Weighted trace cochains; a geometric setup for anomalies

Sylvie PAYCHA

February 14, 2005

Abstract

We extend formulae which measure discrepancies for weighted traces on classical pseudodifferential operators [MN], [CDMP], [PR] to weighted trace cochains, weighted traces corresponding to 0-weighted trace cochains. This extension from 0-cochains to n-cochains is appropriate to handle simultaneously algebraic and geometric discrepancies/anomalies due to the presence of a weight. *Algebraic anomalies* are Hochschild coboundaries of weighted trace cochains on a fixed algebra of pseudodifferential operators weighted by a fixed classical pseudodifferential operator Q with positive order and positive scalar leading symbol. In contrast, *geometric anomalies* arise when considering families of algebras of pseudodifferential operators associated with a smooth fibration of manifolds. They correspond to covariant derivatives (and possibly their curvature) of smooth families of weighted trace cochains, the weight being here an elliptic operator valued form on the base manifold. Both types of discrepancies can be expressed as finite linear combinations of Wodzicki residues. We apply the formulae obtained in the family setting to build Chern-Weil type weighted trace cochains on one hand, and to show that choosing the curvature of a Bismut-Quillen type super connection as a weight, provides covariantly closed weighted trace cochains so that the geometric discrepancies then vanish.

Introduction

Linear forms $A \mapsto \operatorname{tr}\left(Ae^{-\epsilon Q}\right)$ on the algebra $Cl(M,E)$ of classical pseudodifferential operators acting on smooth sections of a hermitian vector bundle E based on a closed Riemannian manifold M where Q is a classical pseudodifferential operator with positive scalar leading symbol and positive order, naturally generalise to multilinear forms, namely JLO cochains [JLO] (see also [P] in relation to anomalies):

$$\begin{aligned}\tilde{\chi}_{n,Q}(\epsilon)(A_0,\cdots,A_n) &= \int_{\Delta_n} du_0\cdots du_n \operatorname{tr}\left(A_0 e^{-\epsilon\cdot u_0 Q} A_1 e^{-\epsilon\cdot u_1 Q}\cdots\right.\\ &\quad \left. A_{n-1}e^{-\epsilon\cdot u_{n-1}Q}A_n e^{-\epsilon\cdot u_n Q}\right),\end{aligned}$$

since $\tilde{\chi}_{0,Q}(\epsilon)(A) = \operatorname{tr}(Ae^{-\epsilon Q})$. Here $\Delta_n := \{(u_0,\cdots,u_n)\in[0,1]^{n+1}, \sum_{i=0}^n u_i = 1\}$.

Recall that [1] the Mellin transform of $t \mapsto \operatorname{tr}\left(Ae^{-tQ}\right)$ defines a meromorphic function $z \mapsto \operatorname{TR}(AQ^{-z})$ (here TR stands for the canonical trace introduced in [KV])

[1] This holds provided Q is invertible, otherwise we turn it into an invertible operator adding the orthogonal projeciton onto its kernel

with simple poles and complex residue given by $\frac{1}{q}\mathrm{res}(A)$ where q is the order of Q and $\mathrm{res}(A)$ the Wodzicki residue of A [Wo]. Similarly, we show that the Mellin transform of $t \mapsto \tilde{\chi}_{n,Q}(t)(A_0, \cdots, A_n)$ yields meromorphic functions $z \mapsto \bar{\chi}_{n,Q}(z)(A_0, \cdots, A_n)$ with simple poles and complex residue given by $\frac{1}{q}\mathrm{res}(A_0 \cdots A_n)$.
On the other hand, the finite part of $z \mapsto \mathrm{TR}(AQ^{-z})$ defines a very useful linear map $A \mapsto \mathrm{tr}^Q(A)$ which we refer to as *Q-weighted trace of A* [P], [CDMP], [MN]. In the same way, the finite part of $z \mapsto \bar{\chi}_{n,Q}(z)(A_0, \cdots, A_n)$ defines *Q-weighted trace cochains* $\chi_n^Q(A_0, \cdots, A_n)$ which yield cyclic cochains that differ from the weighted trace $\mathrm{tr}^Q(A_0 A_1 \cdots A_n)$ of the product of the A_i's (which is not cyclic) by a finite linear combination of Wodzicki residues (Proposition 2):

$$\chi_n^Q(A_0, \cdots, A_n) = \mathrm{tr}^Q(A_0 A_1 \cdots A_n) + \frac{1}{q} \sum_{|k|=1}^{[|a|]+\mathrm{dim} M} \frac{(-1)^{|k|}(|k|-1)!}{(k+1)!} \mathrm{res}\left(A_0 A_1^{(k_1)} \cdots A_n^{(k_n)} Q^{-|k|}\right). \qquad (1)$$

Here $A^{(j)} := \mathrm{ad}_Q^j(A)$, $|a| := a_0 + \cdots + a_n$ is the order of the product $A_0 \cdots A_n$, $[|a|]$ its integer part and for any multiindex $k = (k_1, \cdots, k_n)$, we set $|k| = k_1 + \cdots + k_n$, $(k+1)! = (k_1+1)! \cdots (k_n+1)!$.

The presence of a weight Q leads to discrepancies which are often responsible for the occurence of (local) anomalous phenomena in physics and infinite dimensional geometry [CDMP], [CDP], [PR]. We consider two types of discrepancies, algebraic and geometric ones; the first type arises as Hochschild coboundaries of weighted trace cochains, the second type as covariant derivatives (and possibly the corresponding curvature) of families of weighted trace cochains. Working with families of cochains offers a natural geometric setting that brings together these two types of anomalies in a common framework.
We express the Hochschild coboundary of a weighted trace $2p$-cochain as a finite linear combination of a finite number of Wodzicki residues involving powers of the weight Q (Theorem 1):

$$b\,\chi_{2p}^Q(A_0, \cdots, A_{2p+1}) = \frac{1}{q} \sum_{|k|=0}^{[|a|]+\dim M-1} \frac{(-1)^{|k|}\,|k|!}{(k+1)!} \sum_{j=0}^{p} \mathrm{res}\left(A_0\, A_1^{(k_1)} \cdots \right.$$
$$\left. \cdots A_{2j}^{(k_{2j})}\, A_{2j+1}^{(k_{2j+1}+1)}\, A_{2j+2}^{(k_{2j+2})} \cdots A_{2p+1}^{(k_{2p+1})} Q^{-|k|-1}\right). \qquad (2)$$

When $p = 0$, this yields

$$\left(b\,\mathrm{tr}^Q\right)(A, B) = \frac{1}{q} \sum_{|k|=0}^{[a+b]+\dim M-1} \frac{(-1)^k}{k+1} \mathrm{res}\left(AB^{(k+1)}\, Q^{-k-1}\right),$$

where a is the order of A, b that of B. Equation (2), which is very close in spirit to formulae derived in [H], [CM] (see Appendix A for some analogies [2]to compute character cocycles, shows the expected locality of the algebraic anomaly since the Wodzicki residue has an explicit local description in terms of the (positively) homogeneous symbol $\sigma_{-\mathrm{dim} M}$ of order $-\dim M$ of the operator:

$$\mathrm{res}(A) = \frac{1}{(2\pi)^{\dim M}} \int_{S^*M} dx\, d_S\xi\, \mathrm{tr}_x\left(\sigma_{-\dim \mathrm{M}}(A)(x,\xi)\right)$$

[2]The essential difference lies in the fact that the weighted trace forms are cyclic and hence generally not (b, B) closed, in contrast to the Chern character

where S^*M denotes the unit cotangent sphere and $d_S\xi$ the canonical volume measure on it. It also follows from this formula that Q-weighted trace $2p$- cochains yield $2p$-cocycles on the algebra $Cl_{\leq -\frac{\mathrm{dimM}}{(2p+2)}}(M,E)$ which (strictly) includes the algebra $Cl_{< -\frac{\mathrm{dimM}}{(2p+2)}}(M,E)$ of classical pseudodifferential operators that lie in the Schatten class $\mathcal{I}_{2p+2}\left(L^2(M,E)\right)$. Here $L^2(M,E)$ is the L^2-completion of $C^\infty(M,E)$ w.r.t. to the hermitian metric on E and the Riemannian metric on M.
Weighted trace cochains vary with the weight; if $\mathbb{Q}: x \to Q_x \in Cl(M,E)$ is a smooth family of weights parametrized by a smooth manifold X then (Theorem 2):

$$\begin{aligned}
&\left(d\,\chi_n^{\mathbb{Q}}\right)(A_0,\cdots,A_n)\\
=\;& \frac{1}{q}\cdot\sum_{|k|=0}^{[|a|]+\dim M}\frac{(-1)^{|k|+1}k!}{(k+1)!}\sum_{j=1}^{n+1}\mathrm{res}\Big(A_0\,A_1^{(k_1)}\cdots\\
&\cdot A_{j-1}^{(k_{j-1})}\,(d\,\mathbb{Q})^{(k_j)}\,A_j^{(k_{j+1})}\cdots A_n^{(k_{n+1})}\mathbb{Q}^{-|k|-1}\Big) \qquad (3)
\end{aligned}$$

For $n=0$, this gives:

$$\left(d\,\mathrm{tr}^{\mathbb{Q}}\right)(A)=\frac{1}{q}\cdot\sum_{k=0}^{[a]+\dim M}\frac{(-1)^{k+1}}{k+1}\mathrm{res}\Big(A\;(d\,\mathbb{Q})^{(k)}\,\mathbb{Q}^{-k-1}\Big).$$

In the family setup, to a fibration $\pi:\mathbb{M}\to X$ of closed Riemannian manifolds $\{M_x,\,x\in X\}$ modelled on M and based on a smooth manifold X together with a hermitian vector bundle $\mathbb{E}\to\mathbb{M}$, we can associate a smooth fibration of algebras $Cl(\mathbb{M},\,\mathbb{E})$ modelled on $Cl(M,E)$ with fibre over $x\in X$ given by $Cl(M_x,E_{|_{M_x}})$. Given a smooth family of weights $\mathbb{Q}\in\Omega^{even}(X,Cl(\mathbb{M},\,\mathbb{E}))$, we define corresponding smooth families of $\mathbb{Q}$-weighted trace cochains $(\alpha_0,\cdots,\alpha_n)\mapsto\chi_{2p}^{\mathbb{Q}}(\alpha_0,\cdots,\alpha_n)$ on $\Omega\left(X,Cl(\mathbb{M},\,\mathbb{E})\right)$. When the fibration $Cl(\mathbb{M},\,\mathbb{E})$ is equipped with a connection ∇ such that locally, $\nabla\alpha=d\alpha+ad_\theta$ where θ is a local $Cl(M,E)$-valued on form, we express the covariant derivative of a weighted trace $2p$-cochain as a linear combination of a finite number of Wodzicki residues involving powers of the weight $\mathbb{Q}$ (Theorem 3) [3]:

$$\begin{aligned}
&\left(\nabla\,\chi_n^{\mathbb{Q}}\right)(\alpha_0,\cdots,\alpha_n)\\
=\;& \frac{1}{q}\cdot\sum_{|k|=0}^{[|a|]+\dim M}\frac{k!}{(k+1)!}\sum_{j=1}^{n+1}(-1)^{|\alpha_0|+\cdots+|\alpha_{j-1}|+|k|+1}\mathrm{res}\Big(\alpha_0\wedge\alpha_1^{(k_1)}\wedge\cdots\\
&\wedge\alpha_{j-1}^{(k_{j-1})}\wedge\left(\nabla^{End}\mathbb{Q}\right)^{(k_j)}\wedge\alpha_j^{(k_{j+1})}\wedge\cdots\wedge\alpha_n^{(k_{n+1})}\wedge\mathbb{Q}^{-|k|-1}\Big). \qquad (4)
\end{aligned}$$

For $n=0$ this yields

$$\left(\nabla\,\mathrm{tr}^{Q}\right)(\alpha)=\frac{1}{q}\cdot\sum_{k=0}^{[a]+\dim M}\frac{(-1)^{|\alpha|+k+1}}{k+1}\mathrm{res}\Big(\alpha\wedge\left(\nabla^{End}\mathbb{Q}\right)^{(k)}\wedge\mathbb{Q}^{-k-1}\Big).$$

Equation (4) shows that geometric discrepancies are also local in as far as they can be written in terms of a finite number of Wodzicki residues, thus generalizing

[3] Here again, we are assuming that $\mathbb{Q}$ is invertible, otherwise, provided its kernel defines a vector bundle, we can turn it into an invertible operator adding the orthogonal projection onto its kernel

observations already made previously [PR] on the locality of obstructions of the type $\nabla \mathrm{tr}^{\mathbb{Q}} = \nabla \chi_0^{\mathbb{Q}}$.
Chern-Weil type $\mathbb{Q}$-weighted trace cochains $\chi_n^{\mathbb{Q}}(f_0(\Omega), \cdots, f_n(\Omega))$, where the f_i's are polynomial functions and $\Omega = \nabla^2$ the curvature of ∇ generalize the $\mathbb{Q}$-weighted Chern forms $\mathrm{tr}^{\mathbb{Q}}(f(\Omega)) = \chi_0^{\mathbb{Q}}(f(\Omega))$ discussed in [PR] which occur in [F] in a disguised form, see [CDMP]. The above formula measures the obstruction to their closedness:

$$\begin{aligned} & d\,\chi_n^{\mathbb{Q}}(f_0(\Omega), \cdots, f_n(\Omega)) \\ = \;& \frac{1}{q} \cdot \sum_{|k|=0}^{[|d|\cdot\omega]+n} \frac{k!}{(k+1)!} \sum_{j=1}^{n+1} (-1)^{|k|+1} \mathrm{res} \Big(f_0(\Omega) \wedge (f_1(\Omega))^{(k_1)} \wedge \cdots \wedge (f_{j-1}(\Omega))^{(k_{j-1})} \\ & \wedge \left(\nabla^{End}\mathbb{Q}\right)^{(k_j)} \wedge (f_j(\Omega))^{(k_{j+1})} \wedge \cdots \wedge (f_n(\Omega))^{(k_{n+1})}\, \mathbb{Q}^{-|k|-1} \Big) \end{aligned} \tag{5}$$

where $|d| = \sum_{i=0}^n d_i$ with d_i the degree of the polynomial f_i, and ω is the order of the operator valued 2-form Ω. As expected, these obstruction vanish in the context of families of Dirac operators, replacing ∇ by a Quillen-Bismut type superconnection $\mathbb{A} = \nabla + D$ ([Q], [B], see also [BGV]) and setting $\mathbb{Q} = \mathbb{A}^2$.This leads to characteristic classes built along a line suggested by Scott's work [Sc] and further developped in [PS].

Acknowledgements

I would like to thank Matilde Marcolli for interesting discussions which encouraged me to write this down. I am also grateful to the Max Planck Institute in Bonn and Matilde Marcolli for inviting me for a two months stay during which this article was written.

1 JLO cochains on weighted trace algebras

Let $(\mathcal{A}, \cdot)$ be an associative algebra over $\mathbb{C}$ and let $C^n(\mathcal{A})$ denote the space of continuous $\mathbb{C}$-multilinear valued forms on $\mathcal{A}^{\otimes n+1}$, which corresponds to the space of n-cochains on $\mathcal{A}$. The Hochschild coboundary of an n-cochain χ_n is an $n+1$-cochain defined by:

$$\begin{aligned} b\chi_n(A_0, \cdots, A_{n+1}) &= \sum_{j=0}^{n} (-1)^j \chi_n(A_0, \cdots, A_j \cdot A_{j+1}, \cdots, A_{n+1}) \\ &+ (-1)^{n+1} \chi_n(A_{n+1} \cdot A_0, \cdots, A_n). \end{aligned}$$

Since $b^2 = 0$, this defines a cohomology, called the Hochschild cohomology of $\mathcal{A}$. In the following, we shall drop the explicit $\cdot$ in the product notation writing simply AB for $A \cdot B$.

Definition 1 *A weighted tracial algebra is a triple $(\mathcal{A}, Q, T)$ where $\mathcal{A}$ is a topological unital associative algebra, T is a continuous linear map on a non trivial ideal $\mathcal{I}$ of $\mathcal{A}$ and Q an element of $\mathcal{A}$ such that*

- *T is a trace on $\mathcal{I}$,*

- *the equation* $\frac{d}{dt}U_t = Q\,U_t$ *with initial condition* $U_0 = 1_{\mathcal{A}}$ *has a unique solution* $U_t = e^{-tQ} \in \mathcal{A}$, $t \geq 0$,
- $e^{-tQ} \in \mathcal{I}$ *for any* $0 < t$.

Q is called the weight.
Given a weighted tracial algebra $(\mathcal{A}, Q, T)$, we define a JLO type cochain [JLO] (see also [G-BVF]):

$$\chi_{n,Q}(A_0, \cdots, A_n) := \int_{\Delta_n} du_0 \cdots du_n\, T\left(A_0 e^{-u_0 Q} A_1 e^{-u_1 Q} \cdots A_{n-1} e^{-u_{n-1}Q} A_n e^{-u_n Q}\right).$$

Lemma 1 *Setting* $\mathrm{ad}_A B := [A, B]$ *we have:*

$$\mathrm{ad}_A e^{-uQ} = -u \int_0^1 dt\, e^{-u(1-t)Q} \left(\mathrm{ad}_A Q\right) e^{-utQ} = u \int_0^1 dt\, e^{-u(1-t)Q} \left(\mathrm{ad}_Q A\right) e^{-utQ}.$$

Proof: (see e.g. [G-BVF]) Differentiating the map $u \mapsto [e^{-uQ}, A]$, we get

$$\left(\frac{d}{du} + Q\right)[e^{-uQ}, A] = [A, Q]e^{-uQ}.$$

Solving this equation by the usual Duhamel formula for first order inhomogeneous linear differential equations gives $[e^{-uQ}, A] = \int_0^u e^{-sQ}[Q, A]e^{-(u-s)Q} ds$. Substituting $s = ut$ yields the above formula.

As a consequence, we have

Proposition 1 *If $n = 2k$ is even then*

$$\begin{aligned} & b\,\chi_{2k,Q}(A_0, \cdots, A_{2k+1}) \\ = & \sum_{j=0}^{k} \chi_{2k+1,Q}\left(A_0, A_1, \cdots, A_{2j}, \mathrm{ad}_Q A_{2j+1}, A_{2j+2}, \cdots, A_{2k+1}\right) \end{aligned}$$

If $n = 2k + 1$ is odd,

$$\begin{aligned} & b\,\chi_{2k+1,Q}(A_0, \cdots, A_{2k+2}) \\ = & \sum_{j=0}^{k} \chi_{2k+2,Q}\left(A_0, A_1, \cdots, \cdots, A_{2j}, \mathrm{ad}_Q A_{2j+1}, A_{2j+2}, \cdots, A_{2k+2}\right) \\ - & \chi_{2k+1,Q}\left(A_{2k+2} \cdot A_0, A_1, \cdots, A_{2k+1}\right). \end{aligned}$$

Proof: We carry out the proof in the even case. The odd case can be derived similarly. First observe that given weights $Q_0, \cdots, Q_n \in \mathcal{A}$, we can define the n-cochain $\chi_{n,Q_0,\cdots,Q_n}$ by:

$$\chi_{n,Q_0,\cdots,Q_n}(A_0, \cdots, A_n) := T\left(A_0 e^{-Q_0} A_1 e^{-Q_1} \cdots A_{n-1} e^{-Q_{n-1}} A_n e^{-Q_n}\right).$$

A straightforward computation shows

$$\begin{aligned} & \chi_{n,Q_0,\cdots,Q_n}(A_0, \cdots, A_{j-1}A_j, \cdots, A_n) - \chi_{n,Q_0,\cdots,Q_n}(A_0, \cdots, A_j A_{j+1}, \cdots, A_n) \\ = \; & T(A_0 e^{-Q_0} \cdots A_{j-1}[A_j, e^{-Q_{j-1}}]\, e^{-Q_j} A_{j+1} \cdots A_n\, e^{-Q_n}). \end{aligned}$$

Setting $Q_j = u_j\, Q$ it follows that

$$\begin{aligned}
& b\,\chi_{2k,Q}(A_0, \cdots, A_{2k+1}) \\
= & -\sum_{j=0}^{k} \int_{\Delta_{2k}} du_0 \cdots du_{2k}\, u_{2j}\, T\left(A_0 e^{-u_0 Q} A_1 e^{-u_1 Q} \cdots A_{2j} \cdot\right. \\
\cdot & \left. \left(\int_0^1 dt\, e^{-u_{2j}(1-t)Q}[A_{2j+1}, Q] e^{-u_{2j} t Q}\right) \cdots A_{2k+1} e^{-u_{2k} Q}\right) \\
= & -\sum_{j=0}^{k} \int_{\Delta_{2k}} du_0 \cdots du_{2k} \int_0^{u_{2j}} du\, T\left(A_0 e^{-u_0 Q} A_1 e^{-u_1 Q} \cdots A_{2j} e^{(-u_{2j}+u)Q}[A_{2j+1}, Q]\right. \\
\cdot & \left. e^{-uQ} \cdots A_{2k+1} e^{-u_{2k} Q}\right) \\
= & -\sum_{j=0}^{k} \int_{\Delta_{2k+1}} du_0 \cdots du_{2k+1} T\left(A_0 e^{-u_0 Q} \cdots A_{2j} e^{-u_{2j} Q}[A_{2j+1}, Q]\right. \\
\cdot & \left. e^{-u_{2j+1} Q} A_{2j+2} \cdots A_{2k+1} e^{-u_{2k+1} Q}\right) \\
= & \sum_{j=0}^{k} \chi_{2k+1,Q}\left(A_0, A_1, \cdots, [Q, A_{2j+1}], \cdots, A_{2k+1}\right).
\end{aligned}$$

2 Weighted trace cochains on pseudodifferential operators

Let $\mathcal{A} = Cl(M, E)$ be the algebra of classical pseudodifferential operators (PDOs) acting on smooth sections of a vector bundle E based on a closed Riemannian manifold M and let $\mathcal{I}$ the ideal of smoothing operators equipped with the ordinary trace $T = \mathrm{tr}$. Let $Q \in Cl(M, E)$ be an operator with positive scalar leading symbol and positive order q. Since its leading symbol is invertible, Q is elliptic. Then $(\mathcal{A}, Q, \mathrm{tr})$ is a weighted tracial algebra. We call the couple $(Cl(M, E), Q)$ a *weighted PDO algebra*. Replacing Q by $Q_\epsilon := \epsilon\, Q$ for some fixed $0 < \epsilon < 1$ in the JLO type cochains associated to the weighted trace algebra $(\mathcal{A}, Q, \mathrm{tr})$, we define:

$$\begin{aligned}
\tilde{\chi}_{n,Q}(\epsilon)(A_0, \cdots, A_n) \quad = \quad & \int_{\Delta_n} du_0 \cdots du_n\, \mathrm{tr}\left(A_0 e^{-\epsilon\, u_0 Q} A_1\, e^{-\epsilon\, u_1 Q} \cdots\right. \\
& \left. \cdots A_{n-1} e^{-\epsilon\, u_{n-1} Q} A_n e^{-\epsilon\, u_n Q}\right).
\end{aligned}$$

The following lemma is an immediate consequence of Proposition 1 applied to $Q_t = tQ$:

Lemma 2 *Given a weighted pseudodifferential algebra $(Cl(M, E), Q)$, for any pseudodifferential operators $A_0, \cdots, A_{2p+1}$ in $Cl(M, E)$*

$$\begin{aligned}
& b\,\tilde{\chi}_{2p,Q}(t)(A_0, \cdots, A_{2p+1}) \\
= \quad & t \sum_{j=0}^{p} \chi_{2p+1,Q}(t)\left(A_0, A_1, \cdots, A_{2j}, \mathrm{ad}_Q A_{2j+1}, A_{2j+2}, \cdots, A_{2p+1}\right).
\end{aligned}$$

Definition 2 *Whenever Q is invertible, the Mellin transform of $\tilde{\chi}_{n,Q}$ is given by:*

$$\bar{\chi}_{2k,Q}(z)(A_0, \cdots, A_n)$$

$$
\begin{aligned}
&:= \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} \tilde{\chi}_{2k,Q}(t)(A_0, \cdots, A_n)\, dt \\
&= \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} dt \int_{\Delta_n} du_0 \cdots du_n \operatorname{tr}\left(A_0 e^{-t\cdot u_0 Q} A_1 e^{-t\cdot u_1 Q} \cdots \right. \\
&\quad \left. A_{n-1} e^{-t\cdot u_{n-1} Q} A_n e^{-t\cdot u_n Q}\right).
\end{aligned}
$$

Remark 1 *When Q is not invertible, since it has a finite dimensional kernel being an elliptic operator on a closed manifold, assuming that E comes equipped with a hermitian structure, we can take instead:*

$$
\tilde{Q} := Q + \pi_{\mathrm{KerQ}}
$$

where π_{KerQ} is the orthogonal projection onto the kernel of Q. In the following, we therefore assume Q is invertible unless otherwise specified.

To state the next result, we need some notations.

Definition 3 *For $j \in \mathbb{N}$ and A in the weighted PDO algebra $(Cl(M,E), Q)$, we set as in the introduction:*

$$
A_Q^{(0)} := \mathrm{ad}_Q^j(A), \quad \text{where} \quad \mathrm{ad}_Q(B) = [Q, B],
$$

so that $A_Q^{(0)} = I, \quad A_Q^{(j+1)} = \mathrm{ad}_Q(A^{(j)}) = [Q, A^{(j)}]$.

We shall often drop the subscript Q, writing $A^{(j)}$ instead of $A_Q^{(j)}$. It is useful to notice that since Q has scalar leading symbol then $A^{(j)}$ has order $a + j(q-1)$ if a is the order of A where a the order of a.

Proposition 2 *Given any $A_0, \cdots, A_n \in Cl(M,E)$, the map $z \mapsto \bar{\chi}_{n,Q}(z)(A_0, \cdots, A_n)$ is meromorphic with simple poles. Its complex residue at $z = 0$ is given by:*

$$
\mathrm{Res}_{z=0} \bar{\chi}_{n,Q}(z)(A_0, \cdots, A_n) = \frac{1}{q} \operatorname{res}(A_0 A_1 \cdots A_n).
$$

Its finite part at $z = 0$ called the **Q-weighted n-trace cochain** *is given by:*

$$
\chi_0^Q(A_0) := \mathrm{tr}^Q(A_0) := \mathrm{fp}_{z\to 0} \bar{\chi}_{0,Q}(z)(A_0)
$$

when $n = 0$ and for any $n \in \mathbb{N}$ by

$$
\begin{aligned}
\chi_n^Q(A_0, \cdots, A_n) &:= \mathrm{fp}_{z=0} \bar{\chi}_{n,Q}(z)(A_0, \cdots, A_n) \\
&= \mathrm{tr}^Q(A_0 A_1 \cdots A_n) \\
&+ \frac{1}{q} \sum_{|k|=1}^{[|a|]+\dim M} \frac{(-1)^{|k|}(|k|-1)!}{(k+1)!} \operatorname{res}\left(A_0 A_1^{(k_1)} \cdots A_n^{(k_n)} Q^{-|k|}\right)
\end{aligned}
$$

where $q > 0$ is the order of Q, $|a| := a_0 + \cdots + a_n$ the order of the product $A_0 \cdots A_n$, $[|a|]$ its integer part and where for any multiindex $k = (k_1, \cdots, k_n)$ we have set $|k| = k_1 + \cdots + k_n$, $(k+1)! = (k_1+1)! \cdots (k_n+1)!$.

Remark 2 *This proposition shows that* $\chi_n^Q(A_0,\cdots,A_n)$ *and* $\mathrm{tr}^Q\left(A_0A_1\cdots A_n\right)$*, which are two different ways of defining a "regularized trace" of the product* $A_0\cdots A_n$*, differ by a finite linear combination of Wodzicki residues. The first part of the proposition says that on the level of residues, it does not make any difference what regularization one considers.*

Proof: First observe that (see Appendix B) for each $J\in\mathbb{N}$, there exist positive integers $N_1,N_2,\cdots,N_n$ such that

$$\begin{aligned}&\tilde{\chi}_{n,Q}(t)\left(A_0,A_1,\cdots,A_n\right)\\ =\;&\sum_{k_1=0}^{N_1-1}\cdots\sum_{k_n=0}^{N_n-1}\frac{(-1)^{|k|}t^{|k|}}{(k+1)!}\mathrm{tr}\left(A_0\,A_1^{(k_1)}\cdots A_n^{(k_n)}e^{-tQ}\right)+o(t^J).\end{aligned}\tag{6}$$

Hence, for each $\mathrm{Re}(z)\le A$, we can choose the N_i's large enough so that the rest term vanishes in the following computations:

$$\begin{aligned}&\bar{\chi}_{n,Q}(z)(A_0,\cdots,A_n)\\ :=\;&\frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}\tilde{\chi}_{n,Q}(t)(A_0,\cdots,A_{2k+1})dt\\ =\;&\frac{1}{\Gamma(z)}\sum_{k_1=0}^{N_1-1}\cdots\sum_{k_n=0}^{N_n-1}\frac{(-1)^{|k|}\,t^{|k|}}{(k+1)!}\int_0^\infty dt\,t^{z-1}\,\mathrm{tr}\left(A_0\,A_1^{(k_1)}\cdots A_n^{(k_n)}e^{-tQ}\right)\\ =\;&\frac{1}{\Gamma(z)}\sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty}\frac{(-1)^{|k|}}{(k+1)!}\int_0^\infty dt\,t^{z+|k|-1}\,\mathrm{tr}\left(A_0\,A_1^{(k_1)}\cdots A_n^{(k_n)}e^{-tQ}\right)\\ =\;&\sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty}\frac{(-1)^{|k|}}{(k+1)!}\frac{\Gamma(z+|k|)}{\Gamma(z)}\mathrm{TR}\left(A_0\,A_1^{(k_1)}\cdots A_n^{(k_n)}Q^{-|k|-z}\right),\end{aligned}\tag{7}$$

where TR stands for the canonical trace on all non integer order pseudodifferential operators. If $|k|=0$ then $\frac{\Gamma(z+|k|)}{\Gamma(z)}=1$ and this sum reduces to

$$\mathrm{TR}\left(A_0\,A_1^{(k_1)}\cdots A_n^{(k_n)}Q^{-z}\right)$$

which is meromorphic with simple pole at 0 given by

$$\mathrm{Res}_{z=0}\mathrm{TR}\left(A_0\,A_1\cdots A_nQ^{-z}\right)=\frac{1}{q}\mathrm{res}\left(A_0\,A_1\cdots A_n\right).$$

If $|k|\neq 0$ then $\frac{\Gamma(z+|k|)}{\Gamma(z)}\sim_0(|k|-1)!z$.
Since the the map $z\mapsto\mathrm{TR}\left(A_0\,A_1^{(k_1)}\cdots A_n^{(k_n)}Q^{-|k|-z}\right)$ is also meromorphic with a simple pole at zero, it follows that when $|k|\neq 0$, the expression
$\frac{\Gamma(z+|k|)}{\Gamma(z)}\mathrm{TR}\left(A_0\,A_1^{(k_1)}\cdots A_n^{(k_n)}Q^{-|k|-z}\right)$ converges when z tends to zero to $(|k|-1)!$ times the complex residue at zero of $\mathrm{TR}\left(A_0\,A_1\cdots A_n\,Q^{|k|-z}\right)$ given by:

$$\mathrm{Res}_{z=0}\mathrm{TR}\left(A_0\,A_1^{(k_1)}\cdots A_n^{(k_n)}Q^{-|k|-z}\right)=\frac{1}{q}\mathrm{res}\left(A_0\,A_1^{(k_1)}\cdots A_n^{(k_n)}Q^{-|k|}\right),$$

and finite part $\mathrm{tr}^Q(A_0\cdots A_n)$.
Let us now check that the infinite sums are in fact finite. Since Q has scalar leading

symbol, the operators $A_0 A_1^{(k_1)} \cdots A_n^{(k_n)} Q^{-|k|}$ have order $\sum_{i=1}^n a_i + \sum_{i=1}^n k_i(q-1) - |k|q = \sum_{i=1}^n a_i - \sum_{i=1}^n k_i = |a| - |k|$ which decreases as the indices k_i increase. Since the Wodzicki residue vanishes on operators of order smaller than $-\mathrm{dim}M$, only the terms in the the sum on the r.h.s such that $|k| \leq |a| + \mathrm{dim}M$ survive so that the sum is indeed finite and stops at $[|a|] + \mathrm{dim}M$. This ends the proof of the proposition.

Theorem 1 *Given any $A_0, \cdots, A_{2p+1} \in Cl(M,E)$, the coboundary*

$$z \mapsto b\, \bar{\chi}_{2p,Q}(z)(A_0, \cdots, A_{2p+1})$$

is holomorphic at zero. Its value at $z=0$ is given by:

$$\begin{aligned} b\, \chi_{2p}^Q(A_0, \cdots, A_{2p+1}) &= \frac{1}{q} \sum_{|k|=0}^{[|a|]+\dim M-1} (-1)^{|k|} \frac{|k|!}{(k+1)!} \sum_{j=0}^{p} \mathrm{res} \Big(A_0\, A_1^{(k_1)} \cdots \\ &\quad \cdots A_{2j}^{(k_{2j})}\, A_{2j+1}^{(k_{2j+1}+1)}\, A_{2j+2}^{(k_{2j+2})} \cdots A_{2p+1}^{(k_{2p+1})} Q^{-|k|-1} \Big) \end{aligned}$$

where $|k| := k_1 + \cdots + k_{2p+1}$, $(k+1)! = (k_0+1)! \cdots (k_n+1)!$ and $[|a|]$ is the integer part of $|a| := a_0 + \cdots + a_{2p+1}$ which corresponds to the order of the product $A_0 \cdots A_{2p+1}$.

Proof: By Lemma 2 we have

$$\begin{aligned} & b\, \tilde{\chi}_{2p,Q}(t)(A_0, \cdots, A_{2p+1}) \\ = \; & t \sum_{j=0}^{p} \chi_{2p+1,Q}(t)\, (A_0, A_1, \cdots, A_{2j}, [Q, A_{2j+1}], A_{2j+2}, \cdots, A_{2p+1})\,. \end{aligned}$$

Using the results of Appendix B this yields:

$$\begin{aligned} & b\, \bar{\chi}_{2k,Q}(z)(A_0, \cdots, A_{2p+1}) \\ = \; & \frac{1}{\Gamma(z)} \int_0^\infty dt\, t^{z-1}\, b\, \tilde{\chi}_{2k,Q}(t)(A_0, \cdots, A_{2p+1}) dt \\ = \; & \frac{1}{\Gamma(z)} \sum_{j=0}^{p} \sum_{k_1=0}^{\infty} \cdots \sum_{k_{2p+1}=0}^{\infty} \frac{(-1)^{|k|}}{(k+1)!} \int_0^\infty dt\, t^{z+|k|}\, \mathrm{tr} \Big(A_0\, A_1^{(k_1)} \cdots [Q, A_{2j+1}]^{(k_{2j+1})} \\ & A_{2j+2}^{(k_{2j+2})} \cdots A_{2p+1}^{(k_{2p+1})} e^{-tQ} \Big) \\ = \; & \sum_{j=0}^{p} \sum_{k_1=0}^{\infty} \cdots \sum_{k_{2p+1}=0}^{\infty} \frac{(-1)^{|k|+1}}{(k+1)!} \frac{\Gamma(z+|k|+1)}{\Gamma(z)} \mathrm{TR} \Big(A_0\, A_1^{(k_1)} \cdots A_{2j+1}^{(k_{2j+1}+1)} \\ & A_{2j+2}^{(k_{2j+2})} \cdots A_{2p+1}^{(k_{2p+1})} Q^{-(z+|k|+1)} \Big)\,. \end{aligned}$$

Since $\frac{\Gamma(z+|k|+1)}{\Gamma(z)} = |k|! \cdot z$ for any multiindex k (including the case $|k|=0$) and since $\mathrm{TR} \Big(A_0\, A_1^{(k_1)} \cdots A_{2j}^{(k_{2j})}\, A_{2j+1}^{(k_{2j+1}+1)} \cdots A_{2p+1}^{(k_{2p+1})} Q^{-(z+|k|+1)} \Big)$ has a simple pole at zero given by $\frac{1}{q} \mathrm{res} \Big(A_0\, A_1^{(k_1)} \cdots A_{2j+1}^{(k_{2j+1}+1)} \cdots A_{2p+1}^{(k_{2p+1})} Q^{-(|k|+1)} \Big)$, it follows that $b\, \bar{\chi}_{2k,Q}(z)(A_0, \cdots, A_{2p+1})$ converges to

$$\begin{aligned} \lim_{z\to 0} b\, \bar{\chi}_{2k,Q}(z)(A_0, \cdots, A_{2p+1}) &= \frac{1}{q} \sum_{k=0}^{\infty} (-1)^{|k|} \frac{k!}{(k+1)!} \sum_{j=0}^{p} \mathrm{res} \Big(A_0\, A_1^{(k_1)} \cdots A_{2j}^{(k_{2j})} \\ &\quad \cdot A_{2j+1}^{(k_{2j+1}+1)}\, A_{2j+2}^{(k_{2j+2})} \cdots A_{2p+1}^{(k_{2p+1})} Q^{-|k|-1} \Big)\,. \end{aligned}$$

To finish, let us check that the sum on the r.h.s is finite. Because Q was chosen with scalar leading symbol, only the terms such that $\sum_{i=0}^{2p+1} a_i + q - 1 + \sum_{i=0}^{2p+1} k_i(q-1) - q|k| - q = |a| - |k| - 1 \geq -\mathrm{dim}M$ survive i.e. such that $|k| \leq |a| + \mathrm{dim}M - 1$, which ends the proof.

Remark 3 *For $p = 0$, the above proposition yields:*

$$\begin{aligned} b\,\mathrm{tr}^Q(A,B) &= b\,\chi_0^Q(A,B) \\ &= \frac{1}{q}\sum_{k=0}^{[a+b]+\mathrm{dim}M-1} \frac{(-1)^k}{k+1}\ \mathrm{res}\left(AB^{(k+1)}\,Q^{-k-1}\right) \end{aligned}$$

where a is the order of A, b the order of B. This formula which seems a priori more complicated than the more compact formula [MN], [CDMP]:

$$b\,\mathrm{tr}^Q(A,B) = -\frac{1}{q}\mathrm{res}(A\,[\log Q, B])$$

bares over the latter formula the advantage that it does not require computing the symbol of the logarithm of Q.

Corollary 1 *Given any $A_0, \cdots, A_{2p+1} \in Cl(M,E)$ with orders $a_0, \cdots, a_{2p+1}$ such that $[|a|] \leq -\mathrm{dim}M$ then*

$$b\,\chi_{2p}^Q(A_0, \cdots, A_{2p+1}) = 0.$$

In particular Q-weighted trace $2p$-cochains yield Hochschild cocycles on the subalgebra $Cl_{\leq -\frac{\mathrm{dim}M}{2p+2}}(M,E)$.

Proof: This follows from the fact that the residue terms in Theorem 1 then vanish because a Wodzicki residue vanishes on pseudodifferential operators of order smaller than $-\dim M$.

Remark 4 *Notice that, with the notations of the introduction, the inclusion $Cl(M,E) \cap \mathcal{I}_{2p+2}(L^2(M,E)) \subset Cl_{\leq -\frac{\mathrm{dim}M}{2p+2}}(M,E)$ is strict.*

3 Varying the weight

Let us now consider a fixed associative unital topological algebra $\mathcal{A}$ together with an ideal $\mathcal{I}$ equipped with a trace T and a smooth family of weights $\mathbb{Q} : x \mapsto Q_x \in \mathcal{A}$ so that for any $x \in X$, the triple $(\mathcal{A}, Q_x, T)$ is a weighted tracial algebra. We set for $\chi_n \in C^n(\mathcal{A})$:

$$\begin{aligned} (d\,\chi_n)\,(A_0, \cdots, A_n) &:= d\,(\chi_n(A_0, \cdots, A_n)) \\ &- \sum_{j=0}^{n} \chi_n(A_0, \cdots, d\,A_j, A_{j+1}, \cdots, A_n) \end{aligned}$$

where $A_0, \cdots, A_n \in C^\infty(X, Cl(M,E))$. This defines a $C^n(\mathcal{A})$-valued 1-form on X.

Lemma 3

$$d\,e^{-t\,\mathbb{Q}} = -t\int_0^1 du e^{-t(1-u)\,\mathbb{Q}} \wedge d\,\mathbb{Q} \wedge e^{-ut\,\mathbb{Q}}.$$

Proof: Take $V_u = e^{-uQ}\, e^{-(1-u)(\mathbb{Q}+d\,\mathbb{Q})}$. Then

$$\frac{d}{du} V_u = -\mathbb{Q}\, e^{-u\,\mathbb{Q}}\, e^{-(1-u)(\mathbb{Q}+d\,\mathbb{Q})} + e^{-u\,\mathbb{Q}}\, (\mathbb{Q} + d\,\mathbb{Q})\, e^{-(1-u)(\mathbb{Q}+d\,\mathbb{Q})}.$$

Integrating this from 0 to 1 yields

$$de^{-\,\mathbb{Q}} := e^{-\,\mathbb{Q}} - e^{-(\,\mathbb{Q}+d\,\mathbb{Q})} = \int_0^1 du\, e^{-u\,\mathbb{Q}} d\mathbb{Q}\, e^{-(1-u)(\,\mathbb{Q}+d\,\mathbb{Q})}.$$

Replacing $\mathbb{Q}$ by $t\,\mathbb{Q}$ and making the adequate change of variable then yields the result.

Proposition 3 *For any* $A_0, \cdots, A_n \in \Omega(X, \mathcal{A})$,

$$(d\,\tilde{\chi}_{n,\,\mathbb{Q}})\,(t)(A_0, \cdots, A_n) = -t \sum_{j=1}^{n+1} \tilde{\chi}_{n+1,\,\mathbb{Q}}\,(A_0, A_1, \cdots, A_{j-1}, d\,\mathbb{Q}, A_j, \cdots, A_n)\,,$$

where $d\,\mathbb{Q}$ *stands at the* j*-th position.*

Proof: Since any power $e^{-u\,\mathbb{Q}}, 0 < u < 1$ lies in $\mathcal{I}$, so does the product $A_0\, e^{-u_0\,\mathbb{Q}}\, A_1\, e^{-u_1\,\mathbb{Q}}\, \cdots\, A_n\, e^{-u_n\,\mathbb{Q}}$ lie in $\mathcal{I}$.
We can push d through the trace T in the subsequent computation and use Lemma 3 to express $d\,e^{-u\,\mathbb{Q}}$:

$$\begin{aligned}
& d\, \left(T\left(A_0\, e^{-u_0\,\mathbb{Q}}\, A_1\, e^{-u_1\,\mathbb{Q}}\, \cdots A_j\, \epsilon^{-u_j\,\mathbb{Q}}\, A_{j+1} \cdots A_n\, e^{-u_n\,\mathbb{Q}}\right)\right) \\
- & \sum_{j=1}^{n} T\left(A_0\, e^{-u_0\,\mathbb{Q}}\, A_1\, e^{-u_1\,\mathbb{Q}}\, \cdots d\, A_j\, e^{-u_j\,\mathbb{Q}}\, A_{j+1}\, \cdots\, \alpha_n\, e^{-u_n\,\mathbb{Q}}\right) \\
= & T\left(d\, \left(A_0\, e^{-u_0\,\mathbb{Q}}\, A_1\, e^{-u_1\,\mathbb{Q}} \cdots A_j\, e^{-u_j\,\mathbb{Q}}\, A_{j+1} \cdots A_n\, e^{-u_n\,\mathbb{Q}}\right)\right) \\
- & \sum_{j=1}^{n+1} T\left(A_0\, e^{-u_0\,\mathbb{Q}}\, A_1\, e^{-u_1\,\mathbb{Q}} \cdots d\, A_j\, e^{-u_j\,\mathbb{Q}}\, A_{j+1} \cdots A_n\, e^{-u_n\,\mathbb{Q}}\right) \\
= & \sum_{j=1}^{n+1} T\left(A_0\, e^{-u_0\,\mathbb{Q}}\, A_1\, e^{-u_1\,\mathbb{Q}} \cdots A_{j-1}\, d\, \left(e^{-u_{j-1}\,\mathbb{Q}}\right)\, A_j \cdots A_n\, e^{-u_n\,\mathbb{Q}}\right) \\
= & -\sum_{j=1}^{n+1} u_{j-1}\; T\left(A_0\, e^{-u_0\,\mathbb{Q}}\, A_1\, e^{-u_1\,\mathbb{Q}} \cdots\right. \\
& \left. \cdots A_{j-1}\, \left(\int_0^1 du\, e^{-u_{j-1}(1-u)\,\mathbb{Q}}\, d\,\mathbb{Q}\, e^{-u\, u_{j-1}\,\mathbb{Q}}\right)\, A_j\, e^{-u_j\,\mathbb{Q}}\, A_{j+1} \cdots A_n\, e^{-u_n\,\mathbb{Q}}\right).
\end{aligned}$$

Replacing $\mathbb{Q}$ by $t\,\mathbb{Q}$ we then integrate over the simplex Δ_n; the integration

$$\int_0^1 du\, e^{-u_{j-1}(1-u)\,\mathbb{Q}}\, d\,\mathbb{Q}\, e^{-u_{j-1} u\,\mathbb{Q}}$$

inside the above expression gives rise to an integration on the simplex Δ_{n+1} and yields the result.

4 Comparing weighted trace cochains for different weights

Let us consider a fixed pseudodifferential algebra $Cl(M, E)$, where as before $\pi : E \to M$ is a hermitian vector bundle on a fixed closed Riemannian manifold M, and a smooth family $\mathbb{Q} : x \mapsto Q_x \in Cl(M, E)$ of pseudodifferential operators with positive scalar leading symbol and constant order q parametrized by a smooth manifold X. Their leading symbol being invertible, these operators are elliptic. This gives rise to a smooth family of weighted pseudodifferential algebras

$$(\mathcal{A} = Cl(M, E), \mathbb{Q}) := (\mathcal{A}, Q_x, x \in X)\,.$$

Given any $A_0, \cdots, A_n \in C^\infty(X, \mathcal{A})$, for any $\epsilon > 0$, $\tilde{\chi}_{n,\mathbb{Q}}(\epsilon)(A_0, \cdots, A_n)$ is a smooth function on X defined at point $x \in X$ by $\tilde{\chi}_{n,Q_x}(\epsilon)(A_0, \cdots, A_n)$ and $\bar{\chi}_{n,\mathbb{Q}}(A_0, \cdots, A_n)$ is a smooth function on X defined at point x by $\bar{\chi}_{n,Q_x}(A_0, \cdots, A_n)$.
Let us assume that $\mathbb{Q}$ is invertible. Otherwise, provided $\mathrm{Ker}\,\mathbb{Q}$ defines a vector bundle, we can replace $\mathbb{Q}$ by $\mathbb{Q} + \pi_{Ker\ \mathbb{Q}}$ where $\pi_{Ker\ \mathbb{Q}}$ is otrhogonal projection onto the kernel bundle. We can therefore apply the result of the previous section to every $Q_x, x \in X$, which yield that for any $A_0, \cdots, A_n \in C^\infty\left(X, Cl(\mathbb{M}, \mathbb{E})\right)$, the map $z \mapsto \bar{\chi}_{n,\mathbb{Q}}(z)(A_0, \cdots, A_n)$ is meromorphic with simple poles. Its complex residue at $z = 0$, which is independent of $x \in X$ is given by:

$$\mathrm{Res}_{z=0}\bar{\chi}_{n,\mathbb{Q}}(z)(A_0, \cdots, A_n) = \frac{1}{q}\,\mathrm{res}\,(A_0\, A_1, \cdots, A_n)$$

and its finite part, which we call the **$\mathbb{Q}$-weighted trace cochain** is a smooth function on X given at each point $x \in X$ by

$$\begin{aligned} & \chi_n^{Q_x}(A_0, \cdots, A_n) \\ = \ & \mathrm{tr}^{Q_x}\,(A_0\, A_1 \cdots\, A_n) + \frac{1}{q} \sum_{|k|=1}^{[|a|]+\mathrm{dim} M} \frac{(-1)^{|k|}(|k|-1)!}{(k+1)!}\,\mathrm{res}\left(A_0\, A_1^{(k_1)}\, \cdots\, A_n^{(k_n)}\, Q_x^{-|k|}\right), \end{aligned}$$

where $|a| = \sum_{i=0}^n \mathrm{ord}(A_i)$ and where for any multiindex $k = (k_1, \cdots, k_n)$ we have set $|k| = k_1 + \cdots + k_n$, $k! = k_1! \cdots k_n!$.

Theorem 2 *The map $d\,\bar{\chi}_{n,\mathbb{Q}}$ is holomorphic at zero and for any $A_0, \cdots, A_n \in C^\infty(X, \mathcal{A})$, it is by:*

$$\begin{aligned} & \left(d\,\chi_n^{\mathbb{Q}}\right)(A_0, \cdots, A_n) = \lim_{z\to 0}\left(d\,\bar{\chi}_{n,\mathbb{Q}}\right)(z)(\alpha_0, \cdots, \alpha_n) \\ = \ & \frac{1}{q} \cdot \sum_{|k|=0}^{[|a|]+\mathrm{dim} M} \frac{(-1)^{|k|+1}k!}{(k+1)!} \sum_{j=1}^{n+1} \mathrm{res}\left(A_0\, A_1^{(k_1)}\, \cdots A_{j-1}^{(k_{j-1})}\, (d\,\mathbb{Q})^{(k_j)}\, A_j^{(k_{j+1})} \right. \\ & \left. \cdots\, A_n^{(k_{n+1})}\, \mathbb{Q}^{-|k|-1}\right) \end{aligned}$$

where we have set $|a| = \sum_{i=0}^n \mathrm{ord}(A_i)$, $[|a|]$ to be its integer part, and $|k| = k_1 + \cdots + k_{n+1}$.

Proof: By Proposition 3 we have:

$$\begin{aligned} & (d\,\tilde\chi_{n,\mathbb{Q}}(t))\,(A_0,\cdots,A_n) \\ = \quad & -t\sum_{j=1}^{n+1}\tilde\chi_{n+1,\mathbb{Q}}(t)\,(A_0,A_1,\cdots,A_{j-1},d\,\mathbb{Q},A_j,\cdots,A_n)\,, \end{aligned}$$

where we have inserted $d\,\mathbb{Q}$ at the j-th position. It follows that

$$\begin{aligned} & (d\,\bar\chi_{n,\mathbb{Q}}(z))\,(A_0,\cdots,A_n) \\ = \quad & -\frac{1}{\Gamma(z)}\sum_{j=1}^{n+1}\int_0^\infty t^z dt\,\tilde\chi_{n+1,\mathbb{Q}}(t)(A_0,\cdots,A_{j-1},d\,\mathbb{Q},A_j\cdots,A_n)\,dt \\ = \quad & -\frac{1}{\Gamma(z)}\sum_{j=1}^{n+1}\sum_{k_1=0}^\infty\cdots\sum_{k_{n+1}=0}^\infty\frac{(-1)^{|k|}}{(k+1)!}\int_0^\infty dt\,t^{z+|k|}\,\mathrm{tr}\Big(A_0\,A_1^{(k_1)}\cdots \\ & A_{j-1}^{(k_{j-1})}\,(d\,\mathbb{Q})^{(k_j)}\,A_j^{(k_{j+1})}\cdots A_n^{(k_{n+1})}\,e^{-t\,\mathbb{Q}}\Big)\,dt \\ = \quad & \sum_{j=1}^{n+1}\sum_{k_1=0}^\infty\cdots\sum_{k_{n+1}=0}^\infty\frac{(-1)^{|k|+1}}{(k+1)!}\frac{\Gamma(z+|k|+1)}{\Gamma(z)}\mathrm{TR}\Big(A_0\,A_1^{(k_1)}\cdots \\ & \cdots A_{j-1}^{k_{j-1}}\,(d\,\mathbb{Q})^{(k_j)}\,A_j^{(k_{j+1})}\cdots A_n^{(k_{n+1})}\,\mathbb{Q}^{-(z+|k|+1)}\Big)\,, \end{aligned}$$

where we have used Appendix B in the first equation.
For any multiindex k (including $|k|=0$), $\frac{\Gamma(z+|k|+1)}{\Gamma(z)}=|k|!\cdot z$. Since

$$\mathrm{TR}\Big(A_0\,A_1^{(k_1)}\cdots A_j^{(k_{j-1})}\,(d\,\mathbb{Q})^{(k_j)}\,A_j^{(k_{j+1})}\cdots A_n^{(k_{n+1})}\mathbb{Q}^{-(z+|k|+1)}\Big)$$

has a simple pole at zero given by

$$\frac{1}{q}\,\mathrm{res}\Big(A_0\,A_1^{(k_1)}\cdots A_j^{(k_{j-1})}\,(d\mathbb{Q})^{(k_j)}\,A_j^{(k_{j+1})}\cdots A_n^{(k_{n+1})}\mathbb{Q}^{-(|k|+1)}\Big)\,,$$

it follows that $(d\,\bar\chi_{n,\mathbb{Q}})\,(z)(A_0,\cdots,A_n)$ converges to

$$\begin{aligned} & \lim_{z\to 0}(d\,\bar\chi_{n,\mathbb{Q}}(z))\,(A_0,\cdots,A_n) \\ = \quad & \frac{1}{q}\cdot\sum_{k_1=0}^\infty\cdots\sum_{k_{n+1}=0}^\infty(-1)^{|k|+1}\frac{k!}{(k+1)!}\sum_{j=1}^{n+1}\mathrm{res}\Big(A_0\,A_1^{(k_1)}\cdots \\ & \cdots A_{j-1}^{(k_{j-1})}\,(d\mathbb{Q})^{(k_j)}\,A_j^{(k_{j+1})}\cdots A_n^{(k_{n+1})}\mathbb{Q}^{-|k|-1}\Big)\,. \end{aligned}$$

Let us now check that the sum over k is finite. Since $d\,\mathbb{Q}$ has order bounded by q, the operator valued 1-form $A_0\,A_1^{(k_1)}\,\cdots\,A_{j-1}^{(k_{j-1})}\,(d\,\mathbb{Q})^{(k_j)}\,A_j^{(k_{j+1})}\cdots\,A_n^{(k_{n+1})}\mathbb{Q}^{-|k|-1}$ has order with upper bound $|a|+q+|k|(q-1)-q(|k|+1)=|a|-|k|$. The sum over the multiindices k therefore goes up to $[|a|]+\mathrm{dim}M$.

Corollary 2 *Let $(Cl(M,E),Q_0)$ and $(Cl(M,E),Q_1)$ be two weighted pseudodifferential algebras with Q_0 and Q_1 of same order $q>0$. Then*

$$\chi_n^{Q_1}(A_0,\cdots,A_n)-\chi_n^{Q_0}(A_0,\cdots,A_n)$$

$$= \frac{1}{q} \cdot \sum_{|k|=0}^{[|a|]+\dim M} (-1)^{|k|+1} \frac{k!}{(k+1)!} \sum_{j=1}^{n+1} \int_0^1 dt \, \mathrm{res} \Big(A_0 \, A_1^{(k_1)} \cdots$$

$$A_{j-1}^{(k_{j-1})} \left(\frac{d}{dt} Q_t \right)^{(k_j)} \cdots A_n^{(k_{n+1})} Q_t^{-|k|-1} \Big)$$

Proof: This follows from applying the above theorem to $X = [0,1]$ and a smooth family Q_t interpolating Q_0 and Q_1.

Remark 5 *The one parameter family* $Q_t := Q_0^{1-t} Q_1^t, t \in [0,1]$ *interpolates* Q_0 *and* Q_1 *and each* Q_t *is an elliptic operator with positive scalar symbol* $\sigma_L(Q_t) = \sigma_L(Q_0)^{1-t} \, \sigma_L(Q_1)^t$. *Since*

$$\frac{d}{dt} Q_t = -\log Q_0 \, Q_t + Q_0^{1-t} \, \log Q_1 Q_1^t = -\log Q_0 \, Q_t + Q_t \, \log Q_1,$$

integrating on $[0,1]$ *we get:*

$$\chi_{n,Q_1}(A_0, \cdots, A_n) - \chi_{n,Q_0}(A_0, \cdots, A_n)$$

$$= \frac{1}{q} \cdot \sum_{|k|=0}^{[|a|]+\dim M} (-1)^{|k|+1} \frac{k!}{(k+1)!} \sum_{j=1}^{n+1} \int_0^1 dt \, \mathrm{res} \Big(A_0 \, A_1^{(k_1)} \cdots$$

$$A_{j-1}^{(k_{j-1})} \left(-\log Q_0 \, Q_t + Q_t \, \log Q_1 \right)^{(k_j)} \, A_j^{(k_{j+1})} \cdots A_n^{(k_{n+1})} Q_t^{-|k|-1} \Big)$$

$$= \frac{1}{q} \cdot \sum_{|k|=0}^{[|a|]+\dim M} (-1)^{|k|+1} \frac{k!}{(k+1)!} \sum_{j=1}^{n+1} \int_0^1 dt \, \mathrm{res} \Big(A_0 \, A_1^{(k_1)} \cdots$$

$$A_{j-1}^{(k_{j-1})} \Big(-(\log Q_0)^{(k_j)} \, Q_t + Q_t \, (\log Q_1)^{(k_j)} \Big) \, A_j^{(k_{j+1})} \cdots A_n^{(k_{n+1})} Q_t^{-|k|-1} \Big)$$

$$= \frac{1}{q} \cdot \sum_{|k|=0}^{[|a|]+\dim M} (-1)^{|k|} \frac{k!}{(k+1)!} \int_0^1 dt \, \mathrm{res} \Big(A_0 \, A_1^{(k_1)} \cdots$$

$$A_{j-1}^{(k_{j-1})} \, (\log Q_0)^{(k_j)} \, A_j^{(k_{j+1}+1)} \cdots A_n^{(k_{n+1})} Q_t^{-|k|-1} \Big)$$

$$+ \frac{1}{q} \cdot \sum_{|k|=0}^{[|a|]+\dim M} (-1)^{|k|} \frac{k!}{(k+1)!} \int_0^1 dt \, \mathrm{res} \Big(A_0 \, A_1^{(k_1)} \cdots$$

$$A_{j-1}^{(k_{j-1})} \, (\log Q_0)^{(k_j)} \, A_j^{(k_{j+1})} \, A_{j+1}^{(k_{j+2}+1)} \, A_{j+2}^{(k_{j+3})} \cdots A_n^{(k_{n+1})} Q_t^{-|k|-1} \Big)$$

$$+ \frac{1}{q} \cdot \sum_{|k|=0}^{[|a|]+\dim M} (-1)^{|k|} \frac{k!}{(k+1)!} \int_0^1 dt \, \mathrm{res} \Big(A_0 \, A_1^{(k_1)} \cdots$$

$$A_{j-1}^{(k_{j-1})} \, (\log Q_0)^{(k_j)} \, A_j^{(k_{j+1})} \, A_{j+1}^{(k_{j+2})} \cdots A_n^{(k_{n+1})} Q_t^{-|k|} \Big)$$

$$+ \frac{1}{q} \cdot \sum_{|k|=0}^{[|a|]+\dim M} (-1)^{|k|} \frac{k!}{(k+1)!} \int_0^1 dt \, \mathrm{res} \Big(A_0 \, A_1^{(k_1)} \cdots$$

$$A_{j-1}^{(k_{j-1}+1)} \, (\log Q_1)^{(k_j)} \, A_j^{(k_{j+1})} \cdots A_n^{(k_{n+1})} Q_t^{-|k|-1} \Big)$$

$$
\begin{aligned}
+ \quad & \frac{1}{q}\cdot \sum_{|k|=0}^{[|a|]+\dim M} (-1)^{|k|}\,\frac{k!}{(k+1)!}\int_0^1 dt\,\mathrm{res}\Big(A_0\,A_1^{(k_1)}\,\cdots A_{j-2}^{(k_{j-2}+1)} \\
& A_{j-1}^{(k_{j-1})}\,(\log Q_1)^{(k_j)}\,A_j^{(k_{j+1})}\,A_{j+1}^{(k_{j+2})}\,A_{j+2}^{(k_{j+3})}\,\cdots\,A_n^{(k_{n+1})}Q_t^{-|k|-1}\Big) \\
+ \quad & \cdots + \\
+ \quad & \frac{1}{q}\cdot \sum_{|k|=0}^{[|a|]+\dim M} (-1)^{|k|}\,\frac{k!}{(k+1)!}\int_0^1 dt\,\mathrm{res}\Big(A_0\,A_1^{(k_1)}\,\cdots \\
& A_{j-1}^{(k_{j-1})}\,(\log Q_1)^{(k_j)}\,A_j^{(k_{j+1})}\,A_{j+1}^{(k_{j+2})}\cdots\,A_n^{(k_{n+1})}Q_t^{-|k|}\Big)
\end{aligned}
$$

where we have used the fact that $[Q_t, \log\, Q_0\, Q_t] = [Q_t, \log Q_0]\, Q_t$ *and* $[Q_t, Q_t\, \log Q_1] = Q_t\,[Q_t, \log Q_1]$. *Since* Q_t *has scalar leading symbol, and since* $\sigma(\log Q_t) = q\, log|\xi| + \sigma_0(\log Q_t)$ *where* $\sigma_0(\log Q_t)$ *is a classical (local) symbol of of order* 0, $(\log Q_0)^{(k_j)}$ *has order* $k_j(q-1)$. *The order of the expression inside the Wodzicki residue is therefore* $|a| + |k|\,(q-1) - q\,|k| = |a| - |k|$. *As a consequence, the sum over* $|k|$ *stops at* $[|a|] + \dim M$ *since beyond that* $|a| - |k| < -\dim M$.

5 Fibrations of weighted trace algebras

Let now $\mathcal{A} \to X$ be a smooth fibration of associative algebras over $\mathbb{C}$ based on a manifold X equipped with a connection ∇. The previous section took care of the case when the fibration is trivial and $\nabla = d$. Let $C^n(\mathcal{A}) \to X$ denote the corresponding fibration of spaces of n-cochains on $\mathcal{A}$. It is equipped with the induced connection:

$$
\begin{aligned}
[\nabla\, \chi_n](\alpha_0, \cdots, \alpha_n) \quad := \quad & d\,[\chi_n(\alpha_0, \cdots, \alpha_n)] \\
- \quad & \sum_{j=0}^{n} (-1)^{|\alpha_0|+\cdots+|\alpha_{j-1}|}\chi_n(\alpha_0, \cdots, \nabla\,\alpha_j, \alpha_{j+1}, \cdots, \alpha_n),
\end{aligned}
$$

where $\alpha_0, \cdots, \alpha_n \in \Omega(X, \mathcal{A})$, the space of forms on X with values in $\mathcal{A}$. It takes k-forms with values in n-cochains to $k+1$-forms with values in n-cochains.

Definition 4 *A smooth fibration of weighted tracial algebras is a triple* $(\mathcal{A}, \mathbb{Q}, T)$ *where* $T : \mathcal{I} \to X \times \mathbb{C}$ *is a bundle linear mophism acting on a bundle* $\mathcal{I} \to X$ *of fibrewise ideals of* $\mathcal{A}$ *and* $\mathbb{Q}$ *an element of* $\Omega^{2p}(X, \mathcal{A})$ *for some non negative integer* p, *that satisfies the following requirements:*

1. *T is fibrewise a trace on $\mathcal{I}$,*
2. *For each $x \in X$, and for any $U_1, \cdots, U_{2p} \in T_xX$, $(\mathcal{A}_x, Q_x(U_1, \cdots, U_{2p}), T_x)$ is a weighted trace algebra,*

where we have extended T *to* $\mathcal{I}$*-valued forms on* X. $\mathbb{Q}$ *is called the weight.*

Remark 6 *When the fibration* $\mathcal{A} \to X$ *is trivial,* T *can be taken constant so that choosing* $p = 0$, *we recover the framework of the previous section, namely* $(\mathcal{A}, \mathbb{Q}, T)$ *where* $\mathcal{A}$ *is a fixed topological unital associative algebra,* T *a trace on a non trivial ideal* $\mathcal{I}$ *of* $\mathcal{A}$ *and* $\mathbb{Q} : X \to C^\infty(X, \mathcal{A})$ *a smooth family of weights.*

To a smooth fibration of weighted tracial algebras we associate smooth families of cochains:

$$\begin{aligned} & \chi_{n,\mathbb{Q}}(\alpha_0,\cdots,\alpha_n) \\ := & \int_{\Delta_n} du_0\cdots du_n\, T\left(\alpha_0\wedge e^{-u_0\,\mathbb{Q}}\wedge\alpha_1\wedge e^{-u_1\,\mathbb{Q}}\wedge\cdots\wedge\alpha_{n-1}\wedge e^{-u_{n-1}\,\mathbb{Q}}\wedge\alpha_n\wedge e^{-u_n\,\mathbb{Q}}\right)\end{aligned}$$

which are fibrewise defined using T_x on the fibre.

Lemma 4 *If the connection ∇ is induced by a principal bundle connection so that locally, $\nabla = d + \mathrm{ad}_\theta$ where θ is a local $\mathcal{A}$-valued one form, then*

$$\nabla\, e^{-u\,\mathbb{Q}} = -u\int_0^1 dt\, e^{-u(1-t)\,\mathbb{Q}}\wedge\nabla\,\mathbb{Q}\wedge e^{-ut\,\mathbb{Q}}.$$

Proof: Combining Lemma 3 with Lemma 1 yields:

$$\begin{aligned}\nabla\, e^{-u\,\mathbb{Q}} &= d\, e^{-u\,\mathbb{Q}} + [\theta, e^{-u\,\mathbb{Q}}] \\ &= -u\int_0^1 dt\, e^{-u(1-t)\,\mathbb{Q}}\wedge d\mathbb{Q}\wedge e^{-ut\,\mathbb{Q}} - u\int_0^1 dt\, e^{-u(1-t)\,\mathbb{Q}}\wedge \mathrm{ad}_\theta\mathbb{Q}\wedge e^{-ut\,\mathbb{Q}} \\ &= -u\int_0^1 dt\, e^{-u(1-t)\,\mathbb{Q}}\wedge d\mathbb{Q}\wedge e^{-ut\,\mathbb{Q}} - u\int_0^1 dt\, e^{-u(1-t)\,\mathbb{Q}}\wedge [\theta,\mathbb{Q}]\wedge e^{-ut\,\mathbb{Q}} \\ &= -u\int_0^1 dt\, e^{-u(1-t)\,\mathbb{Q}}\wedge \nabla\mathbb{Q}\wedge e^{-ut\,\mathbb{Q}}.\end{aligned}$$

As a consequence,

Proposition 4 *Given a smooth fibration of weighted trace algebras $(\mathcal{A},\mathbb{Q},T)$ based on X equipped with a connection ∇ such that $[\nabla T]$ vanishes on $\mathcal{I}$, then for any $\alpha_0,\cdots,\alpha_n\in\Omega(X,\mathcal{A})$,*

$$\begin{aligned} & [\nabla\,\chi_{n,\mathbb{Q}}]\,(\alpha_0,\cdots,\alpha_n) \\ = & \sum_{j=1}^{n+1}(-1)^{|\alpha_0|+\cdots+|\alpha_{j-1}|+1}\,\chi_{n+1,\mathbb{Q}}\,(\alpha_0,\alpha_1,\cdots,\alpha_{j-1},\nabla\,\mathbb{Q},\alpha_j,\cdots,\alpha_n)\end{aligned}$$

where $\nabla\,\mathbb{Q}$ stands at the j-th position.

Proof: Since any $e^{-u\,\mathbb{Q}}, 0<u<1$ lies in $\mathcal{I}$, so does $\alpha_0\wedge e^{-u_0\,\mathbb{Q}}\wedge\alpha_1\wedge e^{-u_1\,\mathbb{Q}}\wedge\cdots\wedge\alpha_j\wedge e^{-u_j\,\mathbb{Q}}\wedge\alpha_{j+1}\wedge\cdots\wedge\alpha_n\wedge e^{-u_n\,\mathbb{Q}}$ lie in $\mathcal{I}$. Since ∇T vanishes on $\mathcal{I}$, it follows that we can push ∇ through the trace T in the subsequent computation using Lemma 4 to express $\nabla e^{-u_j\,\mathbb{Q}}$:

$$\begin{aligned} & \nabla\left(T\left(\alpha_0\cdot e^{-u_0\,\mathbb{Q}}\wedge\alpha_1\cdot e^{-u_1\,\mathbb{Q}}\wedge\cdots\wedge\alpha_j\wedge e^{-u_j\,\mathbb{Q}}\wedge\alpha_{j+1}\wedge\cdots\wedge\alpha_n\cdot e^{-u_n\,\mathbb{Q}}\right)\right) \\ - & \sum_{j=1}^{n+1}(-1)^{|\alpha_0|+\cdots+|\alpha_{j-1}|}T\left(\alpha_0\cdot e^{-u_0\,\mathbb{Q}}\wedge\alpha_1\wedge e^{-u_1\,\mathbb{Q}}\wedge\cdots\wedge\nabla\alpha_j\wedge e^{-u_j\,\mathbb{Q}}\wedge\alpha_{j+1}\wedge\cdots\right. \\ & \left.\cdots\wedge\alpha_n\wedge e^{-u_n\,\mathbb{Q}}\right) \\ = & T\left(\nabla\left(\alpha_0\wedge e^{-u_0\,\mathbb{Q}}\wedge\alpha_1\wedge e^{-u_1\,\mathbb{Q}}\wedge\cdots\wedge\alpha_j\wedge e^{-u_j\,\mathbb{Q}}\wedge\alpha_{j+1}\wedge\right.\right. \\ & \left.\left.\cdots\wedge\alpha_n\wedge e^{-u_n\,\mathbb{Q}}\right)\right)\end{aligned}$$

$$
\begin{aligned}
- & \sum_{j=1}^{n+1}(-1)^{|\alpha_0|+\cdots+|\alpha_{j-1}|}T\left(\alpha_0\wedge e^{-u_0\mathbb{Q}}\wedge\alpha_1\cdot e^{-u_1\mathbb{Q}}\wedge\cdots\right.\\
& \left.\cdots\wedge\nabla\alpha_j\wedge e^{-u_j\mathbb{Q}}\wedge\alpha_{j+1}\wedge\cdots\wedge\alpha_n\cdot e^{-u_n\mathbb{Q}}\right)\\
= & \sum_{j=1}^{n+1}(-1)^{|\alpha_0|+\cdots+|\alpha_{j-1}|}T\left(\alpha_0\wedge e^{-u_0\mathbb{Q}}\wedge\alpha_1\cdot e^{-u_1\mathbb{Q}}\wedge\cdots\right.\\
& \left.\cdots\wedge\alpha_{j-1}\wedge\nabla e^{-u_{j-1}\mathbb{Q}}\wedge\alpha_j\wedge\cdots\wedge\alpha_n\wedge e^{-u_n\mathbb{Q}}\right)\\
= & \sum_{j=1}^{n+1}(-1)^{|\alpha_0|+\cdots+|\alpha_{j-1}|+1}u_{j-1}\;T\left(\alpha_0\wedge e^{-u_0\mathbb{Q}}\wedge\alpha_1\wedge e^{-u_1\mathbb{Q}}\wedge\cdots\wedge\alpha_{j-1}\right.\\
& \wedge\left(\int_0^1 dt\,e^{-u_{j-1}(1-t)\mathbb{Q}}\wedge\nabla\mathbb{Q}\wedge e^{-u_{j-1}t\mathbb{Q}}\right)\wedge\alpha_j\wedge e^{-u_j\mathbb{Q}}\wedge\alpha_{j+1}\wedge\\
& \left.\cdots\wedge\alpha_n\wedge e^{-u_n\mathbb{Q}}\right).
\end{aligned}
$$

Integrating over the simplex Δ_n, the integration $\int_0^1 dt\,e^{-u_{j-1}(1-t)\mathbb{Q}}\wedge\nabla\mathbb{Q}\wedge e^{-u_{j-1}t\mathbb{Q}}$ inside the above expression gives rise to an integration on the simplex Δ_{n+1} and yields the result.

Corollary 3 *Given a smooth fibration of weighted trace algebras $(\mathcal{A},\mathbb{Q},T)$ based on X equipped with a connection ∇ such that $[\nabla T]$ vanishes on $\mathcal{I}$, then for any $\alpha_0,\cdots,\alpha_n\in\Omega(X,\mathcal{A})$,*

$$
\begin{aligned}
& d\left(\chi_{n,\mathbb{Q}}(\alpha_0,\cdots,\alpha_n)\right)\\
= & \sum_{j=1}^{n}(-1)^{|\alpha_0|+\cdots|\alpha_{j-1}|}\,\chi_{n,\mathbb{Q}}\left(\alpha_0,\alpha_1,\cdots,\alpha_{j-1},\nabla\alpha_j,\alpha_{j+1},\cdots,\alpha_n\right)\\
+ & \sum_{j=1}^{n+1}(-1)^{|\alpha_0|+\cdots+|\alpha_j|+1}\,\chi_{n+1,\mathbb{Q}}\left(\alpha_0,\alpha_1,\cdots,\alpha_{j-1},\nabla\mathbb{Q},\alpha_j,\cdots,\alpha_n\right)
\end{aligned}
$$

where $\nabla\,\mathbb{Q}$ stands at the j-th position

Proof: This follows from the above proposition combined with the fact that:

$$
\begin{aligned}
& d\left(\chi_{n,\mathbb{Q}}(\alpha_0,\cdots,\alpha_n)\right)\\
= & \sum_{j=1}^{n}(-1)^{|\alpha_0|+\cdots+|\alpha_{j-1}|}\,\chi_{n,\mathbb{Q}}\left(\alpha_0,\alpha_1,\cdots,\alpha_{j-1},\nabla\alpha_j,\cdots,\alpha_{2k+1}\right)\\
+ & [\nabla\,\chi_{n,\mathbb{Q}}]\left(\alpha_0,\cdots,\alpha_n\right).
\end{aligned}
$$

6 Weighted trace cochains for families of pseudodifferential operators

Consider a smooth fibration of smooth closed Riemannian manifolds $\pi:\mathbb{M}\to X$ with fibre M_x above $x\in X$ and a smooth (possibly $\mathbb{Z}_2$-graded) vector bundle $\mathbb{E}$ on $\mathbb{M}$. Let $\mathcal{E}:=\pi_*\,\mathbb{E}$ denote the (possibly $\mathbb{Z}_2$-graded) infinite rank vector bundle with

fibre above x given by $C^\infty(M_x,\, \mathbb{E}_{|_{M_x}})$. Assuming that $\mathbb{E}$ is a hermitian bundle and combining the hermitian structure on $\mathbb{E}_{|_{M_x}}$ with the Riemannian structure on the fibres M_x yields an L^2 structure on the fibres of $\mathcal{E}$.
This geometric setting also gives rise to a smooth fibration $\mathcal{A} := Cl(\mathbb{M},\, \mathbb{E}) \to X$ of vertical classical pseudodifferential operators with fibre above $x \in X$ given by $\mathcal{A}_x := Cl\left(M_x,\, \mathbb{E}_{|_{M_x}}\right)$. There is a natural fibration of smoothing operators $\mathcal{I} \to X$ with fibre above $x \in X$ given by the algebra $\mathcal{I}_x$ of smoothing operators on M_x and there is a smooth fibre bundle morphism $\mathrm{tr} : \mathcal{I} \to X \times \mathbb{C}$ defined by the ordinary trace tr_x on the ideal $\mathcal{I}_x$.
Let $\mathbb{Q} \in \Omega^{2p}(X, \mathcal{A})$ for some non negative integer p, be a $Cl(\mathbb{M},\, \mathbb{E})$-valued even form such that

$$\forall x \in X,\, \forall\, U_1, \cdots, U_{2p} \in T_xX, \quad \mathbb{Q}(U_1, \cdots, U_{2p}) \quad \text{is a weight of constant order}\, q.$$

$(\mathcal{A}, \mathbb{Q}, \mathrm{tr})$ defines a smooth fibration of weighted trace algebras since

1. tr is fibrewise a trace on $\mathcal{I}$,
2. for each $x \in X$, and for any $U_1, \cdots, U_{2p} \in T_xX$, $(\mathcal{A}_x, Q_x(U_1, \cdots, U_{2p}), \mathrm{tr}_x)$ is a weighted trace algebra.

We get this way a smooth fibration $(\mathcal{A}, \mathbb{Q})$ of weighted pseudodifferential algebras $(\mathcal{A}_x, Q_x)\,,\, x \in X$.

Definition 5 *Given any $\alpha_0, \cdots, \alpha_n \in \Omega(X, \mathcal{A})$ set for any $\epsilon > 0$*

$$\begin{aligned}\tilde{\chi}_{n,\,\mathbb{Q}}(\epsilon)(\alpha_0, \cdots, \alpha_n) \quad := \quad & \int_{\Delta_n} du_0 \cdots du_n \,\mathrm{tr}\left(\alpha_0 \wedge e^{-\epsilon u_0\, \mathbb{Q}} \wedge \alpha_1 \wedge e^{-\epsilon u_1\, \mathbb{Q}} \cdots\right. \\ & \left.\alpha_{n-1} \wedge e^{-\epsilon u_{n-1}\, \mathbb{Q}} \wedge \alpha_n \wedge e^{-\epsilon u_n\, \mathbb{Q}}\right)\end{aligned}$$

and let $\bar{\chi}_{n,Q}$ be its Mellin transform:

$$\begin{aligned}& \bar{\chi}_{n,\,\mathbb{Q}}(z)(\alpha_0, \cdots, \alpha_n) \\ := \quad & \frac{1}{\Gamma(z)} \int_0^\infty dt\, t^{z-1} \int_{\Delta_n} du_0 \cdots du_n \,\mathrm{tr}\left(\alpha_0 \wedge e^{-tu_0\, \mathbb{Q}} \wedge \alpha_1 \wedge e^{-tu_1\, \mathbb{Q}} \cdots\right. \\ & \left.\alpha_{n-1} \wedge e^{-tu_{n-1}\, \mathbb{Q}} \wedge \alpha_n \wedge e^{-tu_n\, \mathbb{Q}}\right).\end{aligned}$$

A connection ∇ on $\mathcal{E}$ induces a connection ∇^{End} on $Cl(\mathbb{M},\, \mathbb{E})$ which relates to ∇ by $\nabla^{\mathrm{End}}\alpha = [\nabla, \alpha]\,,\, \forall \alpha \in \Omega(X, Cl\,(\mathbb{M},\, \mathbb{E}))$ where the bracket is a $\mathbb{Z}_2$-graded bracket.

Proposition 5 *For any $\alpha_0, \cdots, \alpha_n \in \Omega(X, Cl(\mathbb{M},\, \mathbb{E}))$,*

$$\begin{aligned}& \left(\nabla\, \tilde{\chi}_{n,\,\mathbb{Q}}\right)(t)(\alpha_0, \cdots, \alpha_n) \\ = \quad & t \sum_{j=1}^{n+1} (-1)^{|\alpha_0| + \cdots + |\alpha_{j-1}| + 1} \tilde{\chi}_{n+1,\,\mathbb{Q}}(\alpha_0, \cdots, \alpha_{j-1}, \nabla^{End}\mathbb{Q}, \alpha_j \cdots, \alpha_n)\end{aligned}$$

where $\nabla^{End}\mathbb{Q}$ is at the j-th entry.

Proof: Since $\mathcal{I}$ is equipped with the ordinary trace tr which commutes with exterior differentiation and vanishes on brackets, for all $\alpha \in \Omega(X, \mathcal{I})$ we have

$$\begin{aligned}(\nabla\, \mathrm{tr})\,(\alpha) &:= d\mathrm{tr}(\alpha) - \mathrm{tr}\left(\nabla^{\mathrm{End}}\alpha\right)\\ &= \mathrm{tr}(d\,\alpha) - \mathrm{tr}\left([\nabla, \alpha]\right)\\ &= \mathrm{tr}(d\,\alpha) + \mathrm{tr}([\Theta, \alpha]) - \mathrm{tr}\left([\nabla, \alpha]\right)\\ &= 0\end{aligned}$$

where we have locally written $\nabla = d + \Theta$, with Θ a local one form with values in $Cl(M, E)$. Hence ∇ commutes with tr on $\mathcal{I}$. Applying Proposition 4 to $t\,\mathbb{Q}$ then yields the result.
When $\mathbb{Q}$ is a family of invertible operators we can apply fibrewise the results of the previous sections to prove the following result.

Theorem 3 *Given any $\alpha_0, \cdots, \alpha_n \in \Omega\left(X, Cl(\mathbb{M}, \mathbb{E})\right)$, the map $z \mapsto \bar{\chi}_{n,\mathbb{Q}}(z)(\alpha_0, \cdots, \alpha_n)$ is meromorphic with simple poles. Its complex residue at $z = 0$ is given by:*

$$\mathrm{Res}_{z=0}\left(\bar{\chi}_{n,\mathbb{Q}}(z)(\alpha_0, \cdots, \alpha_n)\right) = \frac{1}{q}\,\mathrm{res}\left(\alpha_0 \wedge \alpha_1 \wedge \cdots \wedge \alpha_n\right)$$

*and its finite part, which we call the Q-**weighted trace cochain** by*

$$\begin{aligned}\chi_n^{\mathbb{Q}}(\alpha_0, \cdots, \alpha_n) &:= \mathrm{fp}_{z=0}\bar{\chi}_{n,\mathbb{Q}}(z)(\alpha_0, \cdots, \alpha_n)\\ &= \mathrm{tr}^{\mathbb{Q}}\left(\alpha_0 \wedge \alpha_1 \cdots \wedge \alpha_n\right)\\ &+ \frac{1}{q}\sum_{|k|=1}^{[|a|]+\mathrm{dim}M} \frac{(-1)^{|k|}(|k|-1)!}{(k+1)!}\,\mathrm{res}\left(\alpha_0 \wedge \alpha_1^{(k_1)} \wedge \cdots \wedge \alpha_n^{(k_n)} \wedge \mathbb{Q}^{-|k|}\right),\end{aligned}$$

where $|a| = \sum_{i=0}^n \mathrm{ord}(\alpha_i)$ and where for any multiindex $k = (k_1, \cdots, k_n)$ we have set $|k| = k_1 + \cdots + k_n$, $k! = k_1! \cdots k_n!$.
Furthermore the map $(\nabla\,\bar{\chi}_{n,\mathbb{Q}}(z))$ is holomorphic at zero and for any $\alpha_0, \cdots, \alpha_n$ in $\Omega(X, Cl(\mathbb{M}, \mathbb{E}))$

$$\begin{aligned}&\left(\nabla\,\chi_n^{\mathbb{Q}}\right)(\alpha_0, \cdots, \alpha_n)\\ =\;& \lim_{z\to 0}\left(\nabla\,\bar{\chi}_{n,\mathbb{Q}}(z)\right)(\alpha_0, \cdots, \alpha_n)\\ =\;& \frac{1}{q}\cdot \sum_{|k|=0}^{[|a|]+\mathrm{dim}M} \frac{k!}{(k+1)!}\sum_{j=1}^{n+1}(-1)^{|\alpha_0|+\cdots+|\alpha_{j-1}|+|k|+1}\mathrm{res}\Big(\alpha_0 \wedge \alpha_1^{(k_1)} \wedge \cdots\\ &\cdots \wedge\, \alpha_{j-1}^{(k_{j-1})} \wedge \left(\nabla^{End}\mathbb{Q}\right)^{(k_j)} \wedge \alpha_j^{(k_j+1)} \wedge \cdots \wedge \alpha_n^{(k_{n+1})}\,\mathbb{Q}^{-|k|-1}\Big)\end{aligned}$$

where we have set $|a| = \sum_{i=0}^n \mathrm{ord}(\alpha_i)$, and $[|a|]$ to be its integer part, and $|k| = k_1 + \cdots + k_{n+1}$.

Proof: The first part of the theorem follows from Proposition 2 applied to each fibre. Proposition 5 yields the second part of the theorem. By Propostion 5 we have:

$$\begin{aligned}&(\nabla\,\tilde{\chi}_{n,\mathbb{Q}}(t))\,(\alpha_0, \cdots, \alpha_n)\\ =\;& t\sum_{j=1}^{n+1}(-1)^{|\alpha_0|+\cdots+|\alpha_{j-1}|+1}\tilde{\chi}_{n+1,\mathbb{Q}}(t)\left(\alpha_0, \alpha_1, \cdots, \alpha_{j-1}, \nabla^{End}\mathbb{Q}, \alpha_j, \cdots, \alpha_n\right),\end{aligned}$$

where we have inserted $\nabla^{End}\mathbb{Q}$ at the j-th position. It follows that

$$
\begin{aligned}
&\left(\nabla\,\bar{\chi}_{n,\mathbb{Q}}(z)\right)(\alpha_0,\cdots,\alpha_n)\\
= \;& \frac{1}{\Gamma(z)}\sum_{j=1}^{n+1}(-1)^{|\alpha_0|+\cdots+|\alpha_{j-1}|+1}\int_0^\infty t^z dt\,\tilde{\chi}_{n+1,\mathbb{Q}}(\alpha_0,\cdots,\alpha_{j-1},\nabla^{End}\mathbb{Q},\alpha_j,\cdots,\alpha_n)\\
= \;& \frac{1}{\Gamma(z)}\sum_{j=1}^{n+1}(-1)^{|\alpha_0|+\cdots+|\alpha_{j-1}|+1}\sum_{k_1=0}^\infty\cdots\sum_{k_{n+1}=0}^\infty\frac{(-1)^{|k|}}{(k+1)!}\int_0^\infty dt\,t^{z+|k|}\,\mathrm{tr}\Big(\alpha_0\wedge\alpha_1^{(k_1)}\wedge\cdots\\
&\cdots\wedge\alpha_{j-1}^{(k_{j-1})}\wedge\left(\nabla^{End}\mathbb{Q}\right)^{(k_j)}\wedge\alpha_j^{(k_{j+1})}\wedge\cdots\wedge\alpha_n^{(k_{n+1})}\wedge e^{-t\,\mathbb{Q}}\Big)\\
= \;& \sum_{j=1}^{n+1}(-1)^{|\alpha_0|+\cdots+|\alpha_{j-1}|+1}\sum_{k_1=0}^\infty\cdots\sum_{k_{n+1}=0}^\infty\frac{(-1)^{|k|}}{(k+1)!}\frac{\Gamma(z+|k|+1)}{\Gamma(z)}\mathrm{TR}\Big(\alpha_0\wedge\alpha_1^{(k_1)}\wedge\cdots\\
&\cdots\wedge\alpha_{j-1}^{k_{j-1}}\wedge\left(\nabla^{End}\mathbb{Q}\right)^{(k_j)}\wedge\alpha_j^{(k_{j+1})}\wedge\cdots\wedge\alpha_n^{(k_{n+1})}\wedge\mathbb{Q}^{-(z+|k|+1)}\Big).
\end{aligned}
$$

For any multiindex k (including $|k|=0$), $\frac{\Gamma(z+|k|+1)}{\Gamma(z)}=|k|!\cdot z$. Since

$$
\mathrm{TR}\Big(\alpha_0\wedge\alpha_1^{(k_1)}\wedge\cdots\wedge\left(\nabla^{End}\mathbb{Q}\right)^{(k_j)}\wedge\alpha_j^{(k_{j+1})}\wedge\cdots\wedge\alpha_n^{(k_{n+1})}\wedge\mathbb{Q}^{-(z+|k|+1)}\Big)
$$

has a simple pole at zero given by

$$
\frac{1}{q}\,\mathrm{res}\Big(\alpha_0\wedge\alpha_1^{(k_1)}\wedge\cdots\wedge\alpha_j^{(k_{j-1})}\wedge\left(\nabla^{End}\mathbb{Q}\right)^{(k_j)}\wedge\alpha_j^{(k_j+1)}\wedge\cdots\wedge\alpha_n^{(k_{n+1})}\wedge\mathbb{Q}^{-(|k|+1)}\Big),
$$

it follows that $\left(\nabla\,\bar{\chi}_{n,\mathbb{Q}}(z)\right)(\alpha_0,\cdots,\alpha_n)$ converges to

$$
\begin{aligned}
&\lim_{z\to 0}\left(\nabla\,\bar{\chi}_{n,\mathbb{Q}}(z)\right)(\alpha_0,\cdots,\alpha_n)\\
= \;& \frac{1}{q}\cdot\sum_{k_1=0}^\infty\cdots\sum_{k_{n+1}=0}^\infty(-1)^{|k|}\frac{k!}{(k+1)!}\sum_{j=1}^{n+1}(-1)^{|\alpha_0|+\cdots+|\alpha_{j-1}|+1}\mathrm{res}\Big(\alpha_0\wedge\alpha_1^{(k_1)}\wedge\cdots\\
&\cdots\wedge\alpha_{j-1}^{(k_{j-1})}\wedge\left(\nabla^{End}\mathbb{Q}\right)^{(k_j)}\wedge\alpha_j^{(k_j+1)}\wedge\cdots\wedge\alpha_n^{(k_{n+1})}\wedge\mathbb{Q}^{-|k|-1}\Big).
\end{aligned}
$$

Let us now check that the sum over k is finite. Since $\nabla^{End}\mathbb{Q}$ has order bounded by q, the operator valued form $\alpha_0\wedge\alpha_1^{(k_1)}\wedge\cdots\wedge\alpha_{j-1}^{(k_{j-1})}\wedge\left(\nabla^{End}\mathbb{Q}\right)^{(k_j)}\wedge\alpha_j^{(k_j+1)}\wedge\cdots\wedge\alpha_n^{(k_{n+1})}\wedge\mathbb{Q}^{-|k|-1}$ has order with upper bound $|a|+q+|k|(q-1)-q(|k|+1)=|a|-|k|$. The sum over the multiindices k therefore goes up to $[|a|]+\dim M$.

Remark 7 *For $n=0$ this yields*

$$
\begin{aligned}
\left(\nabla\mathrm{tr}^{\mathbb{Q}}\right)(\alpha) &= \left(\nabla\chi_0^{\mathbb{Q}}\right)(\alpha)\\
&= \sum_{k=0}^{[a]+\dim M}\frac{(-1)^{|\alpha|+k+1}\,k!}{q\,(k+1)!}\,\mathrm{res}\Big(\alpha\wedge\left(\nabla^{End}\mathbb{Q}\right)^{(k)}\wedge\mathbb{Q}^{-k-1}\Big).
\end{aligned}
$$

It compares with a formula derived in [CDMP] by other means and used in [PR]:

$$
\left(\nabla\chi_0^{\mathbb{Q}}\right)(\alpha)=\frac{(-1)^{|\alpha|+1}}{q}\mathrm{res}\left(\alpha\wedge\nabla^{End}\log\mathbb{Q}\right)
$$

which is more compact but more awkward to handle because of the presence of a logarithm.

Corollary 4 *Given any $\alpha_0, \cdots, \alpha_n \in \Omega\left(X, Cl(\mathbb{M}, \mathbb{E})\right)$,*

$$\begin{aligned} & d\left(\chi_n^{\mathbb{Q}}(\alpha_0, \cdots, \alpha_n)\right) \\ = & \sum_{j=0}^{n} (-1)^{|\alpha_0|+\cdots+|\alpha_{j-1}|} \chi_n^{\mathbb{Q}}(\alpha_0, \cdots, \alpha_{j-1}, \nabla^{End}\alpha_j, \alpha_{j+1}, \cdots \alpha_n) \\ + & \frac{1}{q} \cdot \sum_{|k|=0}^{[|a|]+\dim M} \frac{k!}{(k+1)!} \sum_{j=1}^{\dim M+1} (-1)^{|\alpha_0|+\cdots+|\alpha_{j-1}|+|k|+1} \mathrm{res}\left(\alpha_0 \wedge \alpha_1^{(k_1)} \wedge \cdots \right. \\ & \left. \cdots \wedge \alpha_{j-1}^{(k_{j-1})} \wedge \left(\nabla^{End}\mathbb{Q}\right)^{(k_j)} \wedge \alpha_j^{(k_{j+1})} \wedge \cdots \wedge \alpha_n^{(k_{n+1})} \wedge \mathbb{Q}^{-|k|-1}\right). \end{aligned}$$

In particular, when $[a] < -n$, then ∇ "commutes" with $\chi_n^{\mathbb{Q}}$:

$$\begin{aligned} & d\left(\chi_n^{\mathbb{Q}}(\alpha_0, \cdots, \alpha_n)\right) \\ = & \sum_{j=1}^{n} (-1)^{|\alpha_0|+\cdots+|\alpha_{j-1}|} \chi_n^{\mathbb{Q}}(\alpha_0, \cdots, \alpha_{j-1}, \nabla^{End}\alpha_j, \alpha_{j+1}, \cdots \alpha_n). \end{aligned}$$

Proof: The first part of the Corollary follows from the above theorem combined with the fact that

$$\begin{aligned} d\left(\chi_n^{\mathbb{Q}}(\alpha_0, \cdots, \alpha_n)\right) & = \sum_{j=0}^{n} (-1)^{|\alpha_0|+\cdots+|\alpha_{j-1}|} \chi_n^{\mathbb{Q}}(\alpha_0, \cdots, \alpha_{j-1}, \nabla^{End}\alpha_j, \alpha_{j+1}, \cdots \alpha_n) \\ & + \left[\nabla\, \chi_n^{\mathbb{Q}}\right](\alpha_0, \cdots, \alpha_n). \end{aligned}$$

The last part of the corollary then follows.

Remark 8 *In particular, this formula yields that $\chi_n^{\mathbb{Q}}$ is covariantly constant on $\Omega\left(X, Cl_{<-\frac{\dim M}{n+1}}(\mathbb{M}, \mathbb{E})\right)$ which corresponds to classical pseudodifferential valued forms with values in operators that lie in the Schatten class $\mathcal{I}_{n+1}\left(L^2(\mathbb{M}, \mathbb{E})\right)$ where $\left(L^2(\mathbb{M}, \mathbb{E})\right) \to X$ is the smooth fibration with fibre above $x \in X$ given by $L^2(M_x, E_{|_{M_x}})$.*

7 Chern-Weil type weighted trace cochains

We continue in the same geometric setting as in the previous section keeping the same notations.

Theorem 4 *Let $f_i, i = 0, \cdots n$ be polynomial functions, then the Chern-Weil type weighted trace cochain $\chi_n^{\mathbb{Q}}(f_0(\Omega), \cdots, f_n(\Omega))$ is generally not closed and we have:*

$$\begin{aligned} & d\,\chi_n^{\mathbb{Q}}(f_0(\Omega), \cdots, f_n(\Omega)) \\ = & \frac{1}{q} \cdot \sum_{|k|=0}^{[|d|\cdot\omega]+\dim M} \frac{(-1)^{|k|+1}\,k!}{(k+1)!} \sum_{j=1}^{n+1} \mathrm{res}\left(f_0(\Omega) \wedge (f_1(\Omega))^{(k_1)} \wedge \cdots \right. \\ & \left. \wedge (f_{j-1}(\Omega))^{(k_{j-1})} \wedge \left(\nabla^{End}\mathbb{Q}\right)^{(k_j)} \wedge (f_j(\Omega))^{(k_j+1)} \cdots \wedge (f_n(\Omega))^{(k_{n+1})} \wedge \mathbb{Q}^{-|k|-1}\right) \end{aligned}$$

where $|d| = \sum_{i=0}^n d_i$ *with* d_i *the degree of the polynomial* f_i*, and* ω *is the order of the operator valued* 2*-form* Ω*.*

Remark 9

- *For* $n = 0$*, the weighted Chern-Weil type forms investigated in [PR] read:*

$$\mathrm{tr}^{\mathbb{Q}}(f(\Omega)) = \chi_0^{\mathbb{Q}}(f(\Omega)).$$

The obstruction to their closedness which follows from the above theorem is given by:

$$d\,\mathrm{tr}^{\mathbb{Q}}(f(\Omega)) = \frac{1}{q} \cdot \sum_{k=0}^{[|d|\,\omega]+\dim M} \frac{(-1)^{k+1}\, k!}{k+1} \mathrm{res}\left(f_0(\Omega) \wedge \left(\nabla^{End}\mathbb{Q}\right)^{(k)} \wedge \mathbb{Q}^{-k-1}\right)$$

which compares with the more compact but maybe less tractable formula [PR]:

$$d\,\chi_0^{\mathbb{Q}}(f(\Omega)) = -\frac{1}{q}\mathrm{res}\left(f_0(\Omega) \wedge \nabla^{End} \log \mathbb{Q}\right).$$

- *As already observed in [PR] in the case of ordinary weighted Chern-Weil type forms, the above theorem tells us that the more negative* $\omega \cdot |d|$ *becomes, the fewer will be the terms that obstruct the closedness. We come back to this below.*

Proof: Differentiating $\chi_n^{\mathbb{Q}}(f_0(\Omega), \cdots, f_n(\Omega))$ yields by Corollary 4:

$$\begin{aligned} & d\,\chi_n^{\mathbb{Q}}(f_0(\Omega), \cdots, f_n(\Omega)) \\ = \;& \frac{1}{q} \cdot \sum_{|k|=0}^{[|a|]+\dim M} \frac{k!}{(k+1)!} \sum_{j=1}^{n+1} (-1)^{|k|+1} \mathrm{res}\left(f_0(\Omega) \wedge (f_1(\Omega))^{(k_1)} \wedge \cdots \wedge (f_{j-1}(\Omega))^{(k_{j-1})}\right. \\ & \wedge \left(\nabla^{End}\mathbb{Q}\right)^{(k_j)} \wedge (f_j(\Omega))^{(k_j+1)} \wedge \cdots \\ & \left.\cdots \wedge (f_n(\Omega))^{(k_{n+1})} \wedge \mathbb{Q}^{-|k|-1}\right) \end{aligned}$$

where we have used the Bianchi identity $\nabla^{End}\Omega = 0$ to cancel the terms $\chi_n^{\mathbb{Q}}(f_0(\Omega), \cdots, f_{j-1}(\Omega))$ and $\nabla^{End} f_j(\Omega), f_{j+1}(\Omega), \cdots f_n(\Omega))$.

Corollary 5 *Let* Ω *have integer order* $\omega < 0$*, then any polynomials* $f_i, i = 0, \cdots n$ *of orders* $d_i, i = 0, \cdots, n$ *large enough so that* $|d| > -\frac{\dim M}{\omega}$ *give rise to closed Chern-Weil type weighted trace cochains* $\chi_n^{\mathbb{Q}}(f_0(\Omega), \cdots, f_n(\Omega))$*. Moreover, if* $\mathbb{Q}_t$*,* $t \in]0,1[$ *is a smooth one parameter families of weights with constant order* q *and* ∇_t*,* $t \in]0,1[$ *a smooth family of connections, the curvatures of which have constant order* ω*, then*

$$\frac{d}{dt}\chi_n^{\mathbb{Q}_t}(f_0(\Omega_t), \cdots, f_n(\Omega_t)) = d \sum_{j=1}^{n} \sum_{i=0}^{|d_j|} \mathrm{tr}^{\mathbb{Q}_t}\left(\Omega_t^{d_0} \wedge \Omega_t^{d_1} \wedge \cdots \wedge \dot{\nabla}_t \wedge \cdots \wedge \Omega_t^{d_n} \wedge \mathbb{Q}_t^{-|k|}\right),$$

where $\dot{\nabla}_t$ *stands in the* j*-th position. It follows that Chern-Weil type weighted trace cochains* $\chi_n^{\mathbb{Q}}(f_0(\Omega), \cdots, f_n(\Omega))$ *define topological characteristic classes.*

Proof: The first part of the corollary follows from the above theorem with $|a| = |d| \cdot \omega$. If $|d| > -\frac{\dim M}{\omega}$, we have $[|a|] = |d| \cdot \omega + \dim M < 0$ so that the sum involving the residues vanishes. This implies that $d\left(\chi_n^{\mathbb{Q}}(f_0(\Omega), \cdots, f_n(\Omega))\right) = 0$.

Let now $\mathbb{Q}_t$, $t \in]0,1[$ be a smooth one parameter families of weights with constant order q and ∇_t, $t \in]0,1[$ a smooth family of connections, the curvatures of which have constant order ω. Replacing ∇ by $\frac{d}{dt}$ in Theorem 3, we first find that $\frac{d}{dt}\bar{\chi}_{n,\mathbb{Q}_t}(z)$ is holomorphic at zero and for any $\alpha_0, \cdots, \alpha_n \in \Omega(X, Cl(\mathbb{M}, \mathbb{E}))$

$$\begin{aligned}
& \left(\frac{d}{dt}\chi_n^{\mathbb{Q}_t}\right)(\alpha_0, \cdots, \alpha_n) \\
= \quad & \lim_{z \to 0}\left(\frac{d}{dt}\,\bar{\chi}_{n,\mathbb{Q}_t}(z)\right)(\alpha_0, \cdots, \alpha_n) \\
= \quad & \frac{1}{q} \cdot \sum_{|k|=0}^{[|a|]+\dim M} \frac{k!}{(k+1)!} \sum_{j=1}^{n+1} (-1)^{|\alpha_0|+\cdots+|\alpha_{j-1}|+|k|+1} \mathrm{res}\left(\alpha_0 \wedge \alpha_1^{(k_1)} \wedge \cdots\right. \\
& \left.\cdots \wedge \alpha_{j-1}^{(k_{j-1})} \wedge \left(\frac{d}{dt}\mathbb{Q}_t\right)^{(k_j)} \cdots \wedge \alpha_n^{(k_{n+1})} \wedge \mathbb{Q}_t^{-|k|-1}\right).
\end{aligned}$$

Setting $\alpha_i = f_i(\Omega_t)$, we have $a_i = d_i \cdot \omega$ and this tells us that provided $[|a|] = [|d| \cdot \omega] < -\dim M$,

$$\left(\frac{d}{dt}\chi_n^{\mathbb{Q}_t}\right)(f_0(\Omega_t), \cdots, f_n(\Omega_t)) = 0. \tag{8}$$

Combining equation (8) and Corollary 4 then yields provided $[|a|] = [|d| \cdot \omega] < -\dim M$ (here as before, $\alpha_i = f_i(\Omega_t)$),

$$\begin{aligned}
& \frac{d}{dt}\left(\chi_n^{\mathbb{Q}_t}(\alpha_0, \cdots, \alpha_n)\right) \\
= \quad & \left(\frac{d}{dt}\,\chi_n^{\mathbb{Q}_t}\right)(\alpha_0, \cdots, \alpha_n) + \sum_{j=0}^{n} \chi_n^{\mathbb{Q}_t}(\alpha_0, \cdots, \frac{d}{dt}\alpha_j, \alpha_{j+1}, \cdots \alpha_n) \\
= \quad & \sum_{j=0}^{n} \mathrm{tr}^{\mathbb{Q}_t}\left(\alpha_0 \wedge \alpha_1 \wedge \cdots \wedge \frac{d}{dt}\alpha_j \wedge \cdots \wedge \alpha_n \wedge \mathbb{Q}_t^{-|k|}\right) \\
+ \quad & \frac{1}{q}\sum_{j=0}^{n}(-1)^{|\alpha_0|+\cdots+|\alpha_{j-1}|+|k|+1|} \sum_{|k|=1}^{[|a|]+\dim M} \frac{k!}{(k+1)!}\,\mathrm{res}\left(\alpha_0 \wedge \alpha_1^{(k_1)} \wedge \cdots\right. \\
& \left.\cdots \wedge \left(\frac{d}{dt}\alpha_j\right)^{(k_j)} \wedge \alpha_j^{(k_j+1)} \cdots \wedge \alpha_n^{(k_n)} \wedge \mathbb{Q}_t^{-|k|}\right).
\end{aligned}$$

We saw that the first term $\left(\frac{d}{dt}\,\chi_n^{\mathbb{Q}_t}\right)(\alpha_0, \cdots, \alpha_n))$ in the first equation vanishes provided $[|a|] = [|d| \cdot \omega] < -\dim M$..

We claim that the second term in the last equation also vanishes for similar reasons in that case. We are therefore left with the first term in the last equation, namely $\sum_{j=0}^{\dim M} \mathrm{tr}^{\mathbb{Q}_t}\left(\alpha_0 \wedge \alpha_1 \wedge \cdots \wedge \frac{d}{dt}\alpha_j \wedge \cdots \wedge \alpha_n \wedge \mathbb{Q}_t^{-|k|}\right)$. Since the f_i's are polynomials, by linearity we can assume that they are of the form $f_i(X) = X^{d_i}$, in which case we have:

$$\frac{d}{dt}\alpha_j \quad = \quad \frac{d}{dt}f_j(\Omega_t)$$

$$\begin{aligned}
&= \sum_{i=1}^{d_j} \Omega_t^{i-1} \wedge \left(\frac{d}{dt}\Omega_t\right) \wedge \Omega_t^{d_j-i-1} \\
&= \sum_{i=1}^{d_j} \Omega_t^{i-1} \wedge \left(\nabla_t \dot{\nabla}_t - \dot{\nabla}_t \nabla_t\right) \wedge \Omega_t^{d_j-i-1} \\
&= \sum_{i=1}^{d_j} \Omega_t^{i-1} \wedge \left(\nabla_t^{End} \dot{\nabla}_t\right) \wedge \Omega_t^{d_j-i-1} \\
&= \sum_{i=1}^{d_j} \nabla_t^{End} \left(\Omega_t^{i-1} \wedge \dot{\nabla}_t \wedge \Omega_t^{d_j-i-1}\right),
\end{aligned}$$

where we have used Bianchi identity $\nabla_t^{End}\Omega_t = 0$ to "pull out" ∇^{End}. Since we know by that ∇_t "commutes" with $\chi_n^{\mathbb{Q}_t}$ provided $[a] < -\dim M$, whenever $\frac{\dim M}{d} > -\omega$, we can write:

$$\begin{aligned}
& \mathrm{tr}^{\mathbb{Q}_t} \left(\alpha_0 \wedge \alpha_1 \wedge \cdots \wedge \frac{d}{dt}\alpha_j \wedge \alpha_{j+1} \wedge \cdots \wedge \alpha_n \wedge \mathbb{Q}_t^{-|k|}\right) \\
& \sum_{i=0}^{d_j} \mathrm{tr}^{\mathbb{Q}_t} \left(\nabla_t^{End} \left(\Omega_t^{d_0} \wedge \Omega_t^{d_1} \wedge \cdots \wedge \dot{\nabla}_t \wedge \cdots \wedge \Omega_t^{d_n} \wedge \mathbb{Q}_t^{-|k|}\right)\right) \\
= \; & d \sum_{i=0}^{|d_j|} \mathrm{tr}^{\mathbb{Q}_t} \left(\Omega_t^{d_0} \wedge \Omega_t^{d_1} \wedge \cdots \wedge \dot{\nabla}_t \wedge \cdots \wedge \Omega_t^{d_n} \wedge \mathbb{Q}_t^{-|k|}\right),
\end{aligned}$$

where $\dot{\nabla}_t$ is in the j-th position. Since this term is exact, summing over j yields the exactness of $\sum_{j=0}^{n} \chi_n^{\mathbb{Q}_t}(\alpha_0, \cdots, \frac{d}{dt}\alpha_j, \alpha_{j+1}, \cdots \alpha_n)$ and ends the proof.

We now specialize to a Quillen-Bismut superconnection setting. Let $D \in C^\infty(X, \mathcal{A})$ be a smooth (possibly odd) section of self-adjoint elliptic pseudodifferential operators with constant order. A superconnection ∇ on $\mathcal{E}$ induces another superconnection $\mathbb{A} := \nabla + \gamma D$ (where $\gamma = 1$ in the $\mathbb{Z}_2$-graded case and $\gamma^2 = 1$ in the non graded case). Its curvature is a two form $\mathbb{A}^2$ with values in $\mathcal{A}$ so that at any point $x \in X$, $\mathbb{A}_x^2$ is a positive elliptic pseudodifferential operator of order 2 valued two form. Taking $\mathbb{Q} := \mathbb{A}^2$ gives rise to covariantly constant weighted trace cochains:

Theorem 5 *The $\mathbb{A}^2$-weighted trace cochains $\chi_n^{\mathbb{A}^2}$ are covariantly constant:*

$$\left(\mathbb{A}\chi_n^{\mathbb{A}^2}\right)(\alpha_0, \cdots, \alpha_n) = 0 \quad \forall \alpha_0, \cdots, \alpha_n \in \Omega(X, Cl(\mathbb{M}, \mathbb{E})).$$

Also, for any polynomial functions $f_0, \cdots, f_n$, the $\chi_n^{\mathbb{A}^2}\left(f_0(\mathbb{A}^2), \cdots, f_n(\mathbb{A}^2)\right)$ define closed characteristic classes which are independent of the choice of connection ∇ from which $\mathbb{A}$ is defined.

Proof: This follows from the above theorem combined with the fact that $\mathbb{A}^{End}\mathbb{Q} = [\mathbb{A}, \mathbb{Q}] = 0$ when $\mathbb{Q} = \mathbb{A}^2$.

Remark 10 *The second part of the theorem is proven in [PS] for $n = 0$. The generalization $n > 0$ does not bring anything new since clearly, the expression $\chi_n^{\mathbb{A}^2}\left(f_0(\mathbb{A}^2), \cdots, f_n(\mathbb{A}^2)\right)$ is a linear combination of terms of the type $\mathrm{tr}^{\mathbb{A}^2}(f(\mathbb{A}^2)) = \chi_0^{\mathbb{A}^2}\left(f(\mathbb{A}^2)\right)$.*

Appendix A: Relation to Higson's cochain $(A_0, \cdots, A_n) \mapsto \langle A_0, \cdots, A_n \rangle_z$

The methods used in this paper to compute the various anomalies/discrepancies, are somewhat similar in spirit to methods used in [CM] and [H] in the computation of a local representative of the Chern character. This appendix points out to some of the relations. N. Higson introduces in [H] formula (4.1) a multilinear form $(A_0, \cdots, A_n) \mapsto \langle A_0, \cdots, A_n \rangle_z$ which relates to $\bar{\chi}_{n,Q}$ as follows:

Proposition 6

$$\begin{aligned} \langle A_0, \cdots, A_n \rangle_z &:= (-1)^n \frac{\Gamma(z)}{2\pi i} \int \lambda^{-z} d\lambda \, \mathrm{tr}\left(A^0 (\lambda - Q)^{-1} A_1 (\lambda - Q)^{-1} \cdots A_n (\lambda - Q)^{-1} \right) \\ &= \Gamma(z+n) \cdot \bar{\chi}_{n,Q}(z+n)(A_0, \cdots, A_n) \\ &\simeq \Gamma(z+n)\, \mathrm{TR}\left(A_0\, A_1 \cdots A_n\, Q^{-n-z} \right) \\ &+ \sum_{|k|=1}^{\infty} \frac{(-1)^{|k|}}{(k+1)!} \Gamma(z+n+|k|)\, \mathrm{TR}\left(A_0\, A_1^{(k_1)} \cdots A_n^{(k_n)} Q^{-n-|k|-z} \right). \end{aligned}$$

Remark 11 *This last expression compares with*

$$\langle A_0, \cdots, A_n \rangle_z \quad \simeq \quad \sum_{|k|=0}^{\infty} \frac{(-1)^{|k|}}{(|k|+n)!} c(k)\, \mathrm{TR}\left(A_0\, A_1^{(k_1)} \cdots A_n^{(k_n)}\, Q^{-n-|k|-z} \right)$$

derived in [H], in the proof of Proposition 4.14.

Proof: By Lemma A.2 in [H] we have:

$$\tilde{\chi}_{n,Q}(t)(A_0, \cdots, A_n) = \frac{(-1)^n}{t^n\, 2\pi i} \int d\lambda\, e^{-t\lambda}\, \mathrm{tr}\left(A_0 (\lambda - Q)^{-1} A_1 (\lambda - Q)^{-1} \cdots A_n (\lambda - Q)^{-1} \right).$$

The result then follows from the fact that $\bar{\chi}_{n,Q}(z)$ is the Mellin transform of $\tilde{\chi}_{n,Q}$:

$$\begin{aligned} &\bar{\chi}_{n,Q}(z)(A_0, \cdots, A_n) \\ = &\frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} \tilde{\chi}_{n,Q}(t)(A_0, \cdots, A_n) \\ = &\frac{(-1)^n}{2\pi i\, \Gamma(z)} \int_0^\infty dt\, t^{-n+z-1} \int d\lambda\, e^{-t\lambda}\, \mathrm{tr}\left(A_0 (\lambda - Q)^{-1} A_1 (\lambda - Q)^{-1} \cdots A_n (\lambda - Q)^{-1} \right) \\ = &\frac{(-1)^n}{2\pi i} \frac{\Gamma(z-n)}{\Gamma(z)} \int d\lambda\, \lambda^{n-z} \mathrm{tr}\left(A_0 (\lambda - Q)^{-1} A_1 (\lambda - Q)^{-1} \cdots A_n (\lambda - Q)^{-1} \right) \\ = &\frac{1}{\Gamma(z)} \langle A_0, \cdots, A_n \rangle_{z-n} \end{aligned}$$

Using formula (7) then yields the last identity in the proposition.

Appendix B: Proof of formula (6)

For any $A, Q \in Cl(M, E)$ such that Q has scalar top order symbol, the following holds [Le](Lemma 4.2):

Lemma 5 *If $p, \epsilon, N > 0$ satisfy $\frac{N-a}{q} - p - \epsilon < 0$ then*

$$e^{-tQ} A = \sum_{j=0}^{N-1} \frac{(-t)^j}{j!} A^{(j)} e^{-tQ} + R_N(A, Q, t)$$

where, for any $c > 0$ such that $Q + c$ is invertible, there exists $C > 0$ such that $\|R_N(A, Q, t)\,(Q + c)^p\| \leq C\, t^{\frac{N-a}{q} - p - \epsilon}$.

Remark 12 *Writing this for short:*

$$e^{-tQ}\, A \simeq \sum_{j=0}^{\infty} \frac{(-t)^j}{j!} A^{(j)}\, e^{-tQ},$$

and taking a Mellin transform $Q^{-z} = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} e^{-tQ} dt$, we find (compare with [H] Lemma 4.30):

$$Q^{-z}\, A \simeq \sum_{j=0}^{\infty} C_{-z}^j A^{(j)}\, Q^{-z-j}.$$

This can also be derived from (compare with [H] Lemma 4.20):

$$(\lambda - Q)^{-1}\, A \simeq \sum_{j=0}^{\infty} A^{(j)}\, (\lambda - Q)^{-(j+1)}$$

using a Cauchy formula $C_z^p L^{z-p} = \frac{1}{2\pi i} \int \mu^z (\mu - L)^{-p-1} d\mu$ applied to $L = \lambda - Q$.

Iterating the above lemma yields [PR] Proposition B.3:

Proposition 7 *Given any $A_0, \cdots, A_n \in Cl(M, E)$, for any $J \in \mathbb{N}$, there exist positive integers $N_1, \cdots, N_n$ such that for $t > 0$*

$$\begin{aligned} & \tilde{\chi}_{n,Q}(t)(A_0, \cdots, A_n) \\ = \quad & \sum_{k_1=0}^{N_1-1} \cdots \sum_{k_n=0}^{N_n-1} \frac{(-t)^{|k|}}{(k+1)!} \mathrm{tr}\left(A_0 A_1^{(k_1)} \cdots A_n^{(k_n)} e^{-tQ}\right) + o(t^J). \end{aligned}$$

This slightly differs from the statement of [PR] Proposition B.3 where the rest term is an $o(t)$. But it can easily be seen from [PR] Lemma B.2 that the integers $N_1, \cdots, N_n$ can in fact be chosen large enough so that the rest term is an $o(t^J)$.
Also in [PR] proposition B.3, the formula involves a constant

$$C_k = \int_{\Delta_n} u_0^{k_1} (u_0 + u_1)^{k_2} \cdots (u_0 + \cdots + u_{n-1})^{k_n}\, du_0 \cdots du_n$$

where $\Delta_n = \{(u_0, \cdots, u_n), 0 \leq u_i \leq 1, \sum_{i=0}^n u_i = 1\}$ is the n-th simplex. But setting $u_0 + \cdots + u_{n-1} = 1 - u_n$ and integrating over u_n yields $C_k = \frac{1}{k_n+1} \int_{\Delta_n} u_0^{k_1} (u_0 + u_1)^{k_2} \cdots (u_0 + \cdots + u_{n-2})^{k_{n-1}}\, du_0 \cdots du_n$ which by induction shows that

$$C_k = \frac{1}{(k_n + 1)\,(k_{n-1} + 1) \cdots (k_0 + 1)} = \frac{k!}{(k+1)!}$$

where we have set $(k + 1)! = (k_n + 1)!(k_{n-1} + 1)! \cdots (k_0 + 1)!$ and $k! = k_n! \cdots k_0!$.

References

[B] J.-M. Bismut, *The Atiyah-Singer index theorem for families of Dirac operators: two equation proofs*, Inv. Math. **83**, p.91-151 (1986)

[BGV] N. Berline, E. Getzler, M. Vergne, **Heat kernels and Dirac operators**, Springer-Verlag, 1992

[CDP] A. Cardona, C. Ducourtioux, S. Paycha, *From tracial anomalies to anomalies in quantum field theory*, Comm. Math. Phys. **242**, 31–65 (2003)

[CDMP] A. Cardona, C. Ducourtioux, J-P. Magnot, S. Paycha, *Weighted traces on algebras of pseudo-differential operators and geometry on loop groups*, Infinite Dim. Anal. Quant. Prob. Rel Top. **5**, No. 4 (2002) 503–540

[CM] A. Connes, H. Moscovici, *The local index formula in non commutative geometry*, Geom. Funct. Anal. **5** (2) 174– 243 (1995)

[F] D. Freed, *The geometry of loop groups*, Journ. Diff. Geom. **28** 223–276 (1988)

[G-BVF] J.M. Gracia Bondia, J.C. Varilly, H. Figueroa, **Elements of noncommutative geometry**, Birkhäuser Advanced Texts, Boston MA (2001)

[H] N.Higson, *The local index formula in non commutative geometry*, Preprint 2004

[JLO] A. Jaffe, A. Lesniewski, K. Osterwalder, *Quantum K-theory. The Chern character*, Comm. Math. Phys. **118** 1–14 (1988)

[KV] M. Kontsevich, S. Vishik, *Determinants of elliptic pseudo-differential operators*, Max Planck Institut preprint, 1994

[Le] M. Lesch, *On the non commutative residue for pseudo-differential operators with log-polyhomogeneous symbols*, Annals of Global Analysis and Geometry, **17** (1999) 151–187

[MN] R. Melrose, V. Nistor, *Homology of pseudo-differential operators I. Manifolds with boundary*, funct-an/9606005, june 1999

[P] S. Paycha, *Renormalized traces as a looking glass into infinite dimensional geometry*, Inf. Dim. Anal. Quant.Prob. Rel. Top., **4**, N.2, p.221-266 (2001)

[PR] S. Paycha, S. Rosenberg, *Curvature on determinant bundles and first Chern forms*, Journ. of Geom. Phys. **45**, p. 393–429 (2003)

[PS] S. Paycha, S. Scott, in preparation

[Q] D. Quillen, *Superconnections and the Chern character*, Topology **24** p.89-95 (1985)

[Sc] S.Scott, *Zeta-Chern forms and the local family index theorem*, Preprint 2003

[Wo] M. Wodzicki, *Non commutative residue* in Lecture Notes in Math. **1283**, Springer Verlag 1987