THE SYMMETRIC ACTION ON SECONDARY HOMOTOPY GROUPS

HANS-JOACHIM BAUES AND FERNANDO MURO

Abstract. We show that the symmetric track group $\mathrm{Sym}_\square(n)$, which is an extension of the symmetric group $\mathrm{Sym}(n)$ associated to the second Stiefel-Withney class, acts as a crossed module on the secondary homotopy group of a pointed space.

Introduction

Secondary homotopy operations like Toda brackets [Tod62] or cup-one-products [BJM83], [HM93], are defined by pasting tracks, where tracks are homotopy classes of homotopies. Since secondary homotopy operations play a crucial role in homotopy theory it is of importance to develop the algebraic theory of tracks. We do this by introducing secondary homotopy groups of a pointed space X

$$\Pi_{n,*}X = \left(\Pi_{n,1}X \xrightarrow{\partial} \Pi_{n,0}X\right)$$

which have the structure of a quadratic pair module, see Section 1. Here ∂ is a group homomorphism with cokernel $\pi_n X$ and kernel $\pi_{n+1}X$ for $n \geq 3$.

We define $\Pi_{n,*}X$ for $n \geq 2$ directly in terms of maps $S^n \to X$ and tracks from such maps to the trivial map. For $n \geq 0$ the functor $\Pi_{n,*}$ is an additive version of the functor $\pi_{n,*}$ studied in [BM05a]. The homotopy category of $(n-1)$-connected $(n+1)$-types is equivalent via $\Pi_{n,*}$ to the homotopy category of quadratic pair modules for $n \geq 3$.

In this paper we consider the "generalized coefficients" of secondary homotopy groups $\Pi_{n,*}X$ obtained by the action of the symmetric group $\mathrm{Sym}(n)$ on $S^n = S^1 \wedge \overset{n}{\cdots} \wedge S^1$ via permutation of coordinates. For a permutation $\sigma \in \mathrm{Sym}(n)$ the map $\sigma\colon S^n \to S^n$ has degree $\operatorname{sign}\sigma \in \{\pm 1\}$. The group $\{\pm 1\}$ also acts on S^n by using the topological abelian group structure of S^1 and suspending $(n-1)$ times. This shows that there are tracks $\sigma \Rightarrow \operatorname{sign}\sigma$ which by definition are the elements of the symmetric track group $\mathrm{Sym}_\square(n)$. Also these tracks act on $\Pi_{n,*}X$. We clarify this action by showing that the group $\mathrm{Sym}_\square(n)$ gives rise to a crossed module which acts as a crossed module on the quadratic pair module $\Pi_{n,*}X$.

The symmetric track group is a central extension

$$\mathbb{Z}/2 \hookrightarrow \mathrm{Sym}_\square(n) \overset{\delta}{\twoheadrightarrow} \mathrm{Sym}(n)$$

1991 *Mathematics Subject Classification.* 55Q25, 55S45.

Key words and phrases. secondary homotopy groups, crossed module, square group, cup-one-product.

The second author was partially supported by the project MTM2004-01865 and the MEC postdoctoral fellowship EX2004-0616.

which, as we show, represents the second Stiefel-Withney class pulled back to $\mathrm{Sym}(n)$. The symmetric track group is computed in Section 6. We actually compute a faithful positive pin representation of $\mathrm{Sym}_\square(n)$ from which we derive a finite presentation of this group. This group also arose in a different way in the work of Schur [Sch11] and Serre [Ser84].

In [BM05c] we describe the smash product operation on secondary homotopy groups $\Pi_{n,*}X$. This operation endows $\Pi_{*,*}$ with the structure of a lax symmetric monoidal functor where the crossed module action of $\mathrm{Sym}_\square(n)$ on $\Pi_{n,*}X$ is of crucial importance. This leads to an algebraic approximation of the symmetric monoidal category of spectra by secondary homotopy groups, see [BM05b]. As an example we prove a formula for the unstable cup-one-product $\alpha \cup_1 \alpha \in \pi_{2n+1}S^{2m}$ of an element $\alpha \in \pi_n S^m$ where n and m are even. We show that

$$2(\alpha \cup_1 \alpha) = \frac{n+m}{2}(\alpha \wedge \alpha)(\Sigma^{2(n-1)}\eta)$$

where $\eta\colon S^3 \to S^2$ is the Hopf map. If $n/2$ is odd and $m/2$ is even then this formula was achieved by totally different methods in [BJM83].

1. Square groups and quadratic pair modules

In this section we describe the algebraic concepts needed for the structure of secondary homotopy groups.

Definition 1.1. A *square group* X is a diagram

$$X = (X_e \underset{H}{\overset{P}{\leftrightarrows}} X_{ee})$$

where X_e is a group with an additively written group law, X_{ee} is an abelian group, P is a homomorphism, H is a function such that the crossed effect

$$(a|b)_H = H(a+b) - H(b) - H(a)$$

is linear in a and $b \in X_e$, and the following relations are satisfied, $x, y \in X_{ee}$,

(1) $(Px|b)_H = 0$, $(a|Py) = 0$,
(2) $P(a|b)_H = -a - b + a + b$,
(3) $PHP(x) = P(x) + P(x)$.

The function

$$T = HP - 1\colon X_{ee} \longrightarrow X_{ee}$$

is an involution, i. e. a homomorphism with $T^2 = 1$.

A morphism of square groups $f\colon X \to Y$ is given by homomorphisms

$$f_e\colon X_e \longrightarrow Y_e,$$
$$f_{ee}\colon X_{ee} \longrightarrow Y_{ee},$$

commuting with P and H.

Let **SG** be the category of square groups. A square group X with $X_{ee} = 0$ is the same as an abelian group X_e. This yields the full inclusion of categories $\mathbf{Ab} \subset \mathbf{SG}$ where **Ab** is the category of abelian groups.

Square groups were introduced in [BP99] to describe quadratic endofunctors of the category **Gr** of groups. More precisely, any square group X gives rise to a quadratic functor

$$-\otimes X\colon \mathbf{Gr} \longrightarrow \mathbf{Gr}.$$

Given a group G the group $G \otimes X$ is generated by the symbols $g \otimes x$ and $[g,h] \otimes z$, $g, h \in G$, $x \in X_e$, $z \in X_{ee}$ subject to the relations

$$\begin{array}{rcl} (g+h)\otimes x &=& g\otimes x + h\otimes x + [g,h]\otimes H(x), \\ {[g,g]}\otimes z &=& g\otimes P(z), \end{array}$$

where $g \otimes x$ is linear in x and $[g,h]\otimes z$ is central and linear in each variable g, h, z. If X is an abelian group then $G \otimes X = G_{ab} \otimes X_e$. In fact, any quadratic functor $F\colon \mathbf{Gr} \to \mathbf{Gr}$ which preserves reflexive coequalizers and filtered colimits has the form $F = - \otimes X$, see [BP99]. The theory of square groups is discussed in detail in [BJP05].

There is a natural isomorphism

$$X_e \xrightarrow{\cong} \mathbb{Z}\otimes X, \quad x \mapsto 1 \otimes x.$$

In particular the homomorphism $n\colon \mathbb{Z} \to \mathbb{Z}$ induces a homomorphism $n^*\colon X_e \to X_e$ fitting into the following commutative diagram

(1.2)
$$\begin{array}{ccc} \mathbb{Z}\otimes X & \xrightarrow{n\otimes X} & \mathbb{Z}\otimes X \\ \cong\big\uparrow & & \big\uparrow\cong \\ X_e & \xrightarrow{\quad n^* \quad} & X_e \end{array}$$

The homomorphism n^* is explicitly given by the following formula,

$$n^* x = n\cdot a + \binom{n}{2} PH(x).$$

Here we set $\binom{n}{2} = \frac{n(n-1)}{2}$ and for any additively written group G and any $n \in \mathbb{Z}$, $g \in G$,

$$n\cdot g = \begin{cases} g + \overset{n}{\cdots} + g, & \text{if } n \geq 0; \\ \\ -g - \overset{-n}{\cdots} - g, & \text{if } n < 0. \end{cases}$$

The function $n\cdot : G \to G$ in general is not a homomorphism, but if G is abelian then $n\cdot$ is a homomorphism. This homomorphism is generalized by n^* in (1.2) for square groups.

Definition 1.3. A *quadratic pair module* C is a morphism $\partial\colon C_{(1)} \to C_{(0)}$ between square groups

$$\begin{array}{rcl} C_{(0)} &=& (C_0 \underset{H}{\overset{P_0}{\leftrightarrows}} C_{ee}), \\ C_{(1)} &=& (C_1 \underset{H_1}{\overset{P}{\leftrightarrows}} C_{ee}), \end{array}$$

such that $\partial_{ee} = 1\colon C_{ee} \to C_{ee}$ is the identity homomorphism. In particular ∂ is completely determined by the diagram

(1.4)
$$\begin{array}{ccc} & C_{ee} & \\ {\scriptstyle P}\swarrow & & \nwarrow{\scriptstyle H} \\ C_1 & \xrightarrow[\partial]{\qquad} & C_0 \end{array}$$

where $\partial = \partial_e$, $H_1 = H\partial$ and $P_0 = \partial P$.

Morphisms of quadratic pair modules $f\colon C \to D$ are therefore given by group homomorphisms $f_0\colon C_0 \to D_0$, $f_1\colon C_1 \to D_1$, $f_{ee}\colon C_{ee} \to D_{ee}$, commuting with H, P and ∂ in (1.4) as in the diagram

$$\begin{array}{ccccccc} C_0 & \xrightarrow{H} & C_{ee} & \xrightarrow{P} & C_1 & \xrightarrow{\partial} & C_0 \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_{ee}} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\ D_0 & \xrightarrow{H} & D_{ee} & \xrightarrow{P} & D_1 & \xrightarrow{\partial} & D_0 \end{array}$$

They form a category denoted by **qpm**.

Quadratic pair modules are also the objects of a bigger category **wqpm** given by weak morphisms. A *weak morphism* $f\colon C \to D$ between quadratic pair modules is given by three homomorphisms f_0, f_1, f_{ee} as above, but we only require the following two diagrams to be commutative

$$\begin{array}{ccc} C_{ee} & \xrightarrow{T} & C_{ee} \\ \downarrow{\scriptstyle f_{ee}} & & \downarrow{\scriptstyle f_{ee}} \\ D_{ee} & \xrightarrow{T} & D_{ee} \end{array} \qquad \begin{array}{ccccccc} \otimes^2(C_0)_{ab} & \xrightarrow{(-|-)_H} & C_{ee} & \xrightarrow{P} & C_1 & \xrightarrow{\partial} & C_0 \\ \downarrow{\scriptstyle \otimes^2(f_0)_{ab}} & & \downarrow{\scriptstyle f_{ee}} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\ \otimes^2(D_0)_{ab} & \xrightarrow{(-|-)_H} & D_{ee} & \xrightarrow{P} & D_1 & \xrightarrow{\partial} & D_0 \end{array}$$

Here $\otimes^2 A = A \otimes A$ denotes the tensor square of an abelian group. Therefore $\mathbf{qpm} \subset \mathbf{wqpm}$ is a subcategory with the same objects.

Let $(\mathbb{Z}, \cdot)$ be the multiplicative (abelian) monoid of the integers $\mathbb{Z}$.

Definition 1.5. Any quadratic pair module C admits an action of $(\mathbb{Z}, \cdot)$ given by the morphisms $n^*\colon C \to C$ in **wqpm**, $n \in \mathbb{Z}$, defined by the equations

- $n^*x = n \cdot x + \binom{n}{2}\partial PH(x)$ for $x \in C_0$,
- $n^*y = n \cdot y + \binom{n}{2}PH\partial(y)$ for $y \in C_1$,
- $n^*z = n^2 z$ for $z \in C_{ee}$.

We point out that $n^*\colon C \to C$ is an example of a weak morphism which is not a morphism in **qpm** since n^* is not compatible with H. Notice that $n^*\colon C_0 \to C_0$ and $n^*\colon C_1 \to C_1$ are induced by the square group morphisms $n \otimes C_{(0)}$ and $n \otimes C_{(1)}$ respectively, see diagram (1.2). We emphasize that this action is always defined for any quadratic pair module C and it is natural in the following sense, for any morphism $f\colon C \to D$ in **qpm** and any $n \in \mathbb{Z}$, the equality

$$fn^* = n^*f$$

holds. This property does not hold if f is a weak morphism. The existence of this action should be compared to the fact that abelian groups are $\mathbb{Z}$-modules.

The category **squad** of stable quadratic modules is described in [Bau91] IV.C and [BM05a]. Quadratic modules in general are discussed in [Bau91] and [Ell93], they are special 2-crossed modules in the sense of [Con84]. There is a faithful forgetful functor from quadratic pair modules and weak morphisms to stable quadratic modules

$$\mathbf{wqpm} \longrightarrow \mathbf{squad} \tag{1.6}$$

sending C as in Definition 1.3 to the stable quadratic module

$$\otimes^2(C_0)_{ab} \stackrel{P(-|-)_H}{\longrightarrow} C_1 \stackrel{\partial}{\longrightarrow} C_0. \tag{1.7}$$

In this paper G_{ab} is the abelianization of a group G and G_{nil} is its projection of G to the variety of groups of nilpotency class 2.

A track category is a groupoid-enriched category, which is also a 2-category where all 2-morphisms (also termed tracks) are vertically invertible. The category $\mathbf{Top}^*$ of pointed spaces is known to be a track category with tracks given by homotopy classes of homotopies. The vertical composition in track categories is denoted by $\square$, and the vertical inverse of a track α is $\alpha^{\boxminus}$.

The forgetful functor (1.6) can be used to pull-back to **wqpm** the track category structure on **squad** introduced in [BM05a] 6. The track structure on **squad** was already a pull-back along the forgetful functor

$$\mathbf{squad} \longrightarrow \mathbf{cross} \tag{1.8}$$

from stable quadratic modules to crossed modules considered also in [BM05a] 6.

Definition 1.9. We recall that a *crossed module* $\partial\colon M \to N$ is a group homomorphism such that N acts on the right of M (the action will be denoted exponentially) and the homomorphism ∂ satisfies the following two properties ($m, m' \in M, n \in N$):

(1) $\partial(m^n) = -n + \partial(m) + n$,
(2) $m^{\partial(m')} = -m' + m + m'$.

The crossed module associated via (1.6) and (1.8) to a quadratic pair module C is given by the homomorphism

$$\partial\colon C_1 \longrightarrow C_0,$$

where C_0 acts on the right of C_1 by the formula, $x \in C_1$, $y \in C_0$,

$$x^y = x + P(\partial(x)|y)_H. \tag{1.10}$$

Definition 1.11. A *track* $\alpha\colon f \Rightarrow g$ between two morphisms $f, g\colon C \to D$ in **wqpm** is a function

$$\alpha\colon C_0 \longrightarrow D_1$$

satisfying the equations, $x, y \in C_0$, $z \in C_1$,

(1) $\alpha(x + y) = \alpha(x)^{f_0(y)} + \alpha(y)$,
(2) $g_0(x) = f_0(x) + \partial\alpha(x)$,
(3) $g_1(z) = f_1(z) + \alpha\partial(z)$.

Tracks in **qpm** are tracks in **wqpm** between morphisms in the subcategory **qpm** $\subset$ **wqpm**.

Proposition 1.12. *The categories* **wqpm** *and* **qpm** *are track categories with the tracks in Definition 1.11.*

This proposition is a direct consequence of [BM05a] 6.4. Vertical and horizontal compositions are defined in the proof of [BM05a] 6.4.

The following result shows that the weak action of $(\mathbb{Z}, \cdot)$ defined above is also natural with respect to tracks in **qpm**.

Proposition 1.13. *Let $f, g\colon C \to D$ be morphisms in* **qpm** *and let $\alpha\colon g \Rightarrow f$ be a track as in Definition 1.11. Then the following diagram commutes*

$$\begin{array}{ccc} C_0 & \xrightarrow{\alpha} & D_1 \\ {\scriptstyle n^*}\downarrow & & \downarrow{\scriptstyle n^*} \\ C_0 & \xrightarrow{\alpha} & D_1 \end{array}$$

Given a pointed set E with base point $* \in E$ we denote by $\langle E\rangle_{nil}$ and $\mathbb{Z}[E]$ to the free group of nilpotency class 2 and to the free abelian group generated by E with $* = 0$ respectively.

Definition 1.14. A quadratic pair module C is said to be 0-*free* if $C_0 = \langle E\rangle_{nil}$, $C_{ee} = \otimes^2\mathbb{Z}[E]$ and H is determined by the equalities $H(e) = 0$ for any $e \in E$ and $(s|t)_H = t \otimes s$ for any $s, t \in \langle E\rangle_{nil}$.

The next lemma shows that 0-free stable quadratic modules are in the image of the forgetful functor in (1.6).

Lemma 1.15. *Any* 0*-free stable quadratic module*

$$\otimes^2\mathbb{Z}[E] \xrightarrow{\omega} M \xrightarrow{\partial} \langle E\rangle_{nil}$$

gives rise to a 0*-free quadratic pair module*

$$\begin{array}{ccccc} & & \otimes^2\mathbb{Z}[E] & & \\ & \swarrow^{P} & & \nwarrow^{H} & \\ M & & \xrightarrow[\partial]{\qquad\qquad} & & \langle E\rangle_{nil} \end{array}$$

with $P(a \otimes b) = \omega(b \otimes a)$.

Later we will need the following technical lemma which measures the lack of compatibility of certain tracks in **wqpm** with the action of $(\mathbb{Z}, \cdot)$.

Lemma 1.16. *Let* C *be a* 0*-free quadratic pair module with* $C_0 = \langle E\rangle_{nil}$, *let* $f\colon C_0 \to C_0$ *be an endomorphism induced by a pointed map* $E \to E$, *and let* $\alpha\colon C_0 \to C_1$ *be a map satisfying*

$$\begin{aligned} \alpha(x+y) &= \alpha(x)^{f(y)} + \alpha(y), \\ m^*x &= f(x) + \partial\alpha(x), \end{aligned}$$

for some $m \in \mathbb{Z}$ *and any* $x, y \in C_0$. *Then the following formula holds for any* $n \in \mathbb{Z}$ *and* $x \in C_0$.

$$\alpha(n^*x) = n^*\alpha(x) + \binom{m}{2}\binom{n}{2}P(x|x)_H.$$

Proof. We first check that the lemma holds for $x+y$ provided it holds for $x, y \in C_0$.

$$\begin{aligned} \alpha n^*(x+y) &= \alpha(n^*x + n^*y) \\ &= \alpha(n^*x)^{f(n^*y)} + \alpha(n^*y) \\ &= n^*\alpha(x) + n^*\alpha(y) + \binom{m}{2}\binom{n}{2}P(x|x)_H \\ &\quad \binom{m}{2}\binom{n}{2}P(y|y)_H + P(-f(n^*x) + n^*m^*x|f(n^*y))_H \\ &= n^*(\alpha(x) + \alpha(y)) + n^2P(-f(x) + m^*x|fy)_H \\ &\quad + \binom{m}{2}\binom{n}{2}P(x+y|x+y)_H \\ &= n^*(\alpha(x)^{f(y)} + \alpha(y)) + \binom{m}{2}\binom{n}{2}P(x+y|x+y)_H \\ &= n^*\alpha(x+y) + \binom{m}{2}\binom{n}{2}P(x+y|x+y)_H. \end{aligned}$$

Here we use that f is compatible with the action of $(\mathbb{Z}, \cdot)$ and that $P(x|x)_H$ is linear in x.

Now since $C_0 = \langle E \rangle_{nil}$ we only need to check that the proposition holds for $e \in E$. But $H(e) = 0$, so we have $n^*e = n \cdot e$. The equality

$$\alpha(n \cdot e) \quad = \quad n \cdot \alpha(e) + \binom{n}{2} P(f(e)|f(e))_H + m \binom{n}{2} P(e|f(e))_H$$

follows easily by induction in n from the first equation of the statement and the laws of a quadratic pair module. On the other hand

$$n^*\alpha(e) \quad = \quad n \cdot \alpha(e) + \binom{n}{2} PH\partial(-f(e) + m \cdot e).$$

One can also check by induction that

$$PH(-f(e) + m \cdot e) \quad = \quad P(f(e)|f(e))_H + mP(e|f(e))_H - \binom{m}{2} P(e|e)_H.$$

Now the proof is finished. □

Lemma 1.16 holds under the more general condition that C_0 is generated by elements $x \in C_0$ with $H(x) = 0$ and $Hf(x) = 0$.

2. Homotopy groups and secondary homotopy groups

Let $\mathbf{Top}^*$ be the category of (compactly generated) pointed spaces. Using classical homotopy groups $\pi_n X$ we obtain for $n \geq 0$ the functor

$$\Pi_n \colon \mathbf{Top}^* \longrightarrow \mathbf{Ab}$$

with

$$\Pi_n X = \begin{cases} \pi_n X, & n \geq 2, \\ (\pi_1 X)_{ab}, & n = 1, \\ \mathbb{Z}[\pi_0 X], & n = 0, \end{cases} \tag{2.1}$$

termed *additive homotopy group*. Here G_{ab} is the abelianization of a group G.

One readily checks that the smash product

$$f \wedge g \colon S^n \wedge S^m \longrightarrow X \wedge Y$$

of maps $\{f \colon S^n \to X\} \in \pi_n X$ and $\{g \colon S^m \to Y\} \in \pi_m Y$ induces a well-defined homomorphism

$$\wedge \colon \Pi_n X \otimes \Pi_m Y \longrightarrow \Pi_{n+m}(X \wedge Y). \tag{2.2}$$

This homomorphism is symmetric in the sense that the interchange map $\tau_{X,Y} : X \wedge Y \to Y \wedge X$ yields the equation in $\Pi_{n+m}(Y \wedge X)$

$$(\tau_{X,Y})_*(f \wedge g) = (-1)^{nm} g \wedge f. \tag{2.3}$$

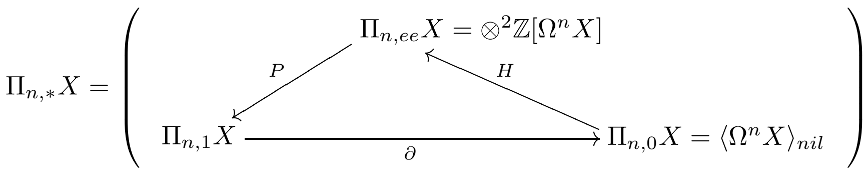
Here the sign $(-1)^{nm}$ is given by the interchange map

$$\tau_{n,m} = \tau_{S^n, S^m} : S^{n+m} \longrightarrow S^{m+n} \tag{2.4}$$

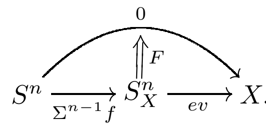
which has degree $(-1)^{nm}$. Here $\tau_{n,m}$ also designates the corresponding element of the symmetric group $\mathrm{Sym}(n + m)$ which acts from the left on S^{n+m}, see Section 5 below.

We want to generalize the smash product operator (2.2) for additive secondary homotopy groups.

Definition 2.5. Let $n \geq 2$. For a pointed space X we define the *additive secondary homotopy group* $\Pi_{n,*}X$ which is the 0-free quadratic pair module given by the diagram



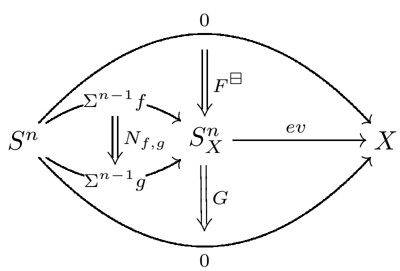
We obtain the group $\Pi_{n,1}X$ and the homomorphisms P and ∂ as follows. The group $\Pi_{n,1}X$ is given by the set of equivalence classes $[f, F]$ represented by a map $f\colon S^1 \to \vee_{\Omega^n X} S^1$ and a track



Here the pointed space

$$S^n_X = \vee_{\Omega^n X} S^n = \Sigma^n \Omega^n X$$

is the n-fold suspension of the n-fold loop space $\Omega^n X$, where $\Omega^n X$ is regarded as a pointed set with the discrete topology. Hence S^n_X is the coproduct of n-spheres indexed by the set of non-trivial maps $S^n \to X$, and $ev\colon S^n_X \to X$ is the obvious evaluation map. Moreover, for the sake of simplicity given a map $f\colon S^1 \to \vee_{\Omega^n X} S^1$ we will denote $f_{ev} = ev(\Sigma^{n-1}f)$, so that F in the previous diagram is a track $F\colon f_{ev} \Rightarrow 0$. The equivalence relation $[f, F] = [g, G]$ holds provided there is a track $N\colon \Sigma^{n-1}f \Rightarrow \Sigma^{n-1}g$ with $\mathit{Hopf}(N) = 0$ if $n \geq 3$ or $\bar{\sigma}\mathit{Hopf}(N) = 0$ if $n = 2$, see (2.6) and (2.7) below, such that the composite track in the following diagram is the trivial track.



That is $F = G\square(ev\, N_{f,g})$. The map ∂ is defined by the formula

$$\partial[f, F] = (\pi_1 f)_{nil}(1),$$

where $1 \in \pi_1 S^1 = \mathbb{Z}$.

The Hopf invariant of a track $N\colon \Sigma^{n-1}f \Rightarrow \Sigma^{n-1}g$ as above is defined in [BM05a] 3.3 by the homomorphism

$$(2.6)\quad H_2(IS^1, S^1 \vee S^1) \stackrel{ad(N)_*}{\longrightarrow} H_2(\Omega^{n-1}S^n_X, \vee_{\Omega^n X} S^1) \cong \begin{cases} \hat{\otimes}^2\mathbb{Z}[\Omega^n X], & n \geq 3, \\ \otimes^2\mathbb{Z}[\Omega^n X], & n = 2. \end{cases}$$

which carries the generator $1 \in \mathbb{Z} \cong H_2(IS^1, S^1 \vee S^1)$ to $\mathit{Hopf}(N)$. Here $ad(N)_*$ is the homomorphism induced in homology by the adjoint of the homotopy

$$N\colon \Sigma^{n-1}IS^1 \cong IS^n \to S^n_X.$$

The reduced tensor square is given by

$$\hat{\otimes}^2 A = \frac{A \otimes A}{a \otimes b + b \otimes a \sim 0},$$

and

$$\bar{\sigma}\colon \otimes^2 A \twoheadrightarrow \hat{\otimes}^2 A \tag{2.7}$$

is the natural projection. The isomorphism in (2.6) is induced by the Pontrjaging product. We refer the reader to [BM05a] 3 for a complete definition of the Hopf invariant for tracks and for the elementary properties which will be used in this paper. For the sake of simplicity we define the *reduced Hopf invariant* as $\overline{Hopf} = Hopf$ if $n \geq 3$ and $\overline{Hopf} = \bar{\sigma} Hopf$ if $n = 2$. A *nil-track* in this paper will be a track in $\mathbf{Top}^*$ with trivial reduced Hopf invariant. In particular the equivalence relation defining elements in $\Pi_{n,1}X$ is determined by nil-tracks.

This completes the definition of $\Pi_{n,1}X$, $n \geq 2$, as a set. The group structure of $\Pi_{n,1}X$ is induced by the comultiplication $\mu\colon S^1 \to S^1 \vee S^1$, compare [BM05a] 4.4.

We now define the homomorphism P for additive secondary homotopy groups $\Pi_{n,*}X$ with $n \geq 2$. Consider the diagram

$$S^n \xrightarrow[\Sigma^{n-1}\beta]{} S^n \vee S^n \qquad (\text{with } B\colon \Sigma^{n-1}\beta \Rightarrow 0)$$

where $\beta\colon S^1 \to S^1 \vee S^1$ is given such that $(\pi_1\beta)_{nil}(1) = -a - b + a + b \in \langle a, b\rangle_{nil}$ is the commutator. The track B is any track with $\overline{Hopf}(B) = -\sigma(a \otimes b) \in \hat{\otimes}^2\mathbb{Z}[a, b]$. Given $x \otimes y \in \otimes^2\mathbb{Z}[\Omega^n X]$ let $\tilde{x}, \tilde{y}\colon S^1 \to \vee_{\Omega^n X} S^1$ be maps with $(\pi_1\tilde{x})_{ab}(1) = x$ and $(\pi_1\tilde{y})_{ab}(1) = y$. Then the diagram

$$S^n \xrightarrow[\Sigma^{n-1}\beta]{} S^n \vee S^n \xrightarrow[\Sigma^{n-1}(\tilde{y},\tilde{x})]{} S^n_X \xrightarrow[ev]{} X \qquad (\text{with } B\colon \Sigma^{n-1}\beta \Rightarrow 0) \tag{2.8}$$

represents an element

$$P(x \otimes y) = [(\tilde{y}, \tilde{x})\beta, ev\,(\Sigma^{n-1}(\tilde{y}, \tilde{x}))B] \in \Pi_{n,1}X.$$

This completes the definition of the quadratic pair module $\Pi_{n,*}X$ for $n \geq 2$. For $n = 0, 1$ we define the additive secondary homotopy groups $\Pi_{n,*}X$ by the following remark. In this way we get for $n \geq 0$ a functor

$$\Pi_{n,*}\colon \mathbf{Top}^* \longrightarrow \mathbf{qpm}$$

which is actually a track functor.

Remark 2.9. Considering maps $f\colon S^n \to X$ together with tracks of such maps to the trivial map, we introduced in [BM05a] the secondary homotopy group $\pi_{n,*}X$, which is a groupoid for $n = 0$, a crossed module for $n = 1$, a reduced quadratic module for $n = 2$, and a stable quadratic module for $n \geq 3$. Let **squad** be the category of stable quadratic modules.

Then using the adjoint functors Ad_n of the forgetful functors ϕ_n as discussed in [BM05a] 6 we get the *additive secondary homotopy group* track functor

$$\Pi_{n,*}\colon \mathbf{Top}^* \longrightarrow \mathbf{squad}$$

given by

$$\Pi_{n,*}X = \begin{cases} \pi_{n,*}X, & \text{for } n \geq 3, \\ \mathsf{Ad}_3\pi_{2,*}X, & \text{for } n = 2, \\ \mathsf{Ad}_3\mathsf{Ad}_2\pi_{1,*}X, & \text{for } n = 1, \\ \mathsf{Ad}_3\mathsf{Ad}_2\mathsf{Ad}_1\pi_{0,*}X, & \text{for } n = 0. \end{cases}$$

This is the secondary analogue of (2.1).

Here the category **squad** of stable quadratic modules is not appropriate to study the smash product of secondary homotopy groups since we do not have a symmetric monoidal structure in **squad**. Therefore we introduced above the category **qpm** of quadratic pair modules and we observe that $\Pi_{n,*}X$ in **squad** yields a functor to the category **qpm** as follows. A map $f\colon X \to Y$ in $\mathbf{Top}^*$ induces a homomorphism $\Pi_{n,0}f\colon \Pi_{n,0}X \to \Pi_{n,0}Y$ between free nil-groups which carries generators in $\Pi_{n,0}X$ to generators in $\Pi_{n,0}Y$ and therefore $\Pi_{n,*}f$ is compatible with H. This shows that Lemma 1.15 gives rise to a canonical lift

$$\begin{array}{ccc} & \overset{\Pi_{n,*}}{\nearrow} & \mathbf{qpm} \\ & & \downarrow \\ \mathbf{Top}^* & \xrightarrow[\Pi_{n,*}]{} & \mathbf{squad} \end{array}$$

Here the vertical arrow, which is the forgetful track functor given by (1.7), is faithful but not full at the level of morphisms.

The definition of $\Pi_{2,*}X$ given above coincides with the lifting of $\mathsf{Ad}_3\pi_{2,*}X$ to **qpm** by the claim (*) in the proof of [BM05a] 4.9.

In this paper we are concerned with the properties of the track functor $\Pi_{n,*}$, mapping to the category **qpm**. The category **qpm** is, in fact, a symmetric monoidal category, defined by a tensor product $\odot$ in **qpm**, see [BJP05], and the smash product yields the operator

$$\wedge\colon \Pi_{n,*}X \odot \Pi_{m,*}Y \longrightarrow \Pi_{n+m,*}(X \wedge Y) \tag{2.10}$$

constructed in [BM05c]. Equation (2.3) has now a secondary analogue given by the right action of the symmetric group $\mathrm{Sym}(n+m)$ on the object $\Pi_{n+m,*}(X \wedge Y)$ in **qpm**. More precisely the following diagram commutes in **qpm**.

$$\begin{array}{ccc} \Pi_{n,*}X \odot \Pi_{m,*}Y & \xrightarrow{\wedge} & \Pi_{n+m,*}(X \wedge Y) \\ & & \cong\big\downarrow (\tau_{X,Y})_* \\ \tau_\odot \big\downarrow \cong & & \Pi_{n+m,*}(Y \wedge X) \\ & & \cong\big\uparrow \tau^*_{n,m} \\ \Pi_{m,*}Y \odot \Pi_{n,*}X & \xrightarrow{\wedge} & \Pi_{m+n,*}(Y \wedge X) \end{array}$$

Here $\tau_\odot$ on the left hand side is given by the symmetry of the tensor product $\odot$ in **qpm** and $\tau^*_{n,m}$ is defined by the action of $\mathrm{Sym}(n+m)$. For this reason we define and study in this paper the properties of the symmetric group action on secondary homotopy groups.

3. Actions of monoid-groupoids in track categories

In this paper we deal with actions on additive secondary homotopy groups. Additive secondary homotopy groups are objects in a track category. In ordinary categories a monoid action is given by a monoid-morphism mapping to an endomorphism monoid in the category. In track categories endomorphism objects are monoids in the monoidal category of groupoids, where the monoidal structure is given by the (cartesian) product. Therefore one can define accordingly actions of such monoids. We make explicit this structure in the following definition.

Definition 3.1. Let $\mathbb{I}$ be the category with only one object $*$ and one morphism $1\colon * \to *$. A *monoid-groupoid* $\mathbf{G}$ is a groupoid together with a multiplication functor $\cdot\colon \mathbf{G} \times \mathbf{G} \to \mathbf{G}$ and a unit functor $u\colon \mathbb{I} \to \mathbf{G}$, satisfying the laws of a monoid in the symmetric monoidal category of groupoids. We usually identify $* = u(*)$. The opposite $\mathbf{G}^{op}$ of a monoid-groupoid is the underlying groupoid $\mathbf{G}$ with its unit functor and multiplication functor given by

$$\mathbf{G} \times \mathbf{G} \xrightarrow{T} \mathbf{G} \times \mathbf{G} \xrightarrow{\cdot} \mathbf{G}.$$

Here T is the interchange of factors in the product. A monoid-groupoid *morphism* $f\colon \mathbf{G} \to \mathbf{H}$ is a functor preserving the multiplication and the unit.

Monoid-groupoids are also termed strict monoidal groupoids. The weaker versions of this concept will not be considered in this paper, therefore we abbreviate the terminology.

The canonical example of a monoid-groupoid is obtained by the endomorphisms of an object X in a track category $\mathbf{C}$, denoted

$$\mathbf{End}_{\mathbf{C}}(X).$$

The multiplication is given by composition in $\mathbf{C}$, and the unit is given by the identity morphism $1_X\colon X \to X$. In fact a monoid-groupoid as defined above is exactly the same thing as a track category with only one object, the opposite monoid-groupoid coincides with the the opposite of the corresponding track category and monoid-groupoid morphisms correspond to 2-functors.

Definition 3.2. Let X be an object in a track category $\mathbf{C}$ and let $\mathbf{G}$ be a monoid-groupoid. A *right action* of $\mathbf{G}$ on X is a monoid-groupoid morphism $\mathbf{G}^{op} \to \mathbf{End}_{\mathbf{C}}(X)$.

An important example of monoid-groupoids arises from crossed modules. The monoid-groupoid $M(\partial)$ associated to a crossed module $\partial\colon T \to G$ has object set G and morphism set the semidirect product $G \ltimes T$. Here we write the groups T and G with a multiplicative group law. An element $(g,t) \in G \ltimes T$ is a morphism $(g,t)\colon g \cdot \partial(t) \to g$ in $M(\partial)$. The composition law $\circ$ is given by the formula $(g,t) \circ (g \cdot \partial(t), t') = (g, t \cdot t')$. Multiplication in the groups G and $G \ltimes T$ defines the multiplication of $M(\partial)$ and the unit is given by the unit elements in G and $G \ltimes T$. Indeed this correspondence determines an equivalence between crossed modules and group objects in the category of groupoids. This example can be used to define crossed module actions.

Definition 3.3. Let X be an object in a track category $\mathbf{C}$ and let $\partial\colon T \to G$ be a crossed module. A *right action* of $\partial\colon T \to G$ on X is a monoid-groupoid morphism $M(\partial)^{op} \to \mathbf{End_C}(X)$.

We are interested in right actions of crossed modules in the track category **wqpm**. We explicitly describe such actions as follows.

Definition 3.4. A *right action of a crossed module* $\partial\colon T \to G$ on a quadratic pair module C in the category **wqpm** consists of a group action of G on the right of C given by morphisms in **wqpm**,

$$g^*\colon C \longrightarrow C, \;\; g \in G,$$

together with a bracket

$$\langle\!\langle -,-\rangle\!\rangle\colon C_0 \times T \longrightarrow C_1$$

satisfying the following properties, $x, y \in C_0$, $z \in C_1$, $s, t \in T$, $g \in G$,

(1) $\langle\!\langle x+y,t\rangle\!\rangle = \langle\!\langle x,t\rangle\!\rangle^{\partial(t)^*y} + \langle\!\langle y,t\rangle\!\rangle$,
(2) $x = \partial(t)^*x + \partial\langle\!\langle x,t\rangle\!\rangle$,
(3) $z = \partial(t)^*z + \langle\!\langle \partial(z),t\rangle\!\rangle$,
(4) $\langle\!\langle x,s\cdot t\rangle\!\rangle = \langle\!\langle \partial(s)^*x,t\rangle\!\rangle + \langle\!\langle x,s\rangle\!\rangle = \partial(t)^*\langle\!\langle x,s\rangle\!\rangle + \langle\!\langle x,t\rangle\!\rangle$,
(5) $\langle\!\langle x,t^g\rangle\!\rangle = g^*\langle\!\langle (g^{-1})^*x,t\rangle\!\rangle$.

We point out that the second equality in (4) follows from (1)–(3). Indeed these are the two possible definitions of the horizontal composition $\langle\!\langle -,s\rangle\!\rangle\langle\!\langle -,t\rangle\!\rangle\colon \partial(st)^* \Rightarrow 1$ of the tracks $\langle\!\langle -,t\rangle\!\rangle\colon \partial(t)^* \Rightarrow 1$ and $\langle\!\langle -,s\rangle\!\rangle\colon \partial(s)^* \Rightarrow 1$ in the track category **wqpm**.

The notion of action defined above corresponds to an action in Norrie's sense ([Nor90]) of a crossed module on the underlying crossed module of a quadratic pair module, however Norrie considers left actions.

The very special kind of action introduced in the following definition will be of importance to describe the symmetric action on additive secondary homotopy groups in Section 5.

Definition 3.5. Let $\{\pm 1\}$ be the multiplicative group of order 2. A *sign group* $G_\square$ is a diagram of group homomorphisms

$$\{\pm 1\} \overset{\imath}{\hookrightarrow} G_\square \overset{\partial}{\twoheadrightarrow} G \overset{\varepsilon}{\longrightarrow} \{\pm 1\}$$

where the first two morphisms form an extension. Here all groups have a multiplicative group law and the composite $\varepsilon\partial$ is also denoted by $\varepsilon\colon G_\square \to \{\pm 1\}$. Moreover, we define the element $\omega = \imath(-1) \in G_\square$.

A sign group $G_\square$ *acts on the right of a quadratic pair module* C if G acts on the right of C by morphisms

$$g^*\colon C \longrightarrow C, \;\; g \in G, \quad \text{in } \mathbf{qpm},$$

and there is a bracket

$$\langle -,-\rangle\colon C_0 \times G_\square \longrightarrow C_1$$

satisfying the following properties, $x, y \in C_0$, $z \in C_1$, $s, t \in G_\square$, were $\varepsilon(t)^*$ is given by the action of $(\mathbb{Z}, \cdot)$ in Definition 1.5,

(1) $\langle x+y,t\rangle = \langle x,t\rangle^{\partial(t)^*y} + \langle y,t\rangle$,
(2) $\varepsilon(t)^*(x) = \partial(t)^*(x) + \partial\langle x,t\rangle$,
(3) $\varepsilon(t)^*(z) = \partial(t)^*(z) + \langle \partial(z),t\rangle$,
(4) $\langle x,s\cdot t\rangle = \langle \partial(s)^*(x),t\rangle + \langle \varepsilon(t)^*x,s\rangle$,

(5) the *ω-formula*:

$$\langle x, \omega\rangle = P(x|x)_H.$$

Notice that the ω-formula corresponds to the k-invariant, see [BM05a] 8.

Remark 3.6. A sign group $G_\square$ gives rise to a crossed module

$$\partial_\square = (\varepsilon, \partial)\colon G_\square \longrightarrow \{\pm 1\}\times G,$$

where $\{\pm 1\}\times G$ acts on $G_\square$ by the formula

$$g^{(x,h)} = \bar{h}^{-1} g \bar{h}\imath\left(\varepsilon(g)^{\binom{x}{2}}\right).$$

Here $g\in G_\square$, $x\in\{\pm 1\}$, $h\in G$ and $\bar{h}\in G_\square$ is any element with $\partial(\bar{h}) = h$. This action is well defined since $G_\square$ is a central extension of G by $\{\pm 1\}$.

Lemma 3.7. *The sign group action in Definition 3.5 corresponds to an action of the crossed module $\partial_\square$ on C in the sense of Definition 3.4 such that $\{\pm 1\}$ acts on C by the action of $(\mathbb{Z}, \cdot)$ in Definition 1.5, G acts by morphisms in* **qpm**, *and the ω-formula holds. The correspondence is given by the formula*

$$\langle\!\langle x, t\rangle\!\rangle = \langle \varepsilon(t)^* x, t\rangle,\ \ x\in C_0,\ \ t\in G_\square.$$

The proof of this lemma is straightforward. We just want to point out that Definition 3.4 (5) follows in this case from Definition 3.5 (4), (5), and Lemma 1.16.

Remark 3.8. A sign group $G_\square$ is *trivial* if G is a trivial group. Notice that a trivial sign group acts on any quadratic pair module in a unique way.

4. The action of $\operatorname{End}_*(S^n)$ on $\Pi_{n,*}X$

Let S^n be the n-sphere and let $\operatorname{End}_*(S^n) = \Omega^n S^n$ be the topological monoid of maps $S^n \to S^n$ in **Top***. Then the fundamental groupoid of $\operatorname{End}_*(S^n)$, denoted by $\pi_{0,*}\operatorname{End}_*(S^n)$, is a monoid-groupoid in the sense of Definition 3.1. It is well known that the monoid of path components of $\operatorname{End}_*(S^n)$ coincides with the multiplicative monoid $(\mathbb{Z}, \cdot)$.

We now consider the right action of $\pi_{0,*}\operatorname{End}_*(S^n)$ on $\Pi_{n,*}X$ for $n\geq 2$. That is, we define for each pointed map $f\colon S^n\to S^n$ an induced map in **qpm**

$$f^*\colon \Pi_{n,*}X \longrightarrow \Pi_{n,*}Y$$

and we define for each track $H\colon f\Rightarrow g$ with $f, g\colon S^n\to S^n$ a track in **qpm**

$$H^*\colon f^*\Rightarrow g^*.$$

This yields a right action of the fundamental groupoid $\pi_{0,*}\operatorname{End}_*(S^n)$ on the secondary homotopy group $\Pi_{n,*}X$ in the track category **qpm** of quadratic pair modules in the sense of Definition 3.2.

Theorem 4.1. *Let X be a pointed space. For any $n\geq 2$ there is a natural action of the monoid-groupoid $\pi_{0,*}\operatorname{End}_*(S^n)$ on the quadratic pair module $\Pi_{n,*}X$.*

The rest of this section is devoted to the proof of this theorem, which is carried out in several steps.

The discrete monoid $\pi_{0,0}\operatorname{End}_*(S^n)$, which is the underlying set of the topological monoid $\operatorname{End}_*(S^n)$, acts on the right of the pointed set $\Omega^n X$ of pointed maps $S^n\to X$ by precomposition, i. e. given $f\colon S^n\to S^n$ the induced endomorphism is

$$f^*\colon \Omega^n X\longrightarrow \Omega^n X,\ \ f^*(g) = gf.$$

This induces a right action of $\pi_{0,0}\operatorname{End}_*(S^n)$ on the free group $\pi_{n,*}X = \langle \Omega^n X\rangle_{nil}$ of nilpotency class 2 which will be denoted in the same way.

In order to extend this action to $\Pi_{n,1}X$ we consider the submonoid

$$\tilde{\pi}_{0,1}\operatorname{End}_*(S^n) \subset \pi_{0,1}\operatorname{End}_*(S^n) \tag{4.2}$$

of the monoid $\pi_{0,1}\operatorname{End}_*(S^n)$ of morphisms in $\pi_{0,*}\operatorname{End}_*(S^n)$ given by tracks between self-maps of S^n of the form

$$\gamma\colon f \Rightarrow \Sigma^{n-1}(\cdot)^{\deg f} = (\cdot)_n^{\deg f}. \tag{4.3}$$

Here $\deg f \in \mathbb{Z}$ denotes the degree of $f\colon S^n \to S^n$ and for $k \in \mathbb{Z}$

$$(\cdot)^k\colon S^1 \longrightarrow S^1\colon z \mapsto z^k$$

is given by the (multiplicative) topological abelian group structure of S^1.

We need a bracket operation

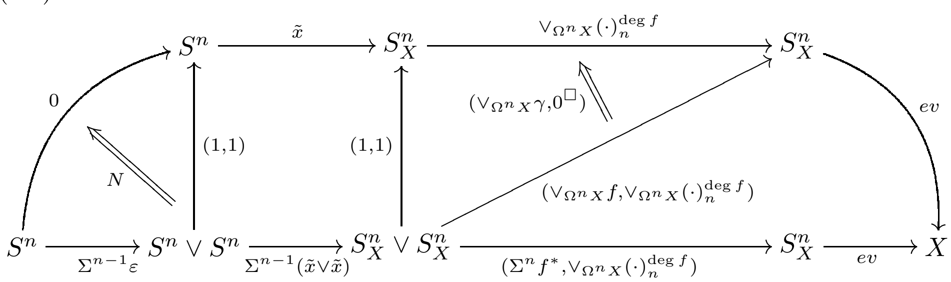
$$\langle -,-\rangle\colon \Pi_{n,0}X \times \tilde{\pi}_{0,1}\operatorname{End}_*(S^n) \longrightarrow \Pi_{n,1}X, \tag{4.4}$$

defined as follows. Let $x \in \pi_{n,0}X = \langle \Omega^n X\rangle_{nil}$ and $\gamma\colon f \Rightarrow (\cdot)_n^{\deg f}$ in $\tilde{\pi}_{0,1}\operatorname{End}_*(S^n)$. We choose maps $\tilde{x}\colon S^1 \to \vee_{\Omega^n X}S^1$, $\varepsilon\colon S^1 \to S^1 \vee S^1$ with $(\pi_1\tilde{x})_{nil}(1) = x$ and $(\pi_1\varepsilon)_{nil} = -a+b \in \langle a,b\rangle_{nil}$. Then $\langle x,\gamma\rangle \in \Pi_{n,1}X$ is the element represented by the map

$$S^1 \xrightarrow{\varepsilon} S^1 \vee S^1 \xrightarrow{\tilde{x}\vee\tilde{x}} (\vee_{\Omega^n X}S^1) \vee (\vee_{\Omega^n X}S^1) \xrightarrow{(\Sigma f^*, \vee_{\Omega^n X}(\cdot)^{\deg f})} \vee_{\Omega^n X}S^1$$

and the track

(4.5)



Here N is a nil-track.

The main properties of the bracket operation in (4.4) are listed in the following proposition.

Proposition 4.6. *The bracket $\langle -,-\rangle$ in (4.4) satisfies the following formulas for any $x,y \in \Pi_{n,0}X$ and $\gamma\colon f \Rightarrow (\cdot)_n^{\deg f}$, $\delta\colon g \Rightarrow (\cdot)_n^{\deg g}$ in $\tilde{\pi}_{0,1}\operatorname{End}_*(S^n)$.*

1. $\langle x+y,\gamma\rangle = \langle x,\gamma\rangle^{f^*y} + \langle y,\gamma\rangle$,
2. $(\deg f)^*x = f^*x + \partial\langle x,\gamma\rangle$,
3. $\langle x,\gamma\delta\rangle = \langle f^*x,\delta\rangle + \langle(\deg g)^*x,\gamma\rangle$,
4. *if* $\omega\colon 1_{S^n} \Rightarrow 1_{S^n}$ *is a track with* $0 \neq \overline{Hopf}(\omega) \in \hat{\otimes}^2\mathbb{Z} = \mathbb{Z}/2$ *then* $\langle x,\omega\rangle = P(x|x)_H$.

Moreover, this bracket operation is natural in X.

Proof. With the notation in [BM05a] 7.4 we have $\langle x,\gamma\rangle = r(ev(\vee_{\Omega^n X}\gamma)(\Sigma^{n-1}\tilde{x}))$ for the track

$$ev(\vee_{\Omega^n X}\gamma)(\Sigma^{n-1}\tilde{x})\colon ev(\Sigma^n f^*)(\Sigma^{n-1}\tilde{x}) = ev(\vee_{\Omega^n X}f)(\Sigma^{n-1}\tilde{x}) \Rightarrow ev(\vee_{\Omega^n X}(\cdot)_n^{\deg f})(\Sigma^{n-1}\tilde{x}),$$

therefore (1) and (2) follow from [BM05a] 7.6 and 7.5 (2).

It is easy to see that the formula

$$ev(\vee_{\Omega^n X}\gamma\delta) = (ev(\vee_{\Omega^n X}\gamma)(\vee_{\Omega^n X}(\cdot)_n^{\deg g}))\Box(ev(\vee_{\Omega^n X}\delta)(\Sigma^n f^*))$$

holds, therefore (3) follows from [BM05a] 7.5 (3).

If we evaluate $\langle x,-\rangle$ at ω then the composite track obtained from (4.5) by going from the lower left S^n to the upper right S^n_X has the same reduced Hopf invariant as the track from S^n to S^n_X in (2.8). Indeed the formula for both reduced Hopf invariants is (c) in the proof of Proposition 4.8. Therefore (4) follows. □

The next result follows from the algebraic properties of the bracket (4.4) which are proved in the previous proposition together with Lemma 1.16.

Proposition 4.7. *The monoid $\tilde{\pi}_{0,1}\operatorname{End}_*(S^n)$ acts on the right of $\Pi_{n,1}X$ by the following formula, $n\geq 2$: given $x\in\Pi_{n,1}X$ and $\gamma\colon f\Rightarrow(\cdot)_n^{\deg f}$*

$$\gamma^* x = (\deg f)^* x - \langle\partial(x),\gamma\rangle.$$

This action satisfies $\partial\gamma^* = f^*\partial$, $\gamma^* P = P(\otimes^2 f^*_{ab})$, and $Hf^* = (\otimes^2 f^*_{ab})H$, therefore it defines an action of $\tilde{\pi}_{0,1}\operatorname{End}_*(S^n)$ on the right of the quadratic pair module $\Pi_{n,*}X$ in the category **qpm**. *This action is natural in X.*

Proof. The equality $Hf^* = (\otimes^2 f^*_{ab})H$ follows from the fact that the endomorphism f^* carries generators to generators in $\langle\Omega^n X\rangle_{nil}$. The equality $\partial\gamma^* = f^*\partial$ follows from Proposition 4.6 (2). Let us check $\gamma^* P = P(\otimes^2 f^*_{ab})$. Given $a,b\in\Omega^n X$

$$\begin{aligned}
\gamma^* P(a\otimes b) &= (\deg f)^* P(a\otimes b) - \langle -a-b+a+b,\gamma\rangle\\
&= P(\deg f)^2(a\otimes b) - \langle b,\gamma\rangle - \langle a,\gamma\rangle + \langle b,\gamma\rangle + \langle a,\gamma\rangle\\
&\quad +P(-f^*(a)+(\deg f)^* a|f^* b)_H - P(-f^*(b)+(\deg f)^* b|f^* a)_H\\
&= P(\deg f)^2(a\otimes b) + P(\partial\langle b,\gamma\rangle|\partial\langle a,\gamma\rangle)_H\\
&\quad +P(-f^*(a)+(\deg f)^* a|f^* b)_H - P(-f^*(b)+(\deg f)^* b|f^* a)_H\\
&= -P(\deg f)^2(a|b)_H\\
&\quad -P(-f^*(a)+(\deg f)^* a|-f^*(b)+(\deg f)^* b)_H\\
&\quad +P(-f^*(a)+(\deg f)^* a|f^* b)_H + P(f^* a|-f^*(b)+(\deg f)^* b)_H\\
&= -P(f^*(a)|f^*(b))_H\\
&= P(f^*(b)|f^*(a))_H\\
&= P(f^*(a)\otimes f^*(a)).
\end{aligned}$$

Here we use Proposition 4.6 (1) and (2) and the fact that $H(a)=0=H(b)$.

Finally given $\delta\colon g\Rightarrow(\cdot)_n^{\deg g}$

$$\begin{aligned}
\gamma^*\delta^*(x) &= \gamma^*((\deg g)^* x - \langle\partial(x),\delta\rangle)\\
&= (\deg f)^*(\deg g)^* x - (\deg f)^*\langle\partial(x),\delta\rangle - \langle g^*\partial(x),\gamma\rangle\\
&= ((\deg f)(\deg g))^* x - \langle(\deg f)^*\partial(x),\delta\rangle - \langle g^*\partial(x),\gamma\rangle\\
&\quad +\binom{\deg f}{2}\binom{\deg g}{2}P(\partial(x)|\partial(x))_H\\
&= (\deg fg)^* x - \langle\partial(x),\gamma\delta\rangle\\
&= (\gamma\delta)^*(x).
\end{aligned}$$

Here we use Proposition 4.6 (1), (2) and (3), Lemma 1.16 and the fact that $P(\partial(x)|\partial(x))_H = -x-x+x+x=0$. □

Proposition 4.8. *For $n \geq 2$ the right action of the monoid $\tilde{\pi}_{0,1}\operatorname{End}_*(S^n)$ on the group $\Pi_{n,1}X$ given by Proposition 4.7 factors through the boundary homomorphism*

$$q\colon \tilde{\pi}_{0,1}\operatorname{End}_*(S^n) \twoheadrightarrow \pi_{0,0}\operatorname{End}_*(S^n), \quad q(\gamma\colon f \Rightarrow (\cdot)_n^{\deg f}) = f,$$

that is, the homomorphism $\gamma^* = f^*$ only depends on the boundary $q(\gamma) = f$.

Proof. Let $\gamma\colon f \Rightarrow (\cdot)_n^{\deg f}$ be any element in $\tilde{\pi}_{0,1}\operatorname{End}_*(S^n)$ and let $\delta\colon (\cdot)_n^{\deg f} \Rightarrow (\cdot)_n^{\deg f}$ be any track. We know that all elements in $q^{-1}(f)$ are of the form $\delta\square\gamma$, therefore we only have to check that for any $[g,G] \in \Pi_{n,1}X$

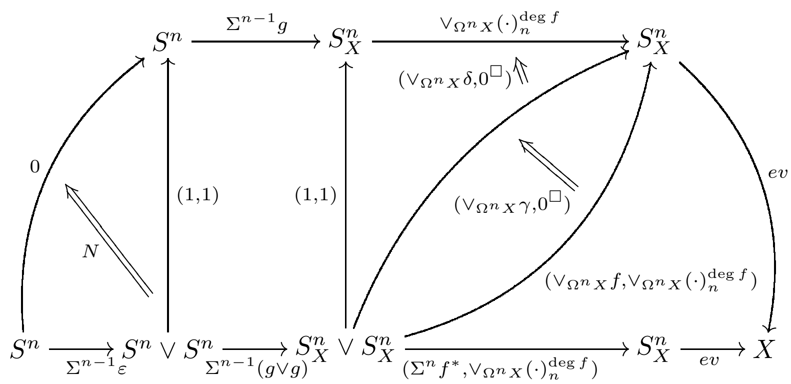
$$\gamma^*[g,G] = (\delta\square\gamma)^*[g,F],$$

or equivalently

$$\langle \partial[g,G], \gamma\rangle = \langle \partial[g,G], \delta\square\gamma\rangle.$$

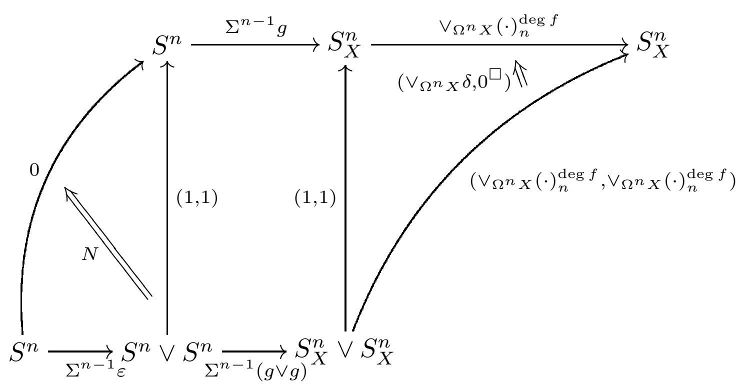
The element $\langle \partial[g,G], \delta\square\gamma\rangle$ is represented by the following diagram

(a)



Let us now pay special attention to the following subdiagram of (a)

(b)

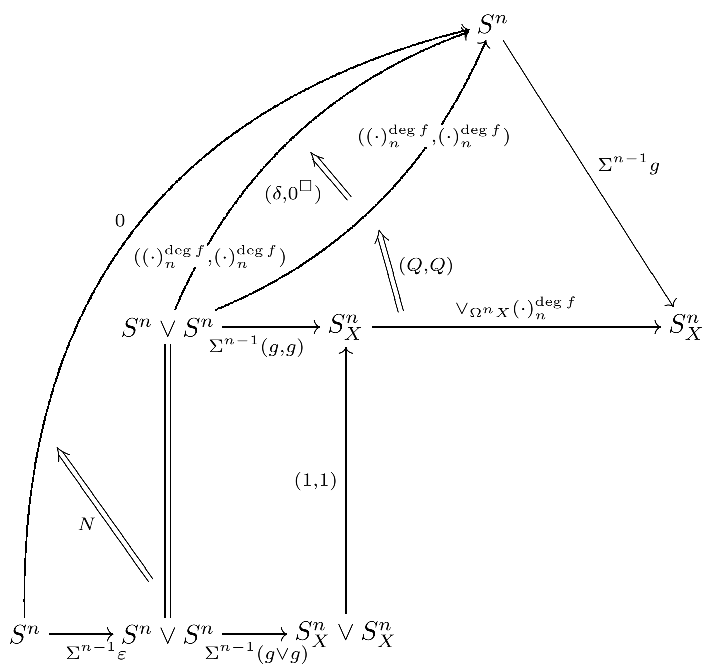


This is a composite track, termed (b), between $(n-1)$-fold suspensions. The reduced Hopf invariant of (b) is trivial if $\overline{Hopf}(\delta) = 0$. If $0 \neq \overline{Hopf}(\delta) \in \hat{\otimes}^2\mathbb{Z} = \mathbb{Z}/2$ then the reduced Hopf invariant of (b) is given by the following formula where $(\pi_1 g)_{ab}(1) = \sum_{i=0}^{k} n_i a_i \in \mathbb{Z}[\Omega^n X]$ for some $a_i \in \Omega^n X$ and $n_i \in \mathbb{Z}$,

(c) $$\overline{Hopf}(\text{b}) = \sum_{i=0}^{k} n_i a_i \hat{\otimes} a_i \in \hat{\otimes}^2\mathbb{Z}[\Omega^n X].$$

Here we have used the elementary properties of the Hopf invariant for tracks described in [BM05a] 3. By using again these properties the reader can easily check that the following composite track has the same reduced Hopf invariant as (b)

(d)

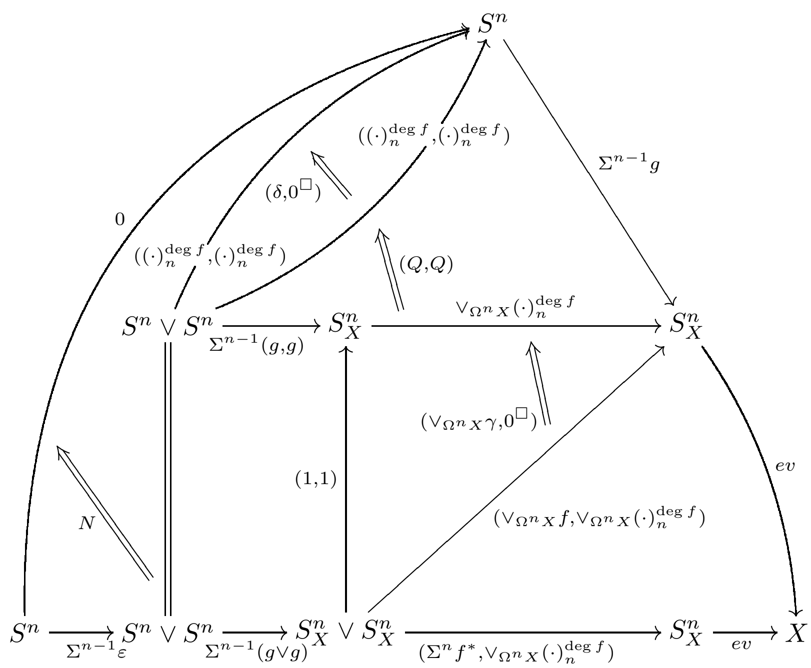


Here

$$Q\colon (\vee_{\Omega^n X}(\cdot)_n^{\deg f})(\Sigma^{n-1}g) \Rightarrow (\Sigma^{n-1}g)(\cdot)_n^{\deg f}$$

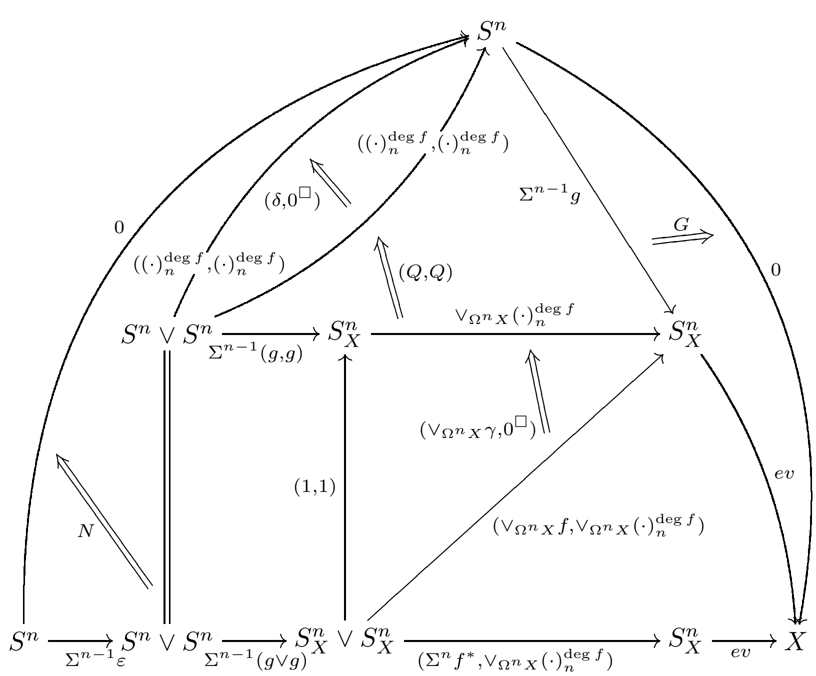
can be any track. Since (b) and (d) have the same reduced Hopf invariantthen (a) represents the same element in $\Pi_{n,1}X$ as

(e)

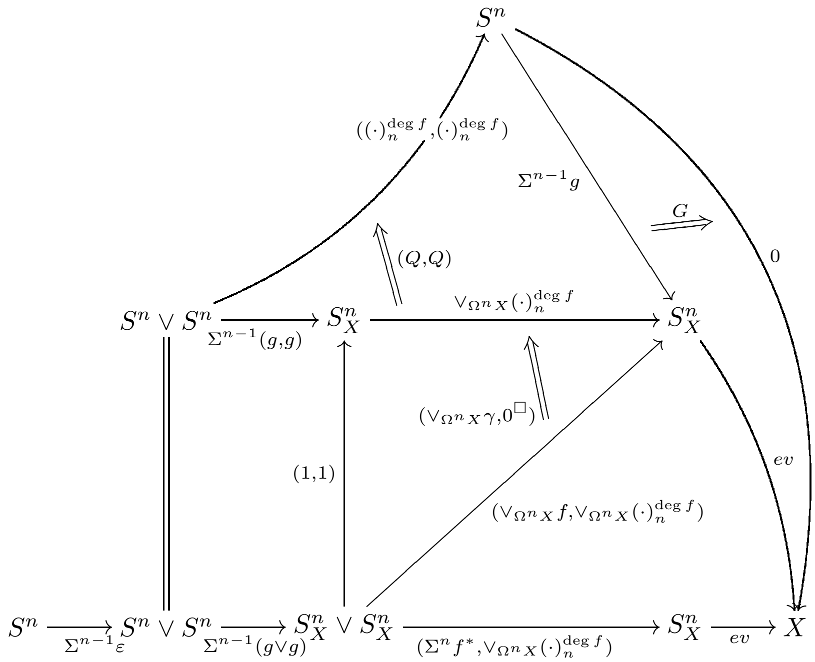


The composite track (e) is the same as

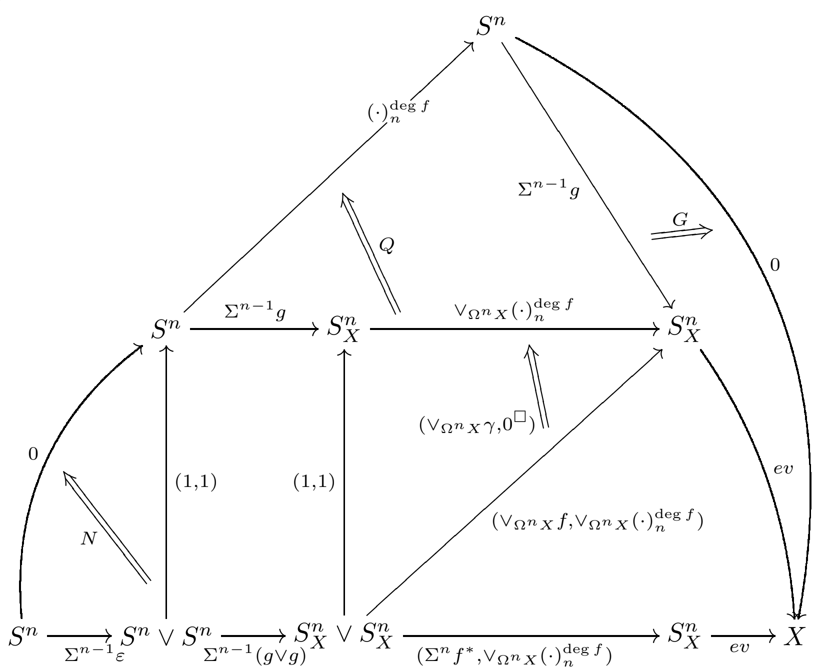
(f)



And (f) is the same as
(g)

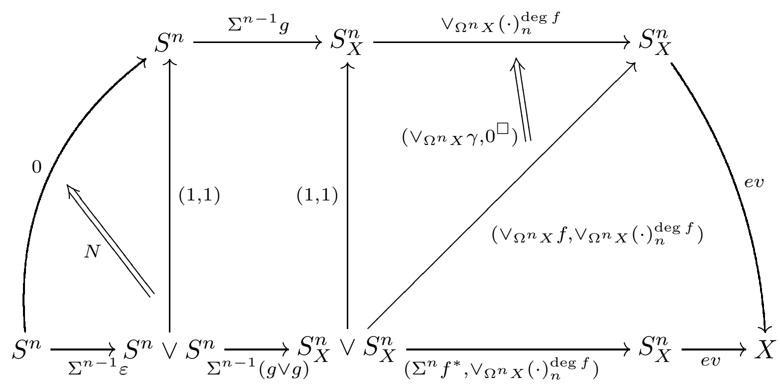


Obviously (g) coincides with
(h)



And (h) is the same as

(i)



Notice that this last composite track (i) represents $\langle \partial[g, G], \gamma \rangle$, hence we are done. □

The next corollary follows from the two previous propositions.

Corollary 4.9. *For any pointed space X and $n \geq 2$ the monoid $\pi_{0,0}\operatorname{End}_*(S^n)$ acts on the right of the quadratic pair module $\Pi_{n,*}X$. This action is natural in X.*

Now Theorem 4.1 is a consequence of the next result.

Proposition 4.10. *The action of $\pi_{0,0}\operatorname{End}_*(S^n)$ on the right of $\Pi_{n,*}X$ given by Corollary 4.9 extends to an action of the whole monoid-groupoid $\pi_{0,*}\operatorname{End}_*(S^n)$, $n \geq 2$.*

Proof. A morphism H in $\pi_{0,*}\operatorname{End}_*(S^n)$ is a track $H \colon f \Rightarrow g$ between maps $f, g \colon S^n \to S^n$, in particular $\deg f = \deg g = k \in \mathbb{Z}$. In order to define a track

$$H^* \colon f^* \Rightarrow g^*$$

between the quadratic pair module morphisms

$$f^*, g^* \colon \Pi_{n,*}X \longrightarrow \Pi_{n,*}X$$

we choose tracks in $\tilde{\pi}_{0,1}\operatorname{End}_*(S^n)$

$$\alpha \colon f \Rightarrow (\cdot)_n^k,$$
$$\beta \colon g \Rightarrow (\cdot)_n^k,$$

such that

$$H = \beta^{\boxminus} \square \alpha.$$

By Proposition 4.6 the maps

$$\langle -, \alpha \rangle, \langle -, \beta \rangle \colon \Pi_{n,0}X \longrightarrow \Pi_{n,1}X$$

are tracks

$$\langle -, \alpha \rangle \colon f^* \Rightarrow k^*,$$
$$\langle -, \beta \rangle \colon g^* \Rightarrow k^*,$$

in the category **wqpm**, therefore we can define H^* as the vertical composition

$$H^* = \langle -, \beta \rangle^{\boxminus} \square \langle -, \alpha \rangle,$$

i. e. H^* is the map

$$H^* \colon \Pi_{n,0}X \longrightarrow \Pi_{n,1}X$$

defined by

$$H^*(x) = \langle x, \alpha\rangle - \langle x, \beta\rangle.$$

By the proof of Proposition 4.6 and by [BM05a] 7.5 (3) the element $H(x)$ coincides with $r(ev(\vee_{\Omega^n X} H)(\Sigma^{n-1}\tilde{x}))$ for $\tilde{x}\colon S^1 \to \vee_{\Omega^n X} S^1$ any map with $(\pi_1\tilde{x})_{nil}(1) = x$ in the sense of [BM05a] 7.4. The reader can now use the properties of the bracket (4.4) described in Proposition 4.6 together with [BM05a] 7.5 (3) to check that this yields a monoid-groupoid action. □

Later we will consider the quotient monoid $\bar{\pi}_{0,1}\operatorname{End}_*(S^2)$ of $\tilde{\pi}_{0,1}\operatorname{End}_*(S^2)$ defined as follows: two elements $\gamma\colon f \Rightarrow (\cdot)_2^{\deg f}$, $\bar{\gamma}\colon g \Rightarrow (\cdot)_2^{\deg g}$ in $\tilde{\pi}_{0,1}\operatorname{End}_*(S^2)$ represent the same element in $\bar{\pi}_{0,1}\operatorname{End}_*(S^2)$ provided $\deg f = \deg g$ and

$$0 = \overline{Hopf}(\bar{\gamma}\square\gamma^{\boxminus}) \in \hat{\otimes}^2\mathbb{Z} = \mathbb{Z}/2.$$

Proposition 4.11. *The bracket operation (4.4) factors for $n = 2$ through the natural projection $\tilde{\pi}_{0,1}\operatorname{End}_*(S^2) \twoheadrightarrow \bar{\pi}_{0,1}\operatorname{End}_*(S^2)$.*

$$\langle -,-\rangle\colon \Pi_{2,0} \times \bar{\pi}_{0,1}\operatorname{End}_*(S^2) \longrightarrow \Pi_{2,1}X.$$

Proof. Two tracks γ and $\bar{\gamma}$ in $\tilde{\pi}_{0,1}\operatorname{End}_*(S^2)$ represent the same element in $\bar{\pi}_{0,1}\operatorname{End}_*(S^2)$ if and only if $\bar{\gamma} = \delta\square\gamma$ for some $\delta\colon (\cdot)_2^k \Rightarrow (\cdot)_2^k$ with $\overline{Hopf}(\delta) = 0$, so we only need to check that $\langle x, \gamma\rangle = \langle x, \delta\square\gamma\rangle$. The element $\langle x, \delta\square\gamma\rangle$ is represented by diagram (a) in the proof of Proposition 4.8 where we assume that g is a map with $(\pi_1 g)_{nil}(1) = x$. As we mention in that proof diagram (b) is a nil-track in these circumstances, therefore we can drop δ from (a) and still obtain the same element in $\Pi_{2,1}X$. But if we drop δ we obtain $\langle x, \gamma\rangle$, hence we are done. □

5. The symmetric action on secondary homotopy groups

The permutation of coordinates in $S^n = S^1 \wedge \cdots \wedge S^1$ induces a left action of the symmetric group $\operatorname{Sym}(n)$ on the n-sphere S^n. This action induces a monoid inclusion

$$\operatorname{Sym}(n) \subset \pi_{0,0}\operatorname{End}_*(S^n). \tag{5.1}$$

We define the *symmetric track group* for $n \geq 3$

$$\operatorname{Sym}_\square(n) \subset \tilde{\pi}_{0,1}\operatorname{End}_*(S^n)$$

as the submonoid of tracks of the from

$$\alpha\colon \sigma \Rightarrow (\cdot)_n^{\operatorname{sign}(\sigma)},$$

where $\sigma \in \operatorname{Sym}(n)$ and $\operatorname{sign}(\sigma) \in \{\pm 1\}$ is the sign of the permutation. Compare the notation in (1.6) and (4.3).

The submonoid defined as above for $n = 2$ will be called the *extended symmetric track group*

$$\overline{\operatorname{Sym}_\square}(2) \subset \tilde{\pi}_{0,1}\operatorname{End}_*(S^2).$$

For $n = 2$ the symmetric track group $\operatorname{Sym}_\square(2)$ is the image of $\overline{\operatorname{Sym}_\square}(2)$ by the natural projection $\tilde{\pi}_{0,1}\operatorname{End}_*(S^2) \twoheadrightarrow \bar{\pi}_{0,1}\operatorname{End}_*(S^2)$ in Proposition 4.11.

$$\operatorname{Sym}_\square(2) \subset \bar{\pi}_{0,1}\operatorname{End}_*(S^2). \tag{5.2}$$

Proposition 5.3. *The symmetric track group is indeed a group. Moreover, it fits into a central extension, $n \geq 2$,*

$$\mathbb{Z}/2 \hookrightarrow \mathrm{Sym}_\square(n) \stackrel{\delta}{\twoheadrightarrow} \mathrm{Sym}(n)$$

with $\delta(\alpha) = \sigma$, which splits if and only if $n = 2$ or 3.

This proposition follows from Corollary 6.9 and Remarks 6.10 and 6.12 below.

For $n = 0$ and $n = 1$ we define $\mathrm{Sym}(n)$ to be the trivial group, and $\mathrm{Sym}_\square(n)$ the trivial sign group. Then the symmetric track group $\mathrm{Sym}_\square(n)$ is a sign group ($n \geq 0$)

$$\{\pm 1\} \hookrightarrow \mathrm{Sym}_\square(n) \stackrel{\delta}{\twoheadrightarrow} \mathrm{Sym}(n) \stackrel{\mathrm{sign}}{\longrightarrow} \{\pm 1\}$$

as in Definition 3.5.

Theorem 5.4. *Let X be a pointed space. For $n \geq 0$ the symmetric group* $\mathrm{Sym}(n)$ *acts naturally on the right of the additive secondary homotopy group $\Pi_{n,*}X$ in the category* **qpm** *of quadratic pair modules. Moreover, the restriction*

$$\langle -, - \rangle \colon \Pi_{n,0}X \times \mathrm{Sym}_\square(n) \longrightarrow \Pi_{n,1}X$$

of the bracket defined in (4.4) if $n \geq 3$ and in Proposition 4.11 if $n = 2$ yields a natural right action of the sign group $\mathrm{Sym}_\square(n)$ *on $\Pi_{n,*}X$ in the sense of Definition 3.5.*

The action of $\mathrm{Sym}(n)$ is given by Corollary 4.9 and the inclusion (5.1) if $n \geq 3$ or (5.2) if $n = 2$. The rest of the statement follows from Proposition 4.6. The cases $n = 0, 1$ are trivial consequences of Remark 3.8.

6. The structure of the symmetric track groups

In this section we construct a positive pin representation for the symmetric track group $\mathrm{Sym}_\square(n)$. By using this representation we obtain a finite presentation of $\mathrm{Sym}_\square(n)$.

The action of $\mathrm{Sym}(n)$ on S^n can be extended to a well-known action of the orthogonal group $O(n)$ which we now recall. Let $[-1,1]^n \subset \mathbb{R}^n$ be the hypercube centered in the origin whose vertices have all coordinates in $\{\pm 1\}$, $D^n \subset \mathbb{R}^n$ the Euclidean unit ball and S^{n-1} its boundary. There is a homeomorphism $\phi \colon [-1,1]^n \to D^n$ fixing the origin defined as follows

$$\phi(\underline{x}) = \frac{\max\limits_{1 \leq i \leq n} |x_i|}{\|\underline{x}\|}\, \underline{x}.$$

Here $\underline{x} \in [-1,1]^n$ is an arbitrary non-trivial vector in the hypercube and $\|\cdot\|$ is the Euclidean norm. This homeomorphism projects the hypercube onto the ball from the origin. There is also a map collapsing the boundary

$$\varrho \colon [-1,1]^n \longrightarrow S^1 \wedge \stackrel{n}{\cdots} \wedge S^1 = S^n,$$

$$\varrho(x_1, \dots, x_n) = (\exp(i\pi(1 + x_1)), \dots, \exp(i\pi(1 + x_n))).$$

The composite

$$\varrho\phi^{-1} \colon D^n \longrightarrow S^n$$

induces a homeomorphism

$$D^n / S^{n-1} \cong S^1 \wedge \stackrel{n}{\cdots} \wedge S^1 = S^n$$

that we fix.

The orthogonal group $O(n)$ acts on the left of the unit ball D^n. This action induces an action of $O(n)$ on the quotient space $S^n = D^n/S^{n-1}$ preserving the basepoint. The interchange of coordinates action of the symmetric group $\mathrm{Sym}(n)$ on $\mathbb{R}^n$ preserves the Euclidean scalar product, and therefore induces a homomorphism

$$i\colon \mathrm{Sym}(n) \hookrightarrow O(n). \tag{6.1}$$

The pull-back of the action of $O(n)$ along this homomorphism is the action of $\mathrm{Sym}(n)$ on S^n given by the smash product decomposition of S^n.

Remark 6.2. The action of $O(n)$ on S^n defines an inclusion $O(n) \subset \mathrm{End}_*(S^n)$. The induced homomorphism on π_1 is the Whitehead-Hopf J-homomorphism

$$J\colon \pi_1 O(n) \cong \pi_1 \mathrm{End}_*(S^n) = \pi_{n+1} S^n \tag{6.3}$$

which is known to be an isomorphism for $n \geq 2$. Let $\pi_{0,*}O(n)$ be the fundamental groupoid of the Lie group $O(n)$. Then, considering elements $A, B \in O(n)$ as pointed maps

$$A, B\colon S^n \longrightarrow S^n$$

the isomorphism in (6.3) allows to identify all morphisms $\gamma\colon A \to B$ in $\pi_{0,*}O(n)$ with all tracks

$$\gamma\colon A \Rightarrow B$$

in $\pi_{0,1}\mathrm{End}_*(S^n)$. Let $\mathrm{Id}_n \in O(n)$ be the identity matrix. The order 2 matrix

$$\begin{pmatrix} \mathrm{Id}_{n-1} & 0 \\ 0 & -1 \end{pmatrix} \in O(n)$$

will be denoted by $\mathrm{Id}_{n-1} \oplus (-1)$. By using the action of $O(n)$ on S^n we have by the notation in (4.3) that

$$\mathrm{Id}_{n-1} \oplus (-1) = (\cdot)_n^{-1}\colon S^n \longrightarrow S^n.$$

Obviously $\mathrm{Id}_n = (\cdot)_n^1 = 1_{S^n}\colon S^n \to S^n$.

The topological group structure of $O(n)$ induces an internal group structure on the fundamental groupoid $\pi_{0,*}O(n)$ in the category of groupoids. In particular the set $\pi_{0,1}O(n)$ of morphisms in $\pi_{0,*}O(n)$ forms a group. We define the subgroup

$$\widetilde{O}(n) \subset \pi_{0,1}O(n)$$

consisting of all the morphisms with target Id_n or $\mathrm{Id}_{n-1} \oplus (-1)$. By Remark 6.2 the symmetric track group is the subgroup

$$\mathrm{Sym}_\square(n) \subset \widetilde{O}(n)$$

of morphisms with source in the image of i in (6.1), $n \geq 3$. The subgroup $\widetilde{O}(n)$ is embedded in an extension

$$\mathbb{Z}/2 \hookrightarrow \widetilde{O}(n) \stackrel{q}{\twoheadrightarrow} O(n), \quad n \geq 3. \tag{6.4}$$

The projection q sends a morphism in $\widetilde{O}(n) \subset \pi_{0,1}O(n)$ to the source, and the kernel is clearly $\pi_1 O(n) = \mathbb{Z}/2$ for $n \geq 3$. The case $n = 2$ will be considered in Remark 6.12 below.

There is also a well-known extension

$$\mathbb{Z}/2 \hookrightarrow Pin_+(n) \stackrel{\rho}{\twoheadrightarrow} O(n) \tag{6.5}$$

given by the positive pin group. Let us recall the definition of this extension.

Definition 6.6. The positive Clifford algebra $C_+(n)$ is the unital $\mathbb{R}$-algebra generated by e_i, $1 \leq i \leq n$, with relations

(1) $e_i^2 = 1$ for $1 \leq i \leq n$,
(2) $e_i e_j = -e_j e_i$ for $1 \leq i < j \leq n$.

Clifford algebras are defined for arbitrary quadratic forms on finite-dimensional vector spaces, see for instance [BtD85] 6.1. The Clifford algebra defined above corresponds to the quadratic form of the standard positive-definite scalar product in $\mathbb{R}^n$. We identify the sphere S^{n-1} with the vectors of Euclidean norm 1 in the vector subspace $\mathbb{R}^n \subset C_+(n)$ spanned by the generators e_i. The vectors in S^{n-1} are units in $C_+(n)$. Indeed for any $v \in S^{n-1}$ the square $v^2 = 1$ is the unit element in $C_+(n)$, so that $v^{-1} = v$. The group $Pin_+(n)$ is the subgroup of units in $C_+(n)$ generated by S^{n-1}. Any $x \in Pin_+(n)$ defines an automorphism of $\mathbb{R}^n \subset C_+(n)$ given by conjugation in $C_+(n)$ as follows

$$\mathbb{R}^n \longrightarrow \mathbb{R}^n \colon w \mapsto -xwx^{-1}.$$

If $x \in S^{n-1}$ then this automorphism is the reflection along the hyperplane orthogonal to the unit vector x. This endomorphism always preserves the scalar product, therefore this defines a homomorphism

$$\rho\colon Pin_+(n) \twoheadrightarrow O(n).$$

This homomorphism is surjective since all elements in $O(n)$ are products of $\leq n$ reflections. It is easy to see that the kernel of ρ is $\mathbb{Z}/2$ generated by $-1 \in C_+(n)$. This is the extension in (6.5).

The Clifford algebra $C_+(n)$ has dimension 2^n. A basis is given by the elements

$$e_{i_1} \cdots e_{i_k}, \quad 1 \leq i_1 < \cdots < i_k \leq n.$$

We give $C_+(n)$ the topology induced by the Euclidean norm associated to this basis. The positive pin group inherits a topology turning (6.5) into a Lie group extension.

Proposition 6.7. *The extension (6.4) is isomorphic to (6.5).*

Proof. Since $Pin_+(n)$ is a topological group $\pi_{0,*}Pin_+(n)$ is a group object in the category of groupoids. We define

$$\widetilde{Pin}_+(n) \subset \pi_{0,1}Pin_+(n)$$

to be the subgroup given by morphisms $x \to y$ in $\pi_{0,*}Pin_+(n)$ with target 1 or e_n. This is well defined since $\{1, e_n\} \subset Pin_+(n)$ is a subgroup. This observation is indeed the key step of the proof, and it shows for example why the negative pin group does not occur as (6.4). Moreover, $\pi_{0,*}\rho$ induces a homomorphism

(a) $$\widetilde{Pin}_+(n) \longrightarrow \widetilde{O}(n).$$

It is well-known that $Pin_+(n)$ has two components. The two components are separated by the function

$$Pin_+(n) \overset{\rho}{\twoheadrightarrow} O(n) \overset{\det}{\twoheadrightarrow} \{\pm 1\}.$$

In particular 1 and e_n lie in different components, hence the homomorphism

(b) $$\widetilde{Pin}_+(n) \longrightarrow Pin_+(n)$$

is surjective. Moreover, it is injective since the two components of $Pin_+(n)$ are known to be simply connected, therefore (b) is an isomorphism. The inverse

$$\text{(c)} \qquad Pin_+(n) \longrightarrow \widetilde{Pin}_+(n)$$

sends an element $x \in Pin_+(n)$ to the image by $\pi_{0,*}\rho$ of the unique morphism $x \to y$ in $\pi_{0,*}Pin_+(n)$ with $y = e_n$ provided $\det \rho(x) = -1$ or $y = 1$ otherwise.

Obviously the composite of (c) and (a) is compatible with the projections onto $O(n)$ in (6.4) and (6.5), so we only need to check that the composite of (c) and (a) induces an isomorphism between the kernels. The kernel of ρ is -1. A path $\gamma\colon [0,1] \to Pin_+(n)$ from -1 to 1 is defined by

$$\gamma(t) = (-\cos(t\pi)e_2 + \sin(t\pi)e_1)e_2 = -\cos(t\pi) + \sin(t\pi)e_1e_2.$$

Now it is an easy exercise to check that $\rho\gamma\colon [0,1] \to O(n)$ is a generator of $\pi_1 O(n)$, and hence we are done.

□

Remark 6.8. We recall that the extension (6.5), and therefore (6.4), represents the second Stiefel-Whitney class $w_2 \in H^2(BO(n), \mathbb{Z}/2)$, compare [Tei92] page 21.

By definition of (6.4) and Proposition 6.7 we obtain the following corollary.

Corollary 6.9. *For $n \geq 3$ the symmetric track group $\mathrm{Sym}_\square(n)$ is the pull back of the central extension for the positive pin group $Pin_+(n)$ in (6.5) along the inclusion $i\colon \mathrm{Sym}(n) \subset O(n)$, in particular there is a central extension*

$$\mathbb{Z}/2 \hookrightarrow \mathrm{Sym}_\square(n) \stackrel{\delta}{\twoheadrightarrow} \mathrm{Sym}(n)$$

classified by the pull-back of the second Stiefel-Whitney class $i^*w_2 \in H^2(\mathrm{Sym}(n), \mathbb{Z}/2)$.

Remark 6.10. The low-dimensional mod 2 cohomology groups of symmetric groups $\mathrm{Sym}(n)$ are as follows, $n \geq 3$,

$$H^1(\mathrm{Sym}(n), \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2\,\chi \oplus \mathbb{Z}/2\,i^*w_1, & \text{for } n = 3; \\ \mathbb{Z}/2\,i^*w_1, & \text{for } n > 3; \end{cases}$$

$$H^2(\mathrm{Sym}(n), \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2\,i^*w_1^2, & \text{for } n = 3; \\ \mathbb{Z}/2\,i^*w_1^2 \oplus \mathbb{Z}/2\,i^*w_2, & \text{for } n > 3. \end{cases}$$

Here we write $w_j \in H^j(BO(n), \mathbb{Z}/2)$ for the j^{th} Stiefel-Whitney class, $j = 1, 2$. The pull-back i^*w_1 corresponds to the sign homomorphism

$$i^*w_1 = \mathrm{sign}\colon \mathrm{Sym}(n) \longrightarrow \{\pm 1\} \cong \mathbb{Z}/2,$$

The pull-back of the second Stiefel-Whitney class is trivial for $n = 3$, therefore $\mathrm{Sym}_\square(3)$ is a split extension of $\mathrm{Sym}(3)$ by $\mathbb{Z}/2$, and $\chi\colon \mathrm{Sym}_\square(3) \twoheadrightarrow \mathbb{Z}/2$ is a retraction.

The following structure theorem follows from Corollary 6.9.

Theorem 6.11. *The symmetric track group $\mathrm{Sym}_\square(n)$ is the subgroup of $Pin_+(n)$ formed by the units $x \in C_+(n)$ such that for any $1 \leq i \leq n$ there exists $1 \leq \sigma(i) \leq n$ with $-xe_ix^{-1} = e_{\sigma(i)}$. The boundary homomorphism $\delta\colon \mathrm{Sym}_\square(n) \twoheadrightarrow \mathrm{Sym}(n)$ sends* *x above to the permutation $\delta(x) = \sigma$. The group* $\mathrm{Sym}_\square(n)$ *has a presentation given by generators ω, t_i, $1 \leq i \leq n-1$, and relations*

$$\begin{aligned} t_1^2 &= 1 \text{ for } 1 \leq i \leq n-1, \\ (t_i t_{i+1})^2 &= 1 \text{ for } 1 \leq i \leq n-2, \\ \omega^2 &= 1, \\ t_i \omega &= \omega t_i \text{ for } 1 \leq i \leq n-1, \\ t_i t_j &= \omega t_j t_i \text{ for } 1 \leq i < j-1 \leq n-1; \end{aligned}$$

with $\omega \mapsto -1$ and $t_i \mapsto \frac{1}{\sqrt{2}}(e_i - e_{i+1})$. In particular $\delta(\omega) = 0$ and $\delta(t_i) = (i\; i+1)$.

This is a group considered by Schur in [Sch11] and by Serre in [Ser84].

Remark 6.12. In case $n = 2$ we have $O(2) = \{\pm 1\} \ltimes S^1$ with $\{\pm 1\}$ acting on S^1 exponentially, $\widetilde{O}(2) = \{\pm 1\} \ltimes \mathbb{R}$ with $\{\pm 1\}$ acting on $\mathbb{R}$ multiplicatively, and the projection $q\colon \widetilde{O}(2) \twoheadrightarrow O(2)$ defined as in (6.4) is the identity in $\{\pm 1\}$ and the exponential map in the second coordinate $\mathbb{R} \twoheadrightarrow S^1\colon x \mapsto \exp(2\pi i x)$. In particular we have an abelian extension

$$\mathbb{Z} \hookrightarrow \widetilde{O}(2) \stackrel{q}{\twoheadrightarrow} O(2).$$

The induced action of $O(2)$ on $\mathbb{Z}$ is given by the determinant $\det\colon O(2) \twoheadrightarrow \{\pm 1\}$. By Remark 6.2 the extended symmetric track group $\overline{\mathrm{Sym}}_\square(2)$ is the pull-back of $i\colon \mathrm{Sym}(2) \subset O(2)$ along q, therefore we have an abelian extension

$$\mathbb{Z} \hookrightarrow \overline{\mathrm{Sym}}_\square(2) \stackrel{q}{\twoheadrightarrow} \mathrm{Sym}(2), \tag{6.13}$$

where $\mathrm{Sym}_\square(2)$ acts on $\mathbb{Z}$ by the unique isomorphism $\mathrm{Sym}_\square(2) \cong \{\pm 1\}$. Now the symmetric track group $\mathrm{Sym}_\square(2)$ can be identified with the push-forward of the extension (6.13) along the natural projection $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2$, therefore we get a central extension

$$\mathbb{Z}/2 \hookrightarrow \mathrm{Sym}_\square(2) \stackrel{q}{\twoheadrightarrow} \mathrm{Sym}(2). \tag{6.14}$$

The cohomology group $H^2(\mathrm{Sym}_\square(2), \mathbb{Z}) = 0$ is trivial, so (6.13) is a splitting extension. Moreover (6.14) is also splitting since it is the push-forward of (6.13).

7. An application to the cup-one-product

Let $n \geq m > 1$ be even integers. The cup-one-product operation

$$\pi_n S^m \longrightarrow \pi_{2n+1} S^{2m}\colon \alpha \mapsto \alpha \cup_1 \alpha$$

is defined in the following way, compare [HM93] 2.2.1. Let k be any positive integer and let $\tau_k \in \mathrm{Sym}(2k)$ be the permutation exchanging the first and the second block of k elements in $\{1, \dots, 2k\}$. If k is even then $\operatorname{sign} \tau_k = 1$. We choose for any even integer $k > 1$ a track $\hat{\tau}_k\colon \tau_k \Rightarrow 1_{S^{2k}}$ in $\mathrm{Sym}_\square(2k)$. Consider the following diagram in the track category $\mathbf{Top}^*$ of pointed spaces where $a\colon S^n \to S^m$ represents α.

$$\begin{array}{ccccccc} & & S^{2n} & \xrightarrow{a \wedge a} & S^{2m} & & \\ 1_{S^{2n}} & \stackrel{\hat{\tau}_n^{\boxminus}}{\Longrightarrow} & \downarrow \tau_n & & \tau_m \downarrow & \stackrel{\hat{\tau}_m}{\Longrightarrow} & 1_{S^{2m}} \\ & & S^{2n} & \xrightarrow{a \wedge a} & S^{2m} & & \end{array} \tag{7.1}$$

By pasting this diagram we obtain a self-track of $a \wedge a$

$$(\hat{\tau}_m(a \wedge a))\Box((a \wedge a)\hat{\tau}_n^{\boxminus})\colon a \wedge a \Rightarrow a \wedge a. \tag{7.2}$$

The set of self-tracks $a \wedge a \Rightarrow a \wedge a$ is the automorphism group of the map $a \wedge a$ in the track category $\mathbf{Top}^*$. The element $\alpha \cup_1 \alpha \in \pi_{2n+1} S^{2m}$ is given by the track (7.2) via the well-known Barcus-Barratt-Rutter isomorphism

$$\mathrm{Aut}(a \wedge a) \cong \pi_{2n+1} S^{2m},$$

see [BB58], [Rut67] and also [Bau91] VI.3.12 and [BJ01] for further details.

The following theorem generalizes [BJM83] 6.5.

Theorem 7.3. *The formula*

$$2(\alpha \cup_1 \alpha) = \frac{n+m}{2}(\alpha \wedge \alpha)(\Sigma^{2(n-1)}\eta)$$

holds, where $\eta\colon S^3 \to S^2$ *is the Hopf map.*

The proof of Theorem 7.3 is based on the following lemma.

Lemma 7.4. *The following formula holds in* $\mathrm{Sym}_\Box(2k)$

$$\hat{\tau}_k^2 = \omega^{\binom{k}{2}}.$$

Proof. Here we use the representation of $\mathrm{Sym}_\Box(2k)$ in $Pin_+(2k)$ given in Theorem 6.11 and the relations (1) and (2) in the definition of the Clifford algebra $C_+(2k)$, see Definition 6.6.

The permutation τ_k can be expressed as a product of transpositions as follows

$$\tau_k = (1\ \ k)(2\ \ k+1)\cdots(k-1\ \ 2k-1)(k\ \ 2k).$$

The element $\frac{1}{\sqrt{2}}(e_i - e_{i+k}) \in S^{2k-1} \subset Pin_+(2k)$ acts on $\mathbb{R}^{2k}$ (with coordinates e_i, $1 \leq i \leq 2k$) by reflection along the plane orthogonal to $\frac{1}{\sqrt{2}}(e_i - e_{i+k})$, see Definition 6.6. This pane is $e_i = e_{i+k}$, therefore the action of $\frac{1}{\sqrt{2}}(e_i - e_{i+k})$ on $\mathbb{R}^{2k}$ interchanges the coordinates in e_i and e_{i+k} and preserves all the other ones.

Now by Theorem 6.11 $\frac{1}{\sqrt{2}}(e_i - e_{i+k})$ lies in the positive pin representation of $\mathrm{Sym}_\Box(2k)$ and $\delta(\frac{1}{\sqrt{2}}(e_i - e_{i+k})) = (i\ \ i+k)$, so

$$\hat{\tau}_k = \pm\frac{1}{2^{\frac{k}{2}}}(e_1 - e_{k+1})(e_2 - e_{k+2})\cdots(e_{k-1} - e_{2k-1})(e_k - e_{2k}).$$

The following equalities hold in the Clifford algebra $C_+(2k)$, see the defining relations in Definition 6.6, $i \neq j$,

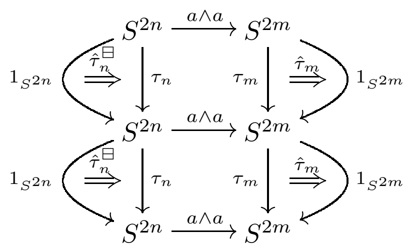
$$\begin{aligned}
(e_i - e_{i+k})^2 &= e_i^2 - e_i e_{i+k} - e_{i+k} e_i + e_{i+k}^2 \\
&= 1 - e_i e_{i+k} + e_i e_{i+k} + 1 \\
&= 2, \\
(e_i - e_{i+k})(e_j - e_{j+k}) &= e_i e_j - e_i e_{j+k} - e_{i+k} e_j + e_{i+k} e_{j+k} \\
&= -e_j e_i + e_{j+k} e_i + e_j e_{i+k} - e_{j+k} e_{i+k} \\
&= -(e_j - e_{j+k})(e_i - e_{i+k}).
\end{aligned}$$

Hence we observe that

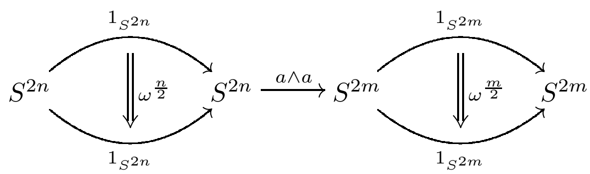
$$\begin{aligned}\hat{\tau}_k^2 &= \frac{1}{2^{\frac{k}{2}}}\frac{1}{2^{\frac{k}{2}}}(-1)^{k-1}2(-1)^{k-2}2\cdots(-1)^1 2(-1)^0 2\\ &= (-1)^{k-1}(-1)^{k-2}\cdots(-1)^1(-1)^0\\ &= (-1)^{\binom{k}{2}}.\end{aligned}$$

The proof is now finished. □

Proof of Theorem 7.3. The element $2(\alpha\cup_1\alpha)$ corresponds to the pasting of the following diagram



By Lemma 7.4 and using that n and m are even this composite track coincides with



therefore $2(\alpha\cup_1\alpha)$ corresponds to the self-track

(a) $$((\omega^{\frac{m}{2}})(a\wedge a))\square((a\wedge a)(\omega^{\frac{n}{2}})).$$

The self-track $\omega^{\frac{m}{2}}(a\wedge a)$ corresponds to the homotopy class

(b) $$(\frac{m}{2}(\Sigma^{2(m-1)}\eta))(\Sigma(\alpha\wedge\alpha)).$$

Since $\Sigma(\alpha\wedge\alpha)=\pm(\Sigma^{m+1}\alpha)(\Sigma^{n+1}\alpha)$ which is a composite of two triple suspensions (b) is

(c) $$(\alpha\wedge\alpha)(\frac{m}{2}(\Sigma^{2(n-1)}\eta)).$$

Moreover, the self-track $(a\wedge a)(\omega^{\frac{n}{2}})$ corresponds to

(d) $$(\alpha\wedge\alpha)(\frac{n}{2}(\Sigma^{2(n-1)}\eta)),$$

so the self-track (a) corresponds to the sum of (c) and (d)

$$(\alpha\wedge\alpha)(\frac{n+m}{2}(\Sigma^{2(n-1)}\eta)).$$

□

References

[Bau91] H.-J. Baues, *Combinatorial Homotopy and 4-Dimensional Complexes*, Walter de Gruyter, Berlin, 1991.

[BB58] W. D. Barcus and M. G. Barratt, *On the homotopy classification of the extensions of a fixed map*, Trans. Amer. Math. Soc. **88** (1958), 57–74.

[BJ01] H.-J. Baues and M. Jibladze, *Suspension and loop objects and representability of tracks*, Georgian Math. J. **8** (2001), no. 4, 683–696.

[BJM83] M. G. Barratt, J. D. S. Jones, and M. E. Mahowald, *The Kervaire invariant problem*, Proceedings of the Northwestern Homotopy Theory Conference (Evanston, Ill., 1982) (Providence, RI), Contemp. Math., vol. 19, Amer. Math. Soc., 1983, pp. 9–22.

[BJP05] H.-J. Baues, M. Jibladze, and T. Pirashvili, *Quadratic algebra of square groups*, Preprint, 2005.

[BM05a] H.-J. Baues and F. Muro, *Secondary homotopy groups*, Preprint MPIM, 2005.

[BM05b] ———, *Secondary homotopy groups of spectra*, In preparation, 2005.

[BM05c] ———, *Smash products for secondary homotopy groups*, Preprint MPIM, 2005.

[BP99] H.-J. Baues and T. Pirashvili, *Quadratic endofunctors of the category of groups*, Advances in Math. **141** (1999), 167–206.

[BtD85] T. Bröcker and T. tom Dieck, *Representations of compact Lie groups*, Graduate Texts in Mathematics, vol. 98, Springer-Verlag, New York, 1985.

[Con84] D. Conduché, *Modules croisés généralisés de longueur* 2, J. Pure Appl. Algebra **34** (1984), no. 2-3, 155–178.

[Ell93] Graham Ellis, *Crossed squares and combinatorial homotopy*, Math. Z. **214** (1993), no. 1, 93–110.

[HM93] K. A. Hardie and M. E. Mahowald, *On cup-one and Toda bracket*, Quaestiones Math. **16** (1993), no. 2, 193–207.

[Nor90] K. Norrie, *Actions and automorphisms of crossed modules*, Bull. Soc. Math. France **118** (1990), no. 2, 129–146.

[Rut67] John W. Rutter, *A homotopy classification of maps into an induced fibre space*, Topology **6** (1967), 379–403.

[Sch11] I. Schur, *Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen*, J. Reine Angew. Math. **139** (1911), 155–250.

[Ser84] J.-P. Serre, *L'invariant de Witt de la forme* $\mathrm{Tr}(x^2)$, Comment. Math. Helv. **59** (1984), no. 4, 651–676.

[Tei92] P. Teichner, *Topological four-manifolds with finite fundamental group*, Dissertation zur Erlangung des Grades Doktor der Naturwissenschaften, Johannes-Gutenberg Universität in Mainz, Mainz, März 1992.

[Tod62] H. Toda, *Composition methods in homotopy groups of spheres*, Annals of Mathematics Studies, No. 49, Princeton University Press, Princeton, N.J., 1962.

Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany

E-mail address: baues@mpim-bonn.mpg.de, muro@mpim-bonn.mpg.de