A PROBABILISTIC WEYL LAW FOR TWO-DIMENSIONAL FLAT TORI

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Abstract. We prove that the error term $R(\lambda)$ in Weyl's law for two-dimensional flat tori satisfies Hardy's conjecture ($= \mathcal{O}(\lambda^{1/4+\epsilon})$) for all tori with the exception of a set of arbitrarily small measure. The proof is spectral in nature.

1. Introduction

Let (M^n, g) be a compact Riemannian manifold of dimension n with Laplace-Beltrami operator Δ and spectral counting function

$$N(\lambda) := \# \{\lambda_j \in \mathrm{Spec}(\Delta);\, \lambda_j \leq \lambda\}.$$

Then, a celebrated theorem of Hörmander [HO] asserts that

$$N(\lambda) = \mathrm{vol}(M)\, \lambda^{n/2} + \mathcal{O}(\lambda^{(n-1)/2}). \tag{1.1}$$

Moreover, the estimate in (1.1) is sharp as can be seen by considering the round sphere, S^n. The question of determining the optimal bound for the error term

$$R(\lambda) := N(\lambda) - \mathrm{vol}(M)\lambda^{n/2}$$

in any given example is a difficult problem which depends on the properties of the associated geodesic flow, and is far from being understood in detail. Nevertheless, there are several important results along these lines: One of the first is a result of Duistermaat-Guillemin [DG] which asserts that in the case where the geodesic flow is clean and the set of unit-speed geodesics in S^*M has null Liouville measure, then one can improve the Hörmander bound in (1.1) to

$$R(\lambda) = o(\lambda^{(n-1)/2}).$$

Subsequently, Ivrii [I1], [I2] gave a different proof of this result and extended it to manifolds with boundary. There are some additional improvements in $R(\lambda)$ that are known for some specific examples. For instance, in the case of hyperbolic manifolds, a result of Bérard [B] gives

$$R(\lambda) = \mathcal{O}\left(\lambda^{(n-1)/2}/\log\lambda\right).$$

This is in all likelihood far from optimal. Indeed, it is conjectured that, at least in the non-arithmetic case $R(\lambda) = \mathcal{O}(\lambda^{\epsilon})$ for all $\epsilon > 0$. In fact, even for noncompact arithmetic surfaces with cusps, such as $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathbb{H}$, one has

$$R(\lambda) = c \cdot \lambda^{1/2}\log\lambda + \mathcal{O}(\lambda^{1/2}),$$

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see [He, 2.21, p. 511].

In the opposite case of completely integrable geodesic flow there are several cases where improved error terms are known: For (generic) convex surfaces of revolution, Colin de Verdière [CdV] has shown that

$$R(\lambda) = \mathcal{O}(\lambda^{1/3}) \tag{1.2}$$

which agrees with a result of Van der Corput and Sierpinski [SI] for the classical circle problem in 2 dimensions, i.e., for the torus $\mathbb{R}^2/\mathbb{Z}^2$.

There are also additional more general results of Volovoy [V] under long-time recurrence estimates for the geodesic flow, but they are difficult to quantify. The geometrically simplest example of an integrable geodesic flow on a surface is the 2-dimensional flat torus. In this case, Hardy's Careful Conjecture [HA] states that 'it is not unlikely that

$$R(\lambda) = \mathcal{O}_\epsilon(\lambda^{1/4+\epsilon}) \tag{1.3}$$

for all positive ϵ'. There is much evidence, both numerical and otherwise to suggest that the bound above is indeed optimal. For instance, a classical result of Cramér [CR] says that

$$\lim_{T\to\infty} \frac{1}{T^{3/2}} \int_0^T |R(\lambda)|^2 \, d\lambda = c > 0 \tag{1.4}$$

for a constant c.

Despite the fact that there has been much work devoted to improving the estimate for $R(\lambda)$, deterministic results are still far away from (1.3). The best result that we are aware of is $\mathcal{O}(\lambda^{23/73} \log^{315/146} \lambda)$ and is due to Huxley [HU]. The problem for general flat tori in dimensions $n \geq 4$ is solved, in fact $R(\lambda) = \mathcal{O}(\lambda^{n/2-1})$ for $n \geq 5$, while $R(\lambda) = \mathcal{O}(\lambda \log \lambda)$ for $n = 4$. These estimates include all tori, rational and irrational. For $n = 3$ estimating the sharp error term is not known and the corresponding bound to (1.2) is $R(\lambda) = \mathcal{O}(\lambda^{5/6})$.

The purpose of this note is to establish the Hardy bound (1.3) in a probabilistic sense, where we average over metric deformations of 2-dimensional tori. More precisely, we prove

Theorem 1.1. *(i) Fix ϵ with $0 < \epsilon < 1$ and let $I := [1-\epsilon, 1+\epsilon]$ and $N(\lambda; \vec{u})$ denote the spectral counting function for the corresponding Laplace-Beltrami operator on the flat two-torus $M(\vec{u})$, $\vec{u} = (u_1, u_2)$ with metric form*

$$H(p, \sigma; u_1, u_2) = u_1 \sigma^2 + u_2 p^2.$$

Then, for any $\delta > 0$,

$$\int_{I^2} \left| N(\lambda; \vec{u}) \; - \; \mathrm{vol}\,(M(\vec{u}))\,\lambda \right| \, d\vec{u} \; = \; \mathcal{O}_\delta(\lambda^{1/4+\delta}). \tag{1.5}$$

(ii) Consider the general metric Hamiltonian for a flat two-dimensional torus $M(\vec{u})$, $\vec{u} = (u_1, u_2, u_3)$ given by:

$$H(p, \sigma; u_1, u_2, u_3) := u_1\sigma^2 + u_3 p\sigma + u_2 p^2. \tag{1.6}$$

Then the following holds:

$$\int_{I^3} \Big| N(\lambda;\vec{u}) \; - \; \mathrm{vol}\,(M(\vec{u}))\,\lambda \Big| \; d\vec{u} \; = \; \mathcal{O}_\delta(\lambda^{1/4+\delta}). \tag{1.7}$$

An immediate application of Chebyshev's inequality gives the following corollary.

Corollary 1. For all $\delta, \epsilon_1 > 0$ small the estimate

$$N(\lambda;\vec{u}) - \mathrm{vol}(M(\vec{u}))\lambda = \mathcal{O}_{\delta,\epsilon_1}(\lambda^{1/4+\delta})$$

holds for all tori $M(\vec{u})$ in the complement of a set A_{δ,ϵ_1} of measure less than ϵ_1 in the moduli space of all tori.

Remark 1.2. (1) We should emphasize that our result in Theorem 1.1 is local. Consequently, it differs significantly from the global result of Randol [R]. Indeed, Randol shows that for $G/\Gamma := \mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$,

$$\int_{G/\Gamma} \Big| N(\lambda;g) - (\zeta(2))^{-1}\pi\lambda \Big|^2 dg = \zeta(2)^{-1}\pi\lambda + \mathcal{O}_N(\lambda|\log\lambda|^{-N}), \tag{1.8}$$

for any integer N. Thus, when estimating over *all* lattices of fixed volume, one gets an asymptotic formula for $R(\lambda)$ which is consistent with the Hörmander error in (1.1) and no better. This global result thus differs considerably from our Theorem 1.1. The point here is that, since our result is purely local, it does not take into account degenerate tori on the boundary of the moduli space $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$ which appear to make a large contribution to (1.8).

Remark 1.3. Results involving spectral averaging on shorter scales than (1.4) are known. For instance, Iosevich, Sawyer, Seeger [ISS] have recently proved

$$\int_T^{T+\log T} R(\mu^2)^2\, d\mu \ll T\log T.$$

Huxley [HU2] proved the estimate

$$\int_T^{T+1} R(\mu^2)^2\, d\mu \ll T\log T.$$

The distribution of $R(\lambda^2)/\sqrt{\lambda}$ for the standard torus $\mathbb{R}^2/\mathbb{Z}^2$ was studied by Heath-Brown [HB] and for shifted convex ovals has been studied by Bleher in a series of papers starting with [Bl].

Remark 1.4. A number of results are known for almost all irrational tori, where 'almost' is interpreted in the sense of Lebesgue measure in an appropriate parameter space. The dimension is, however, large, bigger than 4. For instance, in the case of the torus with eigenvalues given by the quadratic form

$$Q(p_1,p_2,\dots,p_n) = ap_1^2 + Q'(p_2,\dots,p_n)$$

with $n \geq 10$ and with Q' a rational quadratic form,

$$R(\lambda) = \mathcal{O}(\lambda^{n/2-6/5}\log^{1/4}\lambda)$$

for almost all a, see [WA]. In general, one gets only $o(\lambda^{n/2-1})$ for such forms with $n \geq 10$. For diagonal forms

$$Q(p_1, p_2, \dots, p_n) = a_1 p_1^2 + a_2 p_2^2 + \cdots + a_n p_n^2 \tag{1.9}$$

Jarnik [JA1] proved

$$R(\lambda) = \mathcal{O}(\lambda^{n/2-1}), \quad n \geq 5, \quad R(\lambda) = \mathcal{O}(\lambda \log^2 \lambda), \quad n = 4,$$

while for irrational diagonal with $n \geq 5$ we get $R(\lambda) = o(\lambda^{n/2-1})$. These bounds cannot be improved, unless one specialises even further. In a series of papers [JA2], [JA3], [JA4] Jarnik proved estimates of the form

$$R(\lambda) = \mathcal{O}(\lambda^{n/2-\mu+\epsilon})$$

for an explicit $\mu > 1$. For almost all forms of type

$$Q(p_1, p_2, \dots, p_n) = a p_1^2 + (p_2^2 + p_3^2 + \cdots + p_n^2)$$

we can take $\mu = 5/4$. For forms of type

$$Q(p_1, p_2, \dots, p_n) = b_1(p_1^2 + \cdots p_k^2) + b_2(p_{k+1}^2 + \cdots + p_n^2)$$

with k, $n-k \geq 4$, the μ depends on the diophantine properties of b_2/b_1. One conjectures that for almost all diagonal forms (1.9)

$$R(\lambda) = \mathcal{O}(\lambda^{(n-1)/4+\epsilon}).$$

Remark 1.5. Our method is spectral in nature and avoids estimates on exponential sums and the Hardy-Littlewood circle method. It avoids also the sophisticated techniques developed for the circle problem and (rational and irrational) tori.

Remark 1.6. It would be of interest to determine what happens to the error term $R(\lambda)$ in other completely integrable situations. For example, in the case of convex surfaces of revolution, one can perhaps improve the $\mathcal{O}(\lambda^{1/3})$ bound for $R(\lambda)$ by averaging over appropriate moduli-spaces. According to [CdV], this is related to counting shifted lattice points inside a convex curve. We hope to address this point in future work. An interesting related result due to Kendall, see [KE], is the following

$$\int_0^1 \int_0^1 R(\lambda, \alpha, \beta)^2 \, d\alpha \, d\beta = \mathcal{O}(\lambda^{1/4}),$$

where (α, β) represents a shift of the standard lattice $\mathbb{Z}^2$.

We first take up the case where $H(p, \sigma; u_1, u_2) = u_1 \sigma^2 + u_2 p^2$. For such metrics the proof of Theorem 1.1 (i) is rather computational and so we split it up into three parts: an analysis of the averaged density of states (Section 2), mean-square density of states (Section 3) and a rescaled Tauberian argument (Section 4).

2. Averaged density of states

In this section we give an asymptotic estimate for the averaged density of states of the eigenvalues, λ_j, $j = 1, 2, \dots$ of the Laplacian Δ on the torus, T^2. We put $\hbar^{-1} = \sqrt{\lambda}$. Most of our estimates will be given in terms of $\hbar$ but the reader should have no difficulty in expressing them in terms of λ. We define the average density of states as follows:

$$\begin{aligned} \mathrm{AV}(\phi) &= \sum_{j=1}^{\infty} \int_{I^2} \phi(\lambda_j(\vec{u}) - \lambda)\, d\vec{u} \\ &= \hbar \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \int_{I^2} \int_{\mathbb{R}} e^{is[H(m\hbar,n\hbar;\vec{u})-1]/\hbar}\, \check{\phi}(\hbar s)\, ds\, d\vec{u}, \end{aligned} \tag{2.1}$$

where ϕ is even, belongs to the Schwartz space $S(\mathbb{R})$ and its Fourier transform $\check{\phi} \in C_0^\infty(\mathbb{R})$. In fact, we take $\check{\phi}$ such that its support contains only 0 among the lattice points. The second equality follows from the Fourier inversion formula. The density $\mathrm{AV}(\phi)$ is a function of λ, or, equivalently, $\hbar$ and we are interested in estimating it as $\hbar \to 0$. Integration by parts in the s variable gives for any $\delta > 0$ and $\hbar \leq \hbar_0$,

$$\mathrm{AV}(\phi) = \hbar \sum_{|m|,|n|=0}^{\hbar^{-1-\delta}} \int_{I^2} \int_{\mathbb{R}} e^{is[H(m\hbar,n\hbar;\vec{u})-1]/\hbar}\, \check{\phi}(\hbar s)\, ds\, d\vec{u} \,+\, \mathcal{O}(\hbar^\infty). \tag{2.2}$$

We decompose the last expression in (2.2) into a sum of the two terms:

$$d\rho_0(\phi) := \hbar \sum_{|m|,|n|=0}^{\hbar^{-1-\delta}} \int_{I^2} \int_{\mathbb{R}} e^{is[H(m\hbar,n\hbar;\vec{u})-1]/\hbar}\, \zeta(s)\, \check{\phi}(\hbar s)\, ds\, d\vec{u}, \tag{2.3}$$

where $\zeta(s)$ is $\equiv 1$ close to 0 and is in $C_0^\infty(\mathbb{R})$, and

$$d\rho_+(\phi) := \hbar \sum_{|m|,|n|=0}^{\hbar^{-1-\delta}} \int_{I^2} \int_{\mathbb{R}} e^{is[H(m\hbar,n\hbar;\vec{u})-1]/\hbar}\, (1-\zeta(s))\, \check{\phi}(\hbar s)\, ds\, d\vec{u}. \tag{2.4}$$

Using (6.5) from the Appendix (see also [U, Section 1.3]), we get

$$d\rho_0(\phi) = c_0\, \check{\phi}(0) + \mathcal{O}(\hbar^\infty), \tag{2.5}$$

and it remains to estimate the long-time term, $d\rho_+(\phi)$. We can assume that either $|m\hbar| \geq C > 0$ or that $|n\hbar| \geq C > 0$. Otherwise $\lambda_j(u) - \lambda \gg \lambda$. Since $\phi \in S(\mathbb{R})$, we have $\phi(x) \ll |x|^{-N}$ for $|x|$ large and every $N > 0$, while the number of such summands in (2.2) is $\mathcal{O}(\lambda)$. So such terms contribute $\mathcal{O}(\hbar^\infty)$. Assume first that $|m\hbar| \geq C > 0$. Then, by integrating by parts in the u_1 variable in (2.4), we get that

$$\begin{aligned} &\hbar \sum_{|m|=C\hbar^{-1}}^{\hbar^{-1-\delta}} \sum_{|n|=0}^{\hbar^{-1-\delta}} \int_{I^2} \int_{\mathbb{R}} e^{is[H(m\hbar,n\hbar;\vec{u})-1]/\hbar}\, (1-\zeta(s))\, \check{\phi}(\hbar s)\, ds\, d\vec{u} \\ \ll\, &\hbar \sum_{|m|=C\hbar^{-1}}^{\hbar^{-1-\delta}} \sum_{|n|=0}^{\hbar^{-1-\delta}} \int_{\mathbb{R}} \left(\frac{1}{m^2\hbar}\right) \frac{1}{s} (1-\zeta(s))\, \check{\phi}(\hbar s)\, ds \,\ll\, |\log\hbar|\, \hbar^{-\delta}, \end{aligned} \tag{2.6}$$

since the summation over m contributes $\ll \hbar$ and the integration in the s variable contributes $\ll \log\hbar$. When $|n\hbar| \geq C > 0$ we repeat the above argument, except that we integrate by parts with respect to the u_2 variable. Since $\delta > 0$ is arbitrarily small, a combination of (2.6) with the stationary phase estimate for $d\rho_0(\phi)$ gives the following proposition.

Proposition 2.1. Let $\phi \in S(\mathbb{R})$ with $\check{\phi} \in C_0^\infty(\mathbb{R})$. Then, given the two-parameter family of metric forms $H(p,\sigma;u_1,u_2) := u_1\sigma^2 + u_2 p^2$ on the torus, we have that for $\hbar$ sufficiently small and any $\delta > 0$,

$$|\mathrm{AV}(\phi;\hbar)| \;=\; \mathcal{O}_\delta(\hbar^{-\delta}).$$

Remark 2.2. The result of Proposition 2.1, i.e., the estimate $|\mathrm{AV}(\phi;\hbar)| = \mathcal{O}_\delta(\hbar^{-\delta})$ also holds for the three-parameter family of metric forms $H(p,\sigma;u_1,u_2,u_3) := u_1\sigma^2 + u_3 p\sigma + u_2 p^2$. The proof is the same.

3. Mean-square density of states

The mean square density of states is given by the expression

$$\begin{aligned}\mathrm{MS}(\phi) &= \int_{I^2} |d\rho(\phi;\vec{u},\hbar) - c_0(\vec{u})\check{\phi}(0)|^2\, d\vec{u} \\ &= \int_{I^2} |d\rho_+(\phi;\vec{u},\hbar)|^2\, d\vec{u} + \mathcal{O}(\hbar^{2-\delta}|\log\hbar|).\end{aligned} \tag{3.1}$$

Here, the last estimate in (3.1) is an immediate consequence of (2.5) and (2.6). By introducing $\chi(y)$ a function which is ≥ 1 on $[1-\epsilon, 1+\epsilon]$ and with Fourier transform $\hat{\chi}$ of compact support, we are thus reduced to estimating

$$\int_{I^2} |d\rho_+(\phi;\vec{u},\hbar)|^2\, d\vec{u} \ll \hbar^2 \sum_{m_i,n_i} \int_{\mathbb{R}^4} e^{i\,\Phi(m_1,n_1,m_2,n_2;\vec{s},\hbar)/\hbar}\, a(\vec{s};\hbar)\,\chi(u_1)\,\chi(u_2)\, d\vec{s}d\vec{u}, \tag{3.2}$$

where, by the integration by parts argument in section 2 it suffices, modulo $\mathcal{O}(\hbar^\infty)$ errors, to sum over quadruples with $\max(|m_j|,|n_j|) \leq \hbar^{-1-\delta};\ j=1,2$. We set $\vec{s} = (s_1,s_2)$. To simplify the writing in (3.2) we have put

$$\Phi(m_1,n_1,m_2,n_2;\vec{s},\hbar) = \; H(m_1\hbar,n_1\hbar;\vec{u})\,s_1 - H(m_2\hbar,n_2\hbar;\vec{u})\,s_2,$$

and

$$a(\vec{s};\hbar) := (1-\zeta(s_1))\,(1-\zeta(s_2))\,\check{\phi}(\hbar s_1)\,\check{\phi}(\hbar s_2)\,e^{i(s_2-s_1)/\hbar}.$$

To estimate the integral in (3.2), we fix $\delta > 0$ and consider the set

$$\Omega(m_1,n_1,m_2,n_2;\hbar) := \{u \in I^2;\ |H(m_j\hbar,n_j\hbar;\vec{u}) - 1| \leq \hbar^{1-\delta}\,;\ j=1,2\,\}.$$

By an integration by parts in the s_1,s_2 variables in (3.2), it suffices to assume, modulo $\mathcal{O}(\hbar^\infty)$ errors, that we only sum over quadruples (m_1,n_1,m_2,n_2) with the property that for $\hbar \leq \hbar_0$,

$$\Omega(m_1,n_1,m_2,n_2;\hbar) \;\neq\; \emptyset.$$

So, we have that $\int_{I^2} |d\rho_+(\phi;\vec{u},\hbar)|^2\, d\vec{u}$ is bounded by

$$C\hbar^2 \sum_{\Omega(m_1,n_1,m_2,n_2;\hbar)\neq\emptyset} \int_{\mathbb{R}^4} e^{i\,\Phi(m_1,n_1,m_2,n_2;\vec{s},\hbar)/\hbar}\, a(\vec{s};\hbar)\,\chi(u_1)\,\chi(u_2)\, d\vec{s}d\vec{u} + \mathcal{O}(\hbar^\infty).$$

Furthermore, by integrating by parts in u_1,u_2, we can assume, modulo $\mathcal{O}(\hbar^\infty)$ errors, that we can find $(s_1,s_2) \in \operatorname{supp} a$ such that

$$\frac{\partial}{\partial u_j}\Phi(m_1,n_1,m_2,n_2;\vec{s},\hbar) \ll \hbar^{1-\delta}, \quad j=1,2. \tag{3.3}$$

Written out explicitly, the inequality in (3.3) reads

$$\begin{aligned} |m_1\hbar|^2\, s_1 \; - |m_2\hbar|^2\, s_2 \; &\ll \hbar^{1-\delta} \\ |n_1\hbar|^2\, s_1 \; - |n_2\hbar|^2\, s_2 \; &\ll \hbar^{1-\delta}, \end{aligned} \tag{3.4}$$

Since $\min(|s_1|,|s_2|) \gg 1$ on supp $a(\vec{s};\hbar)$, for a given (m_1,n_1,m_2,n_2) we need to be able to solve (3.4) for some s_1, s_2 with $\min(|s_1|,|s_2|) \gg 1$. By inverting the matrix equation in (3.4) using Cramer's rule and the estimate $\max(|m_j|,|n_j|) \ll \hbar^{-1-\delta}$, we get

$$|m_1\hbar|^2\,|n_2\hbar|^2 \;-\; |m_2\hbar|^2\,|n_1\hbar|^2 \;\ll\; \hbar^{1-3\delta}. \tag{3.5}$$

On the other hand, the condition $\Omega(m_1,n_1,m_2,n_2;\hbar) \neq \emptyset$ means that for some $u_1 = u_1(m_1,n_1,m_2,n_2;\hbar)$, $u_2 = u_2(m_1,n_1,m_2,n_2;\hbar)$,

$$\begin{aligned} (u_1,u_2)\,\cdot\,(|m_1\hbar|^2,|n_1\hbar|^2) \;&=\; 1+\mathcal{O}(\hbar^{1-\delta}) \\ (u_1,u_2)\,\cdot\,(|m_2\hbar|^2,|n_2\hbar|^2) \;&=\; 1+\mathcal{O}(\hbar^{1-\delta}), \end{aligned} \tag{3.6}$$

where $\cdot$ is the standard inner product in $\mathbb{R}^2$. Resubstituting (3.6) back into (3.5) or using Cramer's rule, we get

$$\frac{1}{u_2}\,(|m_1\hbar|^2 - |m_2\hbar|^2) = \mathcal{O}(\hbar^{1-3\delta}), \tag{3.7}$$

and

$$\frac{1}{u_1}\,(|n_1\hbar|^2 - |n_2\hbar|^2) = \mathcal{O}(\hbar^{1-3\delta}). \tag{3.8}$$

Since δ is arbitrarily small, we will abuse notation somewhat and will write δ instead of 3δ. Consequently, the two estimates in (3.8) and (3.7) imply that:

$$|m_1| - |m_2| \ll \frac{\hbar^{-1-\delta}}{|m_1|+|m_2|} \quad \text{and} \quad |n_1| - |n_2| \ll \frac{\hbar^{-1-\delta}}{|n_1|+|n_2|}. \tag{3.9}$$

Thus, we have shown that

$$\begin{aligned} \mathrm{MS}(\phi) \quad &\ll \quad \hbar^2 \sum_{m_2,n_2} \; \sum_{m_1\in\beta(m_2,\hbar)} \; \sum_{n_1\in\gamma(n_2,\hbar)} \int_{\mathbb{R}^2} \check{\phi}(\hbar s_1)\,\check{\phi}(\hbar s_2) \\ &\times \hat{\chi}(\hbar[s_1 m_1^2 - s_2 m_2^2])\,\hat{\chi}(\hbar[s_1 n_1^2 - s_2 n_2^2])\,d\vec{s}, \end{aligned} \tag{3.10}$$

where

$$\beta(m_2,\hbar) := \left\{ m_1;\, |m_1| - |m_2| \ll \frac{\hbar^{-1-\delta}}{|m_1|+|m_2|} \right\}, \tag{3.11}$$

and

$$\gamma(n_2,\hbar) := \left\{ n_1;\, |n_1| - |n_2| \ll \frac{\hbar^{-1-\delta}}{|n_1|+|n_2|} \right\}. \tag{3.12}$$

We make either the change of variables

$$S = \hbar(m_1^2 s_1 - m_2^2 s_2), \quad T = s_1 + s_2,$$

or, alternatively,

$$S = \hbar(n_1^2 s_1 - n_2^2 s_2), \quad T = s_1 + s_2.$$

We notice that, since $\check{\phi}$ has compact support, $\hbar(s_1+s_2) \ll 1$, which gives $T \ll \hbar^{-1}$. Equation (3.10) gives:

$$\begin{aligned}
|\mathrm{MS}(\phi)| \ll\ & \hbar^{2-2-2\delta} \sum_{m_2,n_2\neq 0} \frac{1}{m_2 n_2} \min\left(\frac{1}{\hbar m_2^2}, \frac{1}{\hbar n_2^2}\right) \int_{|T|\ll \hbar^{-1}} dT \\
& + \hbar^{2-1-1/2-2\delta} \sum_{m_2\neq 0, n_2=0} \frac{1}{m_2} \frac{1}{\hbar m_2^2} \int_{|T|\ll\hbar^{-1}} dT \\
& + \hbar^{2-1-1/2-2\delta} \sum_{n_2\neq 0, m_2=0} \frac{1}{n_2} \frac{1}{\hbar n_2^2} \int_{|T|\ll\hbar^{-1}} dT \\
& + \hbar^2 \sum_{m_1\neq 0, m_2=0}^{\hbar^{-1/2-\delta}} \sum_{n_1\neq 0, n_2=0}^{\hbar^{-1/2-\delta}} \min\left(\frac{1}{\hbar m_1^2}, \frac{1}{\hbar n_1^2}\right) \int_{|T|\ll\hbar^{-1}} dT.
\end{aligned} \tag{3.13}$$

For the estimate on the first line of (3.13) we used the range of m_1 and m_2 in (3.11) and (3.12). For the second estimate we notice that, when $n_2 = 0$, (3.12) gives $n_1^2 \ll \hbar^{-1-\delta}$. The third estimate is deduced using (3.12) and the fourth using (3.11) and (3.12).

Thus, since $|m_2\hbar|^2 + |n_2\hbar|^2 \geq C > 0$ and $|m_1\hbar|^2 + |n_1\hbar|^2 \geq C > 0$, it follows that:

$$\min\left(\frac{1}{\hbar m_2^2}, \frac{1}{\hbar n_2^2}\right) \ll \hbar \quad \text{and} \quad \min\left(\frac{1}{\hbar m_1^2}, \frac{1}{\hbar n_1^2}\right) \ll \hbar,$$

and so,

$$\begin{aligned}
|\mathrm{MS}(\phi)| \ll\ & \hbar^{-2\delta} \sum_{m_2,n_2\neq 0}^{\hbar^{-1-\delta}} \frac{1}{m_2 n_2} + \hbar^{1/2-2\delta} \sum_{m_2\neq 0}^{\hbar^{-1-\delta}} \frac{1}{m_2} \\
& + \hbar^{1/2-2\delta} \sum_{n_2\neq 0}^{\hbar^{-1-\delta}} \frac{1}{n_2} + \hbar^{2-1/2-\delta-1/2-\delta} \\
& \ll \hbar^{-2\delta} |\log \hbar|^2 + \hbar^{1/2-2\delta} |\log \hbar| + \hbar^{1-\delta}.
\end{aligned} \tag{3.14}$$

Since $\delta > 0$ in (3.14) is arbitrarily small, we have proved:

Proposition 3.1. Consider the two-parameter family of flat metrics on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ given by the Hamiltonian functions $H(p,\sigma;u_1,u_2) := u_1\sigma^2 + u_2 p^2$. Then, for any $\delta > 0$,

$$\mathrm{MS}(\phi) := \int_{I^2} |d\rho(\phi;\vec{u},\hbar) - c_0(\vec{u})\check{\phi}(0)|^2 \, d\vec{u} = \mathcal{O}_\delta(\hbar^{-\delta}).$$

We now turn to the proof of our main theorem. For this we will need to combine the estimates for the averaged and mean-square density of states with a rescaled, well-known Tauberian decomposition, see [DG], [V].

4. The Tauberian Decomposition: Proof of Theorem 1.1(i)

Let $\phi \in \mathbb{S}(\mathbb{R})$ with $\phi(\lambda) > 0$, and $\check{\phi} \in C_0^\infty(\mathbb{R})$ with $\check{\phi}(0) = 1$. We start with a rescaled Tauberian argument used by Duistermaat and Guillemin [DG], but applied to the eigenvalues of Δ rather than $\sqrt{\Delta}$. Taking into account this rescaling we will naturally encounter semiclassical density of states on scales of order $\sim \hbar^2$ where

$\hbar^{-1} = \sqrt{\lambda}$. Our starting point is the following basic Tauberian decomposition (see [DG], [V]):

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\lambda}\phi(x-\lambda')\,dx\,dN(\lambda';\vec{u}) &= \int_{\lambda'\geq\lambda+1}\int_{-\infty}^{\lambda}\phi(x-\lambda')\,dx\,dN(\lambda';\vec{u}) \\ &+\int_{|\lambda-\lambda'|<1}\int_{-\infty}^{\lambda}\phi(x-\lambda')\,dx\,dN(\lambda';\vec{u}) \\ &+\int_{\lambda'<\lambda-1}\int_{-\infty}^{\infty}\phi(x-\lambda')\,dx\,dN(\lambda';\vec{u}) \\ &-\int_{\lambda'\leq\lambda-1}\int_{\lambda}^{\infty}\phi(x-\lambda')\,dx\,dN(\lambda';\vec{u}).\end{aligned} \tag{4.1}$$

In the case at hand, all of the quantities in (4.1) depend on the external parameters $(u_1,u_2)\in I^2$ and our task is to estimate the (integrated) asymptotics of both sides of (4.1). First, we turn to the simplest term, which is the third term on the right-hand side of (4.1). Indeed, since $1=\int_{-\infty}^{\infty}\phi(s)ds$, it follows that

$$\int_{\lambda'<\lambda-1}\int_{-\infty}^{\infty}\phi(x-\lambda')\,dx\,dN(\lambda';\vec{u}) = N(\lambda-1;\vec{u}).$$

To estimate the other terms in (4.1), we will need the following result which hinges on the estimates for the averaged and mean-square density of states in sections 2 and 3:

Proposition 4.1. Let $H(p,\sigma;u_1,u_2)=\sigma^2u_1+p^2u_2$. Then, for any $\phi\in S(\mathbb{R})$ as above, we have that:

$$\int_{I^2}\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\lambda}\phi(x-\lambda')\,dx\,dN(\lambda';\vec{u}) \;-\; (2\pi)^{-2}\,\lambda\,\mathrm{vol}(M(\vec{u}))\right| d\vec{u} = \mathcal{O}_\delta(\lambda^{\frac{1}{4}+\delta}).$$

Proof. First, by the Fubini Theorem,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\lambda}\phi(x-\lambda')\,dx\,dN(\lambda';\vec{u}) \;=\; \int_{-\infty}^{\lambda}\int_{-\infty}^{\infty}\phi(x-\lambda')\,dN(\lambda';\vec{u})\,dx.$$

Since $\phi\in S(\mathbb{R})$ and $0\leq\lambda_j\in\mathrm{Spec}(\Delta)$,

$$\int_{-\infty}^{\infty}\phi(x-\lambda')\,dN(\lambda';\vec{u}) \;=\; \mathcal{O}(|x|^{-\infty}), \tag{4.2}$$

as $x\to-\infty$. Thus, from (4.2) it follows that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\lambda}\phi(x-\lambda')\,dx\,dN(\lambda';\vec{u}) \;=\; \int_{-\infty}^{\infty}\int_{0}^{\lambda}\phi(x-\lambda')\,dx\,dN(\lambda';\vec{u}) \;+\; \mathcal{O}(1). \tag{4.3}$$

Put $\hbar^{-1}=\sqrt{\lambda}$ and consider the rescaled operator $H:=\hbar^2\Delta$ with eigenvalues $\lambda_j(\hbar)=\hbar^2\lambda_j : j=1,2,\ldots.$ Then, making the change of variables $\Lambda:=\hbar^2 x$, and taking (4.3) into account, it follows that, modulo $\mathcal{O}(1)$ errors, we are reduced to estimating

$$\begin{aligned} I(\vec{u},\hbar) &= \hbar^{-2}\sum_{j=1}^{\infty}\int_0^1 \phi\left(\frac{\lambda_j(\hbar;\vec{u})-\Lambda}{\hbar^2}\right)d\Lambda \\ &= \hbar^{-2}\sum_{m,n}^{\hbar^{-1-\delta}}\int_0^1 \phi\left(\frac{H(m\hbar,n\hbar;\vec{u})-\Lambda}{\hbar^2}\right)d\Lambda \,+\, \mathcal{O}(\hbar^\infty). \end{aligned} \tag{4.4}$$

Next, we apply the Fourier inversion formula to right-hand side in (4.4) and split the resulting integral into two pieces corresponding to the zero and non-trivial period spectrum. More precisely, let $\zeta \in C_0^\infty(\mathbb{R})$ be non-negative, equal to 1 close to 0. Its support should be small enough as explained in the Appendix. Then,

$$I(\vec{u},\hbar) \,=\, I_0(\vec{u},\hbar) \,+\, I_+(\vec{u},\hbar), \tag{4.5}$$

where

$$I_0(\vec{u},\hbar) = \hbar^{-1}\mathrm{Trace}\int_0^1\int_{-\infty}^{\infty} e^{is[H(Q_1,Q_2;\vec{u})-\Lambda]/\hbar}\zeta(s)\check{\phi}(\hbar s)\,ds\,d\Lambda,$$

and

$$I_+(\vec{u},\hbar) = \hbar^{-1}\mathrm{Trace}\int_0^1\int_{-\infty}^{\infty} e^{is[H(Q_1,Q_2;\vec{u})-\Lambda]/\hbar}(1-\zeta(s))\check{\phi}(\hbar s)\,ds\,d\Lambda,$$

and $Q_1 = m\hbar$, $Q_2 = n\hbar$. First, we estimate $I_0(\vec{u},\hbar)$ on the right-hand side of (4.5) by exactly the same method as in the Appendix, where we estimated $d\rho_0(\phi)$. We get

$$I_0(\vec{u},\hbar) = (2\pi)^{-2}\mathrm{vol}(M(\vec{u}))\,\hbar^{-2} + \mathcal{O}(\hbar^\infty). \tag{4.6}$$

To prove Proposition 4.1, we need to estimate

$$\int_{I^2}\left|I(\vec{u},\hbar) - I_0(\vec{u},\hbar)\right|d\vec{u} = \int_{I^2}\,|I_+(\vec{u},\hbar)|\,d\vec{u}.$$

To do this, we appeal to Proposition 3.1 on the mean-square density of states and apply the Cauchy-Schwartz inequality. First,

$$\begin{aligned} &\int_{I^2}|I_+(\vec{u},\hbar)|^2 d\vec{u} \\ &= \hbar^{-2}\sum_{m_1,n_1}\sum_{m_2,n_2}\int e^{i\Phi(m_1,n_1,m_2,n_2;\vec{s},\hbar)/\hbar}\, a(\vec{s};\Lambda_1,\Lambda_2,\hbar)d\Lambda_1\,d\Lambda_2\,d\vec{s}\,d\vec{u}, \end{aligned} \tag{4.7}$$

and

$$a(\vec{s},\Lambda_1,\Lambda_2;\hbar) = e^{i[\Lambda_1 s_1 - \Lambda_2 s_2]/\hbar}(1-\zeta(s_1))\check{\phi}(\hbar s_1)(1-\zeta(s_2))\check{\phi}(\hbar s_2).$$

Note that since $\check{\phi} \in C_0^\infty(\mathbb{R})$ the summations in (4.7) are restricted to indices with $\max(|m_i|,|n_i|) \ll \hbar^{-1-\delta}$. Moreover, since $\min(|s_1|,|s_2|) \geq 1$ on supp a, by an iterated integration in the Λ_1, Λ_2 variables, we get

$$\int_0^1\int_0^1 a(\vec{s},\Lambda_1,\Lambda_2;\hbar)\,d\Lambda_1\,d\Lambda_2 = \left(\frac{\hbar^2}{s_1 s_2}\right)\cdot b(s_1,s_2;\hbar),$$

where $b \in C^\infty$ with $b(s_1, s_2; \hbar) = \mathcal{O}(1)$ as for $\hbar \leq \hbar_0$. So, the end result is that

$$\int_{I^2} |I_+(\vec{u}, \hbar)|^2 d\vec{u}$$
$$= \sum_{|m_i|,|n_i|>0}^{C\hbar^{-1-\delta}} \int e^{i\Phi(m_1,n_1,m_2,n_2;\vec{s},\hbar)/\hbar}\, b(s_1, s_2; \hbar) \frac{1}{s_1 s_2} \chi(u_1)\, \chi(u_2) d\vec{s}\, d\vec{u}. \quad (4.8)$$

Since

$$s_1 s_2 \geq \frac{1}{2}(s_1 + s_2) \quad \text{when} \quad \min(s_1, s_2) \geq 1,$$

it follows by the argument in Proposition 3.1 that the right-hand side in (4.8) is

$$\ll \sum_{m_2,n_2} \sum_{m_1 \in \beta(m_2,\hbar)} \sum_{n_1 \in \gamma(n_2,\hbar)} \int_{|s_1|,|s_2| \geq C} \frac{\check{\phi}(\hbar s_1)\check{\phi}(\hbar s_2)}{|s_1| + |s_2|}$$
$$\times \hat{\chi}(\hbar[s_1 m_1^2 - s_2 m_2^2])\hat{\chi}(\hbar[s_1 n_1^2 - s_2 n_2^2])\, d\vec{s},$$

which is bounded by

$$\hbar^{-2-2\delta} \sum_{m_2,n_2} \frac{1}{m_2 n_2} \min\left(\frac{1}{\hbar m_2^2}, \frac{1}{\hbar n_2^2}\right) \int_{C<|T|\ll\hbar^{-1}} \frac{dT}{T} \ll \hbar^{-1-2\delta}\, |\log \hbar|^3.$$

The sets $\beta(m_2, \hbar)$ and $\gamma(n_2, \hbar)$ are defined in (3.11), (3.12) and the argument follows as in (3.13) with the only change being the appearance of the denominator $|s_1|+|s_2|$. Now, by the Cauchy-Schwartz inequality, we have

$$\int_{I^2} |I_+(\vec{u}, \hbar)| d\vec{u} \ll \left(\int_{I^2} |I_+(\vec{u}, \hbar)|^2\, d\vec{u}\right)^{1/2}$$
$$\ll \left(\hbar^{-1-3\delta}\right)^{1/2} \ll \lambda^{1/4+3\delta}. \quad (4.9)$$

This completes the proof of Proposition 4.1. $\square$

By reshuffling the terms in the Tauberian decomposition, it is clear that

$$\begin{aligned} |N(\lambda - 1; \vec{u}) - I_0(\vec{u}, \hbar)| \leq\ & |I_+(\vec{u}, \hbar)| \\ & + \left| \int_{\lambda' \geq \lambda+1} \int_{-\infty}^{\lambda} \phi(x - \lambda')\, dx\, dN(\lambda'; \vec{u}) \right| \\ & + \left| \int_{|\lambda-\lambda'|<1} \int_{-\infty}^{\lambda} \phi(x - \lambda')\, dx\, dN(\lambda'; \vec{u}) \right| \\ & + \left| \int_{\lambda' \leq \lambda-1} \int_{\lambda}^{\infty} \phi(x - \lambda')\, dx\, dN(\lambda'; \vec{u}) \right|. \quad (4.10) \end{aligned}$$

Integrating both sides of (4.10) over the deformation parameters u_1, u_2, we get

$$\begin{aligned}
\int_{I^2} |N(\lambda-1;\vec{u}) - I_0(\vec{u},\hbar)|\, d\vec{u} \quad &\leq \quad \int_{I^2} |I_+(\vec{u},\hbar)| d\vec{u} \\
&+ \int_{I^2} \left| \int_{\lambda' \geq \lambda+1} \int_{-\infty}^{\lambda} \phi(x-\lambda')\, dx\, dN(\lambda';\vec{u}) \right| d\vec{u} \\
&+ \int_{I^2} \left| \int_{|\lambda-\lambda'|<1} \int_{-\infty}^{\lambda} \phi(x-\lambda')\, dx\, dN(\lambda';\vec{u}) \right| d\vec{u} \\
&+ \int_{I^2} \left| \int_{\lambda' \leq \lambda-1} \int_{\lambda}^{\infty} \phi(x-\lambda')\, dx\, dN(\lambda';\vec{u}) \right| d\vec{u} \\
&= T_0 + T_1 + T_2 + T_3 \qquad (4.11)
\end{aligned}$$

From Proposition 4.1 we have that for any $\delta > 0$,

$$T_0 = \int_{I^2} |I_+(\vec{u},\hbar)|\, d\vec{u} \;=\; \mathcal{O}_\delta(\hbar^{-1/2-\delta}).$$

Consequently, it remains to estimate the terms T_1, T_2, T_3 on the right-hand side of the inequality (4.11). We start with T_1:

$$\begin{aligned}
|T_1| \quad &= \quad \int_{I^2} \left| \int_{\lambda' \geq \lambda+1} \int_{-\infty}^{\lambda} \phi(x-\lambda')\, dx\, dN(\lambda';\vec{u}) \right| d\vec{u} \\
&\leq C_N \sum_{m=1}^{\infty} \left(\int_{I^2} \int_{m \leq \lambda'-\lambda \leq m+1} dN(\lambda',\vec{u}) \right) m^{-N} \\
&\leq C_N \sum_{m=1}^{\infty} \left(\int_{I^2} N(\lambda+m+1;\vec{u}) - N(\lambda+m;\vec{u})\, d\vec{u} \right) m^{-N}. \quad (4.12)
\end{aligned}$$

One gets the first inequality above by splitting up the range of $\lambda' - \lambda \in [0,\infty)$ and using the fact $\phi \in S(\mathbb{R})$. We let $\tilde{\lambda} = \lambda + m$. It follows from Proposition 2.1 for the averaged density of states that for any $\delta > 0$,

$$\int_{I^2} N(\lambda+m+1;\vec{u}) - N(\lambda+m;\vec{u})\, d\vec{u} \;\ll\; (m+\lambda)^{\delta},$$

so that the last line in (4.12) is bounded by

$$C_N \lambda^{\delta} \sum_{m=1}^{\infty} \; m^{-N+\delta} = \mathcal{O}_\delta(\lambda^{\delta}).$$

The end result is that

$$|T_1| = \int_{I^2} \left| \int_{\lambda' \geq \lambda+1} \int_{-\infty}^{\lambda} \phi(x-\lambda')\, dx\, dN(\lambda';\vec{u}) \right| du_1 du_2 \;=\; \mathcal{O}_\delta(\lambda^{\delta}). \tag{4.13}$$

The term T_3 can be estimated in the same way as T_1. So, in particular it follows that:

$$|T_3| = \mathcal{O}(\lambda^{\delta}) \tag{4.14}$$

for any $\delta > 0$. We now turn to the estimating the term T_2. Here we also need the bound for the averaged density of states given in Proposition 2.1:

$$\begin{aligned} T_2 &= \int_{I^2} \left| \int_{|\lambda-\lambda'|<1} \int_{-\infty}^{\lambda} \phi(x-\lambda')\, dx\, dN(\lambda';\vec{u}) \right| d\vec{u} \\ &\leq \int_{I^2} \left| \int_{|\lambda-\lambda'|<1} \int_{-\infty}^{\infty} \phi(x-\lambda')\, dx\, dN(\lambda';\vec{u}) \right| d\vec{u} \\ &= \int_{I^2} |N(\lambda+1;\vec{u}) - N(\lambda-1;\vec{u})\,|\, d\vec{u}, \end{aligned}$$

since $\phi(x) \geq 0$ and $\int_{-\infty}^{\infty} \phi(x)dx = 1$.

Now, let $\Theta \in S(\mathbb{R})$ with $\Theta \geq 0$ and $\Theta(x) \geq 1$ when $|x| \leq 1$. Then

$$\int_{I^2} |N(\lambda+1;\vec{u}) - N(\lambda-1;\vec{u})\,|\, d\vec{u} \,\leq \sum_{j=1}^{\infty} \int_{I^2} \Theta(\lambda_j - \lambda)\, d\vec{u} = \mathcal{O}_\delta(\hbar^{-\delta}),$$

where the estimate is a consequence of Proposition 2.1 for the averaged density of states. Therefore, we have that

$$|T_2| = \mathcal{O}_\delta(\hbar^{-\delta}). \tag{4.15}$$

Combining the estimates (4.9), (4.13), (4.14), (4.15) and (4.6) with the integrated Tauberian decomposition in (4.11) gives

$$\int_{I^2} \left| N(\lambda-1;\vec{u}) - c_0(\vec{u})\,\lambda \right| d\vec{u} \,=\, \mathcal{O}_\delta(\lambda^{1/4+\delta}) + \mathcal{O}_\delta(\lambda^{\delta}) + \mathcal{O}(1)$$

for any $\delta > 0$. This completes the proof of Theorem 1.1 (i). □

5. Generic flat tori: Proof of Theorem 1.1 (ii)

Here we take up the case where

$$H(p,\sigma;u_1,u_2,u_3) := u_1\sigma^2 + u_3 p\sigma + u_2 p^2.$$

Then, just as in the proof of Theorem 1.1 (i), we get from the eigenfunction equation that for some $\vec{u} = \vec{u}(m_1,n_1,m_2,n_2;\hbar)$,

$$\begin{aligned} u_1 m_1^2 \,+\, u_3 m_1 n_1 \,+\, u_2 n_1^2 &= \hbar^{-2} + \mathcal{O}(\hbar^{-1-\delta}), \\ u_2 m_2^2 \,+\, u_3 m_2 n_2 \,+\, u_2 n_2^2 &= \hbar^{-2} + \mathcal{O}(\hbar^{-1-\delta}), \end{aligned} \tag{5.1}$$

Furthermore, since there are now three parameters, there are three inequalities analogous to the estimates in (3.5) and (3.6). These inequalities come from solving the system of three equations that we get by computing the Fourier transforms in the u_1, u_2, u_3 variables in the expression for $\mathrm{MS}(\phi;\hbar)$. These are

$$\begin{aligned} m_1^2\, n_2^2 \,-\, m_2^2\, n_1^2 &\ll \hbar^{-3-\delta}, \\ m_1\, m_2\,(\, m_2 n_1 - m_1 n_2\,) &\ll \hbar^{-3-\delta}, \\ n_1\, n_2\,(\, m_2 n_1 - m_1 n_2\,) &\ll \hbar^{-3-\delta}. \end{aligned} \tag{5.2}$$

Multiplying the first equation in (5.1) by m_2^2 , the second by m_1^2 and subtracting gives

$$\begin{aligned} \hbar^{-2}(m_2^2 - m_1^2) &= u_3 m_1 m_2 \,(m_2 n_1 - m_1 n_2) \\ &\quad + u_2(n_1^2 m_2^2 - n_2^2 m_1^2) + \mathcal{O}(\hbar^{-3-\delta}) \\ &= \mathcal{O}(\hbar^{-3-\delta}), \end{aligned} \tag{5.3}$$

where the last inequality in (5.3) follows from the first two estimates in (5.2).

Similarily, multiplying the first equation in (5.1) by n_2^2, the second by n_1^2 and subtracting gives

$$\begin{aligned} \hbar^{-2}(n_2^2 - n_1^2) &= u_3 n_1 n_2 \,(m_2 n_1 - m_1 n_2) \\ &\quad + u_2(n_1^2 m_2^2 - n_2^2 m_1^2) + \mathcal{O}(\hbar^{-3-\delta}) \\ &= \mathcal{O}(\hbar^{-3-\delta}), \end{aligned} \tag{5.4}$$

which follows from the first and last estimates in (5.2). Combining the inequalities in (5.3) and (5.4) gives

$$|m_1| - |m_2| \ll \frac{\hbar^{-1-\delta}}{|m_1| + |m_2|} \quad \text{and} \quad |n_1| - |n_2| \ll \frac{\hbar^{-1-\delta}}{|n_1| + |n_2|}.$$

Restricting to quadruples satisfying the inequalities (3.9), the proof of Theorem 1.1 (ii) then proceeds exactly as before.

6. Appendix: Proof of (2.5)

We set

$$\check{\chi}(s, \hbar) = \zeta(s) \cdot \check{\phi}(\hbar s). \tag{6.1}$$

From (2.3) it is clear that, up to $\mathcal{O}(\hbar^{\infty})$ errors, we have

$$d\rho_0(\phi) = \hbar \cdot \text{Trace} \int_{\mathbb{R}} e^{is[-\hbar^2\Delta - 1]/\hbar} \check{\chi}(s, \hbar)\, ds.$$

We consider the operator $P = (-\hbar^2\Delta - 1)/\hbar$. The one-parameter group of operators $U(s) = e^{isP}$ satisfies the equation

$$\frac{\partial}{\partial s} U(s) = (-i\hbar\Delta - i\hbar^{-1}) U(s)$$

and its Schwartz kernel $\tilde{U}(s, x, y; \hbar)$ is given by

$$\tilde{U}(s, x, y; \hbar) = \frac{1}{(2\pi\hbar)^2} \int e^{i[(x-y)\cdot\xi + s|\xi|^2 - s]/\hbar}\, d\xi.$$

This calculation holds for $\mathbb{R}^2$. Since the torus is covered by $\mathbb{R}^2$, we have

$$U(s, x, y; \hbar) = \sum_{\gamma \in \Gamma} \tilde{U}(s, x + \gamma, y; \hbar).$$

The trace formula

$$\sum_j \chi(\lambda_j) = \int \int U(s, x, x; \hbar) \check{\chi}(s)\, ds\, d\mathrm{vol}(x)$$

gives

$$d\rho_0(\phi) = c_0 \hbar^{-1} \sum_{\gamma \in \Gamma} \int_{\mathbb{R}^2} \int_{\mathbb{R}} e^{i\gamma\cdot\xi/\hbar} e^{is(|\xi|^2-1)/\hbar} \check{\chi}(s,\hbar)\, ds\, d\xi. \tag{6.2}$$

We choose the support of $\check{\chi}(s;\hbar)$, or, equivalently $\zeta(s)$ to be sufficiently small, as explained in (6.3) below. The main contribution in (6.2) comes from the range $|\xi|^2 \sim 1$, because on the complement we can integrate by parts in s as many times as needed and get $\mathcal{O}(\hbar^\infty)$ error. We look first at the summands with $\gamma \neq 0$. We integrate by parts in the (ξ_1, ξ_2) variables. Since $|\xi_j| \leq 2$, say, and $\gamma \neq 0$, we can find a constant $c > 0$, independent of $\vec{u}$, such that $\max(|\gamma_j|) \geq c$. We take

$$\mathrm{supp}(\check{\chi}) \subset (-c/5, c/5). \tag{6.3}$$

Then

$$\left| \frac{\partial}{\partial \xi_j} (\gamma \cdot \xi + |\xi|^2 s) \right| = |\gamma_j + 2\xi_j s| \geq c/5.$$

With the choice (6.3) for the support of $\check{\chi}$ we can ignore the contribution of $\gamma \neq 0$, up to $\mathcal{O}(\hbar^\infty)$ errors. We get

$$d\rho_0(\phi) = c_0 \hbar^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}^2} e^{is(|\xi|^2-1)/\hbar} \check{\chi}(s;\hbar)\, ds\, d\xi + \mathcal{O}(\hbar^\infty). \tag{6.4}$$

We set $\phi_\hbar(\tau) = \hbar^{-1}\phi(\tau/\hbar)$ and from (6.1) it follows that $\chi = \hat{\zeta} * \phi_\hbar$. We change to polar coordinates in (6.4) and substitute $r^2 - 1 = \hbar x$ to get

$$\begin{aligned} d\rho_0(\phi) &= c_0 \hbar^{-2} \int_0^\infty \int_{\mathbb{R}} \hat{\zeta}((r^2-1)/\hbar - y)\phi(y/\hbar)\, r\; dy dr + \mathcal{O}(\hbar^\infty) \\ &= c_0 \hbar^{-1} \int_{-1/\hbar}^\infty \int_{\mathbb{R}} \hat{\zeta}(x-y)\phi(y/\hbar)\, dy\, dx + \mathcal{O}(\hbar^\infty). \end{aligned}$$

By Fubini, we do the x-integration first using

$$\int_{-1/\hbar}^\infty \hat{\zeta}(x-y)\, dx = \int_{\mathbb{R}} \hat{\zeta}(x-y)\, dx + \mathcal{O}(\hbar^\infty) = \zeta(0) + \mathcal{O}(\hbar^\infty).$$

Since $\zeta(0) = 1$, we get finally

$$d\rho_0(\phi) = c_0 \int_{\mathbb{R}} \phi(u)\, du + \mathcal{O}(\hbar^\infty). \tag{6.5}$$

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