THE BFK-GLUING FORMULA FOR ZETA-DETERMINANTS AND THE VALUE OF RELATIVE ZETA FUNCTIONS AT ZERO

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Abstract. The purpose of this paper is to discuss the constant term appearing in the BFK-gluing formula for the zeta-determinants of Laplacians on a complete Riemannian manifold when the warped product metric is given on a collar neighborhood of a cutting hypersurface. If the dimension of a hypersurface is odd, generally this constant is known to be zero. In this paper we describe this constant by using the heat kernel asymptotics and compute it explicitly when the dimension of a hypersurface is 2 and 4. As a byproduct we obtain some results for the value of relative zeta functions at $s = 0$.

§1. **Introduction**

The gluing formula for zeta-determinants of Laplacians on a compact Riemannnian manifold with boundary had been given by Burghelea, Friedlander and Kappeler in [2] and later was extended by Carron in [3]. Their formula, however, contains a constant term which is expressed by the zero coefficient of some asymptotic expansions ([2], [8]). If the dimension of a cutting hypersurface is odd, this constant is known to be zero ([7]). If the product metric is given on a collar neighborhood of a cutting hypersurface, this constant was computed explicitly in [3] and [9]. The BFK-gluing formula also contains some informations about the value of relative zeta functions at $s = 0$. In this paper we discuss this constant term when the warped product metric is given on a collar neighborhood of a cutting hypersurface. More precisely, we describe this constant in terms of heat kernel asymptotics and compute it explicitly when the dimension of a cutting hypersurface is 2 and 4. As a byproduct we obtain some informations about the value of relative zeta functions at $s = 0$, which we discuss in the last section.

Let (M, g) be either a complete oriented Riemannian manifold or a compact oriented Riemannian manifold with boundary W (W is possibly empty) with dimension $m + 1$. Suppose that Y is a hypersurface of M with $Y \cap W = \emptyset$. Choose a collar neighborhood $N_{-1,1}$ of Y such that $N_{-1,1}$ is diffeomorphic to $([-1, 1] \times Y)$ with $N_{-1,1} \cap W = \emptyset$ and Y is identified with $\{0\} \times Y$. We assume that g is a warped product metric on $N_{-1,1}$, *i.e.*,

$$g|_{N_{-1,1}} = (du^2 + e^{h(u)} g_Y), \tag{1.1}$$

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where g_Y is a Riemannian metric on Y, u is the normal direction to Y and $h : [-1,1] \to \mathbb{R}$ is a smooth function with $h(0) = 0$. We denote by $\frac{\partial}{\partial u}$ the unit vecror field on $N_{-1,1}$ which is normal to $Y_u := \{u\} \times Y$. Let $E \to M$ be a complex vector bundle on M having the product structure on $N_{-1,1}$, which means that $E|_{N_{-1,1}} = p^* E|_Y$, where $p : [-1,1] \times Y \to [-1,1]$ is the projection on the first component. Then the Laplacian Δ_M corresponding to the metric g is described on $N_{-1,1}$ as follows.

$$\Delta_M|_{N_{-1,1}} = -\frac{\partial^2}{\partial u^2} - \frac{m}{2} h'(u) \frac{\partial}{\partial u} + e^{-h(u)} \Delta_Y, \tag{1.2}$$

where $m = dim Y$ and Δ_Y is a Laplacian on Y. We impose a boundary condition P_W on W so that Δ_M can be extended to a non-negative self-adjoint operator. We denote by M_{cut} the manifold with boundary $W \cup Y \cup Y$ obtained by cutting M along Y $(= Y_0)$. We also denote by $Y_{0,1}$ $(Y_{0,2})$ the component of the boundary of M_{cut} which is the copy of Y and $\frac{\partial}{\partial u}$ points outward (inward). We impose the Dirichlet boundary condition on $Y_{0,1} \cup Y_{0,2}$ and denote by $\Delta_{M_{cut},\gamma_0}$ the realization of $\Delta_{M_{cut}}$ with respect to P_W on W and the Dirichlet boundary condition on $Y_{0,1} \cup Y_{0,2}$, where $\Delta_{M_{cut}}$ is the natural extension of Δ_M to M_{cut}. Then $\left(e^{-t(\Delta_M+\lambda)} - e^{-t\left(\Delta_{M_{cut},\gamma_0}+\lambda\right)}\right)$ is a trace class operator (see [3]) and we define the relative zeta function and relative zeta-determinant for $(\Delta_M + \lambda,\ \Delta_{M_{cut},\gamma_0} + \lambda)$ by

$$\begin{aligned} \zeta\left(s; \Delta_M + \lambda,\ \Delta_{M_{cut},\gamma_0} + \lambda\right) &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} Tr\left(e^{-t(\Delta_M+\lambda)} - e^{-t\left(\Delta_{M_{cut},\gamma_0}+\lambda\right)}\right) dt \\ \log Det\left(\Delta_M + \lambda,\ \Delta_{M_{cut},\gamma_0} + \lambda\right) &= -\zeta'\left(0; \Delta_M + \lambda,\ \Delta_{M_{cut},\gamma_0} + \lambda\right). \end{aligned} \tag{1.3}$$

Throughout this paper we assume that for $\lambda \in \mathbb{R}^+$ the Dirichlet boundary value problem for $\Delta_{M_{cut}} + \lambda$ has a unique solution, *i.e.* for $(f,g) \in C^\infty(Y_{0,1} \cup Y_{0,2})$ there exists a unique solution ϕ such that

$$(\Delta_{M_{cut}} + \lambda)\phi = 0, \quad \phi|_{Y_{0,1}} = f, \quad \phi|_{Y_{0,2}} = g, \quad P_W(\phi)|_W = 0.$$

Then Burghelea, Friedlander and Kappeler ([2]) had studied $\log Det\left(\Delta_M + \lambda,\ \Delta_{M_{cut},\gamma_0} + \lambda\right)$ for $\lambda \in \mathbb{R}^+$ on a compact manifold and later Carron ([3]) extended their result to the case of a complete non-compact manifold.

To state their results, we introduce elliptic ΨDO's $Q_1(\lambda)$, $Q_2(\lambda)$ and $R(\lambda)$ acting on $C^\infty(Y)$, smooth sections on Y, as follows. For $f \in C^\infty(Y)$, we choose unique sections ϕ, $\psi \in C^\infty(M_{cut})$ satisfying

$$\begin{aligned} &(\Delta_{M_{cut}} + \lambda)\phi = 0, \quad \phi|_{Y_{0,1}} = f, \quad \phi|_{Y_{0,2}} = 0, \quad P_W(\phi)|_W = 0, \\ &(\Delta_{M_{cut}} + \lambda)\psi = 0, \quad \psi|_{Y_{0,1}} = 0, \quad \psi|_{Y_{0,2}} = f, \quad P_W(\psi)|_W = 0. \end{aligned}$$

Then we define

$$Q_1(\lambda)(f) = \left(\frac{\partial}{\partial u}\phi\right)|_{Y_{0,1}} - \left(\frac{\partial}{\partial u}\phi\right)|_{Y_{0,2}}, \qquad Q_2(t)(\lambda) = \left(\frac{\partial}{\partial u}\psi\right)|_{Y_{0,1}} - \left(\frac{\partial}{\partial u}\psi\right)|_{Y_{0,2}} \tag{1.4}$$

and define the Dirichlet-to-Neumann operator $R(\lambda) : C^\infty(Y) \to C^\infty(Y)$ by

$$R(\lambda) = Q_1(\lambda) + Q_2(\lambda). \tag{1.5}$$

It is a well-known fact that for $\lambda \in \mathbb{R}^+$, $R(\lambda)$ is a positive elliptic ΨDO of order 1. Then they proved the following equality. For some real coefficients a_j $(0 \leq j \leq [\frac{m}{2}])$,

$$\log Det\left(\Delta_M + \lambda,\ \Delta_{M_{cut},\gamma_0} + \lambda\right) = \sum_{j=0}^{[\frac{m}{2}]} \pi_j \lambda^j + \log Det R(\lambda). \tag{1.6}$$

When M is compact, it is known in [2] that $\log Det\,(\Delta_M+\lambda, \Delta_{M_{cut},\gamma_0}+\lambda)$ and $\log DetR(\lambda)$ have asymptotic expansions as $\lambda\to\infty$, whose coefficients can be computed by integral of some densities determined by the symbols of operators. Moreover, the zero coefficient in the asymptotic expansion of $\log Det\,(\Delta_M+\lambda, \Delta_{M_{cut},\gamma_0}+\lambda)$ is zero (Lemma 3.2 or [21]). Hence $-\pi_0$ in (1.6) is, in fact, the zero coefficient in the asymptotic expansion of $\log DetR(\lambda)$. This fact enables us to comput π_0 in some cases. It is known that $\pi_0=0$ when $dimM$ is even ([7]). If the metric g is a product one on $N_{-1,1}$ so that $\Delta_M=-\partial_u^2+\Delta_Y$, it is known that $\pi_0=\log 2\cdot(\zeta_{\Delta_Y}(0)+dimker\Delta_Y)$ ([3], [9]).

On the other hand, if M is compact or the essential spectrum $\sigma_{ess}(\Delta_M)$ of Δ_M has a positive lower bound, the coefficient of $\log\lambda$ in the asymptotic expansion of $\log Det\,(\Delta_M+\lambda,\ \Delta_{M_{cut},\gamma_0}+\lambda)$ is $\zeta(0;\Delta_M,\Delta_{M_{cut},\gamma_0})+dimker\Delta_M$ (Lemma 3.2). Hence, the comparison of the $\log\lambda$-coefficients in the asymptotic expansions of (1.6) gives some informations about the value of the relative zeta functions at zero.

In this paper we are going to discuss the constant π_0 in (1.6), or equivalently the polynomial $\sum_{j=0}^{[\frac{m}{2}]}\pi_j\lambda^j$, and the value of relative zeta functions at $s=0$ when the Laplacian is given by (1.2) on $N_{-1,1}$. We next discuss, using the result in [12], the value of the zeta function at $s=0$ for a compatible Dirac Laplacian with the Atiyah-Patodi-Singer (APS) boundary condition, which extends the result given by the author in the Appendix of [17].

§2. **Description of the Dirichlet-to-Neumann operator** $R(\lambda)$

In this section we are going to describe the Dirichlet-to-Neumann operator $R(\lambda)$ into a more useful form by using the variations of $Q_1(\lambda)$, $Q_2(\lambda)$. To do this we first define the operator $Q_{i,u}(\lambda)$ $(i=1,2)$ on $Y_u:=\{u\}\times Y$ for each u, $-1\leq u\leq 1$, in the same way as $Q_i(\lambda)$ on Y_0. More precisely, let $M_{cut,u}$ be the manifold with boundary obtained by cutting M along Y_u and $Y_{u,1}$ ($Y_{u,2}$) be the boundary of $M_{cut,u}$ such that $\frac{\partial}{\partial u}$ points outward (inward). For $f\in C^\infty(Y_u)$ choose ϕ_u, $\psi_u\in C^\infty(M_{cut,u})$ satisfying

$$\begin{aligned}&(\Delta_{M_{cut}}+\lambda)\,\phi_u=0,\quad \phi_u|_{Y_{u,1}}=f,\quad \phi_u|_{Y_0}=0,\quad P_W(\phi_u)|_W=0,\\ &(\Delta_{M_{cut}}+\lambda)\,\psi_u=0,\quad \phi_u|_{Y_0}=0,\quad \psi_u|_{Y_{u,2}}=f,\quad P_W(\psi_u)|_W=0.\end{aligned}$$

Then we define

$$\begin{aligned}Q_{1,u}(\lambda)(f)&=\left(\frac{\partial}{\partial u}\phi_u\right)|_{Y_{u,1}}-\left(\frac{\partial}{\partial u}\phi_u\right)|_{Y_0}=:\widetilde{Q}_{1,u}(\lambda)(f)+\Psi_{1,u}(\lambda)(f),\\ Q_{2,u}(\lambda)(f)&=\left(\frac{\partial}{\partial u}\psi_u\right)|_{Y_0}-\left(\frac{\partial}{\partial u}\psi_u\right)|_{Y_{u,2}}=:\Psi_{2,u}(\lambda)(f)+\widetilde{Q}_{2,u}(\lambda)(f),\end{aligned}\tag{2.1}$$

where $\widetilde{Q}_{1,u}(\lambda)(f):=\left(\frac{\partial}{\partial u}\phi_u\right)|_{Y_{u,1}}$ and $\Psi_{1,u}(\lambda)(f):=-\left(\frac{\partial}{\partial u}\phi_u\right)|_{Y_0}$. $\widetilde{Q}_{2,u}(\lambda)(f)$ and $\Psi_{2,u}(\lambda)(f)$ are defined similarly.

Now for $f\in C^\infty(Y_0)$, choose $\phi(u,y)\in C^\infty(M_{cut})$ such that

$$(\Delta_{M_{cut}}+\lambda)\,\phi(u,y)=0,\quad \phi|_{Y_{0,1}}=f,\quad \phi|_{Y_{0,2}}=0,\quad P_W(\phi)|_W=0.$$

Then for each u, $-1\leq u\leq 1$, we have

$$\left(\frac{\partial}{\partial u}\phi(u,y)\right)|_{Y_u}=\widetilde{Q}_{1,u}(\lambda)\left(\phi(u,y)|_{Y_u}\right).$$

Taking the derivative with respect to u,

$$\left(\frac{\partial^2}{\partial u^2}\phi(u,y)\right)|_{Y_u} = \frac{\partial \widetilde{Q}_{1,u}(\lambda)}{\partial u}\left(\phi(u,y)|_{Y_u}\right) + \widetilde{Q}_{1,u}(\lambda)\left(\frac{\partial \phi(u,y)}{\partial u}|_{Y_u}\right)$$
$$= \left(\frac{\partial}{\partial u}\widetilde{Q}_{1,u}(\lambda) + \widetilde{Q}_{1,u}(\lambda)^2\right)\left(\phi(u,y)|_{Y_u}\right). \tag{2.2}$$

Since ϕ satisfies $(\Delta_{M_{cut}} + \lambda)\,\phi = 0$, (1.2) leads to

$$\left(-\frac{m}{2}h'(u)\widetilde{Q}_{1,u}(\lambda) + e^{-h(u)}\Delta_Y + \lambda\right)\left(\phi(u,y)|_{Y_u}\right) = \left(\frac{\partial}{\partial u}\widetilde{Q}_{1,u}(\lambda) + \widetilde{Q}_{1,u}(\lambda)^2\right)\left(\phi(u,y)|_{Y_u}\right),$$

which shows that

$$\left(\widetilde{Q}_{1,u}(\lambda) + \frac{m}{4}h'(u)\right)^2 = e^{-h(u)}\Delta_Y + \lambda + \frac{m^2}{16}h'(u)^2 - \frac{\partial}{\partial u}\widetilde{Q}_{1,u}(\lambda). \tag{2.3}$$

Similarly, we have

$$\left(\widetilde{Q}_{2,u}(\lambda) - \frac{m}{4}h'(u)\right)^2 = e^{-h(u)}\Delta_Y + \lambda + \frac{m^2}{16}h'(u)^2 + \frac{\partial}{\partial u}\widetilde{Q}_{2,u}(\lambda). \tag{2.4}$$

We next note that $\Psi_{1,u}(\lambda)\left(\phi(u,y)|_{Y_u}\right) = -\left(\frac{\partial}{\partial u}\phi(u,y)\right)|_{Y_{0,2}}$. Taking the derivative, we have

$$\left(\frac{\partial}{\partial u}\Psi_{1,u}(\lambda)\right)\phi(u,y)|_{Y_u} + \Psi_{1,u}(\lambda)\left(\widetilde{Q}_{1,u}(\lambda)\phi(u,y)|_{Y_u}\right) = 0, \tag{2.5}$$

which shows that $\frac{\partial}{\partial u}\Psi_{1,u}(\lambda) + \Psi_{1,u}(\lambda)\widetilde{Q}_{1,u}(\lambda) = 0$. Since $\frac{\partial}{\partial u}\Psi_{1,u}(\lambda)$ and $\Psi_{1,u}(\lambda)$ are operators of same order and $\widetilde{Q}_{1,u}(\lambda)$ is an elliptic operator of order 1, this equality implies that $\Psi_{1,u}(\lambda)$ is a smoothing operator. As the same way, $\Psi_{2,u}(\lambda)$ is also a smoothing operator. Setting $R_u(\lambda) = Q_{1,u}(\lambda) + Q_{2,u}(\lambda)$, we have the following lemma.

Lemma 2.1. *Let $\Omega_u(\lambda) = Q_{1,u}(\lambda) - Q_{2,u}(\lambda) + \frac{m}{2}h'(u)$. Then $R_u(\lambda)^2$ is expressed as follows.*

$$R_u(\lambda)^2 = 4\left(e^{-h(u)}\Delta_Y + \lambda + \frac{m^2}{16}h'(u)^2 + \frac{m}{4}h''(u)\right) - 2\frac{\partial}{\partial u}\Omega_u(\lambda) - \Omega_u(\lambda)^2$$
$$+ \ \textit{a smoothing operator.}$$

Remark : It is well-known ([2], [3]) that $R(\lambda)^{-1} = \gamma_0\left(\Delta_M + \lambda\right)^{-1}(\cdot \otimes \delta_Y)$ and this fact implies that $R(\lambda)$ is positive definite for $\lambda \in \mathbb{R}^+$, where γ_0 is the restriction map to Y.

We now discuss the asymptotic symbols of $\frac{\partial}{\partial u}\Omega_u(\lambda)$ and $\Omega_u(\lambda)^2$. The equations (2.3), (2.4) and (2.5) show that

$$\begin{aligned} Q_{1,u}(\lambda)^2 &= e^{-h(u)}\Delta_Y + \lambda - \frac{m}{2}h'(u)Q_{1,u}(\lambda) - \frac{\partial}{\partial u}Q_{1,u}(\lambda) + \text{ a smoothing operator,} \\ Q_{2,u}(\lambda)^2 &= e^{-h(u)}\Delta_Y + \lambda + \frac{m}{2}h'(u)Q_{2,u}(\lambda) + \frac{\partial}{\partial u}Q_{2,u}(\lambda) + \text{ a smoothing operator.} \end{aligned} \tag{2.6}$$

We denote the asymptotic symbols of $Q_{1,u}(\lambda)$, $Q_{2,u}(\lambda)$ and $e^{-h(u)}\Delta_Y + \lambda$ as follows.

$$\begin{aligned} \sigma(Q_{1,u}(\lambda))(y,\xi,\lambda) &\sim \alpha_{1,u}(y,\xi,\lambda) + \alpha_{0,u}(y,\xi,\lambda) + \alpha_{-1,u}(y,\xi,\lambda) + \cdots \\ \sigma(Q_{2,u}(\lambda))(y,\xi,\lambda) &\sim \beta_{1,u}(y,\xi,\lambda) + \beta_{0,u}(y,\xi,\lambda) + \beta_{-1,u}(y,\xi,\lambda) + \cdots \\ \sigma(e^{-h(u)}\Delta_Y + \lambda) &= (e^{-h(u)}|\xi|^2 + \lambda) + e^{-h(u)}p_1(y,\xi) + e^{-h(u)}p_0(y,\xi). \end{aligned}$$

Then the asymptotic symbols of $Q_{1,u}(\lambda)^2$ and $Q_{2,u}(\lambda)^2$ are given by

$$\begin{aligned}
\sigma(Q_{1,u}(\lambda)^2)(y,\xi,\lambda) &\sim \sum_{k=0}^{\infty} \sum_{\substack{|\omega|+i+j=k \\ i,j\geq 0}} \frac{1}{\omega!} d_\xi^\omega \alpha_{1-i,u}(y,\xi,\lambda) \cdot D_y^\omega \alpha_{1-j,u}(y,\xi,\lambda), \\
\sigma(Q_{2,u}(t)^2)(y,\xi,\lambda) &\sim \sum_{k=0}^{\infty} \sum_{\substack{|\omega|+i+j=k \\ i,j\geq 0}} \frac{1}{\omega!} d_\xi^\omega \beta_{1-i,u}(y,\xi,\lambda) \cdot D_y^\omega \beta_{1-j,u}(y,\xi,\lambda),
\end{aligned} \tag{2.7}$$

which leads to

$$\begin{aligned}
\alpha_{1,u} &= \beta_{1,u} = \sqrt{e^{-h(u)}|\xi|^2+\lambda} \\
\alpha_{0,u} &= \frac{1}{2}\alpha_{1,u}^{-1}\left(-d_\xi\alpha_{1,u}\cdot D_y\alpha_{1,u} + e^{-h(u)}p_1 - \frac{m}{2}h'(u)\alpha_{1,u} + \frac{h'(u)e^{-h(u)}|\xi|^2}{2\sqrt{e^{-h(u)}|\xi|^2+\lambda}}\right) \\
\beta_{0,u} &= \frac{1}{2}\beta_{1,u}^{-1}\left(-d_\xi\beta_{1,u}\cdot D_y\beta_{1,u} + e^{-h(u)}p_1 + \frac{m}{2}h'(u)\beta_{1,u} - \frac{h'(u)e^{-h(u)}|\xi|^2}{2\sqrt{e^{-h(u)}|\xi|^2+\lambda}}\right).
\end{aligned} \tag{2.8}$$

Using the relation (2.7) we can compute the homogeneous components $\alpha_{1-k,u}$ and $\beta_{1-k,u}$ for any $k \geq 1$, and hence the asymptotic symbols of $\frac{\partial}{\partial u}\Omega_u(\lambda)$ and $\Omega_u(\lambda)^2$. For instance, the principal symbols of $\frac{\partial}{\partial u}\Omega_u(\lambda)$ and $\Omega_u(\lambda)^2$ are given as follows.

$$\begin{aligned}
\sigma_L\left(2\frac{\partial}{\partial u}\Omega_u(\lambda)\right) &= \frac{\left(h''(u)-h'(u)^2\right)e^{-h(u)}|\xi|^2}{e^{-h(u)}|\xi|^2+\lambda} + \frac{h'(u)^2e^{-2h(u)}|\xi|^4}{\left(e^{-h(u)}|\xi|^2+\lambda\right)^2}, \\
\sigma_L\left(\Omega_u(\lambda)^2\right) &= \frac{h'(u)^2e^{-2h(u)}|\xi|^4}{4\left(e^{-h(u)}|\xi|^2+\lambda\right)^2}.
\end{aligned} \tag{2.9}$$

Corollary 2.2. *$\frac{\partial}{\partial u}\Omega_u(\lambda)$ and $\Omega_u(\lambda)^2$ are ΨDO's of order 0 and each homogeneous part of the asymptotic symbol tends to 0 pointwisely as $\lambda \to \infty$.*

We next discuss the heat kernel asymptotics of $R_u(\lambda)^2$ at $u = 0$. We assume that $h(0) = 0$ and denote by

$$c_0 = \frac{m^2}{16}h'(0)^2 + \frac{m}{4}h''(0), \qquad \mathfrak{S}_0(\lambda) = -2\frac{\partial}{\partial u}\Omega_u(\lambda)|_{u=0} - \Omega_0(\lambda)^2. \tag{2.10}$$

We suppose that as $t \to 0^+$,

$$\begin{aligned}
Tr e^{-t(\Delta_Y+c_0)} &\sim \sum_{j=0}^{\infty} a_j t^{-\frac{m}{2}+j}, \\
Tr e^{-t\left(\Delta_Y+c_0+\frac{1}{4}\mathfrak{S}_0(\lambda)\right)} &\sim \sum_{j=0}^{\infty} a_j(\lambda)\, t^{-\frac{m}{2}+j} + \sum_{k=1}^{\infty} b_k(\lambda)\, t^k \log t.
\end{aligned} \tag{2.11}$$

Then each coefficient $a_j(\lambda)$ and a_j in (2.11) can be computed by the following formula (*cf.* [5], [20]). Let Γ be a contour in the complex plane $\mathbb{C}$ defined by

$$\Gamma = \{re^{i\pi} | \infty > r \geq \epsilon\} \cup \{\epsilon e^{i\phi} | \pi \geq \phi \geq -\pi\} \cup \{re^{-i\pi} | \epsilon \leq r < \infty\} \tag{2.12}$$

for sufficiently small $\epsilon > 0$ and oriented counterclockwise. Let us fix a finite number of local coordinate charts covering Y, local trivializations for E, and a partition of unity subordinate to this covering. In each coordinate chart we denote the asymptotic symbol of $\left(\Delta_Y + c_0 + \frac{1}{4}\mathfrak{S}_0(\lambda)\right)$ by

$$\sigma\left(\Delta_Y + c_0 + \frac{1}{4}\mathfrak{S}_0(\lambda)\right) \sim p_2(y,\xi) + p_1(y,\xi) + (p_0(y,\xi) + q_0(y,\xi,\lambda)) + q_{-1}(y,\xi,\lambda) + \cdots, \quad (2.13)$$

where $p_2(y,\xi) = |\xi|^2$ and $\sum_{j=0}^{2} p_j(y,\xi)$, $\sum_{j=0}^{\infty} q_j(y,\xi,\lambda)$ are the (asymptotic) symbols of $\Delta_Y + c_0$ and $\frac{1}{4}\mathfrak{S}_0(\lambda)$, respectively. Then the asymptotic symbol $\sum_{j=0}^{\infty} r_{-2-j}(y,\xi,\lambda,\mu)$ of the resolvent $\left(\mu - \left(\Delta_Y + c_0 + \frac{1}{4}\mathfrak{S}_0(\lambda)\right)\right)^{-1}$ is given recursively as follows.

$$r_{-2}(y,\xi,\lambda,\mu) = (\mu - |\xi|^2)^{-1},$$

$$r_{-2-j}(y,\xi,\lambda,\mu) = -(\mu - |\xi|^2)^{-1} \sum_{k=0}^{j-1} \sum_{|\omega|+l+k=j} \frac{1}{\omega!} \partial_\xi^\omega \widetilde{p}_{2-l}(y,\xi,\lambda) \cdot D_y^\omega r_{-2-k}(y,\xi,\lambda,\mu), \quad (2.14)$$

where

$$\widetilde{p}_j(y,\xi,\lambda) = \begin{cases} p_j(y,\xi) & \text{for } j = 1,2 \\ p_0(y,\xi) + q_0(y,\xi,\lambda) & \text{for } j = 0 \\ q_j(y,\xi,\lambda) & \text{for } j < 0. \end{cases}$$

Then $a_j(\lambda)$ $(0 \le j \le [\frac{m}{2}])$ can be computed by the following integral (*cf.* Formula (12) in [5] or Theorem 13.1 in [20]).

$$a_j(\lambda) = \begin{cases} \frac{\Gamma(\frac{m}{2}-j)}{2\cdot 2\pi i} \int_Y dvol(Y) \int_{|\xi|=1} \frac{1}{(2\pi)^m} d\xi' \int_\Gamma \mu^{-\frac{m}{2}+j} r_{-2-2j}(y,\xi,\lambda,\mu) d\mu & \text{for } 0 \le j < \frac{m}{2} \\ \frac{1}{2} \int_Y dvol(Y) \int_{|\xi|=1} \frac{1}{(2\pi)^m} d\xi' \int_0^\infty r_{-2-m}(y,\xi,\lambda,-\nu) d\nu & \text{for } j = \frac{m}{2} \in \mathbb{Z}^+, \end{cases} \quad (2.15)$$

where $d\xi'$ is the usual surface element of the sphere $|\xi| = 1$. Similarly, let us denote the asymptotic symbol of $(\mu - (\Delta_Y + c_0))^{-1}$ by

$$\sigma\left((\mu - (\Delta_Y + c_0))^{-1}\right) \sim \sum_{j=0}^{\infty} \delta_{-2-j}(y,\xi,\mu), \quad (2.16)$$

where $\delta_{-2-j}(y,\xi,\mu)$ is defined recursively by the same formula as (2.14). Then a_j can be computed by the integral (2.15) with the integrand $\delta_{-2-j}(y,\xi,\mu)$. We here note that

$$r_{-2-j}(y,\xi,\lambda,\mu) = \delta_{-2-j}(y,\xi,\mu) + \widetilde{r}_{-2-j}(y,\xi,\lambda,\mu), \quad (2.17)$$

where $\widetilde{r}_{-2-j}(y,\xi,\lambda,\mu)$ tends to 0 pointwisely as $\lambda \to \infty$. In particular, $r_{-2}(y,\xi,\lambda,\mu) = \delta_{-2}(y,\xi,\mu)$. Then we have the following result.

Lemma 2.3. *Each $a_j(\lambda)$, $0 \le j \le [\frac{m}{2}]$, has an asymptotic expansion for $\lambda \to \infty$ of the following form.*

$$a_j(\lambda) \sim a_j + \sum_{k=1}^{\infty} a_{j,\frac{k}{2}} \lambda^{-\frac{k}{2}}.$$

In particular,

$$a_0(\lambda) = a_0 \quad \textit{and} \quad \lim_{\lambda\to\infty} a_j(\lambda) = a_j, \quad (1 < j \le [\frac{m}{2}]).$$

Proof : In view of (2.14), $r_{-2-j}(y,\xi,\lambda,\mu)$ can be expressed by

$$r_{-2-j}(y,\xi,\lambda,\mu) \;=\; \sum_{k\geq 0} \frac{f_k(y,\xi,\mu)}{(|\xi|^2+\lambda)^{\frac{k}{2}}},$$

where $f_k(y,\xi,\mu)$ is a homogeneous polynomial of degree $k-2-j$ with respect to ξ and μ. Hence, (2.15) shows that for $0 \leq j < \frac{m}{2}$,

$$a_j(\lambda) = \sum_k \frac{1}{(\lambda+1)^{\frac{k}{2}}} \frac{1}{2\cdot 2\pi i} \int_Y dvol(Y) \int_{|\xi|=1} \frac{1}{(2\pi)^m} d\xi' \int_\Gamma \mu^{-\frac{m}{2}-j} f_k(y,\xi,\mu) d\mu,$$

from which the result follows. The case of $j = \frac{m}{2}$ can be treated in the same way. □

§3. **The main results and their proofs**

We begin this section with the following lemma, which is straightforward (*cf.* [8] or [21]).

Lemma 3.1. *Let P be an elliptic ΨDO of order > 0 on a compact manifold and $\{\alpha_j\}$, $\{\beta_j\}$ be increasing sequences with $\beta_0 > 0$, and tending to ∞. Suppose that*

$$Tr e^{-tP} \;\sim\; \sum_{j=0}^{\infty} a_j t^{\alpha_j} \;+\; \sum_{j=0}^{\infty} b_j t^{\beta_j} \log t \quad \textit{for} \quad t \to 0^+.$$

Then, as $\lambda \to \infty$,

$$\log Det\,(P+\lambda) \;\sim\; -\sum_{j=0}^{\infty} a_j \frac{d}{ds}\left(\frac{\Gamma(s+\alpha_j)}{\Gamma(s)}\right)_{s=0} \cdot \lambda^{-\alpha_j} + \sum_{j=0}^{\infty} a_j \left(\frac{\Gamma(s+\alpha_j)}{\Gamma(s)}\right)_{s=0} \cdot \lambda^{-\alpha_j} \log\lambda$$
$$+ \sum_{j=0}^{\infty} b_j \Gamma(\beta_j) \cdot \lambda^{-\beta_j} \log\lambda \;-\; \sum_{j=0}^{\infty} b_j \int_0^\infty x^{\beta_j - 1} e^{-x} \log x \; dx \cdot \lambda^{-\beta_j} \;+\; O\left(e^{-c\lambda}\right).$$

In particular, the constant term does not appear and the coefficient of $\log\lambda$ *is* $(\zeta_P(0) + dimkerP)$.

We next consider the asymptotic expansions of $\log Det\,(\Delta_M + \lambda,\ \Delta_{M_{cut},\gamma_0} + \lambda)$ and $\log Det R(\lambda)$ as $\lambda \to \infty$. Let K be a collar neighborhood of Y whose closure is a compact subset of M. Then it is shown in [1] (see also [3]) that for some positive constant C,

$$|\; Tr\left(1_{M-K}\left(e^{-t\Delta_M} - e^{-t\Delta_{M_{cut},\gamma_0}}\right) 1_{M-K}\right) \;| \leq e^{-\frac{C}{t}}.$$

It is a well-known fact that for $t \to 0$,

$$Tr\left(1_K\left(e^{-t\Delta_M} - e^{-t\Delta_{M_{cut},\gamma_0}}\right) 1_K\right) \;\sim\; \sum_{j=0}^{\infty} a_j t^{\frac{j-m}{2}}.$$

Hence, for $t \to 0$,

$$Tr\left(e^{-t\Delta_M} - e^{-t\Delta_{M_{cut},\gamma_0}}\right) \;\sim\; \sum_{j=0}^{\infty} a_j t^{\frac{j-m}{2}}, \tag{3.1}$$

where $dimM = m+1$. This fact together with Lemma 3.1 yields the following result.

Lemma 3.2. *As $\lambda \to \infty$, we have the following asymptotic expansion.*

$$\log Det\left(\Delta_M + \lambda,\ \Delta_{M_{cut},\gamma_0} + \lambda\right) \sim -\sum_{j=0}^{\infty} a_j \left(\frac{d}{ds} \left(\frac{\Gamma(s - \frac{m-j}{2})}{\Gamma(s)} \right) \right)_{s=0} \lambda^{\frac{m-j}{2}}$$
$$+ \sum_{j=0}^{m} a_j \left(\frac{\Gamma(s - \frac{m-j}{2})}{\Gamma(s)} \right)_{s=0} \lambda^{\frac{m-j}{2}} \cdot \log \lambda + O(e^{-c\lambda}).$$

In particular, the constant term does not appear. If M is compact or complete with $\sigma_{ess}(\Delta_M)$ having a positive lower bound, the coefficient of $\log \lambda$ is $\zeta\left(0; \Delta_M,\ \Delta_{M_{cut},\gamma_0}\right) + dimker\Delta_M$.

The above lemma shows that $-\pi_0$ in (1.6) is, in fact, the zero coefficient in the asymptotic expansion of $\log DetR(\lambda)$ as $\lambda \to \infty$. If $dimY$ is odd, this zero coefficient is known to be zero ([7]) and hence we need to discuss the case of $dim\, Y$ even. By Lemma 2.1 and Lemma 3.1 with (2.11) we have the following result.

Lemma 3.3. *Let m be the dimension of Y. Then :*

$$\log DetR(\lambda) = \log 2 \cdot \zeta_{\left(\Delta_Y + c_0 + \frac{1}{4}\mathfrak{S}_0(\lambda) + \lambda\right)}(0) - \frac{1}{2} \sum_{j=0}^{[\frac{m}{2}]} a_j(\lambda) \frac{d}{ds} \left(\frac{\Gamma(s - \frac{m}{2} + j)}{\Gamma(s)} \right)_{s=0} \lambda^{\frac{m}{2} - j}$$
$$+ \frac{1}{2} \sum_{j=0}^{[\frac{m}{2}]} a_j(\lambda) \left(\frac{\Gamma(s - \frac{m}{2} + j)}{\Gamma(s)} \right)_{s=0} \lambda^{\frac{m}{2} - j} \log \lambda + O\left(\lambda^{-c}\right), \quad (c > 0),$$

where

$$\zeta_{\left(\Delta_Y + c_0 + \frac{1}{4}\mathfrak{S}_0(\lambda) + \lambda\right)}(0) = \begin{cases} 0 & \text{if } m \text{ is odd} \\ \sum_{j=0}^{\frac{m}{2}} \frac{(-1)^{\frac{m}{2} - j}}{(\frac{m}{2} - j)!}\, a_j(\lambda)\, \lambda^{\frac{m}{2} - j} & \text{if } m \text{ is even.} \end{cases}$$

Specially, if $dimY$ is even, Lemma 3.3 together with Lemma 2.3 leads to the following corollary.

Corollary 3.4. *Let $m = dim\, Y$ be even. Then $\log DetR(\lambda)$ has the following asymptotic expansion for $\lambda \to \infty$.*

$$\log DetR(\lambda) \sim \sum_{j=0}^{\infty} \mathfrak{p}_j \lambda^{\frac{m-j}{2}} + \sum_{j=0}^{\infty} \mathfrak{q}_j \lambda^{\frac{m-j}{2}} \log \lambda,$$

where

$$\mathfrak{p}_m = \log 2 \sum_{j=1}^{\frac{m}{2}} \frac{(-1)^{\frac{m}{2} - j}}{(\frac{m}{2} - j)!} a_{j, \frac{m}{2} - j} - \frac{1}{2} \sum_{j=1}^{\frac{m}{2} - 1} \frac{(-1)^{\frac{m}{2} - j}}{(\frac{m}{2} - j)!} \left(1 + \frac{1}{2} + \cdots + \frac{1}{\frac{m}{2} - j} \right) a_{j, \frac{m}{2} - j},$$
$$\mathfrak{q}_m = \frac{1}{2} \sum_{j=1}^{\frac{m}{2}} \frac{(-1)^{\frac{m}{2} - j}}{(\frac{m}{2} - j)!} a_{j, \frac{m}{2} - j}.$$

On the other hand, let $P(\eta)$ be a non-negative classical ΨDO of order k with parameter η of weight χ ($\chi > 0$) on a d-dimensional compact closed manifold X. We refer to [2] or [20] for the definitions. Suppose that the asymptotic symbol of $P(\eta)$ is given as follows.

$$\sigma(P(\eta)) = \sum_{j=0}^{\infty} p_{k-j}(\eta, x, \xi),$$

where for $\tau > 0$, $p_{k-j}(\tau^{\chi}\eta, x, \tau\xi) = \tau^{k-j} p_{k-j}(\eta, x, \xi)$. Then it was shown in the Appendix of [2] that $\log DetP(\eta)$ has the following asymptotic expansion as $|\eta| \to \infty$.

$$\log DetP(\eta) \ \sim \ \sum_{j=0}^{\infty} \kappa_j |\eta|^{\frac{d-j}{\chi}} \ + \ \sum_{j=0}^{d} \theta_j |\eta|^{\frac{j}{\chi}} \log|\eta|, \tag{3.2}$$

where each κ_j and θ_j can be computed in terms of the symbol of $P(\frac{\eta}{|\eta|})$. Specially, κ_d and θ_0 can be computes as follows. Let us denote the symbol of $(\mu - P(\eta))^{-1}$ by

$$\sigma\left((\mu - P(\eta))^{-1}\right)(\mu, \eta, x, \xi) \ \sim \ \sum_{j=0}^{\infty} r_{-k-j}(\mu, \eta, x, \xi).$$

We define $J_d(s, \frac{\eta}{|\eta|}, x)$ by

$$J_d(s, \frac{\eta}{|\eta|}, x) \ = \ \frac{1}{2\pi i} \int_{\mathbb{R}^d} d\xi \int_{\Gamma} \mu^{-s} r_{-k-d}(\mu, \frac{\eta}{|\eta|}, x, \xi) d\mu, \tag{3.3}$$

where Γ is a contour in $\mathbb{C}$ defind in (2.12). Then,

$$\kappa_d \ = \ \frac{d}{ds}\left(\frac{1}{(2\pi)^d} \int_M J_d(s, \frac{\eta}{|\eta|}, x) \ dvol(x)\right)|_{s=0}, \quad \theta_0 \ = \ \left(\frac{1}{(2\pi)^d} \frac{k}{\chi} \int_M J_d(s, \frac{\eta}{|\eta|}, x) \ dvol(x)\right)|_{s=0}. \tag{3.4}$$

If the symbol of $P(\eta)$ satisfies the following property

$$p_{k-j}(\eta, x, -\xi) \ = \ (-1)^j p_{k-j}(\eta, x, \xi), \tag{3.5}$$

and $d = dimX$ is odd, then $J_d(s, \frac{\eta}{|\eta|}, x) = 0$ and hence $\kappa_d = \theta_0 = 0$. It is known that the Dirichlet-to-Neumann operator $R(\lambda)$ is a classical ΨDO of order 1 with parameter λ of weight 2 and satisfies (3.5) (*cf.* [7]). This fact with Corollary 3.4 leads to the following result.

Corollary 3.5. *Suppose that $dimM = m+1$ and either M is compact or $\sigma_{ess}(\Delta_M)$ has a positive lower bound, then*

$$\zeta\left(0; \Delta_M, \ \Delta_{M_{cut}, \gamma_0}\right) + dimker\Delta_M \ = \ \begin{cases} 0 & \text{if } m \text{ is odd} \\ \frac{1}{2} \sum_{j=1}^{\frac{m}{2}} \frac{(-1)^{\frac{m}{2}-j}}{(\frac{m}{2}-j)!} a_{j, \frac{m}{2}-j} & \text{if } m \text{ is even.} \end{cases}$$

Since $-\pi_0$ in (1.6) is the zero coefficient in the asymptotic expansion of $\log Det(R(\lambda))$, Corollary 3.4 leads to the following result, which is the main result of this paper.

Theorem 3.6. *Let $dimM = m+1$, $c_0 = \frac{m^2}{16}h'(0)^2 + \frac{m}{4}h''(0)$ and $l = dimker\left(\Delta_Y + c_0\right)$. Then the constant π_0 in (1.6) is the following.*
(1) If m is odd, then $\pi_0 = 0$.
(2) If $m = 2$, then $\pi_0 = -\log 2 \cdot \left(\zeta_{(\Delta_Y + c_0)}(0) + l\right)$.
(3) If $m = 4$, $\pi_0 = -\log 2 \cdot \left(\zeta_{(\Delta_Y + c_0)}(0) + l\right) - \left(\log 2 - \frac{1}{2}\right) \frac{vol(Y)}{64\pi^2}\left(h''(0) - h'(0)^2\right)$, where $vol(Y)$ is the volume of Y.
(4) Generally, if $m \geq 4$ and even, then we have

$$\begin{aligned} \pi_0 \ &= \ -\mathfrak{p}_m \\ &= \ -\log 2 \cdot \left(\zeta_{(\Delta_Y + c_0)}(0) + l\right) \ - \sum_{j=1}^{\frac{m}{2}-1} \frac{(-1)^{\frac{m}{2}-j}}{(\frac{m}{2}-j)!} a_{j, \frac{m}{2}-j} \left(\log 2 \ - \ \frac{1}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{\frac{m}{2}-j}\right)\right). \end{aligned}$$

Proof : The assertion (4) follows from Corollary 3.4. It's enough to prove the assertion (3). If $m=4$, the assertion (4) shows that

$$\pi_0 \;=\; -\log 2\cdot\left(\zeta_{(\Delta_Y+c_0)}(0)+l\right)+a_{1,1}\left(\log 2-\frac{1}{2}\right).$$

By (2.15) and Lemma 2.3, $a_{1,1}$ can be computed as follows. We note that

$$\begin{aligned} r_{-2-1}(y,\xi,\lambda,\mu) &= \delta_{-2-1}(y,\xi,\mu)\\ r_{-2-2}(y,\xi,\lambda,\mu) &= \delta_{-2-2}(y,\xi,\mu)-(\mu-|\xi|^2)^{-2}\cdot q_0(y,\xi,\lambda)\\ &= \delta_{-2-2}(y,\xi,\mu)-(\mu-|\xi|^2)^{-2}\left(-\frac{1}{4}\frac{\left(h''(0)-h'(0)^2\right)|\xi|^2}{|\xi|^2+\lambda}-\frac{5}{16}\frac{h'(0)^2|\xi|^4}{\left(|\xi|^2+\lambda\right)^2}\right).\end{aligned}$$

Then by (2.15) we have

$$\begin{aligned} a_1(\lambda) &= a_1-\frac{1}{2}\frac{vol(Y)\cdot vol(S^3)}{16\pi^4}\left(\frac{1}{4}\frac{h''(0)-h'(0)^2}{(\lambda+1)}+\frac{5}{16}\frac{h'(0)^2}{(\lambda+1)^2}\right)\\ &\sim a_1-\frac{vol(Y)}{16\pi^2}\left(\frac{1}{4}\left(h''(0)-h'(0)^2\right)\frac{1}{\lambda}+\frac{1}{16}\left(-4h''(0)+9h'(0)^2\right)\frac{1}{\lambda^2}+\cdots\right),\end{aligned}$$

which shows that $a_{1,1}=-\frac{vol(Y)}{64\pi^2}\left(h''(0)-h'(0)^2\right)$. This completes the proof of Theorem 3.6. $\square$

Remark : Replacing Δ_M and $\Delta_{M_{cut},\gamma_0}$ by $\Delta_M+\nu$ and $\Delta_{M_{cut},\gamma_0}+\nu$ for $\nu\in\mathbb{R}^+$, we can rewrite (1.6) as follows.

$$\log Det\left(\Delta_M+\nu+\lambda,\ \Delta_{M_{cut},\gamma_0}+\nu+\lambda\right) \;=\; \sum_{j=0}^{[\frac{m}{2}]}\pi_j(\nu)\lambda^j \;+\; \log DetR(\nu+\lambda). \tag{3.6}$$

Since $\Delta_M+\nu$, $\Delta_{M_{cut},\gamma_0}+\nu$ and $R(\nu)$ are invertible operators, we have

$$\log Det\left(\Delta_M+\nu,\ \Delta_{M_{cut},\gamma_0}+\nu\right) \;=\; \pi_0(\nu) \;+\; \log DetR(\nu), \tag{3.7}$$

where $\pi_0(\nu)$ is the polynomial part in (1.6). Similarly, we can express $\pi_0(\nu)$ in terms of coefficients in Lemma 2.3. For instance, $\pi_0(\nu)=0$ when $dimM$ is even.

We next discuss what data determines π_0. We consider the collar neighborhood $N_{-1,1}$ of Y and denote by $\Delta_{N_{-1,1}}$, $\Delta_{N_{-1,0}}$, $\Delta_{N_{0,1}}$ the restrictions of Δ_M to $N_{-1,1}$, $N_{-1,0}$, $N_{0,1}$. We impose the Dirichlet boundary condition on each boundary and denote the realizations by $\Delta_{N_{-1,1},\gamma_0}$, $\Delta_{N_{-1,0},\gamma_0}$, $\Delta_{N_{0,1},\gamma_0}$. In this case (1.6) can be written by

$$\begin{aligned}\log Det\left(\Delta_{N_{-1,1},\gamma_0}+\lambda\right)-\log Det\left(\Delta_{N_{-1,0},\gamma_0}+\lambda\right)-\log Det\left(\Delta_{N_{0,1},\gamma_0}+\lambda\right)&\\ = \sum_{j=0}^{[\frac{m}{2}]}\widetilde{\pi}_j\lambda^j+\log DetR_{N_{-1,1}}(\lambda),&\end{aligned} \tag{3.8}$$

where $R_{N_{-1,1}}(\lambda)=Q_{N_{-1,0}}(\lambda)+Q_{N_{0,1}}(\lambda)$ is defined as follows. For $f\in C^\infty(Y)$, choose $\phi\in C^\infty(N_{-1,0})$, $\psi\in C^\infty(N_{0,1})$ such that

$$\left(\Delta_{N_{-1,0}}+\lambda\right)\phi=0,\quad \left(\Delta_{N_{0,1}}+\lambda\right)\psi=0,\quad \phi|_{Y_{-1}}=\psi|_{Y_1}=0,\quad \phi|_{Y_0}=\psi|_{Y_0}=f.$$

We define

$$Q_{N_{-1,0}}(\lambda)f=\left(\partial_u\phi\right)|_{Y_0},\quad Q_{N_{0,1}}(\lambda)f=-\left(\partial_u\psi\right)|_{Y_0}. \tag{3.9}$$

Then $Q_{N_{-1,0}}(\lambda)$ and $Q_{N_{0,1}}(\lambda)$ satisfy the equations (2.3) and (2.4), respectively, which shows that $Q_{N_{-1,0}}(\lambda)$ ($Q_{N_{0,1}}(\lambda)$) has the same asymptotic symbol as $Q_1(\lambda)$ ($Q_2(\lambda)$) and hence $R(\lambda)-R_{N_{-1,1}}(\lambda)$ is a smoothing operator. This fact shows that $\log DetR(\lambda)$ and $\log DetR_{N_{-1,1}}(\lambda)$ have the same asymptotic expansions as $\lambda\to\infty$. In particular, they have the same zero- and $\log\lambda$- coefficients, which leads to the following result.

Corollary 3.7. *The coefficients* $\mathfrak{p}_m$ *and* $\mathfrak{q}_m$ *in Corollary 3.4 are determined by the data on a collar neighborhood of* Y.

Finally, we discuss the behaviors of $\log Det(\Delta_M + \lambda,\ \Delta_{M_{cut},\gamma_0} + \lambda)$ and $\log DetR(\lambda)$ as $\lambda \to 0$ when $\sigma_{ess}(\Delta_M)$ has a positive lower bound. Let $\{\phi_1, \cdots, \phi_l\}$ be an orthonormal basis for $ker\Delta_M \cap L^2(M)$ with $l = dim\ ker\Delta_M \cap L^2(M)$. We put $a_{ij} = \langle \phi_i|_Y, \phi_j|_Y \rangle_Y$ and $A_0 = (a_{ij})$. Then, Lemma 2.2 in [15] shows that

$$\log Det(\Delta_M + \lambda,\ \Delta_{M_{cut},\gamma_0} + \lambda) \ = \ l \cdot \log \lambda + \log Det(\Delta_M,\ \Delta_{M_{cut},\gamma_0}) + o(\lambda), \tag{3.10}$$

as $\lambda \to 0$. We assume that $dim\ kerR = l$, equivalently $kerR = \{\phi|_Y \mid \phi \in ker\Delta_M \cap L^2(M)\}$. This is the case when M is compact. Then, the following equality can be shown as the same way as Theorem 2.4 in [11].

$$\log DetR(\lambda) \ = \ l \cdot \log \lambda - \log detA_0 + \log DetR(0) + o(\lambda). \tag{3.11}$$

Combining these two equalities we have the following result.

Theorem 3.8. *We assume that* $\sigma_{ess}(\Delta_M)$ *has a positive lower bound and* $dim\ ker\Delta_M \cap L^2(M) = dim\ kerR$. *Then,*

$$\log Det(\Delta_M,\ \Delta_{M_{cut},\gamma_0}) \ = \ \pi_0 - \log detA_0 + \log DetR(0).$$

§4. **The value of relative zeta functions at** $s = 0$

Theorem 3.6 gives some informations about the value of relative zeta functions at $s = 0$ and in this section we discuss this fact. Corollary 3.5, Theorem 3.6 and Corollary 3.7 lead to the following result.

Theorem 4.1. *Let* (M, g), M_{cut} *be as above and* g, Δ_M *be given by (1.1), (1.2) on* N. *Suppose that* $dimM = m + 1$, $c_0 = \frac{m^2}{16} h'(0)^2 + \frac{m}{4} h''(0)$ *and* $l = dimker(\Delta_Y + c_0)$. *We assume that either* M *is compact or complete with* $\sigma_{ess}(\Delta_M)$ *having a positive lower bound. Then :*
(1) If $dimM$ *is even,* $\zeta\,(0; \Delta_M,\ \Delta_{M_{cut},\gamma_0}) + dimker\Delta_M \ = \ 0.$
(2) If $dimM$ *is odd,* $\zeta\,(0; \Delta_M,\ \Delta_{M_{cut},\gamma_0}) + dimker\Delta_M \ = \ \frac{1}{2} \sum_{j=1}^{\frac{m}{2}} \frac{(-1)^{\frac{m}{2}-j}}{(\frac{m}{2}-j)!} a_{j,\frac{m}{2}-j}$, *which is determined by the data on a collar neighborhood of* Y.
(3) If $dimM = 3$, $\zeta\,(0; \Delta_M,\ \Delta_{M_{cut},\gamma_0}) + dimker\Delta_M \ = \ \frac{1}{2}\,(\zeta_{\Delta_Y + c_0}(0) + l).$
(4) If $dimM = 5$, $\zeta\,(0; \Delta_M,\ \Delta_{M_{cut},\gamma_0}) + dimker\Delta_M \ = \ \frac{1}{2}\left(\zeta_{\Delta_Y + c_0}(0) + l + \frac{vol(Y)}{64\pi^2}\left(h''(0) - h'(0)^2\right)\right).$

Remark : Since the coefficient θ_0 in (3.2) vanishes when $dimY$ is odd, the assertion (1) holds generally.

As an application of Theorem 4.1 we consider the case of $h(u) \equiv 0$, *i.e.* the product metric on N and $\Delta_M|_N = -\partial_u^2 + \Delta_Y$. Then it is known that $\pi_0 = -\log 2 \cdot (\zeta_{\Delta_Y}(0) + dimker\Delta_Y)$ and it is not difficult to check that $a_{j,\frac{m}{2}-j} = 0$ for $1 \le j \le \frac{m}{2} - 1$, which gives the following result.

Corollary 4.2. *Suppose that M is a compact closed manifold and the metric g is the product one on N so that $\Delta_M = -\partial_u^2 + \Delta_Y$. Then :*
(1) $\zeta\left(0; \Delta_M,\ \Delta_{M_{cut},\gamma_0}\right) + dimker\Delta_M \ = \ \frac{1}{2}\left(\zeta_{\Delta_Y}(0) + dimker\Delta_Y\right)$.
(2) If $dimM$ is odd, $\zeta_{\Delta_{M_1,\gamma_0}}(0) = -\frac{1}{4}\left(\zeta_{\Delta_Y}(0) + dimker\Delta_Y\right)$.
(3) If $dimM$ is even, $\zeta_{\Delta_{M_1,\gamma_0}}(0) = \frac{1}{2}\left(\zeta_{\Delta_{\widetilde{M_1}}}(0) + dimker\Delta_{\widetilde{M_1}}\right)$*, where $\widetilde{M_1}$ is the double of M_1 and $\Delta_{\widetilde{M_1}}$ is the natural extension of Δ_{M_1} to $\widetilde{M_1}$.*

Let M_1 be the compact manifold with boundary Y and $M_{1,\infty} = M_1 \cup_Y [0,\infty) \times Y$. We give the product metric on the cylinder part of $M_{1,\infty}$ so that on the cylinder part $\Delta_{M_{1,\infty}} = -\partial_u^2 + \Delta_Y$. Suppose that $\mu_1 > 0$ is the smallest positive eigenvalue of Δ_Y and we denote the scattering matrix by

$$S(s) : ker\Delta_Y \to ker\Delta_Y, \qquad |s| < \sqrt{\mu_1}.$$

Then it was shown in [16] that

$$Tr\left(e^{-t\Delta_{M_{1,\infty}}} - e^{-t\left(-\partial_u^2+\Delta_Y\right)_{[0,\infty)\times Y,\gamma_0}}\right) \ = \ b_0 + O(t^{-\rho}), \tag{4.1}$$

where $b_0 = dimker\Delta_{M_{1,\infty}} + \frac{1}{4}\left(TrS(0) + dimker\Delta_Y\right)$ and $\rho > 0$. Then we have the following result.

Theorem 4.3.

$$\begin{aligned} & \zeta(0; \Delta_{M_{1,\infty}}, \left(-\partial_u^2 + \Delta_Y\right)_{[0,\infty)\times Y,\gamma_0}) \\ = \ & \zeta_{\Delta_{M_1,\gamma_0}}(0) - dimker\Delta_{M_{1,\infty}} + \frac{1}{2}\left(\zeta_{\Delta_Y}(0) + dimker\Delta_Y\right) - b_0 \\ = \ & \zeta_{\Delta_{M_1,\gamma_0}}(0) - 2dimker\Delta_{M_{1,\infty}} + \frac{1}{2}\zeta_{\Delta_Y}(0) + \frac{1}{2}dimker\left(Id + S(0)\right). \end{aligned}$$

We next discuss the case of the Neumann boundary condition. We still assume the product metric and product structure near the boundary. Let $\mathfrak{N}$ be the Neumann boundary condition imposed on the boundary of M_1 and $\Delta_{M_1,\mathfrak{N}}$ be the realization. Then it can be shown by the method presented in [12] (*cf.* [18]) that

$$\log Det\left(\Delta_{M_1,\mathfrak{N}} + \lambda\right) - \log Det\left(\Delta_{M_1,\gamma_0} + \lambda\right) \ = \ \sum_{k=0}^{[\frac{m}{2}]} a_k \lambda^k + \log DetQ_1(\lambda), \tag{4.2}$$

where $Q_1(\lambda)$ differs from $\sqrt{\Delta_Y + \lambda}$ by a smoothing operator and hence $\log Det$ of these two operators have the same asymptotic expansions. Since the coefficient of $\log\lambda$ in the asymptotic expansion of $\log Det\left(\sqrt{\Delta_Y + \lambda}\right)$ is $\frac{1}{2}\left(\zeta_{\Delta_Y}(0) + dimker\Delta_Y\right)$, this fact together with Corollary 4.2 leads to the following result.

Theorem 4.4. *Let (M_1, g) be a compact manifold with boundary and the metric g is the product one near the boundary so that $\Delta_M = -\partial_u^2 + \Delta_Y$. Then :*
(1) $\zeta_{\Delta_{M_1,\mathfrak{N}}}(0) + dimker\Delta_{M_1,\mathfrak{N}} - \zeta_{\Delta_{M_1,\gamma_0}}(0) \ = \ \frac{1}{2}\left(\zeta_{\Delta_Y}(0) + dimker\Delta_Y\right)$.
(2) If $dimM_1$ is odd, $\zeta_{\Delta_{M_1,\mathfrak{N}}}(0) + dimker\Delta_{M_1,\mathfrak{N}} = \frac{1}{4}\left(\zeta_{\Delta_Y}(0) + dimker\Delta_Y\right)$.
(3) If $dimM_1$ is even, $\zeta_{\Delta_{M_1,\mathfrak{N}}}(0) + dimker\Delta_{M_1,\mathfrak{N}} = \zeta_{\Delta_{M_1,\gamma_0}}(0)$.

Finally, we discuss the value of the zeta function associated with a compatible Dirac Laplacian with the Atiyah-Patodi-Singer (APS) boundary condition on a compact manifold with boundary.

Let (M, g) be a compact oriented $(m+1)$-dimensional Riemannian manifold and $E \to M$ be a Clifford module bundle. Suppose that Y is a hypersurface of M such that $M - Y$ has two components whose closures are denoted by M_1, M_2. We denote by M_{cut} the compact manifold with boundary obtained by cutting M along Y as before, *i.e.* $M_{cut} = M_1 \cup_Y M_2$. We choose a collar neighborhood $N_{-1,1}$ of Y which is diffeomorphic to $[-1,1] \times Y$ and assume that the metric g is the product one on N and the bundle E has the product structure on N. Suppose that D_M is a compatible Dirac operator acting on smooth sections of E, having the following form on $N_{-1,1}$

$$D_M = G(\partial_u + B),$$

where $G : E|_Y \to E|_Y$ is a bundle automorphism, ∂_u is the outward normal derivative to M_1 on $N_{-1,1}$ and B is a Dirac operator on Y. We further assume that G and B are independent of the normal coordinate u and satisfy

$$\begin{aligned} &G^* = -G, \qquad G^2 = -I, \qquad B^* = B, \qquad GB = -BG, \\ &dim\ \left(ker(G-i) \cap kerB\right) = dim\ \left(ker(G+i) \cap kerB\right). \end{aligned} \tag{4.3}$$

Then we have, on N, the Dirac Laplacian

$$D_M^2 = -\partial_u^2 + B^2.$$

We also denote by $D_{M_{cut}}$, D_{M_i} $(i = 1, 2)$ the extension and restriction of D_M to M_{cut} and M_i. We denote by $\Pi_<$ $(\Pi_>)$ the orthogonal projection onto the space spanned by negative (positive) eigensections of B and by $\sigma : kerB \to kerB$ a unitary operator satisfying

$$\sigma G = -G\sigma, \quad \sigma^2 = Id_{kerB}.$$

We define the generalized APS boundary condition $\Pi_{<,\sigma^-}$, $\Pi_{>,\sigma^+}$ by

$$\Pi_{<,\sigma^-} = \Pi_< + \frac{1}{2}(I - \sigma)|_{kerB}, \quad \Pi_{>,\sigma^+} = \Pi_> + \frac{1}{2}(I + \sigma)|_{kerB}$$

and denote by $D_{M_1,\Pi_{<,\sigma^-}}$, $D^2_{M_1,\Pi_{<,\sigma^-}}$ $(D_{M_2,\Pi_{>,\sigma^+}}, D^2_{M_2,\Pi_{>,\sigma^+}})$ the realizations of D_{M_i}, $D^2_{M_i}$ with respect to the boundary condition $\Pi_{<,\sigma^-}$ $(\Pi_{>,\sigma^+})$. Then it was shown in [12] that

$$\begin{aligned} \log Det \left(D^2_{M_1,\Pi_{<,\sigma^-}} + \lambda\right) - \log Det \left(D^2_{M_1,\gamma_0} + \lambda\right) \ &= \sum_{j=1}^{[\frac{m}{2}]} a_{1,j}\lambda^j \\ &\quad + \ \log Det \left(\Pi_{>,\sigma^+} \left(Q_1(\lambda) + |B|\right) \Pi_{>,\sigma^+}\right), \\ \log Det \left(D^2_{M_2,\Pi_{>,\sigma^+}} + \lambda\right) - \log Det \left(D^2_{M_1,\gamma_0} + \lambda\right) \ &= \sum_{j=1}^{[\frac{m}{2}]} a_{2,j}\lambda^j \\ &\quad + \ \log Det \left(\Pi_{<,\sigma^-} \left(Q_2(\lambda) + |B|\right) \Pi_{<,\sigma^-}\right), \end{aligned} \tag{4.4}$$

where $Q_1(\lambda)$ and $Q_2(\lambda)$ are defined by the same way as in (1.4). We now consider the first equation of (4.4) and we can treat the second equation in the same way. As before, $\log Det \left(D^2_{M_1,\Pi_{<,\sigma^-}} + \lambda\right)$, $\log Det \left(D^2_{M_1,\gamma_0} + \lambda\right)$ and $\log Det \left(\Pi_{>,\sigma^+} \left(Q_1(\lambda) + |B|\right) \Pi_{>,\sigma^+}\right)$ have asymptotic expansions for $\lambda \to \infty$. Moreover, the coefficients of $\log \lambda$ in the asymptotic expansions of $\log Det \left(D^2_{M_1,\Pi_{<,\sigma^-}} + \lambda\right)$

and $\log Det\left(D^2_{M_1,\gamma_0}+\lambda\right)$ are $\left(\zeta_{D^2_{M_1,\Pi_{<,\sigma^-}}}(0)+dimker D^2_{M_1,\Pi_{<,\sigma^-}}\right)$ and $\zeta_{D^2_{M_1,\gamma_0}}(0)$. Now let us consider the asymptotic expansion of $\log Det\left(\Pi_{>,\sigma^+}\left(Q_1(\lambda)+|B|\right)\Pi_{>,\sigma^+}\right)$. It was shown in [9] that

$$Q_i(\lambda) \;=\; \sqrt{B^2+\lambda} \;+\; \text{a smoothing operator}\;, \quad i=1,2,$$

which shows that $\log Det\left(\Pi_{>,\sigma^+}\left(Q_1(\lambda)+|B|\right)\Pi_{>,\sigma^+}\right)$ and $\frac{1}{2}\log Det\left(\sqrt{B^2+\lambda}+|B|\right)$ have the same asymptotic expansions. We compute the $\log\lambda$-coefficient in the asymptotic expansion of $\log Det\left(\sqrt{B^2+\lambda}+|B|\right)$ as follows.

Lemma 4.5. *For $1\leq k\in\mathbb{Z}$ let us denote $f_k(s,\lambda)$ by*

$$f_k(s,\lambda)=\frac{1}{\Gamma(s)}\int_0^\infty t^{\frac{s+k}{2}-1}\,Tr\left(|B|^k e^{-t(B^2+\lambda)}\right)dt.$$

Then the coefficients of $\log\lambda$ in the asymptotic expansions of $f_k(0,\lambda)$ and $-f_k'(0,\lambda)$, as $\lambda\to\infty$, are zero.

Proof : We first note that $Tr\left(|B|^k e^{-tB^2}\right)$ has the following asymptotic expansion for $t\to 0^+$ (Theorem 2.7 in [4]).

$$Tr\left(|B|^k e^{-tB^2}\right)\;\sim\;\sum_{j=0}^\infty b_j^{(k)}t^{\frac{j-m-k}{2}}\;+\;\sum_{j=0}^\infty\left(c_j^{(k)}\log t+d_j^{(k)}\right)t^j. \tag{4.5}$$

Then direct computation shows that the coefficients of $\log\lambda$ in the asymptotic expansions of $f_k(0,\lambda)$ and $-\zeta_k'(0,\lambda)$ for $\lambda\to\infty$ are zero and $b_m^{(k)}$. On the other hand, we note that

$$\zeta_{|B|}(s)\;=\;\frac{1}{\Gamma(\frac{s+k}{2})}\int_0^\infty t^{\frac{s+k}{2}-1}\,Tr\left(|B|^k e^{-tB^2}\right)dt. \tag{4.6}$$

The equation (4.5) shows that the RHS of (4.6) has a pole at $s=0$ with residue $\frac{2b_m^{(k)}}{\Gamma(\frac{k}{2})}$. Since $\zeta_{|B|}(s)$ has a regular value at $s=0$, this fact implies that each $b_m^{(k)}=0$ for $k\geq 1$, which completes the proof of the lemma. $\square$

Lemma 4.6. *The coefficient of $\log\lambda$ in the asymptotic expansion of $\log Det\left(\sqrt{B^2+\lambda}+|B|\right)$ for $\lambda\to\infty$ is $\frac{1}{2}\left(\zeta_{B^2}(0)+dimker B\right)$.*

Proof : We first note that

$$\begin{aligned}
\zeta_{(\sqrt{B^2+\lambda}+|B|)}(s) \;&=\; \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\,Tr e^{-t(\sqrt{B^2+\lambda}+|B|)}dt\\
&=\; \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\sum_{q=0}^\infty\frac{t^q}{q!}Tr\left(\left(\sqrt{B^2+\lambda}-|B|\right)^q e^{-2t\sqrt{B^2+\lambda}}\right)dt\\
&=\; \sum_{q=0}^\infty\frac{1}{q!}\frac{1}{\Gamma(s)}\int_0^\infty t^{s+q-1}\,Tr\left(\left(\sqrt{B^2+\lambda}-|B|\right)^q e^{-2t\sqrt{B^2+\lambda}}\right)dt.
\end{aligned}$$

We now set

$$\zeta_q(s,\lambda)\;=\;\frac{1}{\Gamma(s)}\int_0^\infty t^{s+q-1}\,Tr\left(\left(\sqrt{B^2+\lambda}-|B|\right)^q e^{-2t\sqrt{B^2+\lambda}}\right)dt.$$

In case of $q = 0$, the coefficient of $\log \lambda$ in the asymptotic expansion of $-\zeta_0'(0,\lambda)$ for $\lambda \to \infty$ is $\frac{1}{2}\left(\zeta_{B^2}(0) + dimkerB\right)$ (*cf.* Lemma 3.1). For $q \geq 1$,

$$\begin{aligned}
\zeta_q(s,\lambda) &= \sum_{k=0}^{q} (-1)^k \begin{pmatrix} q \\ k \end{pmatrix} \frac{1}{\Gamma(s)} \int_0^\infty t^{s+q-1}\, Tr\left(\left(\sqrt{B^2+\lambda} \right)^{q-k} |B|^k e^{-2t\sqrt{B^2+\lambda}} \right) dt \\
&= 2^{-s-q} \sum_{k=0}^{q} (-1)^k \begin{pmatrix} q \\ k \end{pmatrix} \frac{\Gamma(s+q)}{\Gamma(\frac{s+k}{2})} \frac{1}{\Gamma(s)} \int_0^\infty t^{\frac{s+k}{2}-1}\, Tr\left(|B|^k e^{-t(B^2+\lambda)} \right) dt.
\end{aligned}$$

Then Lemma 4.5 shows that each $\log \lambda$-coefficient of $-\zeta_q'(0,\lambda)$ is zero, which completes the proof of the lemma. $\square$

Remark : In the asymptotic expansion of $\log Det\left(\sqrt{B^2+\lambda} + |B|\right)$, the computation of the constant term given in Section 3 in [10] has some error and we can correct it as follows. In Lemma 4.5, the constant terms in the asymptotic expansions of $f_k(0,\lambda)$ and $-f_k'(0,\lambda)$, as $\lambda \to \infty$, are zero. Using this fact, we can show that the constant terms in the asymptotic expansions of $-\zeta_0'(0,\lambda)$ and $-\zeta_q'(0,\lambda)$ are $\log 2 \cdot (\zeta_{B^2}(0) + dimkerB)$ and $-\frac{1}{q\cdot 2^q} \cdot (\zeta_{B^2}(0) + dimkerB)$, respectively, which shows that the constant term in the asymptotic expansion of $\log Det\left(\sqrt{B^2+\lambda} + |B|\right)$ is zero.

We can say the similar assertions for $D^2_{M_2,\Pi_{>,\sigma^+}}$. The comparison of $\log \lambda$-coefficients in the asymptotic expansions of (4.4) and Corollary 4.2 leads to the following result.

Theorem 4.7. *Let M, M_1 and M_2 be as above. We denote $dimkerD^2_M$, $dimkerD^2_{M_1,\Pi_{<,\sigma^-}}$, $dimkerD^2_{M_2,\Pi_{>,\sigma^+}}$ by l_M, $l_{M_1,\Pi_{<,\sigma^-}}$, $l_{M_2,\Pi_{>,\sigma^+}}$, respectively. Then :*
(1) $\zeta_{D^2_{M_1,\Pi_{<,\sigma^-}}}(0) + l_{M_1,\Pi_{<,\sigma^-}} - \zeta_{D^2_{M_1,\gamma_0}}(0) \;=\; \frac{1}{4}\left(\zeta_{B^2}(0) + dimkerB\right)$.
(2) If $dimM_1$ is odd, $\zeta_{D^2_{M_1,\Pi_{<,\sigma^-}}}(0) + l_{M_1,\Pi_{<,\sigma^-}} \;=\; 0$.
(3) If $dimM_1$ is even, $\zeta_{D^2_{M_1,\Pi_{<,\sigma^-}}}(0) + l_{M_1,\Pi_{<,\sigma^-}} \;=\; \zeta_{D^2_{M_1,\gamma_0}}(0)$.
(4) $\zeta_{D^2_M}(0) - \zeta_{D^2_{M_1,\Pi_{<,\sigma^-}}}(0) - \zeta_{D^2_{M_2,\Pi_{>,\sigma^+}}}(0) \;=\; -l_M + l_{M_1,\Pi_{<,\sigma^-}} + l_{M_2,\Pi_{>,\sigma^+}}$.

Remark : (1) The second assertion was proved earlier in the appendix of [17] by the author.
(2) The zeta-determinant of a compatible Dirac operator is defined by

$$DetD_M \;=\; e^{-\frac{1}{2}\zeta'_{D^2_M}(0)} e^{\frac{\pi i}{2}\left(\zeta_{D^2_M}(0) - \eta_{D_M}(0)\right)} \;=\; \sqrt{DetD^2_M}\; e^{\frac{\pi i}{2}\left(\zeta_{D^2_M}(0) - \eta_{D_M}(0)\right)}, \tag{4.7}$$

where $\eta_{D_M}(0)$ is the eta-invariant for D_M (*cf.* [19]). The gluing formula for the eta-invariant of a Dirac operator with respect to the APS boundary condition is given in [6] (or [13]) and the gluing formula for the zeta-determinant of a Dirac Laplacian with respect to the APS boundary condition is given in [12] (or [13]). Hence, the assertion (4) together with the results in [6] and [12] completes the gluing formula for the zeta-determinant of a compatible Dirac operator.

In view of Theorem 4.3 we conclude this section with the computation of the value of relative zeta function for Dirac Laplacian with the APS boundary condition on a manifold with cylindrical end. As before, we denote by $M_{1,\infty} := M_1 \cup_Y [0,\infty) \times Y$ and by $D_{M_{1,\infty}}$ the natural extension of D_{M_1} to $M_{1,\infty}$. Then,

$$\begin{aligned}
&\zeta(s, D^2_{M_{1,\infty}}, (-\partial_u^2 + B^2)_{[0,\infty)\times Y, \Pi_{>,\sigma^+}}) \\
&= \zeta(s, D^2_{M_{1,\infty}}, (-\partial_u^2 + B^2)_{[0,\infty)\times Y, \gamma_0}) + \zeta(s, (-\partial_u^2 + B^2)_{[0,\infty)\times Y, \gamma_0}, (-\partial_u^2 + B^2)_{[0,\infty)\times Y, \Pi_{>,\sigma^+}}) \\
&= \zeta(s, D^2_{M_{1,\infty}}, (-\partial_u^2 + B^2)_{[0,\infty)\times Y, \gamma_0}) - \frac{1}{4\sqrt{\pi}} \frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+1)} \zeta_{B^2}(s).
\end{aligned} \tag{4.8}$$

Here we used the fact (*cf.* [12]) that

$$\zeta(s, (-\partial_u^2 + B^2)_{[0,\infty)\times Y,\gamma_0}, (-\partial_u^2 + B^2)_{[0,\infty)\times Y,\Pi_{>,\sigma^+}}) \;=\; -\frac{1}{4\sqrt{\pi}}\frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+1)}\zeta_{B^2}(s). \tag{4.9}$$

Since $dimker\,(Id + S(0)) = \frac{1}{2}dimkerB$ on a manifold with cylindrical end ([14]), Theorem 4.3 together with (4.8) yields the following result.

Corollary 4.8.

$$\begin{aligned} &\zeta(0, D^2_{M_{1,\infty}}, (-\partial_u^2 + B^2)_{[0,\infty)\times Y,\Pi_{>,\sigma^+}}) \\ =\;& \zeta_{D^2_{M_1,\gamma_0}}(0) - 2\; dimkerD^2_{M_{1,\infty}} + \frac{1}{4}\;\zeta_{B^2}(0) + \frac{1}{2}\; dimker\,(Id + S(0)) \\ =\;& \zeta_{D^2_{M_1,\gamma_0}}(0) - 2\; dimkerD^2_{M_{1,\infty}} + \frac{1}{4}\;(\zeta_{B^2}(0) + dimkerB)\,. \end{aligned}$$

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