INVARIANT METRICS OF POSITIVE RICCI CURVATURE ON PRINCIPAL BUNDLES

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Abstract. Let Y be a compact connected Riemannian manifold with a metric of positive Ricci curvature. Let $\pi : P \to Y$ be a principal bundle over Y with compact connected structure group G. If the fundamental group of P is finite, we show that P admits a G invariant metric with positive Ricci curvature so that π is a Riemannian submersion.

§0 Introduction

Throughout this paper, let Y be a compact connected Riemannian manifold which admits a metric g_Y with positive Ricci curvature. This implies that the fundamental group of Y is finite. The converse does not hold; the K^3 surface is a simply connected 4 manifold which does not admit a metric of positive scalar curvature and hence which does not admit a metric of positive Ricci curvature. Let P be a principal bundle over Y with compact connected structure group G. If G admits a metric of positive Ricci curvature, then P admits a G invariant metric which has positive Ricci curvature so that the fibers of P are totally geodesic and so that the projection is a Riemannian submersion, see Bérard Bergery [4] for details. For general G, if P admits a metric of positive Ricci curvature, then $\pi_1(P)$ is finite. The following converse is the main result of this paper:

0.1 Theorem. *Let (Y, g_Y) be a compact connected Riemannian manifold with positive Ricci curvature. Let P be a principal bundle over Y with compact connected structure group G so that $\pi_1(P)$ is finite. Then there exists a G invariant metric g_P on P so that g_P has positive Ricci curvature and so that $\pi : (P, g_P) \mapsto (Y, g_Y)$ is a Riemannian submersion.*

Bérard Bergery [4] showed that if P is a principal torus bundle over Y with $\pi_1(P)$ finite, then P admits a metric with positive Ricci curvature. His argument uses a result of Aubin [3] and does not yield a G invariant metric nor a metric so that π is a Riemannian submersion. The case of a circle bundle is the crucial one and we summarize his argument as follows. Let L be a complex line bundle over Y

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with associated circle bundle S; $\pi_1(S)$ is finite if and only if $0 \neq c_1(L)$ in $H^2(Y;\mathbb{R})$. Choose a connection on L so the associated curvature 2-form $\mathcal{F}$ is harmonic. Use the connection to define a splitting of $T(S) = \mathcal{V} \oplus \mathcal{H}$ into the vertical and horizontal subspaces of the projection $\pi : S \to Y$. Let $ds^2_{\mathcal{H}}$ be the pull-back of the metric on Y to the horizontal distribution $\mathcal{H}$. Let e^1 be the 1-form which is dual to the circular flow ∂_t and consider the metric $ds^2_S = e^{-2C_1}(e^1 \circ e^1) + ds^2_{\mathcal{H}}$. Then for $C_1 >> 0$, the Ricci tensor of g_S is positive semi-definite. Since $\pi_1(S)$ is finite, $c_1(L) \neq 0$ so there exists $y_0 \in Y$ so that $\mathcal{F}(y_0) \neq 0$. At this point, the Ricci curvature of g_S is positive definite. Aubin [3] has shown that a manifold S which admits a metric with positive semi-definite Ricci tensor which is positive definite at a point admits a metric with positive Ricci curvature everywhere; this result then completes the proof.

Here is a brief historical summary of some related work on what might be called the Kaluza-Klein ansatz. Nash [16], Poor [18], and Bérard-Bergery [4] used Riemannian submersions with totally geodesic fibers to construct metrics of positive Ricci curvature on the total spaces of certain compact fiber bundles and on certain vector bundles of rank at least 2. The same construction was used by D'Atri & Ziller [9], Jensen [13], and by Wang & Ziller [24] to construct left invariant and bi invariant Einstein metrics with positive scalar curvature on certain compact Lie groups G where G was viewed as a principal H bundle $H \to G \to G/H$. Several other authors [7, 10, 14, 22] constructed Einstein metrics with positive scalar curvature on the total spaces of principal torus bundles over products of Kaehler Einstein manifolds with positive first Chern class. Cheeger [8], Derdzinski & Rigas [11], and Strake & Walschap [20] used the same method to construct metrics of non negative sectional curvature on certain manifolds. Weinstein [21] studied symplectic structures on fat bundles. A principal ingredient in the constructions used by all these authors involved rescaling the fibers by a constant factor.

Non constant warping has also been used. Nabonnand [15] gave examples of (non-compact) complete manifolds with positive Ricci curvature and infinite fundamental group. Bérard Bergery [5] showed that if M is complete and admits a metric with non-negative Ricci curvature, then $M \times \mathbb{R}^k$ carries a metric with positive Ricci curvature for $k \geq 3$. For example, if M is the K^3 surface, then $M \times \mathbb{R}^3$ admits a metric with positive Ricci curvature but not positive sectional curvature. Gromoll & Meyer [12] used warped products to construct metrics of positive Ricci curvature on certain non compact manifolds. Anderson [1, 2] and Sha & Yang [19] used warped products to construct metrics of positive Ricci curvature on the connected sum of certain simply connected manifolds with positive Ricci curvature. Wei [25] showed that if L is a simply connected nilpotent Lie group, then $L \times \mathbb{R}^p$ admits a complete Riemannian metric with positive Ricci curvature for p sufficiently large.

The proof of the main result of this paper will combine the above mentioned warping techniques. We outline our approach as follows. We first consider the case of a principal circle bundle S. We begin as in the paper [4] and choose a connection on S with harmonic curvature $\mathcal{F}$ to split the tangent space of S into horizontal and vertical subspaces. However, we consider a metric on S of the form $ds^2_S := e^{2(-C_1+\varrho\phi)}(e^1)^2 + ds^2_{\mathcal{H}}$; i.e. we rescale the volume of the circle fibers by a varying conformal factor. Here ϕ is a smooth function on Y and C_1 and ϱ are parameters. Let $\mathcal{O}$ be a non-empty open set so that $|\mathcal{F}|^2 \geq \epsilon_1 > 0$ on $\mathcal{O}$. On $\mathcal{O}$, the Ricci tensor is positive for $C_1 >> 0$. On $\mathcal{O}^c$, we will choose ϕ to give the positivity of the Ricci tensor for ϱ small. Thus a crucial feature of our construction, in contrast to that given in [4] is that the fibers have *non-constant* volumes so the fibers are not minimal and hence are not totally geodesic. This proves Theorem 0.1 in the case that $G = S^1$. The general case will follow by considering first G/H where H is the connected component of the center of G and then by considering a tower of circle bundles; see §1 for details.

§1 is devoted to the proof of Theorem 0.1. In Lemma 1.1, we discuss the geometry of circle bundles, in Lemma 1.2, we discuss the curvature tensor of a circle bundle S, and in Lemma 1.2 we discuss the Ricci curvature of a circle bundle. We then use these Lemmas to give the proof of Theorem 0.1 in the special case of S^1 bundles. We then generalize this argument to torus bundles of higher rank and finally to general compact connected Lie groups to complete the proof of Theorem 0.1. We will postpone the proof of Lemmas 1.1 and 1.2 until §2 to avoid interrupting the flow of our agrument. Both these Lemmas follow easily from the O'Neill formulas [17], see Besse [6] as well. We give a self-contained proof for the convenience of the reader as it is quite straightforward and in any event no more difficult than adapting the O'Neill formulas to the setting at hand. We conclude the paper in §3 by presenting some more specialized results on torus bundles. It is a pleasant task to thank S. Stolz for many helpful conversations relating to this paper.

§1 The geometry of torus bundles

We begin with the rank 1 case. Let L be a complex line bundle over a Riemannian manifold Y. We assume L is equipped with a positive definite pointwise inner product $\langle\cdot,\cdot\rangle$ and a unitary connection ${}^L\nabla$. Let s be a local unitary section to L and let ${}^L\nabla s = \sqrt{-1}\mathcal{A}_s s$ define the normalized connection 1 form $\mathcal{A}_s$. Let $S = S(L)$ be the associated unit circle bundle and let $\pi : S \to Y$ be the associated projection. The connection ${}^L\nabla$ defines a splitting $TS = \mathcal{V} \oplus \mathcal{H}$. Introduce local coordinates $y = (y^1, ..., y^m)$ on Y. Let $(t, y) \mapsto e^{\sqrt{-1}t}s(y)$ give local coordinates (t, y) to S. We refer to §2 for the proof of the following result.

1.1 Lemma.

(1) *∂_t is an invariantly defined tangent vector spanning $\mathcal{V}$.*

(2) *If $\tilde{s} = e^{\sqrt{-1}\Phi}s$, then $\partial_t = \partial_{\tilde{t}}$, $\partial_a^y = \partial_a^{\tilde{y}} - \partial_a^y\Phi\partial_t$, and $\tilde{\mathcal{A}}_s = \mathcal{A}_s + d_Y\Phi$.*

(3) *The horizontal lift of a vector field Ψ on Y is given by $\mathcal{H}\Psi := \Psi - \mathcal{A}_s(\Psi)\partial_t$.*

(4) *We have $e^1 := dt + \pi^*\mathcal{A}_s$ is dual to ∂_t and spans $\mathcal{V}^*$.*

(5) *The normalized curvature $\mathcal{F} := d_Y\mathcal{A}_s$ is invariantly defined and $de^1 = \pi^*\mathcal{F}$.*

Let $f \in C^\infty(Y)$ and let ∇ be a unitary connection on L. We define a metric $ds_S^2 = ds_S^2(f, \mathcal{A})$ on $S(L)$ by requiring that $\pi_* : \mathcal{H} \to TY$ is an isometry, that $\mathcal{V} \perp \mathcal{H}$, and that $|\partial_t|^2 = e^{2f}$. Denote the metric on Y by $ds_Y^2 = g_{ab}dy^a \circ dy^b$. Let $\{F_a\}$ be an orthonormal frame for the horizontal space.

Lemma 1.2.

(1) $R^S(\partial_t, F_a, F_a, \partial_t) = e^{2f}\{-f_{;aa} - f_{;a}f_{;a} + \sum_b e^{2f}\mathcal{F}_{ab}\mathcal{F}_{ab}/4\}$.

(2) $R^S(F_a, F_b, F_b, \partial_t) = e^{2f}\{\mathcal{F}_{ab;b} + 3f_{;b}\mathcal{F}_{ab}\}/2$.

(3) $R^S(F_a, F_b, F_b, F_a) = R^Y(\partial_a, \partial_b, \partial_b, \partial_a) - 3e^{2f}\mathcal{F}_{ab}\mathcal{F}_{ab}/4$.

Let ext denote exterior multiplication and int the dual, interior multiplication. Let $\Delta = \delta d$ be the scalar Laplacian on functions. Let ρ_S and ρ_Y be the Ricci tensors of S and Y. We use the metric to identify vectors and covectors. The next Lemma follows directly from Lemma 1.2.

1.3 Lemma. *Let F be a unit horizontal tangent vector. Let $\mathcal{T} := e^{-f}\partial_t$.*

(1) $\rho_S(\mathcal{T}, \mathcal{T}) = \Delta f - |df|^2 + e^{2f}|\mathcal{F}|^2/4$.

(2) $\rho_S(\mathcal{T}, F) = e^f\{-\delta\mathcal{F}(F) + 3\mathcal{F}(F, df)\}/2$.

(3) $\rho_S(F, F) = \rho_Y(F, F) - e^{2f}|\operatorname{int}(F)\mathcal{F}|^2/2 - f_{;FF} - F(f)^2$.

(4) *The metric g_S has positive Ricci curvature if and only if*
$\forall F,\ 4\rho_S(\mathcal{T}, \mathcal{T})\rho_S(F, F) > \rho_S(\mathcal{T}, F)^2$.

Proof of Theorem 0.1 if $r = 1$. Choose a unitary connection on the complex line bundle L so the curvature $\mathcal{F}$ is harmonic. Since $c_1(L)$ does not vanish in $H^2(Y; \mathbb{R})$, $\mathcal{F}$ does not vanish identically. Since $\mathcal{F}$ is harmonic, $\delta\mathcal{F} = 0$.

(1) Choose $\epsilon > 0$ so $\rho_S(F, F) \geq \epsilon\ \forall F$.

(2) Choose a non-empty open set $\mathcal{O}$ and $\epsilon_1 > 0$ so that $|\mathcal{F}(y)|^2 \geq \epsilon_1\ \forall y \in \mathcal{O}$.

(3) Choose a smooth function ϕ so that $\Delta\phi(y) \geq 1$ for $y \notin \mathcal{O}$.

(4) Let $f := -C_1 + \varrho\phi$. Choose C_1 so $e^{2f}|\operatorname{int}(F)\mathcal{F}|^2 < \epsilon/3\ \forall\ \varrho \in (0, 1)$.

(5) Choose $\epsilon_2 > 0$ so $e^{2f}/4 \geq \epsilon_2\ \forall\ \varrho \in (0, 1)$.

(6) Choose $\varrho_0 > 0$ so $|\varrho\phi_{;FF}| + |\varrho^2 F(\phi)^2| \leq \epsilon/3\ \forall F$ if $\varrho \in (0, \varrho_0)$. Then $\rho_S(F, F) \geq \epsilon/3$.

(7) Choose K so that $\rho_S(\mathcal{T}, \mathcal{T}) \geq \epsilon_1\epsilon_2 - K\varrho$ on $\mathcal{O}$, $\rho_S(\mathcal{T}, \mathcal{T}) \geq \varrho - K\varrho^2$ on $\mathcal{O}^c$, and so that $\rho_S(\mathcal{T}, F)^2 \leq K\varrho^2$.

It now follows that $\rho_S(\mathcal{T}, \mathcal{T})\rho_S(F, F)$ dominates $\rho_S(\mathcal{T}, F)^2$ for small values of ϱ and thus ρ_S is positive definite. $\square$

Proof of Theorem 0.1 for toroidal fibers. To deal with the higher rank case, we use induction on r and proceed inductively. Let L_1, ..., L_r be the complex line bundles defined by the torus bundle P. Let $T^r := S^1 \times ... \times S^1$. Assume $\pi_1(P)$ is finite. Let P_{r-1} be the torus bundle of rank $r - 1$ defined by the complex line bundles L_1, ..., L_{r-1}. We use the long exact sequence of the fibration $S^1 \to P \to P_{r-1}$ to see $\pi_1(P_{r-1})$ is finite. By induction, we may find a metric g_{r-1} on P_{r-1} so $\pi_{r-1} : P_{r-1} \to Y$ is a Riemannian submersion, so g_{r-1} has positive Ricci tensor, and so that g_{r-1} is invariant under the action of T^{r-1}.

We work equivariantly. Let $\tilde{L} := \pi_{r-1}^* L_r$; this complex line bundle admits a natural T^{r-1} action. Let $\tilde{\nabla}_0$ be an arbitrary unitary connection on $\tilde{L}$. We average $\tilde{\nabla}_0$ over the action of T^{r-1} to construct a T^{r-1} invariant connection $\tilde{\nabla}_1$. Let $\mathcal{F}_1$ be the curvature of $\tilde{\nabla}_1$; this is invariant under the T^{r-1} action. Let $\mathcal{F}$ be the harmonic representative of $c_1(\tilde{L})$. Since T^{r-1} is a connected Lie group, it acts trivially on the cohomology of P_{r-1}. Since T^{r-1} acts by isometries, it preserves the harmonic forms and thus $\mathcal{F}$ is T^{r-1} invariant. We decompose $\mathcal{F}=\mathcal{F}_1+d\omega$; by averaging ω over T^{r-1}, we may assume ω is T^{r-1} invariant. Let $\tilde\nabla:=\tilde\nabla_1+\omega$. Then $\tilde\nabla$ is a T^{r-1} invariant connection with harmonic curvature $\mathcal{F}$. Choose $\mathcal{O}$ so that $|\mathcal{F}|^2\ge\epsilon_1>0$ on $\mathcal{O}$. Since $\mathcal{F}$ is T^{r-1} invariant, we can choose $\mathcal{O}$ to be T^{r-1} invariant and hence $\mathcal{O}^c$ is T^{r-1} invariant. Choose ϕ so $\Delta\phi\ge1$ on $\mathcal{O}^c$; by averaging ϕ over the action of T^{r-1}, we can assume ϕ is T^{r-1} invariant. Thus the metric constructed in the case $r=1$ yields a metric on P which is both S^1 and T^{r-1} invariant and hence T^r invariant. Since the composition of Riemannian submersions is a Riemannian submersion, the remaining assertions of Theorem 0.1 now follow if G is a torus. $\square$

Proof of Theorem 0.1 for general G. Let C be the connected component of the center of a compact connected Lie group G. Since C is a compact connected Abelian Lie group, $C=T^r$ is a torus of rank r for some r. Let $P\to Y$ be a principal G bundle with $\pi_1(P)$ finite. Let $\tilde G:=G/C$ and let $\tilde Y:=P/C$. Then the natural projection from $\tilde Y$ to Y defines a principal $\tilde G$ bundle. Since G is reductive, $\tilde G$ is semisimple so $\tilde G$ admits a metric of positive Ricci curvature. The argument in Bérard Bergery [4] shows that $\tilde Y$ admits a $\tilde G$ invariant metric with positive Ricci curvature so that $\pi:(\tilde Y,g_{\tilde Y})\to(Y,g_Y)$ is a Riemannian submersion with totally geodesic fibers. The natural projection from P to $\tilde Y$ defines a principal T^r bundle. The structure group G acts on this principal bundle since C is central. We average over the group G at each stage in applying the construction used to prove Theorem 0.1 in the case $r>1$ and the remainder of the argument is the same. $\square$

§2 Proof of Technical Lemmas

Proof of Lemma 1.1. The flow $v\to e^{\sqrt{-1}t}v$ for $v\in S(L)$ and $t\in\mathbb{R}$ is invariantly defined; ∂_t is the associated vertical vector field. Assertion (1) now follows. Since ${}^L\nabla$ is unitary, $\mathcal{A}_s$ is a real 1-form. If $\tilde s=e^{\sqrt{-1}\Phi(y)}s$, then $(\tilde y,\tilde t)=(y,t-\Phi)$; assertion (2) now follows. We show that $\mathcal{H}$ is invariantly defined by computing:

$$\partial_a^y-\mathcal{A}_s(\partial_a^y)\partial_t=\partial_a^{\tilde y}-\partial_a^y\Phi\partial_t-\mathcal{A}_s(\partial_a^y)\partial_t=\partial_a^{\tilde y}-\mathcal{A}_{\tilde s}(\partial_a^y)\partial_t.$$

Fix $y_0\in Y$ and choose Φ so $(\mathcal{A}_s+d_Y\Phi)(y_0)=0$. Since $\mathcal{A}_{\tilde s}(y_0)=0$, the $\partial_a^{\tilde y}$ are horizontal. Thus at y_0, $\mathcal{H}\partial_a^y=\partial_a^{\tilde y}$ is horizontal. Since $\mathcal{H}\Psi$ is invariantly defined, $\mathcal{H}\Psi$ is the horizontal lift. Since $e^1(\mathcal{H}\Psi)=0$ for all Ψ and since $e^1(dt)=1$, e^1 is the vertical projection of dt and is invariantly defined. By (2), $d_Y\mathcal{A}_{\tilde s}=d_Y\mathcal{A}_s$ so the curvature $\mathcal{F}$ is invariantly defined. Clearly $de^1=\pi^*\mathcal{F}$. $\square$

To prove Lemma 1.2, we shall need a number of technical Lemmas. We begin by computing the Christoffel symbols. Let $\Gamma_{ttt}=g_S(\nabla_{\partial_t}\partial_t,\partial_t)$, $\Gamma_{tta}=g_S(\nabla_{\partial_t}\partial_t,\partial_a)$ etc. Let $\mathcal{A}_a:=\mathcal{A}_s(\partial_a)$; we suppress the role of s for notational convenience.

Lemma 2.1.

(1) $g_S(\partial_t,\partial_t)=e^{2f}$, $g_S(\partial_t,\partial_a)=e^{2f}\mathcal{A}_a$, *and* $g_S(\partial_a,\partial_b)=g^Y_{ab}+e^{2f}\mathcal{A}_a\mathcal{A}_b$.

(2) $\Gamma_{ttt}=0$, $\Gamma_{tta}=-e^{2f}\partial_af$, $\Gamma_{att}=\Gamma_{tat}=e^{2f}\partial_af$,
$\Gamma_{atb}=\Gamma_{tab}=\{\partial_a(e^{2f}\mathcal{A}_b)-\partial_b(e^{2f}\mathcal{A}_a)\}/2$, *and*
$\Gamma_{abt}=\{\partial_a(e^{2f}\mathcal{A}_b)+\partial_b(e^{2f}\mathcal{A}_a)\}/2$.

Proof of Lemma 2.1. We prove the first assertion by computing:

$$\begin{aligned}
g_S(\partial_t,\partial_t) =& e^{2f},\\
g_S(\partial_t,\partial_a) =& g_S(\partial_t,\partial_a-\mathcal{A}_a\partial_t)+g_S(\partial_t,\mathcal{A}_a\partial_t)=e^{2f}\mathcal{A}_a, \text{ and}\\
g_S(\partial_a,\partial_b) =& g_S(\partial_a-\mathcal{A}_a\partial_t,\partial_b-\mathcal{A}_b\partial_t)+g_S(\partial_a-\mathcal{A}_a\partial_t,\mathcal{A}_b\partial_t)\\
&+g_S(\mathcal{A}_a\partial_t,\partial_b-\mathcal{A}_b\partial_t)+\mathcal{A}_a\mathcal{A}_b g_S(\partial_t,\partial_t)\\
=& g^Y_{ab}+e^{2f}\mathcal{A}_a\mathcal{A}_b.
\end{aligned}$$

The Christoffel symbols are given by $2\Gamma_{ijk}=g_{ik/j}+g_{jk/i}-g_{ij/k}$. The second assertion now follows. □

We fix $y_0\in Y$ and choose a coordinate system y centered at y_0. We further normalize the coordinate system by assuming that $g_{ab}=\delta_{ab}+O(|y|^2)$; for example, we could take geodesic normal coordinates centered at y_0. We choose s so that $({}^L\nabla s)(y_0)=0$ or equivalently so $\mathcal{A}=O(|y|)$.

Lemma 2.2.

(1) $g_S^{tt}=e^{-2f}+O(|y|^2)$, $g_S^{tb}=-\mathcal{A}_b+O(|y|^2)$, *and* $g_S^{ab}=\delta_{ab}+O(|y|^2)$.
(2) ${\Gamma_{tt}}^t(y_0)=0$, $\partial_t\Gamma\equiv 0$, ${\Gamma_{ab}}^c(y_0)=0$, ${\Gamma_{ab}}^t(y_0)=(\partial_b\mathcal{A}_a+\partial_a\mathcal{A}_b)(y_0)/2$, *and* ${\Gamma_{at}}^b(y_0)=e^{2f}(\partial_a\mathcal{A}_b-\partial_b\mathcal{A}_a)(y_0)/2$.
(3) ${\Gamma_{ta}}^t=\partial_a(f)-e^{2f}\mathcal{A}_b(\partial_a\mathcal{A}_b-\partial_b\mathcal{A}_a)/2+O(|y|^2)$.
(4) $\partial_a{\Gamma_{ta}}^t(y_0)=\{\partial_a\partial_a(f)-e^{2f}(\partial_a\mathcal{A}_b)\mathcal{F}_{ab})/2\}(y_0)$.
(5) $\partial_a{\Gamma_{bb}}^t(y_0)-\partial_b{\Gamma_{ab}}^t(y_0)=\{\mathcal{F}_{ab;b}/2+2\partial_b f\cdot\partial_a\mathcal{A}_b-\partial_a f\cdot\partial_b\mathcal{A}_b-\partial_b f\cdot\partial_b\mathcal{A}_a\}(y_0)$.
(6) $\partial_a({\Gamma_{bb}}^a-{\Gamma^Y_{bb}}^a)(y_0)=-(e^{2f}\mathcal{F}_{ab}\partial_a\mathcal{A}_b)(y_0)$.
(7) $\partial_b({\Gamma_{ab}}^a-{\Gamma^Y_{ab}}^a)(y_0)=-(e^{2f}\mathcal{F}_{ab}\partial_b\mathcal{A}_a)(y_0)/2$.

Proof of Lemma 2.2. Let $C(y):=C_0(y)+C_1(y)$ where $C_i(y)$ are matrix valued functions with $C_1(y_0)=0$. Then

$$\begin{aligned}
C^{-1}(y) =& \{C_0(y)(I+C_0(y)^{-1}C_1(y))\}^{-1}\\
=& \{I-C_0(y)^{-1}C_1(y)+O(|y|^2)\}C_0(y)^{-1}\\
=& C_0(y)^{-1}-C_0(y)^{-1}C_1(y)C_0(y)^{-1}+O(|y|^2).
\end{aligned}$$

Applying this to $C_0=diag(e^{2f},1,...,1)$ and to $C_1=ds_S^2-C_0$ yields assertion (1). The first two assertions of (2) are immediate. We use (1) to raise indices and derive (2) from Lemma 2.1; note that $\Gamma_{abc}(y_0)=0$. To prove (3), we compute

$${\Gamma_{ta}}^t=\Gamma_{tat}g_S^{tt}+\Gamma_{tab}g_S^{tb}=\partial_a(f)-e^{2f}\mathcal{A}_b(\partial_a\mathcal{A}_b-\partial_b\mathcal{A}_a)/2+O(|y|^2).$$

We differentiate this expression with respect to a and evaluate at y_0 to prove (4). The proof of (5) is similar:

$$\begin{aligned}
&{\Gamma_{ab}}^t=\Gamma_{abt}g_S^{tt}+\Gamma_{abc}g_S^{ct}\\
&\quad=e^{-2f}\{\partial_a(e^{2f}\mathcal{A}_b)+\partial_b(e^{2f}\mathcal{A}_a)\}/2+O(|y|^2)\\
&\quad=(\partial_a\mathcal{A}_b+\partial_b\mathcal{A}_a+2\partial_a(f)\mathcal{A}_b+2\partial_b(f)\mathcal{A}_a)/2+O(|y|^2), \text{ and}\\
&\partial_a{\Gamma_{bb}}^t(y_0)-\partial_b{\Gamma_{ab}}^t(y_0)\\
&\quad=\{2\partial_a\partial_b\mathcal{A}_b-\partial_b\partial_a\mathcal{A}_b-\partial_b\partial_b\mathcal{A}_a\\
&\qquad+4\partial_b f\cdot\partial_a\mathcal{A}_b-2\partial_a f\cdot\partial_b\mathcal{A}_b-2\partial_b f\cdot\partial_b\mathcal{A}_a)\}(y_0)/2.
\end{aligned}$$

We prove assertions (6) and (7) by computing:

$$\begin{aligned}
\Gamma_{bb}{}^a =& g^{ad}\Gamma_{bbd} + g^{at}\Gamma_{bbt} = \Gamma_{bba} - \mathcal{A}_a\Gamma_{bbt} + O(|y|^2)\\
=&\Gamma^Y_{bb}{}^a + e^{2f}\{2\partial_b(\mathcal{A}_b\mathcal{A}_a) - \partial_a(\mathcal{A}_b^2) - 2\mathcal{A}_a\partial_b\mathcal{A}_b)\}/2 + O(|y|^2), \text{ and}\\
\Gamma_{ab}{}^a =& g^{ad}\Gamma_{abd} + g^{at}\Gamma_{abt} = \Gamma_{aba} - \mathcal{A}_a\Gamma_{abt} + O(|y|^2)\\
=&\Gamma^Y_{ab}{}^a + e^{2f}\{\partial_b(\mathcal{A}_a^2) - \mathcal{A}_a(\partial_a\mathcal{A}_b + \partial_b\mathcal{A}_a)\}/2. \quad \square
\end{aligned}$$

Proof of Lemma 1.2. Fix y_0. Adopt the normalizations of Lemma 2.2 where the coordinates on Y are chosen so that $F = \partial_a$ at y_0 for some a. We use Lemma 2.2 to compute at y_0:

$$\begin{aligned}
&R^S(\partial_t, F_a, F_a, \partial_t)\\
&\quad= g_S(\partial_t\Gamma_{aa}{}^b\partial_b + \partial_t\Gamma_{aa}{}^t\partial_t - \partial_a\Gamma_{ta}{}^b\partial_b - \partial_a\Gamma_{ta}{}^t\partial_t, \partial_t)\\
&\qquad + g_S(\Gamma_{tb}{}^c\Gamma_{aa}{}^b\partial_c + \Gamma_{tb}{}^t\Gamma_{aa}{}^b\partial_t + \Gamma_{tt}{}^c\Gamma_{aa}{}^t\partial_c + \Gamma_{tt}{}^t\Gamma_{aa}{}^t\partial_t, \partial_t)\\
&\qquad - g_S(\Gamma_{ab}{}^c\Gamma_{ta}{}^b\partial_c + \Gamma_{ab}{}^t\Gamma_{ta}{}^b\partial_t + \Gamma_{at}{}^c\Gamma_{ta}{}^t\partial_c + \Gamma_{at}{}^t\Gamma_{ta}{}^t\partial_t, \partial_t)\\
&\quad=\{e^{2f}(-\partial_a\Gamma_{ta}{}^t - \Gamma_{ab}{}^t\Gamma_{ta}{}^b - \Gamma_{at}{}^t\Gamma_{ta}{}^t)\}(y_0)\\
&\quad=\{e^{2f}(-\partial_a\partial_a f + e^{2f}(\partial_a\mathcal{A}_b)(\partial_a\mathcal{A}_b - \partial_b\mathcal{A}_a)/2\\
&\qquad - e^{2f}(\partial_a\mathcal{A}_b + \partial_b\mathcal{A}_a)(\partial_a\mathcal{A}_b - \partial_b\mathcal{A}_a)/4 - (\partial_a f)^2)\}(y_0)\\
&\quad=\{e^{2f}(-\partial_a\partial_a f - (\partial_a f)^2 + e^{2f}(\partial_a\mathcal{A}_b - \partial_b\mathcal{A}_a)^2/4)\}(y_0).
\end{aligned}$$

This proves the first assertion. Next we compute at y_0

$$\begin{aligned}
R^S(F_a, F_b, F_b, \partial_t) =& e^{2f}\{\partial_a\Gamma_{bb}{}^t - \partial_b\Gamma_{ab}{}^t + \Gamma_{at}{}^t\Gamma_{bb}{}^t - \Gamma_{bt}{}^t\Gamma_{ab}{}^t\}\\
=& e^{2f}\{\mathcal{F}_{ab;b}/2 + 2\partial_b f\cdot\partial_a\mathcal{A}_b - \partial_a f\cdot\partial_b\mathcal{A}_b - \partial_b f\cdot\partial_b\mathcal{A}_a)\\
&+ \partial_a f\cdot\partial_b\mathcal{A}_b - \partial_b f(\partial_a\mathcal{A}_b + \partial_b\mathcal{A}_a)/2\}\\
=& e^{2f}\{\mathcal{F}_{ab;b} + 3f_{;b}\mathcal{F}_{ab}\}/2.
\end{aligned}$$

This proves the second assertion. We prove assertion (3) by computing at y_0:

$$\begin{aligned}
&R^S(F_a, F_b, F_b, F_a)\\
=&\partial_a\Gamma_{bb}{}^a - \partial_b\Gamma_{ab}{}^a + \Gamma_{at}{}^a\Gamma_{bb}{}^t - \Gamma_{bt}{}^a\Gamma_{ab}{}^t\\
=&R^Y(\partial_a, \partial_b, \partial_b, \partial_a) + e^{2f}\{-\partial_a\mathcal{A}_b\mathcal{F}_{ab} + \partial_b\mathcal{A}_a\mathcal{F}_{ab}/2\\
&+ (\partial_b\mathcal{A}_a + \partial_a\mathcal{A}_b)\mathcal{F}_{ab}/4\}.
\end{aligned}$$

The final assertion of Lemma 1.2 follows from the first three and from the quadratic formula. $\square$

§3 Torus bundles

In this section, we present a few additional results on torus bundles. We begin with the following Lemma which characterizes those bundles with finite fundamental groups in terms of first Chern classes.

3.1 Lemma. *Let (Y, g_Y) be a compact connected Riemannian manifold with positive Ricci curvature. Let P be a principal torus bundle over Y with associated complex line bundles L_1, ..., L_r. Then $\pi_1(P)$ is finite if and only if the first Chern classes $c_1(L_i)$ are linearly independent in $H^2(Y;\mathbb{R})$.*

Proof. The fundamental group of Y is finite. By passing to a finite cover, we may assume that $\pi_1(Y) = 0$. The long exact sequence of a fibration yields:

$$...\pi_2(Y) \to \pi_1(T^r) \to \pi_1(P) \to 0.$$

Thus $\pi_1(P)$ is Abelian and $\pi_1(P)$ is finite if and only if the map $\delta : \pi_2(Y) \to \pi_1(T^r)$ is a rational surjection. We dualize this map and apply the Hurewicz isomorphism to construct $\delta^* : H^1(T^r;\mathbb{R}) \to H^2(Y;\mathbb{R})$; δ is a rational surjection if and only if δ^* is injective. Let BT^r be the classifying space of the r dimensional torus. We identify $H^1(T^r;\mathbb{R})$ with $H^2(BT^r;\mathbb{R})$ to regard $\delta^* : H^2(BT^r;\mathbb{R}) \to H^2(Y;\mathbb{R})$. We use naturality to see then that δ^* is given by the first Chern class. More precisely, if $x_1, ..., x_r$ are the canonical generators of $H^2(BT^r;\mathbb{R})$, then $\delta^*(\Sigma_i a_i x_i) = \Sigma_i a_i c_1(L_i)$. Consequently $\pi_1(P)$ is finite if and only if $\{c_1(L_1), ..., c_1(L_r)\}$ are linearly independent in $H^2(Y;\mathbb{R})$. $\square$

Let Y denote the connected sum of k copies of $S^2 \times S^2$. Then Y admits a metric of positive Ricci curvature, see Anderson [1] or Sha and Yang [19] for details. Then $\dim H^2(Y;\mathbb{Z}) = 2k$ and $H^2(Y;\mathbb{Z})$ embeds in $H^2(Y;\mathbb{R})$. For $r \leq 2k$, we can choose line bundles L_1,..., L_r so that $\{c_1(L_i)\}$ are linearly independent in $H^2(Y;\mathbb{R})$. Let P be the corresponding torus bundle; this implies that $\pi_1(P)$ is finite.

This example is interesting in at least one respect. Instead of proceeding by induction on r, it is natural to try to perform the construction of §1 on the torus bundle all at once. One easily sees that a necessary condition for this to work is that there exist a point y_0 so that the forms $\mathcal{F}_i(y_0)$ are linearly independent for $1 \leq i \leq r$. Let $k \geq 4$ so $r = 2k \geq 8$. Choose the $c_1(L_i)$ linearly independent as cohomology classes. Since $\dim \Lambda^2 Y = 6$, there is no point of Y where the $\mathcal{F}_i$ are linearly independent.

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