mathcal{R}$, that is, $\mathcal{R}_k = \{x \in \mathcal{R} \mid \dim Tx = k\}$. Analogously, one has $\Lambda_k \subset \Lambda$ and $E_k \subset E(\overline{T})$.

A normal irreducible variety X is called an *embedding* of G, or a *group embedding*, if X is a $G \times G$-variety containing an open orbit isomorphic to G. Due to the Bruhat decomposition, group embeddings are spherical $G \times G$-varieties. Substantial information about the topology of a group embedding can be obtained by restricting one's attention to the induced action of $T \times T$. When $G = B = T$, we get back the notion of toric varieties. Let M be a reductive monoid with zero and unit group G. Then there exists a central one-parameter subgroup $\epsilon : \mathbb{G}_m^* \to T$, with image Z, such that $\lim_{t \to 0} \epsilon(t) = 0$. Moreover, the quotient space

$$\mathbb{P}_\epsilon(M) := (M \setminus \{0\})/Z$$

is a normal projective variety on which $G \times G$ acts via $(g, h) \cdot [x] = [gxh^{-1}]$. Hence, $\mathbb{P}_\epsilon(M)$ is a normal projective embedding of the quotient group G/Z. These varieties were introduced by Renner in his study of algebraic monoids ([R2], [R3]). Notably, normal projective embeddings of connected reductive groups are exactly the projectivizations of normal algebraic monoids [Ti].

Now let $X = \mathbb{P}_\epsilon(M)$ be a (projective) group embedding. In [G2] we compute the finite GKM data coming from the $T \times T$-fixed points and $T \times T$-invariant curves of X in terms of the combinatorial invariants of M. These computations are independent of whether or not X is rationally smooth.

Theorem 6.1 ([G2, Theorems 3.1, 3.5]). *Let $X = \mathbb{P}_\epsilon(M)$ be a projective group embedding. Then its natural $T \times T$-action*

$$\mu : T \times T \times \mathbb{P}_\epsilon(M) \to \mathbb{P}_\epsilon(M), \quad (s, t, [x]) \mapsto [sxt^{-1}]$$

is $T \times T$-skeletal. Indeed, after identifying the elements x of $\mathcal{R}_1$ with their corresponding images $[x]$ in X, the set $X^{T \times T}$ corresponds to $\mathcal{R}_1$. As for the closed $T \times T$-curves of X, they fall into three types:

(1) $\overline{U_\alpha[ew]}$, *$e \in E_1(\overline{T})$, $s_\alpha \notin C_W(e)$ and $w \in W$.*
(2) $\overline{[we]U_\alpha}$, *$e \in E_1(\overline{T})$, $s_\alpha \notin C_W(e)$ and $w \in W$.*
(3) $\overline{[TxT]} = \overline{[Tx]} = \overline{[xT]}$, *where $x \in \mathcal{R}_2$.*

The curves of type 1 and 2 lie entirely in closed $G \times G$-orbits, whereas the curves of type 3 do not. Curves of type 3 can be further separated into whether or not the corresponding $T \times T$-fixed points are in the same closed $G \times G$-orbit. In [G2, Section 4], we identify explicitly the $T \times T$-characters associated to these curves. With such data at our disposal, Theorem 5.4 yields an immediate translation of [G2, Theorem 4.10] into the language of equivariant operational K-theory. Furthermore, as Theorem 5.4 does not require any conditions on the singular locus, the result (Theorem 6.2) applies to all projective group embeddings. This description coincides with that of Uma ([U, Theorem 2.1]) when $X = \mathbb{P}_\epsilon(M)$ is smooth, and it extends our previous work on rationally smooth group embeddings [G2]. To state it, we record a few extra facts. Let Λ_1 be the set of rank-one idempotents of the cross-section lattice Λ. Each closed $G \times G$-orbit of $X = \mathbb{P}_\epsilon(M)$ can be written uniquely as $G[e]G \simeq G/P_e \times G/P_e^-$, where $e \in \Lambda_1$, and P_e, P_e^- are opposite parabolic subgroups (see e.g. [R1]).

Theorem 6.2. *Let $X = \mathbb{P}_\epsilon(M)$ be a group embedding. Then the natural map*

$$\mathrm{op}K_{T\times T}(X) \longrightarrow \mathrm{op}K_{T\times T}\left(\bigsqcup_{e\in\Lambda_1} G[e]G\right) = \bigoplus_{e\in\Lambda_1} K_{T\times T}(G[e]G)$$

is injective. In fact, its image consists of all tuples $(\varphi_e)_{e\in\Lambda_1}$, indexed over Λ_1 and with $\varphi_e \in K_{T\times T}(G[e]G)$, subject to the additional conditions:

(a) If $f \in E_2(\overline{T})$ and there is a (necessarily unique) reflection s_{α_f} such that $s_{\alpha_f} f = f s_{\alpha_f} \neq f$, then

$$\varphi_{e_f}(f_1 u) \equiv \varphi_{e_f}(f_2 u) \mod 1 - e^{-\alpha_f} \otimes e^{-(\alpha_f \circ \mathrm{int}(u))},$$

for all $u \in W$. Here, f_1 and $f_2 = s_{\alpha_f} \cdot f_1 \cdot s_{\alpha_f}$ are the two idempotents in $E_1(\overline{T})$ below f, the root α_f corresponds to the reflection s_{α_f}, and $e_f \in \Lambda_1$ is the unique element of Λ_1 which is conjugate to f_1.

(b) If $f \in E_2(\overline{T})$ and $sf = fs = f$ for every reflection $s \in W$, then

$$\varphi_{e_1}(f_1 u) \equiv \varphi_{e_2}(f_2 u) \mod 1 - e^{-\lambda_f} \otimes e^{-(\lambda_f \circ int(u))},$$

for all $u \in W$. Here, λ_f is the character of T defined by the composition

$$T \to Tf \to Tf/k^* \simeq k^*,$$

the idempotents f_1, f_2 are the unique idempotents below f, and $e_i \in \Lambda_1$ is conjugate to f_i, for $i = 1, 2$.

Proof. Since $X^{T\times T} \subset \bigsqcup_{e\in\Lambda_1} G[e]G$, Corollary 3.8 renders the natural map $\mathrm{op}K_{T\times T}(X) \to \mathrm{op}K_{T\times T}\left(\bigsqcup_{e\in\Lambda_1} G[e]G\right)$ injective. Moreover, $G[e]G$ is smooth, so $\mathrm{op}K_{T\times T}(G[e]G)$ is isomorphic to $K_{T\times T}(G[e]G)$. Finally, we apply Theorem 5.4, taking into account that:
(i) the curves of type 1 and 2 in Theorem 6.1 are contained in $\bigsqcup_{e\in\Lambda_1} G[e]G$ and these curves describe $K_{T\times T}(G[e]G)$ via Example 5.5,
(ii) the characters associated to the curves of type (3) give assertions (a) and (b), as in [G2, Theorem 4.10]. □

If $X = \mathbb{P}_\epsilon(M)$ is a group embedding, then X is $G \times G$-spherical. If moreover $\pi_1(G)$ is torsion free, then Corollary A.4 states that $\mathrm{op}K_{G\times G}(X)$ can be read off from $\mathrm{op}K_{T\times T}(X)$ by computing invariants:

$$K_{G\times G}(X) \simeq \mathrm{op}K_{T\times T}(X)^{W\times W}.$$

Corollary 6.3. *Let $X = \mathbb{P}_\epsilon(M)$ be a group embedding. If $\pi_1(G)$ is torsion free, then the ring* $\mathrm{op}K^*_{G\times G}(X)$ *consists of all tuples $(\Psi_e)_{e\in\Lambda_1}$, where*

$$\Psi_e : WeW \to (R(T)\otimes R(T))^{C_W(e)\times C_W(e)},$$

such that

(a) If $f \in E_2(\overline{T})$ and $H_f = \{f, s_{\alpha_f} f\}$, then

$$\Psi_e(f_1) \equiv \Psi_e(f_2) \mod 1 - e^{-\alpha_f}\otimes e^{-\alpha_f},$$

where $e \in \Lambda_1$ is conjugate to f_1, $f_2 = s_{\alpha_f}\cdot f_1 \cdot s_{\alpha_f}$, the reflection $s_{\alpha_f} \in C_W(f)$ is associated with the root α_f, and $f_i \leq f$.

(b) If $f \in E_2$ and $H_f = \{f\}$, then

$$\Psi_e(f_1) \equiv \Psi_{e'}(f_2) \mod 1 - e^{-\lambda_f}\otimes e^{-\lambda_f},$$

where λ_f is character of T defined by f, and $f_1, f_2 \leq f$ are conjugate to e and e', respectively.

Proof. Simply adapt the proof of [G2, Corollary 4.11], using Theorem 6.2 and Corollary A.4. ☐

Associated to $X = \mathbb{P}_\epsilon(M)$, there is a projective torus embedding $\mathcal{Y}$ of T/Z, namely,

$$\mathcal{Y} = \mathbb{P}_\epsilon(\overline{T}) = [\overline{T}\setminus\{0\}]/Z.$$

By construction, $\mathcal{Y}$ is a normal projective toric variety and $\mathcal{Y} \subseteq X$. Our next theorem allows to compare the equivariant operational K-theories of X and its associated torus embedding $\mathcal{Y} \subseteq X$. The situation for general group embeddings contrasts deeply with the corresponding one for regular embeddings ([Br2, Corollary 3.1.2], [U, Corollary 2.2.3]).

Theorem 6.4. *Notation being as above, if $\pi_1(G)$ is torsion free, then the inclusion of the associated torus embedding $\iota : \mathcal{Y} \hookrightarrow X$ induces an injection:*

$$\iota^* : \mathrm{op}K^*_{G\times G}(X) \hookrightarrow \mathrm{op}K_{T\times T}(\mathcal{Y})^W \simeq (\mathrm{op}K_T(\mathcal{Y})\otimes R(T))^W,$$

where the W-action on $\mathrm{op}K_{T\times T}(\mathcal{Y})$ is induced from the action of $\mathrm{diag}(W)$ *on $\mathcal{Y}$. Moreover, ι^* is an isomorphism whenever $C_W(e) = \{1\}$ for every $e \in \Lambda_1$.*

Proof. The argument here is an adaptation of [G2, Proof of Theorem 4.12]. First, consider the commutative diagram

$$\begin{array}{ccc} \mathrm{op}K_{T\times T}(X) & \hookrightarrow & \mathrm{op}K_{T\times T}(X^{T\times T}) \\ \downarrow{\scriptstyle \iota^*} & & \downarrow{\scriptstyle \iota^*} \\ \mathrm{op}K_{T\times T}(\mathcal{Y}) & \hookrightarrow & \mathrm{op}K_{T\times T}(\mathcal{Y}^{T\times T}), \end{array}$$

where both horizontal maps are injective (Proposition 3.7). On the other hand, recall that Λ_1 provides a set of representatives of both the $W \times W$-orbits in $X^{T\times T} = \mathcal{R}_1$ and the W-orbits in $\mathcal{Y}^{T\times T} = E_1(T)$. Thus, after taking invariants, we obtain an injection

$$\begin{array}{c} \mathrm{op}K_{T\times T}(\mathcal{R}_1)^{W\times W} = \bigoplus_{e\in\Lambda_1}((R(T)\otimes R(T))^{C_W(e)\times C_W(e)} \\ \Big\downarrow \iota^* \\ \mathrm{op}K_{T\times T}(E_1(T))^W = \bigoplus_{e\in\Lambda_1}(R(T)\otimes R(T))^{C_W(e)}. \end{array}$$

Placing this information into the commutative square above renders the map

$$\iota^* : \mathrm{op}K_{T\times T}(X)^{W\times W} \longrightarrow \mathrm{op}K_{T\times T}(\mathcal{Y})^W$$

injective. Now observe that $\mathrm{op}K_{T\times T}(\mathcal{Y})^W \simeq (\mathrm{op}K_T(\mathcal{Y})\otimes R(T))^W$. Truly, we have a split exact sequence

$$1 \longrightarrow diag(T) \longrightarrow T\times T \xrightarrow{(t_1,t_2)\mapsto t_1t_2^{-1}} T \longrightarrow 1,$$

where the splitting is given by $t \mapsto (t,1)$. It follows that $T\times T$ is canonically isomorphic to $diag(T)\times(T\times 1)$. Clearly, $diag(T)$ acts trivially on $\mathcal{Y}$. Hence, by [AP, Corollary 5.5], we have a ring isomorphism $\mathrm{op}K_{T\times T}(\mathcal{Y}) \simeq \mathrm{op}K_{diag(T)} \otimes \mathrm{op}K_T(\mathcal{Y})$. This isomorphism is in fact W-invariant (because the W-action on the operational rings is induced from the action of $diag(W)$ on $\mathcal{Y}$).

For the second assertion, assume that $C_W(e) = \{1\}$ for all $e \in \Lambda_1$. We need to show that ι^* is surjective. To achieve our goal, we modify slightly an argument of [LP], Section 4.1, and Brion [Br2], Corollary 3.1.2. Define the $T\times T$-variety

$$\mathcal{N} = \bigcup_{w\in W} w\mathcal{Y}.$$

We claim that this union is, in fact, a disjoint union. Indeed, observe that $\mathcal{N}$ contains all the $T\times T$-fixed points of X. That is, $\mathcal{N}$ has $|\mathcal{R}_1|$ fixed points. On the other hand, each $w\mathcal{Y}$ has $|E_1|$ fixed points (for its corresponding T-action). Now, if it were the case that there is a pair of distinct subvarieties $w\mathcal{Y}$ and $w'\mathcal{Y}$ with non-empty intersection, then this intersection should also contain $T\times T$-fixed points. But then a simple counting argument would yield $|\mathcal{R}_1| < |E_1||W|$. This is impossible, by our assumptions and [G2, Lemma 4.14]. Hence,

$$\mathcal{N} = \bigsqcup_{w\in W} w\mathcal{Y}.$$

In this setup, Corollary 3.8 implies that the restriction map

$$\mathrm{op}K_{T\times T}(X) \to \mathrm{op}K_{T\times T}(\mathcal{N}).$$

is injective. From Theorem 6.1 we know that all the $T\times T$-curves of X are contained either in closed $G\times G$-orbits (curves of type 1. and 2.) or in $\mathcal{N}$

(curves of type 3.). Moreover, note that the curves of type 3. are exactly the $T \times T$-invariant curves of $\mathcal{N}$, so $\mathcal{N}$ is $T \times T$-skeletal and Theorem 5.4 applies to it. After taking $W \times W$-invariants (cf. Corollary 6.3), we see that the aforementioned map induces an isomorphism

$$\mathrm{op}K_{T\times T}(X)^{W\times W} \simeq \mathrm{op}K_{T\times T}(\mathcal{N})^{W\times W} \simeq \left(\bigoplus_{w\in W} \mathrm{op}K_{T\times T}(\mathcal{Y})\right)^{W\times W} \simeq \mathrm{op}K_{T\times T}(\mathcal{Y})^{W}.$$

This concludes the proof. □

Lemma 6.5 ([G2, Lemma 4.14 and Corollary 4.15].)**.** *Let $X = \mathbb{P}_\epsilon(M)$ be a group embedding. Then the following are equivalent:*
(a) $C_W(e) = \{1\}$ for every $e \in E_1(\overline{T})$.
(b) All closed $G \times G$-orbits in X are isomorphic to $G/B \times G/B^-$. □

Group embeddings satisfying the equivalent conditions of Lemma 6.5 are called **toroidal** embeddings (see e.g. [Ti, Chapter 5]). Furthermore, smooth toroidal embeddings are exactly the regular embeddings of reductive groups [Ti, Theorem 29.2].

Theorem 6.4 gives an explicit relation between our results and those of [AP]. Indeed, if $X = \mathbb{P}_\epsilon(M)$ is a toroidal group embedding and $\pi_1(G)$ is torsion free, then $\mathrm{op}K_{G\times G}(X)$ is isomorphic to the subring of W-invariants in $\mathrm{op}K_T(\mathcal{Y}) \otimes R(T)$, where $\mathrm{op}K_T(\mathcal{Y})$ is the ring of integral piecewise exponential functions on the fan of $\mathcal{Y}$.

7. Further remarks

(1) *Extending the results to equivariant operational Chow groups. Poincaré duality for singular schemes.* Kimura's cohomological descent for envelopes [Ki, Theorems 2.3 and 3.1] has also been established for equivariant operational Chow groups $\mathrm{op}A^*_T(-)$ [EG-1, Section 2.6], the operational cohomology groups associated to Edidin and Graham's equivariant Chow groups $A^T_*(-)$. On the category of smooth schemes, the functors $\mathrm{op}A^*_T(-)$ and $A^T_*(-)$ are known to agree (*op. cit.*). Furthermore, on the subcategory of smooth projective T-schemes, corresponding versions of the localization theorem and CS property hold for $A^T_*(-)_\mathbb{Q}$ [Br1, Section 3]. Since these are the intersection theory analogues of our main tools, our arguments are readily translated into the language of equivariant operational Chow groups with $\mathbb{Q}$-coefficients, yielding versions of Theorems 4.1, 4.4 and 5.4 applicable to all singular complete T-schemes. See [G3] for a slightly different approach in the case of T-linear varieties, and [G4] for some applications to characterizing Poincaré duality on the Chow groups of singular T-schemes. Moreover, when $k = \mathbb{C}$ and X is a rationally smooth T-skeletal variety, there is a natural isomorphism between $\mathrm{op}A^*_T(X)_\mathbb{Q}$ and $H^*_T(X)_\mathbb{Q}$, as their images on $A^*_T(X^T)_\mathbb{Q} = H^*_T(X^T)_\mathbb{Q}$ are canonically isomorphic [GKM], [G1].

In particular, the equivariant operational Chow groups of (complex) rationally smooth T-skeletal varieties are free modules over $\mathrm{Sym}[\Delta]_{\mathbb{Q}}$. From this point of view, equivariant operational Chow groups behave like equivariant intersection cohomology, though the former are somewhat more combinatorial and easier to compute on T-skeletal varieties. Notably, for projective group embeddings, the results of [G2] remain valid when translating them into the context of rational smoothness in Chow groups [G4] and equivariant operational Chow rings. The results will appear in [G5].

(2) *Equivariant multiplicities in K-theory.* Let X be a complete T-scheme with finitely many fixed points. In virtue of Thomason's localization theorem for $K^T(-)$ [Th2, Theorem 2.1], the following identity holds in $\mathcal{Q}(\Delta)$, the quotient field of $\mathbb{Z}[\Delta]$:

$$[\mathcal{O}_X] = \sum_{x \in X^T} EK(x,X)[\mathcal{O}_x],$$

where the various $EK(x,X)$ are (possibly zero) rational functions on $(\Delta_{\mathbb{Q}})^*$. Following the nomenclature of [Br1, Section 4.2] we call $EK(x,X)$ the *K-theoretic equivariant multiplicity of X at x*. If all the $EK(x,X)$ are non-zero, then, by Theorem 4.1, the Poincaré duality map

$$\mathrm{op}K_T(X) \to K^T(X), \quad c \mapsto c_{id_X}(\mathcal{O}_X)$$

is injective (cf. proof of [Br3, Theorem 4.1]). We anticipate that the $EK(x,X)$'s are non-zero whenever x is an attractive fixed point of X, because, in that case, $EK(x,X)$ is related to the Hilbert series of $\mathrm{Proj}(k[X_x])$, where X_x is the unique open affine T-stable neighborhood of x (cf. [Br1], [R4], [BV]). The notion of K-theoretic equivariant multiplicity at attractive fixed points is already present in the study of flag varieties (see e.g. [BBM]). For complete toric varieties and simple group embeddings, our claim would imply that the natural map $\mathrm{op}K_T(X) \to K^T(X)$ is always injective (this deeply contrasts with the behaviour of the map $K_T(X) \to K^T(X)$, whose kernel could be rather large, cf. [AP]). In contrast, surjectivity of the Poincaré duality map on singular schemes is a more delicate property, and quite often it does not hold. For instance, consider the $\mathbb{G}_m$-action on $\mathbb{P}^3$ given by $t \cdot [x,y,z,w] = [t^2x, t^4y, t^3z, w]$. Now let $Y \subset \mathbb{P}^3$ be the projective surface $z^2 = xy$. Clearly Y is $\mathbb{G}_m$-invariant, $\mathrm{op}K_T(Y)$ is torsion free, but $K^T(Y)$ has $R(\mathbb{G}_m)$-torsion coming from the fact that $\mu_2 \subset \mathbb{G}_m$ fixes two lines in Y. We shall develop these ideas and explore the behaviour of K-theoretic equivariant multiplicities in a subsequent paper.

Appendix A. G-equivariant Künneth formula for spherical varieties

Recall that a G-variety is called *spherical* if it contains a dense B-orbit. Examples include flag varieties, symmetric spaces, and $G \times G$-equivariant embeddings of G (e.g. toric varieties are spherical). For an up-to-date discussion of spherical varieties, see [Ti] and the references therein.

The following is a result of Merkurjev [M1].

Theorem A.1. *Let G be a connected reductive group. Suppose that $\pi_1(G)$ is torsion-free. Then the following hold:*

(i) $R(T)$ is a free $R(G)$-module of rank $|W|$, and $R(T) \simeq R(G) \otimes \mathbb{Z}[W]$.
(ii) If X is a G-scheme, then

$$R(B) \otimes_{R(G)} K^G(X) \simeq K^G(X \times G/B) \simeq K^B(X) \simeq K^T(X).$$

In particular, $K^G(X) \simeq K^T(X)^W$. □

Theorem A.2. *Let G be a connected reductive group with $\pi_1(G)$ torsion free. Let X be a G-spherical variety. Then for any G-variety Y the exterior product map, or Künneth map,*

$$Ku_G : K_G(X) \otimes_{R(G)} K_G(Y) \to K_G(X \times Y)$$

is an isomorphism.

Proof. Consider the commutative diagram

$$\begin{array}{ccc}
K_B(X) \otimes_{R_B} K_B(Y) & \xrightarrow{\mathrm{Ku}_B} & K_B(X \times Y) \\
\Big\uparrow{\scriptstyle p_1 \otimes p_2} & & \Big\uparrow{\scriptstyle p} \\
K_G(X \times G/B) \otimes_{R_B} K_G(Y \times G/B) & & K_G((X \times Y) \times G/B) \\
\Big\uparrow{\scriptstyle res_X \otimes res_Y} & & \Big\uparrow{\scriptstyle res_{X \times Y}} \\
[R_B \otimes_{R_G} K_G(X)] \otimes_{R_B} [R_B \otimes_{R_G} K_G(Y)] & & R_B \otimes_{R_G} K_G(X \times Y) \\
\Big\uparrow{\scriptstyle natural} & & \Big\uparrow{\scriptstyle \mathrm{id}} \\
[R_B \otimes_{R_G} K_G(X)] \otimes_{R_G} K_G(Y)] & \xrightarrow{id_{R_B} \otimes \mathrm{Ku}_G} & R_B \otimes_{R_G} K_G(X \times Y).
\end{array}$$

The vertical maps are isomorphisms due to Theorem A.1, and Ku_B is an isomorphism by [AP, Proposition 6.4] and the fact that the functors $\mathrm{op}K_B(-)$ and $\mathrm{op}K_T(-)$ agree on B-schemes. Therefore, the bottom horizontal map is also an isomorphism. But this morphism is a faithfully flat extension of Ku_G, because $R(B) \simeq R(T)$ is a free $R(G)$-module. We conclude that Ku_G is an isomorphism of R_G-modules. □

From Theorem A.2 one formally deduces, as in the T-equivariant case (cf. [AP, Proposition 6.3]), the following.

Corollary A.3. *Let G be a connected reductive group with $\pi_1(G)$ torsion free. If X be a complete G-spherical variety, then the G-equivariant Kronecker duality map*

$$\mathrm{op}K_G(X) \longrightarrow \mathrm{Hom}_{R(G)}(K^G(X), R(G)).$$

is an isomorphism. □

As a byproduct, our main theorems on torus actions can be used to calculate the G-equivariant operational K-theory of spherical varieties.

Corollary A.4. *Let G be a connected reductive group with $\pi_1(G)$ torsion free. If X is complete G-spherical variety, then*

$$\mathrm{op}K_T(X) \simeq \mathrm{op}K_G(X) \otimes_{R(G)} R(T).$$

Consequently, $(\mathrm{op}K_T(X))^W \simeq \mathrm{op}K_G(X)$.

Proof. In light of Corollary A.3 we have the isomorphism

$$\mathrm{op}K_G(X) \simeq \mathrm{Hom}_{R(G)}(K^G(X), R(G)).$$

Since the $R(G)$-modules $R(T)$ and $K^G(X)$ are, respectively, free and finitely generated, tensoring with $R(T)$ both sides of the identity above yields

$$\mathrm{Hom}_{R(T)}(K^G(X) \otimes_{R(G)} R(T), R(G) \otimes_{R(G)} R(T)).$$

The latter expression identifies, in turn, to $\mathrm{op}K_T(X)$, due to T-equivariant Kronecker duality (Property (f), Introduction) and the fact that $K^G(X) \otimes_{R(G)} R(T) \simeq K^T(X)$. The second assertion follows from our previous argument once we recall that $R(T) \simeq R(G) \otimes \mathbb{Z}[W]$. □

Next we show that the functors $\mathrm{op}K_G(-)$ and $K_G(X)$ agree on smooth projective G-spherical varieties. For this, a few extra facts need to be brought to our attention. The natural map $\mathrm{op}K_G(X \to pt) \to K^G(X)$ is always surjective. This stems from the proof of [AP, Proposition 4.4]. In fact, this map is an isomorphism when G is a torus. Nonetheless, for more general reductive groups G, injectivity of such map is a more delicate issue, and it does not follow from the arguments given in [AP, Proposition 4.4]. The main problem is that, unlike to torus case, $K^G(X)$ might not be generated by the classes of the structure sheaves of G-invariant subvarieties. We can overcome this issue in the case of smooth projective G-spherical varieties by appealing to G-equivariant Kronecker duality.

Corollary A.5. *Let G be a connected reductive group with $\pi_1(G)$ torsion free. If X is a smooth projective G-spherical variety, then the natural maps*

$$K_G(X) \to \mathrm{op}K_G(X) \to \mathrm{op}K_G(X \to pt) \to K^G(X)$$

are isomorphisms.

Proof. Note that the displayed diagram is a factorization of the Poincaré duality map $K_G(X) \to K^G(X)$, which is known to be an isomorphism. Moreover, $K_G(X)$ is a free $R(G)$-module of rank $|X^T|$ [U, Lemma 1.6]. Thus, it suffices to show that the last two maps in the array are isomorphisms. Bearing this in mind, observe that the map $\mathrm{op}K_G(X) \to \mathrm{op}K_G(X \to pt)$ is an isomorphism by [AP, Proposition 4.3], (which is independent of their Lemma 2.3, the main technical issue in our setting). On the other hand, by Corollary A.3, $\mathrm{op}K_G(X)$ is a free $R(G)$-module of rank $|X^T|$ (hence so is op$K_G(X \to pt)$). It follows that op$K_G(X \to pt) \to K^G(X)$, being a surjective map of free modules of the same rank, is an isomorphism. □

References

[AP] Anderson, D.; Payne, S. *Operational K-theory.* arXiv:1301.0425v1.

[AS] Atiyah, M.F.; Segal, G. B. *The index of elliptic operators: II.* The Annals of Mathematics, 2nd Ser., Vol. 87, No. 3. May, 1968, 531-545.

[SGA6] Berthelot, P.; Grothendieck, A.; Illusie, L. *Seminaire de Géométrie Algébrique 6: Theorie des intersections et theoreme de Riemann-Roch.* 1966-1967.

[B1] Bialynicki-Birula, A. *Some theorems on actions of algebraic groups.* The Annals of Mathematics, 2nd Ser., Vol 98, No. 3, Nov. 1973, pp. 480-497.

[BCP] Bifet, E.; De Concini, C.; Procesi,C. *Cohomology of regular embeddings.* Advances in Mathematics, 82, pp. 1-34 (1990).

[BGS] Bloch, S.; Gillet, H.; Soulé, C. *Non-Archimedean Arakelov theory.* J. Algebraic Geom. 4 (1995), no. 3, 427-485.

[Bo1] Borel, A. *Seminar on transformation groups.* Annals of Math Studies, No. 46, Princeton University Press, Princeton, N.J. 1960.

[Bo2] Borel, A. *Linear Algebraic Groups.* Third Edition, Springer-Verlag.

[BBM] Borho, W.; Brylinski, J.L. ; MacPherson, R. *Nilpotent orbits, primitive ideals, and characteristic classes.* Prog. Math. 78, Birkhäuser, 1989.

[Br1] Brion, M. *Equivariant Chow groups for torus actions.* Transf. Groups, vol. 2, No. 3, 1997, pp. 225-267.

[Br2] Brion, M. *The behaviour at infinity of the Bruhat decomposition.* Comment. Math. Helv. 73, 1998, pp. 137-174.

[Br3] Brion, M. *Poincaré duality and equivariant cohomology.* Michigan Math. J. 48, 2000, pp. 77-92.

[BJ] Brion, M., Joshua, R. *Equivariant Chow Ring and Chern Classes of Wonderful Symmetric Varieties of Minimal Rank.* Transformation groups, Vol. 13, N. 3-4, 2008, pp. 471-493.

[BV] Brion, M.; Vergne, M. *An equivariant Riemann-Roch theorem for complete, simplicial toric varieties.*

[C] Carrell, J. *The Bruhat graph of a Coxeter group, a conjecture of Deodhar, and rational smoothness of Schubert varieties.* Proc. Sympos. Pure Math., 56, Part 1, Amer. Math. Soc., Providence, RI, 1994.

[CG] Chriss, N.; Ginzburg, V. *Representation Theory and Complex Geometry.* Birkhäuser, 1997.

[CS] Chang, T., Skjelbred, T. *The topological Schur lemma and related results.* Ann. Math., 2nd. series, vol. 100, No. 2 (1974), pp. 307-321.

[EG-1] Edidin, D., Graham, W. *Equivariant Intersection Theory.* Invent. math. 131, 595-634 (1998).

[EG-2] Edidin, D., Graham, W. *Algebraic cycles and completions of equivariant K-theory.* Duke Math. Jour. Vol. 144, No. 3, 2008.

[F] Fulton, W. *Intersection Theory.* Springer-Verlag, Berlin, 1984.

[FM] Fulton, W., MacPherson, R. *Categorical framework for the study of singular spaces.* Mem. Amer. Math. Soc. 31 (1981), no. 243.

[FMSS] Fulton, W., MacPherson, R., Sottile, F., Sturmfels, B. *Intersection theory on spherical varieties.* J. of Algebraic Geometry, 4 (1994), 181-193.

[Gi] Gillet, H. *Homological descent for the K-theory of coherent sheaves*, in Algebraic K-theory, number theory, geometry and analysis (Bielefeld, 1982), 80-103, Lecture Notes in Math. 1046, Springer, Berlin, 1984.

[G1] Gonzales, R. *Rational smoothness, cellular decompositions and GKM theory.* Geometry and Topology, Vol. 18, No. 1 (2014), 291-326.

[G2] Gonzales, R. *Equivariant cohomology of rationally smooth group embeddings.* To appear in Transformation Groups.
[G3] Gonzales, R. *Equivariant operational Chow rings of T-linear varieties.* arXiv:1304.3645v2.
[G4] Gonzales, R. *On a notion of rational smoothness for intersection theory.* Preprint.
[G5] Gonzales, R. *Equivariant intersection theory on group embeddings.* Preprint.
[GKM] Goresky, M., Kottwitz, R., MacPherson, R. *Equivariant cohomology, Koszul duality, and the localization theorem.* Invent. math. 131, 25-83 (1998)
[Ki] Kimura, S. *Fractional intersection and bivariant theory.* Comm. in Alg. 20:1, pp. 285-302.
[Kr] Krishna, A. *Equivariant cobordism for torus actions.* Adv. Math. 231 (2012), no. 5, 2858-2891.
[LP] Littelmann, P.; Procesi, C. *Equivariant Cohomology of Wonderful Compactifications.* Operator algebras, unitary representations, enveloping algebras, and invariant theory, Paris, 1989.
[M1] Merkurjev, A. *Comparison of the equivariant and the standard K-theory of algebraic varieties.* Algebra i Analiz 9 (1997), no. 4, 175-214.
[M2] Merkurjev, A. *Equivariant K-theory.* Handbook of K-theory. Vol. 1, 2, 925-954, Springer, Berlin, 2005.
[P] Payne, S. *Equivariant Chow cohomology of toric varieties.* Math. Res. Lett. 13 (2006), no. 1, 2941.
[R1] Renner, L. *Linear algebraic monoids.* Encyclopedia of Math. Sciences, vol. 134, Springer, Berlin (2005).
[R2] Renner, L. *The H-polynomial of a semisimple monoid.* J. Alg. 319 (2008), 360-376.
[R3] Renner, L. *Rationally smooth algebraic monoids.* Semigroup Forum 78 (2009), 384-395.
[R4] Renner, L. *Hilbert Series for Torus Actions.* Advances in Mathematics, 76, 19-32 (1989).
[Seg] Segal, G. *Equivariant K-theory.* Publ. math. l'I.H.E.S., tome 34 (1968), p. 129-151.
[Su] Sumihiro, H. *Equivariant completion.* J. Math. Kyoto University 14 (1974), 1-28.
[Th1] Thomason, R. *Algebraic K-theory of group scheme actions.* Ann. Math. Study, vol 113, (1987), 539-563.
[Th2] Thomason, R. *Une formule de Lefschetz en K-théorie équivariante algébrique.* Duke Math. J, vol 68, No. 3 (1992), 447-462.
[Ti] Timashev, D. *Homogeneous spaces and equivariant embeddings.* Encyc. Math. Sci. 138, Springer.
[To] Totaro, B. *Chow groups, Chow cohomology and linear varieties.* To appear in J. Alg. Geometry.
[U] Uma, V. *Equivariant K-theory of compactifications of algebraic groups.* Transf. Groups 12 (2007), No. 2, 371-406.
[VV] Vezzosi, G., Vistoli, A. *Higher algebraic K-theory for actions of diagonalizable algebraic groups.* Invent. Math. 153 (2003), No. 1, 1-44.

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