EVERY FINITE COMPLEX HAS THE HOMOLOGY OF A DUALITY GROUP

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INTRODUCTION

A group G belongs to the class $\mathbb{D}(n)$ if, by definition:

a) There exists a G-module D and a class $e \in H_n(G;D)$ such that the cap product with e :

$$H^i(G;B) \longrightarrow H_{n-i}(G;B \otimes_{\mathbf{Z}} D)$$

is an isomorphism for all i and all G-module B.

b) The Eilenberg space $BG = K(G,1)$ is homotopy equivalent to a finite complex of dimension n .

Part a) of the above definition is the classsical definition of a duality group, given by Bieri and Eckmann [Be1]. There is no known example of a group satisfying a) but not b), but it is not known that a) implies b).

Denote by $\mathbb{D}$ the union of all the classes $\mathbb{D}(n)$, for all n.

The classical list of examples of groups in $\mathbb{D}$ (fundamental group of aspherical manifolds, free groups, cohomology dimension 2 groups with one end, arithmetic groups, etc, see [Bn, VII 10]) was recently enriched by new examples: braid groups [Sq], mapping class [Hr] . This suggests that the class $\mathbb{D}$ is larger than previously expected and inspired to the author the main result of this paper, Theorem A below.

The statement of Theorem A requires two definitions. A map $f : \longrightarrow Y$ is called <u>acyclic</u> if its theoretical fibre is an acyclic space, or, equivalently, if the homomorphism $f_* : H_*(X;B) \longrightarrow H_*(Y;B)$ induces an isomorphism for any $\pi_1(X)$-module B (see [HH] for a survey about acyclic maps). A group P is called <u>locally perfect</u> if any finitely generated subgroup of P is contained in a finitely generated perfect subgroup of P. Acyclic maps f with $\ker\pi_1 f$ locally perfect enjoy interesting geometric properties (see, for instance [HV]).

<u>Theorem A.</u> Let X be a finite complex. Then, there is a group $G \in \mathbb{D}$ and an acyclic map $BG \longrightarrow X$. Moreover, $\ker(G \longrightarrow \pi_1(X))$ is locally perfect.

In particular, the fact that a group G is in $\mathbb{D}$ implies nothing on, for instance, its integral homology, except being finitely generated.

Theorem A has well known predecessors: given a complex X, D. Kan and W. Thurston [KT] have first shown the existence of an acyclic map $BG \longrightarrow X$, for some (very large) group G. If X is finite, G can be taken so that BG is a finite complex, as shown by G. Baumslag, E. Dyer and A. Heller [BDH] A simple proof of these fact given by C.R.F. Maunder [Ma] makes possible to have $\dim BG = \dim X$. All these constructions enjoy strong functoriality properties.

In contrast, although it uses the principle of [Ma], our construction of the group G of Theorem A is not functorial. We strongly suspect that there is no such functorial construction, at least respecting the free products with amalgamation. Also, the following problem remains open:

<u>Problem.</u> Let X be a finite complex. What is the minimal integer $r(X)$ such that there is a group $G \in \mathbb{D}(r(X))$ and an acyclic map $BG \longrightarrow X$.

The construction given here to prove Theorem A is, on this respect, not very efficient and just gives the inequality $r(X) \leq 10m(X)-7$, where $m(X)$ is the minimal number of simplexes of positive dimension of a polyedron homotopy equivalent to X (see Remark (3.3)). For instance: $1 = r(S^1) \leq 23$. So far, there is no counter-example to the possible conjecture that $r(X)$ is equal to the homotopy dimension of X.

On the other hand, given a finite complex X, there are groups $G \in \mathbb{D}(r)$ with acyclic maps $BG \longrightarrow X$ for arbitrarely large r's (see Remark (3.4)).

1. PRELIMINARY RESULTS

The first three lemmas of this section consist of elementary properties of the classes $D(n)$. Lemma (1.4) is a criterium to recognize when a homomorphism between free amalgamated products is injective. These results are used in § 3 for the proof of Theorem A.

(1.1) Lemma. Let A be a subgroup of the groups B and C. Suppose that B and C are in $D(n)$ and A is in $D(n-1)$. Then, the amalgamated product $B*_A C$ is in $D(n)$.

Proof: Condition a) of the definition of $D(n)$ is fullfilled by [BE2, Theorem 3.2]. Condition b) comes from the fact that the space $B(B*_A C)$ is the union over BA of BB and BC [Bn, Theorem 7.3].

□

(1.2) Lemma. Let $G \in D(m)$ and $H \in D(n)$. Then $G \times H \in D(m+n)$.

Proof: Condition a) is classical (see [BE1, Theorem 3.5]). Condition b) is obvious, since $B(G\times H) \cong BG \times BH$.

□

(1.3) Lemma. Let G be a one-relator group $G = \langle a_1, a_2, \dots a_k | r \rangle$ Suppose that:

1) G is not cyclic
2) r is not a proper power
3) G is not a non-trivial free product.

Then, $G \in D(2)$.

Proof: Condition 2) together with [DV, Theorem 2.1] gives that the 2-complex associated with the given presentation of G is homotopy equivalent to BG . This guarantees Condition b) of the defintion of $D(2)$, and implies that G is of cohomology dimension 2. By [BE1, Theorem 5.2], G is then a free product of duality groups of dimension 1 and 2. But Condition 1) and 3) then implies that G is a duality group of dimension 2 and thus satisfies Condition a) for $n = 2$.

□

For our last lemma, let us consider a homomorphism between the following diagrams of groups and subgroups

$$\begin{bmatrix} C \subset A \subset A_1 & \xrightarrow{f} & C' \subset A' \subset A_1' \\ \cap & & \cap \\ B & & B' \end{bmatrix}$$

which therefore induces homomorphisms

$$f : D = A *_C B \longrightarrow D' = A' *_{C'} B'$$

and

$$f : D_1 = A_1 *_C B \longrightarrow D_1' = A_1' *_{C'} B'$$

(1.4) Lemma Suppose that $f|A_1$ and $f|B$ is injective. Suppose also that $C' \cap f(A_1) = C' \cap f(B) = f(C)$. Then :

a) $f : D_1 \longrightarrow D_1'$ is injective and

b) $D' \cap f(D_1) = f(D)$

Proof : Recall the results about the unique writings for elements of a free amalgamated product $G = E*_H F$. Choosing sets of representatives $\bar{E}$ and $\bar{F}$ for the right cosets $H\backslash E$ and $H\backslash F$, which 1 as the representative of H, then any element g of G can be uniquely written as $g = he_1f_1 \ldots e_kf_k$ with $h \in H$, $e_i \in \bar{E}$ and $e_i \neq 1$ if $i > 1$, $f_i \in \bar{F}$ and $f_i \neq 1$ if $i < k$.

In our situation, the corresponding sets of representatives $\bar{A}'_1$ and $\bar{B}'$ for $C'\backslash A'_1$ and $C'\backslash B'$ can be, because of the condition $C' \cap f(A_1) = C' \cap f(B) = f(C)$, chosen of the form

$$A_1 = \bar{A} \amalg T \amalg \hat{A}'_1 \quad , \quad B' = \bar{B} \amalg \hat{B}'$$

where

$$\bar{A} = \text{set of representatives for } C\backslash A$$

$$\bar{A} \amalg T = \text{" \quad " \quad " \quad "} \ C\backslash A_1$$

$$\bar{B} = \text{" \quad \quad \quad "} \ C\backslash B$$

Using the unicity of the writing with such A_1 and B' , it is easy to deduce Conditions a) and b). □

2. SOME ACYCLIC GROUPS IN $\mathbb{D}$.

This section is devoted to the proof of the following proposition:

(2.1) Proposition. For $r \geqq 2$, there is a group $\Omega^r \in \mathbb{D}(r)$ such that:

a) $H_*(B\Omega^r;\mathbb{Z})$ (the integers $\mathbb{Z}$ are endowed with the trivial G-action). We say that Ω^r is an acyclic group.

b) $\Omega^r \times \mathbb{Z}$ is a subgroup of Ω^{r+1} .

Proof: Define a group R by the presentation:

$$R = \langle z,t \mid tzt^{-1} = z^2 \rangle .$$

The group R is not a non-trivial free product. Indeed, as $R/[R,R]$ is infinite cyclic, one of the free summand should be a non-trivial perfect group. But R is solvable, being the semi-direct product of $\mathbb{Z}$ with $\mathbb{Z}[1/2]$. Hence, by Lemma (1.3), $R \in \mathbb{D}(2)$.

Let R_i $(i = 1,2,3,4)$ be copies of R , with generators t_i and z_i. Form the free amalgamated products:

$$S = R_1 *_{(z_1 = t_2)} R_2$$
$$T = R_3 *_{(t_3 = z_4)} R_4$$

By Lemma (2.1), the groups S and T are in $\mathbf{D}(2)$. The free group $F = \langle u,v \rangle$ admits monomorphisms $j_S : F \longrightarrow S$ and

$j_T : F \longrightarrow T$ as follows:

$$j_S(u) = t_1 \qquad j_T(u) = z_3$$

$$j_S(v) = z_2 \qquad j_T(v) = t_4$$

Define $\Omega^2 = S *_F T$. As $F \in \mathbf{D}(1)$, the group Ω^2 is in $\mathbb{D}(2)$ by Lemma (1.1). One checks by Mayer-Vietoris sequences that Ω^2 is acyclic. (The reader may have recognize in Ω^2 the group invented by Higman [Hi]).

The group:

$$A = \langle a,b,c,d \mid [a,b][c,d]d^{-1} = 1 \rangle$$

is not a non-trivial free product. Indeed, the relator $[a,b][c,d]d^{-1}$ is the product of the disjoint minimal word $[a,b]$ and $[c,d]d^{-1}$. The conclusion then follows from [Sh, Theorem 1 and 2]. Therefore, $A \in \mathbf{D}(2)$, by Lemma (1.2).

Let $1 \neq w \in \Omega^2$. Define

$$B = A * \Omega^2/\{e = w\}$$

$$C_a = B * \Omega^2/\{b = w\}$$

$$C_b = B * \Omega^2/\{a = w\}$$

One has:

a) B, C_a and C_b are groups in $D(2)$

b) $H_1(B) = Z \oplus Z$, generated by a and b

$H_1(C_a) = Z$, generated by a

$H_1(C_b) = Z$, generated by b

c) $H_2(B) = H_2(C_a) = H_2(C_b) = 0$.

Thus, C_a and C_b are homology circles in $D(2)$. They can be embedded in homology circles $D_a, D_b \in D(3)$ by forming the push-out diagram

$$\begin{array}{ccc} \langle u\rangle \times \langle v\rangle & \longrightarrow & \Omega^2 \times \langle v\rangle \\ \downarrow & & \downarrow \\ \langle u\rangle \times C_a & \rightarrowtail & D_a \end{array}$$

where, in the left vertical arrow, v is sent to a . The same construction is used for D_b . The inclusions $C_a \rightarrowtail D_a$ and $C_b \rightarrowtail D_b$ are homology isomorphisms. Consider the two composed inclusions $B \rightarrowtail C_a \rightarrowtail D_a$ and $B \rightarrowtail C_b \rightarrowtail D_b$. Define Ω^3 by the push-out diagram

$$\begin{array}{ccc} B & \longrightarrow & D_a \\ \downarrow & & \downarrow \\ D_b & \longrightarrow & \Omega^3 \end{array}$$

The properties of Ω^3 are the following:

1) $\Omega^3 \in \mathbf{D}(3)$, by Lemma (1.1)

2) Ω^3 is acyclic (use Mayer-Vietoris sequence)

3) One has $Z \times \Omega^2 \rightarrowtail D_a \longrightarrow \Omega^3$.

Suppose now by induction that Ω^i is defined for $2 \leqq i \leqq n-1 \geqq 3$. Define Ω^n by the push-out diagram

$$\begin{array}{ccc} \langle w \rangle \times \Omega^{n-2} & \longrightarrow & \langle w \rangle \times \Omega^{n-1} \\ \downarrow & & \downarrow \\ \Omega^2 \times \Omega^{n-2} & \longrightarrow & \Omega^n \end{array}$$

It is easily checked that this sequence of Ω^i enjoys all the properties of Proposition (2.1).

□

3. PROOF OF THEOREM A.

For X a polyedron, we denote by $S(X)$ the category whose objects are the subpolyedron of X and whose morphisms are inclusions. We denote by m_X the number of simplexes of X of dimension ≥ 1 .

For each integer $r \geq 0$, we consider the following category $C(r)$: an object of $C(r)$ is a polydron U such that each connected component U_i of U has the homotopy type of $B\pi_1(U_i)$ and $\pi_1(U_i)$ is in $D(r)$. We shall write $U \in C(r)$ to say that U is an object of $C(r)$. A morphism of $C(r)$ between U and V is an inclusion of U as a subpolyedron of V such that the induced homomorphism from $\pi_1(U,u)$ to $\pi_1(V,u)$ is injective for all $u \in U$.

Theorem A is a direct consequence of the following proposition:

(3.1) PROPOSITION. Let X be a polyedron, then there exists an integer r and

a) a covariant functor $L : S(X) \longrightarrow C(r)$

b) for each connected subpolyedron Y of X , there is an acyclic map $\beta_Y : L(Y) \longrightarrow Y$. The group $\ker(\pi_1(L(Y) \longrightarrow \pi_1(Y))$ is locally perfect. Moreover, if $Y \subset Y'$ is an inclusion of connected subpolyedra of X , the following diagram

$$\begin{array}{ccc} L(Y) & \subset & L(Y') \\ \Big\downarrow{\scriptstyle \beta_Y} & & \Big\downarrow{\scriptstyle \beta_{Y'}} \\ Y & \subset & Y' \end{array}$$

is commutative.

c) a covariant functor $M : S(X) \longrightarrow C(r)$ such that:

c1) For any subpolydron Y of $X, M(Y)$ is an acyclic space (i.e. $H_*(M(Y);Z) = 0$ for $* \geq 1$).

c2) there is a natural transformation of functors $L \longrightarrow M$ (i.e., there are inclusions $L(Y) \subset M(Y)$ in M or $C(r)$ and, when $Y \subset Y'$, the following diagram

$$\begin{array}{ccc} L(Y) & \subset & L(Y') \\ \cap & & \cap \\ M(Y) & \subset & M(Y') \end{array}$$

is commutative.

c3) For each supolyedra $Y \subset Y'$ of X, one has, for any $y \in L(Y)$:

$$\pi_1(M(Y),y) \cap \pi_1(L(Y'),y) = \pi_1(L(Y),y)$$

<u>Proof:</u> The proof is by induction on m_X. The induction start with the case $m_X = 0$ (i.e. X is discrete), where we set $L(Y) = Y$, $B_Y = id_Y$ and $M(Y) = cY$, the cone over Y.

The induction step consists of the following: Suppose that (r, L, β_Y, M) as above is constructed for a poyedron X. We then construct $(\bar{r}, \bar{L}, \bar{\beta}_Y, \bar{M})$ for a polyedron $\bar{X} = X \cup e$, where e is a simplex of dimension ≥ 1. This will be done in several steps.

<u>Step 1.</u> For Y a subpolyedron of $\bar{X}$, define

$$L_1(Y) = \begin{cases} L(Y) \times B\Omega^3 & \text{if } Y \subset X . \\ [L(Y \cap X) \times B\Omega^3] \cup_{L(\partial e) \times B\Omega^2} [M(\partial e) \times B\Omega^3]; & \text{otherwise,} \end{cases}$$

To simplify the notation we set

$$L_1(\overset{\circ}{e}) = M(\partial e) \times B\Omega^3 \quad \text{and} \quad \hat{L}_1(\partial e) = L(\partial e) \times B\Omega^2 .$$

We can thus write, if $Y \subset X$:

$$L_1(Y) = L(Y \cap X) \times B\Omega^3 \cup_{\hat{L}_1(\partial e)} L_1(\overset{\circ}{e}) .$$

It follows from the results of §1 that L_1 is a covariant functor from $S(X)$ to $C(r+3)$. If Y is a connected subpolyedron of X, the acyclic map $\beta_Y^1 : L_1(Y) \longrightarrow Y$ is defined as the composition:

$$L(Y \cap X) \times B^3 \longrightarrow L(Y \cap X) \xrightarrow{\amalg \beta_Z} Y \quad (Z \text{ connected component of } Y \cap X)$$

which can be extended to the part $L_1(\overset{\circ}{e})$ since e is contractible. The group $\ker \pi_1 \beta_Y^1$ is normally generated by $\ker \pi_1 \beta_Y$, copies of Ω^3 and $\pi_1(M(\partial e))$. It is therefore locally perfect. The functor L_1 then satisfies to conditions a) and b). At this stage, the functor M_1 will be defined only on $S(X)$ by $M_1(Y) = M(Y) \times B\Omega^3$ for $Y \subset X$.

Step 2 One defined a space $\check{M}_2(e)$ as follows :

$$\check{M}_2(e) = \left(\hat{L}_1(\partial e) \times B\Omega^3\right) \cup_{\hat{L}_1(\partial e) \times B\Omega^2} \left(L_1(\overset{\circ}{e}) \times B\Omega^2\right)$$

It follows from § 1 that $\check{M}_2(e) \in C(r+5)$. By the Mayer-Vietoris sequence, the space $\check{M}_2(e)$ is acyclic.

To keep some coherence in the notation, write :

$$L_2(Y) = L_1(y) \times B\Omega^2 \text{ , for } Y \subset \bar{X}, \quad L_2(Y) \in C(r+5)$$

$$M_2(Y) = M_1(Y) \times B\Omega^2 \text{ , for } Y \subset X, \quad M_2(Y) \in C(r+5)$$

$$\hat{L}_2(\partial e) = \hat{L}_1(\partial e) \times B\Omega^2 \in C(r+4)$$

$$L_2(\overset{\circ}{e}) = L_1(\overset{\circ}{e}) \times B\Omega^2 \in C(r+5).$$

The inclusion $L_2(\overset{\circ}{e}) \subset \check{M}_2(e)$ induced a monomorphism

$$j_L : \pi_1(L_2(\overset{\circ}{e})) \longrightarrow \pi_1(\check{M}_2(e))$$

On the other hand define a monomorphism

$$j_M : \pi_1(M_2(\partial e)) \longrightarrow \pi_1(\check{M}_2(e))$$

by sending $(x,v) \in \pi_1(M_1(\partial e)) \times \Omega^2 = \pi_1(M_2(\partial e))$ to $(1,a)\overline{(x,v)}(1,a^{-1})$ where $\overline{(x,v)}$ is the image of (x,v) under the identification $\pi_1(M_1(\partial e)) \times \Omega^2 = \pi_1(L_2(\overset{\circ}{e})) \xrightarrow{j_L} \pi_1(\check{M}_2(e))$ and $(1,a) \in \pi_1(\hat{L}_1(\partial e)) \times \Omega^3$ with $a \in \Omega^3$ an element commuting with those of $\Omega^2 \subset \Omega^3$ (such an element a exists, since $\Omega^2 \times Z \subset \Omega^3$ by Proposition (2.1)). This property of a implies that j_L and j_M coincide on $\pi_1(\hat{L}_2(\partial e))$, and then produce a homomorphism j :

$\pi_1(K) \longrightarrow \pi_1(\check{M}_2(e))$, where K is the space

$$K = M_2(\partial e) \cup_{\hat{L}_2(\sigma e)} L_2(\overset{\circ}{e}).$$

Since a commutes with the elements of Ω^2 in Ω^3 , one checks using the unicity of the reduced writing in free amalgamated products, that j is injective. The monomorphism j can be realized by an inclusion of K as a subpolyedron of a polyedron $M_2(e)$ having the same homotopy type as $\check{M}_2(e)$. This inclusion is then a morphism of C(r+5).

<u>Step 3</u> Define, for each subpolyedron Y of X

$$M_3(Y) = \begin{cases} M_2(Y) \times B\Omega^3 & \text{, if } Y \subset X \\ M_2(Y \cap X) \times B\Omega^3 \cup_{M_2(\partial e) \times B\Omega^2} \dot{M}_2(e) \times B\Omega^3, & \text{otherwise} \end{cases}$$

where the inclusion of $M_2(\sigma e)$ into $M_2(e)$ is the one defined in Step 2, via the space K. The space $M_3(Y)$ contains $L_3(Y) = L_2(Y) \times B\Omega^2$. This is obvious when $Y \subset X$. When Y contains the simplex e, one has

$$L_3(Y) = L_2(Y\ X) \times B\Omega^2 \cup_{\hat{L}_2(\partial e) \times B\Omega^2} L_2(\overset{\circ}{e}) \times B\Omega^2$$

and therefore there is an inclusion $L_3(Y) \subset M_3(Y)$ inducing the obvious inclusion $L_2(Y \cap X) \times B\Omega^2 \subset M_2(Y \cap X) \times B\Omega^3$, and the inclusion $L_2(\overset{\circ}{e}) \subset M_2(e)$ defined in Step 2. One check easily that

$$[\pi_1(M_2(\sigma e) \times \Omega^2] \cap \pi_1(L_2(\overset{\circ}{e})) \times \Omega^2 = \pi_1(\hat{L}_2 \sigma e)) \times \Omega^2.$$

On the other hand , one has $\left[\pi_1(M_2(\partial e)) \times \Omega^2\right] \cap \left(\pi_1(L_2(Y \cap X)) \times \Omega^2\right) = \pi_1(\hat{L}_2(\partial e)) \times \Omega^2$, by Condition c3). By Lemma (1.4), this implies that the inclusion $L_3(Y) \subset M_3(Y)$ induces a monomorphism on the fundamental groups. Also, if $Y \subset Y'$ are subpolydra of X , one has a commutative diagram:

$$\begin{array}{ccc} L_3(Y) & \subset & L_3(Y') \\ \cap & & \cap \\ M_3(Y) & \subset & M_3(Y') \end{array}$$

The inclusiôn of the L_3's is a morphism of $C(r+7)$ and the inclusi of the M_3's is a morphism of $C(r+8)$. By Part b) of Lemma (1.4), one deduces that Condition c3) is verified for (M_3,L_3) . But we need at last adaption in order to have a $\bar{L}(Y)$ and a $\bar{M}(Y)$ in the same class $C(\bar{r})$.

<u>Step 4</u> For Y a subpolyedron of $\bar{X}$, define $\bar{L}(Y) = L_3(Y)\times B\Omega^3$ and define $M(Y)$ by the push-out diagram

$$\begin{array}{ccc} L_2(Y) \times B\Omega^2 & \subset & \bar{L}(Y) \\ \cap & & \cap \\ M_3(Y) \times B\Omega^2 & \subset & \bar{M}(Y) \end{array}$$

It follows from § 1 that the inclusion $\bar{L}(Y) \subset \bar{M}(Y)$ is a morphism of $C(r+10)$. Set $\bar{r} = r + 10$. Condition c1) is checked by Mayer-Vietoris sequence. If $Y \subset Y'$ are subpolyedra of X , there is a commutative diagram of inclusions as in c2). The following

$$[\P_1(L_3(Y'))\times\Omega^2] \cap [\P_1(L_3(Y))\times\Omega^3] = \P_1(L_3(Y))\times\Omega^2$$

is obvious and the following

$$[\P_1(M_3(Y))\times\Omega^2] \cap [\P_1(L_3(Y'))\times\Omega^2] = \P_1(L_3(Y))\times\Omega^2$$

comes from Condition c3) for (M_3,L_3) which was established in step 3. Therefore, using Part a) of Lemma (1.4), one deduces that the inclusion $\bar{M}(Y) \subset \bar{M}(Y')$ is a morphism of $C(r)$. Part b) of Lemma (1.4) permits us to check Condition c3) for $(\bar{M},\bar{L})$. Observe that $\bar{L}(Y) = L_1(Y)\times B\Omega^2\times B\Omega^2\times B\Omega^3$, and therefore there is a projection $\bar{L}(Y)\longrightarrow L_1(Y)$. The composition of this projection with β_Y^1 , for y connected, gives the required acyclic map $\bar{\beta}_Y : \bar{L}(Y)\longrightarrow Y$. Therefore, $(\bar{r},\bar{L},\bar{\beta}_Y,\bar{M})$ is constructed for $\bar{X}$, which acheaves the proof of $\square$.

REMARKS

(3.2) The principle of the proof of Proposition (3.1) is essentially the same as in [Ma] but a much stronger control of the successive amalgamation is necessary in order to stay in the classes $C(r)$.

(3.3) The proof of Proposition (3.1) gives a (presumably very weak) majoration of the integer $r(X)$ defined in a problem stated in the introduction. Suppose that X is a polyedron such that $n_X = m(X)$, the minimal number of simplexes of positive dimension of any poledron in the homotopy type of X . Write $X^{(0)} = X_0 \subset X_1 \subset \ldots \subset X_{r(X)} = X$, where $X_i = X_{i-1} \cup e$, with e a simplex of positive dimension. Then, the spaces $L(X_i)$ of the proof of (3.1) are in $C(10i)$. Observe that for $L(X_{m(X)})$, we only need the first step, since there is no need of the space $M(X)$. Therefore, the proof of (3.1) produces a group $G \in D(r)$ with an acyclic map $BG \longrightarrow X$ with $r = 10(m(X)-1) + 3 = 10m(X)-7$. This gives the inequality $r(X) \leq 10m(X) - 7$.

(3.4) For connected CW-complex X , let $P(X)$ be the subset of integers r such that there exists $G_r \in D(r)$ with an acyclic map $BG \longrightarrow X$. One has

(3.4.1) $P(pt) = \{0\} \cup [2,\infty[$, by Lemma (2.1) and the fact that $D(1)$ is the class of finitely generated free groups therefore contains no acyclic group).

(3.4.2) If $\{r,r+1\} \subset P(X)$ and $G_r \subset G_r X_1$, then $[r,\infty[\subset P(X)$. This comes the last argument of the proof of Proposition (2.1).

(3.4.3) $P(S^1) = [1,\infty[$ by (3.3.2) and the proof of Proposition (2.1)

<u>Problem</u> : If $X \neq pt$, is $P(X) = [r(X),\infty[$?

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