MODULES AND MORITA THEOREM FOR OPERADS

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Dedicated to A. N. Tyurin on the occasion of his 60th birthday

§0. Introduction and summary.

(0.1) Morita theory. Let A, B be two commutative rings. If their respective categories of modules are equivalent, then A and B are isomorphic. This is not anymore true if A and/or B are not assumed to be commutative. Morita theory describes the relashionship between A and B in this case. The fact that a ring is not determined uniquely by its category of modules has deep implications for non–commutative geometry which tends to substitute an elusive non–commutative space by the category of sheaves on it.

In this note we construct a fragment of Morita theory for operads. More precisely, let k be a field. For a k–linear operad $\mathcal{P}$, we construct matrix operads $\mathrm{Mat}\,(n, \mathcal{P})$ and prove that their respective categories of representations (that is, algebras) are equivalent.

In order to compare the situation with the classical one, let us remind the exact statement of the Morita theorem. For A, B as above, denote by A–Mod, B–Mod their respective categories of left modules. They are equivalent iff one can find a (B, A)–bimodule M which is finitely generated and projective as B–module. Then the functor A–Mod $\to$ B–Mod :

$$N \mapsto M \otimes_A N \tag{0.1.1}$$

establishes an equivalence, and A is isomorphic to an algebra of the type $e\,\mathrm{Mat}(n, B)e$ where e is the idempotent defining M.

A k–linear operad $\mathcal{P}$ can be considered as an associative ring in the linear monoidal category $\mathbf{S}$–Vect whose objects are families of representations of all the symmetric groups $\mathbf{S}_n, n \geq 0$. The plethysm monoidal product in this category (denoted $\circ$ below) is not symmetric. This adds a new dimension of non–commutativity to the situation. In particular, the notions of left and right modules become asymmetric, and in our Morita theorem we replace left modules by $\mathcal{P}$–algebras, whereas right modules remain right $\mathcal{P}$–modules in $\mathbf{S}$–Vect. Denote the categories of these objects $\mathcal{P}$–Alg and Mod–$\mathcal{P}$ respectively.

Our argument proceeds as follows.

(i) We use a kind of relative tensor product operation $\circ_{\mathcal{P}} : \mathrm{Mod}–\mathcal{P} \times \mathcal{P}–\mathrm{Alg} \to \mathrm{Vect}$ (see sec. 1.3 below) to construct an operadic version of the functor (0.1.1). Let $\mathcal{Q}$ be another k–linear operad. Consider an $\mathbf{N}$–graded $\mathcal{Q}$–algebra M which is simultaneously a right $\mathcal{P}$–module such that both structures are compatible in the sense that will be made explicit in sec. 1.4. Then for any A in $\mathcal{P}$–Alg the product $M \circ_{\mathcal{P}} A$ is in $\mathcal{Q}$–Alg.

(ii) Given an M in Mod–$\mathcal{P}$, we can construct its endomorphism operad $\mathcal{Q} = \mathrm{Op\,End}_{\mathcal{P}}(M)$. It consists of the part of $\bigoplus_n \mathrm{Hom}\,(M^{\otimes n}, M)$ compatible with the grading and the action of $\mathcal{P}$. Then M becomes a $\mathcal{Q}$–algebra, so that the construction of (i) provides the functor $\mathcal{P}–\mathrm{Alg} \to \mathcal{Q}–\mathrm{Alg}$.

(iii) Finally, if we take for M a free module of rank n in Mod–$\mathcal{P}$, its endomorphism operad is denoted $\mathrm{Mat}\,(n, \mathcal{P})$, and the functor produced in (ii) turns out to be an equivalence of categories: see (1.8.1).

Hopefully, this construction can be extended to a fuller statement providing the necessary conditions for equivalence as well.

Moreover, the multifaceted analogies between linear operads and rings suggest several other Morita–like contexts. For example, one can ask which equivalences may exist between the categories of right modules for different operads. Notice that equivalences between the categories of modules over algebras of various operadic types can be studied in principle by passing to the universal enveloping algebras and applying Morita theory for associative algebras.

(0.2) Plan of the paper. The first section is devoted to the Morita theorem. We first remind the generalities on operads, operadic modules, and algebras, in order to fix notation. We proceed by describing several examples and constructions related to the operadic modules which deserve to be better known: see e. g. Theorem (1.6.3) for the construction of a Lie algebra, which provides in particular a family of canonical vector fields on any formal Frobenius manifold. Finally, we prove the operadic Morita theorem described above.

The simplest application of the classical Morita theory to supergeometry furnishes the following fact: *the category of D–modules on a supermanifold M is equivalent to the one on its underlying manifold* M_{red}. (To get a more sensible statement, $\mathbf{Z}_2$–grading must be introduced; for more sophisticated generalizations see [P]). In fact, the algebra of differential operators of the $\mathbf{Z}_2$–graded exterior algebra $W := \wedge_k(V)$ of a finite dimensional vector space V is isomorphic to the matrix algebra $\mathrm{End}_k(W)$. Therefore, sheaves of algebras Diff_M and $\mathrm{Diff}_{M_{\mathrm{red}}}$ are Morita equivalent. In the second section of this paper we discuss some operadic versions of this remark.

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§1. Modules over operads and abstract Morita theory.

(1.1) Monoidal structures on S–Vect. We work over a fixed base field k and denote by Vect the category of k–vector spaces. We denote by $\mathbf{S}$ the category whose objects are the standard finite sets $[n] = \{1, 2, ..., n\}$, $n \geq 0$ and morphisms are bijections, i.e., $\mathrm{Hom}_{\mathbf{S}}([n], [n]) = \mathbf{S}_n$ is the symmetric group. By an $\mathbf{S}$–space we mean a contravariant functor $\mathbf{S} \to \mathrm{Vect}$, i.e., a collection $\mathcal{V} = \{\mathcal{V}(n)\}_{n\geq 0}$ where $\mathcal{V}(n)$ is a vector space with a right $\mathbf{S}_n$-action. Let $\mathbf{S}$–Vect be the category of $\mathbf{S}$–spaces. It possesses three important monoidal structures. Let $\mathcal{V} = \{\mathcal{V}(n)\}$ and $\mathcal{W} = \{\mathcal{W}(n)\}$ be two $\mathbf{S}$–spaces. Set

$$(\mathcal{V} \otimes \mathcal{W})(n) = \mathcal{V}(n) \otimes_k \mathcal{W}(n), \tag{1.1.1}$$

$$(\mathcal{V} \boxtimes \mathcal{W})(n) = \bigoplus_{i+j=n} \mathrm{Ind}_{S_i \times S_j}^{S_n} (\mathcal{V}(i) \otimes_k \mathcal{W}(j)), \tag{1.1.2}$$

$$\mathcal{V} \circ \mathcal{W} = \bigoplus_{n\geq 0} \mathcal{V}(n) \otimes_{S_n} \mathcal{W}^{\boxtimes n}. \tag{1.1.3}$$

Note that as a vector space,

$$(\mathcal{V} \circ \mathcal{W})(n) = \bigoplus_{p\geq 0} \bigoplus_{a_1+...+a_p=n} \mathcal{V}(p) \otimes_k \mathcal{W}(a_1) \otimes_k ... \otimes_k \mathcal{W}(a_p). \tag{1.1.4}$$

Also, for $\mathcal{V} \in \mathbf{S}$–Vect and $X \in \mathrm{Vect}$ define the vector space

$$\mathcal{V}\langle X\rangle = \bigoplus_{n\geq 0} \mathcal{V}(n) \otimes_{S_n} X^{\otimes n}. \tag{1.1.5}$$

The products (1.1.2–3) are analogous to the operations of multiplication and composition of fornal power series $\sum v_n x^n/n!$, and (1.1.5) is analogous to the evaluation of such series. The product (1.1.3) is known as plethysm. Each of the three products makes $\mathbf{S}$–Vect into a k–linear monoidal category, with $\otimes$ and $\boxtimes$ giving in fact symmetric monoidal structures, whereas $\circ$ is non–symmetric. The unit objects with respect to the three structures are:

$$\mathcal{C}om : \quad \mathcal{C}om(n) = k, n \geq 0, \tag{1.1.6}$$

$$\mathbf{1} : \quad \mathbf{1}(0) = k,\ \mathbf{1}(n) = 0, n \neq 0, \tag{1.1.7}$$

$$\mathbf{I}: \quad \mathbf{I}(1) = k,\, \mathbf{I}(n) = 0, n \neq 1. \tag{1.1.8}$$

Note that we have canonical identifications

$$(\mathcal{V} \circ \mathcal{W})\langle X \rangle = \mathcal{V}\langle \mathcal{W}\langle X \rangle \rangle, \tag{1.1.9}$$

$$(\mathcal{V} \boxtimes \mathcal{W}) \circ \mathcal{X} = (\mathcal{V} \circ \mathcal{X}) \boxtimes (\mathcal{W} \circ \mathcal{X}), \tag{1.1.10}$$

$$(\mathcal{V} \boxtimes \mathcal{W})\langle X \rangle = \mathcal{V}\langle X \rangle \otimes_k \mathcal{W}\langle X \rangle, \tag{1.1.11}$$

which correspond to the familiar rules of dealing with power series. In addition, we have the natural morphisms of **S**–spaces

$$(\mathcal{V}_1 \otimes \mathcal{V}_2) \circ (\mathcal{W}_1 \otimes \mathcal{W}_2) \to (\mathcal{V}_1 \circ \mathcal{W}_1) \otimes (\mathcal{V}_2 \circ \mathcal{W}_2) \tag{1.1.12}$$

and of vector spaces

$$(\mathcal{V}_1 \otimes \mathcal{V}_2)\langle X_1 \otimes X_2 \rangle \to \mathcal{V}_1\langle X_1 \rangle \otimes \mathcal{V}_2\langle X_2 \rangle. \tag{1.1.13}$$

For example, (1.1.12) is the "diagonal map" sending the summand

$$\mathcal{V}_1(p) \otimes \mathcal{V}_2(p) \otimes \mathcal{W}_1(a_1) \otimes \mathcal{W}_2(a_1) \otimes ... \otimes \mathcal{W}_1(a_p) \otimes \mathcal{W}_2(a_p) \subset$$

$$\subset \big((\mathcal{V}_1 \otimes \mathcal{V}_2) \circ (\mathcal{W}_1 \otimes \mathcal{W}_2)\big)(n), \quad a_1 + ... + a_p = n,$$

see (1.1.4), into the tensor product of the summand

$$\mathcal{V}_1(p) \otimes \mathcal{W}_1(a_1) \otimes ... \otimes \mathcal{W}_1(a_p) \subset (\mathcal{V}_1 \circ \mathcal{W}_1)(n)$$

and of the summand

$$\mathcal{V}_2(p) \otimes \mathcal{W}_2(a_1) \otimes ... \otimes \mathcal{W}_2(a_p) \subset (\mathcal{V}_2 \circ \mathcal{W}_2)(n).$$

In particular, this map is injective. The morphism (1.1.13) is defined in a similar way but it is not injective in general.

(1.2) Operads and modules. A k-linear operad $\mathcal{P}$ can be defined as a monoid object in (**S**–Vect, $\circ$). This means that $\mathcal{P}$ is an **S**–space endowed with associative multiplication (which is a morphism in **S**–Vect) and a distinguished element

$$\mu_{\mathcal{P}} : \mathcal{P} \circ \mathcal{P} \to \mathcal{P}, \; 1 \in \mathcal{P}(1) \tag{1.2.1}$$

with the usual properties. Spelling these data out with the help of (1.1.4), we get the maps

$$\mu_{m_1,\dots,m_l} : \mathcal{P}(l) \otimes \mathcal{P}(m_1) \otimes \cdots \otimes \mathcal{P}(m_l) \to \mathcal{P}(m_1 + \cdots + m_l) \tag{1.2.2}$$

satisfying May's axioms (see [M], [S], [Gi–K]).

A right, resp. left $\mathcal{P}$–module is an $\mathbf{S}$–space M together with a multiplication morphism

$$^{M}\mu : M \circ \mathcal{P} \to M, \quad \text{resp.} \quad \mu^{M} : \mathcal{P} \circ M \to M \tag{1.2.3}$$

satisfying the usual asociativity and unit requirements (cf. [Mar1-2], [Re], [BJT]). We denote by Mod–$\mathcal{P}$, resp. $\mathcal{P}$–Mod, the categories of right, resp. left $\mathcal{P}$-modules. Note that Mod–$\mathcal{P}$ is an abelian k-linear category, because the plethysm product $\circ$ is linear in the first argument, but $\mathcal{P}$–Mod is not even additive.

Similarly, a $\mathcal{P}$–algebra is a vector space A together with a morphism of vector spaces

$$\mu^{A} : \mathcal{P}\langle A\rangle \to A \tag{1.2.4}$$

satisfying the usual requirements. We denote by $\mathcal{P}$–Alg the category of $\mathcal{P}$–algebras. It is also not additive. A left $\mathcal{P}$–module can be seen as a graded $\mathcal{P}$–algebra.

If V is a vector space, then $\mathcal{P}\langle V\rangle$ is a $\mathcal{P}$–algebra called *the free algebra generated by V*.

(1.2.5) Examples. (a) Any vector space V determines the operad $\mathrm{Op\,End}\,(V)$ with

$$\mathrm{Op\,End}\,(V)(n) = \mathrm{Hom}_k\,(V^{\otimes n}, V) \tag{1.2.6}$$

and the multiplication law (1.2.2) given by the composition of multilinear maps. A structure of the $\mathcal{P}$–algebra on V is the same as a morphism of operads $\mathcal{P} \to \mathrm{Op\,End}(V)$.

More generally, if $(\mathcal{C}, \otimes)$ is any k-linear symmetric monoidal category, any object $V \in \mathcal{C}$ gives rise to an operad $\mathrm{Op\,End}_{\mathcal{C}}(V)$ defined in a similar way.

(b) The $\mathbf{S}$–space $\mathcal{C}om$ has a natural operad structure with all the maps (1.2.2) being the canonical identifications $k \otimes \dots \otimes k \to k$. A $\mathcal{C}om$–algebra is the same as a commutative algebra in the usual sense. Note that $\mathcal{C}om = \mathrm{Op\,End}(k)$.

(c) The unit object $\mathbf{I} \in (\mathbf{S}\text{–Vect}, \circ)$ is an operad called the trivial operad. An $\mathbf{I}$–algebra is the same as a vector space and an $\mathbf{I}$-module (left or right) is the same as an $\mathbf{S}$–space.

(d) Any operad $\mathcal{P}$ is a left and a right module over itself. It follows that for any vector space V the $\mathbf{S}$–space $V \otimes \mathcal{P} = \{V \otimes_k \mathcal{P}(n)\}_{n \geq 0}$ is a right $\mathcal{P}$–module. Such modules will be called free (of finite rank, if dim $V < \infty$). A right module will be called projective, if it is a direct summand of a free module.

(e) An example of a right $\mathrm{Op\,End}(V)$–module is given by $V^\star = \{(V^*)^{\otimes n}\}_{n \geq 0}$. The module structure is given by taking the superposition of an n–linear form $V^{\otimes n} \to k$ with n multilinear maps $V^{\otimes a_i} \to V$, $i = 1, ..., n$. Notice that if $\dim(V) = d < \infty$, then $\mathrm{Op\,End}(V)$, as a right module over itself, is isomorphic to the direct sum of d copies of $V^\star$, so $V^\star$ is a projective module.

(f) Let Γ be the category whose objects are the finite sets $[n]$, $n \geq 0$ as above, but morphisms are all maps of finite sets. Thus $\mathbf{S}$ is the subcategory of Γ formed by all objects and all isomorphisms. By a Γ–space, cf. [Se], we mean a contravariant functor $\Gamma \to \mathrm{V}ect$. Such a functor gives, in particular, an $\mathbf{S}$–space. We denote by Γ–Vect the category of Γ–spaces.

Claim. *The category* Mod–$\mathcal{C}om$ *of right* $\mathcal{C}om$*–modules is equivalent to* Γ–Vect *so that the forgetful functor* Mod–$\mathcal{C}om \to \mathbf{S}$–Vect *is identified with the restriction functor* Γ–Vect $\to \mathbf{S}$–Vect.

Before proving this, let us recall the connection between Γ-spaces and simplicial spaces. Let $\Delta \subset \Gamma$ be the subcategory with the same objects $[n]$ and only monotone maps as morphisms. Thus contravariant functors $\Delta \to$ Vect are simplicial vector spaces; their category will be denoted Δ–Vect. Each morphism in Δ has the form

$$(1.2.7)\quad p_{a_1,...,a_n} : [a_1 + ... + a_n] \to [n], \quad \{1, ..., a_1\} \mapsto 1,\ \{a_1 + 1, ..., a_2\} \mapsto 2, \text{ etc.}$$

for some uniquely defined $a_1, ..., a_n \geq 0$. Such a morphism is surjective when all $a_i > 0$ and injective when all $a_i \leq 1$. The category Γ is obviously generated by the subcategories Δ and $\mathbf{S}$.

Let us now prove our Claim. As $\mathcal{C}om\,(n) = k$ for each $n \geq 0$, a right $\mathcal{C}om$–module is a vector $\mathbf{S}$–space M together with maps $\mu_{a_1,...,a_n} : M(n) \to M(a_1 + ... + a_n)$ given for any $a_1, ..., a_n \geq 0$ and satisfying the associativity conditions coming from the compositions in $\mathcal{C}om$. The latter compositions are all reduced to the canonical identifications $k \otimes ... \otimes k \to k$. Hence the associativity conditions mean that the rule $p_{a_1,...,a_n} \mapsto \mu_{a_1,...,a_n}$ defines a contravariant functor $\Delta \to$ Vect. Next, the equivariance conditions (that ${}^M\mu$ is a morphism of $\mathbf{S}$–modules) mean that the two functors defined on $\mathbf{S}, \Delta \subset \Gamma$ give rise to a well–defined functor $\Gamma \to$ Vect.

(g) Let $\mathcal{A}ss$ be the operad whose algebras are associative algebras with unit ([Gi–K]). Thus $\mathcal{A}ss(n)$ is the regular representation of $\mathbf{S}_n$. Let us describe right $\mathcal{A}ss$–modules. Denote by $\widetilde{\Gamma}$ the category whose objects are the finite sets $[n]$ as before and a morphism $\Phi : [m] \to [n]$ consists, first, of a map $\phi : [m] \to [n]$ in the ordinary sense and, second, of a choice of a total order on each fiber $\phi^{-1}(i)$. The composition of morphisms in $\widetilde{\Gamma}$ is defined using the lexicographic ordering of the fibers of a composition. Thus we have a functor $\widetilde{\Gamma} \to \Gamma$ surjective on morphisms; both Δ and $\mathbf{S}$ are realized as subcategories of Γ which generate $\mathrm{Mor}(\Gamma)$. As before, we define a $\widetilde{\Gamma}$–space as a contravariant functor $\widetilde{\Gamma} \to \mathrm{Vect}$. The proofs of the following two propositions are straightforward and left to the reader.

Claim. *(a) If $\mathcal{V}$ is a $\widetilde{\Gamma}$–space, then the collection of the $\mathcal{V}(n)_{S_n}$ forms a simplicial space.*

(b) If $V_\bullet = \{V_n\}_{n\geq 0}$ is a simplicial space, then the collection of $V_n \otimes_k k[S_n]$ forms a $\widetilde{\Gamma}$–space.

Claim. *The category of right $\mathcal{A}ss$–modules is equivalent to the category of $\widetilde{\Gamma}$–spaces.*

(h) Denote by $\overline{M}_{g,n+1}$ the Deligne–Mumford stack classifying stable curves with $n+1$ points (x_i) labeled by $(0,1,\dots,n) = [n] \cup \{0\}$, cf. [Kn]. The point x_0 will play the special role. In the dual graph of the curve it will define a flag called *the root.* If $g = 0$, the choice of the root induces the orientation of the dual tree "from the root upwards".

The group $\mathbf{S}_n$ acts upon $\overline{M}_{g,n+1}$ by renumbering all points except for x_0. Moreover, consider the boundary morphisms

$$\overline{M}_{g,l+1} \times \overline{M}_{0,m_1+1} \times \dots \overline{M}_{0,m_l+1} \to \overline{M}_{g,m_1+\dots+m_l+1} \tag{1.2.8}$$

gluing the root point of the universal curve parametrized by $\overline{M}_{0,m_i+1}$ to the i–th labeled point of the universal curve parametrized by $\overline{M}_{g,l+1}$, $i = 1,\dots,l$.

If $g = 0$, we get an operad in the monoidal category of smooth projective manifolds with n–th component $\overline{M}_{0,n+1}$ for $n \geq 2$ and multiplications (1.2.8) (cf. (1.2.2)). To produce a k–linear operad we put $H_*\overline{M}(n) := H_*(\overline{M}_{0,n+1}, k)$ and define the structure maps via pushforward of the geometric maps. We can also formally add $H_*\overline{M}(1) := k$ and $H_*\overline{M}(0) = 0$.

Algebras over $H_*\overline{M}$ (more precisely, cyclic algebras, cf. below) are called Cohomological Field Theories. Formal completion of such an algebra H at zero has a natural structure of the formal Frobenius manifold. The theory of Gromov–Witten invariants produces such a structure on the cohomology of any smooth projective algebraic manifold. For all of this see e.g. [Man2].

(k) Now consider the diagrams (1.2.8) for a fixed $g \geq 1$. They define on the $\mathbf{S}$–stack $\overline{M}_g$, $\overline{M}_g(n) := \overline{M}_{g,n+1}$, the structure of right module over the geometric operad $\overline{M}_0$. Again, in order to pass to the k–linear situation, it suffices to apply any homology theory with coefficients in k. One can even consider Chow groups because the Künneth formula holds for the left hand side of (1.2.8) in the Chow theory.

Actually, both sides of (1.2.8) admit compatible morphisms onto $\overline{M}_{g,1}$: at the left hand side, project onto $\overline{M}_{g,l+1}$, forget $x_1, \dots, x_l$ and stabilize, at the right hand side forget $x_1, \dots, x_{m_1+\dots+m_l}$ and stabilize. Let (C,x) be a stable curve of genus g with one labeled point. Denote by $C_x^{[l]}$ the fiber of the forgetful morphism $\overline{M}_{g,l+1} \to \overline{M}_{g,1}$. Generally, this is a stack, but if (C,x) has no non–trivial automorphisms, it is an algebraic variety. If moreover C is smooth, it can be expressed via Beilinson–Ginzburg–Fulton–MacPherson "resolution of diagonals", see in particular [BG], Proposition 3.8.

Our discussion shows that the $\mathbf{S}$–space $\{\{C_x^{[n]}\}\}$ is a geometric module over the geometric operad $\overline{M}_0$. Again, one can produce from it a k–linear homology module. Such modules form a local system over $\overline{M}_{g,1}$.

The last remark concerns enlarged symmetry of $\overline{M}_{g,l+1}$. In fact, the whole $\mathbf{S}_{l+1}$ acts upon this space rather than its subgroup $\mathbf{S}_l$. The axiomatization of this symmetry leads to the notion of the *cyclic operad* introduced and studied in [Ge–K]. Algebras and modules over a cyclic operad may also admit a cyclic structure, and we may ask for a cyclic version of Morita theory. We leave this question for a future research, restricting ourselves to the remark that the resolutions of diagonals $C^{[n+1]}$ for any smooth curve C form a cyclic module over our operad $\overline{M}_0$.

(1.3) Relative plethysm. We will now review the relative plethysm (or circle-over construction) for modules over an operad ([Re]).

Let $\mathcal{P}$ be an operad, M a right $\mathcal{P}$–module and N a left $\mathcal{P}$–module. Their relative plethysm $M \circ_{\mathcal{P}} N \in \mathbf{S}$–Vect is defined as the cokernel of (the difference of) the two morphisms

$$(1.3.1) \qquad \partial_0, \partial_1 : M \circ \mathcal{P} \circ N \underset{\longrightarrow}{\overset{\longrightarrow}{}} M \circ N, \quad \partial_0 = {}^M\mu \circ \mathrm{Id}_N, \ \partial_1 = \mathrm{Id}_M \circ \mu^N.$$

This construction is similar to the usual tensor product of a right and a left module over an algebra and in fact can be performed for "modules" over any monoid object in any monoidal category (provided the cokernels exist). Similarly, let A be a $\mathcal{P}$–algebra. The relative evaluation $M \circ_{\mathcal{P}} \langle A \rangle \in$ Vect is, by definition, the cokernel of the two morphisms

$$(1.3.2) \qquad \partial_0, \partial_1 : (M \circ \mathcal{P})\langle A \rangle \underset{\longrightarrow}{\overset{\longrightarrow}{}} M\langle A \rangle, \quad \partial_0 = {}^M\mu\langle \mathrm{Id}_A \rangle, \ \partial_1 = \mathrm{Id}_M\langle \mu_A \rangle.$$

The following canonical identifications are proved by mimicking the standard arguments for modules over an algebra:

$$(1.3.3) \qquad \mathcal{P} \circ_{\mathcal{P}} N = N, \quad M \circ_{\mathcal{P}} \mathcal{P} = \mathcal{P}, \quad \mathcal{P} \circ_{\mathcal{P}} \langle A \rangle = A.$$

(1.3.4) Example. Let V be a finite–dimensional k–vector space and $\mathcal{P} = \operatorname{Op}\operatorname{End}(V)$, $A = V$, $M = V^\star$, see Example 1.2.5(e). Then we claim that

$$V^\star \circ \operatorname{Op}\operatorname{End}(V)\langle V\rangle = k.$$

Indeed, we have a morphism of vector spaces

$$\widetilde{\phi}: \quad V^\star\langle V\rangle = \bigoplus_{n\geq 0}(V^*)^{\otimes n} \otimes_{S_n} V^{\otimes n} \quad \to \quad k, \tag{1.3.5}$$

which on the n–th summand is induced by the n–th tensor power of the canonical pairing. Clearly $\widetilde{\phi}$ descends to a surjective morphism $\phi : V^\star \circ_{\operatorname{Op}\operatorname{End}(V)} \langle V\rangle \to k$. Let us prove that ϕ is injective. For this, notice first that the image in $V^\star \circ_{\operatorname{Op}\operatorname{End}(V)} \langle V\rangle$ of the nth summand in $V^\star\langle V\rangle$ factors through

$$\left((V^*)^{\otimes n} \otimes_{End(V)^{\otimes n}} V^{\otimes n}\right)_{S_n} = k.$$

In other words, we have morphisms $\psi_n : k \to V^\star \circ_{\operatorname{Op}\operatorname{End}(V)} \langle V\rangle$ such that $\sum_{n\geq 0}\psi_n$ is surjective. Let us now prove that the images of the ψ_n coincide with each other. For this, choose an identification $V \to k^d$, $d = \dim V$ and use this to make V into a commutative associative algebra (the direct sum of d copies of k). We get an element $m \in \operatorname{Op}\operatorname{End}(V)(2)$ giving this algebra structure. Now, using the $\operatorname{Op}\operatorname{End}(V)$–linearity conditions for m and several copies of 1, we identify the $\operatorname{Im}(\psi_n)$ with each other.

(1.4) Bimodules and functors between categories of operadic algebras. Let $\mathcal{P}$, $\mathcal{Q}$ be two operads. A $(\mathcal{Q}, \mathcal{P})$–bimodule ([Mar1] [Re]) is, by definition, a space $M \in \mathbf{S}$–Vect together with a right $\mathcal{P}$–module structure and a left $\mathcal{Q}$–module structure on M, which commute, i.e., give rise to a well-defined map

$$\mathcal{P} \circ M \circ \mathcal{Q} \to M. \tag{1.4.1}$$

(1.4.2) Examples. (a) Any operad $\mathcal{P}$ is a $(\mathcal{P}, \mathcal{P})$-bimodule.

(b) Let V, W be two vector spaces. Define the $\mathbf{S}$–space $\operatorname{Op}\operatorname{Hom}(V, W)$ by

$$\operatorname{Op}\operatorname{Hom}(V, W)(n) = \operatorname{Hom}_k(V^{\otimes n}, W).$$

Then $\operatorname{Op}\operatorname{Hom}(V, W)$ is an $(\operatorname{Op}\operatorname{End}(W), \operatorname{Op}\operatorname{End}(V))$–bimodule.

Similarly, if $(\mathcal{C}, \otimes)$ is any symmetric monoidal category and V, W are objects of $\mathcal{C}$, we define an $(\operatorname{Op}\operatorname{End}_{\mathcal{C}}(W), \operatorname{Op}\operatorname{End}_{\mathcal{C}}(V))$–bimodule $\operatorname{Op}\operatorname{Hom}_{\mathcal{C}}(V, W)$

Let L be a right $\mathcal{Q}$–module, N a right $\mathcal{P}$–module and M, as before, a $(\mathcal{Q},\mathcal{P})$–bimodule. Then $L\circ_{\mathcal{Q}} M$ is naturally a right $\mathcal{P}$-module, $M\circ_{\mathcal{P}} N$ is naturally a left $\mathcal{Q}$–module and we have a canonical associativity isomoprhism

$$(L\circ_{\mathcal{Q}} M)\circ_{\mathcal{P}} N \simeq L\circ_{\mathcal{Q}} (M\circ_{\mathcal{P}} N), \tag{1.4.3}$$

which allows us to write more complicated iterated relative plethysm without parentheses. Similarly, if A is a $\mathcal{P}$–algebra, we have a canonical isomoprhism

$$(L\circ_{\mathcal{Q}} M)\circ_{\mathcal{P}} \langle A\rangle \simeq L\circ_{\mathcal{Q}} \langle M\circ_{\mathcal{P}} \langle A\rangle\rangle.$$

As a consequence, note a formula for the relative evaluation on a free algebra:

$$M\circ_{\mathcal{Q}} \langle \mathcal{Q}\langle A\rangle\rangle \simeq M\langle A\rangle \tag{1.4.4}$$

(take $\mathcal{P}=\mathbf{I}$ to be the trivial operad).

(1.4.5) Examples. (a) Let V be a finite–dimensional vector space. Then $V\otimes \mathcal{C}om=\{V\}_{n\geq 0}$ is naturally an $(\operatorname{Op}\operatorname{End}(V),\mathcal{C}om)$–bimodule. The arguments used in Example 1.3.4 establish an identification

$$V^\star \circ_{\operatorname{Op}\operatorname{End}(V)} (V\otimes \mathcal{C}om)\simeq \mathcal{C}om.$$

(b) Let V,W,X be finite–dimensional k–vector spaces. Then we have an identification

$$\operatorname{Op}\operatorname{Hom}(W,X)\circ_{\operatorname{Op}\operatorname{End}(W)} \operatorname{Op}\operatorname{Hom}(V,W)\simeq \operatorname{Op}\operatorname{Hom}(V,X)$$

(as bimodules). This follows easily from the above example and Example 1.3.4, because $\operatorname{Op}\operatorname{Hom}(W,X)$ is isomorphic, as a right $\operatorname{Op}\operatorname{End}(W)$–module, to the direct sum of $\dim X$ copies of $W^\star$, and $\operatorname{Op}\operatorname{End}(W)$ is isomorphic, as a right module over itself, to the direct sum of $\dim W$ copies of $W^\star$.

The importance of $(\mathcal{Q},\mathcal{P})$–bimodules for us is that any such bimodule M defines a functor $f_M:\mathcal{P}\text{–Alg}\to\mathcal{Q}\text{–Alg}$,

$$f_M(A)=M\circ_{\mathcal{P}} A. \tag{1.4.6}$$

The action of $\mathcal{Q}$ on $f_M(A)$ is transferred from the action on M. Relative plethysm of bimodules corresponds to the composition of functors:

$$f_{N\circ_{\mathcal{Q}} M}=f_N\circ f_M, \tag{1.4.7}$$

as it follows from (1.4.4).

(1.4.8) Example. Let $\mathcal{P}=\mathbf{I}$ be the trivial operad, so that $\mathcal{P}\text{–Alg}=\text{Vect}$. An $(\mathbf{I},\mathbf{I})$–bimodule M is just an $\mathbf{S}$–space. The functor $f_M:\text{Vect}\to\text{Vect}$ is then

$$V\mapsto M\langle V\rangle=\bigoplus_{n\geq 0} M(n)\otimes_{S_n} V^{\otimes n}.$$

Functors of this kind are called analytic in [J].

The following corollary of Example 1.4.5(b) is the first instance of the general Morita–type theorem to be proved later.

(1.4.9) Theorem. *Let V, W be any two finite–dimensional k–vector spaces. Then the functor $f_{\mathrm{Op\,Hom}\,(V,W)}$ induces an equivalence between the categories of algebras over* $\mathrm{Op\,End}\,(V)$ *and over* $\mathrm{Op\,End}\,(W)$*. In particular, all these categories are equivalent to the category of commutative algebras, as* $\mathcal{C}om = \mathrm{Op\,End}\,(k)$*.*

In fact, the functor $f_{\mathrm{Op\,Hom}(V,W)}$ can be described in a more explicit way, using the tensor product of modules over a ring.

(1.4.10) Proposition. *Let V, W be as before and A be an* $\mathrm{Op\,End}(V)$*–algebra. Then, as a vector space,*

$$\mathrm{Op\,Hom}\,(V,W) \circ_{\mathrm{Op\,End}\,(V)} \langle A\rangle \simeq \mathrm{Hom}_k(V,W) \otimes_{\mathrm{End}\,(V)} A,$$

where we view A as a left module over the ring $\mathrm{End}(V) = \mathrm{Op\,End}(V)(1)$*.*

Proof. As we pointed out already, $\mathrm{Op\,Hom}(V,W)$ is isomorphic, as a right $\mathrm{Op\,End}(V)$–module, to a direct sum of $\dim V$ copies of $V^\star$. Notice also that both sides of the proposed equality are additive in W. Thus, it is enough to treat the particular case $W = k$, so that $\mathrm{Op\,Hom}(V,W) = V^\star$. Next, since the functor

$$f_{\mathrm{Op\,Hom}\,(k,V)} : \mathrm{Op\,End}\,(k)\text{–Alg} = \mathcal{C}om\text{–Alg} \to \mathrm{Op\,End}\,(V)\text{–Alg}$$

is an equivalence, we can assume that A lies in the image of this functor. However, $\mathrm{Op\,Hom}(k,V)$ is isomorphic to $V \otimes_k \mathcal{C}om$ as a right $\mathcal{C}om$–module. Therefore, for a commutative algebra B we have

$$f_{\mathrm{Op\,Hom}(k,V)}(B) = (V \otimes_k \mathcal{C}om) \circ_{\mathcal{C}om} \langle B\rangle = V \otimes_k B.$$

Now, the left and the right hand sides of the proposed equality are, respectively

$$V^\star \circ_{\mathrm{Op\,End}(V)} \langle V \otimes_k B\rangle, \quad V^* \otimes_{\mathrm{End}(V)} (V \otimes_k B).$$

But both these spaces are canonically identified with B: the first one in virtue of Theorem 1.4.9 and Example 1.4.5, the second one by elementary linear algebra. Proposition is proved.

(1.5) Right modules as a tensor category: the role of the product $\boxtimes$. Consider the symmetric monoidal structure $\boxtimes$ on $\mathbf{S}$–Vect, see (1.1.2). The identification (1.1.10) shows that any vector space X gives rise to a fiber functor ("evaluation at X")

$$\mathrm{Ev}_X : (\mathbf{S}\text{–Vect}, \boxtimes) \to (\mathrm{Vect}, \otimes), \quad \mathcal{V} \mapsto \mathcal{V}\langle X\rangle. \tag{1.5.1}$$

In particular, taking $X = k$, we get a vector space

$$\underline{\mathcal{V}} := \mathcal{V}\langle k\rangle = \bigoplus \mathcal{V}(n)_{S_n} \tag{1.5.2}$$

which depends on $\mathcal{V}$ is a multiplicative (with respect to $\boxtimes$) way. The first part of the following proposition was observed in [F].

(1.5.3) Proposition. *Let $\mathcal{P}$ be any operad. Then the operation $\boxtimes$ makes* Mod–$\mathcal{P}$*, the category of right $\mathcal{P}$–modules, into a symmetric monoidal abelian category. For any vector space X the functor Ev_X is a fiber functor on* $(\mathrm{Mod}\text{–}\mathcal{P}, \boxtimes)$*, i.e., it takes $\boxtimes$ into $\otimes$.*

Proof. Follows from (1.1.9).

Let now M be a right $\mathcal{P}$–module. We define the operad $\mathrm{Op\,End}_{\mathcal{P}}(M)$ as the endomorphism operad of M as an object of the symmetric monoidal category Mod–$\mathcal{P}$:

$$\mathrm{Op\,End}_{\mathcal{P}}(M)(n) = \mathrm{Hom}_{\mathrm{Mod}-\mathcal{P}}(M^{\boxtimes n}, M). \tag{1.5.4}$$

Similarly, for two right $\mathcal{P}$–modules M, N we define an $(\mathrm{Op\,End}_{\mathcal{P}}(N), \mathrm{Op\,End}_{\mathcal{P}}(M))$–bimodule $\mathrm{Op\,Hom}_{\mathcal{P}}(M,N)$ by

$$\mathrm{Op\,Hom}_{\mathcal{P}}(M,N)(n) = \mathrm{Hom}_{\mathrm{Mod}-\mathcal{P}}(M^{\boxtimes n}, N). \tag{1.5.5}$$

(1.5.6) Proposition. *We have an isomorphism of operads $u : \mathcal{P} \to \mathrm{Op\,End}_{\mathcal{P}}(\mathcal{P})$.*

Proof. The construction of u is straightforward: the composition $\mu : \mathcal{P} \circ \mathcal{P} \to \mathcal{P}$ gives, upon restriction to the nth summand in (1.1.4), a $\mathbf{S}_n$–equivariant morphism of right $\mathcal{P}$–modules $\mathcal{P}(n) \otimes_k \mathcal{P}^{\boxtimes n} \to \mathcal{P}$, and the associativity of μ implies that u is a morphism of operads. To see that it is injective, look at the action of $u(p)$, $p \in \mathcal{P}(n)$, on $1 \otimes \cdots \otimes 1 \in \mathcal{P}(1)^{\otimes n}$. To check the surjectivity, consider any element $\varphi = (\varphi_{l_1,\ldots,l_n}) \in \mathrm{Op\,End}_{\mathcal{P}}(\mathcal{P})(n)$. Put $p = \varphi_{1,\ldots,1}(1 \otimes \cdots \otimes 1) \in \mathcal{P}(n) \subset \mathcal{P}^{\boxtimes n}(n)$. Then $\varphi = u(p)$. In fact, for $q_i \in \mathcal{P}(l_i)$ we have

$$\varphi_{l_1,\ldots,l_n}(q_1 \otimes \cdots \otimes q_n) = \varphi_{l_1,\ldots,l_n}(1(q_1) \otimes \cdots \otimes 1(q_n))$$

$$= (\varphi_{1,\ldots,1}(1 \otimes \cdots \otimes 1))(q_1, \ldots, q_n) = p(q_1, \ldots, q_n).$$

Proposition is proved.

(1.6) A Lie algebra associated to an operad. Let us recall the main ideas of the Tannaka–Krein duality using [De] as a reference. If $(\mathcal{C}, \otimes)$ is a symmetric monoidal k-linear category and $\Phi : \mathcal{C} \to \mathrm{Vect}$ is a fiber functor, then one can form the group $\mathrm{Aut}(\Phi)$ of all $\otimes$-automorphisms of Φ. If $\mathcal{C}$ satisfies additional properties listed in [De] (i.e., $\mathcal{C}$ is Tannakian, in the terminology of *loc. cit.*), then $\mathrm{Aut}(\Phi)$ naturally becomes (the group of k-points of) a group scheme over k and, moreover, $\mathcal{C}$ becomes identified with the category of regular representations of this group scheme. Working at the infinitesimal level, we have then the following definition.

(1.6.1) Definition. *Let $(\mathcal{C}, \otimes)$ be a k–linear symmetric monoidal category, $\Phi : \mathcal{C} \to Vect$ a fiber functor. A derivation of Φ is a natural transformation $L : \Phi \to \Phi$ such that for any two objects $A, B \in \mathcal{C}$ the morphism $L_{A\otimes B} : \Phi(A\otimes B) \to \Phi(A\otimes B)$ coincides, after the identification $\Phi(A \otimes B) \simeq \Phi(A) \otimes \Phi(B)$, with $L_A \otimes 1_{\Phi(B)} + 1_{\Phi(A)} \otimes L_B$.*

All the derivations of Φ form, obviously, a Lie algebra which we denote $\mathrm{Der}(\Phi)$.

We now fix an operad $\mathcal{P}$ and specialize the above to the symmetric monoidal category $(\mathrm{Mod} - \mathcal{P}, \boxtimes)$ and the fiber functor Φ given by $\Phi(M) = \underline{M} = M\langle k\rangle$, see Proposition 1.5.3. This category is not Tannakian because it lacks dual objects. Nevertheless, Definition 1.6.1 is applicable and gives some Lie algebra depending only on $\mathcal{P}$. We start with a down–to–earth description of this algebra.

For $p \in \mathcal{P}(l)$, $q \in \mathcal{P}(n)$ and $1 \le i \le l$ set

$$p \circ_i q = p(1, ..., 1, q, 1, ..., 1) \in \mathcal{P}(l + n - 1), \tag{1.6.2}$$

where q on the right hand side is at the ith position, cf. [Mar1]. In a similar way, if M is a right $\mathcal{P}$–module, $m \in M(l)$ and $q \in \mathcal{P}(n)$, we define $m \circ_i q \in M(l + n - 1)$, $1 \le i \le l$.

(1.6.3) Theorem. *(a) Let $\mathcal{P}$ be an operad and set for $p \in \mathcal{P}(a+1)$, $q \in \mathcal{P}(b+1)$:*

$$[p, q] = \sum_{i=1}^{a+1} p \circ_i q \quad - \quad \sum_{j=1}^{b+1} q \circ_j p \quad \in \quad \mathcal{P}(a + b + 1).$$

Then the operation $[-, -]$ induces on the vector space $\underline{\mathcal{P}}$ (see (1.5.2)) the structure of a graded Lie algebra, which we denote $\mathcal{L}(\mathcal{P})$.

(b) For every right $\mathcal{P}$–module M the operation

$$(m, q) \mapsto mq = \sum_{i=1}^{a+1} m \circ_i q \in M(a + b + 1), \quad m \in M(a + 1),\ q \in \mathcal{P}(b + 1),$$

induces on $\underline{M}$ the structure of a graded right $\mathcal{L}(\mathcal{P})$–module. If M_1, M_2 are two right $\mathcal{P}$–modules, then $\underline{M_1 \boxtimes M_2}$, is identified with $\underline{M_1} \otimes \underline{M_2}$ as a $\mathcal{L}(\mathcal{P})$–module.

(c) For any finite–dimensional $\mathcal{P}$–algebra A the Lie algebra $\mathcal{L}(\mathcal{P})$ acts on (the affine space underlying) A by polynomial vector fields.

Proof. (a) First of all, $\underline{\mathcal{P}}$ is the quotient of $\bigoplus \mathcal{P}(n)$ by the subspace J spanned by elements of the form $p(1 - s)$, $p \in \mathcal{P}(a + 1), s \in \mathbf{S}_{a+1}$. Using the functional notation, one easily checks that J is the two–sided ideal with respect to the bracket $[\,,]$. Moreover, $p \circ_i q \equiv p \circ_j q \bmod J$. The skew symmetry of the bracket is obvious.

To see the Jacobi identity, it is enough to prove it already in $\bigoplus \mathcal{P}(a+1)$. Choose $p \in \mathcal{P}(a+1), q \in \mathcal{P}(b+1), r \in \mathcal{P}(c+1)$. To explain the pattern of cancellation, write the Jacobi tensor $[[p,q],r]] - [p,[q,r]] - [[p,r],q]$ and look, say, at the terms of the type $(p \circ_i q) \circ_j r$ coming from the first summand. Each of these terms represents, geometrically, a "nested composition" in which the three vertices of the naturally arising tree are linearly ordered with respect to the partial order present on any rooted tree. Each such nested composition will find its counterpart in $[p,[q,r]]$ with which it will cancel. However, $[p,[q,r]]$ will give rise to more terms of the form $(p \circ_i q) \circ_j r$ than come from $[[p,q],r]$; these new terms correspond to non-nested compositions in which the arising tree has two vertices whose outputs are both connected with the input of the third vertex. But these same compositions will appear as well in $[[p,r],q]$ because operadic associativity implies the commutativity of non-nested compositions.

(b) This statement is checked similarly, and we omit the details.

(c) Notice that $A^\star = \{(A^*)^{\otimes n}\}_{n\geq 0}$ is naturally a right $\mathcal{P}$–module, and $\underline{A^\star} = S(A^*)$ is the algebra of polynomial functions on (the affine space underlying) A. So $S(A^*)$ becomes a $\mathcal{L}(\mathcal{P})$-module, and to prove our assertion we need only to check that $\mathcal{L}(\mathcal{P})$ acts by algebra derivations. But this follows from the compatibility with tensor products.

(1.6.4) Remark. The proof above shows that the direct sum $\Lambda(\mathcal{P}) = \bigoplus \mathcal{P}(a+1)$ also forms a graded Lie algebra, of which $\mathcal{L}(\mathcal{P})$ is a quotient. The Gerstenhaber bracket on the Hochschild complex of an associative algebra A, see [Ger], is a particular case of the construction of $\Lambda(\mathcal{P})$ but for the operad Op $\mathrm{End}(A[-1])$ in the category of graded vector spaces.

(1.6.5) Theorem. *The Lie algebra $\mathcal{L}(\mathcal{P})$ is identified with* $\mathrm{Der}(\Phi)$ *for the fiber functor described above.*

Proof. Part (b) of Theorem 1.6.3 means that we have a morphism of Lie algebras $u : \mathcal{L}(\mathcal{P}) \to \mathrm{Der}(\Phi)$. To see that u is injective it is enough to consider an action of $p \in \mathcal{P}(a+1)$ on $\underline{\mathcal{P}}$ where $\mathcal{P}$ is considered as a module over itself. It remains to prove surjectivity, i.e., that any derivation $D : \Phi \to \Phi$ comes from some element of $\mathcal{L}(\mathcal{P})$. To see this, consider the D–action on $\underline{\mathcal{P}}$ and let $q = D(1)$ be the image of $1 \in \mathcal{P}(1)$. We claim that $D = u(q)$. Indeed, let M be an arbitrary right $\mathcal{P}$–module and $m \in M(a+1)$. Then the associativity of the right $\mathcal{P}$–action on M can be expressed as follows. Consider the natural morphism

$$M(a+1) \otimes_k \mathcal{P}^{\boxtimes(a+1)} \to M \tag{1.6.6}$$

given by restricting the action $M \circ \mathcal{P} \to M$, see (1.1.3). Let us view the source of this morphism as a right $\mathcal{P}$–module by considering $M(n)$ as just the vector space of multiplicities of the $\mathcal{P}$–module $\mathcal{P}^{\boxtimes(a+1)}$. Then (1.6.6) is a morphism of modules.

Now, because D is a derivation, applying D to $m \otimes 1 \otimes ... \otimes 1$ in the source of (1.6.6) we get

$$\sum_{i=1}^{a+1} m \otimes 1 \otimes ... \otimes 1 \otimes q \otimes 1 \otimes ... \otimes 1$$

with q at the i–th place in the i–th term. Since (1.6.6) is a morphism of modules, we find that the D–action on m is given by the formula in (1.6.3)(b). Theorem is proved.

(1.6.7) Examples. (a) Let V be a finite-dimensional vector space and $\mathcal{P} = \operatorname{Op}\operatorname{End}(V)$. Then

$$\underline{\mathcal{P}} = \bigoplus_{n \geq 0} S^n(V^*) \otimes V$$

can be naturally identified with $\operatorname{Der} k[V^*]$, the space of polynomial vector fields on V. Moreover, the bracket $[p, q]$ is identified with the usual Lie bracket of vector fields, so $\mathcal{L}(\operatorname{Op}\operatorname{End}(V)) \simeq \operatorname{Der} k[V^*]$ as a Lie algebra, the action (1.6.3)(c) being the tautological one.

(b) Let $\mathcal{P} = \mathcal{A}ss$ be the associative operad. Then $\mathcal{P}(n)_{S_n} = k = \mathcal{C}om(n)$, so $\mathcal{L}(\mathcal{A}ss) = \mathcal{L}(\mathcal{C}om) = \operatorname{Der} k[x]$ is the Lie algebra of polynomial vector fields on the line (example (a) for $V = k$). Explicitly, the canonical generator $e_n \in \mathcal{C}om(n)$ corresponds to the vector field $x^n(d/dx)$, so that

$$[e_m, e_n] = (m - n)e_{m+n-1}.$$

As a corollary, we get part (a) of the following fact.

(1.6.8) Theorem. *(a) Let A be a finite–dimensional associative algebra. Then the Lie algebra $Der k[x]$ acts by polynomial vector fields on A. Explicitly, $e_n = x^n(d/dx)$ acts by the vector field*

$$E_n = \left(X^n, \frac{\partial}{\partial X} \right).$$

Here X^n is the A–valued polynomial function on A which raises every element to the n–th power, $\frac{\partial}{\partial X}$ is the canonical A^*–valued vector field on A (corresponding to $Id \in A^* \otimes_k A$) and $(-, -)$ is the structure pairing between A and A^*.

(b) Let $A^\times \subset A$ be the set of invertible elements. Let $\operatorname{Der} k[x, x^{-1}]$ *be the Witt algebra of regular vector fields on $k^\times = k - \{0\}$, with the basis $e_n = x^n(d/dx), n \in \mathbf{Z}$. Then the operators E_n defined similarly to (a), give an action of* $\operatorname{Der} k[x, x^{-1}]$ *by regular vector fields on $A^\times$.*

This fact for $A = \operatorname{Mat}(r, \mathbf{C})$ was noticed by N. Wallach ([W]). The proof of part (b) in general is left to the reader.

As we saw in (1.2.5) (g), any simplicial vector space gives rise to a right $\mathcal{A}ss$–module and thus, by the above, it defines a representation of $\operatorname{Der} k[x]$. Let us describe this action explicitly. Let $V_\bullet$ be a simplicial vector space, with face and degeneracy operators

$$\partial_{n,i} : V_n \to V_{n-1},\ 0 \le i \le n, \quad s_{n,i} : V_n \to V_{n+1},\ 0 \le i \le n.$$

Let $C(V_\bullet) = \bigoplus_n V_n$ be the graded vector space associated to V. Define the linear operators $L_p : C(V_\bullet) \to C(V_\bullet)$ of degree $p-1$, $p \ge 0$, as follows. On the summand $V_n \subset C(V_\bullet)$ we set

$$L_0 = \sum_{i=0}^{n} \partial_{n,i}, \quad L_1 = (n+1)\cdot \mathrm{Id}, \quad L_p = \sum_{i=0}^{n} s_{n-p+1,i} \dots s_{n+1,i} s_{n,i}, \ \ p \ge 1.$$

(1.6.9) Theorem. *The operators L_p satisfy the commutation relations $[L_p, L_q] = (p-q)L_{p+q-1}$ and thus make $C(V_\bullet)$ into a $\operatorname{Der} k[x]$–module.*

The proof can be deduced directly from the standard simplicial identities between faces and degenerations.

(1.6.10) Example. Here we describe the Lie algebra $\mathcal{L}(H_*\overline{M})$ for the operad $H_*\overline{M}$ introduced in (1.2.5) (h). Consider a stable tree with $n+1$ tails (flags which are not halves of the edges). Stability means that there are at least three flags at each vertex. Any numbering of flags by $\{0, 1, \dots, n\}$ determines an irreducible closed submanifold of $\overline{M}_{0,n+1}$ whose open dense subset parametrizes curves of genus zero of the respective combinatorial type. The image of the homology class of this submanifold in $\mathcal{L}(H_*\overline{M})$ depends only on the isomorphism class of the respective rooted tree where root is the flag labeled by 0. For any such (isomorphism class) τ denote by $L(\tau)$ the corresponding element of the Lie algebra. Then we have:

(i) $\mathcal{L}(H_*\overline{M})$ is spanned by all $L(\tau)$. Besides the general $\mathbf{Z}$–grading (see (1.6.2)), it has an additional $\mathbf{Z}$–grading by algebraic dimension of $L(\tau)$ which is $n-2$ less the number of edges of τ.

(ii) The bracket is defined by

$$[L(\sigma), L(\tau)] = \sum_i L(\sigma \circ_i \tau) - \sum_j L(\tau \circ_j \sigma). \tag{1.6.11}$$

Here i (resp. j) runs over non–root flags of σ (resp. of τ), and $\sigma \circ_i \tau$ denotes the tree obtained from σ by gluing its flag i to the root of τ.

(iii) The elements $L(\tau)$ are not lineraly independent. If k has characteristic zero, all linear relations between $L(\tau)$ can be obtained by symmetrizing the linear relations between the boundary homology classes described in [Man2], sec. III.3.5.

All of this follows from Keel's description of the Chow and homology groups of $\overline{M}_{0,n+1}$ ([Ke]).

For a general $H_*\overline{M}$–algebra H there is nothing to add to the description of vector fields given in the proof of (1.6.3) (c). Let us therefore consider the case of the cyclic algebra H, for example, quantum cohomology. In this case H comes with a richer structure, namely it is endowed with a non–degenerate symmetric pairing g (Poincaré form), and the algebra structure geometrically appears in the guise of Gromov–Witten $\mathbf{S}_{n+1}$–equivariant maps

$$I_{n+1}:\, H^{\otimes (n+1)} \to H^*(\overline{M}_{0,n+1},k).$$

The operadic action needed to define $L(\tau)$

$$\mu_n:\, H_*(\overline{M}_{0,n+1})\otimes H^{\otimes n}\to H$$

is obtained from I_{n+1} by partial dualization with the help of g.

(1.7) The role of the product $\otimes$. Consider now the monoidal structure $\otimes$ on $\mathbf{S}$–Vect, see (1.1.1).

If $\mathcal{P}_1,\mathcal{P}_2$ are operads, then the distributivity maps (1.1.12) make $\mathcal{P}_1\otimes\mathcal{P}_2$ into an operad. If A_i is a $\mathcal{P}_i$–algebra, $i=1,2$, then $A_1\otimes_k A_2$ is a $\mathcal{P}_1\otimes\mathcal{P}_2$–algebra via (1.1.13). Similarly, if M_i is a left (resp. right) $\mathcal{P}_i$–module, i=1,2, then $M_1\otimes M_2$ is a left (resp. right) $\mathcal{P}_1\otimes\mathcal{P}_2$–module.

Let $\mathcal{P}$ be an operad, M be a right $\mathcal{P}$–module and V be a vector space. Then we have the right $\mathcal{P}$–module $V\otimes M$, see Example 1.2.5(d). Notice that if N,W are another right $\mathcal{P}$–module and a vector space, then we have an identification

$$(V\otimes M)\boxtimes(W\otimes N)=(V\otimes_k W)\otimes(M\boxtimes N), \tag{1.7.1}$$

which implies the following.

(1.7.2) Proposition. *We have an isomorphism of operads*

$$\mathrm{Op}\,\mathrm{End}_{\mathcal{P}}(V\otimes M)\simeq \mathrm{Op}\,\mathrm{End}(V)\otimes\mathrm{Op}\,\mathrm{End}_{\mathcal{P}}(M)$$

and an isomorphism of bimodules

$$\mathrm{Op}\,\mathrm{Hom}_{\mathcal{P}}(V\otimes M,W\otimes N)\simeq \mathrm{Op}\,\mathrm{Hom}(V,W)\otimes\mathrm{Op}\,\mathrm{Hom}_{\mathcal{P}}(M,N).$$

(1.7.3) Definition. *The d by d matrix operad over $\mathcal{P}$ is defined by*

$$\mathrm{Mat}\,(d,\mathcal{P}) = \mathcal{P} \otimes \mathrm{Op\,End}\,(k^d) = \mathrm{Op\,End}_{\mathcal{P}}(\mathcal{P}^{\oplus d})$$

(the last equality being a consequence of (1.7.2) and Proposition 1.5.6).

Next, let $\mathcal{P}_i, i = 1, 2$ be operads, M_i be a right $\mathcal{P}_i$–module and N_i be a left $\mathcal{P}_i$–module. Then the distributivity maps (1.1.11) give rise to a morphism of $\mathbf{S}$–modules

$$(M_1 \otimes M_2) \circ_{\mathcal{P}_1 \otimes \mathcal{P}_2} (N_1 \otimes N_2) \to (M_1 \circ_{\mathcal{P}_1} N_1) \otimes (M_2 \circ_{\mathcal{P}_2} N_2). \tag{1.7.4}$$

In general this need not be an isomoprhism. We will be interested in the case when $\mathcal{P}$ is a fixed operad, $M_1 = N_1 = \mathcal{P}_1 = \mathcal{P}$, while $\mathcal{P}_2 = \mathrm{Op\,End}(V)$, $\dim V < \infty$, $M_2 = V^\star$ and $N_2 = V \otimes \mathcal{C}om$. Notice that in this case $M_2 \circ_{\mathcal{P}_2} N_2 \simeq \mathcal{C}om$, see Example 1.4.5(a). Let us denote $V \otimes \mathcal{C}om$ by $\widetilde{V}$. The specialization of the morphism (1.7.4) to our case is then:

$$u : (\mathcal{P} \otimes V^\star) \circ_{\mathcal{P} \otimes \mathrm{Op\,End}(V)} (\mathcal{P} \otimes \widetilde{V}) \to \mathcal{P} \otimes (V^\star \circ_{\mathrm{Op\,End}(V)} \widetilde{V}) = \mathcal{P}. \tag{1.7.5}$$

(1.7.6) Proposition. *The morphism u in (1.7.5) is an isomorphism of* $\mathbf{S}$–*modules.*

Proof. Let $\dim(V) = d$. It is enough to show that $u^{\oplus d}$, the direct sum of d copies of u, is an isomorphism. On the other hand, $(V^\star)^{\oplus d}$ is identified, as a right $\mathrm{Op\,End}(V)$–module, with $\mathrm{Op\,End}(V)$ itself and hence $(\mathcal{P} \otimes V^\star)^{\oplus d}$ is identified, as a $\mathcal{P} \otimes \mathrm{Op\,End}(V)$–module, with $\mathcal{P} \otimes \mathrm{Op\,End}(V)$ itself. It follows that $u^{\oplus d}$ is identified with the morphism

$$\mathcal{P} \otimes \mathrm{Op\,End}(V) \circ_{\mathcal{P} \otimes \mathrm{Op\,End}(V)} (\mathcal{P} \otimes \widetilde{V}) \to \mathcal{P} \otimes \Big(\mathrm{Op\,End}(V) \circ_{\mathrm{Op\,End}(V)} \widetilde{V}\Big)$$

which is just the identity map $\mathcal{P} \otimes \widetilde{V} \to \mathcal{P} \otimes \widetilde{V}$. Thus $u^{\oplus d}$ is an isomorphism and hence u is an isomorphism.

(1.8) Morita equivalence. LConsider an operad $\mathcal{P}$ and a $\mathcal{P}$–module M. Put $\mathcal{Q} = \mathrm{Op\,End}_{\mathcal{P}}(M)$. Then M is a $(\mathcal{Q}, \mathcal{P})$–bimodule, so that the construction of (1.3) gives us a functor $f_M : \mathcal{P}$–Alg$\to \mathcal{Q}$–Alg.

(1.8.1) Theorem. *If M is free as $\mathcal{P}$–module, that is, isomorphic to $\mathcal{P}^{\oplus d}$, then f_M is an equivalence of categories.*

Proof. As above, put $V = k^d$ so that $M = \mathcal{P} \otimes_k V = \mathcal{P} \otimes \widetilde{V}$. Let $M^* = \operatorname{Op}\operatorname{Hom}_{\mathcal{P}}(M, \mathcal{P}) = \mathcal{P} \otimes V^*$, the last identification following from Proposition 1.7.2. Then M^* is a $(\mathcal{P}, \mathcal{Q})$–bimodule and we have the functor $f_{M^*} : \mathcal{Q}\text{–Alg} \to \mathcal{P}\text{–Alg}$. We claim that f_M and f_{M^*} are mutually inverse.

To prove this, it suffices to construct the isomorphisms

$$M^* \circ_{\mathcal{Q}} M \cong \mathcal{P} \quad (\text{as } (\mathcal{P}, \mathcal{P})\text{–bimodules}). \tag{1.8.2}$$

$$M \circ_{\mathcal{P}} M^* \cong \mathcal{Q} \quad (\text{as } (\mathcal{Q}, \mathcal{Q})\text{–bimodules}). \tag{1.8.3}$$

Now, (1.8.2) is the content of Proposition 1.7.6. As for (1.8.3), we have, first of all, a morphism

$$\psi : M \circ_{\mathcal{P}} M^* = (\mathcal{P} \otimes \widetilde{V}) \circ_{\mathcal{P} \otimes \mathcal{C}om} (\mathcal{P} \otimes V^\star) \to \mathcal{P} \otimes (V \circ_{\mathcal{C}om} V^\star), \tag{1.8.4}$$

a particular case of (1.7.4). Note that the natural morphism of $\operatorname{Op}\operatorname{End}(V)$–bimodules

$$\phi : \widetilde{V} \circ_{\mathcal{C}om} V^\star \to \operatorname{Op}\operatorname{End}(V)$$

is an isomorphism. This is because $\widetilde{V}$ is a free right $\mathcal{C}om$–module, so ϕ is isomorphic to the direct sum of $d = \dim(V)$ copies of the isomorphism $\mathcal{C}om \circ_{\mathcal{C}om} V^\star \to V^\star$. Thus (1.8.4) can be regarded as a morphism

$$\psi : M \circ_{\mathcal{P}} M^* \to \mathcal{P} \otimes \operatorname{Op}\operatorname{End}(V) = \mathcal{Q}.$$

To see that this is an isomorphism, it is enough to notice that ψ is isomorphic (as a morphism of $\mathbf{S}$–modules) to the direct sum of d copies of the morphism

$$\mathcal{P} \circ_{\mathcal{P}} (\mathcal{P} \otimes V^\star) \to \mathcal{P} \otimes V^\star,$$

which is an isomorphism. Theorem 1.8.1 is proved.

(1.9) The super–version. Most of the above constructions can be performed in any k–linear abelian symmetric monoidal category $\mathcal{C}$, instead of Vect. We will be particularly interested in the category SVect of super-vector spaces. Recall that objects of SVect are $\mathbf{Z}/2$–graded k–vector spaces, the tensor product is the usual graded one and the symmetry isomorphism $V \otimes W \to W \otimes V$ involves the Koszul sign.

If $\mathcal{P}$ is an operad in SVect, then $\mathcal{P}$–algebras (in SVect) will be called $\mathcal{P}$–superalgebras and their category will be denoted $\mathcal{P}$–SAlg.

All the statements given above for k–linear operads generalize to SVect without difficulty. In particular, for any super–vector space V we have the endomorphism operad $\operatorname{Op}\operatorname{End}(V)$ in SVect, for any two super–vector spaces V, W we have a bimodule $\operatorname{Op}\operatorname{Hom}(V, W)$ and so on.

We will need a superversion of the Morita theorem and of a more precise statement underlying it. Let $\mathcal{P}$ be any operad in SVect, M any right $\mathcal{P}$–module, $\mathcal{Q} = \operatorname{Op}\operatorname{End}_{\mathcal{P}}(M)$ and $M^* = \operatorname{Op}\operatorname{Hom}_{\mathcal{P}}(M, \mathcal{P})$. Then, as before, we have a morphism

$$M^* \circ_{\mathcal{Q}} M \to \mathcal{P} \tag{1.9.1}$$

of $(\mathcal{P}, \mathcal{P})$-bimodules and a morphism

$$M \circ_{\mathcal{P}} M^* \to \mathcal{Q} \tag{1.9.2}$$

of $(\mathcal{Q}, \mathcal{Q})$-bimodules.

(1.9.3) Theorem. *If M is free, i.e., isomorphic to $\mathcal{P} \otimes_k V$ where V is a finite–dimensional supervector space, then the morphisms (1.9.1–2) are isomorphisms and hence the functor f_M is an equivalence between $\mathcal{P}$–SAlg and $\mathcal{Q}$-SAlg.*

The proof is obtained by performing the same steps as for Theorem 1.8.1, with easy modifications to accomodate the $\mathbf{Z}/2$-grading.

§2. Differential algebras.

(2.1) Operadic approach to nonlinear differential equations. Let X be a smooth complex algebraic variety of dimension p. Denote by $\mathcal{D}(n) = \mathcal{D}_X(n)$ the sheaf of n–linear multidifferential operators

$$(u_1, ..., u_n) \mapsto L(u_1, ..., u_n), \quad u_i \in \mathcal{O}_X$$

acting on regular functions on X. If $(x_1, ..., x_p)$ is a local coordinate system on X, and $\partial^I = \partial_{x_1}^{i_1} ... \partial_{x_p}^{i_p}$, $I = (i_1, ..., i_p)$, is the iterated partial derivative corresponding to a multiindex I, then a local section L of $\mathcal{D}(n)$ acts on functions as follows

$$L(u_1, ..., u_n) = \sum_{I(1),\dots,I(n)} f_{I(1),\dots,I(n)}(x)(\partial^{I(1)}u_1) \cdot \ldots \cdot (\partial^{I(n)}u_n), \quad f_{I(1),\dots,I(n)}(x) \in \mathcal{O}_X.$$

Thus $\mathcal{D}(1)$ is the usual sheaf of rings of linear differential operators on X. The collection $(\mathcal{D}_X(n))_{n\geq 0}$ forms a sheaf of operads on X, with the composition given by superposition of multilinear differential operators. As with any operad, $\mathcal{D}_X(n)$ is endowed with a left $\mathcal{D}_X(1)$–module (in particular, a $\mathcal{O}_X$-module) structure and with n commuting right $\mathcal{D}_X(1)$–module (in particular, $\mathcal{O}_X$–module) structures.

As is well known, the study of sheaves of modules over the sheaf of rings $\mathcal{D}_X(1)$ provides a fruitful algebraic approach to the theory of linear differential equations. The introduction of the sheaf of operads $\mathcal{D}_X$ and its sheaves of algebras provides a similar algebraic language for systems of nonlinear differential equations. More precisely, let us give the following definition.

(2.1.1) Definition. *A system of differential equations on X (polynomial in the derivatives) is a sheaf $\mathcal{A}$ of $\mathcal{D}_X$–algebras which is locally given by finitely many generators and relations. A solution to a system of equations given by $\mathcal{A}$ is a morphism of $\mathcal{D}_X$–algebras $\mathcal{A} \to \mathcal{O}_X$ (note that $\mathcal{O}_X$ is a $\mathcal{D}_X$–algebra by definition).*

To understand how this is related with the intuitive concept of a system of differential equations, consider the case of one unknown function u and a system of differential equations

$$P_\nu\left(u, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j}, \ \dots\right) = 0 \tag{2.1.2}$$

where P_ν is a polynomial in u and its derivatives with coefficients in $\mathcal{O}_X$. We polarize each homogeneous component of P_ν to a multilinear differential operator thus writing each equation from (2.1.2) as

$$\sum_n L_{\nu,n}(u, ..., u) = 0. \tag{2.1.3}$$

where each $L_{\nu,n}$ is an n–ary multilinear differential operator, i.e., a section of $\mathcal{D}_X(n)$. Then we form a $\mathcal{D}_X$–algebra $\mathcal{A}$ on one generator U subject only to the relations (2.1.3). A $\mathcal{D}_X$–algebra homomorphism $\mathcal{A} \to \mathcal{O}_X$ is clearly the same as a solution to (2.1.2), as U should go to some function $u \in \mathcal{O}_X$ so that the defining relations of U in $\mathcal{A}$ are satisfied in $\mathcal{O}_X$. Similarly, a system of equations on n unknown functions is translated into an algebra with n generators.

The next proposition can be checked by straightforward manipulations in local coordinates.

(2.1.4) Proposition. *(a) The filtration $F_\bullet\mathcal{D}_X(n)$ by total number of derivatives is compatible with the operad structure (i.e., makes $\mathcal{D}_X$ into an operad in the category of filtered quasicoherent sheaves of $\mathcal{O}_X$-modules). In particular, $F_0\mathcal{D}_X$ is a sheaf of operads which can be identified with $\mathcal{C}om \otimes_{\mathbf{C}} \mathcal{O}_X$, and $\mathrm{gr}^F_\bullet \mathcal{D}_X$ is an operad in the category of graded sheaves.*

(b) A $\mathcal{D}_X$-algebra is the same as a commutative $\mathcal{O}_X$-algebra (via $\mathcal{C}om \otimes_{\mathbf{C}} \mathcal{O}_X = F_0\mathcal{D}_X \subset \mathcal{D}_X$) which is made into a left $\mathcal{D}_X(1)$-module such that vector fields in $\mathcal{D}_X(1)$ act by algebra derivations.

Part (b) shows that our approach is in fact identical to the classical approach of "differential algebra" of Ritt [Ri]. However, the operadic point of view seems to present several advantages. Let us explain, for example, the analog of the fundamental fact that the algebra $\mathrm{gr}^F_\bullet \mathcal{D}_X(1)$ is commutative and can be identified with the algebra of functions on T^*X. Recall that an algebra A is commutative if and only if the multiplication $A \otimes A \to A$ is an algebra morphism, i.e., A is an algebra in the category of algebras.

(2.1.5) Proposition. *(a) Each sheaf $\mathrm{gr}^F_\bullet \mathcal{D}_X(n)$ has a natural structure of a commutative algebra so that $\mathrm{gr}^F_\bullet \mathcal{D}_X$ becomes an operad in the category of algebras.*

(b) The spectrum of the algebra $\mathrm{gr}^F_\bullet \mathcal{D}_X(n)$ is identified with $(T^*_X)^{\oplus n}$.

(c) The operad structure on $\mathrm{gr}^F_\bullet \mathcal{D}_X$ is induced by a structure of the cooperad (in the category of algebraic varieties) on the collection of $(T^*_X)^{\oplus n}$ given by the maps

$$\nu_{a_1,...,a_n} : (T^*_X)^{\oplus(a_1+...+a_n)} \to (T^*_X)^{\oplus n} \times (T^*_X)^{\oplus a_1} \times ... \times (T^*_X)^{\oplus a_n}, \tag{2.1.6}$$

$$(\xi_1, ..., \xi_{a_1+...+a_n}) \mapsto$$

$$\left(\left(\sum_{i=1}^{a_1} \xi_i, \sum_{i=a_1+1}^{a_1+a_2} \xi_i, ..., \sum_{i=a_1+...+a_{n-1}+1}^{a_1+...+a_n} \xi_i \right), \ (x_1, ..., x_{a_1}), \ ..., \ (x_{a_1+...+a_{n-1}+1}, ..., x_{a_1+...+a_n}) \right).$$

Again, the proof is straightforward.

(2.1.7) Remarks. (a) Note that the construction (2.1.6) can be defined in fact for any abelian group A (instead of T^*_X), making $(A^{\oplus n})_{n\geq 0}$ into a cooperad.

(b) Proposition 2.1.5(c) suggests a natural algebraic approach to microlocalization of nonlinear equations: instead of an open cone in T^*_X in linear theory, the microlocalization should be done with respect to a sub-cooperad in $\{(T^*_X)^{\oplus n}\}$.

(2.2) $\mathcal{D}_X$ as an endomorphism operad. We can view the operad $\mathcal{D}_X$, though not formally as particular case, but as a close analog of the construction of the endomorphism operad $\mathrm{Op\,End}\,(V)$ from (1.2.6): we take for V the space (sheaf) $\mathcal{O}_X$ of functions on X and consider only local endomorphisms. All the abstract constructions of §1 can be performed in this situation and have interesting meaning.

First of all, the role of the dual space V^* is played by ω_X, the sheaf of volume forms on X, and the pairing between V and V^* is replaced by the Serre duality. The analog of the fact that $\mathrm{Op\,End}\,(V)(n) = (V^*)^{\otimes n} \otimes V$ is given by the following coordinate-free definition of $\mathcal{D}(n)$ in terms of local cohomology sheaves, generalizing Sato's definition of linear differential operators [SKK].

(2.2.1) Proposition. *Consider the $(n+1)$-fold Cartesian product X^{n+1} whose factors will be labeled by integers $0, 1, ..., n$ and let $p_i : X^{n+1} \to X$, $i = 0, ..., n$, be the projections. Let also $\Delta \subset X^{n+1}$ be the image of the diagonal embedding of X. Then we have a natural identification*

$$\mathcal{D}_X(n) = \underline{H}_\Delta^{n\dim(X)}\left(\bigotimes_{i=1}^n p_i^*\omega_X\right)$$

Proof. As in the classical case of linear differential operators, the easiest way to prove such a statement is first to construct, independently, an operad structure on the collection of cohomology sheaves in (2.2.1) and an action of this operad on $\mathcal{O}_X$. This is easily done directly in cohomological terms by using the canonical maps

$$H^{nd}_{(x,\dots,x)}(\omega_{X^n}) \to H^{nd}(X^n, \omega_{X^n}) \to \mathbf{C} \tag{2.2.2}$$

provided by the Serre duality. After this, it remains to prove that, (a) the action is indeed by differential operators and, (b) that all the differential operators are obtained in this way. The statement (a) follows from the explicit way of describing (2.2.2) in terms of residues. To see (b), recall that for any smooth subvariety Z in a smooth variety Y, of codimension r, and any algebraic vector bundle $\mathcal{E}$ on Y, the cohomology sheaf $\underline{H}^d_Z(Y, \mathcal{E})$ has a natural filtration with quotients $\det(N_{Z/Y}) \otimes S^j(N^*_{Z/Y}) \otimes E|_Z$ which in the case $Z = X, Y = X^{n+1}$, $\mathcal{E} = \bigotimes_{i=1}^n p_i^*\omega_X$ become identified with the quotients of the natural filtration on $\mathcal{D}_X(n)$ described in (2.1.4).

As we are almost in the endomorphism operad situation, let us further compare it to the framework of (1.2). Writing $\sim$ to denote the analogous objects, we have $\mathcal{P} \sim \mathcal{C}om$, $V \sim \mathcal{O}_X$, $M \sim \mathcal{C}om \otimes_{\mathbf{C}} \mathcal{O}_X$ and $\mathcal{Q} \sim \mathcal{D}_X$. Next, the analog of the module

$M^\star$ from (1.2.5)(e) looks as follows. Consider X^n, with projections $p_i : X^n \to X$, $i = 1, ..., n$ and the diagonal $\Delta \subset X^n$, so that Δ is identified with X. Let

$$\Omega(n) = \underline{H}_\Delta^{(n-1)\dim(X)}\left(\bigotimes_{i=1}^n p_i^* \omega_X\right) = \omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X(n-1). \tag{2.2.2}$$

(2.2.3) Proposition. *The collection $\Omega = \{\Omega(n)\}_{n\geq 1}$ is naturally a right $\mathcal{D}_X$-module in the sense of (1.2).*

The proof is straightforward.

Proposition 2.2.3 generalizes the fact that ω_X is a right module over the algebra $\mathcal{D}_X(1)$.

(2.3) $\mathcal{D}$-algebras on supermanifolds. Let now X be a supermanifold of dimension $(p|q)$, see [Man1]. Ch. 4. Thus X consists of an ordinary smooth complex algebraic variety X_{red} of dimension p and a sheaf $\mathcal{O}_X$ of $\mathbf{Z}/2$-graded supercommutative algebras which is locally isomorphic to $\mathcal{O}_{X_{\mathrm{red}}} \otimes_{\mathbf{C}} \Lambda[\xi_1, \ldots, \xi_q]$. Here $\Lambda[\xi_1, \ldots, \xi_q]$ is the exterior algebra on the generators ξ_i. By ω_X we will understand the sheaf of volume forms in the sense of Berezin, see *loc. cit.* All the constructions and statements of (2.1)–(2.2) can be immediately generalized to the super case, giving a sheaf of operads (in SVect) $\mathcal{D}_X$ on X of which $\mathcal{O}_X$ is a sheaf of superalgebras.

In particular, consider the case $p = 0$, i.e., $X = \mathbf{C}^{(0|q)} = \mathrm{Spec}\,\Lambda$, where $\Lambda = \Lambda[\xi_1, ..., \xi_q]$. Note that Λ is finite–dimensional super–vector space over $\mathbf{C}$, of dimension $(2^{q-1}|2^{q-1})$ and X_{red} is in this case just a point, so a sheaf on it is just a vector space.

(2.3.1) Proposition. *(a) For $X = \mathbf{C}^{(0|q)}$ the sheaf of operads $\mathcal{D}_X$ on $X_{\mathrm{red}} = \{\mathrm{pt}\}$ is $\mathrm{Op\,End}(\Lambda)$, the endomorphism operad of Λ considered as a super-vector space.*

(b) If $X = X_{\mathrm{red}} \times \mathbf{C}^{(0|q)}$ is a split supermanifold, then $\mathcal{D}_X \simeq \mathcal{D}_{X_{\mathrm{red}}} \otimes_{\mathbf{C}} \mathrm{Op\,End}(\Lambda)$ is isomorphic to a matrix operad over $\mathcal{D}_{X_{\mathrm{red}}}$.

Proof. Part (b) follows from (a). To see (a), notice first that $\mathcal{D}$-action on $\mathcal{O}$, i.e., on Λ, gives an operad morphism $\mathcal{D} \mapsto \mathrm{Op\,End}(\Lambda)$. Let ξ^I, $I = (1 \leq i_1 < ... < i_m \leq q)$, be the monomial basis in Λ. Let $F_{I(1),...,I(n),J} \in \mathrm{Hom}_{\mathbf{C}}(\Lambda^{\otimes n}, \Lambda)$ be the "matrix unit" which takes $\xi^{I(1)} \otimes ... \otimes \xi^{I(n)}$ to ξ^J and all the other tensor products of monomials to 0. Such operators form a basis in $\mathrm{Hom}_{\mathbf{C}}(\Lambda^{\otimes n}, \Lambda)$. On the other hand, the space $\mathcal{D}(n)$ of all n–linear differential operators on Λ has a basis formed by operators

$$L_{I(1),...,I(n),J}, \quad u \mapsto \xi^J(\partial^{I(1)}u)...(\partial^{I(n)}u).$$

Consider the filtration of Λ by the powers of the maximal ideal $(\xi_1, ..., \xi_q)$ and the induced filtration on $\mathrm{Hom}_{\mathbf{C}}(\Lambda^{\otimes n}, \Lambda)$. Modulo this filtration, the action of $L_{I(1),...,I(n),J}$ is given by $F_{I(1),...,I(n),J}$, whence the statement.

Theorem 1.9.3 is now formally applicable in the situation of (2.5.1)(b) but we prefer to make a slightly more general statement. Let $\mathcal{D}_X$–SAlg be the category of sheaves of $\mathcal{D}_X$–superalgebras, see (1.9). Note that even if X happens to be purely even, then a $\mathcal{D}_X$-superalgebra is still required to be $\mathbf{Z}/2$-graded.

(2.3.2) Theorem. *The categories $\mathcal{D}_X$–SAlg and $\mathcal{D}_{X_{\mathrm{red}}}$–SAlg are equivalent.*

This is an operadic extension of the result of I. Penkov [P] on $\mathcal{D}_X(1)$-modules.

Proof. Consider the natural embedding of the supermanifolds $i : X_{\mathrm{red}} \to X$. On the level of underlying spaces this is the identity and the source and target of i differ only by the sheaves of rings: $\mathcal{O}_{X_{\mathrm{red}}}$ as opposed to $\mathcal{O}_X$. Accordingly, both $\mathcal{D}_X$ and $\mathcal{D}_{X_{\mathrm{red}}}$ are sheaves of operads in SVect on the same underlying space X_{red}. This means that we can apply the formalism of Op Hom bimodules of Section 1, if we understand them in a sheaf–theoretical sense. Let

$$\mathcal{D}_{\to} = \{D_{\to}(n)\} = \underline{\mathrm{Op\,Hom}}(\mathcal{O}_X, \mathcal{O}_{X_{\mathrm{red}}}).$$

In other words, we set $\mathcal{D}_{\to}(n)$ to be the sheaf of n–linear differential operators $\mathcal{O}_X \times \cdots \times \mathcal{O}_X \to \mathcal{O}_{X_{\mathrm{red}}}$ in the obvious sense. Clearly $\mathcal{D}_{\to}$ is a $(\mathcal{D}_{X_{\mathrm{red}}}, \mathcal{D}_X)$–bimodule. Similarly, let

$$\mathcal{D}_{\leftarrow} = \{D_{\leftarrow}(n)\} = \underline{\mathrm{Op\,Hom}}(\mathcal{O}_{X_{\mathrm{red}}}, \mathcal{O}_X)$$

be the collection of sheaves of n–linear differential operators $\mathcal{O}_{X_{\mathrm{red}}} \times ... \times \mathcal{O}_{X_{\mathrm{red}}} \to \mathcal{O}_X$. This is a $(\mathcal{D}_X, \mathcal{D}_{X_{\mathrm{red}}})$–bimodule. The relative plethysm with these bimodules defines functors from $\mathcal{D}_X$–SAlg to $\mathcal{D}_{X_{\mathrm{red}}}$–SAlg and back. We claim that these functors are mutually inverse. More precisely, we have natural morphisms of sheaves of operads (in SVect)

$$\mathcal{D}_{\leftarrow} \circ_{\mathcal{D}_{X_{\mathrm{red}}}} \mathcal{D}_{\to} \to \mathcal{D}_{X_{\mathrm{red}}}, \quad \mathcal{D}_{\to} \circ_{\mathcal{D}_X} \mathcal{D}_{\leftarrow} \to \mathcal{D}_X, \tag{2.3.3}$$

obtained by sheafification of (1.9.1–2). In virtue of (1.4.7), it is enough to show that the morphisms (2.3.3) are isomorphisms. To verify this, we can work locally, over an affine open set $U \subset X_{\mathrm{red}}$. In this case the supermanifold $(U, \mathcal{O}_X|_U)$ is split, so Proposition (2.3.1)(b) together with Theorem 1.9.3 imply that our morphisms are isomorphisms after restruction on U.

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