Quantum G-manifolds

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1 Introduction

Passing from classical mechanics to quantum mechanics involves replacing the commutative function algebra, $\mathcal{A}$, of classical observables on the appropriate phase space, M, with a noncommutative (deformed) algebra, $\mathcal{A}_t$, of quantum observables (see [BFFLS] where the deformation quantization scheme is developed). The algebra $\mathcal{A}$ is a Poisson algebra and the product in $\mathcal{A}_t$ is given by a power series in the formal parameter t with leading term the original commutative product and with leading term in the commutator given by the Poisson bracket. If the classical system is invariant under a Lie group of symmetries, G, then G acts on the phase space (and thus on $\mathcal{A}$) and the associated quantum system often retains the group of symmetries. In particular, the algebra $\mathcal{A}_t$ is often invariant under the action of G, or under the action of its universal enveloping algebra $U(\mathfrak{g})$.

Modern field-theoretical models and, in particular, the problem of incorporating gravity into a quantum field theory led to the requirement of deforming (quantizing) the group symmetry and the phase space themselves. This is one of the reasons for the interest in quantum groups. The quantum group, $U_h(\mathfrak{g})$, defined by Drinfeld and Jimbo is a deformation of $U(\mathfrak{g})$ as a Hopf algebra. The quantization of the phase space and its symmetry group corresponds to a $U_h(\mathfrak{g})$ invariant deformation of the algebra $\mathcal{A}_t$, which leads us to the problem of two parameter (or double) quantization, $\mathcal{A}_{t,h}$, of the function algebra for a $U(\mathfrak{g})$ invariant Poisson structure. In other words, the problem of two parameter (or double) quantization appears if we want to quantize the Poisson bracket in such a way that multiplication in the quantized algebra is invariant under the quantum group action.

In the present talk we consider examples of one and two parameter invariant quantizations of Poisson function algebras on some natural G-manifolds.

In fact, we will regard $\mathcal{A}$ as the sheaf of algebras and $\mathcal{A}_t$ ($\mathcal{A}_{t,h}$) as a deformation of that sheaf. We call the *quantum* G-manifold the manifold M endowed with the deformed sheaf of non-commutative algebras.

In Section 2 we review some facts on quantum groups related to our problem. Note that we consider only the case of semisimple G. In Section 3 we formulate precisely the problem and reduce it to the problem of a G-invariant non-associative quantization. Note

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that in our setting we suppose that the action of G on the space of function on M does not deform. In Section 4 we describe Poisson brackets admissible for $U_h(\mathfrak{g})$-invariant one and two parameter quantizations.

Beginning from Section 5 we consider examples of $U_h(\mathfrak{g})$-invariant quantization on some G-manifolds. Namely, we consider two-sided and Ad-action G on itself, coadjoint action G on $\mathfrak{g}^*$, and semisimple orbits in $\mathfrak{g}^*$.

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2 Quantum groups

We will consider quantum groups in the sense of Drinfeld, [Dr2], as Hopf algebras being deformed universal enveloping algebras. So we will regard that $U_h(\mathfrak{g}) = U(\mathfrak{g})[[h]]$ as a topological $\mathbb{C}[[h]]$-module and $U_h(\mathfrak{g})/hU_h(\mathfrak{g}) = U(g)$ as a Hopf algebra over $\mathbb{C}$. In particular, the deformed comultiplication in $U_h(\mathfrak{g})$ has the form

$$\Delta_h = \Delta + h\Delta_1 + o(h), \tag{2.1}$$

where Δ is the comultiplication in the universal enveloping algebra $U(\mathfrak{g})$. One can prove, [Dr2], that the map $\Delta_1 : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is such that $\Delta_1 - \sigma\Delta_1 = \delta$ (σ is the usual permutation) being restricted to $\mathfrak{g}$ gives a map $\delta : \mathfrak{g} \to \wedge^2\mathfrak{g}$ which is a 1-cocycle and defines the structure of a Lie coalgebra on $\mathfrak{g}$ (the structure of a Lie algebra on the dual space $\mathfrak{g}^*$). The pair $(\mathfrak{g}, \delta)$ is called a quasiclassical limit of $U_h(\mathfrak{g})$.

In general, a pair $(\mathfrak{g}, \delta)$, where $\mathfrak{g}$ is a Lie algebra and δ is such a 1-cocycle, is called a Lie bialgebra. It is proven, [EK], that any Lie bialgebra $(\mathfrak{g}, \delta)$ can be quantized, i.e., there exists a quantum group $U_h(\mathfrak{g})$ such that the pair $(\mathfrak{g}, \delta)$ is its quasiclassical limit.

A Lie bialgebra $(\mathfrak{g}, \delta)$ is said to be a coboundary one if there exists an element $r \in \wedge^2\mathfrak{g}$, called the classical r-matrix, such that $\delta(x) = [r, \Delta(x)]$ for $x \in \mathfrak{g}$. Since δ defines a Lie coalgebra structure, r has to satisfy the so-called classical Yang-Baxter equation which can be written in the form

$$[\![r, r]\!] = \varphi, \tag{2.2}$$

where $[\![\cdot, \cdot]\!]$ stands for the Schouten bracket and $\varphi \in \wedge^3\mathfrak{g}$ is an invariant element. We denote the coboundary Lie bialgebra by $(\mathfrak{g}, r)$.

In case $\mathfrak{g}$ is a simple Lie algebra, the most known r-matrix is the Sklyanin-Drinfeld one:

$$r = \sum_\alpha X_\alpha \wedge X_{-\alpha}, \tag{2.3}$$

where the sum runs over all positive roots; the root vectors X_α are chosen in such a way that $(X_\alpha, X_{-\alpha}) = 1$ for the Killing form $(\cdot, \cdot)$. This is the only r-matrix of weight zero, [SS], and its quantization is the Drinfeld-Jimbo quantum group. A classification of all r-matrices for simple Lie algebras was given in [BD].

From results of Drinfeld and of Etingof and Kazhdan one can derive the following description of quantum groups associated with semisimple Lie algebras.

Proposition 2.1. *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then*

a) any Lie bialgebra $(\mathfrak{g}, \delta)$ is a coboundary one;

b) the quantization, $U_h(\mathfrak{g}, r)$, of any coboundary Lie bialgebra $(\mathfrak{g}, r)$ exists and is isomorphic to $U(\mathfrak{g})[[h]]$ as a topological $\mathbb{C}[[h]]$-algebra;

c) the comultiplication in $U_h(\mathfrak{g}, r)$ has the form

$$\Delta_h(x) = F_h \Delta(x) F_h^{-1}, \qquad x \in U(\mathfrak{g}), \tag{2.4}$$

where $F_h \in U(\mathfrak{g})^{\otimes 2}[[h]]$ and can be chosen in the form

$$F_h = 1 \otimes 1 + \frac{h}{2} r + o(h). \tag{2.5}$$

Moreover,

$$(\varepsilon \otimes 1)F_h = (1 \otimes \varepsilon)F_h = 1 \otimes 1, \tag{2.6}$$

where $\varepsilon : U_h(\mathfrak{g}, r) \to \mathbb{C}[[h]]$ is the counit in $U_h(\mathfrak{g}, r)$, which coincides with the natural extension of the counit $U(\mathfrak{g}) \to \mathbb{C}$.

It follows from the coassociativity Δ_h that F_h satisfies the equation

$$(F_h \otimes 1) \cdot (\Delta \otimes id)(F_h) = (1 \otimes F_h) \cdot (id \otimes \Delta)(F_h) \cdot \Phi_h \tag{2.7}$$

for some invariant element $\Phi_h \in U(\mathfrak{g})^{\otimes 3}[[h]]$.

It follows from (2.7) that if F_h has the form (2.5), then the coefficient by h in Φ_h vanishes. Moreover, the coefficient by h^2 is up to a factor the element φ from (2.2), i.e.,

$$\Phi_h = 1 \otimes 1 \otimes 1 + h^2 \varphi + o(h^2). \tag{2.8}$$

In addition, it follows from (2.7) that Φ_h satisfies the pentagon identity

$$(id^{\otimes 2} \otimes \Delta)(\Phi_h) \cdot (\Delta \otimes id^{\otimes 2})(\Phi_h) = (1 \otimes \Phi_h) \cdot (id \otimes \Delta \otimes id)(\Phi_h) \cdot (\Phi_h \otimes 1). \tag{2.9}$$

One can suppose that an invariant element Φ_h of the form (2.8) satisfying (2.9) is given in advance. Such Φ_h can be constructed independently, [Dr2]. Then, for any element r obeying (2.2) one can find F_h of the form (2.5) such that (2.7) holds. So, we may take such F_h to define comultiplication (2.4) in the quantum group $U_h(\mathfrak{g}, r)$. In the following we fix Φ_h satisfying the additional relation, [DS2]:

$$\Phi_h \Phi_h^s = 1, \tag{2.10}$$

where s is the antipod in $U(\mathfrak{g})$ defined by $s(x) = -x$ for $x \in \mathfrak{g}$, and $\Phi_h^s = (s \otimes s \otimes s)(\Phi_h)$.

The Hopf algebra $U = U_h(\mathfrak{g}, r)$ is quasitriangular with the universal R-matrix

$$\mathcal{R} = F_{21} e^{\frac{h}{2}\mathbf{t}} F^{-1} = 1 \otimes 1 + h(\frac{\mathbf{t}}{2} - r) + \cdots, \tag{2.11}$$

where $\mathbf{t} \in \mathrm{Sym}^2 \mathfrak{g}$ is an invariant element. $\mathcal{R}$ satisfies the relations, [Dr2]

$$\Delta'(x) = \mathcal{R}\Delta(x)\mathcal{R}^{-1} \tag{2.12}$$

and

$$\begin{aligned} \Delta_1\mathcal{R} &= \mathcal{R}_{13}\mathcal{R}_{23} \\ \Delta_2\mathcal{R} &= \mathcal{R}_{13}\mathcal{R}_{12}. \end{aligned} \tag{2.13}$$

Here subscripts indicate the position of the corresponding tensor factors; for example, if $a = a_1 \otimes a_2$, then $a_{13} = a_1 \otimes 1 \otimes a_3$ and $a_{21} = a_2 \otimes a_1$.

3 $U_h(\mathfrak{g})$-invariant quantization

Let G be a semisimple connected complex Lie group with the Lie algebra $\mathfrak{g}$. Let M be a G-manifold and $\mathcal{A} = \mathcal{A}(M)$ a sheaf of functions on M. It may be the sheaf of analytic, smooth, or algebraic functions, dependingly of the type of M. Then $U(\mathfrak{g})$ acts on sections of $\mathcal{A}$, and the multiplication in $\mathcal{A}$ is $U(\mathfrak{g})$-invariant, i.e.,

$$bm(x \otimes y) = m\Delta(b)(x \otimes y) \qquad b \in U(\mathfrak{g}),\ x, y \in \mathcal{A}.$$

By a deformation quantization of $\mathcal{A}$ we mean a sheaf of associative algebras, $\mathcal{A}_h$, which is equal to $\mathcal{A}[[h]]$ as a $\mathbb{C}[[h]]$-module, with multiplication in $\mathcal{A}_h$ of the form

$$m_h = \sum_{k=0}^{\infty} h^k m_k,$$

where $m_0 = m$ is the usual commutative multiplication in $\mathcal{A}$ and m_k, $k > 0$, are bidifferential operators vanishing on constants. The action $U(\mathfrak{g})$ on $\mathcal{A}$ is naturally extended to the action of the $\mathbb{C}[[h]]$-algebra $U(\mathfrak{g})[[h]] = U_h(\mathfrak{g})$ on the $\mathbb{C}[[h]]$-module $\mathcal{A}[[h]] = \mathcal{A}_h$.

We will also consider two parameter quantizations on M. A two parameter quantization of $\mathcal{A}$ is an algebra $\mathcal{A}_{t,h}$ equal to $\mathcal{A}[[t,h]]$ as a $\mathbb{C}[[t,h]]$-module and having a multiplication of the form

$$m_{t,h} = m_0 + tm_1' + hm_1'' + o(t,h). \tag{3.1}$$

We are going to study quantizations of $\mathcal{A}$ which are invariant under the $U_h(\mathfrak{g})$-action, i.e., under the comultiplication Δ_h. The invariance means that

$$bm_h(x \otimes y) = m_h\Delta_h(b)(x \otimes y) \qquad b \in U(\mathfrak{g}),\ x, y \in \mathcal{A}, \tag{3.2}$$

where m_h means one or two parameter multiplication. Of course, for a two parameter $U_h(\mathfrak{g})$-invariant multiplication, $m_{t,h}$, the multiplication $m_t = m_{t,0}$ is $U(\mathfrak{g})$-invariant.

We call m_h satisfying (3.2) a $U_h(\mathfrak{g})$-invariant quantization of the G-manifold M.

Definition 3.1. A $\mathbb{C}[[h]]$-linear map $\mu_h : \mathcal{A}_h \otimes \mathcal{A}_h \to \mathcal{A}_h$ is called a Φ_h-*associative* multiplication if

$$\mu_h(\Phi_1 x \otimes \mu_h(\Phi_2 y \otimes \Phi_3 z))) = \mu_h(\mu_h(x \otimes y) \otimes z) \quad \text{for } x, y, z \in \mathcal{A},$$

where $\Phi_h = \Phi_1 \otimes \Phi_2 \otimes \Phi_3$ (summation implicit).

We say that the Φ_h-associative multiplication $\mu_h = \sum_{k=0}^{\infty} h^k \mu_k$ gives a Φ_h-associative quantization of $\mathcal{A}$ if $\mu_0 = m_0$, the usual multiplication in $\mathcal{A}$, and μ_k, $k > 0$, are bidifferential operators vanishing on constants.

Proposition 3.1. *There is a natural one-to-one correspondence between $U_h(\mathfrak{g})$-invariant and $U(\mathfrak{g})$-invariant Φ_h-associative quantizations of $\mathcal{A}$. Namely, if μ_h is a $U(\mathfrak{g})$-invariant Φ_h-associative multiplication in $\mathcal{A}[[h]]$, then*

$$m_h = \mu_h F_h^{-1} \tag{3.3}$$

gives a $U_h(\mathfrak{g})$-invariant associative multiplication in $\mathcal{A}[[h]]$.

Proof. This follows immediately from (2.4) and (2.7). This follows also from the categorical interpretation of Φ_h and F_h, [Dr2], [DGS]. □

This proposition shows that given a $U(\mathfrak{g})$-invariant Φ_h-associative quantization of $\mathcal{A}$, we can get the $U_h(\mathfrak{g})$-invariant quantization of $\mathcal{A}$ for any quantum group $U_h(\mathfrak{g})$ from Proposition 2.1 b) by applying F_h from (2.5) to the Φ_h-associative multiplication.

4 Poisson brackets associated with $U_h(\mathfrak{g})$-invariant quantization

A skew-symmetric map $f : \mathcal{A}^{\otimes 2} \to \mathcal{A}$ we call a *bracket* if it satisfies the Leibniz rule: $f(ab, c) = af(b, c) + f(a, c)b$ for $a, b, c \in \mathcal{A}$. It is easy to see that any bracket is presented by a bivector field on M. Further we will identify brackets and bivector fields on M.

For an element $\psi \in \wedge^k \mathfrak{g}$ we denote by ψ_M the k-vector field on M which is induced by the action map $\mathfrak{g} \to \mathrm{Vect}(M)$.

A bracket f is a Poisson one if the Schouten bracket $[\![f, f]\!]$ is equal to zero.

Definition 4.1. A G-invariant bracket f on M we call a φ-*bracket* if

$$[\![f, f]\!] = -\varphi_M, \tag{4.1}$$

where $\varphi \in \wedge^3 \mathfrak{g}$ is an invariant element.

Proposition 4.1. *Let $\mathcal{A}_h$ be a $U(\mathfrak{g})$-invariant Φ_h-associative quantization with multiplication $\mu_h = m_0 + h\mu_1 + o(h)$, where m_0 is the multiplication in $\mathcal{A}$. Then the map $f : \mathcal{A}^{\otimes 2} \to \mathcal{A}$, $f(a, b) = \mu_1(a, b) - \mu_1(b, a)$, is a φ-bracket for φ from (2.8).*

Proof. A direct computation. □

Corollary 4.1. *Let $\mathcal{A}_h$ be a $U_h(\mathfrak{g})$-invariant associative quantization with multiplication $m_h = m_0 + hm_1 + o(h)$. Then the corresponding Poisson bracket $p(a,b) = m_1(a,b) - m_1(b,a)$ has the form*

$$p(a,b) = f(a,b) - r_M(a,b), \tag{4.2}$$

where r is the r-matrix corresponding to $U_h(\mathfrak{g})$ and f is a φ-bracket with $\varphi = [\![r,r]\!]$.

Proof. Follows from Proposition 4.1 and (3.3). □

Now, let us consider a two parameter $U_h(\mathfrak{g})$-invariant quantization (3.1). With such a quantization one associates two Poisson brackets: the bracket $v(a,b) = m_1'(a,b) - m_1'(b,a)$ along t, and the bracket $p(a,b) = m_1''(a,b) - m_1''(b,a)$ along h. It is easy to check that p and v are compatible Poisson brackets, i.e., their Schouten bracket $[\![p,v]\!]$ is equal to zero. So, we have

Corollary 4.2. *Let $\mathcal{A}_{t,h}$ be a $U_h(\mathfrak{g})$-invariant associative quantization of the form (3.1). Then the Poisson bracket $p(a,b) = m_1''(a,b) - m_1''(b,a)$ has the form*

$$p(a,b) = f(a,b) - r_M(a,b),$$

where r is the r-matrix corresponding to $U_h(\mathfrak{g})$ and f is a φ-bracket with $\varphi = [\![r,r]\!]$. The Poisson bracket $s(a,b) = m_1'(a,b) - m_1'(b,a)$ is $U(\mathfrak{g})$-invariant and compatible with f (and thus with p), i.e., f satisfies (4.1) and the additional condition

$$[\![f,s]\!] = 0. \tag{4.3}$$

Proof. Similar to Corollary (4.1). □

So, studying the problem of $U_h(\mathfrak{g})$-invariant quantizations on a G-manifold M is devided into two parts: 1) finding G-invariant bivector fields, f, on M satisfying (4.1) and (4.3) (if a G-invariant Poisson bracket s is given); 2) proving the existence of quantization of brackets (4.2).

5 Quantization on symmetric spaces and on G

5.1 Quantization on symmetric spaces

Let G be a semisimple connected Lie group. Let us recall that a symmentric space over G is a space of the form $M = G/H$, where H is a subgroup of G such that $G_0^\sigma \subset H \subset G^\sigma$, where G^σ is the set of fixed points of σ, an involutive automorphism of G, and G_0^σ is the identity component of G^σ. The automorphism σ induces an automorphism of the Lie algebra $\mathfrak{g}$ of G that we also denote by σ.

Let r be an r-matrix on $\mathfrak{g}$ and $U_h(\mathfrak{g}, r)$ the corresponding quantum group. In [DS1], the following statement is proven.

Theorem 5.1. *Let M be a symmetric space over a semisimple Lie group G, σ the corresponding involution of $\mathfrak{g}$. Let an r-matrix r on $\mathfrak{g}$ be such that the element $[\![r, r]\!] = \varphi$ is σ-invariant. Then there exists a $U_h(\mathfrak{g}, r)$-invariant quantization on M.*

Remark 5.1. Note that the σ-invariance of φ implies that $\varphi_M = 0$ and r_M is a Poisson bracket. So, in the case of symmetric space M the invariant part f of the corresponding Poisson bracket (4.2) may be taken to be zero.

5.2 Quantization of two-sided action G on itself

Consider the group G as a $G \times G$-manifold with the two-sided action of G on itself. Then, the group G may be considered as a symmetric space, $G = (G \times G)/H$, where H is the diagonal. The action of $G \times G$ on G is: $(g_1, g_2) \triangleright g = g_1 g g_2^{-1}$, $(g_1, g_2) \in G \times G$, $g \in G$. In this case σ on $\mathfrak{g} \oplus \mathfrak{g}$, the Lie algebra of $G \times G$, is the usual permutation: $\sigma(x, y) = (y, x)$.

Let r_1 and r_2 be two r-matrices on $\mathfrak{g}$ such that

$$[\![r_1, r_1]\!] = [\![r_2, r_2]\!] = \varphi. \tag{5.1}$$

The quantum group $U_h(\mathfrak{g}, r_1) \otimes U_h(\mathfrak{g}, r_2)$ may be considered as a quantization of the Lie bialgebra $\mathfrak{g} \oplus \mathfrak{g}$ with r-matrix (r_1, r_2). Applying Theorem 5.1 we obtain

Proposition 5.1. *Let G be a semisimple Lie group. Let r_1 and r_2 be r-matrices on its Lie algebra satisfying (5.1). Then there exists a $U_h(\mathfrak{g}, r_1) \otimes U_h(\mathfrak{g}, r_2)$-invariant quantization of the two-sided action G on G.*

Remark 5.2. It is clear that the presence of a $U_h(\mathfrak{g}, r_1) \otimes U_h(\mathfrak{g}, r_2)$-action is equivalent to the presence of two commuting actions: ρ_1 of $U_h(\mathfrak{g}, r_1) \otimes 1$ and ρ_2 of $1 \otimes U_h(\mathfrak{g}, r_2)$. So, the previous proposition states the existence of the invariant quantization on G with respect to ρ_1 and ρ_2 simultaneously.

Remark 5.3. The element Φ_h from (2.9) associated to $U_h(\mathfrak{g}, r_1)$ and $U_h(\mathfrak{g}, r_2)$ induces in the obvious way the corresponding element $\Phi_h \otimes \Phi_h$ associated to $U_h(\mathfrak{g}, r_1) \otimes U_h(\mathfrak{g}, r_2)$. It follows from (2.10) that the image of $\Phi_h \otimes \Phi_h$ by the $G \times G$-action on G is equal to unity. This implies that the usual multiplication in the function algebra on G is $\Phi_h \otimes \Phi_h$-associative. So, accordingly to Proposition (3.1), the quantization from the conclusion of Proposition 5.1 may be given explicitly as

$$m_h = m \rho_1(F_{h1}^{-1}) \rho_2(F_{h2}^{-1}), \tag{5.2}$$

where F_{h1} and F_{h2} are elements from (2.4) associated to $U_h(\mathfrak{g}, r_1)$ and $U_h(\mathfrak{g}, r_2)$, respectively.

Remark 5.4. It is easy to calculate that, up to a factor, the Poisson bracket on G corresponding to the quantization from Proposition 5.1 has the form

$$r_1^L + r_2^R, \tag{5.3}$$

where r_i^L (r_i^R) is the extension of r_i as a left- (right-) invariant bivector field on G. If we take $r_1 = r$, where r is the Sklyanin-Drinfeld r-matrix (2.3) and $r_2 = -r$, then (5.3) becomes

$$r^L - r^R, \tag{5.4}$$

which is the Sklyanin-Drinfeld Poisson bracket on G.

5.3 Quantization of Ad-action G on itself

Consider the group G as a G-manifold with respect to the Ad-action: $g_1 \triangleright g = g_1 g g_1^{-1}$. The corresponding action $U(\mathfrak{g})$ on $\mathcal{A}(G)$ is obtained by the embedding $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ and the two-sided action $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ on $\mathcal{A}(G)$ considered above. In the quantum case, given a $U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$-invariant quantization, m_h, of $\mathcal{A}(G)$, the corresponding embedding $\Delta_h : U_h(\mathfrak{g}) \to U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$ does not give a $U_h(\mathfrak{g})$-invariant quantization of the function algebra, because this embedding is not a morphism of coalgebras. Nevertheless, the multiplication m_h can be modified to give a quantization of the Ad-action.

Namely, the following lemma holds.

Lemma 5.1. *Let $\mathcal{A}$ be an algebra with $U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$-invariant multiplication m_h. Let the action $U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$ on $\mathcal{A}$ is given by two commutative actions ρ_1 and ρ_2 of $U_h(\mathfrak{g})$. Let the action $U_h(\mathfrak{g})$ on $\mathcal{A}$ is induced by the embedding $\Delta_h : U_h(\mathfrak{g}) \to U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$. Then the multiplication*

$$\mu_h'(a \otimes b) = \mu_h(\rho_2(\mathcal{R}_1)a \otimes \rho_1(\mathcal{R}_2)b) \tag{5.5}$$

is associative and $U_h(\mathfrak{g})$-invariant. Here $\mathcal{R} = \mathcal{R}_1 \otimes \mathcal{R}_2$ is R-matrix of the quantum group $U_h(\mathfrak{g})$.

Proof. A direct computation using relations (2.13). □

To construct Ad-invariant quantizations from two-sided $U_h(\mathfrak{g})$-module algebras one may use the following

Lemma 5.2. *Let $\mathcal{A}$ be an algebra with $U_h(\mathfrak{g})^{cop}$-invariant multiplication m_h, where $U_h(\mathfrak{g})^{cop}$ is $U_h(\mathfrak{g})$ with the opposite comultiplication. Then the multiplication*

$$\mu_h'(a \otimes b) = \mu(\mathcal{R}_1 a \otimes \mathcal{R}_2 b) \tag{5.6}$$

is $U_h(\mathfrak{g})$-invariant.

Proof. The same as of Lemma 5.1. □

Corollary 5.1. *There exists a $U_h(\mathfrak{g})$-invariant quantization of the Ad-action G on G.*

Remark 5.5. Using (5.2) and Lemma 5.1, it is easy to calculate that, up to a factor, the invariant part of the Poisson bracket corresponding to the quantization from the previous corollary is:

$$f(a, b) = (t_1^L a)(t_2^R b) - (t_1^L b)(t_2^R a), \tag{5.7}$$

where $\mathbf{t} = t_1 \otimes t_2$ is the invariant element of $Sym^2\mathfrak{g}$. Here x^L (x^R) is the extension of $x \in \mathfrak{g}$ as a left- (right-) invariant vector field on G.

6 Quantization on $\mathfrak{g}^*$

Let $\mathfrak{g}$ be a complex Lie algebra. Then, the symmetric algebra $S\mathfrak{g}$ can be considered as a function (polynomial) algebra on $\mathfrak{g}^*$. The algebra $U(\mathfrak{g})$ is included in the family of algebras $(S\mathfrak{g})_t = T(\mathfrak{g})[t]/J_t$, where J_t is the ideal generated by the elements of the form $x \otimes y - \sigma(x \otimes y) - t[x, y]$, $x, y \in \mathfrak{g}$, σ is the permutation. By the PBW theorem, $(S\mathfrak{g})_t$ is a free module over $\mathbb{C}[t]$. We have $(S\mathfrak{g})_0 = S\mathfrak{g}$, so this family of quadratic-linear algebras gives a $U(\mathfrak{g})$ invariant quantization of $S\mathfrak{g}$ by the Lie bracket s.

It turns out that for $\mathfrak{g} = sl(n)$ this picture can be extended to the quantum case, [Do2]. Namely,

Theorem 6.1. *Let $\mathfrak{g} = sl(n)$. Then, there exist deformations, σ_h and $[\cdot,\cdot]_h$, of both the mappings σ and $[\cdot,\cdot]$ such that the two parameter family of algebras $(S\mathfrak{g})_{t,h} = T(\mathfrak{g})[[h]][t]/J_{t,h}$, where $J_{t,h}$ is the ideal generated by elements of the form $x \otimes y - \sigma_h(x \otimes y) - t[x, y]_h$, $x, y \in \mathfrak{g}$, gives a $U_h(\mathfrak{g})$-invariant quantization of the Lie bracket s on $\mathfrak{g}^*$.*

Using Corollary 4.2, let us describe the Poisson brackets corresponding to $(S\mathfrak{g})_{t,s}$.

Proposition 6.1. *The pair of brackets corresponding to the quantization $(S\mathfrak{g})_{t,h}$ consists of two compatible Poisson brackets:*

s (along t) is the Lie bracket;

p (along h) is a quadratic Poisson bracket of the form $p = f - \{\cdot,\cdot\}_r$, where f is the invariant quadratic bracket which is defined by a unique up to a factor invariant map

$$f : \wedge^2\mathfrak{g} \to S^2\mathfrak{g}, \tag{6.1}$$

and $\{\cdot,\cdot\}_r$ is the r-matrix bracket. Moreover, $[\![s, f]\!] = 0$ and $[\![f, f]\!] = -\overline{\varphi}$, where $\overline{\varphi}$ has the form $\overline{\varphi}(a, b, c) = [\varphi_1, a][\varphi_2, b][\varphi_3, c]$, and $\varphi = \varphi_1 \wedge \varphi_2 \wedge \varphi_3 = [\![r, r]\!]$. Recall that φ is a unique up to a factor invariant element of $\wedge^3\mathfrak{g}$.

Note that the bracket f from the above proposition can be restricted to any orbit of $SL(n)$ in $sl(n)^*$. This follows from the following general statement, [Do1].

Proposition 6.2. *Let G be a semisimple Lie group with its Lie algebra $\mathfrak{g}$, $s = [\cdot,\cdot]$ the Poisson-Lie bracket on $\mathfrak{g}^*$. Let $f = \{\cdot,\cdot\}$ be an invariant bracket on an open set of $\mathfrak{g}^*$ such that the Schouten bracket $[\![s, f]\!]$ is a three-vector field, ψ, which can be restricted to an orbit $\mathcal{O}$ of G. Then f can be restricted to $\mathcal{O}$.*

Note that an invariant bivector field defined on a part of an orbit is in fact defined on the whole orbit.

Remark 6.1. Using quantized Verma modules, one can prove that the family $(S\mathfrak{g})_{t,h}$ can be restricted to any semisimple orbit to give a $U_h(\mathfrak{g})$-invariant quantization of the Kirillov bracket on it.

Remark 6.2. One can prove, [Do1], that for a simple $\mathfrak{g} \neq sl(n)$ there are no φ-brackets on $\mathfrak{g}^*$ compatible with the Poisson-Lie bracket. Hence, in this case a two parameter $U_h(\mathfrak{g})$-invariant quantization on $\mathfrak{g}^*$ does not exist.

It seems that for a simple $\mathfrak{g} \neq sl(n)$ there exist no φ-brackets on $\mathfrak{g}^*$, which implies that on $\mathfrak{g}^*$ even a one parameter $U_h(\mathfrak{g})$-invariant quantization does not exist. This statement could be easily derived from Conjecture 7.1 at the end of the paper.

Remark 6.3. Note that on a neighborhood of zero in $\mathfrak{g}$ a φ-bracket exists. Indeed, due to the map $\exp : \mathfrak{g} \to G$ the bracket (5.7) may be carried over from G to a neighborhood of zero in $\mathfrak{g}$.

Note also that the bracket of the form (5.7) is correctly defined on $gl(n)$ (considered as an associative algebra) and being restricted to $sl(n)$ gives a quadratic bracket which coincides with (6.1).

Remark 6.4. It follows from the previous remark that for any simple $\mathfrak{g}$, a φ-bracket exists on any orbit in $\mathfrak{g}^*$ (not necessarily semisimple). Indeed, multiplying a given orbit by a constant one can suppose that it goes through the neighborhood of zero where the bracket (5.7) is defined. Now, the statement follows from Proposition 6.2.

Since bracket (6.1) is compatible with the Poisson-Lie bracket, it follows that for $\mathfrak{g} = sl(n)$ on any orbit in $\mathfrak{g}^*$ there exists a φ-bracket compatible with the Kirillov bracket.

As we will see in Subsection 7.2, in case $\mathfrak{g} \neq sl(n)$ not all orbits (even semisimple ones) admit φ-brackets compatible with the Kirillov ones.

7 Quantization on semisimple orbits

Let G be a complex connected simple Lie group with the Lie algebra $\mathfrak{g}$. Let $\mathfrak{l}$ be a Levi subalgebra of $\mathfrak{g}$, the Levi factor of a parabolic subalgebra. Let L be a Lie subgroup of G with Lie algebra $\mathfrak{l}$. Such a subgroup is called a Levi subgroup. It is known that L is a closed connected subgroup. Denote $M = G/L$ and let $o \in M$ be the image of the unity by the natural projection $G \to M$. Then L is the stabilizer of o. It is known, that M may be realized as a semisimple orbit of G in the coadjoint representation $\mathfrak{g}^*$. Conversely, any semisimple orbit in $\mathfrak{g}^*$ is a quotient of G by a Levi subgroup. We call *the rank* of M the dimension of the center of $\mathfrak{l}$. So, if M is a maximal orbit, i.e., $\mathfrak{l}$ is equal to the Cartan subalgebra, then the rank of M is equal to the rank of $\mathfrak{g}$.

7.1 One parameter quantization

In [Do3] the following statement is proven.

Theorem 7.1. *Let G be a simple Lie group, $\mathfrak{g}$ its Lie algebra, M a semisimple orbit in $\mathfrak{g}^*$. Then*

a) The set of all φ-brackets on M form an affine nonsingular algebraic manifold, X_M, of dimension $rank(M)$.

b) There exists an analytic universal $U_h(\mathfrak{g})$-invariant family of multiplications $\mu_{f,h}$ on $\mathcal{A}(M)$ of the form

$$\mu_{f,h}(a,b) = ab + (h/2)(f(a,b) - r_M(a,b)) + \sum_{n\geq 2} h^n \mu_{f,n}(a,b), \quad f \in X. \tag{7.1}$$

The universality means that for any $U_h(\mathfrak{g})$-invariant multiplication, m_h, there exists a formal path in X_M, $\psi(h)$, such that m_h is equivalent to $\mu_{\psi(h),h}$, and multiplications corresponding to different paths are not equivalent.

Remark 7.1. Let us consider M as an abstract manifold. Then all nondegenerate G-invariant Poisson brackets on it appear as restrictions of the Poisson-Lie bracket by embeddings M into $\mathfrak{g}^*$ as orbits. In the quantum case, when $\mathfrak{g} = sl(n)$, a dense open set of φ-brackets in X_M can be obtained in the same way, as restrictions of the quadratic bracket (6.1). But in general there are φ-brackets on M which can not be obtained in this way. For example, the Sklyanin-Drinfeld bracket (5.4) can be carried over from G to M considered as a quotient of G, but the obtained bracket can not be realized by embedding M into $\mathfrak{g}^*$ as an orbit.

Note that also for $\mathfrak{g} \neq sl(n)$ the essential part of φ-brackets on M may be obtained with the help of embeddings M into $\mathfrak{g}^*$ as restrictions of the bracket (5.7) carried over to an open set of $\mathfrak{g}^*$ (see Remark 6.4).

7.2 Two parameter quantization

It turns out that φ-brackets compatible with the Kirillov bracket exist not on any orbits in $\mathfrak{g}^*$.

Definition 7.1. Let M be an orbit in $\mathfrak{g}^*$ (not necessarily semisimple). We call M a *good* orbit, if there exists a φ-bracket on it compatible with the Kirillov bracket (we call such a bracket a *good* bracket).

The following proposition, [DGS], gives a classification of good semisimple orbits.

Proposition 7.1. *a) For $\mathfrak{g}$ of type A_n all semisimple orbits are good.*

b) For all other $\mathfrak{g}$, a semisimple orbit M is good if and only if $M = G/L$. Here L is a Levi subgroup whose Lie algebra is generated by the Cartan subalgebra and root vectors $X_{\pm\alpha}$, where α runs all simple roots away of one or two roots which appear in the representation of the maximal root with coefficient 1.

c) The good brackets f on a good orbit, M, form a one-dimensional variety isomorphic to $\mathbb{C}$: all such brackets have the form

$$\pm f_0 + c \cdot s_M,$$

where s_M is the Kirillov bracket on M, $c \in \mathbb{C}$, and f_0 is a fixed good bracket.

So, if $\mathfrak{g} \neq sl(n)$, the good orbits are either Hermitian symmetric spaces or, for $\mathfrak{g}$ of types D_n and E_6, bisymmetric spaces.

It follows from Proposition 7.1 c) that for a good orbit the family

$$h(f_0 - r_M) + ts_M, \tag{7.2}$$

is uniquely defined up to a linear change of parameters (h, t).

Proposition 7.2. *Let M be a good semisimple orbit. Then the family (7.2) can be quantized, i.e. there exists a two parameter $U_h(\mathfrak{g})$-invariant multiplication in $\mathcal{A}(M)$ of the form*

$$m_{t,h}(a,b) = ab + \frac{1}{2}(h(f_0 - r_M) + ts_M) + \sum_{k+l\geq 2} h^k t^l m_{k,l}(a,b).$$

The theorem is proven in [Do3].

7.3 Problems

An open question is to investigate the problem of one and two parameter quantization for non-semisimple orbits, in particular, for nilpotent ones. In this connection, I would like to mention the following problems which seem to be purely from the theory of (simple) Lie algebras:

1. Give a classification of good orbits in case $\mathfrak{g} \neq sl(n)$ (see Remark 6.4).

2. Give a classification of φ- and good brackets for non-semisimple orbits (see the same Remark).

3. Prove the following

Conjecture 7.1. Let $\mathfrak{g}$ be a simple Lie algebra. Then all invariant Poisson brackets on the polynomial algebra on $\mathfrak{g}^*$, $S\mathfrak{g}$, have the form $b[\cdot,\cdot]$, where $[\cdot,\cdot]$ is the Poisson-Lie bracket and b an invariant element of $S\mathfrak{g}$.

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