Duality for Calabi Curves and Minimal Surfaces in $\mathbb{R}^n$

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0. Introduction

An algebraic curve in $\mathbb{P}_n$ that does not lie on any hyperplane is said to be full: its degree must at least equal n. Curves of degree n differ from the rational normal curve, $\mathcal{R}_n$, by a projective automorphism, see [5]. Consequently, if $\psi : M \longrightarrow \mathbf{C}^{n+1}$ is a non-constant holomorphic curve whose Gauss map $\gamma_\psi = [\psi'] : M \longrightarrow \mathbb{P}_n$ takes values on an algebraic curve of degree n, then we may as well suppose that $\gamma_\psi : M \longrightarrow \mathcal{R}_n$. It turns out that there exists a natural lift of such a γ_ψ into the holomorphic line bundle of degree n over $\mathcal{R}_n$, from which ψ can be recovered. This may be understood in terms of the classical duality between curves in $\mathbb{P}_{n+1}$ and $\mathbb{P}^*_{n+1}$, see section 2. This construction generalizes the Lie-Hitchin correspondence for null curves in $\mathbf{C}^3$, [10], [11], and in view of that it is not very surprising to find that such curves in $\mathbf{C}^{n+1}$ possess various *Weierstrass representation formulae.*

Our interest in this correspondence and these formulae derives from the existence of two applications to differential geometry together with a possible application to the theory of Lax equations which we describe in section 4.

Firstly, Calabi, see [1], [7], has shown that if $\gamma : M \longrightarrow \mathbb{P}_n$ is a full holomorphic curve that induces, from the Fubini-Study metric, a metric of constant Gaussian curvature, K, away from branch points on M, then γ must take values on some unitary transformation of $\mathcal{R}'_n$, which is the image of $\rho_n : \mathbb{P}_1 \longrightarrow \mathbb{P}_n$, where

$$\rho_n(\zeta) = [1, \sqrt{n}\zeta, ..., \sqrt{\binom{n}{k}}\zeta^k, ..., \zeta^n],$$

and furthermore that $K = 2/n$. We call a curve $\psi : M \longrightarrow \mathbf{C}^{n+1}$ such that γ_ψ induces a metric of constant Gaussian curvature a *Calabi curve.* The correspondence described here facilitates the study of these curves in $\mathbf{C}^{n+1}$ in terms of holomorphic curves in the holomorphic line bundle of degree n, $\pi : \mathcal{O}(n) \longrightarrow \mathbb{P}_1$. The metric induced by such a curve in $\mathbf{C}^{n+1}$ satisfies a *generalized Ricci condition*, see [7]. (Note that for $n = 1$ the curves in $\mathbf{C}^2$ are not constrained.)

Secondly, for $n \geq 2$, $\mathcal{R}_n$ lies on a non-singular quadric hypersurface in $\mathbb{P}_n$ and a judicious transformation of coordinates of $\mathbf{C}^{n+1}$, and resulting choice of $\mathbb{R}^{n+1}$, determines that the correspondence generates null holomorphic curves in $\mathbf{C}^{n+1}$ and thus minimal surfaces in $\mathbb{R}^{n+1}$. In particular, this facilitates the construction of non-degenerate complete minimal surfaces of finite total Gaussian curvature in $\mathbb{R}^{n+1}$ which are characterized by the fact that the image of their Gauss map has the smallest possible degree. See section 3.

1. Weierstrass Formulae

(1.1) Let M be a Riemann surface and suppose that $\psi : M \longrightarrow \mathbf{C}^{n+1}$ is a full Calabi curve. It follows that there exists $U \in U(n+1)$ and a holomorphic differential ω on M such that

$$\psi = U\tilde{\psi} = U\int(1, \sqrt{n}g, ..., \sqrt{\binom{n}{k}}g^k, ..., g^n)\omega, \tag{1}$$

where $g = \rho_n^{-1} \circ \gamma_{\tilde{\psi}}$.

(Note that the fullness assumption may be dropped here provided the obvious modifications are made.)

Conversely, if $g : M \longrightarrow \mathbf{C}$ is meromorphic and ω is a holomorphic differential on M such that whenever g has a pole of order m at $p \in M$, ω has a zero at p of order nm, then ψ, defined as above, gives a Calabi curve $\psi : M \longrightarrow \mathbf{C}^{n+1}$. Note that the branch points of ψ occur where ω has a zero off the divisor of poles of g. (This is analogous to the usual Weierstrass formulae for null curves, see [8].)

(1.2) Now reparameterize $\tilde{\psi}$ by its Gauss map. I.e. suppose that g^{-1} and F exist on an open set $V \subset \mathbb{P}_1$ and $g^{-1}(V) \subset M$ respectively, such that $\omega = Fd\xi$, and furthermore that $f : V \longrightarrow \mathbf{C}$ holomorphic, satisfies

$$f^{(n+1)}(\zeta) = F \circ g^{-1}(\zeta)\frac{dg}{d\zeta}^{-1}(\zeta),$$

where $f^{(n+1)}$ denotes the (n+1)-st. derivative of f. Substituting $f^{(n+1)}$ into the above formula, and changing the variable to $\zeta = g(\xi)$, we integrate the above by parts to obtain:

$$\tilde{\psi}_k \circ g^{-1}(\zeta) = \sqrt{\binom{n}{k}} \sum_{l=0}^{k} (-1)^l \frac{k!}{(k-l)!} \zeta^{k-l} f^{(n-l)}(\zeta),$$

where $\tilde{\psi} = (\tilde{\psi}_0, ..., \tilde{\psi}_n)$.

(1.3) The metric induced by $\tilde{\psi}$ is given by:

$$ds^2 = |f^{(n+1)}|^2 (1 + |\zeta|^2)^n |d\zeta|^2.$$

Consequently, if $f^{(n+1)}(p) = 0$ then p is a branch point of ds^2. We will see in 2.7 that this has a simple geometric interpretation.

(1.4) If $\psi : M \longrightarrow \mathbf{C}^{n+1}$ is a meromorphic Calabi curve then it is not hard to see from the above formula that it may be encoded into a pair of meromorphic functions (g_ψ, f_ψ), together with a unitary transformation. Suppressing the unitary transformations we have:

$$\begin{aligned} g_\psi(\xi) &= \rho_n^{-1} \circ \gamma_\psi(\xi) \\ f_\psi(\xi) &= \frac{1}{n!} \sum_{k=0}^{n} (-1)^k \sqrt{\binom{n}{k}} g_\psi^{n-k}(\xi) \psi_k(\xi). \end{aligned}$$

Conversely, given a pair of meromorphic functions (g, f) on M, g non-constant, one can invert the above by setting:

$$\psi_k(\xi) = \sqrt{\binom{n}{k}} \sum_{l=0}^{k} (-1)^l \frac{k!}{(k-l)!} g^{k-l}(\xi) f^{(n-l)}(\xi),$$

where

$$f^{(1)} = \frac{df}{dg}, f^{(2)} = \frac{d}{dg}(\frac{df}{dg}), etc..$$

If $f = P \circ g$, where P is a polynomial of degree $d_P \leq n$ then these formula will produce a constant map: this has a simple geometrical explanation, see 2.7.

(1.5) In dimension 3 one can write:

$$\begin{aligned} \psi_0 &= f^{(2)} \\ \psi_1 &= \sqrt{2}(gf^{(2)} - f^{(1)}) \\ \psi_2 &= g^2 f^{(2)} - 2gf^{(1)} + 2f \end{aligned}$$

where, $f = f_\psi$,

$$f_\psi = \frac{1}{2}(\psi_2 - \sqrt{2}g\psi_1 + g^2\psi_0).$$

Note that these are equivalent to the classical Weierstrass formula for minimal surfaces in $\mathbb{R}^3$.

(1.6) Similiarly, any full meromorphic Calabi curve in $\mathbf{C}^4$ may, after unitary transformation, be brought into the following form:

$$\begin{aligned} \psi_0 &= f^{(3)} \\ \psi_1 &= \sqrt{3}(gf^{(3)} - f^{(2)}) \\ \psi_2 &= \sqrt{3}(g^2 f^{(3)} - 2gf^{(2)} + 2f^{(1)}) \\ \psi_3 &= g^3 f^{(3)} - 3g^2 f^{(2)} + 6gf^{(1)} - 6f, \end{aligned}$$

where

$$f = f_\psi = \frac{1}{6}(-\psi_3 + \sqrt{3}g\psi_2 - \sqrt{3}g^2\psi_1 + g^3\psi_0),$$

and g gives the Gauss map of ψ.

The point here is that it is very easy to construct Calabi curves simply by substituting meromorphic functions into the above. We remark on the relevant moduli for Calabi curves in section 2.

(1.7) Note that since the induced metric on M takes the form

$$ds^2 = |f^{(n+1)}|^2(1 + |g|^2)^n |d\xi|^2,$$

it follows that meromorphic Calabi curves induce complete metrics (in the sense that every divergent path has infinite length).

2. Duality

(2.1) Let Y denote the total space of $\pi : \mathcal{O}(n) \longrightarrow \mathbb{P}_1$. A global holomorphic section $\sigma \in H^0(\mathbb{P}_1, \mathcal{O}(n))$ that vanishes to order n at some point of $\mathbb{P}_1$ is said to be *normal.* The set of lines of normal sections forms a curve of degree n, $\mathcal{R} \subset I\!P(H^0(I\!P_1, \mathcal{O}(n)))$.

Note that $q : I\!P_1 \longrightarrow I\!P_n$, given by $q(\zeta) = \{\sigma \in H^0(I\!P_1, \mathcal{O}(n)); \sigma$ vanishes to order n at $\zeta\}$ gives a canonical identification with $\mathcal{R}$.

(2.2) The hyperplane $\Pi_\zeta = \{\sigma \in H^0(I\!P_1, \mathcal{O}(n)); \sigma(\zeta) = 0\}$ enjoys tangential intersection with $C(R)$, the cone over R, along $q(\zeta)$. This follows because if σ vanishes at ζ then it cannot vanish to order n elsewhere on $I\!P_1$. Such a hyperplane is said to be *normal.*

$\Pi = \bigcup_{\zeta \in I\!P_1} \Pi_\zeta$ is the kernel of

$$I\!P_1 \times H^0(I\!P_1, \mathcal{O}(n)) \longrightarrow \mathcal{O}(n), (\zeta, \sigma) \longrightarrow \sigma(\zeta),$$

and thus there is the following isomorphism:

$\mathcal{O}(n) \simeq \{I\!P_1 \times H^0(I\!P_1, \mathcal{O}(n))\}/\Pi =$ affine normal planes in $H^0(I\!P_1, \mathcal{O}(n))$.

Note that $\mu \in \mathcal{O}(n)$ is dual to the affine plane $\Pi_\mu \subset H^0(I\!P_1, \mathcal{O}(n))$ of sections that pass through μ and consequently, μ lies on the image of a global section σ iff σ lies on Π_μ.

Remark Note that for $n = 1$ the normality constraint is vacuous.

(2.3) A *normal curve* $\psi : M \longrightarrow H^0(I\!P_1, \mathcal{O}(n))$ is characterized by the fact that $\psi'(\xi, \zeta)$ is a normal section for each $\xi \in M$. Identifying $\mathcal{R}$ with $I\!P_1$, and thus viewing γ_ψ as a map to $I\!P_1$, for ψ non-constant,

$$\Gamma_\psi : M \longrightarrow \mathcal{O}(n),$$

given by $\Gamma_\psi(\xi) = \psi(\xi, \gamma_\psi(\xi))$, is a globally defined lift of the Gauss map. Note that $\Gamma_\psi(\xi)$ may be viewed as the (unique) affine normal plane, with normal direction $\gamma_\psi(\xi)$ that passes through $\psi(\xi,) \in H^0(I\!P_1, \mathcal{O}(n))$.

(2.4) It is not hard to show that if γ_ψ is non-constant then Γ_ψ determines ψ. Let $Sp\acute{e}(\mathcal{O}(n))$ denote the étalé space of the sheaf of germs of holomorphic sections of $\mathcal{O}(n)$. There is a (canonically defined) holomorphic map

$$\Psi : Sp\acute{e}(\mathcal{O}(n)) \longrightarrow H^0(I\!P_1, \mathcal{O}(n)),$$

which is given on stalks by,

$$\Psi : \mathcal{O}(n)_\zeta \longrightarrow \mathcal{O}(n)_\zeta / (\mathcal{I}_\zeta^n \otimes \mathcal{O}(n)_\zeta) \xrightarrow{\sim} H^0(I\!P_1, \mathcal{O}(n)),$$

where $\mathcal{I}_\zeta$ is the ideal sheaf of holomorphic functions vanishing at ζ.

(2.5) Let $\mathcal{G} \subset Sp\acute{e}(\mathcal{O}(n))$ denote the set of germs of global sections. The following are immediate generalizations of results described in [10]:

Proposition The holomorphic curve $\Psi : Sp\acute{e}(\mathcal{O}(n)) \longrightarrow H^0(I\!P_1, \mathcal{O}(n))$ is normal, and its Gauss curve is given by $\Gamma_\Psi([\sigma]_\zeta) = \sigma(\zeta)$.

Proposition If ψ is normal, with γ_ψ non-constant, then $\psi|_{\tilde{M}} = \Psi \circ \Gamma_\psi^*$, where $\tilde{M} = \{\xi \in M$; there exists some neighbourhood V of ξ such that $\Gamma_\psi(V)$ is transverse to the fibre $\pi^{-1}(\gamma_\psi(\xi))\}$, and $\Gamma_\psi^* : \tilde{M} \longrightarrow Sp\acute{e}(\mathcal{O}(n))$ is the natural natural lift of Γ_ψ over $\tilde{M}$.

(2.6) If $\psi : M \longrightarrow H^0(I\!P_1, \mathcal{O}(n))$ has non-constant Gauss map then locally trivializing $\mathcal{O}(n)$ one can write, away from branch points of γ_ψ,

$$\Gamma_\psi \circ \gamma_\psi^{-1}(\zeta) = f(\zeta),$$

where f is a (locally defined) holomorphic function. Thus if we choose the basis $\beta_0, ..., \beta_n$ for $H^0(I\!P_1, \mathcal{O}(n))$, where

$$\beta_k(\zeta) = \frac{(-1)^k \sqrt{\binom{n}{k}}}{n!} \zeta^{n-k},$$

then

$$f(\zeta) = \psi_0\beta_0(\zeta) + ... + \psi_n\beta_n(\zeta) + \mathcal{O}(\zeta^{n+1})$$

where, as in section 2,

$$\psi_k \circ g^{-1}(\zeta) = \sqrt{\binom{n}{k}} \sum_{l=0}^{k} (-1)^l \frac{k!}{(k-l)!} \zeta^{k-l} f^{(n-l)}(\zeta).$$

This elucidates the significance of f.

Thus normal curves in $H^0(I\!P_1, \mathcal{O}(n))$ give, by unitary transformation, all Calabi curves in $\mathbf{C}^{n+1}$.

(2.7) If $f = P \circ g$ with $d_P \leq n$ then (g, f) describes a global section of $\pi : \mathcal{O}(n) \longrightarrow I\!P_1$ and osculation gives the point corresponding to that section.

Note that branch points, where $f^{(n+1)} = 0$, give points where the curve is *hyperosculated* by the osculating section.

(2.8) Taking the basis $\{\beta_0, ..., \beta_n, \eta\}$ for $H^0(Y, \pi^{-1}\mathcal{O}(n))$, where $\zeta^k = \zeta^k \circ \pi$ and η denotes the tautological section $\eta(\zeta, \eta) = \eta$, the embedding $\iota : Y \longrightarrow \mathbb{P}(H^0(Y, \pi^{-1}\mathcal{O}(n))) \simeq \mathbb{P}_{n+1}$ is given by $\iota(\zeta, \eta) = [\beta_0, ..., \beta_n, \eta]$ and thus Y is compactified to $\mathcal{C}(\mathcal{R})$, the projective cone over $\mathcal{R}$. A hyperplane in $\mathbb{P}_{n+1}$ lying tangent to $\mathcal{R}$ is said to be *normal.* Global sections of $\mathcal{O}(n)$ are cut out by hyperplanes that do not pass through the vertex $\mathbf{v}$ of $\mathcal{C}(\mathcal{R})$, normal sections are cut out by normal hyperplanes which do not pass through $\mathbf{v}$. Normal hyperplanes that pass through $\mathbf{v}$ lie tangent to $\mathcal{C}(\mathcal{R})$ and thus cut out a fibre of $\pi : \mathcal{O}(n) \longrightarrow \mathbb{P}_1$. The set of normal hyperplanes in $\mathbb{P}_{n+1}$ forms a dual projective cone $\mathcal{C}(\mathcal{R}^*) \subset \mathbb{P}^*_{n+1}$, where $\mathcal{R}^*$ is the degree n curve formed by normal hyperpla

(2.9) *It is clear from this that normal curves are characterized by the fact that the hyperplanes of* $\mathbf{C}^{n+1}$ *osculating them lie tangent to the curve on the hyperplane at infinity that is cut out by intersection with* $\mathcal{C}(\mathcal{R}^*)$. Recall that osculation determines a natural correspondence between full curves in $\mathbb{P}_{n+1}$ and $\mathbb{P}^*_{n+1}$, see [5]. The correspondence can be thought of as follows:

Theorem Osculation determines a correspondence between full curves on $\mathcal{C}(\mathcal{R}) \subset \mathbb{P}_{n+1}$ and full curves in $\mathbb{P}^*_{n+1}$ that are normal with respect to $\mathcal{R}^*$.

(2.10) Blowing up the vertex of $\mathcal{C}(\mathcal{R})$ gives the Hirzebruch surface

$$\mathcal{S}_n \simeq \mathbb{P}(\mathcal{O}(n) \oplus \mathcal{O})$$

and $\mathbb{P}^*_{n+1}$ is thus identified with the linear system $|E_0|$, see [4] for details and notation. $\mathcal{R}$ determines a distinguished irreducible element of $|E_0|$ and normality is defined in $|E_0|$ with respect to that curve in the obvious way. Thus one can reformulate the above as:

Theorem There exists a natural correspondence between full algebraic curves on $\mathcal{S}_n$ and full normal algebraic curves curves in $|E_0|$.

(2.11) This is useful in the description of moduli for meromorphic Calabi curves $\psi : M \longrightarrow \mathbf{C}^{n+1}$. In fact, it is clear that $U(n+1) \times |aE_0 + bC|$, where $|aE_0 + bC|$ give the linear systems on $\mathcal{S}_n$, with $a > 0, b \geq 0$, provide natural compactifications of the moduli spaces of such curves.

Remark $S_1 \simeq I\!P_2$ with one point blown up. It is more natural to compactify $\mathcal{O}(1)$ to $I\!P_2$ and then the construction reduces to classical duality.

If $\mathcal{A}$ is an irreducible algebraic curve on $\mathcal{S}_n$ it is determined up to linear equivalence by the intersection numbers:

$\mathcal{A} \cdot C = k$, which gives the degree of $\pi|_{\mathcal{A}}$ and consequently equals the degree, as a branched covering, of the Gauss map of $\psi_{\mathcal{A}}$, the normal curve determined by osculation of A;

$\mathcal{A} \cdot E_0 = c$, which is the *class* of $\psi_{\mathcal{A}}$ that counts (with multiplicity) the number of hyperplanes osculating $\psi_{\mathcal{A}}$ that pass through a point of $\mathbf{C}^{n+1}$.

If $\mathcal{A}$ is not smooth then the domain of definition of $\psi_{\mathcal{A}}$ is its normalization $\tilde{\mathcal{A}} \longrightarrow \mathcal{A}$. The genus of the generic curve of degree $\pi|_{\mathcal{A}} = k$ and class c is given by the adjunction formula as $g = \frac{1}{2}(kn(1-k) + 2kc - 2k - 2c) + 1$.

(2.12) Given a pair of meromorphic functions (g, f), we determine the *end structure* of the corresponding Calabi curve (with Gauss map g). I.e. those points where the curve goes to infinity in $\mathbf{C}^{n+1}$. Viewing the pair as a description of an algebraic curve on $\mathcal{S}_n$ observe that such points are given by points where the curve on $\mathcal{S}_n$ intersects the curve at infinity, E_∞, or points where the curve osculates a fibre of $\mathcal{S}_n \longrightarrow I\!P_1$. These considerations lead to the following:

Theorem Suppose that (g, f) is a pair of meromorphic functions such that g is non-constant and f is not a polynomial in g of degree $\leq n$. Let $\mathcal{D}_\infty(g)$ denote the divisor of poles of g. The end structure of the Calabi curve ψ generated by the formulae of 1.4 is determined as follows:

(i) Those points where ψ osculates the hyperplane at infinity are given, off $\mathcal{D}_\infty(g)$, by the poles of f, and on $\mathcal{D}_\infty(g)$ by the poles of $g^{-n}f$.

(ii) Those points where ψ osculates a finite hyperplane (at infinity) are given off $\mathcal{D}_\infty(g)$ by the poles of

$$\frac{d^n f}{dg^n},$$

and on $\mathcal{D}_\infty(g)$ by the poles of

$$\frac{d^n}{d(\frac{1}{g})^n}(g^{-n}f).$$

Remarks(i) On $\mathcal{D}_\infty(g)$ we must take the twisting of $\pi : \mathcal{O}(n) \longrightarrow \mathbb{P}_1$ into account.

(ii) The geometry of an end in case (ii) above is determined by the first $r \leq n$ for which, off $\mathcal{D}_\infty(g)$,

$$\frac{d^r f}{dg^r} = \infty,$$

and on $\mathcal{D}_\infty(g)$ for which,

$$\frac{d^r}{d(\frac{1}{g})^r}(g^{-n}f) = \infty.$$

(iii) On $\mathcal{S}_n$ there is the linear equivalence $E_\infty \sim E_0 - nC$, see [4] section 4, which gives $\mathcal{A} \cdot E_\infty = c - nk$. Thus the multiplicity of that part of the end structure corresponding to osculation of the hyperplane at infinity itself is determined by (k, c).

3. Minimal Surfaces in $\mathbb{R}^{n+1}$

(3.1) A curve of degree n in $\mathbb{P}_n$ is cut out by quadrics, see [4], and in particular lies on a non-singular quadric hypersurface. A linear transformation of coordinates on $\mathbf{C}^{n+1}$ converts this quadric hypersurface to $(z_0^2 + ... + z_n^2 = 0)$ and thus renders any normal curve *null*. The corresponding Weierstrass formulae for meromorphic data (g, f) yield complete branched minimal surfaces of finite total Gaussian curvature equal to $-2\pi n deg(g)$. This follows immediately from the fact that $ndeg(g)$ gives the homology degree of the Gauss map of the minimal surfaces derived from (g, f).

(3.2) *Note that minimal surfaces constructed in this way are characterized by the geometrical property that the image of their Gauss map has the smallest possible degree (without the surface being degenerate in the sense that the image of its Gauss map lies on a hyperplane, see [6], section 4). Furthermore any such surface in $\mathbb{R}^{n+1}$ may be constructed in this way.*

(3.3) It is easy to write down suitable transformations for arbitary n, but since they are **not** canonical we limit ourselves to illustrating the procedure in 4 and 5 dimensions.

(3.4) It follows from 1.6 that the curves described there satisfy $3\psi_0'\psi_3' - \psi_1'\psi_2' = 0$ and consequently $\omega : M \longrightarrow \mathbf{C}^4$ given by:

$$\begin{aligned}
\omega_0 &= \frac{\sqrt{3}}{2}(\psi_0 + \psi_3) \\
\omega_1 &= -i\frac{\sqrt{3}}{2}(\psi_0 - \psi_3) \\
\omega_2 &= \frac{1}{2}(\psi_1 - \psi_2) \\
\omega_3 &= -\frac{i}{2}(\psi_1 + \psi_2)
\end{aligned}$$

satisfies $\omega_0^2 + \ldots + \omega_3^2 = 0$. Hence the real part of the following gives a minimal surface in $I\!R^4$ of the type described in 3.4:

$$\begin{aligned}
\omega_0 &= \frac{\sqrt{3}}{2}\{(1+g^3)f^{(3)} - 3g^2 f^{(2)} + 6gf^{(1)} - 6f\} \\
\omega_1 &= -i\frac{\sqrt{3}}{2}\{(1-g^3)f^{(3)} + 3g^2 f^{(2)} - 6gf^{(1)} + 6f\} \\
\omega_2 &= \frac{\sqrt{3}}{2}\{(g-g^2)f^{(3)} - (1-2g)f^{(2)} - 2gf^{(1)}\} \\
\omega_3 &= -i\frac{\sqrt{3}}{2}\{(g+g^2)f^{(3)} - (1+2g)f^{(2)} + 2gf^{(1)}\},
\end{aligned}$$

where

$$f^{(1)} = \frac{df}{dg}, f^{(2)} = \frac{d}{dg}(\frac{df}{dg}), etc..$$

Remark There exist Weierstrass formulae in integrated form for general null curves in $\mathbf{C}^4$, see [9], [12]. For integral formulae see [6].

(3.5) Similiarly, full normal curves $\psi : M \longrightarrow \mathbf{C}^5$ satisfy $2\psi_0'\psi_4' + \psi_1'\psi_3' - (\psi_2')^2 = 0$ and hence $\omega : M \longrightarrow \mathbf{C}^5$ given by:

$$\begin{aligned}
\omega_0 &= \frac{1}{\sqrt{2}}(\psi_0 + \psi_4) \\
\omega_1 &= \frac{-i}{\sqrt{2}}(\psi_0 - \psi_4) \\
\omega_2 &= \frac{1}{2}(\psi_1 + \psi_3) \\
\omega_3 &= \frac{-i}{2}(\psi_1 - \psi_3) \\
\omega_4 &= -i\psi_2
\end{aligned}$$

satisfies $\omega_0^2 + ... + \omega_4^2 = 0$. So the real part of the following gives a minimal surface in $\mathbb{R}^5$ of the type described in 3.4:

$$\begin{aligned}
\omega_0 &= \frac{1}{\sqrt{2}}\{(1+g^4)f^{(4)} - 4g^3 f^{(3)} + 12g^2 f^{(2)} - 24gf^{(1)} + 24f\} \\
\omega_1 &= \frac{-i}{\sqrt{2}}\{(1-g^4)f^{(4)} + 4g^3 f^{(3)} - 12g^2 f^{(2)} + 24gf^{(1)} - 24f\} \\
\omega_2 &= (g+g^3)f^{(4)} - (1-3g^2)f^{(3)} + 6gf^{(2)} - 6f^{(1)} \\
\omega_3 &= -i\{(g-g^3)f^{(4)} + (1-3g^2)f^{(3)} - 6gf^{(2)} + 6f^{(1)}\} \\
\omega_4 &= \frac{-i}{\sqrt{6}}\{g^2 f^{(4)} - 2gf^{(3)} + 2f^{(2)}\}
\end{aligned}$$

where

$$f^{(1)} = \frac{df}{dg}, f^{(2)} = \frac{d}{dg}(\frac{df}{dg}), etc..$$

(3.6) **Remark** The examples given in Theorem 3 of [2], which give complete minimal surfaces with total Gaussian curvature $-2\pi n$, are generated by osculation of the curve $\eta = \zeta^{n+1}$ in $\pi : \mathcal{O}(n) \longrightarrow \mathbb{P}_1$. In dimension 3 this gives Enneper's surface. These give the 'simplest' non-trivial examples in each dimension: for any power less than $n+1$, osculation gives a constant map.

The pleasure of constructing new explicit examples is left for the reader.

4. Lax Equations

(4.1) It turns out that it is natural, in quite general circumstances, to associate to a *Lax form with parameter* an algebraic curve C (its spectral curve), together with a dynamical system on the Jacobean of C. C lies on some $\mathcal{O}(n) \longrightarrow \mathbb{P}_1$, where for many interesting examples $n = 2$. This formalism subsumes many examples of finite dimensional completely integrable Hamiltonian systems; it also covers the Nahm equations of monopole theory: see [3] for details and further references.

(4.2) In view of this, the following appears to be a natural question: *Given a Lax form with spectral curve $C \subset \mathcal{O}(n)$, how does the geometry of the Calabi curve C^* reflect the structure of solutions of the Lax equation?*

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