Explicit computations around the Lichtenbaum-Hartshorne vanishing theorem

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1. Introduction

Let I denote a proper ideal of a local (Noetherian) ring (A, M). By [2] and [3] it is of great importance in algebraic geometry and commutative algebra to have vanishing results for the local cohomology modules $H_I^i(X), i \in \mathbb{N}$, for an A-module X. See [2] for the definition and basic results on local cohomology. It is known, see [2], that $H_I^i(X) = 0$ for all $i > \dim X$. The Lichtenbaum-Hartshorne vanishing theorem, see [3], states that $H_I^d(A) = 0, d = \dim A$, provided $\dim \hat{A}/(I\hat{A} + P) > 0$ for all $P \in \operatorname{Ass}\hat{A}$ with $\dim \hat{A}/P = d$. Here $\hat{A}$ denotes the completion of A. Now there are several proofs of this vanishing result, see [1], [3], [5]. In particular, R.Y. Sharp has also shown the necessity of the local condition for the vanishing of $H_I^d(A)$, see [8]. All the proofs use the fact that, under certain circumstances, the I-adic topology on A is equivalent to the topology induced by a filtration $\{J_n\}_{n \in \mathbb{N}}$ of ideals such that A/J_n does not have M-torsion.

The main point of the present paper is an explicit computation of $H_I^d(A)$ and, by the same way, to clarify the equivalence of the topologies involved. It turns out that the local cohomology module $H_I^d(A)$ is the obstruction for the equivalence of these topologies. In fact we extend the vanishing result to $H_I^n(X)$, where X denotes a finitely generated A-module with $n = \dim X$. To this end one has to

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generalize the notion of the canonical module. To be more precise, assume that (A, M) possesses a dualizing complex, see [4], [6] and [7]. Then we introduce the canonical module K_X of X, see Section 3 for the definition and a brief summary of properties needed in the paper. The canonical module K_A of A is nothing else but the ordinary canonical module. Note that $K_{\hat{X}}$ always exists, where $\hat{X}$ denotes the completion of X. Let $\mathrm{Ass}_{\hat{A}} K_{\hat{X}} = \{Q_1, \ldots, Q_t\}$ and $0 = Y(Q_1) \cap \ldots \cap Y(Q_t)$ be a minimal primary decomposition of $K_{\hat{X}}$. Define $T_I(X) = \{P \in \mathrm{Ass}_R X : \dim A/(I+P) = 0\}$.

(1.1) THEOREM. a) *There is an isomorphism*

$$\mathrm{Hom}_A(H_I^n(X), E) \cong \bigcap_{Q_i \notin T_{I\hat{A}}(K_{\hat{X}})} Y(Q_i),$$

where E denotes the injective hull of the residue field.
b) *The $\hat{A}$-module* $\mathrm{Hom}_A(H_I^n(X), E)$ *is finitely generated and*

$$\mathrm{Ass}_{\hat{A}}\ \mathrm{Hom}_A(H_I^n(X), E) = \{P \in T_{I\hat{A}}(\hat{X}) : \dim \hat{A}/P = n\}.$$

c) *The following conditions are equivalent:*
(i) $H_I^n(X) = 0$.
(ii) $\{P \in T_{I\hat{A}}(\hat{X}) : \dim \hat{A}/P = n\} = \emptyset$.
(iii) *The topology defined by* $\{I^n K_{\hat{X}} : \langle \hat{M} \rangle\}$ *on* $K_{\hat{X}}$ *is equivalent to the* $I\hat{A}$*-adic topology on* $K_{\hat{X}}$.

The proof of 1.1 is given in Section 4. Section 2 is concerned with the equivalence of a certain topology on X with the I-adic topology. One of the main points is the computation and the vanishing of $\varprojlim H_M^0(X/I^n X)$, see 2.3. The notation is the same as in [6].

2. On ideal topologies

Let X denote a finitely generated A-module, (A, M) a local ring. For an ideal I of A and a submodule $Y \subseteq X$ the increasing sequence of submodules

$$Y \subseteq Y :_X I \subseteq \ldots \subseteq Y :_X I^n \subseteq \ldots$$

becomes stationary. Denote its ultimate constant value by $Y :_X \langle I \rangle$. Note that

$$Y :_X \langle I \rangle = Y :_X I^n \quad \text{for all large } n.$$

One has $\text{Ass}_A X/Y :_X \langle I\rangle = \text{Ass}_A X/Y \setminus V(I)$. Therefore the primary decomposition of $Y :_X \langle I\rangle$ consists of those primary components of Y whose associated prime ideals do not contain I. Let $\text{Ass}_A X = \{P_1, \ldots, P_s\}$ and $0 = Z(P_1) \cap \ldots \cap Z(P_s)$ a minimal primary decomposition of X. For a prime ideal P of A let $X \to X_P, x \mapsto \frac{x}{1}$, denote the natural homomorphism.

(2.1) LEMMA. *The following submodules of X coincide:*

a) $\bigcap_{n\geq 1} I^n X :_X \langle M\rangle$,

b) $\bigcap_{P\in S_I(X)} \ker (X \to X_P)$, *where* $S_I(X) = \text{Supp}\ X/IX \setminus V(M)$,

c) $\bigcap_{P_i \notin T_I(X)} Z(P_i)$, *where*

$$T_I(X) = \{P \in \text{Ass}_A X : \dim A/(I+P) = 0\}.$$

Proof. First of all note that

$$IX :_X \langle M\rangle = \bigcap_{P\in S_I(X)} (IX_P \cap X).$$

Here $IX_P \cap X$ denotes the inverse image of IX_P under the natural map $X \to X_P$. To this end let $x \in IX :_X \langle M\rangle$. For any $P \in S_I(X) =: S$ choose an element $r_P \in M^n \setminus P$, where n is such that $IX :_X \langle M\rangle = IX :_X M^n$. Hence, $r_P x \in IX$ and $x \in IX_P \cap X$. Conversely let

$$x \in \bigcap_{P\in S} (IX_P \cap X).$$

That is, for every $P \in S$ there is an element $s_P \in A\setminus P$ such that $s_P x \in IX$. Let J denote the ideal of A generated by $s_P, P \in S$, and by $\text{Ann}_A X/IX$. Then $\text{Supp}_A A/J \subseteq \{M\}$ and there is an integer n such that

$$M^n x \subseteq Jx \subseteq IX.$$

Whence $x \in IX :_X \langle M\rangle$, as required.

With the aid of the above formula it follows that

$$\bigcap_{n\geq 1} I^n X :_X \langle M\rangle = \bigcap_{P\in S} \ker(X \to X_P)$$

since $\bigcap_{n\geqslant 1} I^n X_P = 0$ by the Krull intersection theorem. Now the equality of the second and third module is clear since $\ker(X \to X_P)$ is the intersection of those Q-primary components of X with $Q \subseteq P$. □

For the local ring (A, M) denote by $(\hat{A}, \hat{M})$ its M-adic completion. For a finitely generated A-module X the M-adic completion $\hat{X}$ is isomorphic to $X \otimes_A \hat{A}$. Moreover $A \to \hat{A}$ is a faithfully flat extension.

(2.2) LEMMA. *The following conditions are equivalent:*
i) $\bigcap_{n\geqslant 1} I^n \hat{X} :_{\hat{X}} \langle \hat{M} \rangle = 0$.
ii) *For any integer n there is an integer $m = m(n)$ such that $I^m X :_X \langle M \rangle \subseteq I^n X$.*

Proof. By the faithful flatness it is easily seen that (ii) holds if and only if $I^m \hat{X} :_{\hat{X}} \langle \hat{M} \rangle \subseteq I^n \hat{X}$. That is, without loss of generality we may assume X complete. The implication ii) $\Rightarrow$ i) is a consequence of Krull's intersection theorem. In order to prove i) $\Rightarrow$ ii) let us make a slight modification of Chevalley's theorem, see [9], Ch. VIII, Theorem 13. For a fixed integer n the modules

$$E_{mn} = (I^m X : \langle M \rangle + I^n X)/I^n X, \quad m \geqslant n,$$

form a decreasing sequence of modules of finite length. Whence there is an integer $\ell = \ell(n)$ such that $E_{mn} = E_{\ell n}$ for all $m \geqslant \ell$. Put

$$E_n = I^\ell X : \langle M \rangle + I^n X.$$

Then $E_n + I^k X = E_k$ for all $n \geqslant k$. Suppose there is an integer k such that $E_k \neq I^k X$. Then $E_n \neq I^n X$ for all $n \geqslant k$. Now choose elements $y_n \in E_n \setminus I^n X$ such that

$$y_{n+1} \equiv y_n \text{ modulo } I^n X \quad \text{for all } n \geqslant k.$$

Then $\{y_n\}$ is a convergent series with

$$0 \neq z := \lim y_n \in X.$$

Note that, see [9], X is complete with respect to the I-adic topology. For a given $n \in \mathbb{N}$ there is an integer $n_0 \geqslant n$ such that $z - y_m \in I^n X$ for all $m \geqslant n_0$. But then

$$0 \neq z \in \bigcap_{m \geqslant 1} E_m = \bigcap_{n \geqslant 1} I^n X :_X \langle M \rangle,$$

a contradiction. □

The condition ii) of 2.2 means nothing else but the equivalence of the topology defined by $\{I^n X :_X \langle M \rangle\}_{n \geqslant 0}$ to the I-adic topology on X. Therefore $\bigcap_{n \geqslant 1} I^n \hat{X} : \langle \hat{M} \rangle$ gives the obstruction for the equivalence of both of these topologies.

(2.3) LEMMA. *The inverse system $\{X/I^n X\}_{n \geqslant 1}$ with the natural induced maps defines an inverse system $\{H^0_M(X/I^n X)\}_{n \geqslant 1}$ such that*

$$\varprojlim H^0_M(X/I^n X) \simeq \bigcap_{n \geqslant 1} I^n \hat{X} :_{\hat{X}} \langle \hat{M} \rangle.$$

Proof. If we apply the local cohomology functor to $\{X/I^n X\}_{n \geqslant 1}$ we get the desired inverse system. Because $H^0_M(X/I^n X)$ is of finite length it possesses the structure of an $\hat{A}$-module such that

$$H^0_M(X/I^n X) \simeq H^0_M(X/I^n X) \otimes_A \hat{A} \simeq H^0_{\hat{M}}(\hat{X}/I^n \hat{X}).$$

That is, without loss of generality we may assume X as complete. Now $H^0_M(X/I^n X) = I^n X :_X \langle M \rangle / I^n X$ and there is the following short exact sequence of inverse systems

$$0 \to \{I^n X\} \to \{I^n X : \langle M \rangle\} \to \{I^n X : \langle M \rangle / I^n X\} \to 0.$$

By passing to the inverse limit there is an injection

$$0 \to \bigcap_{n \geqslant 1} I^n X : \langle M \rangle \xrightarrow{\varphi} \varprojlim \{I^n X : \langle M \rangle / I^n X\}.$$

Now we claim that φ is surjective. To this end let

$$\{y_n + I^n X\} \in \varprojlim \{I^n X : \langle M \rangle / I^n X\},$$

where $y_n \in I^nX : \langle M \rangle$. Then the sequence defines an element $z \in \varprojlim X/I^nX = X$. Note that X is I-adically complete, see [9], Ch. VIII. That is, for every n there exists an integer $n_0 \geqslant n$ such that $z - y_m \in I^nX$ for all $m \geqslant n_0$. Hence

$$z \in \bigcap_{m \geqslant 1} I^mX : \langle M \rangle,$$

as required. □

By view of 2.3 one might continue with the explicit computation of $\varprojlim H^i_M(X/I^nX), i \in \mathbb{N}$, which is closely related to the cohomology groups of the formal completion of $U = \mathrm{Spec}A \backslash V(M)$ along $V(I)$.

3. The canonical module of a module

In this section let (A, M) denote a local ring possessing a dualizing complex $D^\bullet_A$. See [4], [6], and [7] for basic results on dualizing complexes. If A is complete or, more general, the factor ring of a Gorenstein ring, then $D^\bullet_A$ exists. One may normalize $D^\bullet_A$ such that $D^i_A = 0$ for all $i < -d, d = \dim A$, resp. $i > 0$ and such that

$$D^i_A = \bigoplus_{P \in \mathrm{Spec}A, \dim A/P = -i} E_R(R/P), \quad -d \leqslant i \leqslant 0,$$

where $E_R(R/P)$ denotes the injective hull of R/P. It follows that $0 \neq H^{-d}(D^\bullet_A)$, which is called the canonical module K_A of A. In [6] this concept is generalized to an arbitrary finitely generated A-module X as follows: Consider the complex $\mathrm{Hom}_A(X, D^\bullet_A)$. Then $(\mathrm{Hom}_A(X, D^\bullet_A))^i = 0$ for $i > 0$ and $i < -\dim X$. Define

$$H^{-i}(\mathrm{Hom}_A(X, D^\bullet_A)) = \begin{cases} K^i_X & \text{if } \ 0 \leqslant i < \dim X \text{ and} \\ K_X & \text{if } \ i = \dim X, \end{cases}$$

and call K_X the canonical module of X. Note that K_X is a finitely generated A-module.

We say that a finitely generated A-module X satisfies Serre's condition $S_r, r \in \mathbb{N}$, provided

$$\mathrm{depth}_{A_P} X_P \geqslant \min\{r, \dim_{A_P} X_P\}$$

for all prime ideals $P \in \mathrm{Supp}_A X$. Note that X satisfies always S_0, while S_1 holds if and only if X is unmixed. X is a Cohen-Macaulay

A-module if and only if it satisfies $S_n, n = \dim X$. Next summarize a few basic properties of K_X^i and K_X respectively.

(3.1) PROPOSITION. a) $\dim K_X = \dim X$ *and* $\dim K_X^i \leqslant i$ *for all* $0 \leqslant i < \dim X$.
b) *X satisifies condition* S_r *if and only if* $\dim K_X^i \leqslant i - r$ *for all* $0 \leqslant i < \dim X$.
c) K_X *satisfies condition* S_2.
d) *If* $\dim_{A_P} X_P + \dim A/P = \dim X$ *for* $P \in \operatorname{Supp} X$, *then* $(K_X)_P \simeq K_{X_P}$.

These results are shown in [6], 3.1.1 and 3.2.1. As it is convenient for duality one may relate X in a natural way to K_{K_X}, the canonical module of the canonical module. To this end let $X[n]$ denote the module X considered as a complex concentrated in degree $-n$. Let $n = \dim X$. Then there is a short exact sequence of complexes

$$0 \to K_X[n] \to \operatorname{Hom}_A(X, D_A^\bullet) \to I_X^\bullet \to 0,$$

where $I_X^\bullet$ is defined as the cokernel of the natural embedding. It follows that

$$H^{-i}(I_X^\bullet) = K_X^i, 0 \leqslant i < n, \ H^{-i}(I_X^\bullet) = 0 \quad \text{otherwise} .$$

Applying $\operatorname{Hom}_A(\bullet, D_A^\bullet)$ to the above exact sequence and taking cohomology it yields the following exact sequence

$$0 \to H^{-1}(\operatorname{Hom}_A(I_X^\bullet, D_A^\bullet)) \to X \to K_{K_X} \to H^0(\operatorname{Hom}_A(I_X^\bullet, D_A^\bullet)) \to 0.$$

That is, there is a natural homomorphism

$$\tau_X : X \to K_{K_X}.$$

(3.2) PROPOSITION. *Suppose X is equidimensional. Then the following holds for* τ_X:
a) τ_X *is injective.*
b) τ_X *is an isomorphism if and only if X satisfies condition* S_2.
c) τ_{K_X} *is an isomorphism.*

For the proof of a) and b) see [6], 3.2.2. The statement in c) follows by b) and 3.1. In the following we describe coker τ_X.

(3.3) PROPOSITION. *Suppose X is equidimensional. Then*

$$\text{Supp coker } \tau_X = \{P \in \text{Supp}X : X_P \textit{ does not satisfy } S_2\}$$

and $\dim \text{coker } \tau_X \leqslant \dim X - 2$.

Proof. Let $C = \text{coker } \tau_X$. By 3.2 there is a short exact sequence $0 \to X \to K_{K_X} \to C \to 0$. because X is equidimensional we know that

$$\dim X = \dim A/P + \dim_{A_P} X_P \text{ for } P \in \text{Supp}X$$

and therefore $(K_{K_X})_P \simeq K_{K_{X_P}}$, see 3.1. By the functoriality of τ_X we see that $C_P = 0$ if and only if X_P satisfies S_2, see 3.2. This proves the first part of the statement. For the second let $P \in \text{Supp}X$ with $\dim A/P > \dim X - 1$. Then $\dim_{A_P} X_P \leqslant 1$ and X_P satisfies S_2, i.e., $C_P = 0$ by the previous argument. □

We will end this section by relating the set of associated primes and the annihilator of X to that of K_X.

(3.4) PROPOSITION. a) $\text{Ass}_A K_X = \{P \in \text{Ass}_A X : \dim A/P = \dim X\}$.
b) $\text{Ann}_A K_X = (\text{Ann}_A X)_{\dim X}$, *i.e., the intersection of all P-primary components of* $\text{Ann}_A X$ *such that* $\dim A/P = \dim X$.

Proof. a) is shown in [6], 3.1. b) We have $\text{Ann}_A X \subseteq \text{Ann}_A K_X$ by the definition of K_X. Equality holds provided X satisfies S_2, see 3.2. Now let $P \in \text{Ass}_A X$ be a prime ideal with $\dim A/P = \dim X$. Then $\text{Ann}_{A_P} X_P = \text{Ann}_{A_P} K_{X_P}$. Since K_X is equidimensional it proves $\text{Ann}_A K_X = (\text{Ann}_A X)_{\dim X}$ as it is easily seen by the primary decomposition. □

4. On the vanishing of local cohomology

In the first part of this section let (A, M) denote a local ring possessing a normalized dualizing complex $D_A^{\bullet}$. Let X be a finitely generated A-module with $n = \dim X$. For an ideal I of A it is known, see e.g. [2], that

$$H^i_I(X) = 0 \text{ for all } i > n.$$

The following results concern the structure of $H^n_I(X)$. To this end let $\hat{X} \simeq X \otimes_A \hat{A}$ denote the completion of X. Let $\mathrm{Ass}_{\hat{A}} \hat{X} = \{P_1, \ldots, P_t\}$ and $\mathrm{Ass}_{\hat{A}} K_{\hat{X}} = \{Q_1, \ldots, Q_t\}$. Fix a minimal primary decomposition

$$\begin{array}{lll} 0 = X(P_1) \cap \ldots \cap X(P_s) & \text{of} & \hat{X} \text{ resp.} \\ 0 = Y(Q_1) \cap \ldots \cap Y(Q_t) & \text{of} & K_{\hat{X}}. \end{array}$$

considered as an $\hat{A}$-module resp.

(4.1) THEOREM. a) *Suppose X satisfies S_2. Then there is an isomorphism*

$$\mathrm{Hom}_A(H^n_I(K_X), E) \simeq \bigcap_{P_i \notin T_{I\hat{A}}(\hat{X})} X(P_i).$$

b) *There is an isomorphism*

$$\mathrm{Hom}_A(H^n_I(X), E) \simeq \bigcap_{Q_i \notin T_{I\hat{A}}(K_{\hat{X}})} Y(Q_i).$$

Here E denotes the injective hull of the residue field A/M of A.

Proof. We begin with the proof of the formula claimed in a). By 2.1 and 2.3 it is enough to show the following isomorphism

$$\mathrm{Hom}_A(H^n_I(K_X), E) \simeq \varprojlim H^0_M(X/I^n X).$$

The local duality theorem, see e.g. [4], provides an isomorphism

$$\varprojlim H^0_M(X/I^n X) \simeq \mathrm{Hom}_A(\varinjlim H^0(\mathrm{Hom}_A(X/I^n X, D^\bullet_A)), E)$$

$$\simeq \mathrm{Hom}_A(H^0_I(\mathrm{Hom}_A(X, D^\bullet_A)), E),$$

where we use that $R\Gamma_I(X^\bullet) \simeq \varinjlim R\mathrm{Hom}_A(A/I^n, X^\bullet)$ for a bounded complex $X^\bullet$ with finitely generated cohomology modules, see [4]. As in Section 3 take the short exact sequence

$$0 \to K_X[n] \to \mathrm{Hom}_A(X, D^\bullet_A) \to I^\bullet_X \to 0$$

and apply the derived functor $R\Gamma_I$. The long exact cohomology sequence of the resulting short exact sequence of complexes yields the following four term exact sequence

$$H_I^{-1}(I_X^\bullet) \to H_I^n(K_X) \to H_I^0(\mathrm{Hom}_A(X, D_A^\bullet)) \to H_I^0(I_X^\bullet).$$

Now the statement follows provided $H_I^i(I_X^\bullet) = 0$ for $i = 0, -1$. In order to prove this take the spectral sequence for computing the hypercohomology

$$E_2^{pq} = H_I^p(H^q(I_X^\bullet)) \Rightarrow E^{p+q} = H_I^{p+q}(I_X^\bullet).$$

Because $H^q(I_X^\bullet) = K_X^{-q}$ and $\dim K_X^{-q} \leqslant -q-2$, see 3.1, it follows that $E_2^{pq} = 0$ for all $p + q \in \{0, -1\}$. Therefore $H_I^i(I_X^\bullet) = 0$ for $i = 0, -1$, as required.

In order to prove b) we first show that $H_I^n(X) \simeq H_I^n(K_{K_X})$. To this end let U denote the maximal submodule of X such that $\dim U < n$. Then X/U is equidimensional and

$$H_I^n(X) \simeq H_I^n(X/U) \text{ resp. } K_X \simeq K_{X/U}$$

as easily seen. That is, without loss of generality we may assume X equidimensional. Then the short exact sequence

$$0 \to X \to K_{K_X} \to C \to 0$$

provides $H_I^n(X) \simeq H_I^n(K_{K_X})$ because $\dim C \leqslant n-2$, see 3.2 and 3.3. Finally the statement b) follows now by a) because K_X satisfies condition S_2, see 3.1. □

The following Corollary 4.2 b) is the dual statement of a result shown by R.Y. Sharp in [8] for the case $X = A$.

(4.2) COROLLARY. *Let X denote a finitely generated module over an arbitrary local ring (A, M) with $n = \dim X$.*
a) *There is an isomorphism*

$$\mathrm{Hom}_A(H_I^n(X), E) \simeq \bigcap_{Q_i \notin T_{I\hat{A}}(K_{\hat{X}})} Y(Q_i),$$

where $\mathrm{Ass}_{\hat{A}} K_{\hat{X}} = \{Q_1, \ldots, Q_t\}$ and $0 = Y(Q_1) \cap \ldots \cap Y(Q_t)$ is a minimal primary decomposition.
b) *$H_I^n(X)$ is an Artinian A-module and*

$$\mathrm{Ass}_{\hat{A}} \operatorname{Hom}_A(H_I^n(X), E) = \{P \in T_{I\hat{A}}(\hat{X}) : \dim \hat{A}/P = n\}.$$

c) $H_I^n(X) = 0$ *if and only if* $\{P \in T_{I\hat{A}}(\hat{X}) : \dim \hat{A}/P = n\} = \emptyset$.

Proof. First note that $\operatorname{Hom}_A(H_I^n(X), E)$ admits a natural $\hat{A}$-module structure such that

$$\operatorname{Hom}_A(H_I^n(X), E) \simeq \operatorname{Hom}_{\hat{A}}(H_I^n(X) \otimes_A \hat{A}, E).$$

Because $H_I^n(X) \otimes_A \hat{A} \simeq H_{I\hat{A}}^n(\hat{X})$ and since $\hat{A}$ possesses a dualizing complex a) follows by 4.1. This shows also that $\operatorname{Hom}_A(H_I^n(X), E)$ is a finitely generated $\hat{A}$-module. By Matlis duality the first part of b) is true. The second is an easy consequence of a). Finally c) follows by b). □

By view of the results shown in 2.2 and 4.1 b) now 4.2 proves the statements of 1.1 in the introduction.

Let us call A a quasi-Gorenstein ring, if K_A exists and $K_A \simeq A$. In this case A satisfies S_2.

(4.3) COROLLARY. *Let I denote an ideal of (A, M), a complete local quasi-Gorenstein ring with $d = \dim A$.*

a) $\operatorname{Hom}_A(H_I^d(A), E) \simeq \bigcap_{P \in S_I(A)} \ker(A \to A_P)$.

b) $\operatorname{Hom}_A(H_I^d(A), E) \simeq \bigcap_{P_i \notin T_I(A)} Z(P_i)$

where $\mathrm{Ass}\, A = \{P_1, \ldots, P_r\}$ *and* $0 = Z(P_1) \cap \ldots \cap Z(P_r)$ *is a minimal primary decomposition.*

c) $H_I^d(A) = 0$ *if and only if the topology defined by* $\{I^n : \langle M \rangle\}$ *is equivalent to the I-adic topology.*

Proof. Because $K_A \simeq A$ the statements follow by 4.1 with the aid of 2.1 and 2.2. □

In the case of a complete Gorenstein ring a) of 4.3 is shown by F.W. Call and R.Y. Sharp, see [1]. Under the same assumption they got also a particular case of c). For a one-dimensional prime ideal P of a complete Gorenstein ring A they have shown that $H_P^d(A) = 0$ if and only if the topology defined by the symbolic powers $\{P^{(n)}\}$ is equivalent to the P-adic topology. Now this is true also in the more general situation of 4.3 since $P^{(n)} = P^n : \langle M \rangle$ for a one-dimensional prime ideal P of A.

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