Quasi Potentials and Kähler-Einstein Metrics on Flag Manifolds

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The aim of this paper is to prove the following result.

Proposition. *Let G be a complex reductive group and P, Q parabolic subgroups of G with $P \subset Q$ and K a maximal compact subgroup of G. The K-invariant Kähler-Einstein metric of G/P restricted to any fiber of the fibration $G/P \to G/Q$ is again Kähler-Einstein.*

The proof of this result uses ideas suggested by the theory of linear algebraic groups, in particular §8 of R. Steinberg's Lectures on Chevalley Groups [St 1] and §14 of Borel-Hirzebruch [Bo-Hi]. An essential ingredient is an explicit description of the differential forms dual to the Dynkin lines [St 2]; this description goes back to Borel-Hirzebruch [Bo-Hi, §14]. A special case of this result is proved and used by the first and the second authors in [Az-Ko] to prove the existence of complete Ricci-flat Kähler metrics on the complexification of Riemannian symmetric spaces of compact type. In our attempt of proving our main result, we eventually simplified the arguments in [Bo-Hi, §14].

1. Recollection of Known Results, Quasi-Potentials on G/P

Let G be a complex reductive group, B a Borel subgroup of G, T a maximal torus of G contained in B, R the roots of T in G, R^+ the positive system of roots defined by the pair (B,T) and S the corresponding simple system of roots. One knows that for each $\alpha \in R^+$ there exists $X_\alpha, X_{-\alpha} \in \text{Lie}(G)$ such that the map

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto X_\alpha, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto X_{-\alpha}$$

is an isomorphism of $sl(2,\mathbf{C})$ onto the Lie algebra generated by X_α, $X_{-\alpha}$. Hence there exists a homomorphism ϕ_α from $SL(2,\mathbf{C})$ onto a subgroup L_α of G whose Lie algebra is generated by X_α, $X_{-\alpha}$. We set, for $\alpha \in R^+$,

$$u_\alpha(z) = \phi_\alpha \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, \quad u_{-\alpha}(z) = \phi_\alpha \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}, \quad \check{\alpha}(z) = \phi_\alpha \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}.$$

By a variant of Bruhat's Lemma [St 1, p.99], the group

$$K = < \phi_\alpha(SU(2));\ \alpha \text{ is simple} >$$

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is a maximal compact subgroup of G and we have the decomposition $G = KB$. Let P be a parabolic subgroup of G containing B and L the Levi-complement, i.e., the maximal reductive subgroup of P containing T. The roots R_P of T in P have the decomposition

$$R_P = R'_P \cup R''_P$$

where

$$R'_P = \{\alpha \in R_P;\ -\alpha \in R_P\}$$

and

$$R''_P = \{\alpha \in R_P;\ -\alpha \notin R_P\}.$$

Notice that R''_P consists of positive roots. We have

$$L =< T, U_\alpha, U_{-\alpha};\ \alpha \in R'_P \cap R^+ >$$

and

$$P = L \cdot \mathrm{R_u}(P),$$

where U_α is the root subgroup corresponding to the root α and $\mathrm{R_u}(P)$ is the unipotent radical of P; in fact

$$\mathrm{R_u}(P) = \prod_{\alpha \in R''_P} U_\alpha,$$

the product being taken in any preassigned order. The indecomposable positive roots in L form a system π of the set of simple roots S. We have

$$L = T_1 \cdot L',$$

where L' denotes the commutator subgroup of L' (which is semi-simple) and

$$T_1 = \{ \prod_{\alpha \in S \setminus \pi} \check{\alpha}(z_\alpha);\ \alpha \in \mathbf{C}^*\}.$$

Without loss of generality, we may assume that G is semisimple and simply connected, so that $\pi_1(G) = 0 = \pi_2(G)$. Let ω_α $(\alpha \in S)$ be the fundamental dominant weights and ρ_α the irreducible representation of G with highest weight ω_α and v_α a highest weight vector therein. Choose a hermitian inner product on the representation space invariant under the maximal compact subgroup K and let $\overline{\omega}_\alpha$ be the (1,1)-form defined on G by

$$\overline{\omega}_\alpha = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\|\rho_\alpha(g) \cdot v_\alpha\|^2.$$

Let $\pi : G \to G/P$ be the natural map. The following result is proved in [Al-Pe] for the classical groups and in general in [Az-Lo].

Proposition. *If ω is a closed K-invariant (1,1)-form on G/P, then its pull-back $\pi^*\omega$ is of the form*

$$\pi^*\omega = \sqrt{-1}\partial\bar{\partial}\phi$$

where

$$\phi(g) = \sum_{\alpha \in S\setminus\pi} c_\alpha \log \|\rho_\alpha(g)\cdot v_\alpha\|.$$

Conversely any form $\sqrt{-1}\partial\bar{\partial}\phi$ for such a ϕ on G can be pushed down to a K-invariant closed (1,1)-form ω on G/P. Moreover, ω is a Kähler form if and only if $c_\alpha > 0$ for all $\alpha \in S\setminus\pi$.

Definition. *We call the function ϕ a quasi potential for ω.*

Note that if ω is non-degenerate, it can, by compactness of G/P, never have a potential there. A sufficient condition for a closed (1,1)-form on a complex manifold M to have a potential is that $H^2(M,\mathbf{R}) = 0$ and $H^1(M,\mathcal{O}_M) = 0$, $\mathcal{O}_M$ being the sheaf of holomorphic functions on M. For the convenience of the reader, we recall the main steps of the proof of the above proposition. The idea is that as G is semi-simple and simply connected, every closed (1,1)-form $\widetilde{\omega}$ on G can be written as $\widetilde{\omega} = \sqrt{-1}\partial\bar{\partial}\phi$, and the potential ϕ is determined up to functions of type $h+\overline{h}$, h being holomorphic on G. In particular, if ω is a closed K-invariant (1,1)-form on G/P and $\widetilde{\omega} = \pi^*\omega$, then we may assume that $\widetilde{\omega} = \sqrt{-1}\partial\bar{\partial}\phi$ is left K-invariant and right P-invariant, and therefore the potential ϕ is determined up to a constant. Indeed, as K is maximal compact in G, every K-invariant holomorphic function on G must be constant. Since $G = KP$, the potential ϕ is completely determined on P. Now for $p \in P$, $R_p^*\widetilde{\omega} = \widetilde{\omega}$, R_p being the right translation by p, so $R_p^*\phi - \phi$ is a constant, say, $c(p)$ and so $c(pq) = c(p)+c(q)$ for $p, q \in P$, and also $\phi(p)-\phi(e) = c(p)$. Taking $\phi(e) = 0$, it remains to determine additive characters of P which are invariant under the maximal compact subgroup $K_P = K \cap P$ of P. Since a character vanishes on the commutator subgroup P' of P, using the decomposition $P = P'T_1$, where

$$T_1 = \{ \prod_{\alpha \in S\setminus\pi} \check{\alpha}(z_\alpha);\ z_\alpha \in \mathbf{C}^* \},$$

we just have to determine the additive characters of 1-parameter group $\check{\alpha}$ which are invariant under S^1 and these are clearly of the form

$$\check{\alpha}(z) = c_\alpha \log |z|.$$

Hence the potential is of the form claimed above. For the proof of the fact that the form $\sqrt{-1}\partial\bar{\partial}\phi$ for such a ϕ can be pushed down and it is Kähler if all $c_\alpha > 0$ ($\forall \alpha \in S\setminus\pi$), the reader is referred to [Az-Lo].

2. Description of the Duals of the Dynkin Lines

Let $\xi_0 = eP \in G/P$. With the notations of §1, let $\alpha \in S\setminus\pi$. Then $L_\alpha \cdot \xi_0 \cong \mathbf{P}^1(\mathbf{C})$. The following definition is due to Steinberg [St 2].

Definition. *We denote the copy of the projective line $L_\alpha \cdot \xi_0$ by $\mathbf{P}_\alpha$ and call it the Dynkin line corresponding to the root α.*

Let $\overline{\omega}_\alpha$ be the (1,1)-form on G as defined in §1. Then, as already remarked, $\overline{\omega}$ is the pull-back of a K-invariant closed (1,1)-form ω_α; namely, if s is a local section of $G \to G/P$, then $\omega_\alpha(\xi) = \overline{\omega}_\alpha(s(\xi))$.

Proposition. *The forms $\{\omega_\alpha;\ \alpha \in S\backslash\pi\}$ form a basis of $H^2(G/P, \mathbf{R})$ and*

$$\int_{\mathbf{P}_\beta} \omega_\alpha = \delta_{\alpha\beta}.$$

Proof. As we have assumed G to be simply connected, we have $\pi_1(G) = 0 = \pi_2(G)$. From the homotopy exact sequence of the fibration $P \to G \to G/P$ we obtain $\pi_1(G/P) = 0$, $\pi_2(G/P) \cong \pi_1(P)$. Using the decomposition $P = T_1 L' \mathrm{R}_u(P)$, where $T_1 = \{\prod_{\alpha \in S\backslash\pi} \check{\alpha}(z_\alpha);\ z_\alpha \in \mathbf{C}^*\}$, we see that $\pi_1(P) \cong \pi_1(T_1)$. From $\pi_1(G/P) = 0$ and the Hurewicz theorem we have $\pi_2(G/P) \cong H_2(G/P, \mathbf{Z})$ and so rank of $\pi_2(G/P)$ is equal to the cardinality of $S\backslash\pi$. Let us now show that for $\alpha, \beta \in S\backslash\pi$, $\int_{\mathbf{P}_\beta} \omega_\alpha = \delta_{\alpha\beta}$, where $\mathbf{P}_\beta = L_\beta \cdot \xi_0$ is a Dynkin line. The computation is similar to that in [Az]. A local section of $G \to G/P$ defined in a neighborhood of $\xi_0 = eP$ is $r \cdot \xi_0 \mapsto r$, $r \in \mathrm{R}_u(P)^-$ where $\mathrm{R}_u(P)^-$ is the unipotent radical of the parabolic subgroup opposite to P. We have

$$\mathbf{P}_\beta = U_{-\beta} \cdot \xi_0 \cup n_\beta \cdot \xi_0,$$

where $n_\beta = \phi_\beta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, so

$$\int_{\mathbf{P}_\beta} \omega_\alpha = \int_{U_{-\beta} \cdot \xi_0} \omega_\alpha.$$

By the Gram-Schmidt process applied to the columns of $SL(2, \mathbf{C})$ we have:

$$\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} = k \begin{pmatrix} (1+|z|^2)^{\frac{1}{2}} & 0 \\ 0 & (1+|z|^2)^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & \frac{\overline{z}}{1+|z|^2} \\ 0 & 1 \end{pmatrix},$$

where $k \in SU(2)$, so

$$u_{-\beta}(z) = \phi_\beta(k) \check{\beta}((1+|z|^2)^{\frac{1}{2}}) u_\beta(\frac{\overline{z}}{1+|z|^2}).$$

Hence

$$\begin{aligned} \|u_{-\beta} \cdot v_\alpha\| &= \|\check{\beta}((1+|z|^2)^{\frac{1}{2}}) \cdot v_\alpha\| \\ &= \left((1+|z|^2)^{\frac{1}{2}}\right)^{\omega_\alpha(\check{\beta})} = (1+|z|^2)^{\frac{1}{2}\delta_{\alpha\beta}}. \end{aligned}$$

So

$$\int_{\mathbf{P}_\beta} \omega_\alpha = 0$$

if $\alpha \neq \beta$ and

$$\int_{\mathbf{P}_\beta} \omega_\alpha = \frac{\sqrt{-1}}{2\pi} \int_{\mathbf{C}} \partial\bar{\partial} \log(1+|z|^2) = 1$$

if $\alpha = \beta$. Hence the forms $\{\omega_\alpha\}_{\alpha\in S\setminus\pi}$ are independent generators of $H^2(G/P)$ and their duals are the Dynkin lines $\{\mathbf{P}_\alpha\}_{\alpha\in S\setminus\pi}$.

3. Chern Classes of Line Bundles on G/P

For the proof of the main result, we need a good formula for the Chern class of homogeneous line bundles on G/P. We derive a formula by constructing a norm on such line bundles which is suitable for root-theoretic computations, following the ideas in [Az]. Let (L,π) be a holomorphic line bundle on a complex manifold M. Recall that a norm on L is a differentiable function $N : L \to \mathbf{R}^{\geq 0}$ such that for all $p \in M$, $v \in \pi^{-1}(p)$ and $z \in \mathbf{C}$, $N(zv) = |z|N(v)$, and $N(v) = 0$ if and only if $v = 0$. If s is a local nonvanishing section of L then the form $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log N(s)^{-2}$ is a well-defined form and its cohomology class is independent of s and the chosen norm: this is, by definition, the Chern class $c(L)$ of L. We want to write down the Chern class of homogeneous line bundle on G/P. Let χ be a holomorphic character of P and L_χ the line bundle $G\times_P \mathbf{C} = G\times \mathbf{C}/\sim$, where $(g,z)\sim(g_1,z_1)$ if and only if $g_1 = gp$, $z_1 = \chi(p^{-1})z$ for some $p \in P$. Fix a Borel subgroup $B \subset P$ and a maximal torus $T \subset B$. Let S be the set of simple roots and $\pi \subset S$ the set of indecomposable positive roots of the Levi complement of P. Let K be the maximal compact subgroup of G as defined in §1. Now $G\times_P\mathbf{C} = KP\times_P\mathbf{C} \cong K\times_{K\cap P}\mathbf{C}$, by an isomorphism, say, Θ. the function $f : K\times_{K\cap P}\mathbf{C} \to \mathbf{R}^{\geq 0}$, $f([k\times z]) = |z|$, is well-defined as $\chi(K_P) \subset S^1$ (K_P being $K\cap P$). Define a function $N : G\times_P\mathbf{C} \to \mathbf{R}^{\geq 0}$ by

$$N([g\times z]) = f(\Theta([g\times z])).$$

Here, $[g\times z]$ denotes the equivalence class of (g,z) under the relation $\sim$. By construction, N is K-invariant and

$$c(L_\chi) = \sum_{\alpha\in S\setminus\pi} c_\alpha[\omega_\alpha].$$

To calculate c_α we follow [Az]. A local cross section of L_χ defined near ξ_0 is $r\cdot\xi_0 \mapsto [r\times 1]$, $r \in \mathrm{R_u}(P)^-$. A computation as in loc.cit and §2 shows that

$$\int_{\mathbf{P}_\alpha} c(L_\chi) = -<\chi,\check{\alpha}>.$$

Hence

$$c(L_\chi) = -\sum_{\alpha\in S\setminus\pi} <\chi,\check{\alpha}> [\omega_\alpha].$$

We have thus proved the Chern class formula for homogeneous line bundles over flag manifolds.

4. Proof of the Main Result

Before proving the main result, let us recall the definition of the Ricci form of a volume form [Sh, p.322]. If V is a volume form on an n-dimensional complex manifold, then in local holomorphic coordinates $V = \phi dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n$. The Ricci form of V is by definition

$$\operatorname{Ric} V = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log|\phi|.$$

By a classical computation of E. Calabi [Ca], for a Kähler form ω, its Ricci form (the (1,1)-form associated to the Ricci curvature tensor) $\operatorname{Ric}(\omega)$ coincides with the Ricci form of the volume form ω^n. The first Chern class of a complex manifold M is the Chern class of the anticanonical bundle $K_{G/P}^{-1}$, hence $\operatorname{Ric}(\omega) \in c_1(M)$. Consider now $M = G/P$. Fix a maximal compact subgroup K of G so that $G = KP$ and $K \cap P = K_P$ is a maximal compact subgroup of P. If ω_1 and ω_2 are closed K-invariant (1,1)-forms on G/P and $[\omega_1] = [\omega_2]$ then $\omega_1 - \omega_2 = \sqrt{-1}\partial\bar{\partial}\phi$, [Be, p.85]. By averaging over K, we may assume that ϕ is K-invariant, hence ϕ is a constant so $\omega_1 = \omega_2$. In particular, if ω is a K-invariant Kähler form, then $\operatorname{Ric}(\omega)$ is also K-invariant and reprents $c_1(G/P)$, so if $\omega \in c_1(G/P)$ we must have $\omega = \operatorname{Ric}(\omega)$ and ω would be Kähler-Einstein. It is classical that $c_1(G/P)$ is positive [Bo-Hi, §14]. For the sake of completeness, and as a modification of the argument gives the main result, we give a proof in the present set-up. Now $c_1(G/P)$ is the Chern class of the line bundle on G/P defined by the character

$$\chi = -\sum_{\alpha \in \mathrm{R_u}(P)} \alpha,$$

where $\mathrm{R_u}(P)$ denotes the unipotent radical of P. Indeed, the anticanonical bundle of $K_{G/P}^{-1}$ is the homogeneous line bundle defined by the determinant of the isotropy representation of P at the tangent space of $T_{\xi_0}(G/P) \cong \operatorname{Lie}(\mathrm{R_u}(P)^-)$. The Levi-complement of P is described by a set π of simple roots and $\alpha \in \mathrm{R_u}(P)$ if and only if α is a positive root not supported by π. By the formula in §3:

$$c_1(G/P) = c(L_\chi) = \sum_{\alpha \in S\setminus\pi} < \rho, \check{\alpha} > [\omega_\alpha],$$

where ρ is the sum of all positive roots not supported by π. Let $\alpha \in S\setminus\pi$ and let w_α be the reflection along α. We have $\rho = \alpha + \sigma_1 + \sigma_2$ where σ_1 is the sum of all positive roots $r \neq \alpha$ such that both r and $w_\alpha(r)$ are not supported by π and σ_2 is the sum of all positive roots $t \neq \alpha$ such that t is not supported by π but $w_\alpha(t)$ is supported by π. Now w_α permutes all roots occuring in σ_1, hence it fixes σ_1; and if $t \neq \alpha$ is a positive with t not supported by π but $w_\alpha(t)$ is supported by π, then $w_\alpha(t) = t- < t, \check{\alpha} > \alpha$ shows that $< t, \check{\alpha} >> 0$. Hence

$$<\rho,\check{\alpha}>=2+<\sigma_1,\check{\alpha}>+<\sigma_2,\check{\alpha}>=2+<\sigma_2,\check{\alpha}>\geq 2$$

so

$$c(L_\chi)=\sum_{\alpha\in S\backslash\pi}<\rho,\check{\alpha}>[\omega_\alpha]$$

with $<\rho,\check{\alpha}>>0$ for $\forall\alpha\in S\backslash\pi$. By the proposition in §1, we see that $c(L_\chi)$ is represented by a K-invariant Kähler form whose pull-back to G is

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\Big(\prod_{\alpha\in S\backslash\pi}\|\rho_\alpha(g)\cdot v_\alpha\|^{<\rho,\check{\alpha}>}\Big). \tag{1}$$

Now fix a parabolic subgroup Q such that $P\subset Q$. The fibers of $G/P\to G/Q$ are K-translates of Q/P. It suffices to verify that $c_1(Q/P)$ restricted tp Q/P is $c_1(Q/P)$, for, as we have just seen, $c_1(G/P)$ is represented by a K-invariant positive form whose restriction to Q/P is positive and invariant under the maximal compact group $K_Q=K\cap Q$, so if $c_1(G/P)$ restricts to $c_1(Q/P)$, then the Kähler-Einstein metric of G/P restricted to Q/P would be Kähler-Einstein. Let π and $\tilde{\pi}$ ($\pi\subset\tilde{\pi}$) be the simple sets of roots of the Levi components of P and Q. Now

$$c_1(G/P)=\sum_{\alpha\in S\backslash\pi}<\rho,\check{\alpha}>[\omega_\alpha]$$

where

$$\rho=\sum_{r>0,\mathrm{supp}(r)\not\subset\pi}r=\sigma+\tau$$

where σ is the sum of positive roots with support outside π but in $\tilde{\pi}$ and τ is the sum of positive roots not supported by $\tilde{\pi}$. Now for $\alpha\in\tilde{\pi}$, $w_\alpha(\tau)=\tau$, so $<\tau,\check{\alpha}>=0$. Hence $<\rho,\check{\alpha}>=<\sigma,\check{\alpha}>$ for all $\alpha\in\tilde{\pi}$. On the other hand, $Q/P=L_Q/L_Q\cap P$, L_Q being the Levi-complement of Q, so

$$c_1(Q/P)=\sum_{\alpha\in\tilde{\pi}\backslash\pi}<\sigma,\check{\alpha}>[\omega_\alpha]|_{Q/P}.$$

Finally, we have

$$c_1(G/P)=\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\|s\|^{-2}, \tag{2}$$

where s is a local non-vanishing section of the line bundle $G\times_P\mathbf{C}$, with P operating on C by the character ρ. Since at $\xi_0=eP$ we can take $s(r\cdot\xi_0)=[r\times 1]$, $r\in\mathrm{R_u}(P)^-$, it follows from (1) and (2) that $\omega_\alpha|_{Q/P}=0$ for $\forall\alpha\in S\backslash\tilde{\pi}$ and therefore

$$\begin{aligned} c_1(G/P)|_{Q/P} &= \sum_{\alpha \in \tilde{\pi} \setminus \pi} < \rho, \check{\alpha} > [\omega_\alpha]|_{Q/P} + \sum_{\alpha \in S \setminus \tilde{\pi}} < \rho, \check{\alpha} > [\omega_\alpha]|_{Q/P} \\ &= \sum_{\alpha \in \tilde{\pi} \setminus \pi} < \sigma, \check{\alpha} > [\omega_\alpha]|_{Q/P} \\ &= c_1(Q/P). \end{aligned}$$

This proved the result completely.

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