ON TWISTING OPERATORS AND
NEWFORMS OF HALF-INTEGRAL WEIGHT II
- COMPLETE THEORY OF
NEWFORMS FOR KOHNEN SPACE

Masaru Ueda

Department of Mathematics
Faculty of Science
Nara - Women's University
630 Nara

Japan

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
53225 Bonn

Germany

In this paper, the author formulates and proves a complete theory of newforms for Kohnen space. The newforms satisfy the Strong multiplicity One theorem. The space of oldforms is explicitly described. Kohnen space is a canonical subspace of the space of cusp forms of half-integral weight which corresponds to primitive forms of integral weight and of odd level via Shimura Correspondence.
Masaru Ueda (Dept. of Math., Nara Women's University, Nara 630 Japan)

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MASARU UEDA

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Introduction

The theory of newforms is very important and useful for arithmetical study of modular forms of integral weight. This theory have the following nice properties:

(i) The space of newforms have an orthogonal C-basis consisting of common eigenforms on all Hecke operators and such common eigenforms are uniquely determined up to multiplication of complex numbers. Moreover such common eigenforms satisfy *the Strong Multiplicity One Theorem* (cf. [M, §4.6]).

(ii) The full space of cusp forms, $S(2k, N)$, can be reconstructed by the space of newforms, i. e., we have the following decomposition:

$$S(2k, N) = \bigoplus_{0 < B | N} \bigoplus_{0 < A | (N/B)} S^0(2k, B) \mid [\left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix} \right)]_{2k} ,$$

where $S^0(2k, B)$ is the space of newforms (see below §0 and [M, §4.6]).

(iii) The above operator $f \to f \mid [\left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix} \right)]_{2k}$ (almost) preserve the Fourier coefficients of cusp form. Hence for studying the Fourier coefficients of cusp forms, it is sufficient to study cusp forms only in the space of newforms.

(iv) The theory of newforms has tight relations to both Representation theory and Geometry.

Until now, several authors have attempted to find a similar theory of newforms of half-integral weight which satisfy similar properties like the above (i)–(iv) (cf. [She], [N], [K], [M-R-V], [U1], [She-W]).

In the paper [K], W. Kohnen defined (what is called) Kohnen space which can be considered as the canonical subspace corresponding to cusp forms of integral weight and of odd level via Shimura correspondence. And when the level is a $4 \times$ (odd *squarefree* integer), he also established a theory of newforms for this Kohnen space.

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In the previous paper [U1], the author generalized Kohnen's work and obtained a similar theory of newforms for Kohnen space of arbitrary level ($= 4 \times$ (odd integer), cf. [U1, §3]). But those results are half-way for a technical reason (cf. [U1, §4]).

The aim of this paper is to complete the results in the previous paper and to formulate and state a complete theory of newforms of Kohnen space.

Let us precisely state a formulation of theory of newforms for Kohnen space.

Let k, $N \in \mathbf{Z}_+$ and N divisible by 4. Let χ be an even character modulo N with $\chi^2 = 1$. We denote the p-adic additive valuation for any integer m by $\mathrm{ord}_p(m)$. We decompose N as follows:

$$N = 2^{\mathrm{ord}_2(N)} M_1 M_{2+}, \quad M_1 := \prod_{\substack{p|N, p \neq 2 \\ \mathrm{ord}_p(N)=1}} p, \quad M_{2+} := \prod_{\substack{p|N, p \neq 2 \\ \mathrm{ord}_p(N) \geq 2}} p^{\mathrm{ord}_p(N)} .$$

Denote the set of all prime divisors of M_{2+} by Π.

Kohnen spaces $S(k + 1/2, N, \chi)_K$ can be defined only in the case of $\mathrm{ord}_2(N) = 2$ (See below §0 (d) for the definition).

We shall consider such a case. For simplifying the explanation, we deal with only the case of $k \geq 2$. In the case of $k = 1$, we must slightly modify subspaces (cf. §0).

Define the space of oldform $\mathfrak{O}(k + 1/2, N, \chi)_K$ as follows.

$$\mathfrak{O}(k + 1/2, N, \chi)_K = \sum_{\substack{0 < B|M \\ B \neq M}} \sum_{\substack{0 < A|(M/B) \\ \xi(\frac{A}{\cdot})=\chi}} S(k + 1/2, 4B, \xi)_K \,|\tilde{\delta}_A$$

$$+ \sum_{\substack{0 < B|M \\ B \neq M}} \sum_{\substack{0 < A|(M/B)^2 \\ \xi(\frac{A}{\cdot})=\chi}} \sum_{\substack{0 \leq e_l \leq 2 \\ (l \in \Pi)}} S(k + 1/2, 4B, \xi)_K \,|U(A) \prod_{l \in \Pi} R_l{}^{e_l} .$$

Here, ξ runs over all characters modulo $4B$ such that $\xi\left(\frac{A}{\cdot}\right) = \chi$. The operator $\tilde{\delta}_A$, the shift operator $U(A)$, and the twisting operator R_l ($l \in \Pi$) are defined as follows: For $f = \sum_{n \geq 1} a(n)\mathrm{e}(nz)$,

$$f|\tilde{\delta}_A(z) := A^{k/2+1/4} f(Az) = A^{k/2+1/4} \sum_{n \geq 1} a(n)\mathrm{e}(Anz) ,$$

$$f|U(A)(z) := \sum_{n \geq 1} a(An)\mathrm{e}(nz) , \quad f|R_l(z) := \sum_{n \geq 1} a(n) \left(\frac{n}{l}\right) \mathrm{e}(nz) .$$

The space $\mathfrak{O}(k + 1/2, N, \chi)_K$ is a subspace of $S(k + 1/2, N, \chi)_K$. We denote by $\mathfrak{N}(k + 1/2, N, \chi)_K$ the orthogonal complement of $\mathfrak{O}(k + 1/2, N, \chi)_K$ in $S(k + 1/2, N, \chi)_K$.

The space $\mathfrak{N}(k + 1/2, N, \chi)_K$ is stable by the twisting operators R_p for all $p \in \Pi$. Hence we can decompose this space into common eigen subspaces as follows:

$$\mathfrak{N}(k + 1/2, N, \chi)_K = \oplus_{\kappa \in \{\pm 1\}^\Pi} \mathfrak{N}^{\emptyset, \kappa}(k + 1/2, N, \chi)_K ,$$

$$\mathfrak{N}^{\emptyset, \kappa}(k + 1/2, N, \chi)_K := \{f \in \mathfrak{N}(k + 1/2, N, \chi)_K \;;\; f|R_p = \kappa(p)f \text{ for all } p \in \Pi \} .$$

Here $\{\pm 1\}^{\Pi} := \mathrm{Map}(\Pi, \{\pm 1\})$.

We call these spaces $\mathfrak{N}^{\emptyset,\kappa}(k + 1/2, N, \chi)_K$ ($\kappa \in \{\pm 1\}^{\Pi}$) the spaces of newforms, because these subspaces have the following nice properties (cf. below §3 and especially (3.7)).

(1) $\mathfrak{N}^{\emptyset,\kappa}(k + 1/2, N, \chi)_K$ has an orthogonal **C**-basis consisting of common eigenforms for all Hecke operators $\tilde{T}_{k+1/2,N,\chi}(p^2)$ (p:prime, $p{\not|}M$) and $U(p^2)$ (p:prime, $p \mid M$), which are uniquely determined up to multiplication with non-zero complex numbers. Let f be such a common eigenform and λ_p the eigenvalue of f with respect to $\tilde{T}_{k+1/2,N,\chi}(p^2)$ ($p{\not|}M$) resp. $U(p^2)$ ($p|M$). Then there exist a primitive form $F \in S^0(2k, M)$ of weight $2k$ and of conductor M which is uniquely determined and satisfies the following: For a prime p,

$$F|T_{2k,M}(p) = \lambda_p F \quad \text{if } (p, M) = 1 \quad \text{and} \quad F|U(p) = \lambda_p F \quad \text{if } p|M.$$

Furthermore we can find, by using the trace relation (2.28)(1), which primitive form occurs via the above correspondence. □

(2) **(The Strong Multiplicity One Theorem)**
Let f, g be two non-zero elements of $\mathfrak{N}^{\emptyset,\kappa}(k+1/2, N, \chi)_K$. If f and g are common eigenforms of $\tilde{T}_{k+1/2,N,\chi}(p^2)$ with the same eigenvalue for all prime numbers p prime to some integer A, then $\mathbf{C}f = \mathbf{C}g$. □

(3) The space of oldforms $\mathfrak{O}(k+1/2, N, \chi)_K$ has also an orthogonal **C**-basis consisting of common eigenforms for all operators $\tilde{T}_{k+1/2,N,\chi}(p^2)$ (p:prime, $p{\not|}N$). The system of eigenvalues of such a common eigenform corresponds to a primitive form of weight $2k$ whose conductor is a divisor of M and is less than M (cf. (3.5)). □

(4) The space of oldform $\mathfrak{O}(k + 1/2, N, \chi)_K$ is generated by the spaces of cusp forms of lower level. Hence, by induction, we see that the spaces $S(k + 1/2, N, \chi)_K$ are reconstructed by the spaces of type of $\mathfrak{N}^{\emptyset,\kappa}(k + 1/2, 4B, \xi)_K$ and the operators of type of $\tilde{\delta}_A$, $U(A)$, and R_l.

From the above definition these operators $\tilde{\delta}_A$, $U(A)$, and R_l (almost) preserve Fourier coefficients of cusp forms. Hence for studying Fourier coefficients of cusp forms $\in S(k + 1/2, N, \chi)_K$, it is sufficient to study cusp forms only in the spaces of newforms $\mathfrak{N}^{\emptyset,\kappa}(k + 1/2, N, \chi)_K$. □

Finally the author has some comments.

There exists a case such that $\mathfrak{N}^{\emptyset,\kappa}(k + 1/2, N, \chi)_K \cong \mathfrak{N}^{\emptyset,\kappa'}(k + 1/2, N, \chi)_K$ as modules over Hecke algebra for two distinct κ, $\kappa' \in \{\pm 1\}^{\Pi}$. See (3.8) for such a example.

It seems likely that there exists a similar theory for any full spaces of cusp forms $S(k + 1/2, N, \chi)$ even if $\mathrm{ord}_2(N) \geq 3$.

In the case of $\mathrm{ord}_2(N) \leq 3$, the author thinks that necessary preparations have already done in the author's previous papers [U3-5]. But situations are quite different in the case of $\mathrm{ord}_2(N) \geq 4$ (cf. [U5]).

It seems that its reason is the existence of the twisting operators for characters $\left(\frac{-1}{\cdot}\right)$ and $\left(\frac{2}{\cdot}\right)$. These twisting operators can be defined only if $\mathrm{ord}_2(N) \geq 4$ and $\mathrm{ord}_2(N) \geq 6$, respectively. See the forthcoming papers [U6].

This paper is composed as follows: §0 is general preliminaries. §1 is preparation from representation theory of finite groups. We apply results of §1 to calculations of §2. In §2, we will complete an attempt in the previous paper [U1,§4]. §3 is the main part of this paper. We will formulate and state a complete theory of newforms for Kohnen space in §3. In the Appendix, we will prove several general formulae which are used in calculations of §2.

The author wrote this paper during he was staying at Max-Planck-Institut. The author would like to express his hearty thanks to Max-Planck-Institut and its staff for their warm hospitality.

§0. Notational Preliminary.
Throughout this paper, we use the following notations.

(a) **General notations.** Let A, B be subsets of a set X and $\{A_i\}_{i \in I}$ a family of subsets of X. If $A \cup B$ is a disjoint union, then we denote $A + B := A \cup B$ for simplicity. Similarly, if $\cup_{i \in I} A_i$ is a disjoint union, then we denote $\sum_{i \in I} A_i := \cup_{i \in I} A_i$.

We denote the set of positive integers by $\mathbf{Z}_+$. We denote the additive valuation for any integer m by $\mathrm{ord}_p(m)$.

See [M,p.82] for the definition of the Kronecker symbol $\left(\frac{a}{b}\right)$ (a, b integers with $(a, b) \neq (0, 0)$).

Let N be a positive integer and m an integer $\neq 0$. We write $m | N^\infty$ if every prime factor of m divides N.

Let k denote a non-negative integer. If $z \in \mathbf{C}$ and $x \in \mathbf{C}$, we put $z^x = \exp(x \cdot \log(z))$ with $\log(z) = \log(|z|) + \sqrt{-1} \arg(z)$, $\arg(z)$ being determined by $-\pi < \arg(z) \leq \pi$. Also we put $\mathrm{e}(z) = \exp(2\pi\sqrt{-1}z)$.

Let $\mathfrak{H}$ be the complex upper half plane. For a complex-valued function $f(z)$ on $\mathfrak{H}$, $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbf{R})$, $\gamma = \begin{pmatrix} u & v \\ w & x \end{pmatrix} \in \Gamma_0(4)$ and $z \in \mathfrak{H}$, we define functions $J(\alpha, z)$, $j(\gamma, z)$ and $f|[\alpha]_k(z)$ on $\mathfrak{H}$ by: $J(\alpha, z) = cz + d$, $j(\gamma, z) = \left(\frac{-1}{x}\right)^{-1/2} \left(\frac{w}{x}\right) (wz + x)^{1/2}$ and $f|[\alpha]_k(z) = (\det \alpha)^{k/2} J(\alpha, z)^{-k} f(\alpha z)$.

For $m \in \mathbf{Z}_+$ we define a shift operator $U(m)$ on formal power series in $\mathrm{e}(z)$ by

$$\sum_{n \geq 0} a(n)\mathrm{e}(nz) | U(m) := \sum_{n \geq 0} a(mn)\mathrm{e}(nz) \, .$$

Let χ be a Dirichlet character modulo N. We denote the conductor of χ by $\mathfrak{f}(\chi)$ and the p-primary component of χ by χ_p for each prime divisor p of N.

Let V, V' be finite-dimensional vector spaces over $\mathbf{C}$. We denote the trace of a linear operator T on V by $\mathrm{tr}(T; V)$ and also the kernel of a linear map F from V to V' by $\mathrm{Ker}(F; V)$.

We denote the set of all mapping from a set A to a set B by $\mathrm{Map}(A, B)$. Furthermore we use the abbreviated notation B^A ($= \mathrm{Map}(A, B)$).

Let A be a set of prime numbers and $(\alpha_p)_{p \in A}$ a system of integers. We put the following notation: $A(\alpha)_i := \{p \in A \mid \alpha_p = i\}$ and $A(\alpha)_{i+} := \{p \in A \mid \alpha_p \geq i\}$ for any $i \in \mathbf{Z}$.

(b) **Modular forms of integral weight.**

Let k and N be positive integers. By $S(2k, N)$, we denote the space of all holomorphic cusp forms of weight $2k$ with the trivial character on the group $\Gamma = \Gamma_0(N)$. We also denote the subspace of $S(2k, N)$ spanned by all newforms in $S(2k, N)$ by $S^0(2k, N)$.

Let $\alpha \in GL_2^+(\mathbf{R})$. If Γ and $\alpha^{-1}\Gamma\alpha$ are commensurable, we define a linear operator $[\Gamma\alpha\Gamma]_{2k}$ on $S(2k, N)$ by: $f|[\Gamma\alpha\Gamma]_{2k} = (\det \alpha)^{k-1} \sum_{\alpha_i} f|[\alpha_i]_{2k}$, where α_i runs over a system of representatives for $\Gamma \backslash \Gamma\alpha\Gamma$. For a positive integer n with $(n, N) = 1$, we put $T_{2k,N}(n) = \sum_{ad=n} \left[\Gamma \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \Gamma\right]_{2k}$, where the sum is extended over all pairs of integers (a, d) such that $a, d > 0$, $a|d$, $ad = n$.

Let Q be a positive divisor of N such that $(Q, N/Q) = 1$. Take any element $\gamma_Q \in SL_2(\mathbf{Z})$ which satisfies the conditions:

$$\gamma_Q \equiv \begin{cases} \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) & (\text{mod } Q) \, ; \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & (\text{mod } N/Q) \, . \end{cases}$$

Put $W(Q) = \gamma_Q \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix}$. The following facts are well-known: $W(Q)$ is a normalizer of Γ; $[W(Q)]_{2k}$ induces a C-linear automorphism of order 2 on $S(2k, N)$ and this operator is independent of a choice of an element γ_Q. For $Q = 1$, we can take $\gamma_1 = W(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence we have $[W(1)]_{2k} = 1$. Moreover for the sake of simplicity, we use the following abbreviated notation: Let A be a subset of the set of all prime divisors of N. Then $W_A := W\left(\prod_{p \in A} p^{\text{ord}_p(N)}\right)$. In particular, we simply write $W_l = W_A$ if $A = \{l\}$.

Moreover, if the subscripts are obvious and any confusion does not occur, we simply write $T(n) = T_{2k,N}(n)$ and $W(Q) = [W(Q)]_{2k}$, etc..

For any $f(z) = \sum_{n=1}^{\infty} a(n)e(nz) \in S(2k, N)$ and χ a primitive character modulo $\mathfrak{f} = \mathfrak{f}(\chi)$, put $f|R_\chi(z) := \sum_{n=1}^{\infty} \chi(n)a(n)e(nz)$. From [Sh 3,Prop. 3.64] we have $f|R_\chi \in S(2k, N', \chi^2)$, where N' is the least common multiple of N and $\mathfrak{f}(\chi)^2$. We call this operator R_χ the twisting operator of χ.

(c) Modular forms of half-integral weight.

Let k denote a non-negative integer, N a positive integer divisible by 4, and χ an even character modulo N such that $\chi^2 = 1$. Put $\mu = \text{ord}_2(N)$, $M = 2^{-\mu}N$ and $\Gamma_0 = \Gamma_0(N)$. Then there is a square-free odd positive divisor M_0 of M such that $\chi = \left(\frac{M_0}{\cdot}\right)$ or $\left(\frac{2M_0}{\cdot}\right)$ (the Kronecker symbol).

Let $\mathfrak{G}(k + 1/2)$ be the group consisting of all pairs (α, φ), where $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbf{R})$ and φ is a holomorphic function on $\mathfrak{H}$ satisfying $\varphi(z) = t(\det \alpha)^{-k/2-1/4} J(\alpha, z)^{k+1/2}$ with $t \in \mathbf{C}$ and $|t| = 1$. The group law is defined by: $(\alpha, \varphi(z)) \cdot (\beta, \psi(z)) = (\alpha\beta, \varphi(\beta z)\psi(z))$. For a complex-valued function f on $\mathfrak{H}$ and $(\alpha, \varphi) \in \mathfrak{G}(k + 1/2)$, we define a function $f|(\alpha, \varphi)$ on $\mathfrak{H}$ by: $f|(\alpha, \varphi)(z) = \varphi(z)^{-1}f(\alpha z)$. Moreover if there will be no confusion, we also write $\gamma^* = (\gamma, j(\gamma, z)^{2k+1})$ for all $\gamma \in \Gamma_0(4)$.

By $\Delta_0 = \Delta_0(N, \chi) = \Delta_0(N, \chi)_{k+1/2}$, we denote the subgroup of $\mathfrak{G}(k + 1/2)$ consisting of all pairs (γ, φ), where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma \in \Gamma_0$ and $\varphi(z) = \chi(d)j(\gamma, z)^{2k+1}$ and also denote $\Delta_1 = \Delta_1(N) := \{\gamma^* \mid \gamma \in \Gamma_1(N)\}$.

We denote by $G(k+1/2, N, \chi)$ (resp. $S(k+1/2, N, \chi)$) the space of integral (resp. cusp) forms of weight $k+1/2$ with the character χ on the group Γ_0, namely, the space of all the complex-valued holomorphic functions f on $\mathfrak{H}$ which satisfy $f|\xi = f$ for all $\xi \in \Delta_0$ and which are holomorphic (resp. are holomorphic and vanish) at all cusps of Γ_0. Moreover we also denote by $S(k+1/2, \Delta_1(N))$ the space of cusp forms of weight $k+1/2$ on the group $\Gamma_1(N)$ i.e., the space of all the complex-valued holomorphic functions f on $\mathfrak{H}$ which satisfy $f|\xi = f$ for all $\xi \in \Delta_1$ and which are holomorphic and vanish at all cusps of $\Gamma_1(N)$ ([cf. Sh 1]).

In the case $k = 1$, i.e., the case of weight $3/2$, $S(3/2, N, \chi)$ contains theta series of special type. We know that these theta series correspond to Eisenstein series via Shimura correspondence.

From this reason we define the orthogonal complement $V(N; \chi)$ of the space of such theta series in $S(3/2, N, \chi)$. In the case of weight $3/2$, we deal only with this complement $V(N; \chi)$. See [U1, §0, §1] for the details.

Let $\xi \in \mathfrak{G}(k+1/2)$. If Δ_0 and $\xi^{-1}\Delta_0\xi$ are commensurable, we define a linear operator $[\Delta_0\xi\Delta_0]_{k+1/2}$ on $G(k+1/2, N, \chi)$ and $S(k+1/2, N, \chi)$ by: $f|[\Delta_0\xi\Delta_0]_{k+1/2} = \sum_\eta f|\eta$, where η runs over a system of representatives for $\Delta_0\backslash\Delta_0\xi\Delta_0$. Similarly, if ξ and $\xi^{-1}\Delta_1\xi$ are commensurable, we define a linear operator $[\Delta_1\xi\Delta_1]$ on $S(k+1/2, \Delta_1(N))$ by: $f|[\Delta_1\xi\Delta_1] = \sum_{\eta\in\Delta_1\backslash\Delta_1\xi\Delta_1} f|\eta$.

Then for a positive integer n with $(n, N) = 1$, we put

$$\tilde{T}_{k+1/2,N,\chi}(n^2) = n^{k-3/2} \sum_{ad=n} a\left[\Delta_0\left(\begin{pmatrix} a^2 & 0 \\ 0 & d^2 \end{pmatrix}, (d/a)^{k+1/2}\right)\Delta_0\right]_{k+1/2},$$

where the sum is extended over all pairs of integers (a, d) such that $a, d > 0$, $a|d$ and $ad = n$. We simply write $\tilde{T}(n^2) := \tilde{T}_{k+1/2,N,\chi}(n^2)$ if the subscripts are obvious and any confusion does not occur. These operators $\tilde{T}(n^2)$ $((n, N) = 1)$ are hermitian and commutative with each other on $S(k+1/2, N, \chi)$ (cf. [Sh 2, lemma 5], [Sh 3, Prop.(3.32)], [U1,(1.9)]).

For any $m \in \mathbf{Z}_+$, put $\tilde{\delta}_m := \left(\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}, m^{-k/2-1/4}\right)$.

Let Q be an odd positive divisor of N such that $(Q, N/Q) = 1$. Take any element $\gamma_Q \in SL_2(\mathbf{Z})$ satisfying the conditions:

$$\gamma_Q \equiv \begin{cases} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \pmod{Q}; \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N/Q}. \end{cases}$$

Then $\gamma_Q \in \Gamma_0(N/Q) \subseteq \Gamma_0(4)$. Put $\widetilde{W}(Q) := \gamma_Q{}^*\tilde{\delta}_Q \in \mathfrak{G}(k+1/2)$. See [U1,§1] for the details of properties of these $\tilde{\delta}_m$ and $\widetilde{W}(Q)$.

Let $f(z) = \sum_{n=0}^\infty a(n)e(nz) \in G(k+1/2, N, \chi)$ and ψ a primitive character modulo $\mathfrak{f}(\psi)$. Let N' be the least common multiple of N, $\mathfrak{f}(\psi)^2$, and $\mathfrak{f}(\psi)\mathfrak{f}(\chi)$. Then $f|R_\psi(z) := \sum_{n=0}^\infty \psi(n)a(n)e(nz)$ belongs to the space $G(k+1/2, N', \chi\psi^2)$. In particular, if f is a cusp form, so is $f|R_\psi$ [Sh 1, Lemma 3.6]. We call this operator R_ψ the twisting operator of ψ.

(d) Kohnen space. We keep to the notations in the subsection (c).

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Let k be a positive integer. Suppose that $N = 4M$ and M is an odd natural number. We define the Kohnen space $S(k + 1/2, N, \chi)_K$ as follows:

$$S\left(k + \frac{1}{2}, N, \chi\right)_K = \left\{ \begin{array}{l} S\left(k + \frac{1}{2}, N, \chi\right) \ni f(z) = \sum_{n=1}^{\infty} a(n)\mathbf{e}(nz) \ ; \\ a(n) = 0 \ \text{for} \ \chi_2(-1)(-1)^k n \equiv 2, 3 \pmod 4 \end{array} \right\} ,$$

where χ_2 is the 2-primary component of χ.

In the case of weight 3/2, we define $V(N; \chi)_K := V(N; \chi) \cap S(3/2, N, \chi)_K$. See [U1, §0, §1] for the details.

§1 Representations of SL_2 over finite fields and non-vanishing of Fourier coefficients.

(1.1) We begin with a summary of representations of $\mathbf{G} := SL_2(\mathbf{F}_p)$ (p is an odd prime number).

Define subgroups of $\mathbf{G}$ by:

$$\mathbf{B} := \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbf{F}_p^{\times}, \ b \in \mathbf{F}_p \right\} , \quad \mathbf{U} := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbf{F}_p \right\}$$

and put

$$\psi : \mathbf{F}_p \ni u \bmod p \mapsto \mathbf{e}(u/p) := \exp(2\pi\sqrt{-1}u/p) \in \mathbf{C}^{\times} ,$$

$$\psi_a\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) := \psi(au) \quad (a, u \in \mathbf{F}_p).$$

The set of all irreducible representations of $\mathbf{U}$ is given by $\{\psi_a \mid a \in \mathbf{F}_p\}$. For any Dirichlet character χ modulo p, we define the representations $\underline{\underline{\chi}}$ of $\mathbf{B}$ of degree one by: $\underline{\underline{\chi}} : \mathbf{B} \ni \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mapsto \chi(a) \in \mathbf{C}^{\times}$.

The following facts are well-known.

(1.2) Proposition ([S, Chapter 7, pp. 54–60]).

(1) If $\chi^2 \neq 1$, $\mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}} \underline{\underline{\chi}}$ is an irreducible representation.

(2) If $\chi = 1$ (the trivial representation), $\mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}} 1 = 1 \oplus \mathfrak{C}_p$, where $\mathfrak{C}_p$ is an irreducible representation of $\mathbf{G}$ of degree p which is called Steinberg representation and $\mathrm{Res}_{\mathbf{U}} \mathfrak{C}_p \cong \oplus_{a \in \mathbf{F}_p} \psi_a$.

(3) If $\chi = \left(\frac{\cdot}{p}\right)$ (Legendre symbol), $\mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}} \underline{\underline{\chi}} = \mathfrak{C}_{(p+1)/2} \oplus \mathfrak{C}'_{(p+1)/2}$, where $\mathfrak{C}_{(p+1)/2}$ and $\mathfrak{C}'_{(p+1)/2}$ denote irreducible representations of $\mathbf{G}$ of degree $(p+1)/2$, which are not equivalent to each other and satisfy the following:

$$\mathrm{Res}_{\mathbf{U}}\left(\mathfrak{C}_{(p+1)/2}\right) \cong \psi_0 \oplus \left(\oplus_{a \in \mathbf{F}_p^{\times 2}} \psi_a\right) , \quad \mathrm{Res}_{\mathbf{U}}\left(\mathfrak{C}'_{(p+1)/2}\right) \cong \psi_0 \oplus \left(\oplus_{a \in \mathbf{F}_p^{\times} - \mathbf{F}_p^{\times 2}} \psi_a\right) .$$

We call $\mathfrak{C}_{(p+1)/2}$ (resp. $\mathfrak{C}'_{(p+1)/2}$) the residual (resp. non-residual) representation. □

(1.3) From now on and until the end of the paper, we assume that $\chi^2 = 1$.

We will determine the explicit **C**-basis of the irreducible components 1, $\mathfrak{C}_p$, $\mathfrak{C}_{(p+1)/2}$, and $\mathfrak{C}'_{(p+1)/2}$ in $\mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}} \underline{\chi}$ $\left(\chi = 1, \left(\frac{\cdot}{p} \right) \right)$.

We can identify the induced representation $\mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}} \underline{\chi}$ with the left **C[G]**-module $\mathbf{C}[\mathbf{G}] \otimes_{\mathbf{C[B]}} \underline{\chi}$. Hence, it is sufficient to find the explicit **C**-basis of irreducible components of this left **C[G]**-module.

Let $e := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ be the unit element of **G** and we chose a basis of the representation space of $\underline{\chi}$ by ε, i.e., $\underline{\chi}\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right) \varepsilon = \chi(a)\varepsilon$. A system of all representatives for $\mathbf{G}/\mathbf{B}$ is given by the $p+1$ elements: $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\xi_a := \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $(a \in \mathbf{F}_p)$. Under these notation, a **C**-basis of $\mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}} \underline{\chi} = \mathbf{C}[\mathbf{G}] \otimes_{\mathbf{C[B]}} \underline{\chi}$ is given by the $p+1$ elements: $e \otimes \varepsilon$ and $\xi_a \otimes \varepsilon$ $(a \in \mathbf{F}_p)$.

Take any element $u \in \mathbf{F}_p$ and $h := \alpha(e \otimes \varepsilon) + \sum_{x \in \mathbf{F}_p} \beta_x(\xi_x \otimes \varepsilon) \in \mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}} \underline{\chi}$ $(\alpha, \beta_x \in \mathbf{C}$ $(x \in \mathbf{F}_p))$. We have the identity $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot h = \alpha(e \otimes \varepsilon) + \sum_{x \in \mathbf{F}_p} \beta_x(\xi_{u+x} \otimes \varepsilon)$. Hence any element h belonging to the representation ψ_a $(a \in \mathbf{F}_p)$ is expressed as follows:

$$(1.4) \quad \begin{cases} h = \alpha(e \otimes \varepsilon) + \beta \left(\sum_{x \in \mathbf{F}_p} \xi_x \otimes \varepsilon \right) & (\alpha, \beta \in \mathbf{C}) \quad \text{if } a = 0, \\ h = \beta \sum_{x \in \mathbf{F}_p} \mathrm{e}(-ax/p)(\xi_x \otimes \varepsilon) & (\beta \in \mathbf{C}) \qquad \text{if } a \neq 0. \end{cases}$$

$\square$

We therefore see that for any non-zero a, $\mathbf{C}\left(\sum_{x \in \mathbf{F}_p} \mathrm{e}(-ax/p)(\xi_x \otimes \varepsilon) \right)$ is the ψ_a-component in the **C[U]**-modules $\mathrm{Res}_{\mathbf{U}} \mathfrak{C}_p$, $\mathrm{Res}_{\mathbf{U}} \mathfrak{C}_{(p+1)/2}$, and $\mathrm{Res}_{\mathbf{U}} \mathfrak{C}'_{(p+1)/2}$.

We must determine an explicit basis of each ψ_0-component in the above three **C[U]**-modules.

Put for any $a \in \mathbf{F}_p$, $h_a := \sum_{x \in \mathbf{F}_p} \mathrm{e}(-ax/p)(\xi_x \otimes \varepsilon) \in \mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}} \underline{\chi}$ and put $\mathcal{X} := \sum_{x \in \mathbf{F}_p} \xi_x \in \mathbf{C}[\mathbf{G}]$.

We will calculate the element $\mathcal{X} h_a = \sum_{x,y \in \mathbf{F}_p} \mathrm{e}(-ay/p)(\xi_x \xi_y \otimes \varepsilon)$. Since $\xi_x \xi_0 = \begin{pmatrix} -1 & -x \\ 0 & -1 \end{pmatrix} \in \mathbf{B}$, the part of $y = 0$ in the above sum is

$$\sum_{x \in \mathbf{F}_p} \xi_x \xi_0 \otimes \varepsilon = \sum_{x \in \mathbf{F}_p} e \otimes \underline{\chi}\left(\begin{pmatrix} -1 & -x \\ 0 & -1 \end{pmatrix} \right) \varepsilon = \chi(-1)\, p\, (e \otimes \varepsilon).$$

Next assume that $y \neq 0$. Then

$$\xi_x \xi_y = \begin{pmatrix} xy-1 & -x \\ y & -1 \end{pmatrix} = \xi_{x-y^{-1}} \begin{pmatrix} -y & 0 \\ 0 & -y^{-1} \end{pmatrix} \begin{pmatrix} 1 & -y^{-1} \\ 0 & 1 \end{pmatrix}.$$

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Hence

$$\sum_{\substack{x \in \mathbf{F}_p \\ y \in \mathbf{F}_p^\times}} e(-ay/p)(\xi_x \xi_y \otimes \varepsilon) = \sum_{\substack{x \in \mathbf{F}_p \\ y \in \mathbf{F}_p^\times}} e(-ay/p)\xi_{x-y^{-1}} \otimes \underline{\chi}\left(\begin{pmatrix} -y & 0 \\ 0 & -y^{-1} \end{pmatrix} \begin{pmatrix} 1 & -y^{-1} \\ 0 & 1 \end{pmatrix} \right) \varepsilon$$

$$= \sum_{y \in \mathbf{F}_p^\times} e(-ay/p)\chi(-y) \left(\sum_{x \in \mathbf{F}_p} \xi_{x-y^{-1}} \otimes \varepsilon \right)$$

$$= \left(\sum_{y \in \mathbf{F}_p^\times} e(ay/p)\chi(y) \right) h_0 .$$

From these results,

$$(1.5) \qquad \mathcal{X} h_a = \begin{cases} p(e \otimes \varepsilon) + (p-1)h_0, & \text{if } \chi = \mathbf{1} \text{ and } a = 0, \\ p(e \otimes \varepsilon) - h_0, & \text{if } \chi = \mathbf{1} \text{ and } a \neq 0, \\ \left(\frac{-1}{p} \right) p(e \otimes \varepsilon), & \text{if } \chi = \left(\frac{\cdot}{p} \right) \text{ and } a = 0, \\ \left(\frac{-1}{p} \right) p(e \otimes \varepsilon) + \left(\frac{a}{p} \right) \mathfrak{g}_p h_0, & \text{if } \chi = \left(\frac{\cdot}{p} \right) \text{ and } a \neq 0, \end{cases}$$

where $\mathfrak{g}_p := \sum_{x \in \mathbf{F}_p^\times} e(x/p) \left(\frac{x}{p} \right)$ is the gauss sum. $\square$

Suppose $\chi = \left(\frac{\cdot}{p} \right)$ and take a quadratic residue $a \in \mathbf{F}_p^\times$. Since $h_a \in \mathfrak{C}_{(p+1)/2}$, $\mathcal{X} h_a = \left(\frac{-1}{p} \right) p(e \otimes \varepsilon) + \left(\frac{a}{p} \right) \mathfrak{g}_p h_0 = \mathfrak{g}_p(\mathfrak{g}_p(e \otimes \varepsilon) + h_0) \in \mathfrak{C}_{(p+1)/2}$. From the formula (1.4), $\mathbf{C}(\mathfrak{g}_p(e \otimes \varepsilon) + h_0) \cong \psi_0$ and hence this element gives a basis of the ψ_0-component of $\mathfrak{C}_{(p+1)/2}$.

We can find an explicit basis of the ψ_0-component of $\mathfrak{C}_p$, $\mathfrak{C}'_{(p+1)/2}$ in the same way. The case of $\mathbf{1}$ is trivial. Thus we obtain the following.

(1.6) Proposition. Under the above notation, we have the following explicit expression of irreducible $\mathbf{C}[\mathbf{G}]$-modules in $\mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}} \underline{\chi}$.

$$\begin{aligned} \mathbf{1} \quad &= \mathbf{C}((e \otimes \varepsilon) + h_0) , \\ \mathfrak{C}_p \quad &= \mathbf{C}(p(e \otimes \varepsilon) - h_0) \quad \oplus \quad \left(\oplus_{a \in \mathbf{F}_p^\times} \mathbf{C} h_a \right) , \\ \mathfrak{C}_{(p+1)/2} \quad &= \mathbf{C}(\mathfrak{g}_p(e \otimes \varepsilon) + h_0) \quad \oplus \quad \left(\oplus_{a \in \mathbf{F}_p^{\times 2}} \mathbf{C} h_a \right) , \\ \mathfrak{C}'_{(p+1)/2} \quad &= \mathbf{C}(\mathfrak{g}_p(e \otimes \varepsilon) - h_0) \quad \oplus \quad \left(\oplus_{a \in \mathbf{F}_p^\times - \mathbf{F}_p^{\times 2}} \mathbf{C} h_a \right) . \end{aligned}$$

Here, each direct summand of the right-hand sides is an irreducible $\mathbf{C}[\mathbf{U}]$-module. Precisely speaking, each first direct summand is isomorphic to ψ_0 and the other direct summands $\mathbf{C} h_a$ ($a \neq 0$) are isomorphic to ψ_a respectively, as $\mathbf{C}[\mathbf{U}]$-module. $\square$

(1.7) From now, we study representations of finite groups which are constructed by modular forms of half-integral weight.

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We use following notation until the end of this section.

Let k be a positive integer and $N = 4 \times M$, M is a positive odd integer.

Put $\nu_p = \mathrm{ord}_p(N) = \mathrm{ord}_p(M)$ for any odd prime p. We decompose $M = M_1 M_{2+}$, $M_1 := \prod_{p|M, \nu_p=1} p$ and $M_{2+} := \prod_{p|M, \nu_p \geq 2} p^{\nu_p}$.

Let χ be an even Dirichlet character with $\chi^2 = 1$ and for any prime p, χ_p the p-primary component of χ. Moreover we denote the M_1(resp. $2M_{2+}$)-component of χ by $\chi_1 := \prod_{p|M_1} \chi_p$ (resp. $\tilde{\chi} := \prod_{p|2M_{2+}} \chi_p$).

For any positive integer $\alpha \in \mathbf{Z}_+$, put

$$\mathbf{G}(\alpha) := SL_2(\mathbf{Z}/\alpha\mathbf{Z}), \quad \mathbf{B}(\alpha) := \left\{ \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix} \right) \in \mathbf{G}(\alpha) \right\}, \quad \mathbf{U}(\alpha) := \left\{ \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix} \right) \in \mathbf{G}(\alpha) \right\}.$$

Moreover if $(\alpha, \beta) = 1$, we naturally identify as follows: $\mathbf{G}(\alpha\beta) = \mathbf{G}(\alpha) \times \mathbf{G}(\beta)$ and for simplicity, we shortly write $\mathbf{G}_1 := \mathbf{G}(M_1)$, $\mathbf{B}_1 := \mathbf{B}(M_1)$, and $\tilde{\mathbf{B}} := \mathbf{B}(4M_{2+})$.

For any positive integer α divisible by 4, put $\Delta(\alpha) := \{\gamma^* \mid \gamma \in \Gamma(\alpha)\}$, $\Delta_1(\alpha) := \{\gamma^* \mid \gamma \in \Gamma_1(\alpha)\}$, and $\Delta_0(\alpha) := \{\gamma^* \mid \gamma \in \Gamma_0(\alpha)\}$ (cf. §0(c)).

Under these notation, we have that $\Delta_0(4M_{2+}) \triangleright \Delta(N) = \Delta(4M_1M_{2+})$ and that

$$\Delta_0(4M_{2+}) / \Delta(N) \cong \Gamma_0(4M_{2+}) / \Gamma(N) \cong \tilde{\mathbf{B}} \times \mathbf{G}_1.$$

From this we can define the representation π of the group $\tilde{\mathbf{B}} \times \mathbf{G}_1$ on $S(k + 1/2, \Delta(N))$ as follows:

$$(1.8) \qquad [\pi(\gamma \bmod(N))] f := f \mid \gamma^{*-1}, \quad (f \in S(k + 1/2, \Delta(N))),$$

where $\gamma \bmod(N) \in \Gamma_0(4M_{2+})/\Gamma(N) \simeq \tilde{\mathbf{B}} \times \mathbf{G}_1$. We consider $S(k + 1/2, \Delta(N))$ as a left $\mathbf{C}[\tilde{\mathbf{B}} \times \mathbf{G}_1]$-module by the above representation π.

(1.9) Let π_f be the left $\mathbf{C}[\tilde{\mathbf{B}} \times \mathbf{G}_1]$-module generated by a non-zero cusp form $f \in S(k + 1/2, N, \chi)$, i.e.,

$$\pi_f := \mathbf{C}[\tilde{\mathbf{B}} \times \mathbf{G}_1] f = \langle f | \gamma^* ; \gamma \in \Gamma_0(4M_{2+}) \rangle_{\mathbf{C}}.$$

Moreover we can define the following one-dimensional representation $\underline{\underline{\chi}}$ as in (1.1)

$$\underline{\underline{\chi}} : \mathbf{B}(N) \ni \left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix} \right) \mapsto \chi(a) \in \mathbf{C}^\times, \quad \underline{\underline{\chi_1}} : \mathbf{B}_1 \ni \left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix} \right) \mapsto \chi_1(a) \in \mathbf{C}^\times,$$

$$\underline{\underline{\tilde{\chi}}} : \tilde{\mathbf{B}} \ni \left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix} \right) \mapsto \tilde{\chi}(a) \in \mathbf{C}^\times, \qquad \underline{\underline{\chi_p}} : \mathbf{B}(p^{\nu_p}) \ni \left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix} \right) \mapsto \chi_p(a) \in \mathbf{C}^\times.$$

Then we have a canonical identification $\underline{\underline{\chi}} = \underline{\underline{\tilde{\chi}}} \otimes \underline{\underline{\chi_1}} = \underline{\underline{\tilde{\chi}}} \otimes \left(\otimes_{p|M_1} \underline{\underline{\chi_p}} \right)$.

For any $f \in S(k + 1/2, N, \chi)$ and any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(N)$, $\pi(\gamma \bmod(N)) f = f \mid \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix} \right)^* = \chi(a) f$. Hence we get the isomorphism $\underline{\underline{\chi}} \cong \mathbf{C} f$ as $\mathbf{B}(N)$-modules. We denote the basis of the representation space of $\underline{\underline{\chi}}$ by ε. Then we get the following proposition by easy computation.

(1.10) Proposition. For any non-zero $f \in S(k+1/2, N, \chi)$, the following map gives an surjective homomorphism of $\tilde{\mathbf{B}} \times \mathbf{G}_1$-modules:

$$\Phi = \Phi_f : \mathrm{Ind}_{\tilde{\mathbf{B}} \times \mathbf{B}_1}^{\tilde{\mathbf{B}} \times \mathbf{G}_1} \underline{\chi} \quad \to \quad \pi_f \,, \qquad \left(\sum_\eta a_\eta \eta \otimes \varepsilon \mapsto \sum_\eta a_\eta \pi(\eta) f \right) \,,$$

where $a_\eta \in \mathbf{C}$ and η is taken over all representatives for $\left(\tilde{\mathbf{B}} \times \mathbf{G}_1 \right) / \left(\tilde{\mathbf{B}} \times \mathbf{B}_1 \right)$. We can, therefore, identify π_f with a subrepresentation of $\mathrm{Ind}_{\tilde{\mathbf{B}} \times \mathbf{B}_1}^{\tilde{\mathbf{B}} \times \mathbf{G}_1} \underline{\chi}$. □

We have canonical identity:

$$\mathrm{Ind}_{\tilde{\mathbf{B}} \times \mathbf{B}_1}^{\tilde{\mathbf{B}} \times \mathbf{G}_1} \underline{\chi} = \underline{\tilde{\chi}} \otimes \mathrm{Ind}_{\mathbf{B}_1}^{\mathbf{G}_1} \underline{\chi_1} = \underline{\tilde{\chi}} \otimes \left(\otimes_{p|M_1} \mathrm{Ind}_{\mathbf{B}(p)}^{\mathbf{G}(p)} \underline{\chi_p} \right) \,.$$

From (1.2), we know that each $\mathrm{Ind}_{\mathbf{B}(p)}^{\mathbf{G}(p)} \underline{\chi_p}$ is the direct sum of two distinct irreducible submodules. We therefore have the following decomposition

$$(1.11) \qquad \mathrm{Ind}_{\tilde{\mathbf{B}} \times \mathbf{B}_1}^{\tilde{\mathbf{B}} \times \mathbf{G}_1} \underline{\chi} = \oplus_{(\theta_p)_{p|M_1}} \left(\underline{\tilde{\chi}} \otimes \left(\otimes_{p|M_1} \theta_p \right) \right) \,,$$

where θ_p is taken over the set of irreducible $\mathbf{C}[\mathbf{G}(p)]$-modules $\{\mathbf{1}_{\mathbf{G}(p)}, \mathfrak{C}_p\}$ or $\{\mathfrak{C}_{(p+1)/2}, \mathfrak{C}'_{(p+1)/2}\}$ accordingly to $\chi_p = \mathbf{1}_{\mathbf{G}(p)}$ or $\left(\frac{\cdot}{p} \right)$ (cf. Proposition (1.2)). We note that $\underline{\tilde{\chi}} \otimes \left(\otimes_{p|M_1} \theta_p \right)$ are irreducible $\mathbf{C}[\mathbf{G}]$-modules and not equivalent to each other.

(1.12) Now we more closely study the representation π_f by using the results of (1.1)–(1.6). We need some national preliminaries.

For any prime divisor p of M_1, let ε_p be a basis of one-dimensional representation space of $\underline{\chi_p}$. Put $e_p := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathbf{G}(p)$ and $\xi_p(a) := \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathbf{G}(p)$ for $a \in \mathbf{F}_p$ and $\mathcal{X}_p(a) := \sum_{x \in \mathbf{F}_p} \mathbf{e}(-ax/p) \xi_p(x) \in \mathbf{C}[\mathbf{G}(p)]$. Moreover let $\tilde{\varepsilon}$ be a basis of $\underline{\tilde{\chi}}$.

For any prime divisor p of M_1 and any $x \in \mathbf{F}_p$, we chose an element $\gamma_p(x) \in SL_2(\mathbf{Z})$ such that

$$(1.13) \qquad SL_2(\mathbf{Z}) \ni \gamma_p(x) \equiv \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N/p \,, \\ \xi_p(x) \mod p \,, \end{cases}$$

and define an operator X_p $(p \mid M_1)$ as follows:

$$(1.14) \qquad f \mid X_p := \sum_{x \in \mathbf{F}_p} \pi(\gamma_p(x) \bmod(N)) f \,, \quad f \in S(k+1/2, N, \chi) \,.$$

In [U1, §1], We defined several fundamental operators on $S(k+1/2, N, \chi)$: $U(p)$, $\widetilde{W}(p)$, and Y_p. The above operator X_p can be express by those.

(1.15) Proposition. Let p be any prime divisor of M_1 and f any element of $S(k+1/2, N, \chi)$. Then the following hold.

(1) $f|X_p = p^{-k/2+3/4}f|\widetilde{W}(p)U(p)$. Hence the map $f \mapsto f|X_p$ gives an operator on $S(k+1/2, N, \chi)$.

(2) $f|Y_pU(p) = f|U(p)X_p$.

(3) $f|X_p{}^2 = \begin{cases} (p-1)\ f|X_p + pf, & \text{if } \chi_p = 1, \\ \left(\frac{-1}{p}\right)pf, & \text{if } \chi_p = \left(\frac{\cdot}{p}\right). \end{cases}$

Hence X_p is a semi-simple operator on $S(k+1/2, N, \chi)$.

Proof. (1) We keep to the notation in [U1, §1]. We have from the definition [U1,p.151] $\widetilde{W}(p) = \gamma_p^*\tilde{\delta}_p$, where $\gamma_p \in SL_2(\mathbf{Z})$ is satisfied the following condition:

$$\gamma_p \equiv \begin{cases} \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \pmod{p} \\ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \pmod{N/p}, \end{cases}$$

and $\tilde{\delta}_p = \left(\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right), p^{-k/2-1/4}\right)$. From [U1, (1.19)],

$$f|\widetilde{W}(p)U(p) = p^{k/2-3/4}\sum_{a\in\mathbf{F}_p} f|\widetilde{W}(p)\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right), p^{k/2+1/4}\right)\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)^*$$

$$= p^{k/2-3/4}\sum_{a\in\mathbf{F}_p} f|\gamma_p^*\left(\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right), 1\right)\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)^* = p^{k/2-3/4}\sum_{a\in\mathbf{F}_p} f|\gamma_p^*\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)^*.$$

By using the notation $\gamma_p(a)$ (cf. (1.13)), we can easily calculate for any $a \in \mathbf{F}_p$, $f|\gamma_p^*\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)^* = \pi(\gamma_p(-a) \bmod(N))f$. We therefore have

$$f|\widetilde{W}(p)U(p) = p^{k/2-3/4}\sum_{a\in\mathbf{F}_p} f|\gamma_p^*\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)^*$$

$$= p^{k/2-3/4}\sum_{a\in\mathbf{F}_p} \pi(\gamma_p(-a) \bmod(N))f = p^{k/2-3/4}f|X_p.$$

From this and [U1, (1.22)], $f \mapsto f|X_p$ gives an operator on $S(k+1/2, N, \chi)$.

(2) From the definition of Y_p (cf. [U1, p.155]) and the above (1),

$$p^{k/2-3/4}f|Y_pU(p) = f|U(p)\widetilde{W}(p)U(p) = (f|U(p))|\widetilde{W}(p)U(p) = p^{k/2-3/4}f|U(p)X_p.$$

(3) The map $f \mapsto f|U(p)$ gives an isomorphism from $S(k+1/2, N, \chi)$ onto $S(k+1/2, N, \chi\left(\frac{p}{\cdot}\right))$ (cf. [U1, (1.28)]). Hence it follows from the above (2) that $U(p)^{-1}Y_pU(p) = X_p$ on $S(k+1/2, N, \chi)$. By using this relation, we can deduce properties of X_p from those of Y_p. In particular, from [U1, Proposition (1.27)], we have the above relation (3). $\square$

(1.16) Take any non-zero $f \in S(k+1/2, N, \chi)$ and assume that π_f is irreducible. From (1.10) and (1.11), there exists a system of irreducible representations $(\theta_p)_{p|M_1}$ such that Φ_f: $\underline{\tilde{\chi}} \otimes (\otimes_{p|M_1}\theta_p) \cong \pi_f$. Moreover for any prime divisor q of M_1,

$$\text{Ker}(\Phi_f) \supseteq \underline{\tilde{\chi}} \otimes \left(\otimes_{p|M_1,\, p\neq q} \text{Ind}_{B(p)}^{G(p)}\underline{\chi_p}\right) \otimes \theta_q',$$

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where θ'_q is the irreducible component such that $\mathrm{Ind}_{\mathbf{B}(q)}^{\mathbf{G}(q)} \underline{\underline{\chi_q}} = \theta_q \oplus \theta'_q$.

Set the elements $\alpha_p, \alpha'_p \in \mathbf{C}[\mathbf{G}(p)]$ such that

$$(1.17) \qquad (\alpha_p, \alpha'_p) := \begin{cases} (e_p + \mathcal{X}_p(0), pe_p - \mathcal{X}_p(0)), & \text{if } \theta_p = \mathbf{1}_{\mathbf{G}(p)}, \\ (pe_p - \mathcal{X}_p(0), e_p + \mathcal{X}_p(0)), & \text{if } \theta_p = \mathfrak{C}_p, \\ (\mathfrak{g}_p e_p + \mathcal{X}_p(0), \mathfrak{g}_p e_p - \mathcal{X}_p(0)), & \text{if } \theta_p = \mathfrak{C}_{(p+1)/2}, \\ (\mathfrak{g}_p e_p - \mathcal{X}_p(0), \mathfrak{g}_p e_p + \mathcal{X}_p(0)), & \text{if } \theta_p = \mathfrak{C}'_{(p+1)/2}. \end{cases}$$

Then $\alpha_p \otimes \varepsilon_p \in \theta_p$, $\alpha'_p \otimes \varepsilon_p \in \theta'_p$ and

$$(1.18) \qquad \tilde{\varepsilon} \otimes \left(\otimes_{p|M_1, \, p \neq q} e_p \otimes \varepsilon_p \right) \otimes (\alpha'_q \otimes \varepsilon_q) \in \mathrm{Ker}(\Phi_f).$$

We therefore have
(1.19)

$$0 = \Phi_f \left(\tilde{\varepsilon} \otimes \left(\otimes_{p|M_1, \, p \neq q} e_p \otimes \varepsilon_p \right) \otimes (\alpha'_q \otimes \varepsilon_q) \right) = \begin{cases} qf - f \mid X_q, & \text{if } \theta_q = \mathbf{1}_{\mathbf{G}(q)}, \\ f + f \mid X_q, & \text{if } \theta_q = \mathfrak{C}_q, \\ \mathfrak{g}_q f - f \mid X_q, & \text{if } \theta_q = \mathfrak{C}_{(q+1)/2}, \\ \mathfrak{g}_q f + f \mid X_q, & \text{if } \theta_q = \mathfrak{C}'_{(q+1)/2}. \end{cases}$$

In other wards, f is a common eigenform of X_p for all prime divisors $p|M_1$.

Conversely, assume that f is a common eigenform of X_p, $p|M_1$.

Decompose π_f into irreducible components: $\pi_f = \rho_1 \oplus \cdots \oplus \rho_m$ and let $f = f_1 + \cdots + f_m$, $f_i \in \rho_i$ $(i = 1, \ldots, m)$.

We apply the following operator to the above: $\mathcal{M} := |\tilde{\mathbf{B}} \times \mathbf{B}_1|^{-1} \sum_\gamma \overline{\chi}(d) \gamma^*$, where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is taken over all representatives of $\Gamma_0(N)/\Gamma(N)$.

We easily see that $f = f|\mathcal{M}$ and for any i, $f_i|\mathcal{M} \in \rho_i \cap S(k + 1/2, N, \chi)$. From this, without loss of generality, we can assume that $f_i \in S(k + 1/2, N, \chi)$ and $f_i \neq 0$ $(i = 1, \ldots, m)$. Then ρ_i is generated by a non-zero form f_i for all i. It follows from (1.19) that all f_i's are common eigenforms of X_p, $p|M_1$.

Set $f_i|X_p = \sigma_p^{(i)} f_i$, $(i = 1, \ldots, m, \, p|M_1, \, \sigma_p^{(i)} \in \mathbf{C})$.

Since π_f is identified with subrepresentation of $\mathrm{Ind}_{\tilde{\mathbf{B}} \times \mathbf{B}_1}^{\tilde{\mathbf{B}} \times \mathbf{G}_1} \underline{\underline{\chi}}$, ρ_i $(i = 1, \ldots, m)$ are not equivalent to each other (cf. (1.11)). Hence, from (1.19), the systems of eigenvalues $(\sigma_p^{(1)})_{p|M_1}, \ldots, (\sigma_p^{(m)})_{p|M_1}$ are different from each other. Therefore we see that $m = 1$ and π_f is irreducible.

Thus we get the following results.

(1.20) Proposition Let the notation be the same as above. For $(0 \neq)f \in S(k + 1/2, N, \chi)$, the following hold good.

(1) π_f is an irreducible $\mathbf{C}[\tilde{\mathbf{B}} \times \mathbf{G}_1]$-module if and only if f is a common eigenform of X_p, $p|M_1$.

(2) Suppose that π_f is irreducible. Then $\pi_f \cong \underline{\underline{\chi}} \otimes \left(\otimes_{p|M_1} \theta_p \right)$, where θ_p's are irreducible $\mathbf{C}[\mathbf{G}(p)]$-modules. Let σ_p $(p|M_1)$ be an eigen value of f on X_p: $f|X_p =$

$\sigma_p f$. Then σ_p is given by the following table.

$$(1.21) \qquad \sigma_p = \begin{cases} p, & \text{if } \chi_p = 1 \text{ and } \theta_p = 1_{G(p)}, \\ -1, & \text{if } \chi_p = 1 \text{ and } \theta_p = \mathfrak{C}_p, \\ \mathfrak{g}_p, & \text{if } \chi_p = \left(\frac{\cdot}{p}\right) \text{ and } \theta_p = \mathfrak{C}_{(p+1)/2}, \\ -\mathfrak{g}_p, & \text{if } \chi_p = \left(\frac{\cdot}{p}\right) \text{ and } \theta_p = \mathfrak{C}'_{(p+1)/2}. \end{cases}$$

$\square$

(1.22) Now we will study relations between the above representations and Fourier coefficients.

Let f be a non-zero element of $S(k + 1/2, N, \chi)$ and take any system $(a_p)_{p|M_1} \in \prod_{p|M_1} \mathbf{F}_p$. Then

$$\Phi_f(\tilde{\varepsilon} \otimes (\otimes_{p|M_1} \mathcal{X}_p(a_p) \otimes \varepsilon_p)) = \sum_{\substack{(x_p)_{p|M_1} \\ x_p \in \mathbf{F}_p}} \prod_{p|M_1} \mathbf{e}(-a_p x_p/p) \pi(\gamma((x_p)) \bmod(N)) f \ ,$$

where $\gamma((x_p)) \in SL_2(\mathbf{Z})$ is an element satisfied the following condition:

$$\gamma((x_p)) \equiv \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \bmod 4M_{2+} \\ \xi_p(x_p) & \bmod p \qquad \text{for all prime divisors } p \text{ of } M_1. \end{cases}$$

Choose a matrix $\gamma_{M_1} \in SL_2(\mathbf{Z})$ and an element $x \in \mathbf{Z}$ for $(x_p)_{p|M_1} \in \prod_{p|M_1} \mathbf{F}_p$ such that

$$\gamma_{M_1} \equiv \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \bmod 4M_{2+}, \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \bmod M_1, \end{cases} \quad \text{and} \quad x \equiv \begin{cases} 0 & \bmod 4M_{2+}, \\ x_p & \bmod p \text{ for all } p|M_1. \end{cases}$$

and put $\alpha((x_p)) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. Then we note that $\gamma((x_p)) \gamma_{M_1} \alpha((x_p))^{-1} \in \Gamma(N)$. Moreover set $g := f|\widetilde{W}(M_1) = \sum_{n=1}^{\infty} b(n) \mathbf{e}(nz) \in S(k + 1/2, N, \chi\left(\frac{M_1}{\cdot}\right))$. Hence

$$\Phi_f(\tilde{\varepsilon} \otimes \otimes_{p|M_1} (\mathcal{X}_p(a_p) \otimes \varepsilon_p)) = \sum_{\substack{(x_p)_{p|M_1} \\ x_p \in \mathbf{F}_p}} \prod_{p|M_1} \mathbf{e}(-a_p x_p/p) f|\gamma^*_{M_1} \alpha((x_p))^{*-1}$$

$$= \sum_{\substack{(x_p)_{p|M_1} \\ x_p \in \mathbf{F}_p}} \prod_{p|M_1} \mathbf{e}(-a_p x_p/p) f|\widetilde{W}(M_1) \tilde{\delta}_{M_1}^{-1} \alpha((x_p))^{*-1}$$

$$= M_1^{-k/2-1/4} \sum_{\substack{(x_p)_{p|M_1} \\ x_p \in \mathbf{F}_p}} \prod_{p|M_1} \mathbf{e}(-a_p x_p/p) g((z - x)/M_1)$$

$$= M_1^{-k/2-1/4} \sum_{n=1}^{\infty} b(n) \mathbf{e}(nz/M_1) \sum_{\substack{(x_p)_{p|M_1} \\ x_p \in \mathbf{F}_p}} \left(\prod_{p|M_1} \mathbf{e}(-a_p x_p/p) \right) \mathbf{e}(-nx/M_1) \ .$$

Since the G.C.D. of all M_1/p (p: prime divisor of M_1) is 1, there exist integers $u_p \in \mathbf{Z}$ such that $\sum_{p|M_1} u_p \cdot (M_1/p) = 1$. We therefore have

$$\sum_{\substack{(x_p)_{p|M_1} \\ x_p \in \mathbf{F}_p}} \left(\prod_{p|M_1} \mathrm{e}(-a_p x_p/p) \right) \mathrm{e}(-nx/M_1) = \sum_{\substack{(x_p)_{p|M_1} \\ x_p \in \mathbf{F}_p}} \prod_{p|M_1} \mathrm{e}((-a_p x_p - n x u_p)/p)$$

$$= \prod_{p|M_1} \sum_{x_p \in \mathbf{F}_p} \mathrm{e}((-a_p x_p - n x_p u_p)/p)$$

$$= \begin{cases} M_1, & \text{if } -a_p \equiv n u_p \pmod{p} \text{ for all } p|M_1, \\ 0, & \text{otherwise.} \end{cases}$$

Here, we use the condition: $x \equiv x_p \pmod{p}$.

Since $-a_p \equiv n u_p \pmod{p} \Leftrightarrow -a_p(M_1/p) \equiv n \pmod{p}$, we get the following.

(1.23) Under the above notation,

$$\Phi_f(\tilde{\varepsilon} \otimes \otimes_{p|M_1}(\mathcal{X}_p(a_p) \otimes \varepsilon_p)) = M_1^{-k/2+3/4} \sum_{\substack{n \equiv -a_p(M_1/p) \bmod(p) \\ p|M_1}} b(n)\mathrm{e}(nz/M_1) \ ,$$

where $g := f|\widetilde{W}(M_1) = \sum_{n=1}^{\infty} b(n)\mathrm{e}(nz)$. $\quad\square$

(1.24) Moreover we suppose that π_f is irreducible. In this case, there are irreducible $\mathbf{C}[\mathbf{G}(p)]$-modules θ_p ($p|M_1$) such that $\underline{\tilde{\chi}} \otimes \otimes_{p|M_1}\theta_p \cong \pi_f$ via Φ_f.

We use the notation α_p and α'_p of (1.17). Then in all cases of $\theta_p = \mathbf{1}_{\mathbf{G}(p)}$, $\mathfrak{C}_p$, $\mathfrak{C}_{(p+1)/2}$, or $\mathfrak{C}'_{(p+1)/2}$, we can express that $\mathcal{X}_p(0) = c_0 \times \alpha_p + c_1 \times \alpha'_p$ with a non-zero constant c_0 and a constant c_1. And from the formula (1.18), it follows that for any system $(a_p)_{p|M_1} \in \prod_{p|M_1} \mathbf{F}_p$, there exists a non-zero element c_2 such that

(1.25) $\tilde{\varepsilon} \otimes \otimes_{p|M_1}(\mathcal{X}_p(a_p) \otimes \varepsilon_p)$

$\equiv c_2\, \tilde{\varepsilon} \otimes \left(\otimes_{p|M_1, a_p \neq 0}(\mathcal{X}_p(a_p) \otimes \varepsilon_p) \right) \otimes \left(\otimes_{p|M_1, a_p=0}(\alpha_p \otimes \varepsilon) \right) \pmod{\operatorname{Ker} \Phi_f} \ .$

Now we set the following notation for any prime divisor p of M_1:

(1.26) $\quad S_p^\times := \left\{ \begin{array}{ll} \emptyset, & \text{if } \theta_p = \mathbf{1}_{\mathbf{G}(p)}, \\ \mathbf{F}_p^\times, & \text{if } \theta_p = \mathfrak{C}_p, \\ \mathbf{F}_p^{\times 2}, & \text{if } \theta_p = \mathfrak{C}_{(p+1)/2}, \\ \mathbf{F}_p^\times - \mathbf{F}_p^{\times 2}, & \text{if } \theta_p = \mathfrak{C}'_{(p+1)/2}, \end{array} \right\} \quad \text{and} \quad S_p := S_p^\times \cup \{0\} \ .$

From the formula of (1.6), we see that $S_p^\times = \{a \in \mathbf{F}_p^\times \mid \mathcal{X}_p(a) \otimes \varepsilon_p \in \theta_p\}$.

From (1.25), we see that for any system $(a_p)_{p|M_1} \in \prod_{p|M_1} \mathbf{F}_p$,

$\Phi_f(\tilde{\varepsilon} \otimes (\otimes_{p|M_1} \mathcal{X}_p(a_p) \otimes \varepsilon_p)) \neq 0$

$\Leftrightarrow \Phi_f(\tilde{\varepsilon} \otimes (\otimes_{p|M_1, a_p \neq 0}\mathcal{X}_p(a_p) \otimes \varepsilon_p) \otimes (\otimes_{p|M_1, a_p=0}\alpha_p \otimes \varepsilon_p)) \neq 0$

$\Leftrightarrow \tilde{\varepsilon} \otimes (\otimes_{p|M_1, a_p \neq 0}\mathcal{X}_p(a_p) \otimes \varepsilon_p) \otimes (\otimes_{p|M_1, a_p=0}\alpha_p \otimes \varepsilon_p) \in \underline{\tilde{\chi}} \otimes (\otimes_{p|M_1}\theta_p)$

$\Leftrightarrow \mathcal{X}_p(a_p) \otimes \varepsilon_p \in \theta_p$ for any $p|M_1$ such that $a_p \neq 0$

$\Leftrightarrow a_p \in S_p^\times$ for any $p|M_1$ such that $a_p \neq 0$

$\Leftrightarrow (a_p)_{p|M_1} \in \prod_{p|M_1} S_p \ .$

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Combining the above with the formula (1.23), we obtain the following theorem.

(1.27) Theorem Let f be a non-zero element of $S(k+1/2, N, \chi)$ and $g := f|\widetilde{W}(M_1) = \sum_{n=1}^{\infty} b(n)e(nz) \in S\left(k+1/2, N, \chi\left(\frac{M_1}{\cdot}\right)\right)$. Suppose that π_f is irreducible and $\pi_f \cong \underline{\underline{\tilde{\chi}}} \otimes \otimes_{p|M_1} \theta_p$ with irreducible $\mathbf{C}[\mathbf{G}(p)]$-modules θ_p.

Then for a $(a_p)_{p|M_1} \in \prod_{p|M_1} \mathbf{F}_p$, we have the following:

$$(a_p)_{p|M_1} \in \prod_{p|M_1} S_p \quad \Leftrightarrow \quad \text{`` there exists } n \in \mathbf{Z}_+ \text{ such that } b(n) \neq 0 \text{ and } n \equiv -a_p(M_1/p) \pmod{p} \text{ for all } p|M_1 \text{ ''}.$$

In other words,

$$(a_p)_{p|M_1} \notin \prod_{p|M_1} S_p \quad \Leftrightarrow \quad \text{`` } b(n) = 0 \text{ for all } n\text{'s which satisfy the condition: } n \equiv -a_p(M_1/p) \pmod{p} \text{ for any } p|M_1 \text{ ''}.$$

See (1.26) for the definition of S_p. $\square$

§2. A complete theory of newforms for Kohnen spaces

We will generalize the results of §4 of the previous paper [U1] and will get a complete theory of newforms for Kohnen spaces. So we will consider only Kohnen spaces in this and next sections.

We keep to the notations in §0, §1 and in [U1]. See those for the detail of definitions and notations.

We recall some notations.

Let k be a positive integer and $N = 4 \times M$, M is a positive odd integer. Put $\nu_p := \mathrm{ord}_p(N) = \mathrm{ord}_p(M)$ for any odd prime p. We decompose $M = M_1 M_{2+}$, $M_1 := \prod_{p|M, \nu_p=1} p$ and $M_{2+} := \prod_{p|M, \nu_p \geq 2} p^{\nu_p}$.

Let χ be an even Dirichlet character with $\chi^2 = 1$ and for any prime p, χ_p the p-primary component of χ. Moreover we denote the $M_1(\text{resp. } 2M_{2+})$-component of χ by $\chi_1 := \prod_{p|M_1} \chi_p$ (resp. $\tilde{\chi} := \prod_{p|2M_{2+}} \chi_p$).

We also denote the set of all prime divisors of M_{2+} by Π. For any $l \in \Pi$, we denote the twisting operator of $\left(\frac{\cdot}{l}\right)$ by R_l (cf. §0 (c)). Furthermore for any subset $I \subseteq \Pi$, we put $R_I := \prod_{l \in I} R_l$.

For any $\kappa \in \{\pm 1\}^{\Pi}$ and any $l \in \Pi$, we define operators by $e_l^\kappa := \frac{1}{2}(R_l^2 + \kappa(l)R_l)$ and $e_\Pi^\kappa := \prod_{l \in \Pi} e_l^\kappa$. For any subset I of Π, we set the following notations: $l_I := \prod_{l \in I} l$.

Let $(\alpha_l)_{l \in \Pi}$ be a system of integers such that $0 \leq \alpha_l \leq \nu_l$ for all $l \in \Pi$. For simplicity, put $M_{2+}^{(\alpha)} := \prod_{l \in \Pi} l^{\alpha_l}$, $\tilde{N}(\alpha) := 4M_1 M_{2+}^{(\alpha)}$, $D = \Pi(\alpha)_0 := \{l \in \Pi \mid \alpha_l = 0\}$, $E = \Pi(\alpha)_1 := \{l \in \Pi \mid \alpha_l = 1\}$, and $F = \Pi(\alpha)_{2+} := \{l \in \Pi \mid \alpha_l \geq 2\}$. We note that every positive divisor of M_{2+} is of the form $M_{2+}^{(\alpha)}$ for some system $(\alpha_l)_{l \in \Pi}$.

We choose and fix $\kappa \in \{\pm 1\}^{\Pi}$ until the end of this paper. Let $\kappa|_F \in \{\pm 1\}^F$ be the restriction of κ to F.

We have to study the subspace $\mathfrak{S}^{\emptyset, \kappa|_F}\left(k+1/2, \tilde{N}(\alpha), \chi\right)_K$ of $S(k+1/2, N, \chi)_K$. We recall the definition of this subspace (cf. [U1, (3.3) and (3.5)]).

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We decompose the character χ into two parts as follows:

$$\chi = \eta\eta', \quad \eta = \left(\frac{u}{}\right), \quad \eta' = \left(\frac{u'}{}\right), \quad 0 < u \mid l_D, \quad 0 < u' \mid M_1 l_E l_F.$$

Since the character η' can be defined with modulo $\tilde{N}(\alpha)$, the subspace $\mathfrak{S}^{\emptyset,\kappa'|_F}\left(k + 1/2, \tilde{N}(\alpha), \eta'\right)_K$ is naturally defined, where $\kappa'|_F := \kappa|_F \cdot \left(\frac{u}{}\right) \in \{\pm 1\}^F$ (cf. [U1, p.177]). And then, using this subspace, we define as follows

$$(2.1) \qquad \mathfrak{S}^{\emptyset,\kappa|_F}\left(k + 1/2, \tilde{N}(\alpha), \chi\right)_K := \mathfrak{S}^{\emptyset,\kappa'|_F}\left(k + 1/2, \tilde{N}(\alpha), \eta'\right)_K |U(u).$$

$\mathfrak{S}^{\emptyset,\kappa'|_F}\left(k + 1/2, \tilde{N}(\alpha), \eta'\right)_K$ has a **C**-basis $\mathcal{B}_{\eta'}$ consisting of common eigenforms for all Hecke operators $\tilde{T}(n^2) = \tilde{T}_{k+1/2,\tilde{N}(\alpha),\eta'}(n^2)$ $(n \in \mathbf{Z}_+, (n, \tilde{N}(\alpha)) = 1)$ (cf. [U1, (3.11)]). Moreover $U(u)$ gives an injection on $S\left(k + 1/2, \tilde{N}(\alpha) \cdot l_D, \eta'\right)_K$ ([U1, (1.28)]) and is commutative with all Hecke operators $\tilde{T}(n^2)$'s (cf. [U1, (1.8) and (1.20)]). Therefore the set $\mathcal{B}_\chi := \{g|U(u) ; g \in \mathcal{B}_{\eta'}\}$ gives a **C**-basis of $\mathfrak{S}^{\emptyset,\kappa|_F}\left(k + 1/2, \tilde{N}(\alpha), \chi\right)_K$ consisting of common eigenforms for all Hecke operators $\tilde{T}(n^2) = \tilde{T}_{k+1/2,N,\chi}(n^2)$ $(n \in \mathbf{Z}_+, (n, N) = 1)$. Put

$$B^{(\alpha)} := \sum_{0 < a | l_D} \mathfrak{S}^{\emptyset,\kappa|_F}\left(k + 1/2, \tilde{N}(\alpha), \chi\right)_K |U(a^2) \subseteq \begin{cases} S(k + 1/2, N, \chi)_K & \text{if } k \geq 2, \\ V(N; \chi)_K & \text{if } k = 1. \end{cases}$$

Then from [U1,(3.10)],

$$(2.2) \qquad B^{(\alpha)} = \bigoplus_{0 < a \mid l_D} \mathfrak{S}^{\emptyset,\kappa|_F}\left(k + 1/2, \tilde{N}(\alpha), \chi\right)_K |U(a^2)$$

$$\subseteq \begin{cases} S^{\emptyset,\kappa|_F}\left(k + 1/2, \tilde{N}(\alpha) \cdot l_D, \chi\right)_K & \text{if } k \geq 2, \\ V^{\emptyset,\kappa|_F}\left(\tilde{N}(\alpha) \cdot l_D, \chi\right)_K & \text{if } k = 1. \end{cases}$$

Since every $U(a^2)$ $(0 < a \mid l_D)$ gives an injection on $S\left(k + 1/2, \tilde{N}(\alpha) \cdot l_D, \chi\right)$ (cf. [U1, (1.28)]), we get the following:

$$(2.3) \qquad B^{(\alpha)} = \bigoplus_{f \in \mathcal{B}_\chi} \left(\bigoplus_{0 < a \mid l_D} \mathbf{C}\, f|U(a^2) \right),$$

where f takes over all elements in $\mathcal{B}_\chi$.

Moreover we can see that

$$(2.4) \qquad B^{(\alpha)}|e_\Pi^\kappa \subseteq \mathfrak{S}^{\emptyset,\kappa}(k + 1/2, N, \chi)_K.$$

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In fact, applying the operator e_{Π}^{κ} to the both sides of definition, we see that

$$B^{(\alpha)}|e_{\Pi}^{\kappa} \subseteq \begin{cases} S(k+1/2,N,\chi)_K \,|e_{\Pi}^{\kappa} = S^{\emptyset,\kappa}(k+1/2,N,\chi)_K, & \text{if } k \geq 2, \\ V(N;\chi)_K \,|e_{\Pi}^{\kappa} = V^{\emptyset,\kappa}(N;\chi)_K, & \text{if } k = 1. \end{cases}$$

Every generator of $B^{(\alpha)}|e_{\Pi}^{\kappa}$, $f|U(a^2)e_{\Pi}^{\kappa}$ ($f \in \mathcal{B}_\chi$, $0 < a|l_D$), is a common eigenform on $\tilde{T}(n^2)$ ($n \in \mathbf{Z}_+$, $(n,N) = 1$) and it has the same system of eigen values as f (and so as $g \in \mathcal{B}_{\eta'}$). This system corresponds to a primitive form in $S^0(2k, M_1 M')$, $0 < M'|M_{2+}^{(\alpha)}$ ([U1, (3.11)]).

From [U1,(3.10)] and the Strong Multiplicity One theorem of weight $2k$ ([M, Theorem 4.6.19]), $\mathfrak{S}^{\emptyset,\kappa}(k+1/2,N,\chi)_K$ contains all of generators, $f|U(a^2)e_{\Pi}^{\kappa}$, and therefore contains all elements of $B^{(\alpha)}|e_{\Pi}^{\kappa}$.

We denote for any $f \in \mathcal{B}_\chi$

$$(2.5) \qquad\qquad B_f^{(\alpha)} := \bigoplus_{0 < a \,|\, l_D} \mathbf{C}\, f|U(a^2)\,.$$

These spaces $B_f^{(\alpha)}$ are contained in $S\left(k+1/2, \tilde{N}(\alpha)\cdot l_D, \chi\right)$. We will decompose $B_f^{(\alpha)}$ by the semi-simple operators Y_p, $p|M_1 l_D l_E$, on $S\left(k+1/2, \tilde{N}(\alpha)\cdot l_D, \chi\right)$ (cf. [U1, (1.27)]).

(2.6) First we consider the case of $p \mid M_1 l_E$.

Take $g \in \mathcal{B}_{\eta'}$ such that $f = g \mid U(u)$. Then

$$g \mid Y_p \in \begin{cases} S\left(k+1/2, \tilde{N}(\alpha), \eta'\right)_K, & \text{if } k \geq 2, \\ V\left(\tilde{N}(\alpha);\eta'\right)_K, & \text{if } k = 1, \end{cases}$$

and since R_l ($l \in F$) commutes with Y_p ([U1, (1.24)(2)]), we see

$$g \mid Y_p \in \begin{cases} S^{\emptyset,\kappa'|_F}\left(k+1/2, \tilde{N}(\alpha), \eta'\right)_K, & \text{if } k \geq 2, \\ V^{\emptyset,\kappa'|_F}\left(\tilde{N}(\alpha);\eta'\right)_K, & \text{if } k = 1. \end{cases}$$

Since Y_p commutes with Hecke operators $\tilde{T}(n^2)$, $(n,N) = 1$, both g and $g|Y_p$ belong to the same system of eigen values on $\tilde{T}(n^2)$, $(n,N) = 1$. And since $g \in \mathfrak{S}^{\emptyset,\kappa'|_F}\left(k+1/2, \tilde{N}(\alpha), \eta'\right)_K$, its system of eigenvalues corresponds to a primitive form of weight $2k$ and of conductor $M_1 l_E \times$ (a divisor of $\prod_{l \in F} l^{\alpha_l}$) ([U1, (3.10)(2)]).

From these, first we see that $g|Y_p \in \mathfrak{S}^{\emptyset,\kappa'|_F}\left(k+1/2, \tilde{N}(\alpha), \eta'\right)_K$ ([U1, (3.10)(1)]) and next, by using the Strong Multiplicity One theorem ([U1, (3.11)(2)]), $g|Y_p = \tau_p g$ for some $\tau_p \in \mathbf{C}$.

Combining these with (A.1), we obtain that

$$(2.7) \qquad f|Y_p = g|U(u)Y_p = \eta'_p(u)\left(\frac{u}{p}\right) g|Y_p U(u) = \eta'_p(u)\left(\frac{u}{p}\right)\tau_p f\,,$$

i. e., f is an eigen form on Y_p.

Moreover in this case, Y_p commute with $U(a^2)$ $(0 < a \mid l_D)$ because of (A.1). Hence we see that every element of $B_f^{(\alpha)}$ is a common eigenform on Y_p's $(p \mid M_1 l_E)$ belonging to the same system of eigen values as f. $\square$

As to the case of $p \mid l_D$, we get the following proposition.

(2.8) Proposition Let I be any subset of D and f be any element of $\mathcal{B}_\chi$. And let $A(I)$ be the subset of $\mathrm{Map}(I, \mathbf{C})$ consisting of all functions ρ such that $\rho(p) = \pm \sqrt{\left(\frac{-1}{p}\right) p}$ resp. $\rho(p) \in \{p, -1\}$ according to $\chi_p = 1$ resp. $\left(\frac{\cdot}{p}\right)$. Hence $|A(I)| = 2^{|I|}$. Then we have the following decomposition:

$$\bigoplus_{0 < a \mid l_I} \mathbf{C}\, f|U(a^2) = \bigoplus_{\rho \in A(I)} \mathbf{C}\, f_\rho \,,$$

where each f_ρ $(\rho \in A(I))$ is a non-zero cusp form satisfying the condition:

$$f_\rho \mid Y_p = \rho(p)\, f_\rho \quad \text{(for all } p \in I\text{)} \,.$$

Proof. We will prove with induction on $|I|$.

If $|I| = 0$ ($\Leftrightarrow I = \emptyset$), then $l_\emptyset = 1$ and the assertion is trivial.

Next suppose that $I \neq \emptyset$ and the assertion holds good for any proper subset I' of I.

Take and fix a prime $p \in I$ and decompose $I = \{p\} + I'$, i.e. $I' = I - \{p\}$. By the assumption of induction, we have

$$\bigoplus_{0 < a \mid l_{I'}} \mathbf{C}\, f|U(a^2) = \bigoplus_{\rho' \in A(I')} \mathbf{C}\, f_{\rho'} \,,$$

where $A(I')$ is the subset of $\mathrm{Map}(I', \mathbf{C})$ which is described in the assertion. Hence $|A(I')| = 2^{|I'|}$ and $f_{\rho'}$ $(\rho' \in A(I'))$ is a non-zero cusp form satisfying the condition:

$$f_{\rho'} \mid Y_p = \rho'(p)\, f_{\rho'} \quad \text{(for all } p \in I'\text{)} \,.$$

Since $U(p^2)$ gives an automorphism of $S\left(k + 1/2, \tilde{N}(\alpha) \cdot l_D, \chi\right)_K$, we can calculate as follows:

$$\bigoplus_{0 < a \mid l_I} \mathbf{C}\, f|U(a^2) = \left(\bigoplus_{0 < a \mid l_{I'}} \mathbf{C}\, f|U(a^2)\right) \oplus \left(\bigoplus_{0 < a \mid l_{I'}} \mathbf{C}\, f|U(a^2)\right) \mid U(p^2)$$

$$= \left(\bigoplus_{\rho' \in A(I')} \mathbf{C}\, f_{\rho'}\right) \oplus \left(\bigoplus_{\rho' \in A(I')} \mathbf{C}\, f_{\rho'} \mid U(p^2)\right)$$

$$= \bigoplus_{\rho' \in A(I')} \left(\mathbf{C} f_{\rho'} \oplus \mathbf{C}\, (f_{\rho'}|U(p^2))\right) \,.$$

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From [U1,(1.24)] $f_{\rho'}|U(p^2)Y_q = f_{\rho'}|Y_q U(p^2)$ for any $q \in I'$. Hence, every element of $\mathbf{C} f_{\rho'} \oplus \mathbf{C}(f_{\rho'}|U(p^2))$ is a common eigenform on all Y_q ($q \in I'$) which belongs to ρ'.

Now, we will decompose the space $\mathbf{C} f_{\rho'} \oplus \mathbf{C}(f_{\rho'}|U(p^2))$ into two eigen subspaces on Y_p.

From the definition of $\mathcal{B}_\chi$, there exists $g \in \mathcal{B}_{\eta'}$ such that $f = g \mid U(u)$. Then

$$\bigoplus_{0<a \mid l_{I'}} \mathbf{C}\, f|U(a^2) = \left(\bigoplus_{0<a \mid l_{I'}} \mathbf{C} g \mid U(a^2) \right) \mid U(u)$$

and so for any $f_{\rho'}$ ($\rho' \in A(I')$), there exists $g_{\rho'} \in \bigoplus_{0<a \mid l_{I'}} \mathbf{C} g \mid U(a^2)$ such that $f_{\rho'} = g_{\rho'} \mid U(u)$.

These forms $g_{\rho'}$ satisfy the conditions: $g_{\rho'} \in S\left(k + 1/2, \tilde{N}(\alpha) \cdot l_{I'}, \eta'\right)$. And also we have $(p, \tilde{N}(\alpha) \cdot l_{I'}) = 1$, $(u, 4l_F) = 1$, and $\eta'_u = 1$.

g is an eigen form on $\tilde{T}(p^2)$. From this and [U1, (1.20)], $g_{\rho'}$ is also an eigen form on $\tilde{T}(p^2)$ belonging to the same eigen value as g.

From the above facts, we can apply the formulae (A.9) and (A.15) in the appendix to the forms $f_{\rho'} = g_{\rho'} \mid U(u)$ and $f_{\rho'} \mid U(p^2) = g_{\rho'} \mid U(up^2)$.

It follows that the space $\mathbf{C}\, f_{\rho'} \oplus \mathbf{C}\, f_{\rho'} \mid U(p^2)$ can be decomposed into two distinct one-dimensional eigen subspaces on Y_p:

$$\mathbf{C}\, f_{\rho'} \oplus \mathbf{C}\, f_{\rho'} |U(p^2) = \mathbf{C} h_1 \oplus \mathbf{C} h_2 \,.$$

And also we see that h_1, h_2 are two common eigenforms on Y_q ($q \in I$) whose eigen values are coincide ($= \rho'(q)$) on the operators Y_q ($q \in I'$) and are different only on Y_p. We can extend ρ' to two elements in $\mathrm{Map}(I, \mathbf{C})$ by associating $\{p\}$ with each of such two eigen values.

Combining this and the above formula, we obtain the assertion. $\square$

From this proposition and (2.6), we have the following decomposition:

$$(2.9) \qquad B_f^{(\alpha)} = \bigoplus_{0<a \mid l_D} \mathbf{C}\, f|U(a^2) = \bigoplus_{\rho \in A(D)} \mathbf{C}\, f_\rho \,.$$

Here $A(D)$ is a subset of $\mathrm{Map}(D, \mathbf{C})$ such that $|A(D)| = 2^{|D|}$ and each f_ρ is a non-zero common eigen form on Y_p ($p|M_1 l_D l_E$) satisfying the condition:

$$(2.10) \qquad f_\rho \mid Y_p = c_p f_\rho \,, \qquad c_p := \begin{cases} \lambda_p, & \text{for all } p \mid M_1 l_E, \\ \rho(p), & \text{for all } p \in D, \end{cases}$$

where λ_p is the eigen value of $f \in \mathcal{B}_\chi$ on Y_p.

(2.11) We take such a f_ρ and fix it. Since $f_\rho \in S\left(k + 1/2, \tilde{N}(\alpha) \cdot l_D, \chi\right)_K$, $h_\rho := f_\rho|\widetilde{W}(M_1 l_D l_E)^{-1} \in S\left(k + 1/2, \tilde{N}(\alpha) \cdot l_D, \chi\left(\frac{M_1 l_D l_E}{\cdot}\right)\right)_K$ (cf. [U1, (1.22)]).

Let X_p be an operator defined in (1.14) of §1. The above form h_ρ is a common eigen form of X_p for any $p|M_1 l_D l_E$.

Proof of this claim. From [U1,(1.18)] and Proposition (A.1), it follows that

$$h_\rho \mid \widetilde{W}(M_1 l_D l_E) U(p) \widetilde{W}(p) = c_0 h_\rho \mid \widetilde{W}(p) \widetilde{W}(M_1 l_D l_E/p) U(p) \widetilde{W}(p)$$

$$= h_\rho | \widetilde{W}(p) U(p) \widetilde{W}(M_1 l_D l_E/p) \widetilde{W}(p) = c_1 h_\rho | \widetilde{W}(p) U(p) \widetilde{W}(M_1 l_D l_E) \ ,$$

where $c_0 = \left(\chi\left(\frac{M_1 l_D l_E}{}\right)\right)_{M_1 l_D l_E/p}(p)$ and $c_1 = \left(\chi\left(\frac{M_1 l_D l_E}{}\right)\right)_p (M_1 l_D l_E/p)$.

Hence, for any prime divisor p of $M_1 l_D l_E$,

$$(2.12) \qquad c_p h_\rho = f_\rho | Y_p \widetilde{W}(M_1 l_D l_E)^{-1}$$

$$= p^{-k/2 + 3/4} h_\rho | \widetilde{W}(M_1 l_D l_E) U(p) \widetilde{W}(p) \widetilde{W}(M_1 l_D l_E)^{-1}$$

$$= c_1 h_\rho | X_p \ .$$

Thus we proved the above claim. $\square$

From this claim and Proposition (1.20), $\Delta_0(4 \prod_{l \in F} l^{\alpha_l})/\Delta(4 M_1 l_D M_{2+}^{(\alpha)})$-module π_{h_ρ} is irreducible. We decompose it into primary components:

$$(2.13) \qquad \pi_{h_\rho} \cong \underline{\psi} \otimes \left(\otimes_{p|M_1} \theta_p\right) \otimes \left(\otimes_{p|l_D l_E} \theta_p\right) \ ,$$

where $\psi = \prod_{p \nmid M_1 l_D l_E} \left(\chi\left(\frac{M_1 l_D l_E}{}\right)\right)_p$ and $\underline{\psi}$ is defined by ψ in the same way as (1.1) or (1.9).

(2.14) Claim We can determine each primary components θ_p ($p \in D + E$) by using (1.21) and (2.12) as follows:

[Case 1] ($p|l_D l_E$ and $\chi_p = 1$)

$$\theta_p \cong \begin{cases} \mathfrak{C}_{(p+1)/2}, & \text{if } \lambda_p = \left(\frac{M_1 l_D l_E/p}{p}\right) \mathfrak{g}_p \text{ and } p \in E, \\ \mathfrak{C}'_{(p+1)/2}, & \text{if } \lambda_p = -\left(\frac{M_1 l_D l_E/p}{p}\right) \mathfrak{g}_p \text{ and } p \in E, \\ \mathfrak{C}_{(p+1)/2}, & \text{if } \rho(p) = \left(\frac{M_1 l_D l_E/p}{p}\right) \mathfrak{g}_p \text{ and } p \in D, \\ \mathfrak{C}'_{(p+1)/2}, & \text{if } \rho(p) = -\left(\frac{M_1 l_D l_E/p}{p}\right) \mathfrak{g}_p \text{ and } p \in D. \end{cases}$$

[Case 2] ($p|l_D$ and $\chi_p = \left(\frac{\cdot}{p}\right)$)

$$\theta_p \cong \begin{cases} \mathbf{1}_{G(p)}, & \text{if } \rho(p) = p, \\ \mathfrak{C}_p, & \text{if } \rho(p) = -1. \end{cases}$$

[Case 3] (if $p|l_E$ and $\chi_p = \left(\frac{\cdot}{p}\right)$) $\qquad \theta_p \cong \mathfrak{C}_p$.

Proof of this claim. The assertions in the Case 1 and 2 is easily verified.

We consider the Case 3. Suppose that $\theta_p \cong 1_{G(p)}$. We have $h_\rho | X_p = p h_\rho$ from (1.20)(2). Hence $\lambda_p = p$. This follows from (2.10) and (2.12). Moreover, using (2.7) and the fact: $\eta'_p = \chi_p = \left(\frac{\cdot}{p}\right)$, we get that $g | Y_p = pg$.

Applying [U1,(1.26)] to $g \ (= \sum_{n \geq 1} a(n) e(nz) \in S\left(k + 1/2, \tilde{N}(\alpha), \eta'\right)_K)$ and p, we have $g|Y_p = g|\widetilde{W}(p)^{-1}\tilde{\delta}_p - g + p^{-k/2+3/4} g|U(p)\tilde{\delta}_p$.

Since $p^{-k/2+3/4} g|U(p) = (g|Y_p)|\widetilde{W}(p)^{-1} = (pg)|\widetilde{W}(p)^{-1}$, we therefore have $g = p^{-k/2-1/4} g|U(p)\tilde{\delta}_p = \sum_{n \geq 1} a(pn) e(pnz)$.

From this formula we have that $a(n) = 0$ if $(p, n) = 1$. Hence there is a form $g' \in S\left(k + 1/2, \tilde{N}(\alpha)/p, \eta'\left(\frac{p}{\cdot}\right)\right)_K$ if $k \geq 2$, resp. $V\left(\tilde{N}(\alpha)/p; \eta'\left(\frac{p}{\cdot}\right)\right)_K$ if $k = 1$ such that $g = g'(pz)$ ([U1, (1.11)]).

Both g and g' have the same system of eigen values on the Hecke operators $\tilde{T}(n^2)$, $(n, N) = 1$ (cf. [U1,(1.8)]). Since g is in $\mathfrak{S}^{\emptyset, \kappa'|F}\left(k + 1/2, \tilde{N}(\alpha), \eta'\right)_K$, the system of eigen values corresponds to a primitive forms of weight $2k$ and of conductor $M_1 l_E \times$ (a divisor of $\prod_{l \in F} l^{\alpha_l}$).

On the other hand, from [U1,(3.10)] for the spaces $S\left(k + 1/2, \tilde{N}(\alpha)/p, \eta'\left(\frac{p}{\cdot}\right)\right)_K$ and $V\left(\tilde{N}(\alpha)/p; \eta'\left(\frac{p}{\cdot}\right)\right)_K$, the system of eigen values of g' corresponds to a primitive form of weight $2k$ and of a conductor prime to p.

This is a contradiction. Hence we have $\theta_p \cong \mathfrak{C}_p$. $\square$

Now we will find the condition for $f_\rho | e_{\Pi}^\kappa \neq 0$ by using the results in §1.

(2.15) Claim (1) If $\chi_p = 1$, we have $\lambda_p = \pm \, \mathfrak{g}_p \ (p \in E)$ and $\rho(p) = \pm \, \mathfrak{g}_p \ (p \in D)$. If $\chi_p = \left(\frac{\cdot}{p}\right)$, $\rho(p) = p$ or $-1 \ (p \in D)$.

(2) We have

$$
f_\rho | e_{\Pi}^\kappa \neq 0 \quad \Leftrightarrow \quad
\begin{cases}
\lambda_p = \left(\frac{-1}{p}\right) \kappa(p)\mathfrak{g}_p, & \text{for all } p \in E \text{ with } \chi_p = 1, \\[2mm]
\rho(p) = -1, & \text{for all } p \in D \text{ with } \chi_p = \left(\frac{\cdot}{p}\right), \\[2mm]
\rho(p) = \left(\frac{-1}{p}\right) \kappa(p)\mathfrak{g}_p, & \text{for all } p \in D \text{ with } \chi_p = 1.
\end{cases}
$$

Remark. In the above, the condition: $\lambda_p = \left(\frac{-1}{p}\right) \kappa(p)\mathfrak{g}_p$ depends only on f.

Proof of the claim. The first assertion follow from [U1, (1.27)] and (2.10). We will prove the second assertion.

[⇒] We apply (1.27) to the forms h_ρ and $f_\rho \ (\in S\left(k + 1/2, \tilde{N}(\alpha) \cdot l_D, \chi\right)_K)$. Set $f_\rho(z) := \sum_{n \geq 1} c(n) e(nz)$ and use the notation S_p on π_{h_ρ} in (1.26-27).

From (1.27), we know the following:

(2.16) $\qquad c(n) = 0$ if $n \equiv -x \ (M_1 l_D l_E / p) \mod p$ for some $x \mod p \notin S_p$

We modify the condition. Let first $p \in E$ with $\chi_p = 1$. From the definition of

S_p and the claim (2.14),

$$\text{the condition in (2.16)} \quad \Leftrightarrow \quad -n\left(M_1 l_D l_E/p\right)^{-1} \pmod{p} \notin S_p$$

$$\Leftrightarrow \quad \left(\frac{-nM_1 l_D l_E/p}{p}\right) = \begin{cases} -1, & \text{if } \theta_p \cong \mathfrak{C}_{(p+1)/2}, \\ 1, & \text{if } \theta_p \cong \mathfrak{C}'_{(p+1)/2}. \end{cases}$$

Suppose that $\theta_p \cong \mathfrak{C}_{(p+1)/2}$. From the first assertion, $\lambda_p = \pm \, \mathfrak{g}_p$. If $\lambda_p = -\left(\frac{M_1 l_D l_E/p}{p}\right) \mathfrak{g}_p$, then $\theta_p \cong \mathfrak{C}'_{(p+1)/2}$ (cf. (2.14)). This is a contradiction. Hence $\lambda_p = \left(\frac{M_1 l_D l_E/p}{p}\right) \mathfrak{g}_p$ and so $\left(\frac{-n}{p}\right) \lambda_p \mathfrak{g}_p^{-1} = -1$. Thus we can easily deduce the following from similar argument.

(2.17)
$$\text{the condition in (2.16)} \Leftrightarrow \left(\frac{-n}{p}\right) \lambda_p \mathfrak{g}_p^{-1} = -1 \Leftrightarrow \left(\frac{n}{p}\right) = -\left(\frac{-1}{p}\right) \mathfrak{g}_p \lambda_p^{-1}. \quad \square$$

We suppose that $\lambda_p = -\left(\frac{-1}{p}\right) \kappa(p) \mathfrak{g}_p$ $\left(\Leftrightarrow \kappa(p) = -\left(\frac{-1}{p}\right) \mathfrak{g}_p \lambda_p^{-1}\right)$ for some $p \in E$ with $\chi_p = 1$. If $\left(\frac{n}{p}\right) = \kappa(p)$, we have $c(n) = 0$ from (2.17). On the other hand, if $\left(\frac{n}{p}\right) = -\kappa(p)$ or 0, we have $\left(\frac{n}{p}\right)^2 + \kappa(p)\left(\frac{n}{p}\right) = 0$.

Observing that $f_\rho | e_p^\kappa = \frac{1}{2} \sum_{n \geq 1} c(n) \left(\left(\frac{n}{p}\right)^2 + \kappa(p)\left(\frac{n}{p}\right)\right) e(nz)$, we get $f_\rho | e_p^\kappa = 0$ and so $f_\rho | e_\Pi^\kappa = 0$.

In the same way, we get the assertions in the case of $p \in D$ with $\chi_p = 1$.

Next we consider the case of $p \in D$ with $\chi_p = \left(\frac{\cdot}{p}\right)$. Let $\theta_p \cong \mathbf{1}_{\mathbf{G}(p)}$. Then $S_p = \{0 \bmod p\}$. It follows from (2.16) that $c(n) = 0$ for any n prime to p. Hence we have $f_\rho | e_p^\kappa = 0$. From these $f_\rho | e_\Pi^\kappa \neq 0 \Rightarrow \theta_p \cong \mathfrak{C}_p \Rightarrow \rho(p) = -1$. We use the first assertion at the last implication.

[$\Leftarrow$] From (2.2) and $f_\rho \in B_f^{(\alpha)} \subseteq B^{(\alpha)}$, it follows that $f_\rho | R_p = \kappa(p) f_\rho$ for all $p \in F$. Hence for all $p \in F$, $f_\rho | e_p^\kappa = f_\rho$. Therefore,

$$f_\rho | e_\Pi^\kappa = f_\rho | \prod_{p \in D+E} e_p^\kappa = 2^{-(|D|+|E|)} \sum_{n \geq 1} c(n) \prod_{p \in D+E} \left\{ \left(\frac{n}{p}\right)^2 + \kappa(p)\left(\frac{n}{p}\right) \right\} e(nz).$$

For each prime $p | M_1 l_D l_E$, we choose a residue class $b_p \pmod{p} \in \mathbf{F}_p^\times$ satisfying the following condition:

$$\begin{cases} \left(\frac{b_p}{p}\right) = \left(\frac{M_1 l_D l_E/p}{p}\right) \mathfrak{g}_p \lambda_p^{-1}, & \text{if } p \in E \text{ and } \chi_p = 1, \\ \left(\frac{b_p}{p}\right) = \left(\frac{M_1 l_D l_E/p}{p}\right) \mathfrak{g}_p \rho(p)^{-1}, & \text{if } p \in D \text{ and } \chi_p = 1, \\ \left(\frac{b_p}{p}\right) = \left(\frac{-1}{p}\right) \left(\frac{M_1 l_D l_E/p}{p}\right) \kappa(p), & \text{if } p \in D+E \text{ and } \chi_p = \left(\frac{\cdot}{p}\right), \\ b_p \in S_p, & \text{if } p | M_1. \end{cases}$$

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We can see from the definition S_p and the claim (2.14) that $(b_p)_{p|M_1 l_D l_E} \in \prod_{p|M_1 l_D l_E} S_p$. Hence by using Theorem (1.27), there exists $n_0 \in \mathbf{Z}_+$ such that $c(n_0) \neq 0$ and $-n_0 \equiv b_p \ (M_1 l_D l_E / p) \pmod{p}$ for all prime divisors p of $M_1 l_D l_E$.

For such a n_0, we have by using the assumption in the claim

$$
\left(\frac{n_0}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{b_p}{p}\right)\left(\frac{M_1 l_D l_E / p}{p}\right)
$$

$$
= \begin{cases}
\left(\frac{-1}{p}\right) \mathfrak{g}_p \lambda_p^{-1} = \kappa(p), & \text{if } p \in E \text{ and } \chi_p = 1, \\
\left(\frac{-1}{p}\right) \mathfrak{g}_p \rho(p)^{-1} = \kappa(p), & \text{if } p \in D \text{ and } \chi_p = 1, \\
\kappa(p), & \text{if } p \in D + E \text{ and } \chi_p = \left(\frac{\cdot}{p}\right).
\end{cases}
$$

Thus we see that $\left(\frac{n_0}{p}\right) = \kappa(p)$ and also that the n_0-th Fourier coefficient of $f_\rho | e_\Pi^\kappa$ is equal to $2^{-(|D|+|E|)} c(n_0) \prod_{p \in D+E} \{\kappa(p)^2 + \kappa(p)\kappa(p)\} = c(n_0) \neq 0$. This means that $f_\rho | e_\Pi^\kappa \neq 0$. $\square$

Combining (2.5), (2.9) with (2.15), we get the following claim.

(2.18) Claim Let the notation be the same as above. Then

$$
B_f^{(\alpha)} | e_\Pi^\kappa = \mathbf{C} \, f_{\rho_0} | e_\Pi^\kappa, \quad \rho_0(p) := \begin{cases}
\left(\frac{-1}{p}\right) \kappa(p) \mathfrak{g}_p, & \text{for } p \in D \text{ and } \chi_p = 1, \\
-1, & \text{for } p \in D \text{ and } \chi_p = \left(\frac{\cdot}{p}\right).
\end{cases}
$$

Moreover $f_{\rho_0} | e_\Pi^\kappa \neq 0$ if and only if $f \in \mathcal{B}_\chi$ satisfies the following conditions:

$$
(*) \qquad f | Y_p = \left(\frac{-1}{p}\right) \kappa(p) \mathfrak{g}_p \, f \quad \text{for all } p \in E \text{ with } \chi_p = 1.
$$

We have the following formula:

(2.19) Claim

$$
B^{(\alpha)} | e_\Pi^\kappa = \bigoplus_{f \in \mathcal{B}_\chi, \, f:(*)} \mathbf{C} \, f_{\rho_0} | e_\Pi^\kappa,
$$

$$
\text{and} \quad \rho_0(p) := \begin{cases}
\left(\frac{-1}{p}\right) \kappa(p) \mathfrak{g}_p, & \text{for } p \in D \text{ and } \chi_p = 1, \\
-1, & \text{for } p \in D \text{ and } \chi_p = \left(\frac{\cdot}{p}\right),
\end{cases}
$$

where $(*)$ is the same condition as in the claim (2.18) and f_{ρ_0} is a non-zero common eigenform on Y_p $(p|M_1 l_D l_E)$ such that $f_{\rho_0} | Y_p = \rho_0(p) f_{\rho_0}$ if $p \in D$ resp. $\lambda_p f_{\rho_0}$ if $p \in E$. Here λ_p is the eigen value of f on Y_p.

Proof of the claim. From [U1, (1.20)] and [U3, (1.7)], $f_{\rho_0} | e_\Pi^\kappa \in B_f^{(\alpha)}$ is a common eigenform on all $\tilde{T}(n^2)$ $((n, N) = 1)$ belonging to the same system of eigenvalues as those of f.

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And since there is a $g \in \mathcal{B}_{\eta'}$ such that $f = g|U(u)$, the system of eigenvalues is also the same as those of $g \in \mathcal{B}_{\eta'}$.

By applying the Strong Multiplicity One theorem [U1, (3.11)(2)] to elements in $\mathcal{B}_{\eta'}$, each different element in $\mathcal{B}_{\eta'}$ belongs to a different system of eigenvalues from each other. Therefore, $\{f_{\rho_0}|e_{II}^{\kappa} \; ; \; f \in \mathcal{B}_{\chi}, f$ satisfies the condition $(*)\}$ is a system of linear independent forms. $\square$

We will express the above condition $(*)$ in terms of $g \in \mathcal{B}_{\eta'}$ such that $f = g|U(u)$. Since $\eta'_p = \chi_p$ for all $p \in E$, we have $f|Y_p = \left(\frac{u}{p}\right) g|Y_p U(u)$ for all $p \in E$ with $\chi_p = 1$ (cf. (2.7)).

Hence, the condition $(*) \Leftrightarrow \left(\frac{u}{p}\right) g|Y_p U(u) = \left(\frac{-1}{p}\right) \kappa(p) \mathfrak{g}_p g|U(u)$.

Moreover since $U(u)$ is an isomorphism on $S\left(k + 1/2, \tilde{N}(\alpha) \cdot l_D, \eta'\right)$, we get that the condition $(*) \Leftrightarrow g|Y_p = \left(\frac{-u}{p}\right) \kappa(p) \mathfrak{g}_p g$.

In [U1, (3.6)], we defined the hermitian involutions $\mathbf{w}_p$ $(p \,|\, M_1 l_E)$ on $\mathfrak{S}^{\emptyset, \kappa'|_F}\left(k + 1/2, \tilde{N}(\alpha), \eta'\right)_K$. Then $g|\mathbf{w}_p = p^{-1/2} \left(\frac{-1}{p}\right)^{k+1/2} \eta'(p) g|Y_p$ for all $p \in E$ with $\chi_p = 1$. This follows from [U1, (3.7)] and $\eta'_p = \chi_p$ for all $p \in E$. Hence $f : (*) \Leftrightarrow g|\mathbf{w}_p = \left(\frac{-1}{p}\right)^k \eta'(p) \left(\frac{u}{p}\right) \kappa(p) g = \left(\frac{-1}{p}\right)^k \chi(p) \kappa(p) g$. Here, we note that $\chi(p)$ is meaningful because of $\chi_p = 1$.

We denote this condition for g by $(**)$:

$$(**) \qquad g|\mathbf{w}_p = \left(\frac{-1}{p}\right)^k \chi(p) \kappa(p) g \quad \text{for all } p \in E \text{ with } \chi_p = 1 \,,$$

and we denote

$$\mathfrak{S}_{**}^{\emptyset, \kappa'|_F} = \mathfrak{S}_{**}^{\emptyset, \kappa'|_F}\left(k + 1/2, \tilde{N}(\alpha), \eta'\right)_K$$
$$:= \left\{ \begin{array}{l} h \in \mathfrak{S}^{\emptyset, \kappa'|_F}\left(k + 1/2, \tilde{N}(\alpha), \eta'\right)_K \; ; \\ h|\mathbf{w}_p = \left(\frac{-1}{p}\right)^k \chi(p) \kappa(p) h \text{ for all } p \in E \text{ with } \chi_p = 1 \end{array} \right\} .$$

From [U1, (3.9) and (3.11)], we have $g|\mathbf{w}_p = \pm g$ for all $g \in \mathcal{B}_{\eta'}$ and so $\{g \in \mathcal{B}_{\eta'} \; ; \; g$ satisfies the condition $(**)\}$ is a basis of the space $\mathfrak{S}_{**}^{\emptyset, \kappa'|_F}$.

As we show in the proof of the claim (2.19), all of three forms $f_{\rho_0}|e_{II}^{\kappa}$, f, and g are common eigenforms on $\tilde{T}(n^2)$ $(n \in \mathbf{Z}_+, (n, N) = 1)$ and all of them belong to the same system of eigenvalues.

Therefore, for all $n \in \mathbf{Z}_+$ and $(n, N) = 1$,

(2.20)

$$
\mathrm{tr}\left(\tilde{T}(n^2); B^{(\alpha)}|e_\Pi^\kappa\right) = \mathrm{tr}\left(\tilde{T}(n^2); \bigoplus_{f \in \mathcal{B}_\chi, f:(*)} \mathbf{C}f_{\rho_0}|e_\Pi^\kappa\right)
$$

$$
= \mathrm{tr}\left(\tilde{T}(n^2); \bigoplus_{g \in \mathcal{B}_{\eta'}, g:(**)} \mathbf{C}g\right)
$$

$$
= \mathrm{tr}\left(\tilde{T}(n^2); \mathfrak{S}_{**}^{\emptyset, \kappa'|F}\left(k + 1/2, \tilde{N}(\alpha), \eta'\right)_K\right) .
$$

$\square$

We will describe the right hand side of the above formula in terms of cusp forms of weight $2k$.

Take any element g of the basis of $\mathfrak{S}_{**}^{\emptyset, \kappa'|F}$, i.e. $g \in \mathcal{B}_{\eta'}$ which satisfies the condition $(**)$. Let G be the primitive form of weight $2k$ which corresponds to g in the sense of [U1, Theorem (3.11)]. From [U1, (3.9)], we have $g|U(p^2) = -p^{k-1}\, g|\mathbf{w}_p = -p^{k-1}\left(\frac{-1}{p}\right)^k \chi(p)\kappa(p)\, g$ for all $p \in E$ with $\chi_p = 1$. So from [U1, (3.11)], $G|U(p) = -p^{k-1}\left(\frac{-1}{p}\right)^k \chi(p)\kappa(p)\, G$.

Let $W(p)$ be the Atkin-Lehner operator of weight $2k$ (cf. §0 (b)). From [M, Corollary 4.6.18], $G|U(p) = -p^{k-1}G|W(p)$ and so $G|W(p) = \left(\frac{-1}{p}\right)^k \chi(p)\kappa(p)\, G$ for all $p \in E$ with $\chi_p = 1$. Hence the space $\mathfrak{S}_{**}^{\emptyset, \kappa'|F}$ corresponds to all such G's.

We can deduce the following expression by the trace relation in the paper [U1, (3.10)(2), (A.2)(3), and (A.5)]:

(2.21) For any $n \in \mathbf{Z}_+$ prime to $\tilde{N}(\alpha)$,

$$
\mathrm{tr}\left(\tilde{T}(n^2); \mathfrak{S}_{**}^{\emptyset, \kappa'_F}\left(k + 1/2, \tilde{N}(\alpha), \eta'\right)_K\right)
$$

$$
= \sum_{((\beta_l), \Psi', (\tau', \sigma')) \in P'} \Xi'((\beta_l), \Psi', (\tau', \sigma'))
$$

$$
\times \mathrm{tr}\left(T(n); S^{*(\hat{\tau}', \sigma')}\left(2k, M_1 l_E \prod_{l \in F} l^{\beta_l}\right)|R_{\Psi'}\right) ,
$$

where the notations are as follows: $\sum_{((\beta_l), \Psi', (\tau', \sigma')) \in P'}$ is the sum extended over all elements of the following set:

$$
P' := \left\{
\begin{array}{l}
((\beta_l)_{l \in F}, \Psi', (\tau', \sigma'))\ ;\ (\beta_l) = (\beta_l)_{l \in F}\ \text{is a system of integers} \\
\text{such that } 0 \le \beta_l \le \alpha_l\ \text{for any } l \in F, \Psi' \subseteq F(\beta)_0 + F(\beta)_1, \\
\tau' \in \mathrm{Map}(F, \{\pm 1\}), \sigma' \in \mathrm{Map}(F(\beta)_{2+}, \{\pm 1\})
\end{array}
\right\} .
$$

$F(\beta)_i := \{l \in F \mid \beta_l = i\}$ for $i = 0, 1$ and $F(\beta)_{2+} := \{l \in F \mid \beta_l \ge 2\}$. $\Xi'((\beta_l), \Psi', (\tau', \sigma')) = \prod_{p \in F} \Xi'_p((\beta_l), \Psi', (\tau', \sigma'))$. Each $\Xi'_p((\beta_l), \Psi', (\tau', \sigma'))$ is the

constant determined by the table [U1, (2.22)]. $\hat{\tau}'$ is the extension of τ which is defined as follows:

$$\hat{\tau}'(l) = \begin{cases} \tau'(l), & \text{if } l \in F; \\ \left(\frac{-1}{l}\right)^k \chi(l)\kappa(l) \prod_{q \in \Psi'}\left(\frac{l}{q}\right), & \text{if } l \in E \text{ and } \chi_l = 1. \end{cases}$$

Finally,

$$S^{*(\hat{\tau}',\sigma')}\left(2k, M_1 l_E \prod_{l \in F} l^{\beta_l}\right) := \left\{ \begin{array}{l} f \in S^*\left(2k, M_1 l_E \prod_{l \in F} l^{\beta_l}\right) ; \\ f|W_l = \hat{\tau}'(l)f \text{ for all } l \in F + \{l \in E \mid \chi_l = 1\}, \\ f|R_l W_l = \sigma'(l)f|R_l \text{ for all } l \in F(\beta)_{2+} \end{array} \right\}.$$

See [U1, Appendix 1] for the definition of the space $S^*(2k, M_1 l_E \prod_{l \in F} l^{\beta_l})$. $\square$

In order to compare this formula (2.21) with those in [U1,(3.10)(2)], we must modify the above formula.

We define a system of integers $(\tilde{\beta}_l)_{l \in \Pi}$ for each system of integers $(\beta_l)_{l \in F}$ as follows: $\tilde{\beta}_l = \beta_l$, 1, or 0 according to $l \in F$, E, or D.

Then we have $\Pi(\tilde{\beta})_0 = D + F(\beta)_0$, $\Pi(\tilde{\beta})_1 = E + F(\beta)_1$, $\Pi(\tilde{\beta})_{2+} = F(\beta)_{2+}$, and $l_E \times \prod_{l \in F} l^{\beta_l} = \prod_{l \in \Pi} l^{\tilde{\beta}_l}$.

We want to extend the domain of definition of τ' $(=F)$ to the whole set Π. Since $\{\pm 1\}^{\Pi} = \{\pm 1\}^F \times \{\pm 1\}^{D+E}$, we will append this factor $\{\pm 1\}^{D+E}$ and denote each element of $\{\pm 1\}^{\Pi} = \{\pm 1\}^F \times \{\pm 1\}^{D+E}$ as $\tau = (\tau', \tau'')$.

We note that there are some assumptions on the primes $l \in E$ with $\chi_l = 1$ in the definition of the space $S^{*(\hat{\tau}',\sigma')}(2k, M_1 l_E \prod_{l \in F} l^{\beta_l})$. We remove this condition by replacing the coefficients $\Xi'((\beta_l), \Psi', (\tau', \sigma')) =: \prod_{p \in F} \Xi'_p((\beta_l), \Psi', (\tau', \sigma'))$ with $\prod_{p \in F} \Xi'_p((\beta_l), \Psi', (\tau', \sigma')) \times \prod_{p \in D+E} \Xi''_p$, where the notation is as follows:

$$2 \times \Xi''_p := \begin{cases} 1 + \tau''(p), & \text{if } p \in D; \\ 1 + 1, & \text{if } p \in E \text{ and } \chi_p = \left(\frac{\cdot}{p}\right); \\ 1 + \left(\frac{-1}{p}\right)^k \chi(p)\kappa(p) \prod_{q \in \Psi'}\left(\frac{p}{q}\right)\tau''(p), & \text{if } p \in E \text{ and } \chi_p = 1. \end{cases}$$

Finally replacing σ' with σ and combining the formula (2.20) and (2.21), we can obtain the following expression of $\mathrm{tr}\left(\tilde{T}(n^2); B^{(\alpha)}|e_{\Pi}^{\kappa}\right)$ in terms of primitive forms of weight $2k$.

(2.22) Under the above notation, we get the followings for all $n \in \mathbf{Z}_+$ prime to N,

$$\mathrm{tr}\left(\tilde{T}(n^2); B^{(\alpha)}|e_{\Pi}^{\kappa}\right)$$
$$= \sum_{((\beta_l),\Psi',(\tau',\sigma),\tau'') \in P' \times \{\pm 1\}^{D+E}} \prod_{p \in F} \Xi'_p((\beta_l), \Psi', (\tau', \sigma)) \times \prod_{p \in D+E} \Xi''_p$$
$$\times \mathrm{tr}\left(T(n); S^{*(\tau,\sigma)}\left(2k, M_1 \prod_{l \in \Pi} l^{\tilde{\beta}_l}\right)|R_{\Psi'}\right).$$

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where $\tau := (\tau', \tau'') \in \{\pm 1\}^F \times \{\pm 1\}^{D+E} = \{\pm 1\}^\Pi$, $\sigma \in \{\pm 1\}^{F(\beta)_{2+}} = \{\pm 1\}^{\Pi(\tilde\beta)_{2+}}$, and

$$S^{*(\tau,\sigma)}\left(2k, M_1 \prod_{l\in\Pi} l^{\tilde\beta_l}\right) := \left\{ \begin{array}{l} f \in S^*\left(2k, M_1 \prod_{l\in\Pi} l^{\tilde\beta_l}\right); \\ f|W_p = \tau(p)f \text{ for all } p \in \Pi, \\ f|R_p W_p = \sigma(p)f|R_p \text{ for all } l \in \Pi(\tilde\beta)_{2+} = F(\beta)_{2+} \end{array} \right\}.$$

$\square$

(2.23) Now we will compare the above formula (2.22) with [U1, (3.10)(2)]. And we will see that the formula (2.22) can be considered as a part of the expression formula [U1, (3.10)(2)] of $\mathrm{tr}\left(\tilde{T}(n^2); \mathfrak{S}^{\theta,\kappa}(k+1/2, N, \chi)_K\right)$.

We first note that for any parameter $((\beta_l), \Psi', (\tau', \sigma), \tau'') \in P' \times \{\pm 1\}^{D+E}$, the system $((\tilde\beta_l)_{l\in\Pi}, \Psi', (\tau, \sigma))$ contains in the range P of parameters in the expression formula [U1,(3.10)(2)] of the trace $\mathrm{tr}\left(\tilde{T}(n^2); \mathfrak{S}^{\theta,\kappa}(k+1/2, N, \chi)_K\right)$. This is easily verified.

We second must study for any $(\beta_l)_{l\in F}$, Ψ', $(\tau, \sigma) = ((\tau', \sigma), \tau'')$, whether the coefficient of $\mathrm{tr}\left(T(n); S^{*(\tau,\sigma)}\left(2k, M_1 \prod_{l\in\Pi} l^{\tilde\beta_l}\right) | R_{\Psi'}\right)$ is equal to those in the expression formula [U1, (3.10)(2)].

Both coefficients are defined as products of primary components on $p \in \Pi$. We will compare these primary components.

For $p \in F + E$, we can verify the following identities:

$$(2.24) \qquad \Xi_p((\tilde\beta_l), \Psi', (\tau, \sigma)) = \left\{ \begin{array}{ll} \Xi'_p((\beta_l), \Psi', (\tau', \sigma)), & \text{if } p \in F; \\ \Xi''_p, & \text{if } p \in E, \end{array} \right.$$

where the left-hand side is the constant with respect to $\mathfrak{S}^{\theta,\kappa}(k+1/2, N, \chi)_K$ determined by [U1, (2.22)] and $\tau' := \tau|_F$.

These identities are easily verified, in case by case, from the following facts.

(1) The 2-primary part of χ is $\chi_2 = \eta_2 \eta'_2 = \left(\frac{(-1)}{\cdot}\right)\left(\frac{(-1)}{\cdot}\right)$, $\eta'_2 = \left(\frac{(-1)}{\cdot}\right)$, and also $\eta'_{2M_1 l_E} = \left(\frac{(-1)}{\cdot}\right) \chi_{M_1} \prod_{p\in E} \chi_p$.

(2) For $p \in E + F$, we have $\chi_p = \eta'_p$.

Next we consider the case of $p \in D$. Then if $\tau(p) = -1$, the identity like (2.24) does not hold good.

However, since any prime $p \in D$ does not occurs in the level $M_1 \prod_{l\in\Pi} l^{\tilde\beta_l}$, the Atkin-Lehner operator W_p is the identity operator $\mathbf{1}$. Hence if $\tau(p) = -1$, the space $S^{*(\tau,\sigma)}\left(2k, M_1 \prod_{l\in\Pi} l^{\tilde\beta_l}\right)$ is always equal to $\{0\}$. Therefore we can neglect such cases and eventually without a loss of validity, we can consider that we also have the same identity as (2.24) in the case of $p \in D$. $\square$

Therefore we get the following formula.

(2.25) Proposition Let notation be the same as above and $n \in \mathbf{Z}_+$ such that $(n, N) = 1$.

$$\mathrm{tr}\Big(\tilde{T}(n^2); B^{(\alpha)}|e_\Pi^\kappa\Big) = \sum_{((\beta_l), \Psi', (\tau', \sigma), \tau'') \in P' \times \{\pm 1\}^{D+E}} \prod_{l \in \Pi} \Xi_l((\tilde{\beta}_l), \Psi', (\tau, \sigma))$$
$$\times \mathrm{tr}\Bigg(T(n); S^{*(\tau,\sigma)}\Bigg(2k, M_1 \prod_{l \in \Pi} l^{\tilde{\beta}_l}\Bigg) |R_{\Psi'}\Bigg) ,$$

where each coefficient $\Xi_l((\tilde{\beta}_l), \Psi', (\tau, \sigma))$ has the value either 0 or 1 which is the co-efficient of $\mathrm{tr}\Big(T(n); S^{*(\tau,\sigma)}\Big(2k, M_1 \prod_{l \in \Pi} l^{\tilde{\beta}_l}\Big) |R_{\Psi'}\Big)$ in the expression formula [U1, (3.10)(2)] of the trace of $\tilde{T}(n^2)$ on the space $\mathfrak{S}^{\emptyset,\kappa}(k+1/2, N, \chi)_K$.

In other words, we can consider the above formula as a part of the expression formula [U1,(3.20)(2)]. $\square$

(2.26) For any (α_l), $B^{(\alpha)}|e_\Pi^\kappa$ is a subspace of $\mathfrak{S}^{\emptyset,\kappa}(k + 1/2, N, \chi)_K$ (cf. (2.4)). We define the following subspace of $\mathfrak{S}^{\emptyset,\kappa}(k + 1/2, N, \chi)_K$:

$$(2.27) \qquad \hat{\mathfrak{N}}^{\emptyset,\kappa} = \hat{\mathfrak{N}}^{\emptyset,\kappa}(k + 1/2, N, \chi)_K := \sum_{(\alpha_l) \neq (\nu_l)} B^{(\alpha)}|e_\Pi^\kappa .$$

Here, (α_l) in the above sum runs over all system of integers such that $0 \le \alpha_l \le \nu_l = \mathrm{ord}_l(N)$ $(l \in \Pi)$ and $(\alpha_l)_{l \in \Pi} \neq (\nu_l)_{l \in \Pi}$. We also denote by $\mathfrak{N}^{\emptyset,\kappa} = \mathfrak{N}^{\emptyset,\kappa}(k + 1/2, N, \chi)_K$ the orthogonal complement of $\hat{\mathfrak{N}}^{\emptyset,\kappa}(k + 1/2, N, \chi)_K$ in $\mathfrak{S}^{\emptyset,\kappa}(k+1/2, N, \chi)_K$.

Since any operators $\tilde{T}(n^2)$ $((n, N) = 1)$ fix each subspace $B^{(\alpha)}|e_\Pi^\kappa$ respectively, both $\hat{\mathfrak{N}}^{\emptyset,\kappa}$ and $\mathfrak{N}^{\emptyset,\kappa}$ are stable under the action of $\tilde{T}(n^2)$ $((n, N) = 1)$.

We can generalize [U1, Theorem (4.13)] as follows:

(2.28) Theorem Let the notation be the same as above and let $\kappa \in \{\pm 1\}^\Pi$. We suppose that $\mathrm{ord}_2(N) = 2$. The following assertions hold good.

(1)   For all $n \in \mathbf{Z}_+$ prime to N,

$$\mathrm{tr}\Big(\tilde{T}(n^2); \mathfrak{N}^{\emptyset,\kappa}(k + 1/2, N, \chi)_K\Big)$$
$$= \sum_{\Pi(\nu)_2 = I+J+K} \sum_{\substack{\tau \in \mathrm{Map}(\Pi, \{\pm 1\}) \\ \sigma \in \mathrm{Map}(\Pi - (I+J), \{\pm 1\})}} \Xi((\nu(I, J)_l), I + J, (\tau, \sigma))$$
$$\times \mathrm{tr}\Bigg(T(n); S^{*(\tau,\sigma)}\Bigg(2k, M_1 \prod_{l \in J} l \prod_{l \in \Pi - (I+J)} l^{\nu_l}\Bigg) |R_{I+J}\Bigg) ,$$

where $\Pi(\nu)_2 := \{l \in \Pi \mid \nu_l = 2\}$, $\sum_{\Pi(\nu)_2 = I+J+K}$ is the sum extended over all partitions such that $\Pi(\nu)_2 = I + J + K$, $\nu(I, J)_l$ is a constant which has a value 0, 1, or ν_l ($:=\mathrm{ord}_l(N)$) according to $l \in I$, J, or $\Pi - (I + J)$, $\Xi((\nu(I, J)_l), I + J, (\tau, \sigma))$ are the constants determined by [U1,(2.22)].

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(2) Let $\underline{\underline{B}}$ be an orthogonal basis of $\mathfrak{S}^{\emptyset,\kappa}(k+1/2,N,\chi)_K$ which is stated in [U1,(3.10)(2)]. Let $\underline{\underline{B_0}}$ (resp. $\underline{\underline{B_1}}$) be the set of all $f \in \underline{\underline{B}}$ which correspond to primitive forms $\in S^0(2k,M)$ (resp. $\in S^0(2k,M')$, $0 < M'|M$, and $M' < M$) in the sense of [U1,Theorem (3.11)(1)]. Then $\underline{\underline{B_0}}$ (resp. $\underline{\underline{B_1}}$) generates the space $\mathfrak{N}^{\emptyset,\kappa}(k+1/2,N,\chi)_K$ (resp. $\hat{\mathfrak{N}}^{\emptyset,\kappa}(k+1/2,N,\chi)_K$).

(3) Let f be any element of $\underline{\underline{B_0}}$ and λ_p the eigenvalue of f with respect to $\tilde{T}_{k+1/2,N,\chi}(p^2)$ (p:prime, $p \nmid M$) resp. $U(p^2)$ (p:prime, $p|M$). Then the primitive form F which corresponds to f in the sense of [U1, Theorem (3.11)(1)] satisfies the condition $F|T(p) = \lambda_p F$ resp. $F|U(p) = \lambda_p F$ for all primes with $p \nmid M$ resp. $p|M$.

Proof. Let $\underline{\underline{B}}$ be the same C-basis of $\mathfrak{S}^{\emptyset,\kappa}(k+1/2,N,\chi)_K$ as in the above statement (2) and P the same set of parameters as in the expression formula [U1,(3.10)(2)].

For any (α_l), the space $B^{(\alpha)}|e_{\Pi}^{\kappa}$ is a stable subspace of $\mathfrak{S}^{\emptyset,\kappa}(k+1/2,N,\chi)_K$ under the action of all operators $\tilde{T}(n^2)$ $((n,N)=1)$. Using the Strong Multiplicity One theorem [U1, (3.11)(2)] on $\mathfrak{S}^{\emptyset,\kappa}(k+1/2,N,\chi)_K$, $B^{(\alpha)}|e_{\Pi}^{\kappa}$ is generated by the set $\underline{\underline{B}} \cap (B^{(\alpha)}|e_{\Pi}^{\kappa})$. Similarly, we can see that the spaces $\hat{\mathfrak{N}}^{\emptyset,\kappa}$ and $\mathfrak{N}^{\emptyset,\kappa}$ are generated by $\underline{\underline{B}} \cap \hat{\mathfrak{N}}^{\emptyset,\kappa} = \cup_{(\alpha_l)\neq(\nu_l)} (B^{(\alpha)}|e_{\Pi}^{\kappa}) \cap \underline{\underline{B}}$ and $\mathfrak{N}^{\emptyset,\kappa} \cap \underline{\underline{B}}$ respectively. And also $\underline{\underline{B}} = (\underline{\underline{B}} \cap \mathfrak{N}^{\emptyset,\kappa}) + (\underline{\underline{B}} \cap \hat{\mathfrak{N}}^{\emptyset,\kappa})$. For any $((\rho_l), \Psi, (\tau,\sigma)) \in P$, we denote by $\underline{\underline{B}}_{((\rho_l),\Psi,(\tau,\sigma))}$ the subset of $\underline{\underline{B}}$ which corresponds to the subspace $S^{*(\tau,\sigma)}(2k,M_1 \prod_{l\in\Pi} l^{\rho_l}) |R_{\Psi}$ in the sense of [U1,(3.11)(1)].

Then from (2.25), there exist only two possible cases: either $\underline{\underline{B}}_{((\rho_l),\Psi,(\tau,\sigma))} \subseteq B^{(\alpha)}|e_{\Pi}^{\kappa}$, or $\underline{\underline{B}}_{((\rho_l),\Psi,(\tau,\sigma))} \cap B^{(\alpha)}|e_{\Pi}^{\kappa} = \emptyset$; and whether the former case is or not depends only on the parameter $((\rho_l), \Psi, (\tau,\sigma))$.

We define the subsets P^* and P^{**} of P by:

$$P^* := \left\{ \begin{array}{l} ((\rho_l), \Psi, (\tau,\sigma)) \in P; ((\rho_l), \Psi, (\tau,\sigma)) \text{ does not satisfy} \\ \text{(at least) one of the following three conditions:} \\ \text{(i) } \rho_l = \nu_l \text{ on } \Pi(\rho)_{2+}; \text{ (ii) } \Psi = \Pi(\rho)_0 + \Pi(\rho)_1; \\ \text{(iii) } \nu_l = 2 \text{ on } \Pi(\rho)_0 + \Pi(\rho)_1 \end{array} \right\}$$

and $P^{**} := P - P^*$.

Take a system $(\alpha_l)_{l\in\Pi}$ and a parameter $((\rho_l), \Psi, (\tau,\sigma)) \in P$ such that $(\alpha_l) \neq (\nu_l)$ and $\underline{\underline{B}}_{((\rho_l),\Psi,(\tau,\sigma))} \subseteq B^{(\alpha)}|e_{\Pi}^{\kappa}$. Then $((\rho_l), \Psi, (\tau,\sigma)) \in P^*$. In fact, suppose that $((\rho_l), \Psi, (\tau,\sigma))$ satisfies the above condition (i) and (ii). We get from (2.25) that $\rho_l \leq \alpha_l$ for all $l \in \Pi$ and $\Psi \subseteq \Pi(\alpha)_{2+}$. Hence $\Pi(\rho)_{2+} \subseteq \Pi(\alpha)_{2+}$ and by using the condition (i), $(\rho_l =) \alpha_l = \nu_l$ for all $l \in \Pi(\rho)_{2+}$.

From the condition (ii), $\Pi(\alpha)_{2+} \supseteq \Psi = \Pi - \Pi(\rho)_{2+} \supseteq \Pi - \Pi(\alpha)_{2+}$. This means $\Pi = \Pi(\alpha)_{2+}$.

Since $(\alpha_l) \neq (\nu_l)$, there exists $l \in \Pi$ such that $\alpha_l < \nu_l$. From the above, then $l \in \Pi - \Pi(\rho)_{2+}$ ($\subseteq \Pi(\alpha)_{2+}$) and $2 \leq \alpha_l < \nu_l$. Hence the condition (iii) is not satisfied.

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The contrary is also true. Take any $((\rho_l), \Psi, (\tau, \sigma)) \in P^*$. Put

$$\alpha_l := \begin{cases} \rho_l, & \text{if } l \in \Pi(\rho)_{2+}, \\ 2, & \text{if } l \in \Psi, \\ 1, & \text{if } \Pi(\rho)_1 - \Psi, \\ 0, & \text{if } \Pi(\rho)_0 - \Psi. \end{cases}$$

Since $((\rho_l), \Psi, (\tau, \sigma))$ does not satisfy one of the conditions (i)–(iii), we have $(\alpha_l) \neq (\nu_l)$.

Next we put

$$\beta_l := \begin{cases} \rho_l, & \text{if } l \in \Pi(\rho)_{2+}, \\ 1, & \text{if } l \in \Psi \cap \Pi(\rho)_1, \\ 0, & \text{if } l \in \Psi \cap \Pi(\rho)_0. \end{cases}$$

We also define the system of integers $(\tilde{\beta}_l)_{l \in \Pi}$ by the above (β_l) in the same manner as in (2.21). Then we have $\tilde{\beta}_l = \rho_l$ for all $l \in \Pi$. Moreover observing that $\Psi \subseteq \Pi(\rho)_0 + \Pi(\rho)_1 \subseteq (\Pi(\alpha)_{2+})(\beta)_0 + (\Pi(\alpha)_{2+})(\beta)_1$ and $(\Pi(\alpha)_{2+})(\beta)_{2+} = \Pi(\rho)_{2+}$, we can see that $((\beta_l)_{l \in \Pi(\alpha)_{2+}}, \Psi, (\tau, \sigma))$ occurs in the parameter set $P' \times \{\pm 1\}^{\Pi(\alpha)_1 + \Pi(\alpha)_0}$ of the expression formula (2.25). Hence $\underline{\underline{B}}_{((\rho_l), \Psi, (\tau, \sigma))} \subseteq B^{(\alpha)} | e_{\Pi}^{\kappa}$.

Thus, we see that $\hat{\mathfrak{N}}^{\theta, \kappa}$ is generated by the set $\cup_{((\rho_l), \Psi, (\tau, \sigma)) \in P^*} \underline{\underline{B}}_{((\rho_l), \Psi, (\tau, \sigma))}$ and so $\cup_{((\rho_l), \Psi, (\tau, \sigma)) \in P^{**}} \underline{\underline{B}}_{((\rho_l), \Psi, (\tau, \sigma))}$ generates $\mathfrak{N}^{\theta, \kappa}$. The assertion (1) is easily deduced from this result.

We will prove the assertion (2). For any $((\rho_l), \Psi, (\tau, \sigma)) \in P$, we have from the definition: $S^{*(\tau, \sigma)}\left(2k, M_1 \prod_{l \in \Pi} l^{\rho_l}\right) | R_{\Psi} \subseteq S^0\left(2k, M_1 \prod_{l \in \Pi - \Psi} l^{\rho_l} \prod_{l \in \Psi} l^2\right)$. Hence the assertion (2) follows from the fact:

$$((\rho_l), \Psi, (\tau, \sigma)) \in P^{**} \quad \Leftrightarrow \quad \prod_{l \in \Pi - \Psi} l^{\rho_l} \prod_{l \in \Psi} l^2 = M_{2+} .$$

The assertion (3) can be proved by the same method as in the [U1, Theorem (3.11)(1)]. $\square$

§3 Another simpler definition of newforms for Kohnen spaces.

In this section, we give another simpler definition of newforms for Kohnen spaces than the previous one.

We keep to the notations in the previous sections and the paper [U1]. See those for the details of definitions and notations.

We note that we consider only Kohnen spaces in this section.

(3.1) In this paper and the previous paper [U1], we define several subspaces and give several decompositions. Combining those decompositions, we obtain the

following:

$$\left\{ \begin{array}{ll} S := S(k+1/2, N, \chi)_K & \text{if } k \geq 2 \\ V := V(N; \chi)_K & \text{if } k = 1 \end{array} \right\}$$

$$= \left(\bigoplus_{\kappa \in \{\pm 1\}^{\Pi}} \mathfrak{N}^{\emptyset, \kappa}(k+1/2, N, \chi)_K \right) \oplus \left(\bigoplus_{\kappa \in \{\pm 1\}^{\Pi}} \hat{\mathfrak{N}}^{\emptyset, \kappa}(k+1/2, N, \chi)_K \right)$$

$$\oplus \left(\bigoplus_{\kappa \in \{\pm 1\}^{\Pi}} \bigoplus_{\substack{0 < e, d,\ d \neq M_1 \\ ed|M_1}} \mathfrak{S}^{\emptyset, \kappa}(k+1/2, 4dM_{2+}, \chi)_K | U(e^2) \right)$$

$$\oplus \left\{ \begin{array}{ll} \text{Ker}(R_\Pi, S) & \text{if } k \geq 2 \\ \text{Ker}(R_\Pi, V) & \text{if } k = 1 \end{array} \right\} .$$

This formula follows from [U1, (1.5), (1.7), (3.10)(2)] and (2.27) in this paper. □

(3.2) Definition We put the following notation:

$$\mathfrak{N}(k+1/2, N, \chi)_K := \bigoplus_{\kappa \in \{\pm 1\}^{\Pi}} \mathfrak{N}^{\emptyset, \kappa}(k+1/2, N, \chi)_K ,$$

$$\mathfrak{O}(k+1/2, N, \chi)_K := \left(\bigoplus_{\kappa \in \{\pm 1\}^{\Pi}} \hat{\mathfrak{N}}^{\emptyset, \kappa}(k+1/2, N, \chi)_K \right)$$

$$\oplus \left(\bigoplus_{\kappa \in \{\pm 1\}^{\Pi}} \bigoplus_{\substack{0 < e, d, d \neq M_1 \\ ed|M_1}} \mathfrak{S}^{\emptyset, \kappa}(k+1/2, 4dM_{2+}, \chi)_K | U(e^2) \right)$$

$$\oplus \left\{ \begin{array}{ll} \text{Ker}(R_\Pi, S) & \text{if } k \geq 2 \\ \text{Ker}(R_\Pi, V) & \text{if } k = 1 \end{array} \right\} .$$

Then we rewrite the formula in (3.1) as follows.

$$\left\{ \begin{array}{ll} S := S(k+1/2, N, \chi)_K , & \text{if } k \geq 2 \\ V := V(N; \chi)_K , & \text{if } k = 1 \end{array} \right\} = \mathfrak{N}(k+1/2, N, \chi)_K \oplus \mathfrak{O}(k+1/2, N, \chi)_K.$$

□

(3.3) Proposition

$$\text{Ker}(R_\Pi, S) = \sum_{l \in \Pi} S\left(k+1/2, N/l, \chi\left(\tfrac{l}{\cdot}\right)\right)_K |\tilde{\delta}_l .$$

$$\text{Ker}(R_\Pi, V) = \sum_{l \in \Pi} V\left(N/l; \chi\left(\tfrac{l}{\cdot}\right)\right)_K |\tilde{\delta}_l .$$

Proof. By using [U1,(1.9)], the subspaces of the right-hand sides are contained in S, V respectively. Since $(f|\tilde{\delta}_l)(z) = l^{k/2+1/4} f(lz)$, those subspaces are contained in the subspaces of left-hand sides.

We will prove the contrary. Let I be any non-empty subset of Π.

From [U1,(1.5)(3)], it is sufficient to prove that S^I, V^I are contained in the subspaces of right-hand sides respectively.

From the formula after [U1,(1.4)], we have

$$S^I = \{ \sum_{n \geq 1} b(n) \mathbf{e}(nz) \in S; b(n) = 0 \text{ for all } n \notin L_I \},$$

$$V^I = \{ \sum_{n \geq 1} b(n) \mathbf{e}(nz) \in V; b(n) = 0 \text{ for all } n \notin L_I \},$$

where $L^I := \{ a \in \mathbf{Z}; (a, l_\Pi) = l_I \} \ (\subseteq \mathbf{Z} l_I)$.

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Take a prime $l \in I$ and any element $f = \sum_{n=1}^{\infty} b(n)e(nz)$ in S^I or V^I. Then from the above formula, $b(n) \neq 0 \Rightarrow n \in L_I \subseteq \mathbf{Z}l$. Hence there exists a function $g(z)$ on $\mathfrak{H}$ such that $f(z) = g(lz)$. It follows from [U1,(1.11)] that $g \in S\left(k + 1/2, N/l, \chi\left(\frac{l}{\cdot}\right)\right)_K$ if $f \in S^I$, or $g \in V\left(N/l; \chi\left(\frac{l}{\cdot}\right)\right)_K$ if $f \in V^I$. This prove the contrary inclusion. $\square$

(3.4) Definition For any primitive form F of weight $2k$, we denote by $\{\lambda_F(n); n \in \mathbf{N}\}$ the system of eigenvalues on the Hecke operators $\{T(n); n \in \mathbf{N}\}$. Then we define two subspaces of $S(k + 1/2, N, \chi)_K$ as follows:

$$S(k + 1/2, N, \chi; F)_K := \left\{ \begin{array}{l} f \in S(k + 1/2, N, \chi)_K \, ; f|\tilde{T}(n^2) = \lambda_F(n)f \\ \text{for all } n \in \mathbf{Z}_+ \text{ such that } (n, N) = 1 \end{array} \right\} .$$

$$V(N, \chi; F)_K := \left\{ \begin{array}{l} f \in V(N; \chi)_K \, ; f|\tilde{T}(n^2) = \lambda_F(n)f \\ \text{for all } n \in \mathbf{Z}_+ \text{ such that } (n, N) = 1 \end{array} \right\} .$$

These are considered as the eigen subspaces corresponding to the primitive form F. $\square$

Remark. We can prove that $V(N, \chi; F)_K = S(3/2, N, \chi; F)_K$ for any primitive (cusp) form F.

Under the above notation, we get the following theorem.

(3.5) Theorem (1) For $k \geq 2$, the following decompositions hold good.

$$S(k + 1/2, N, \chi)_K = \bigoplus_{F:(*1)} S(k + 1/2, N, \chi; F)_K \, ,$$

$$\mathfrak{N}(k + 1/2, N, \chi)_K = \bigoplus_{F:(*2)} S(k + 1/2, N, \chi; F)_K \, ,$$

$$\mathfrak{O}(k + 1/2, N, \chi)_K = \bigoplus_{F:(*3)} S(k + 1/2, N, \chi; F)_K \, .$$

(2) For $k = 1$, the following decompositions hold good.

$$V(N; \chi)_K = \bigoplus_{F:(*1)} V(N, \chi; F)_K \, ,$$

$$\mathfrak{N}(3/2, N, \chi)_K = \bigoplus_{F:(*2)} V(N, \chi; F)_K \, ,$$

$$\mathfrak{O}(3/2, N, \chi)_K = \bigoplus_{F:(*3)} V(N, \chi; F)_K \, .$$

Here (*1)–(*3) are the following conditions on primitive forms F of weight $2k$:

(*1)   the conductor of F is a divisor of M.
(*2)   the conductor of F is M.
(*3)   the conductor of F is a divisor of M and less than M.

Proof. From (2.28)(2), the subspaces $\mathfrak{N}^{\theta,\kappa}(k + 1/2, N, \chi)_K$ has an orthogonal **C**-basis consisting of common eigenforms for all Hecke operators $\tilde{T}(n^2)$ ($n \in \mathbf{Z}_+$,

$(n, N) = 1$). Moreover the system of eigenvalues of any element of such basis corresponds to a primitive forms of weight $2k$ and of conductor M.

Similarly, from (2.28)(2) and [U1, (3.10)(2) and (3.5)], the subspaces $\hat{\mathfrak{N}}^{\emptyset,\kappa}(k + 1/2, N, \chi)_K$ and $\mathfrak{S}^{\emptyset,\kappa}(k + 1/2, 4dM_{2+}, \chi)_K | U(e^2)$ have orthogonal $\mathbf{C}$-basis consisting of common eigenforms for all $\tilde{T}(n^2)$ $(n \in \mathbf{Z}_+, (n, N) = 1)$. Moreover the system of eigenvalues of any element of such basis corresponds to a primitive forms of weight $2k$ and of a conductor which divides M and is less than M.

We will prove a similar result for the subspaces $\mathrm{Ker}(R_\Pi, S)$ and $\mathrm{Ker}(R_\Pi, V)$, i. e., these subspaces have orthogonal $\mathbf{C}$-basis consisting of common eigenforms for all $\tilde{T}(n^2)$ $(n \in \mathbf{Z}_+, (n, N) = 1)$ and moreover the system of eigenvalues of any element of such basis corresponds to a primitive forms of weight $2k$ and of a conductor which divides M and is less than M.

We use an induction on M_{2+}.

If $M_{2+} = 1$, $\Pi = \emptyset$ and $R_\Pi = 1$. Hence, $\mathrm{Ker}(R_\Pi, S) = \mathrm{Ker}(R_\Pi, V) = \{0\}$ and the statement holds good.

Let $M_{2+} > 1$ and assume that the above claim holds good if the subspace has a smaller "M_{2+}"-part than M_{2+}.

We consider the spaces $S\left(k + 1/2, N/l, \chi\left(\frac{l}{\cdot}\right)\right)_K$ and $V\left(N/l; \chi\left(\frac{l}{\cdot}\right)\right)_K$ for any $l \in \Pi$. From the assumption of the induction and [U1,(1.8)], these subspaces have orthogonal $\mathbf{C}$-basis consisting of common eigenforms for all $\tilde{T}(n^2)$ $(n \in \mathbf{Z}_+, (n, N) = 1)$ and moreover the system of eigenvalues of any element of such basis corresponds to a primitive forms of weight $2k$ and of conductor dividing M/l.

Combining this and (3.3), the claim for $\mathrm{Ker}(R_\Pi, S)$ and $\mathrm{Ker}(R_\Pi, V)$ follows.

From (3.1) and the Strong Multiplicity One theorem of weight $2k$, the first assertion follows. The second and third assertion are easily seen by using the above results. $\square$

Our next purpose is to rewrite the space of "oldform" $\mathfrak{O}(k + 1/2, N, \chi)_K$.

(3.6) Proposition We have the following formulae.
[The case of $k \geq 2$]

$$\mathfrak{O}(k + 1/2, N, \chi)_K = \sum_{\substack{0 < B | M \\ B \neq M}} \sum_{\substack{0 < A | (M/B) \\ \xi\left(\frac{A}{\cdot}\right) = \chi}} S(k + 1/2, 4B, \xi)_K | \tilde{\delta}_A$$

$$+ \sum_{\substack{0 < B | M \\ B \neq M}} \sum_{\substack{0 < A | (M/B)^2 \\ \xi\left(\frac{A}{\cdot}\right) = \chi}} \sum_{\substack{0 \leq e_l \leq 2 \\ (l \in \Pi)}} S(k + 1/2, 4B, \xi)_K | U(A) \prod_{l \in \Pi} R_l^{e_l}.$$

[The case of $k = 1$]

$$\mathfrak{O}(3/2, N, \chi)_K = \sum_{\substack{0 < B | M \\ B \neq M}} \sum_{\substack{0 < A | (M/B) \\ \xi\left(\frac{A}{\cdot}\right) = \chi}} V(4B; \xi)_K | \tilde{\delta}_A$$

$$+ \sum_{\substack{0 < B | M \\ B \neq M}} \sum_{\substack{0 < A | (M/B)^2 \\ \xi\left(\frac{A}{\cdot}\right) = \chi}} \sum_{\substack{0 \leq e_l \leq 2 \\ (l \in \Pi)}} V(4B; \xi)_K | U(A) \prod_{l \in \Pi} R_l^{e_l}.$$

Here, ξ runs over all (even quadratic) Dirichlet characters defined modulo $4B$ such that $\xi\left(\frac{4}{\cdot}\right) = \chi$.

Remark. We know the relation of the twisting operator: $R_l{}^3 = R_l$ for all $l \in \Pi$. Hence we can replace the range $0 \le e_l \le 2$ with $0 \le e_l \in \mathbf{Z}$.

Proof. From [U1, (1.9), (1.22), (1.23)], we easily see that the right-hand sides of the statements are subspaces of $S(k + 1/2, N, \chi)_K$ resp. $V(N; \chi)_K$.

The subspaces $S(k + 1/2, 4B, \xi)_K$ and $V(4B; \xi)_K$ have orthogonal **C**-basis consisting of common eigenforms on hermitian operator $\tilde{T}(n^2)$ $(n \in \mathbf{Z}_+, (n, 4B) = 1)$, moreover the systems of eigenvalues of such eigenforms correspond to primitive forms of weight $2k$ and of conductors dividing B. (cf. [U1,(3.10)]).

The operators $\tilde{T}(n^2)$ $(n \in \mathbf{Z}_+, (n, N) = 1)$ commute with the operators $\tilde{\delta}_A$, $U(A)$, and R_l $(l \in \Pi)$ (cf. [U1, (1.8), (1.20)] and [U3, (1.7)]). Hence all subspaces in the right-hand sides of the statements have generators which are common eigenforms on $\tilde{T}(n^2)$ $(n \in \mathbf{Z}_+, (n, N) = 1)$. Moreover those systems of eigenvalues correspond to primitive forms of weight $2k$ and of a conductor which divides M and are less than M, because $B|M$ and $B \ne M$.

Combining this with (3.5), we see that $\mathfrak{O}(k+1/2, N, \chi)_K$ contains the right-hand side.

Next we will prove the contrary inclusion.

We have an explicit expression formula (3.2) of $\mathfrak{O}(k+1/2, N, \chi)_K$. We will check each direct summand of the formula (3.2).

It follows from (3.3) that $\text{Ker}(R_\Pi, S)$ and $\text{Ker}(R_\Pi, V)$ occurs in the first term of the right-hand side.

Next we consider the part of the subspaces $\mathfrak{S}^{\emptyset,\kappa}(k + 1/2, 4dM_{2+}, \chi)_K |U(e^2) = \mathfrak{S}^{\emptyset,\kappa'}(k + 1/2, 4dM_{2+}, \chi^{2+})_K |U(m_1 e^2)$ $(0 < e, d, d \ne M_1, ed|M_1)$, where see [U1, (3.1)] for the definition of m_1 and χ^{2+}. We only remark $\kappa' := \kappa \cdot \left(\frac{m_1}{\cdot}\right)$, $0 < m_1|M_1$, and $\chi^{2+}\left(\frac{m_1}{\cdot}\right) = \chi$.

This subspace is contained in the space $S\left(k + 1/2, 4dM_{2+}, \chi^{2+}\right)_K |U(m_1 e^2)$ (when $k \ge 2$), resp. $V\left(4dM_{2+}; \chi^{2+}\right)_K |U(m_1 e^2)$ (when $k = 1$).

We decompose $m_1 = m_2 \cdot m_3$ with $m_2 = (m_1, d)$.

Then the character $\chi^{2+}\left(\frac{m_2}{\cdot}\right)$ can be defined modulo $4dM_{2+}$. From [U1, (1.28)], we have $S\left(k + 1/2, 4dM_{2+}, \chi^{2+}\right)_K |U(m_2) = S\left(k + 1/2, 4dM_{2+}, \chi^{2+}\left(\frac{m_2}{\cdot}\right)\right)_K$ and $V\left(4dM_{2+}; \chi^{2+}\right)_K |U(m_2) = V\left(4dM_{2+}; \chi^{2+}\left(\frac{m_2}{\cdot}\right)\right)_K$.

From this we may assume that $(m_1, d) = 1$.

We decompose that $m_1 = m_4 \cdot m_5$ with $m_4 = (m_1, e)$.

We first consider the case of $m_4 = 1$. In this case, we set $B = dM_{2+}$.

Since $(m_1, ed) = (e, d) = 1$, $m_1 e^2 | (M_1/d)^2 = (M/B)^2$ and so the subspaces $S\left(k + 1/2, 4dM_{2+}, \chi^{2+}\right)_K |U(m_1 e^2)$ (when $k \ge 2$) and $V\left(4dM_{2+}; \chi^{2+}\right)_K |U(m_1 e^2)$ (when $k = 1$) occur in the second term of the right hand side.

Next we consider the case of $m_4 > 1$. Take a prime divisor p of m_4. Then $m_1 e^2 = (m_1/p)(e/p)^2 \times p^3$.

For $k \geq 2$, we have

$$S\left(k + 1/2, 4dM_{2+}, \chi^{2+}\right)_K |U(m_1 e^2)$$

$$= S\left(k + 1/2, 4dM_{2+}, \chi^{2+}\right)_K |U((m_1/p)(e/p)^2)U(p^3)$$

$$\subseteq S\left(k + 1/2, 4M/p, \chi^{2+}\left(\frac{m_1/p}{}\right)\right)_K |U(p^3)$$

$$\subseteq S\left(k + 1/2, 4M/p, \chi^{2+}\left(\frac{m_1/p}{}\right)\right)_K |\tilde{\delta}_p$$

$$+ S\left(k + 1/2, 4M/p, \chi^{2+}\left(\frac{m_1/p}{}\right)\right)_K |U(p) .$$

Here, we use the formula (A.4) at the last inclusion.

For $k = 1$, we can see in the same way,

$$V\left(4dM_{2+}; \chi^{2+}\right)_K |U(m_1 e^2) \subseteq V\left(4M/p; \chi^{2+}\left(\frac{m_1/p}{}\right)\right)_K |\tilde{\delta}_p$$

$$+ V\left(4M/p; \chi^{2+}\left(\frac{m_1/p}{}\right)\right)_K |U(p) .$$

Here, we use that Hecke operator $\tilde{T}(p^2)$ fixes $V\left(4M/p; \chi^{2+}\left(\frac{m_1/p}{}\right)\right)_K$.

We set $B = M/p$, $A = p$ and so $A = p|(M/B) = p$. This means that $S\left(k + 1/2, 4dM_{2+}, \chi^{2+}\right)_K|U(m_1 e^2)$ (when $k \geq 2$) and $V\left(4dM_{2+}; \chi^{2+}\right)_K|U(m_1 e^2)$ (when $k = 1$) are contained in the right-hand side.

We finally consider the part of $\hat{\mathfrak{N}}^{\emptyset,\kappa}(k + 1/2, N, \chi)_K$. For all $(\alpha_l)_{l \in \Pi} \neq (\nu_l)_\Pi$, we have the following (cf. §2 (2.1)):

$$B^{(\alpha)} := \bigoplus_{0 < a | l_D} \mathfrak{S}^{\emptyset,\kappa|_F}\left(k + 1/2, \tilde{N}(\alpha), \chi\right)_K |U(a^2)$$

$$:= \bigoplus_{0 < a | l_D} \mathfrak{S}^{\emptyset,\kappa'|_F}\left(k + 1/2, \tilde{N}(\alpha), \eta'\right)_K |U(ua^2) ,$$

where $D := \Pi(\alpha)_0$, $F := \Pi(\alpha)_{2+}$, and see §2 for the other notation.

It follows from the above that $B^{(\alpha)}$ occurs in the second term of the right-hand side. Here, we note that $B = M_1 M_{2+}^{(\alpha)}$ and so $l_D^2|(M/B)$. Hence $A = ua^2|l_D^3|(M/B)^2$.

Since $e_\Pi^\kappa = \prod_{l \in \Pi} \frac{1}{2}(R_l^2 + \kappa(l)R_l)$ and $\hat{\mathfrak{N}}^{\emptyset,\kappa}(k+1/2, N, \chi)_K = \sum_{(\alpha_l) \neq (\nu_l)} B^{(\alpha)}|e_\Pi^\kappa$, the subspace $\hat{\mathfrak{N}}^{\emptyset,\kappa}(k+1/2, N, \chi)_K$ is contained in the second term of the right-hand side. $\square$

From the above, we can give the following simple reformulation of results of §2 and §3.

(3.7) Final formulation of newform for Kohnen space. Let k be a positive integer and M an odd positive integer. Let χ be an even character modulo $N := 4M$

with $\chi^2 = 1$. We denote the p-adic additive valuation for any integer m by $\mathrm{ord}_p(m)$. We decompose M as follows:

$$M = M_1 M_{2+}, \quad M_1 := \prod_{\substack{\mathrm{ord}_p(M)=1 \\ p|M}} p, \quad M_{2+} := \prod_{\substack{\mathrm{ord}_p(M)\geq 2 \\ p|M}} p^{\mathrm{ord}_p(M)} .$$

Denote the set of all prime divisors of M_{2+} by Π.

Define the space of oldform $\mathfrak{O}(k+1/2, N, \chi)_K$ as follows.

[The case of $k \geq 2$]

$$\mathfrak{O}(k+1/2, N, \chi)_K = \sum_{\substack{0<B|M \\ B\neq M}} \sum_{\substack{0<A|(M/B) \\ \xi(\frac{A}{\cdot})=\chi}} S(k+1/2, 4B, \xi)_K \,|\tilde{\delta}_A$$

$$+ \sum_{\substack{0<B|M \\ B\neq M}} \sum_{\substack{0<A|(M/B)^2 \\ \xi(\frac{A}{\cdot})=\chi}} \sum_{\substack{0\leq e_l\leq 2 \\ (l\in\Pi)}} S(k+1/2, 4B, \xi)_K \,|U(A) \prod_{l\in\Pi} R_l^{e_l} .$$

[The case of $k = 1$]

$$\mathfrak{O}(3/2, N, \chi)_K = \sum_{\substack{0<B|M \\ B\neq M}} \sum_{\substack{0<A|(M/B) \\ \xi(\frac{A}{\cdot})=\chi}} V(4B; \xi)_K \,|\tilde{\delta}_A$$

$$+ \sum_{\substack{0<B|M \\ B\neq M}} \sum_{\substack{0<A|(M/B)^2 \\ \xi(\frac{A}{\cdot})=\chi}} \sum_{\substack{0\leq e_l\leq 2 \\ (l\in\Pi)}} V(4B; \xi)_K \,|U(A) \prod_{l\in\Pi} R_l^{e_l} .$$

Here, ξ runs over all characters modulo $4B$ such that $\xi\left(\frac{A}{\cdot}\right) = \chi$.

$\mathfrak{O}(k+1/2, N, \chi)_K$ is a subspace of $S(k+1/2, N, \chi)_K$ if $k \geq 2$, resp. $V(N; \chi)_K$ if $k = 1$. We denote by $\mathfrak{N}(k+1/2, N, \chi)_K$ the orthogonal complement of $\mathfrak{O}(k+1/2, N, \chi)_K$ in $S(k+1/2, N, \chi)_K$ resp. $V(N; \chi)_K$ according as $k \geq 2$ resp. $k = 1$.

Then the space $\mathfrak{N}(k+1/2, N, \chi)_K$ is stable by the twisting operators R_p for all $p \in \Pi$. Hence we can decompose this space into common eigen subspaces as follows:

$$\mathfrak{N}(k+1/2, N, \chi)_K = \oplus_{\kappa\in\{\pm 1\}^\Pi} \mathfrak{N}^{\emptyset,\kappa}(k+1/2, N, \chi)_K ,$$

$$\mathfrak{N}^{\emptyset,\kappa}(k+1/2, N, \chi)_K := \{f \in \mathfrak{N}(k+1/2, N, \chi)_K \; ; \; f|R_p = \kappa(p)f \text{ for all } p \in \Pi \}.$$

We call these spaces $\mathfrak{N}^{\emptyset,\kappa}(k+1/2, N, \chi)_K$ the spaces of newforms of Kohnen space, because these subspaces $\mathfrak{N}^{\emptyset,\kappa}(k+1/2, N, \chi)_K$ ($\kappa \in \{\pm 1\}^\Pi$) have the following nice properties.

(1) $\mathfrak{N}^{\emptyset,\kappa}(k+1/2, N, \chi)_K$ has an orthogonal C-basis consisting of common eigenforms for all Hecke operators $\tilde{T}_{k+1/2,N,\chi}(p^2)$ (p: prime, $p{\nmid}M$) and $U(p^2)$ (p: prime, $p|M$), which are uniquely determined up to multiplication with non-zero complex numbers. Let f be such a common eigenform and λ_p the eigenvalue of f with

respect to $\tilde{T}_{k+1/2,N,\chi}(p^2)$ $(p{\nmid}M)$ resp. $U(p^2)$ $(p|M)$, then there exists a primitive form $F \in S^0(2k, M)$ of weight $2k$ and of conductor M which is uniquely determined and satisfies the following: For a prime p,

$$F|T_{2k,M}(p) = \lambda_p F \quad \text{if } (p, M) = 1 \qquad \text{and} \qquad F|U(p) = \lambda_p F \quad \text{if } p \mid M.$$

Here, we can find, by using the trace relation (2.28)(1), which primitive form occurs via the above correspondence.

(2) (**The Strong Multiplicity One Theorem**)

Let f, g be two non-zero elements of $\mathfrak{N}^{\emptyset,\kappa}(k+1/2, N, \chi)_K$. If f and g are common eigenforms of $\tilde{T}_{k+1/2,N,\chi}(p^2)$ with the same eigenvalue for all prime numbers p prime to some integer A, then $\mathbf{C}f = \mathbf{C}g$.

(3) The space of oldforms $\mathfrak{O}(k + 1/2, N, \chi)_K$ has also an orthogonal C-basis consisting of common eigenforms for all Hecke operators $\tilde{T}_{k+1/2,N,\chi}(p^2)$ (p: prime, $p{\nmid}N$). The system of eigenvalues of such a common eigenform corresponds to a primitive forms of weight $2k$ whose conductor is a divisor of M and less than M (cf. (3.5)).

(4) The space of oldform $\mathfrak{O}(k + 1/2, N, \chi)_K$ is generated by the spaces of cusp forms of lower level. Hence, by induction, we see that the spaces $S(k + 1/2, N, \chi)_K$ ($k \geq 2$) and $V(N; \chi)_K$ ($k = 1$) are reconstructed by the spaces of type of $\mathfrak{N}^{\emptyset,\kappa}(k + 1/2, 4B, \xi)_K$ and the operators of type of $\tilde{\delta}_A$, $U(A)$, and R_l.

These operators $\tilde{\delta}_A$, $U(A)$, and R_l (almost) fix the Fourier coefficient of cusp forms, i. e., for $f = \sum_{n \geq 1} a(n)\mathbf{e}(nz)$,

$$f|\tilde{\delta}_A(z) := A^{k/2+1/4} f(Az) = A^{k/2+1/4} \sum_{n \geq 1} a(n)\mathbf{e}(Anz) \,,$$

$$f|U(A)(z) := \sum_{n \geq 1} a(An)\mathbf{e}(nz) \,, \quad f|R_l(z) := \sum_{n \geq 1} a(n)\left(\frac{n}{l}\right)\mathbf{e}(nz) \,.$$

From this we claim the following: For studying Fourier coefficients of a cusp form $\in S(k + 1/2, N, \chi)_K$ resp. $V(N; \chi)_K$, it is sufficient to study cusp forms only in the spaces of newforms $\mathfrak{N}^{\emptyset,\kappa}(k + 1/2, N, \chi)_K$. $\square$

(3.8) **Remark** There exists a case such that $\mathfrak{N}^{\emptyset,\kappa}(k + 1/2, N, \chi)_K \cong \mathfrak{N}^{\emptyset,\kappa'}(k + 1/2, N, \chi)_K$ as modules over Hecke algebra for two distinct κ, $\kappa' \in \{\pm 1\}^{\Pi}$. For example we have the following isomorphism: Let p be an odd prime and $\{\kappa, \kappa'\} = \mathrm{Map}(\{p\}, \{\pm 1\})$. Then

$$\mathfrak{N}^{\emptyset,\kappa}(k + 1/2, 4p^2, 1)_K \cong \mathfrak{N}^{\emptyset,\kappa'}(k + 1/2, 4p^2, 1)_K$$

$$\cong \left\{ \bigoplus_{\sigma \in \{\pm 1\}^{\{p\}}} S^{*(1,\sigma)}(2k, p^2) \right\} \oplus \left(1 + \left(\frac{-1}{p}\right)\right)/2 \left\{ S^0(2k, p)|R_p \oplus S(2k, 1)|R_p \right\} .$$

(cf.(2.28)(1)) $\square$

Appendix. We collect several propositions which are use in the section 2.

Let k and N be positive integers such that $4 \mid N$ and χ an even Dirichlet character modulo N with $\chi^2 = 1$.

(A.1) Proposition Let Q and a be two odd positive divisors of N such that $(Q, N/Q) = (a, Q) = 1$. For any $f \in S(k + 1/2, N, \chi)$, the following identity holds good:

$$f \mid U(a)\widetilde{W}(Q) = \chi_Q(a) \, f \mid \widetilde{W}(Q)U(a) \,,$$

where χ_Q is the Q-primary component of χ.

Proof. If $Q = 1$, the assertion is trivial and so we suppose that $Q \neq 1$.

We decompose $a = l_1 l_2 \cdots l_t$ ($l_1, \dots, l_t$ are odd prime numbers) and $Q = p_1^{e_1} \cdots p_n^{e_n}$ (p_i's are distinct prime numbers and $e_i \geq 1$).

First we consider the case of $t = 1$ and will prove this case by using induction on n.

If $n = 1$, the assertion follows from [U1,(1.20)(1)].

Suppose that $n \geq 2$ and the assertion holds good for any m less than n.

Observing $f|U(l_1) \in S\left(k + 1/2, N, \chi\left(\frac{l_1}{\cdot}\right)\right)$, we have the following formula from [U1,(1.18)] and the assumption of induction:

$$
\begin{aligned}
(*) \quad f|U(l_1)\widetilde{W}(Q) &= \left(\chi\left(\frac{l_1}{\cdot}\right)\right)_{p_n} (Qp_n^{-e_n}) \, f|U(l_1)\widetilde{W}(Qp_n^{-e_n})\widetilde{W}(p_n^{e_n}) \\
&= \chi_{p_n}(Qp_n^{-e_n})\chi_{p_1 \cdots p_{n-1}}(l_1) \, f|\widetilde{W}(Qp_n^{-e_n})U(l_1)\widetilde{W}(p_n^{e_n}) \\
&= \chi_{p_n}(Qp_n^{-e_n})\chi_{p_1 \cdots p_{n-1}}(l_1) \left(\chi\left(\frac{Qp_n^{-e_n}}{\cdot}\right)\right)_{p_n} (l_1) \\
&\quad \times f|\widetilde{W}(Qp_n^{-e_n})\widetilde{W}(p_n^{e_n})U(l_1) \\
&= \chi_{p_n}(Qp_n^{-e_n})\chi_{p_1 \cdots p_n}(l_1) \, f|\widetilde{W}(Qp_n^{-e_n})\widetilde{W}(p_n^{e_n})U(l_1) \\
&= \chi_{p_1 \cdots p_n}(l_1) \, f|\widetilde{W}(Q)U(l_1) = \chi_Q(l_1) \, f|\widetilde{W}(Q)U(l_1) \,.
\end{aligned}
$$

Next we suppose that $t \geq 2$ and the assertion holds good for any s less than t. From the above formula $(*)$ and the assumption of induction, we have the following

$$
\begin{aligned}
f|U(a)\widetilde{W}(Q) &= f|U(l_1 \cdots l_{t-1})U(l_t)\widetilde{W}(Q) \\
&= \left(\chi\left(\frac{l_1 \cdots l_{t-1}}{\cdot}\right)\right)_Q (l_t) \, f|U(l_1 \cdots l_{t-1})\widetilde{W}(Q)U(l_t) \\
&= \chi_Q(l_t)\chi_Q(l_1 \cdots l_{t-1}) \, f|\widetilde{W}(Q)U(l_1 \cdots l_{t-1})U(l_t) \\
&= \chi_Q(a) \, f|\widetilde{W}(Q)U(a) \,.
\end{aligned}
$$

Thus we obtain the assertion. $\square$

Now, we choose and fix an odd prime number p satisfying $(p, N) = 1$. Take a form $g \in S(k + 1/2, N, \chi)$. Then g is also contained in $S(k + 1/2, Np, \chi)$. Hence we can consider the form $g \mid Y_p$ (See [U1, p.155] for the definition of Y_p).

Since $(p, N) = 1$ and χ is defined modulo N, we have $\chi_p = 1$. Therefore from [U1, (1.27)], $p^{-k+3/2} \, g \mid U(p)\widetilde{W}(p)U(p)\widetilde{W}(p) = g \mid Y_p^2 = \left(\frac{-1}{p}\right) p \, g.$

Using [U1, (1.18)], $p^{-k+3/2}g \mid U(p)\widetilde{W}(p)U(p) = \left(\frac{-1}{p}\right)pg \mid \widetilde{W}(p)^{-1} = \left(\frac{-1}{p}\right)p\left(\frac{-1}{p}\right)^{k+1/2}\chi_p(-1)\chi_N(p)\,g \mid \widetilde{W}(p).$

Since $g \in S(k+1/2, N, \chi)$ and $\widetilde{W}(p) = \gamma_p{}^*\tilde{\delta}_p$ for some $\gamma_p \in \Gamma(N)$, we get $g \mid \widetilde{W}(p) = g \mid \tilde{\delta}_p$.

Thus we obtain the following formula for any $g \in S(k+1/2, N, \chi)$.

$$(A.2) \qquad g \mid \tilde{\delta}_p = \left(\frac{-1}{p}\right)^{k+1/2}\chi(p)p^{-k/2-1/4}\,g \mid Y_p U(p)\,.$$

Next we consider $g = \sum_{n=1}^{\infty} a(n)e(nz) \in S(k+1/2, N, \chi)$ as an element of $S(k+1/2, Np, \chi)$ and apply [U1, (1.26), (1.18)] to g.

$$g \mid Y_p = \left(\frac{-1}{p}\right)^{k+3/2}\chi(p)g \mid \widetilde{W}(p)\tilde{\delta}_p + \left(\frac{-1}{p}\right)^{-1/2}p^{1/2}\sum_{n\geq 1}a(n)\left(\frac{n}{p}\right)e(nz)$$

$$= \left(\frac{-1}{p}\right)^{k+3/2}\chi(p)g \mid \tilde{\delta}_{p^2} + \left(\frac{-1}{p}\right)^{-1/2}p^{1/2}\sum_{n\geq 1}a(n)\left(\frac{n}{p}\right)e(nz)$$

$$= \left(\frac{-1}{p}\right)^{k-1/2}\chi(p)p^{3/2-k}$$

$$\times \sum_{n\geq 1}\left\{p^{2k-1}a(n/p^2) + p^{k-1}\chi(p)\left(\frac{-1}{p}\right)^k\left(\frac{n}{p}\right)a(n)\right\}e(nz)\,,$$

where $a(n/p^2) = 0$ if $p^2 \nmid n$.

By using [Sh 1, Theorem (1.7)], this formula is expressed by $U(p^2)$ and the Hecke operator $\tilde{T}_{k+1/2, N, \chi}(p^2)$, i.e., for any $g \in S(k+1/2, N, \chi)$,

$$(A.3) \qquad g \mid Y_p = \left(\frac{-1}{p}\right)^{k-1/2}\chi(p)p^{3/2-k}\left(g \mid \tilde{T}_{k+1/2, N, \chi}(p^2) - g \mid U(p^2)\right)\,.$$

□

Combining (A.2) with (A.3), we have for any $g \in S(k+1/2, N, \chi)$,

$$(A.4) \quad g \mid \widetilde{W}(p) = g \mid \tilde{\delta}_p = p^{-3k/2+5/4}\left(g \mid \tilde{T}_{k+1/2, N, \chi}(p^2)\,U(p) - g \mid U(p^3)\right)\,.$$

□

Now we assume moreover that g is an eigen form of $\tilde{T}(p^2)$, i.e.,

$$(A.5) \qquad g \mid \tilde{T}_{k+1/2, N, \chi}(p^2) = \lambda_p\,g \quad (\lambda_p \in \mathbb{C})\,.$$

and let u be a squarefree odd positive integer such that $(u, 4M_{2+}) = 1$ and $\chi_u = 1$. Here, χ_u is the u-primary component of χ.

We will describe the action of Y_p on the forms $f := g \mid U(u)$ and $f \mid U(p^2) = g \mid U(up^2)$. Here, we consider the action of Y_p on the space $S\left(k + 1/2, \tilde{N}, \chi\left(\frac{u}{\cdot}\right)\right)$ ($\tilde{N}$ is the least common multiple of Np and u).

(A.6) First we assume that $(p, u) = 1$.

Since $g \mid U(p)$ in contained in $S\left(k + 1/2, \tilde{N}, \chi\left(\frac{p}{\cdot}\right)\right)$, we can apply Proposition (A.1) to $g \mid U(p)$. Hence we have $f \mid Y_p = p^{-k/2+3/4} g \mid U(p) \, U(u) \, \widetilde{W}(p) = \left(\frac{u}{p}\right) p^{-k/2+3/4} g \mid U(p) \, \widetilde{W}(p) \, U(u) = \left(\frac{u}{p}\right) g \mid Y_p \, U(u)$.

By using (A.3) and (A.5),

$$(A.7) \qquad f \mid Y_p = \left(\frac{u}{p}\right)\left(\frac{-1}{p}\right)^{k-1/2} \chi(p) p^{3/2-k}\left(\lambda_p \, f \, - \, f \mid U(p^2)\right).$$

Applying Y_p to the both sides of the above formula and using [U1, (1.27)],

$$\left(\frac{-1}{p}\right) p\, f = f \mid Y_p{}^2 = \left(\frac{u}{p}\right)\left(\frac{-1}{p}\right)^{k-1/2} \chi(p) p^{3/2-k}\left(\lambda_p \, f \mid Y_p - f \mid U(p^2)\, Y_p\right).$$

Modifying this formula and using (A.7), we have

(A.8)

$$f \mid U(p^2)\, Y_p = -\left(\frac{u}{p}\right)\left(\frac{-1}{p}\right)^{-k+3/2} \chi(p) p^{k-1/2} f \, + \, \lambda_p \, f \mid Y_p$$

$$= \left(\frac{u}{p}\right)\left(\frac{-1}{p}\right)^{k-1/2} \chi(p) p^{3/2-k}\left\{(\lambda_p{}^2 - p^{2k-2}) f \, - \, \lambda_p \, f \mid U(p^2)\right\}.$$

We can represent these relations as a matrix as follows:

$$(A.9) \quad (f \mid Y_p, \; f \mid U(p^2)\, Y_p)$$

$$= (f, \; f \mid U(p^2)) \times \left(\frac{u}{p}\right)\left(\frac{-1}{p}\right)^{k-1/2} \chi(p) p^{3/2-k} \begin{pmatrix} \lambda_p & \lambda_p{}^2 - p^{2k-2} \\ -1 & -\lambda_p \end{pmatrix}.$$

$\square$

The characteristic polynomials of this matrix is $t^2 - \left(\frac{-1}{p}\right) p$. Hence, this matrix has two distinct eigen values.

(A.10) Next we assume that $p \mid u$.

Since $g \mid U(p^2)$ is contained in $S\left(k + 1/2, \tilde{N}, \chi\right)$, we can apply Proposition (A.1) to $g \mid U(p^2)$. Observing $\chi_p = 1$, $f \mid Y_p = p^{-k/2+3/4} g \mid U(p^2) \, U(u/p) \, \widetilde{W}(p) = p^{-k/2+3/4} g \mid U(p^2) \, \widetilde{W}(p) \, U(u/p)$.

By using (A.3) and (A.5), $g \mid U(p^2) = \lambda_p g - \left(\frac{-1}{p}\right)^{-k+1/2} \chi(p) p^{k-3/2} g \mid Y_p$. Applying $\widetilde{W}(p)$ to the both sides,

$$(A.11) \quad g \mid U(p^2)\, \widetilde{W}(p) = \lambda_p g \mid \widetilde{W}(p) - \left(\frac{-1}{p}\right)^{-k+1/2} \chi(p) p^{k-3/2} g \mid Y_p \, \widetilde{W}(p).$$

Observing $g \mid U(p) \in S\left(k + 1/2, \tilde{N}, \chi\left(\frac{2}{\cdot}\right)\right)$, we apply [U1, (1.18)] to $g \mid U(p)$,

(A.12)

$$g \mid Y_p \widetilde{W}(p) = p^{-k/2+3/4} g \mid U(p) \widetilde{W}(p)^2 = p^{-k/2+3/4} \left(\frac{-1}{p}\right)^{k-1/2} \chi(p) g \mid U(p) \,.$$

Combining (A.4), (A.11), and (A.12), we have

$$p^{-k/2+3/4} g \mid U(p^2) \, \widetilde{W}(p) = p^{-2k+2} \{(\lambda_p{}^2 - p^{2k-2}) g \mid U(p) - \lambda_p g \mid U(p^3)\} \,.$$

Therefore

(A.13) $$f \mid Y_p = p^{-2k+2} \{(\lambda_p{}^2 - p^{2k-2}) f - \lambda_p f \mid U(p^2)\} \,.$$

Applying Y_p to the both sides of the above (cf.[U1, (1.27)]),

$$(p-1) f \mid Y_p + p f = f \mid Y_p{}^2 = p^{-2k+2} \{(\lambda_p{}^2 - p^{2k-2}) f \mid Y_p - \lambda_p f \mid U(p^2) \, Y_p\} \,.$$

Modifying this formula and using (A.13),
(A.14)
$$f \mid U(p^2) \, Y_p = p^{-2k+2} \{(\lambda_p{}^3 - \lambda_p(p^{2k-1} + p^{2k-2})) f - (\lambda_p{}^2 - p^{2k-1}) f \mid U(p^2)\} \,.$$

We can represent these relations as a matrix as follows:

(A.15) $(f \mid Y_p, \; f \mid U(p^2) \, Y_p)$

$$= (f, \; f \mid U(p^2)) \times p^{-2k+2} \begin{pmatrix} \lambda_p{}^2 - p^{2k-2} & \lambda_p{}^3 - \lambda_p(p^{2k-1} + p^{2k-2}) \\ -\lambda_p & -(\lambda_p{}^2 - p^{2k-1}) \end{pmatrix} \,.$$

□

The characteristic polynomials of this matrix is $t^2 - (p-1)t - p = (t-p)(t+1)$. Hence, this matrix has two distinct eigen values.

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DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
NARA WOMEN'S UNIVERSITY
Nara 630 Japan
(E-Mail: m-ueda@cc.nara-wu.ac.jp)