Link complements arising form arithemetic group actions

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§1. Introduction

Let $\mathbb{H}^3$ be 3-dimensional hyperbolic space and $\mathrm{Iso}^+(\mathbb{H}^3)$ its group of orientation preserving isometries. Taking the model

$$\mathbb{H}^3 = \mathbb{C} \times \mathbb{R}^+ = \{(z,r) \mid z \in \mathbb{C}, r \in \mathbb{R}, r > 0\}$$

the group $\mathrm{Iso}^+(\mathbb{H}^3)$ may be identified with $\mathbf{PSL}_2(\mathbb{C})$. The action of $\mathbf{PSL}_2(\mathbb{C})$ on $\mathbb{H}^3$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z,r) = \frac{1}{N} \cdot ((az+b)\overline{(cz+d)} + a\bar{c}r^2, r)$$

with

$$N = |cz+d|^2 + |c|^2 r^2.$$

Note that we do not, for this paper, distinguish between a matrix in $\mathbf{SL}_2(\mathbb{C})$ and its image in $\mathbf{PSL}_2(\mathbb{C})$.

A subgroup $\Gamma \leq \mathbf{PSL}_2(\mathbb{C})$ acts properly discontinuously on $\mathbb{H}^3$ if and only if it is discrete in $\mathbf{PSL}_2(\mathbb{C})$. If, furthermore Γ is torsionfree then the quotient space

$$\Gamma\backslash\mathbb{H}^3$$

gets the structure of a hyperbolic 3-dimensional Riemannian manifold from $\mathbb{H}^3$. If $\Gamma\backslash\mathbb{H}^3$ has finite hyperbolic volume then it can be compactified to

$$\Gamma\hat{\backslash}\mathbb{H}^3$$

so that the boundary $\partial\Gamma\hat{\backslash}\mathbb{H}^3$ consists of finitely many tori. We are here particularly concerned with cases where $\Gamma\backslash\mathbb{H}^3$ is homeomorphic to a link complement in S^3. We call a torsionfree discrete subgroup $\Gamma \leq \mathbf{PSL}_2(\mathbb{C})$ a *link complement group* if

$$\Gamma\backslash\mathbb{H}^3 \cong S^3 - L$$

where $L \subseteq S^3$ is some link.

An interesting class of discrete groups $\Gamma \leq \mathbf{PSL}_2(\mathbb{C})$ is the class of arithmetic groups. For the notion of arithmeticity see [B]. A subclass of these is obtained in the following way. Take $d \in \mathbb{N}$ a squarefree natural number and write

$$K_{-d} = \mathbb{Q}(\sqrt{-d}) \leq \mathbb{C}$$

for the corresponding imaginary quadratic numberfield. We define

$$\mathcal{O}_{-d}$$

to be the ring of integers in K_{-d}. For a nonzero ideal $\mathcal{A} \subseteq \mathcal{O}_{-d}$ we introduce the following groups

$$\mathbf{PSL}_2(\mathcal{O}_{-d},\mathcal{A}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{PSL}_2(\mathbb{C}) \;\Big|\; a,d \in \mathcal{O}_{-d}, b \in \mathcal{A}, c \in \mathcal{A}^{-1}\}.$$

The definition of $\mathcal{A}^{-1}$ is:

$$\mathcal{A}^{-1} = \{x \in K_{-d} \;\mid\; x \cdot a \in \mathcal{O}_{-d} \text{ for all } a \in \mathcal{A}\}.$$

Notice that $\mathbf{PSL}_2(\mathcal{O}_{-d}) = \mathbf{PSL}_2(\mathcal{O}_{-d},\mathcal{O}_{-d})$ and that for a fixed d the groups $\mathbf{PSL}_2(\mathcal{O}_{-d},\mathcal{A})$ are all commensurable. It is a well-known fact that two groups $\mathbf{PSL}_2(\mathcal{O}_{-d},\mathcal{A}_1)$ and $\mathbf{PSL}_2(\mathcal{O}_{-d},\mathcal{A}_2)$ are conjugate in $\mathbf{PSL}_2(K_{-d})$ if and only if $\mathcal{A}_1$ and $\mathcal{A}_2$ represent up to a square the same element in the idealclass group of $\mathcal{O}_{-d}$. Hence there are only finitely many conjugacy classes of groups $\mathbf{PSL}_2(\mathcal{O}_{-d},\mathcal{A})$ as $\mathcal{A}$ runs through all nonzero ideals of $\mathcal{O}_{-d}$.

If $\Gamma \leq \mathbf{PSL}_2(\mathcal{O}_{-d})$ is a torsionfree subgroup of finite index then $\Gamma\backslash\mathbb{H}^3$ is the interior of a compact 3-manifold bounded by finitely many tori. If $\Gamma \leq \mathbf{PSL}_2(\mathbb{C})$ is an arithmetic link complement group then Γ is up to conjugacy commensurable with a unique $\mathbf{PSL}_2(\mathcal{O}_{-d})$. This well-known result can be found in [R]. A first result in the search for torsionfree, discrete arithmetic subgroups $\Gamma \leq \mathbf{PSL}_2(\mathbb{C})$ so that $\Gamma\backslash\mathbb{H}^3$ is homeomorphic to a link complement is:

1.1. Theorem: Let $\Gamma \leq \mathbf{PSL}_2(\mathbb{C})$ be a torsionfree, discrete, arithmetic subgroup so that $\Gamma\backslash\mathbb{H}^3$ is homeomorphic to a link complement then Γ is conjugate to a subgroup of finite index in $\mathbf{PSL}_2(\mathcal{O}_{-d},\mathcal{A})$ where $\mathcal{A} \subseteq \mathcal{O}_{-d}$ is a nonzero ideal and

$$d \in \{1,2,3,5,6,7,10,11,14,15,19,23,31,35,39,47,55,71,95,119\}.$$

This result is mainly contained in $[B-N]$. We shall comment on it in paragraph 2. In some previous work $[G,S,2]$ it was proved that if Γ satisfying the hypothesis of Theorem 1.1 is contained in a $\mathbf{PSL}_2(\mathcal{O}_{-d})$ then

$$d \in \{1,2,3,5,6,7,11,15,19,23,31,39,47,71\}.$$

An obvious consequence of Theorem 1.1 is

1.2. Corollary: Up to conjugacy there are only finitely many commensurability classes of torsionfree, discrete, arithmetic subgroups $\Gamma \leq \mathbf{PSL}_2(\mathbb{C})$ so that $\Gamma\backslash\mathbb{H}^3$ is homeomorphic to a link complement in S^3.

For most d which are in the exceptional list in Theorem 1.1 it is not known whether there is a link complement group $\Gamma \leq \mathbf{PSL}_2(\mathbb{C})$ commensurabel with $\mathbf{PSL}_2(\mathcal{O}_{-d})$. For small d like $d \in \{1,2,3,7\}$ there are many examples listed in [H] or [R]. It can be seen from these examples that there are infinitely many conjugacy classes of arithmetic link complement groups $\Gamma \leq \mathbf{PSL}_2(\mathbb{C})$.

Theorem 1.1 shows that all arithmetic link complement groups can be found as subgroups of finitely many groups of the type $\mathbf{PSL}_2(\mathcal{O}_{-d}, \mathcal{A})$. The main result of this paper is a method to single out link complement groups from the subgroups of finite index in a fixed group $\mathbf{PSL}_2(\mathcal{O}_{-d}, \mathcal{A})$.

Let d be fixed and small and let $\mathcal{A} \subseteq \mathcal{O}_{-d}$ be a nonzero ideal. Then usually a nice $\mathbf{PSL}_2(\mathcal{O}_{-d}, \mathcal{A})$ invariant tesselation of $\mathbb{H}^3$ is known, see for example [G, G, M], [Sw], [H], [Sch]. From this a presentation of $\mathbf{PSL}_2(\mathcal{O}_{-d}, \mathcal{A})$ is readily obtained. The presentation can then be used to find all subgroups $\Gamma \leq \mathbf{PSL}_2(\mathcal{O}_{-d}, \mathcal{A})$ of a given (low) index n. This is done with the help of standard computer programs, see [G, S, 1] for more details. Let Γ now be from this list. It can be easily checked whether Γ is torsionfree. We are then left with the problem to find necessary conditions for Γ to be a link complement group. We state some in the following

1.3. Proposition: Let $\Gamma \leq \mathbf{PSL}_2(\mathbb{C})$ be a torsionfree link complement group. Then:
1) $H_1(\Gamma, \mathbb{Z})$ is torsionfree,
2) $H^1_{\mathrm{cusp}}(\Gamma, \mathbb{C}) = 0$,
3) Γ^{ab} is generated by images of unipotent elements,
4) Γ is generated by unipotent elements.

For more comments see paragraph 2. It is clear, see [G,S,1], that all these conditions can be checked for a specific group Γ. To follow the problem whether $\Gamma\backslash\mathbb{H}^3$ is homeomorphic to a link complement we have now to employ more geometric techniques. The tesselation of $\mathbb{H}^3$ leads to a combinatorial description of $\Gamma\backslash\mathbb{H}^3$. From this combinatorial description we then have to decide whether $\Gamma\backslash\mathbb{H}^3$ is a link complement and also find the link. This is in fact the most difficult part of our program. An elementary approuch to visualise the manifold $\Gamma\backslash\mathbb{H}^3$ seems to be very difficult, see [Ba] for an attempt.

A priori the combinatorial structure of $\Gamma\backslash\mathbb{H}^3$ depends on the presentation chosen for the group Γ. Its complexity increases with the index of Γ in $\mathbf{PSL}_2(\mathcal{O}_{-d})$. However, in all cases under consideration, a special type of handle cancellation permits us to simplify the structure and thus to decide that $\Gamma\backslash\mathbb{H}^3$ embeds in S^3. This simplification process is due to Zieschang [Z] and is the geometric counterpart to the Nielsen-transformations of group representations described in [M,K,S], [C,Z,1-3]. It is then routine work to find a system of longitude-meridian pairs on $\partial(\Gamma\backslash\mathbb{H}^3)$ which gives us the desired links. We are indebted to Wolfgang Haken for explaining the Whitehead-Zieschang technique to one of us.

We do not know whether the above process is an algorithm. In particular, we were not able to decide the embeddability of $\Gamma\backslash\mathbb{H}^3$ in S^3 algorithmically. However in all cases we have studied, we were, with some effort, able to do so.

We shall carry out the above program in paragraphs 3 - 5 on the subgroups of index ≤ 12 in $\mathbf{PSL}_2(\mathcal{O}_{-7})$. In paragraph 3 we describe the conjugacy classes of torsionfree subgroups of index $\leq$ 12 in $\mathbf{PSL}_2(\mathcal{O}_{-7})$ which also have a torsionfree first homology group. We then develop in each case a combinatorial description of $\Gamma\backslash\mathbb{H}^3$. We then apply to these combinatorial manifolds the simplification process described above and prove that they are embeddable in S^3 and thus are homeomorphic to link complements. We also were able to find the corresponding links. We have listed them in paragraph 6. Our list contains many examples of links which have not been found before.

We hope that our lists will shed some light on the following two problems.

Problem 1: For which links L is there a torsionfree, discrete arithmetic group $\Gamma \leq \mathbf{PSL}_2(\mathbb{C})$ so that $S^3 - L$ is homeomorphic to $\Gamma\backslash\mathbb{H}^3$?

The corresponding question for just torsionfree, discrete subgroups $\Gamma \leq \mathbf{PSL}_2(\mathbb{C})$ has been answered by Thurston [Th].

The particular case of knot-complements was studied by Reid [R]. He proves that the figure 8 knot is the only knot whose complement is homeomorphic to $\Gamma\backslash\mathbb{H}^3$ with Γ arithmetic.

Problem 2: Which torsionfree arithmetic subgroups $\Gamma \leq \mathbf{PSL}_2(\mathbb{C})$ are link complement groups?

The examples we studied suggest that Proposition 1.3 has a converse. That is, we always found if $\Gamma \leq \mathbf{PSL}_2(\mathbb{C})$ is torsionfree arithmetic, has torsionfree $H_1(\Gamma, \mathbb{Z})$ and is generated by unipotent elements that $\Gamma\backslash\mathbb{H}^3$ is homeomorphic to a link complement. It seems plausible that under these hypothesis it is possible to glue solid tori into the cusps which kill all of the fundamental group. The resulting closed 3-manifold would have trivial fundamental group and should be S^3. Unfortunately we were not able to carry this idea through.

Amongst all subgroups of finite index in $\mathbf{PSL}_2(\mathcal{O}_{-7})$ those which are torsionfree and have torsionfree first homology group and hence are potential link complement groups seem to be quite rare. We put A_n to be the number of conjugacy classes of subgroups of finite index n in $\mathbf{PSL}_2(\mathcal{O}_{-7})$ and T_n to be the number of conjugacy classes of subgroups of index n which are torsionfree and have torsionfree first homology group. Note that $T_n = 0$ if 6 does not divide n. We report the following table:

n	A_n	T_n
6	25	6
12	197	24
18	1877	124

In all cases the links in paragraph 6 are covered by other links under cyclic coverings. By this many other arithmetic link complements can be found. We mention only one example: The link under $\Gamma_{-7}(6,4)$ can be arranged to look like

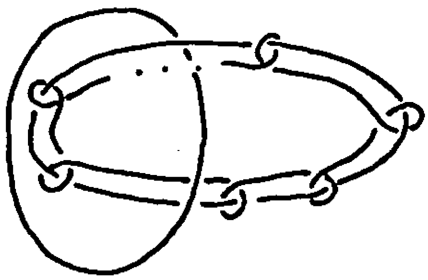


Figure 1.1

The links under Figure 1.2 then cyclicly cover the above link.

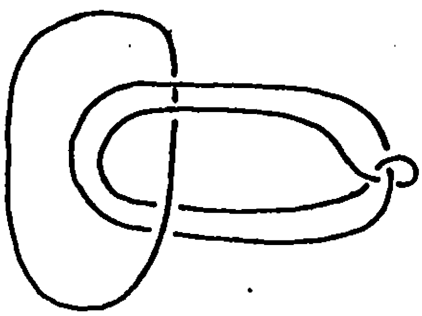


Figure 1.2

Obviously they all have 2 components. However for other links, for example $\Gamma_{-7}(12,1)\backslash\mathbb{H}^3$ there are cyclic coverings with an increasing number of components.

§2 Link complement groups in $\mathbf{PSL}_2(\mathbb{C})$

We start off by discussing the proof of Proposition 1.3. Let $\Gamma \leq \mathbf{PSL}_2(\mathbb{C})$ be a link complement group. That is Γ is torsionfree, discrete and $\Gamma\backslash\mathbb{H}^3$ is homeomorphic to $S^3 - L$ for some link $L \subseteq S^3$. Since

$$\Gamma \cong \pi_1(\Gamma\backslash\mathbb{H}^3) \cong \pi_1(S^3 - L)$$

we may infere the first claim of Proposition 1.3 from standard properties of fundamental groups of link complements [Ro].

The second claim is proved in [Schw], note that

$$H^1_{cusp}(\Gamma, \mathbb{C}) = \ker(H^1(\Gamma\backslash\mathbb{H}^3, \mathbb{C}) \to H^1(\partial(\Gamma\backslash\mathbb{H}^3), \mathbb{C})).$$

This standard condition gives a strong restriction on the groups Γ.

The third claim is obviously implied by the fourth.

To prove the fourth we use that $\pi_1(S^3 - L)$ is generated by Wirtinger elements. They go to unipotent elements under the isomorphism $\pi_1(S^3 - L) \cong \Gamma$. See also [G,S,1]. This proves Proposition 1.3.

To prove Theorem 1.1 let $\Gamma \leq \mathbf{PSL}_2(\mathbb{C})$ be an arithmetic link complement group. Then $\Gamma\backslash\mathbb{H}^3$ is not cocompact and hence a conjugate of Γ is commensurable with a $\mathbf{PSL}_2(\mathcal{O}_{-d})$, [R] Proposition 1. By Proposition 1.3 Γ is generated by unipotent elements and hence we infere from [R] Lemma 1 that $\Gamma \leq \mathbf{PSL}_2(K_{-d})$. From standard properties [B] of arithmetic groups it follows that $\Gamma \leq \mathbf{PSL}_2(\mathcal{O}_{-d}, \mathcal{A})$ for a suitable ideal $\mathcal{A} \leq \mathcal{O}_{-d}$. Since $H^1_{cusp}(\Gamma, \mathbb{C}) = 0$ we get that also $H^1_{cusp}(\mathbf{PSL}_2(\mathcal{O}_{-d}, \mathcal{A}), \mathbb{C}) = 0$. By an application of the Lefschetz-trace formula J.Blume-Nienhaus has proved that this can only happen in rather restricted cases, namely at most for the d listed in the theorem.

§3 Subgroups of index ≤ 12 in $\mathbf{PSL_2}(\mathcal{O}_{-7})$.

Here we shall describe the combinatorial data which will be used to analyse the topology of the quotients $\Gamma\backslash\mathbb{H}^3$ where $\Gamma \leq \mathbf{PSL_2}(\mathcal{O}_{-7})$ are certain subgroups of finite index.

3.1 *The group* $\mathbf{PSL_2}(\mathcal{O}_{-7})$

We start off by describing the group $\mathbf{PSL_2}(\mathcal{O}_{-7})$. We put

$$\omega = \frac{1}{2} + \frac{1}{2}\sqrt{-7}.$$

We then have $\mathcal{O}_{-7} = \mathbb{Z} \oplus \mathbb{Z}\omega$. The group $\mathbf{PSL_2}(\mathcal{O}_{-7})$ is generated by

$$(3.1) \qquad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}$$

and presented by the relations

$$(3.2) \qquad B^2 = (AB)^3 = ACA^{-1}C^{-1} = (BAC^{-1}BC)^2 = 1.$$

We shall also need

$$S_1 = AB = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, U_1 = CBC^{-1} = \begin{pmatrix} \omega & 1-\omega \\ 1 & -\omega \end{pmatrix},$$

$$S_2 = S_1^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, U_2 = AC^{-1}BCA^{-1} = \begin{pmatrix} 1-\omega & \omega+1 \\ 1 & \omega-1 \end{pmatrix}.$$

The groups Δ_1, Δ_2 generated by the following matrices

$$(3.3) \qquad \Delta_1 = < S_1, U_1 >, \qquad \Delta_2 = < S_2, U_2 >$$

are finite and in fact isomorphic to the symmetric group on 3 symbols. We have

$$\Delta_1 \cap \Delta_2 = < S_1 > = \{1, S_1, S_1^2\}.$$

It is well-known [G,S,1] that if $\Gamma \leq \mathbf{PSL_2}(\mathcal{O}_{-7})$ is torsionfree of finite index then

$$6 \,|\, |\mathbf{PSL_2}(\mathcal{O}_{-7}) : \Gamma|.$$

So we shall contend our study here to subgroups of index 6 and 12.

3.2 *The invariant tesselation*

We shall proceed by giving a tesselation of $\mathbb{H}^3$ by hyperbolic prisms. The vertices of these prisms will be contained in $\partial\mathbb{H}^3 = \mathbb{P}^1(\mathbb{C})$. In our model we think of

$$\partial\mathbb{H}^3 = (\mathbb{C} \times \{0\}) \cup \{\infty\}$$

with $\mathbf{PSL_2}(\mathbb{C})$ acting by linear fractional transformations. We define the following points in $\partial\mathbb{H}^3$.

$$(3.4) \qquad \begin{aligned} &a = (0,\ 0), \quad b = (\tfrac{\omega}{2},\ 0), \quad c = (\omega,\ 0), \\ &d = (\tfrac{1}{2} + \tfrac{\omega}{2},\ 0), \quad e = (1,\ 0), \quad f = \infty, \\ &g = (\tfrac{1}{2} - \tfrac{\omega}{2},\ 0), \quad h = (1-\omega,\ 0), \quad i = (1 - \tfrac{\omega}{2},\ 0). \end{aligned}$$

Let $\mathcal{P}$ be the geodesic prism spanned by $a, \cdots, f$ and $\mathcal{P}_1$ that spanned by a, e, f, g, h, i. We have the following (euclidean) picture of the double prism $\mathcal{P} \cup \mathcal{P}_1$ in $\mathbb{H}^3 \cup \partial\mathbb{H}^3$:

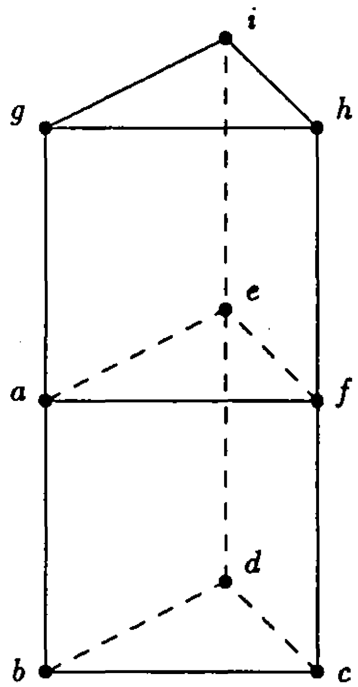


Figure 3.1: $\mathcal{P} \cup \mathcal{P}_1$

We find [G,G,M] that Δ_1 is the normalizer of $\mathcal{P}$ in $\mathbf{PSL}_2(\mathcal{O}_{-7})$ and that Δ_1 acts transitively on the vertices of $\mathcal{P}$ by the formulas:

$$S_1 : \begin{array}{l} a \mapsto f \\ b \mapsto c \\ c \mapsto d \\ d \mapsto b \\ e \mapsto a \\ f \mapsto e \end{array} \qquad U_1 : \begin{array}{l} a \mapsto d \\ b \mapsto e \\ c \mapsto f \\ d \mapsto a \\ e \mapsto b \\ f \mapsto c \end{array} . \tag{3.5}$$

The group Δ_2 is the normalizer of $\mathcal{P}_1$ and also acts transitively on its vertices. We have the formulas

$$S_2 : \begin{array}{l} a \mapsto e \\ e \mapsto f \\ f \mapsto a \\ g \mapsto i \\ h \mapsto g \\ i \mapsto h \end{array} \qquad U_2 : \begin{array}{l} a \mapsto i \\ e \mapsto g \\ f \mapsto h \\ g \mapsto e \\ h \mapsto f \\ i \mapsto a \end{array} . \tag{3.6}$$

It is known that the union

$$\bigcup_{\gamma \in \mathrm{PSL}_2(\mathcal{O}_{-7})} \gamma\mathcal{P} \cup \bigcup_{\gamma \in \mathrm{PSL}_2(\mathcal{O}_{-7})} \gamma\mathcal{P}_1$$

defines a $\mathbf{PSL}_2(\mathcal{O}_{-7})$ invariant tesselation of $\mathbb{H}^3 \cup \partial\mathbb{H}^3$. We put

$$\mathcal{Q} = \mathcal{P} \cup \mathcal{P}_1.$$

The set $\mathcal{Q} \cap \mathbb{H}^3$ is then the fundamental domain for the action of $\mathbf{PSL}_2(\mathcal{O}_{-7})$ on $\mathbb{H}^3$.

3.3 *Subgroups of index 6 in* $\mathbf{PSL}_2(\mathcal{O}_{-7})$

Let $\Gamma \leq \mathbf{PSL}_2(\mathcal{O}_{-7})$ be a torsionfree subgroup of index 6 then both the sets Δ_1, Δ_2 are full sets of representatives for the cosets

$$\Gamma\backslash\mathbf{PSL}_2(\mathcal{O}_{-7}) = \{\Gamma g_1, \cdots, \Gamma g_6\}.$$

This shows that the hyperbolic double prism with vertices removed shown in figure (3.1) is a fundamental domain for the action of Γ on $\mathbb{H}^3$. To get a combinatorial description of the quotient $\Gamma\backslash\mathbb{H}^3$ we need to find the identifications of the boundary of $\mathcal{P} \cup \mathcal{P}_1$ induced by Γ. To do this it is enough to describe the identifications of the oriented edges of $\mathcal{P} \cup \mathcal{P}_1$ induced by Γ. The identifications of the 2-cells then follow. Oriented edges will be denoted by

$$[x, y]$$

where $x, y \in \{a, \cdots, i\}$. This information can be extracted from the permutation representation

$$\sigma : \mathbf{PSL}_2(\mathcal{O}_{-7}) \longrightarrow \mathcal{S}_6 = \mathcal{S}(\{1, \cdots, 6\})$$

on the cosets of Γ. We have for $g \in \mathbf{PSL}_2(\mathcal{O}_{-7})$:

$$g\Gamma g_i = \Gamma g_i g^{-1} = \Gamma \sigma_{g^{-1}}(i).$$

Γ is the stabilizer of 1 under σ in $\mathbf{PSL}_2(\mathcal{O}_{-7})$. With the help of tables (3.5), (3.6) we now pick for every oriented edge E of $\mathcal{P} \cup \mathcal{P}_1$ a $\gamma_E \in \Delta_1 \cup \Delta_2$ so that

$$E = \gamma_E^{-1}[a, f]$$

Because of (3.5), (3.6) γ_E is unique. Two oriented edges E_1, E_2 are then identified by Γ if and only if

$$\gamma_{E_1}^{-1}\gamma_{E_2} \in \Gamma.$$

This can again be checked by the permutation representation.

We shall give now for the relevant subgroups Γ of index 6 in $\mathbf{PSL}_2(\mathcal{O}_{-7})$ generators and the permutation representation on the generators A, B, C of $\mathbf{PSL}_2(\mathcal{O}_{-7})$. This is then sufficient to compute the edge identifications which we will also give.

We infere from [G,S,1] that $\mathbf{PSL}_2(\mathcal{O}_{-7})$ has 6 conjugacy classes of torsionfree subgroups Γ of index 6 with torsionfree $H_1(\Gamma, \mathbb{Z})$. They are represented by the groups $\Gamma_{-7}(6,1)$ up to $\Gamma_{-7}(6,6)$ in the numbering of [G,S,1]. We have

$\Gamma_{-7}(6,1)$:

Generators: C, A^2, BCB , $H_1(\Gamma, \mathbb{Z}) = \mathbb{Z}^3$
Permutation representation: $\sigma_A = (1,2)(3,4)(5,6)$, $\sigma_B = (1,3)(2,5)(4,6)$, $\sigma_C = (1)$
Edge identifications: $\Gamma_{-7}(6,1)\backslash\mathcal{P} \cup \mathcal{P}_1$ has 3 classes of oriented edges which can be read off from:
1) $[a,f],[c,f],[c,d],[i,g],[a,g]$,
2) $[b,d],[h,g],[h,f],[e,f],[e,d]$,
3) $[e,a],[b,a],[b,c],[h,i],[e,i]$.

$\Gamma_{-7}(6,2)$:

Generators: $A, BC^2B, BCA^{-2}B$, $H_1(\Gamma, \mathbb{Z}) = \mathbb{Z}^3$
Permutation representation: $\sigma_A = (2,3,4,5)$, $\sigma_B = (1,2)(6,4)(3,5)$, $\sigma_C = (2,4)(3,5)$
Edge identifications:
1) $[a,f],[c,f],[h,f],[e,f]$,
2) $[e,a],[i,e],[b,c],[a,b],[h,i]$,
3) $[d,c],[d,e],[g,i],[g,a],[d,b],[g,h]$.

$\Gamma_{-7}(6,3)$:

Generators: $A, C, BACB$, $H_1(\Gamma, \mathbb{Z}) = \mathbb{Z}^3$
Permutation representation: $\sigma_A = (2,3,4,5)$, $\sigma_B = (1,2)(3,5)(4,6)$, $\sigma_C = (2,5,4,3)$
Edge identifications:
1) $[a,f],[c,f],[h,f],[e,f]$,
2) $[e,a],[d,e],[c,d],[g,h],[a,g]$,
3) $[b,d],[b,a],[i,h],[i,e],[b,c],[i,g]$.

$\Gamma_{-7}(6,4)$:

Generators: $A^2, CA^{-1}, BCA^{-1}B$, $H_1(\Gamma, \mathbb{Z}) = \mathbb{Z}^3$
Permutation representation: $\sigma_A = (1,2)(3,4)(5,6)$, $\sigma_B = (1,3)(2,5)(4,6)$, $\sigma_C = (1,2)(3,4,)(5,6)$
Edge identifications:
1) $[a,f],[h,f],[d,b],[a,b],[h,i]$,
2) $[c,b],[c,f],[g,i],[e,f],[e,i]$,
3) $[e,a],[g,a],[c,d],[e,d],[g,h]$.

$\Gamma_{-7}(6,5)$:

Generators: C, BAB , $H^1(\Gamma, \mathbb{Z}) = \mathbb{Z}^2$
Permutation representation: $\sigma_A = (1,3,4,5)$, $\sigma_B = (1,2)(6,4)(3,5)$, $\sigma_C = (2,6)$
Edge identifications:
1) $[a,f],[c,f],[i,e],[a,e]$,
2) $[f,e],[d,e],[d,b],[g,h],[f,h]$,
3) $[c,b],[i,h],[c,d],[a,b],[i,g],[a,g]$.

$$\Gamma_{-7}(6,6):$$

Generators: AC^{-1}, BAB , $H_1(\Gamma,\mathbb{Z}) = \mathbb{Z}^2$
Permutation representation: $\sigma_A = (1,3,4,5)$, $\sigma_B = (1,2)(4,6)(3,5)$,
$\sigma_C = (2,6)(1,3,4,5)$
Edge identifications:
1) $[a,f],[d,e],[h,f],[a,e]$,
2) $[f,e],[i,e],[b,c],[f,c],[i,g]$,
3) $[d,c],[h,g],[d,b],[a,b],[h,i],[a,g]$.

3.4 *Subgroups of index 12 in* $\mathbf{PSL}_2(\mathcal{O}_{-7})$

Let $\Gamma \leq \mathbf{PSL}_2(\mathcal{O}_{-7})$ be a torsionfree subgroup of index 12. Then there is a $T \in \mathbf{PSL}_2(\mathcal{O}_{-7})$ so that the sets

$$\Delta_1 \cup T\Delta_1 \quad \text{and} \quad \Delta_2 \cup T\Delta_2$$

are full sets of coset representatives for $\Gamma\backslash\mathbf{PSL}_2(\mathcal{O}_{-7})$. We define

$$\mathcal{P}' = T\mathcal{P}, \mathcal{P}^1 = T\mathcal{P}_1.$$

We find that the union of the two double prisms

$$\mathcal{Q} = \mathcal{P} \cup \mathcal{P}_1 \cup \mathcal{P}' \cup \mathcal{P}_1'$$

with their vertices removed is a fundamental domain for the action of Γ on $\mathbb{H}^3$. For the vertices we adopt the notation shown in the following picture

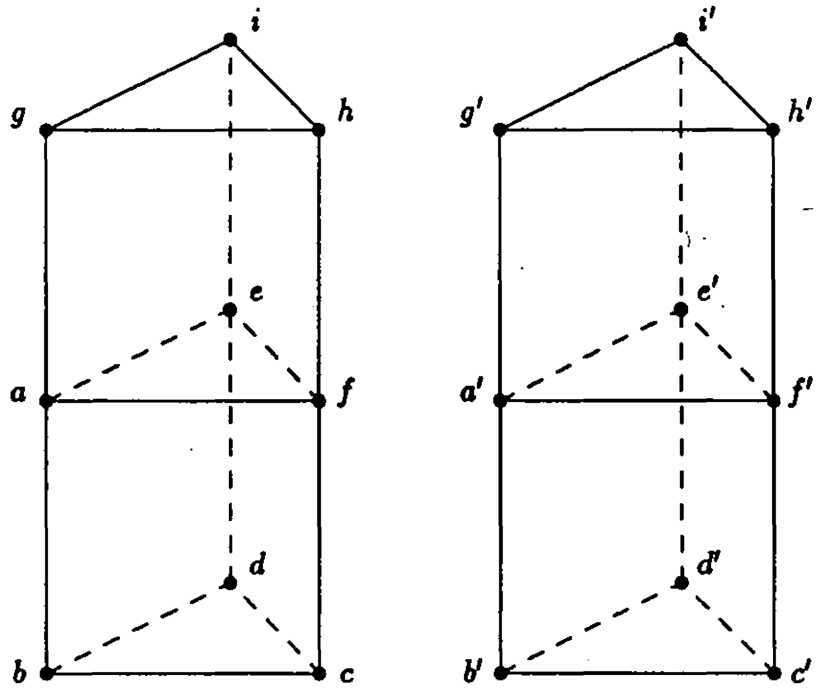


Figure 3.2: $\mathcal{Q} = \mathcal{P} \cup \mathcal{P}_1 \cup \mathcal{P}' \cup \mathcal{P}_1'$

We infere from [G,S,1] that $\mathbf{PSL}_2(\mathcal{O}_{-7})$ has 24 conjugacy classes of subgroups of index 12 which are torsionfree and have torsionfree commutator quotient group. We list them here together with their generators, permutation representations and the corresponding edge identifications for $\mathcal{Q}$.

$\Gamma_{-7}(12,1)$:

Generators: A, C, BC^2B, BCA^3B, $\quad H_1(\Gamma, \mathbb{Z}) = \mathbb{Z}^4$,
Permutation representation: $\sigma_A = (9,3,6,5,2,4)(8,7,12)(10,11)$,
$\sigma_B = (1,2)(4,5)(6,7)(8,9)(3,10)(11,12), \sigma_C = (2,3)(4,6)(12,7,8)(5,9)(10,11)$.
Edge identifications:

1) $[a,f],[c,f],[h,f],[e,f]$,
2) $[d,c],[d,e],[g,a],[i',h'],[g,h],[d',b']$,
3) $[e,a],[h',f'],[f',a'],[a',b']$,
4) $[b,a],[i,h],[c',b'],[g',a'],[e',f']$,
5) $[c,b],[h',g'],[e,i],[a',e'],[f',c']$,
6) $[b,d],[g',i'],[i,g],[c',d'],[e',d'],[e',i']$.

$\Gamma_{-7}(12,2)$:

Generators: A, C, BC^3B, BCA^2B, $\quad H_1(\Gamma, \mathbb{Z}) = \mathbb{Z}^4$,
Permutation representation: $\sigma_A = (2,3,4,5,6,7)(8,9,10)(11,12)$,
$\sigma_B = (1,2)(3,7)(6,8)(4,10)(5,11)(9,12), \sigma_C = (2,6,4)(3,7,5)(8,10,9)$.
Edge identifications:

1) $[a,f],[c,f],[h,f],[e,f]$,
2) $[e,a],[d',e'],[e',f'],[f',h']$,
4) $[d,e],[a',f'],[i',e'],[g,h],[c',d']$,
5) $[b,a],[i,h],[b',d'],[i,e],[b,c],[g',h']$,

3) $[d,c],[c',f'],[i',h'],[g,a],[a',e']$, 6) $[b,d],[b',a'],[g',i'],[g',a'],[i,g],[b',c']$.

$\Gamma_{-7}(12,3)$:

Generators:: $A, C, BC^4B, BCBC^{-1}BC^{-1}B$, $H_1(\Gamma,\mathbb{Z}) = \mathbb{Z}^4$,
Permutation representation: $\sigma_A = (2,3,4,5,6,7,8,9)(10,11)$,
$\sigma_B = (1,2)(3,9)(8,10)(4,11)(5,7)(6,12), \sigma_C = (2,8,6,4)(3,9,7,5)$.

$\Gamma_{-7}(12,4)$:

Generators:: $C, BC^2B, BCA^2B, ABC^2BA^{-1}$, $H_1(\Gamma,\mathbb{Z}) = \mathbb{Z}^4$,
Permutation representation: $\sigma_A = (1,2)(3,4,5,6)(7,8,9,10)(11,12)$,
$\sigma_B = (1,3)(2,7)(4,10)(5,11)(6,8)(9,12), \sigma_C = (3,5)(7,9)(4,6)(8,10)$.

$\Gamma_{-7}(12,5)$:

Generators:: $AC^{-1}, C^2, BACB, A^{-1}BACBA$, $H_1(\Gamma,\mathbb{Z}) = \mathbb{Z}^4$,
Permutation representation: $\sigma_A = (1,2)(3,4,5,6)(7,8,9,10)(11,12)$
$\sigma_B = (1,3)(2,7)(4,10)(6,8)(5,11)(9,12), \sigma_C = (1,2)(3,6,5,4)(7,10,9,8)(11,12)$.

$\Gamma_{-7}(12,6)$:

Generators:: $A, C, BA^2C^2B, BCBCB$, $H_1(\Gamma,\mathbb{Z}) = \mathbb{Z}^4$,
Permutation representation: $\sigma_A = (2,3,4,5,6,7,8,9)(10,11)$,
$\sigma_B = (1,2)(3,9)(5,7)(4,10)(6,12)(8,11), \sigma_C = (2,5,8,3,6,9,4,7)(10,11)$.

$\Gamma_{-7}(12,7)$:

Generators:: $C, A^3, BCB, A^{-1}BA^3BA$, $H_1(\Gamma,\mathbb{Z}) = \mathbb{Z}^3$,
Permutation representation: $\sigma_A = (1,2,3)(4,5,6)(7,8,9)(10,11,12)$,
$\sigma_B = (1,4)(2,7)(3,10)(5,12)(6,8)(9,11), \sigma_C = (7,10,9,12,8,11)$.
Edge identifications:
1) $[a,f],[c,f],[c,d],[i,g][a,g]$,
2) $[b,d],[h,g],[e,d],[h',f'],[e',f']$,
3) $[h,f],[e,f],[a',f'],[c',f']$,
4) $[e,a],[b,a],[d',b'],[h',i'],[e',i']$,
5) $[b,c],[h,i],[e,i],[c',b'],[g',i'],[a',b']$,
6) $[d',c'],[h',g'],[g',a'],[a',e'],[e',d']$.

$\Gamma_{-7}(12,8)$:

Generators:: $C, BACB, BAC^{-2}B$, $H_1(\Gamma,\mathbb{Z}) = \mathbb{Z}^3$,
Permutation representation: $\sigma_A = (1,2,3)(4,5,6)(7,8,9)(10,11,12)$,
$\sigma_B = (1,4)(2,7)(3,10)(5,12)(6,8)(9,11), \sigma_C = (4,6,5)(7,11,9,10,8,12)$.
Edge identifications:

1) $[a,f],[c,f],[g',a'],[f',a']$,
2) $[h,i],[e,i],[e',a'],[b',a'],[c',d']$,
3) $[g,i],[h,f],[e,f],[b',d'],[e',d']$,
4) $[e,a],[b,c],[c',f'],[h',f'],[i',g']$,
5) $[d,c],[b,a],[g,a],[g',g'],[e',f']$,
6) $[d,e],[g,h],[i',h'],[b',c'],[e',i'],[b,d]$.

$\Gamma_{-7}(12,9)$:

Generators:: $C, BACB, ABC^{-3}AB$, $\quad H_1(\Gamma,\mathbb{Z}) = \mathbb{Z}^3$,
Permutation representation: $\sigma_A = (2,3,4,5,6,7,8,9)(1,11)$,
$\sigma_B = (1,4)(2,10)(3,9)(5,7)(6,12)(8,11), \sigma_C = (2,9,8,7,6,5,4,3)(10,12)$.

$\Gamma_{-7}(12,10)$:

Generators:: $AC^{-1}, BAC^{-1}B, BC^2BC^2$, $\quad H_1(\Gamma,\mathbb{Z}) = \mathbb{Z}^3$,
Permutation representation:
$\sigma_A = (1,2,6,5)(3,4,7,8)(9,10)(11,12)$,
$\sigma_B = (1,3)(2,10)(4,12)(5,11)(6,7)(8,9), \sigma_C = (1,2,6,5)(3,4,7,8)(9,11)(10,12)$.

$\Gamma_{-7}(12,11)$:

Generators:: $C, BC^2A^{-1}B, BCBA^{-2}B$, $\quad H_1(\Gamma,\mathbb{Z}) = \mathbb{Z}^3$,
Permutation representation: $\sigma_A = (2,3,4,5,6)(7,8,9,10,11)$,
$\sigma_B = (1,10)(2,12)(3,6)(4,7)(9,11)(5,8), \sigma_C = (2,7,3,8,4,9,5,10,6,11)$.
Edge identifications:
1) $[a,f],[c,f],[h,f],[e,f]$,
2) $[b',d'],[b',a'],[g',i'],[g',a'],[b',c'],[g',h']$,
3) $[c,d],[a,b],[h,i],[a,g],[d',c'],[i',h']$,
4) $[b,d],[g,i],[c',f'],[h',f'],[a',e']$,
5) $[d,e],[i,e],[b,c],[g,h],[f',e']$,
6) $[a,e],[a',f'],[d',e'],[i',e']$.

$\Gamma_{-7}(12,12)$:

Generators:: $C, BCB, A^{-1}BCBA^{-1}$, $\quad H_1(\Gamma,\mathbb{Z}) = \mathbb{Z}^3$,
Permutation representation: $\sigma_A = (1,2,3,4)(5,6,7,8)(9,10)(11,12)$,
$\sigma_B = (1,5)(2,9)(3,7)(4,11)(6,12)(8,10), \sigma_C = (9,11)(10,12)$.

$\Gamma_{-7}(12,13)$:

Generators:: $A, BC^2B, CBCA^2B$, $\quad H_1(\Gamma,\mathbb{Z}) = \mathbb{Z}^3$,
Permutation representation: $\sigma_A = (2,8,5,4)(3,10,6,9)$,
$\sigma_B = (1,2)(3,11)(4,8)(5,12)(6,7)(9,10), \sigma_C = (1,7)(2,3)(4,9)(5,6)(8,10)(11,12)$.

$\Gamma_{-7}(12,14)$:

Generators:: $A, C^2, CBCBC^2B, BACB$, $H_1(\Gamma,\mathbb{Z}) = \mathbb{Z}^3$,
Permutation representation: $\sigma_A = (2,3,4,5,6)(7,8,9,10,11)$,
$\sigma_B = (1,2)(12,7)(3,6)(8,11)(9,4), \sigma_C = (1,12)(2,6,5,4,3)(7,9,11,8,10)$.
Edge identifications:
1) $[a,f],[e,f],[c',f'],[h',f']$,
2) $[e,a],[d,e],[a,g],[c',d'],[g',h']$,
3) $[b,a],[c,d],[i,g],[b',d'],[i',h'],[a',g']$,
4) $[b,d],[h,g],[i,e],[b',c'],[e',d'],[i',g']$,
5) $[f,c],[f,h],[f',e'],[f',a']$,
6) $[b,c],[h,i],[e',a'],[i',e'],[a',b']$.

$\Gamma_{-7}(12,15)$:

Generators:: $A, C^2, BC^{-1}A^2B, CBC^2B$, $H_1(\Gamma,\mathbb{Z}) = \mathbb{Z}^3$,
Permutation representation: $\sigma_A = (3,7,10,9,6,8,5,4)(11,12)$,
$\sigma_B = (1,3)(2,6)(4,7)(5,11)(8,9)(10,12), \sigma_C = (1,2)(3,10,6,5)(4,7,9,8)(11,12)$.

$\Gamma_{-7}(12,16)$:

Generators:: $A, CBACB, BACBC^{-1}$, $H_1(\Gamma,\mathbb{Z}) = \mathbb{Z}^3$,
Permutation representation: $\sigma_A = (2,3,4,5)(6,7,8,9)$,
$\sigma_B = (1,2)(6,10)(3,5)(7,9)(4,11)(8,12), \sigma_C = (1,10)(2,9,4,7)(6,5,8,3)(11,12)$.

$\Gamma_{-7}(12,17)$:

Generators:: $A, CBACB, BA^{-1}C^2B$, $H_1(\Gamma,\mathbb{Z}) = \mathbb{Z}^3$,
Permutation representation: $\sigma_A = (2,3,4,5)(6,7,8,9)$,
$\sigma_B = (10,2)(1,6)(3,5)(8,11)(7,9)(4,12), \sigma_C = (1,10)(2,8,3,9,4,6,5,7)(11,12)$.
Edge identifications:
1) $[a,f],[e,f],[c',f'],[h',f']$,
2) $[f,c],[f,h],[f',e'],[f',a']$,
3) $[a,b],[a,g],[e',a'],[c',d'],[h',i']$,
4) $[d,e],[i,e],[b',d'],[g',i'],[b',c'],[g',h']$,
5) $[e,a],[b,c],[g,h],[d',e'],[i',e']$,
6) $[d,c],[i,h],[d,b],[i,g],[b',a'],[g',a']$.

$\Gamma_{-7}(12,18)$:

Generators:: $AC^{-1}, CBC^{-1}B, BA^3B$, $H_1(\Gamma,\mathbb{Z}) = \mathbb{Z}^3$,
Permutation representation: $\sigma_A = (2,1,3)(4,5,6)(7,8,9)(10,11,12)$,
$\sigma_B = (2,4)(1,7)(3,10)(5,12)(6,8)(9,11), \sigma_C = (2,1,3)(4,7)(5,8)(6,9)(10,12,11)$.
Edge identifications:
1) $[a,f],[h,f],[c,d],[e,d],[h',g']$,
2) $[c,f],[h,g],[e,f],[a,g],[b',d']$,
3) $[b,d],[i,g],[g',i'],[c'd'],[e',d'],[e',i']$,
4) $[b,a],[i,h],[c',b'],[g',a'],[e',f']$,
5) $[a,e],[a',f'],[b',a'],[f',h']$,
6) $[i,e],[b,c],[e',a'],[e',f'],[g',h']$.

$\Gamma_{-7}(12,19)$:

Generators:: $AC^{-1}, A^3, BAC^{-1}B, BA^3B$, $\quad H_1(\Gamma,\mathbb{Z}) = \mathbb{Z}^3$,
Permutation representation: $\sigma_A = (1,2,3)(4,5,6)(7,11,8)(9,12,10)$
$\sigma_B = (1,4)(2,8)(3,9)(5,10)(6,7)(11,12), \sigma_C = (1,2,3)(4,5,6)(7,9)(8,10)(11,12)$.
Edge identifications:

1) $[a,f],[h,f],[d,b],[a,b],[h,i]$,
2) $[c,b],[g,i],[e,i],[c',f'],[e',f']$,
3) $[c,f],[e,f],[a',f'],[h',f']$,
4) $[e,a],[g,a],[c',d'],[e',d'],[i',g']$,
5) $[c,d],[e,d],[g,h],[b',d'],[h',g'],[a',g']$,
6) $[c',b'],[b',a'],[i',h'],[a',e'],[e',i']$.

$\Gamma_{-7}(12,20)$:

Generators:: C, A^4B, A^2BA^2, $\quad H_1(\Gamma,\mathbb{Z}) = \mathbb{Z}^2$,
Permutation representation: $\sigma_A = (1,2,3,4,5,6,7,8)(9,10,11,12)$,
$\sigma_B = (1,5)(2,9)(3,7)(4,10)(6,11)(8,12), \sigma_C = (9,12,11,10)$.
Edge identifications:

1) $[a,f],[c,f],[c,d],[i,g],[a,g]$,
2) $[b,d],[h,g],[e,d],[h',f'],[e',f']$,
3) $[a',f'],[c',f'],[c',d'],[i',g'],[a,g']$,
4) $[b,c],[h,i],[e,i],[b',d'],[h',g'],[e',d']$,
5) $[e,a],[b,a],[b',c'],[h',g'],[e',i']$,
6) $[h,f],[e,f],[e',a'],[b',a']$.

$\Gamma_{-7}(12,21)$:

Generators:: $A, BA^2C^{-2}B, BC^{-1}BC^{-1}$, $\quad H_1(\Gamma,\mathbb{Z}) = \mathbb{Z}^2$,
Permutation representation: $\sigma_A = (2,3,4,5)(6,7,8,9)$
$\sigma_B = (1,2)(3,5)(4,11)(6,10)(7,9)(8,12), \sigma_C = (1,10,11,12)(2,8,4,6)(3,9,5,7)$.
Edge identifications:

1) $[a,f],[e,f],[c',f'],[a',g']$,
2) $[e,a],[b,c],[h,i],[a',b'],[e',i']$,
3) $[d,c],[g,i],[d,b],[g,h],[d',e'],[f',h']$,
4) $[d,e],[f,h],[f',e'],[f',a']$,
5) $[b,a],[i,e],[e',a'],[b',c'],[h',i']$,
6) $[g,a],[f,c],[d',c'],[g',i'],[d',b'],[g',h']$.

$\Gamma_{-7}(12,22)$:

Generators:: $A, BC^{-1}BC^{-1}, BA^{-1}CBC$, $\quad H_1(\Gamma,\mathbb{Z}) = \mathbb{Z}^2$,
Permutation representation: $\sigma_A = (2,3,4,5)(6,7,8,9)$,
$\sigma_B = (1,8)(2,12)(3,5)(4,11)(6,10)(7,9), \sigma_C = (12,10,11,1)(2,9,5,8,4,7,3,6)$.
Edge identifications:

1) $[a,f],[e,f],[c',f'],[e',i']$,
2) $[b,a],[f,h],[f',e'],[f',a']$,
3) $[i,e],[e,d],[e',a'],[b',c'],[g',h']$,
4) $[g,a],[f,c],[d',c'],[i',h'],[d',b'],[i',g']$,
5) $[b,d],[g,i],[b,c],[g,h],[d',e'],[f',h']$,
6) $[d,c],[i,h],[a,e],[b',a'],[a',g']$.

$\Gamma_{-7}(12,23)$:

Generators:: $A, BC^2B, BACBC^{-1}$, $\quad H_1(\Gamma,\mathbb{Z}) = \mathbb{Z}^2$,
Permutation representation: $\sigma_A = (2,3,4,5)(6,7,8,9)$,
$\sigma_B = (1,2)(10,6)(12,8)(11,4)(3,5)(9,7), \sigma_C = (1,10,11,12)(2,9)(6,3)(5,8)(4,7)$.

Edge identifications:

1) $[a,f],[f,e],[c',f'],[e',i']$,
2) $[e,a],[c,d],[g,h],[d',e'],[g',a']$,
3) $[b,d],[i,h],[b,c],[i,g],[b',a'],[f',h']$,
4) $[b,a],[f,h],[f',e'],[f',a']$,
5) $[e,d],[a,g],[e',a'],[c',d'],[g',h']$,
6) $[i,e],[f,c],[b',d'],[i',h'],[b',c'],[i',g']$.

$$\Gamma_{-7}(12,24):$$

Generators:: AC^{-1}, BC^4, AC^3B, $\quad H_1(\Gamma,\mathbb{Z})=\mathbb{Z}^2$,
Permutation representation: $\sigma_A=(1,2,3,4,5,6,7,8)(9,10,11,12)$,
$\sigma_B=(1,5)(2,9)(8,12)(3,7)(6,11)(4,10), \sigma_C=(1,2,3,4,5,6,7,8)(9,11)(10,12)$.
Edge identifications:

1) $[a,f],[h,f],[d,b],[a,b],[h,i]$,
2) $[c,b],[g,i],[e,i],[c',f'],[e',f']$,
3) $[a',f'],[h',f'],[d',b'],[a',b'],[h',i']$,
4) $[c,d],[e,d],[g,h],[c',b'],[g',i'],[e',i']$,
5) $[e,a],[g,a],[c',d'],[e',d'],[g',h']$,
6) $[c,f],[e,f],[e',a'],[h',a']$.

From the permutation representations it follows that the following inclusion relations are up to $\mathbf{PSL}_2(\mathcal{O}_{-7})$ conjugacy the only inclusions between the groups $\Gamma_{-7}(12,1),\ldots,\Gamma_{-7}(12,6)$ and the groups $\Gamma_{-7}(6,1),\cdots,\Gamma_{-7}(6,4)$:

$$\Gamma_{-7}(12,3)\leq\Gamma_{-7}(6,2);\ \Gamma_{-7}(12,4)\leq\Gamma_{-7}(6,1),\Gamma_{-7}(6,2);$$
$$\Gamma_{-7}(12,5)\leq\Gamma_{-7}(6,3),\Gamma_{-7}(6,4);\ \Gamma_{-7}(12,6)\leq\Gamma_{-7}(6,3).$$

Since the groups $\Gamma_{-7}(6,1),\ldots,(6,4)$ are pairwise isomorphic [G,S,1], it follows that the groups

$$\Gamma_{-7}(12,3),\Gamma_{-7}(12,4),\Gamma_{-7}(12,5),\Gamma_{-7}(12,6)$$

are pairwise isomorphic. From the Mostow rigidity theorem [Ma] we infer that the corresponding manifolds are pairwise homeomorphic.

Correspondingly the inclusions up to $\mathbf{PSL}_2(\mathcal{O}_{-7})$ conjugacy between the groups $\Gamma_{-7}(12,7),\ldots,\Gamma_{-7}(12,19)$ and $\Gamma_{-7}(6,1),\ldots,\Gamma_{-7}(6,4)$ are:

$$\Gamma_{-7}(12,9)\leq\Gamma_{-7}(6,3);\ \Gamma_{-7}(12,10)\leq\Gamma_{-7}(6,4);\ \Gamma_{-7}(12,12)\leq\Gamma_{-7}(6,1);$$
$$\Gamma_{-7}(12,13)\leq\Gamma_{-7}(6,2);\ \Gamma_{-7}(12,15)\leq\Gamma_{-7}(6,2);\ \Gamma_{-7}(12,16)\leq\Gamma_{-7}(6,3).$$

It follows that the manifolds corresponding to the groups:

$$\Gamma_{-7}(12,9),\Gamma_{-7}(12,10),\Gamma_{-7}(12,12),\Gamma_{-7}(12,13),\Gamma_{-7}(12,15),\Gamma_{-7}(12,16),$$

are pairwise homeomorphic.

§4. Handle decompositions of $\Gamma\backslash\mathbb{H}^3$

In this and in the next paragraph we study the topological type of the quotient manifold $M = \Gamma\backslash\mathbb{H}^3$ where $\Gamma = \Gamma_{-d}(m,n)$ and d,m,n take the following values:

$$d = 7\,,\ m = 6\,, n = 1,\ldots,5,$$
$$d = 7\,,\ m = 12\,, n = 1,\ldots,24$$

It turns out that all these manifolds are link complements. The corresponding links are identified and listed; see paragraph 6.

In order to obtain M we manipulate the identification complex Q of Γ described in paragraph 3. For that we first assign to Q what we call its Whitehead graph $G(Q)$. Then, as a next step, we modify $G(Q)$ in two ways. These modifications simplify the graph and finally allow us to visualize the manifold M as a link complement, see paragraph 6.

One of these simplifications diminishes the number of edges of $G(Q)$. Following a suggestion of W. Haken, we refer to it as the *Whitehead-Zieschang reduction process*. Topologically, this process means that the complex Q is replaced by a simpler one which is obtained from Q by a cut-and-paste process. By the other simplification we reduce the number of vertices of $G(Q)$. Topologically, this corresponds to cancelling a pair of complementary handles in M, see 4.2.

We stress that both kinds of simplification alter the manifold M but not its topological type. This fact implies that the links constructed by us are only determined up to homeomorphisms. In general, a link complement admits many homeomorphisms which cannot be extended to a self-homeomorphism of S^3.

4.1 The Whitehead graph of Q

As before let Q be the identification complex obtained from Q as described in paragraph 3. Thus Q is either a double prism or a pair of double prisms. When Q consists of two pieces, we make it connected by picking two Γ-equivalent faces, one of each component, and glue them together.

Now we label the oriented edges of Q by numbers $1, 2, \ldots$, where Γ-equivalent edges get the same label. Similarly, we label the faces of ∂Q by capital letters $A^+, A^-, B^+, B^-, \ldots$, indicating that the face A^+ has to be identified with the face A^-, etc. It turns out that, once the edge labels are given, there is exactly one possibility to glue pairs of faces together in such a way that the edge labels are respected.

Canonical handle decomposition of M

The polyhedron Q admits a natural handle decomposition $\mathcal{H} = \mathcal{H}_0 \cup \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3$ (see [RS; p. 74] for notation). The class of 3-handles $\mathcal{H}_3$ is formed by the stars of the vertices of Q with respect to its second normal subdivision. The 2-handles $\mathcal{H}_2$ are given as product neighbourhoods of the edges, and the class of 1-handles $\mathcal{H}_1$ consists of product neighbourhoods of the faces of Q lying in its boundary. Finally, there is a single 0-handle, namely Q minus the union of 1-, 2-, and 3-handles.

We observe that the handle decomposition $\mathcal{H}$ of Q induces canonically a handle decomposition of M. Denote by Q_0 the polyhedron arising from Q by removing the interior of the 3-handles belonging to $\mathcal{H}_3$. We then have a quotient map $\rho : Q_0 \to M$, and the i-handles of M are composed by the images of the i-handles of Q_0. To be more precise, each 1-handle of M is of the form $[-1,1] \times \mathcal{D}^2$ and is the union of two 1-handles $[0,1] \times \mathcal{D}^2$ and $[-1,0] \times \mathcal{D}^2$ of Q associated with a pair of faces F^+, F^- and glued together along $\{0\} \times \mathcal{D}^2$. We refer to this handle as an *F-handle*. Similarly, every 2-handle of M is defined as the union of a class of 2-handles of Q which belong to equivalent edges and which are glued together along boundary disks of the form $J \times I$ where $J \subset \partial\mathcal{D}^2$. Thus the number of 2-handles of M equals the number of equivalence classes of edges of Q under the action of Γ.

A flat presentation of Q

As the next step we visualize the handle decomposition $\mathcal{H}$ of Q on the topological 3-ball $(\mathbb{R}^2 \times (-\infty, 0]) \cup \{\infty\}$. This is done in such a way that all edges of Q come to lie on $\mathbb{R}^2 \times \{0\}$. To see what is meant by that first deform Q so that it becomes a convex subset of $\mathbb{R}^3$, then apply stereographic projection. Figure 4.2 und Figure 4.4 show such a flat presentation of Q in the cases $\Gamma_{-7}(6,4)$ and $\Gamma_{-7}(12,22)$, respectively.

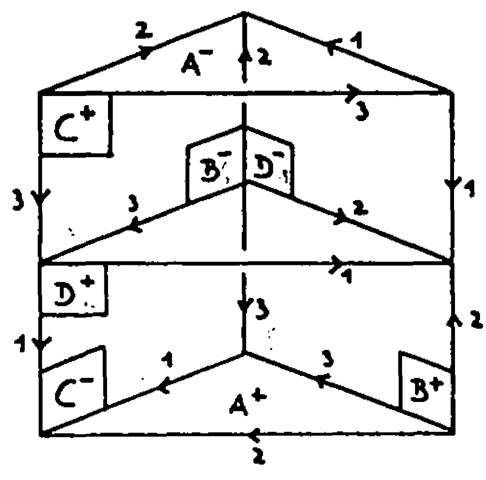


Figure 4.1: Q in the case $\Gamma_{-7}(6,4)$.

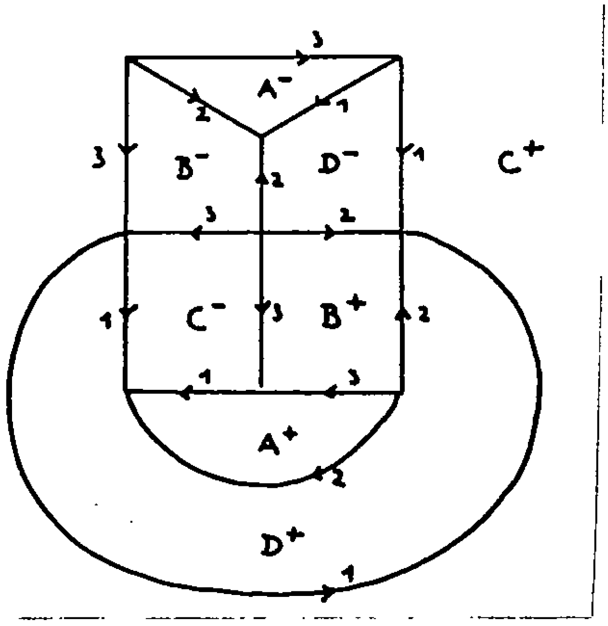


Figure 4.2: Q made flat

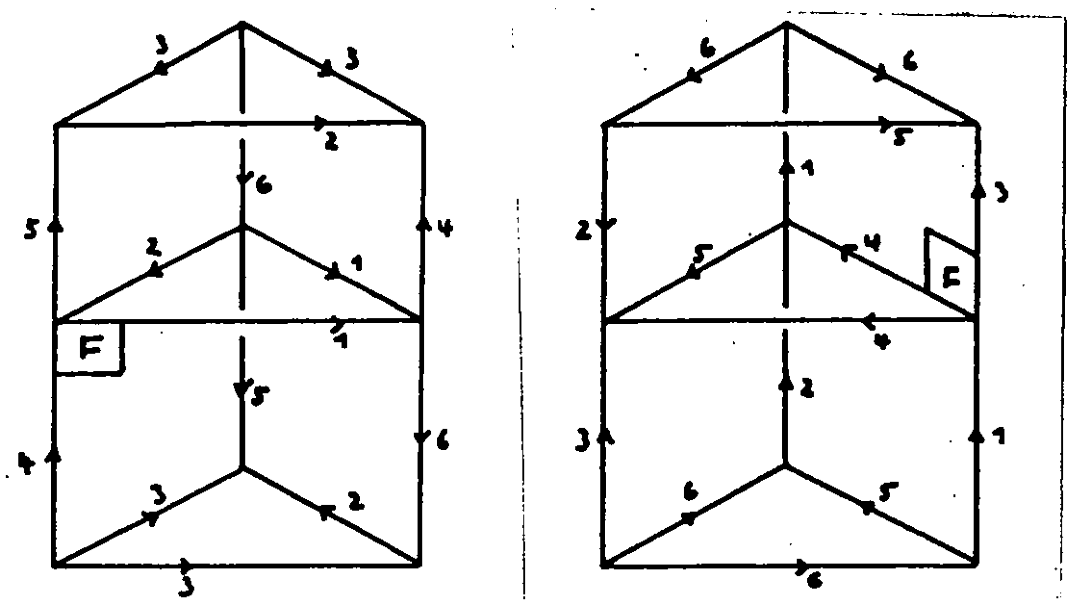


Figure 4.3: Q in the case $\Gamma_{-7}(12, 22)$

Whitehead graphs

A *Whitehead graph* G is a finite planar graph together with the following data:
(1) A fixed point free involution of the vertex set of G which preserves degree,
(2) a fixed embedding $G \hookrightarrow \mathbb{R}^2$.

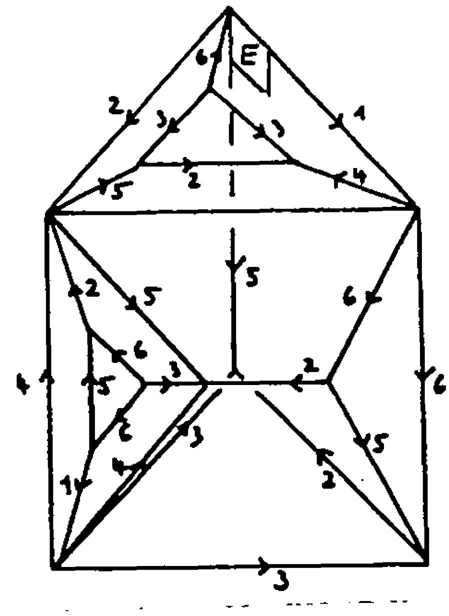


Figure 4.4: Q made convex

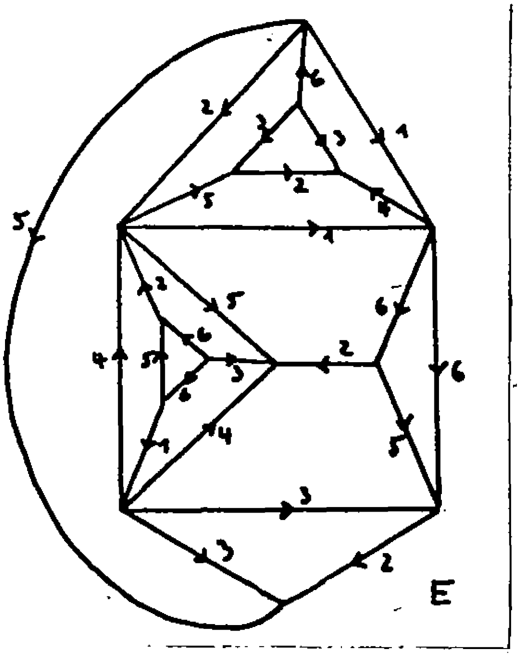


Figure 4.4 a): Q made flat
the face E contains the point ∞

A Whitehead graph can be used to construct compact 3-manifolds with boundary. To see how this works we choose for each vertex v of $G \subset \mathbb{R}^2$ a small disk D_v in $\mathbb{R}^2$ centered at v where all these disks should have the same radius. Then ∂D_v meets the edges of G in a set of points whose number equals the degree d of v. If necessary, after an isotopy of G in $\mathbb{R}^2$ we may assume that $\partial D_v \cap G$ is invariant under the rotation of D_v through the angle $2\pi/d$. Now for each pair of vertices (v, v') of G corresponding under the involution given by (1) we attach a 1-handle $I \times D_v$ along D_v and $d_{v'}$. This is done in such a way that for each $x \in \partial D_v \cap G$ the arc $I \times \{x\}$ connects x with $x' \in \partial D_{v'} \cap G$.

On the boundary of the handle body H_g constructed in this way we have a system of (pairwise disjoint) circles arising from the edges of G, together with the various arcs $I \times \{x\}$ on the attached 1-handles. Now the compact 3-manifold we have in mind is obtained from $H - g$ by attaching a 2-handle $D^2 \times I$ along a tubular neighbourhood of every of these circles in ∂H_g.

Two observations are in place here:

1. The handle body $H - g$ can be realized in $\mathbb{R}^2 \times \mathbb{R}$ where the attached 1-handles lie in $\mathbb{R}^2 \times [0, \infty)$ and are unknotted and unlinked.
2. The topological type of the 3-manifold constructed above depends only on the isotopy type of G in $S^2 = \mathbb{R}^2 \cup \{\infty\}$. It is fixed once we have specified, for each pair of disks D_v, $D_{v'}$, corresponding base points on $\partial D_v \cap G$ resp. $\partial D_{v'} \cap G$ and opposite orientations for ∂D_v and $\partial D_{v'}$.

Examples of Whitehead graphs arise in a natural way from a flat presentation of any identification complex Q. Namely, take $G = G(Q)$ to be the dual graph of the 1-skeleton of Q in S^2. Clearly, conditions (1) and (2) of the definition are satisfied. Figure 4.5 shows

the Whitehead graph $G(Q)$ for $\Gamma_{-7}(6,4)$. The vertices are already indicated as disks whose boundaries are oriented. Here and later on we adopt the convention that vertices $A^-, B^-, \ldots$ get clockwise orientation and vertices $A^+, B^+, \ldots$ get counter-clockwise orientation. Also base points are already marked. Note that for every pair F^+, F^- the choice of one of the two base points is free whereas after this the other base point is fixed according to the identification prescription on ∂Q. Orientations and labels for the edges of G are obtained by the following rule:
Take any edge e, give it any of the two possible orientations, and label it by 1.1. Now to the endpoint x of e, lying in some $\partial F^{\pm}$, corresponds a point $x' \in \partial F^{\mp}$, and x' belongs to a unique edge e'. Give e' the label 1.2 and orient it so that x' becomes its starting point. Proceeding in this way one comes back to the edge 1.1 after finitely many steps. Next take any edge which is still unlabeled and repeat the same procedure thus creating 2.1, 2.2, ... As G is finite, this process stops.

We remark that orienting the edges is not necessary. However, it will turn out to be useful.

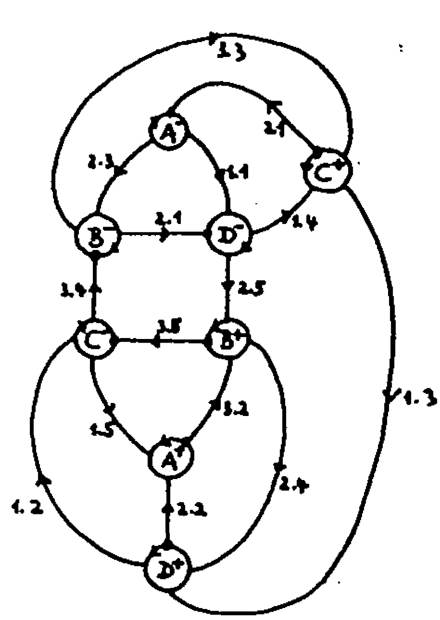


Figure 4.5: $G(Q)$ for $\Gamma_{-7}(6,4)$

Denote by $M(G(Q))$ the 3-manifold obtained from the Whitehead graph $G(Q)$.

4.1. Proposition: Let $M = \Gamma\backslash\mathbb{H}^3$ be constructed via the identification complex Q. Then $M = M(G(Q))$.

Proof. Recall the canonical handle decomposition of M originating from the natural handle decomposition of Q. We observe that this decomposition arises from the graph $G(Q) \subset S^2 = \partial((\mathbb{R}^2 \times (-\infty, 0]) \cup \{\infty\})$ by attaching 1 - and 2-handles in just the same way as we did it when constructing $M(G(Q))$. The point is that, after attaching an F-handle for every pair of vertices F^+, F^-, the edges of G (together with the connecting arcs on the F-handles) become the attaching spheres of the 2-handles of both M and $M(G(Q))$. Thus M and $M(G(Q))$ are homeomorphic. □

4.2 Whitehead-Zieschang reduction

In this section we shall see how a given marked and orientated Whitehead graph can be modified into a simpler one giving a homeomorphic 3-manifold. After finitely many such simplifications we will be able to visualize the corresponding 3-manifold as a link complement.

The *complexity* of a Whitehead graph G, denoted $c(G)$, is defined to be the pair (g, n) where g is half the number of vertices of G and n is the number of edges. We order the pairs (g, n) lexicographically.

Diminishing the number of edges

Assume we have found a simple closed curve α in $\mathbb{R}^2$ with the following properties:

(1) α does not meet the vertices $A^{\pm}, B^{\pm}, \ldots,$ of G.

(2) α meets the edges of G transversely.

(3) α seperates some pair of vertices F^+ and F^- and $|\alpha \cap G| <$ degree F^+ (where $|\ldots|$ denotes cardinality).

Clearly, α bounds a disk $\tilde{F}$ which is properly embedded in $M_0 = (\mathbb{R}^2 \times (-\infty, 0] \cup \{\infty\}$. We give α the clockwise orientation and fix any point of $\alpha \cap G$ as base point. Now we split M_0 along $\tilde{F}$ and identify the two resulting 3-balls along F^+ and F^- in the obvious way. This operation gives us a new Whitehead graph G' satisfying

$$\begin{aligned} c(G') &= (g, n - \text{ degree}\, F^+ + \text{ degree}\, \tilde{F}^+) \\ &= (g, n - \text{ degree}\, F^+ + |\alpha \cap G|) \\ &< c(G), \quad \text{by}\,(3) \end{aligned}$$

Moreover, G' is again marked and oriented, and a labelling of the edges of G yields a labelling of the edges of G' in an almost canonical manner. Fig. 4.6 a) reproduces Fig. 4.5 where a curve α has been drawn in. The vertices A^+, B^+, C^-, D^+ are seperated by α from A^-, B^-, C^+, D^-, respectively. We select the pair $D^{\pm}$. (The pairs $B^{\pm}$ or $C^{\pm}$ could be taken either way; the pair $A^{\pm}$ would not lead to a reduction of complexity.) Fig. 4.6 b) shows the new Whitehead graph.

4.2. Proposition: If G and G' are Whitehead graphs as above then $M(G) = M(G')$.

This proposition is due to Zieschang; see [Z]. It is the geometric counterpart to a corresponding result on equivalences of group presentations attributed to Nielsen (see [M,K,S], [C,Z,1-3] for instance).

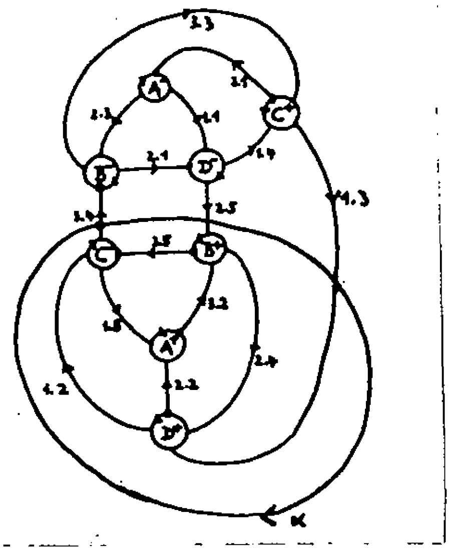


Figure 4.6 a)

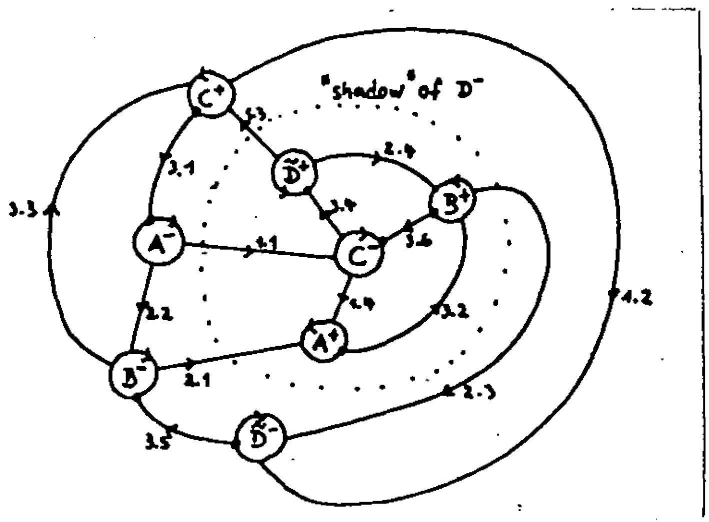


Figure 4.6 b)

With regard to Proposition 4.2. we refer to the process of constructing the graph G' out of G as the *Whitehead-Zieschang reduction process.*

Handle cancelling

Next we describe a second modification a Whitehead graph G can be submitted to. Its effect again is a reduction of complexity; this time the number of vertices is lowered. The process is motivated by the geometric operation of cancelling a pair of complementary handles in the manifold $M(G)$. This implies that the new Whitehead graph determines a manifold that is homeomorphic to $M(G)$. In order to understand how G is modified and why this is done in such a way we have to recall how handle cancelling proceeds. Before we can omit the F-handle H_1 and the 2-handle H_2 we have to check that the attaching sphere s of H_2 intersects the belt sphere of H_1 in exactly one point. But s arises from some class of edges $s.1, s.2, \ldots s$. Thus, in terms of G, we have to verify that there is exactly one edge $s.\kappa$ leaving or entering F^+. If this is the case then we push H_1 with one of its "feet", say F^+, along s until we come to F^-. Of course, in doing so, we have to pull along with F^+ all the other cocores of 2-handles running over the F-handle H_1. This explains how the remaining edges of G adjacent to F^+ or F^- must be changed before F^+, F^- and $s.1, s.2, \ldots$ are erased. Note that the new graph requires a new edge labelling (and sometimes also a marking of new base points for some pairs of vertices).

Figure 4.7 illustrates the Whitehead graph G' we obtain from the graph G depicted in Fig. 4.6 after cancelling $A^{\pm}$ and the edges 1.1, 1.2, 1.3, 1.4. (Other choices of vertices and edges would be admissible as well.) In this case we have

$$c(G') = (3, 14) < (4, 15) = c(G).$$

However, in general, the second component can increase when handles are cancelled. Note also that in more complicated examples than $\Gamma_{-7}(6, 4)$ we have to carry out several handle cancellations and also more than only one Whitehead-Zieschang reduction before we arrive at a Whitehead graph which does not allow any further simplification.

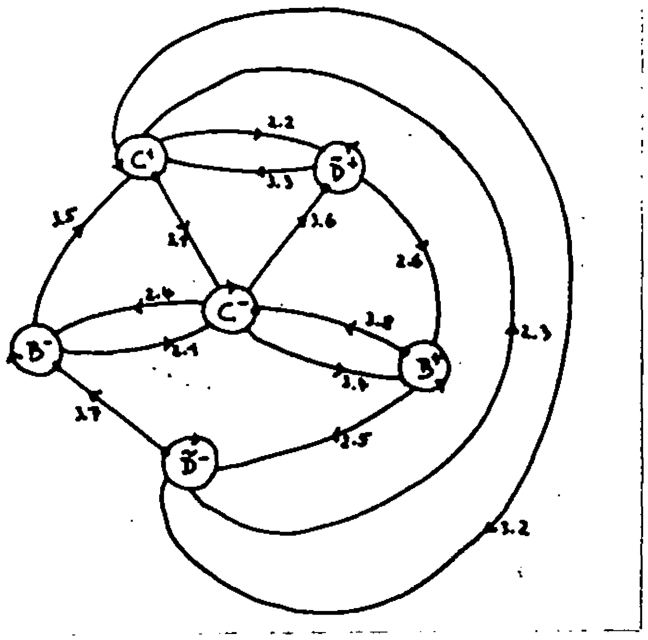


Figure 4.7: The A-handle and the curve 1 are cancelled

§5 Link complements

Our goal in this paragraph is to show that for all values of d, m and n in the list at the beginning of paragraph 4 the manifold $M = \Gamma_{-d}(m,n)\backslash\mathbb{H}^3$ is a link complement. Moreover, we determine the links explicitly, see paragraph 6. To begin with we establish a general criterion for the embeddability of M in S^3.

A criterion for embeddability

Recall that the manifold $M(G)$ arises from the Whitehead graph G in two steps. First we attach an F-handle for every pair F^+, F^- of vertices of G. The result of this is a standard handle body H_g in S^3, together with a system S of orientated circles on ∂H_g. Then $M(G)$ is obtained from H_g by attaching 2-handles along S. The question now is whether these 2-handles can be realized within S^3. Note that this is exactly the case if every circle of S spans a disk in the complementary handle body $H_g' = \overline{S^3 - H_g}$. In order to decide whether disks of such a kind exist or not we dispose of the following algebraic criterion.

For each F-handle H_1 we choose a longitude f, i.e. a simple closed oriented curve on ∂H_h which spans a disk in H_g' and which is of the form $f = f_0 f_1$ where f_0 runs over H_1 from F^- to F^+ and f_1 goes back on $\mathbb{R}^2$ to the starting point. Furthermore, these longitudes should be disjoint and meet the system S transversely and only in $\mathbb{R}^2$. Now to every system $s = s.1, s.2, \ldots, s.k_s$ of edges of G we assign a word $w(s)$ in the symbols f as follows. Starting with $s.1$ we write f if we traverse f from right to left, and f^{-1} if we traverse f from left to right. Then we proceed, still on $s.1$ or with $s.2, \ldots$ until we reach a second longitude, and so on. Finally going along $s.k_s$ we complete the definition of the word $w(s) = f^{\epsilon} \dots (\epsilon = \pm 1)$.

5.1. Proposition : If for every edge system s of G the word $w(s)$ is trivial as an element of the free group generated by $\{f\}$ then $M(G)$ embeds in S^3.

Proof: Recalling that the edge systems s of G are in canonical 1-1 correspondence with the system S of attaching spheres on ∂H_g, the condition $w(s) = 1$ means precisely that the curve s on $\partial H'_g$ is null-homotopic in H'_g. As every null-homotopic simple closed curve on $\partial H'_g$ bounds a disk in H'_g, we see that we can attach the 2-handle along s inside of H'_g. A standard procedure finally shows that these handles can be found disjoint. □

Finding the right longitudes

Of course, if the condition for embeddability formulated in Prop. 5.1. is not fulfilled then this does not necessarily mean that the manifold M cannot be embedded in S^3. It might as well be that our choice of longitudes was not good. To see what can happen let us have a look at Fig. 5.1. The diagram reproduces Fig. 4.7, with longitudes b, c, d drawn in addition (it suffices to specify the part of the longitude lying in $\mathbb{R}^3$. First disregarding d_1, the curves 2 and 3 satisfy $w(2) = bb^{-1} = 1$ and $w(3) = c^{-1}c = 1$. But if we replace the longitude d by d_1 then we get $w(2) = bd_1^{-1}b^{-1}d_1 \neq 1$ and $w(3) = d_1c^{-1}d_1^{-1}c \neq 1$.

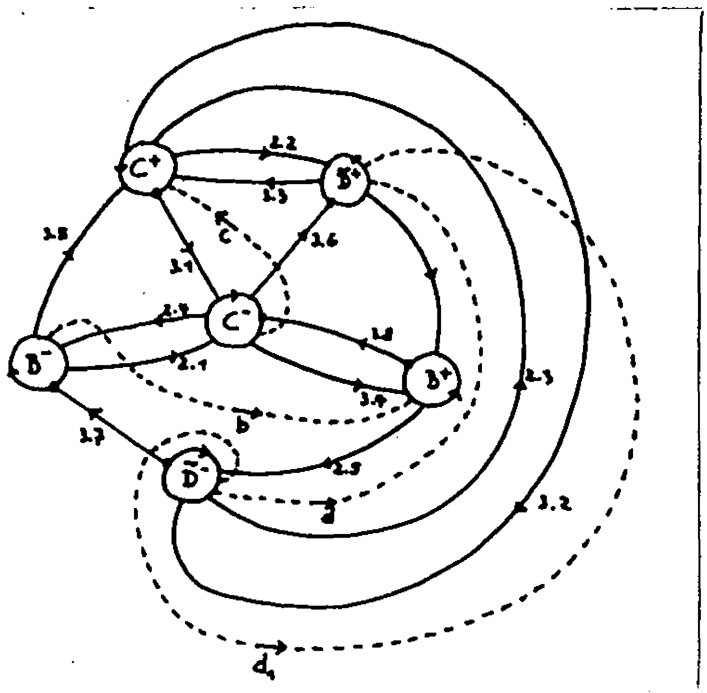


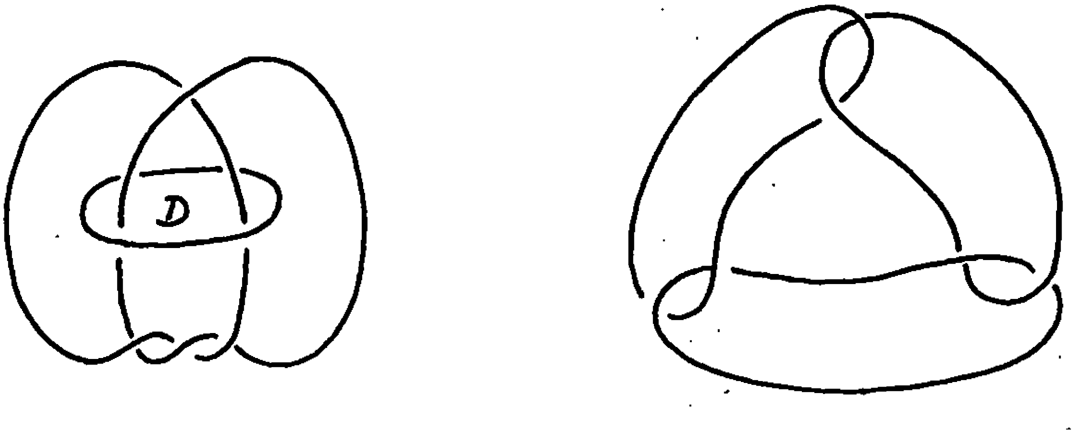
Figure 5.1:

Thus our discussion proves that the manifold $\Gamma_{-7}(6,4)\backslash\mathbb{H}^3$ embeds in S^3. The embedding itself depends upon the choice of the longitudes b, c, d. Indeed, there are other good choices of longitudes easily found in this example. These lead to different links with homeomorphic complements. Notice also that a compact 3-manifold whose boundary consists of tori and which can be embedded in S^3 is in fact a link complement.

5.2 How to determine the link

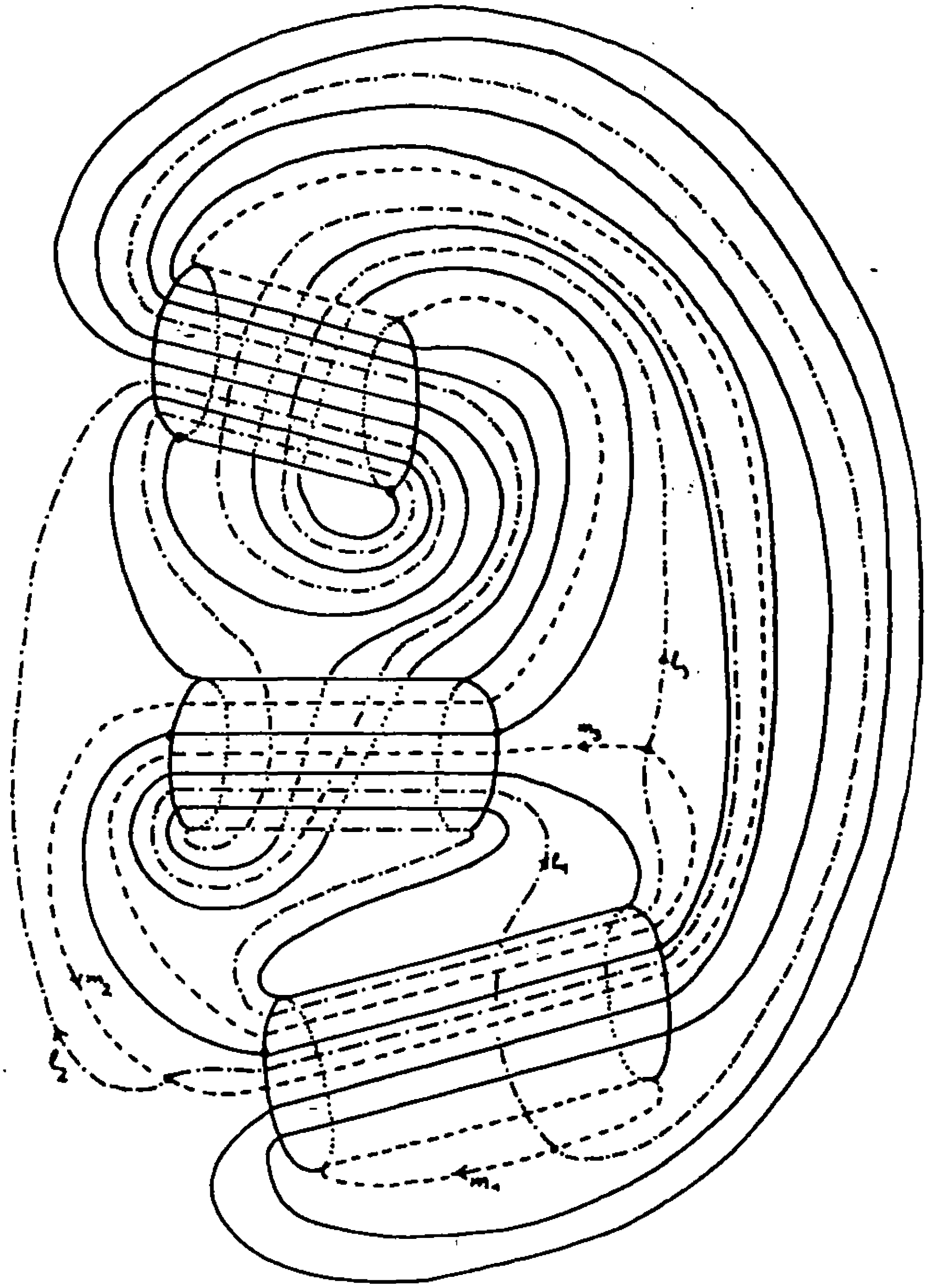
Now suppose we are given an explicit embedding of the manifold M into S^3 as described in section 5.1. Then a final task remains to be done. We have to determine the link L. The components of L are provided by longitudes of the various boundary components of $M \subset S^3$. Therefore, our problem consists in finding for each boundary torus T_i a pair m_i, l_i of simple closed curves in T_i which intersect (geometrically) in exactly one point and such that m_i (the meridian of T_i) bounds a disc in $S^3 - M$. The curves l_i then form the link we are looking for. It is convenient to search for m_i and l_i in the Whitehead graph G which, together with the system $\{f\}$ of longitudes on H_g, gives us the embedding of M into S^3. $\mathbb{R}^2 - G$ consists of classes of regions which belong to the same T_i. Within these classes we have to find m_i and l_i. A necessary condition for m_i to be a meridian of T_i again is that the word $w(m_i)$ in $f\}$ obtained by going along m_i in the same way as before when the words $w(s)$ were defined must be trivial.

Figure 5.2 shows the system of meridians and longitudes for $\Gamma_{-7}(6,4)\backslash\mathbb{H}^3$. The picture is obtained by first deforming the graph of Fig. 5.1 so that the longitudes b, c and d become straight lines. Then the $B-, C-$, and $\tilde{D}$-handle should be imagined as a bridge lying directly over the corresponding lines. Figure 5.3 a) shows the desired link L we obtain by simplifying the longitudes l_1, l_2, l_3 isotropically. Finally, a Dehn twist along the central disk D yields the symmetric 3-component link depicted in Fig. 8b).



a) b)

Figure 5.3:



—— meridians

-.-.-.- longitudes

- - - - system S

Figure 5.2

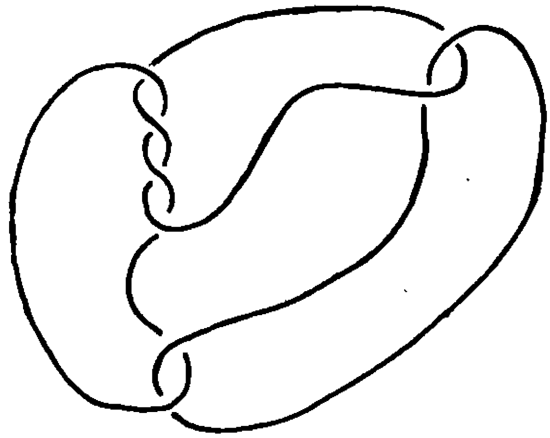
§6. The list of links

This paragraph contains pictures of the links which come out of the program described above carried out for the subgroups of index ≤ 12 in $\mathbf{PSL}_2(\mathcal{O}_{-7})$. The details of the computations are somewhat messy, we omit them here.

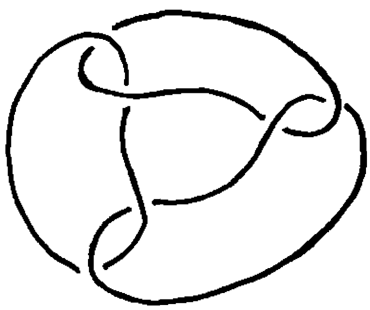
The manifolds $\Gamma_{-7}(6,1)\backslash\mathbb{H}^3, \cdots, \Gamma_{-7}(6,4)\backslash\mathbb{H}^3$ are pairwise homeomorphic, [G,S,1]. We only describe the case:

$$\Gamma_{-7}(6,4):$$

The manifold $\Gamma_{-7}(6,4)\backslash\mathbb{H}^3$ is by our analysis in paragraph 5 homeomorphic to $S^3 - L$ where L is the following link with 3 components:



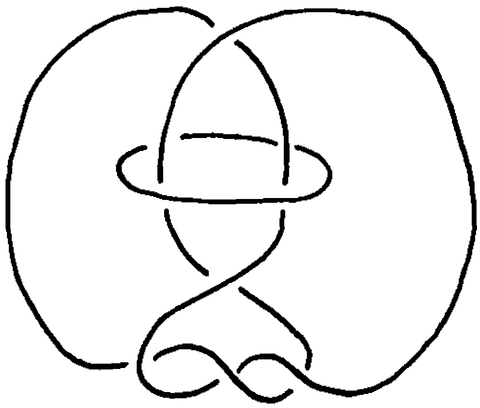
Under an obvious Dehn-twist $S^3 - L$ is homeomorphic to $S^3 - L_1$ where L_1 is the following link:



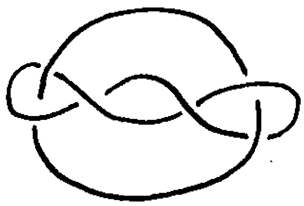
These results are known, see the references in [G,S,1].

$\Gamma_{-7}(6,5)$, $\Gamma_{-7}(6,6)$:

Our method shows that $\Gamma_{-7}(6,5)\backslash\mathbb{H}^3$ is homeomorphic to $S^3 - L$ where L is the following link:



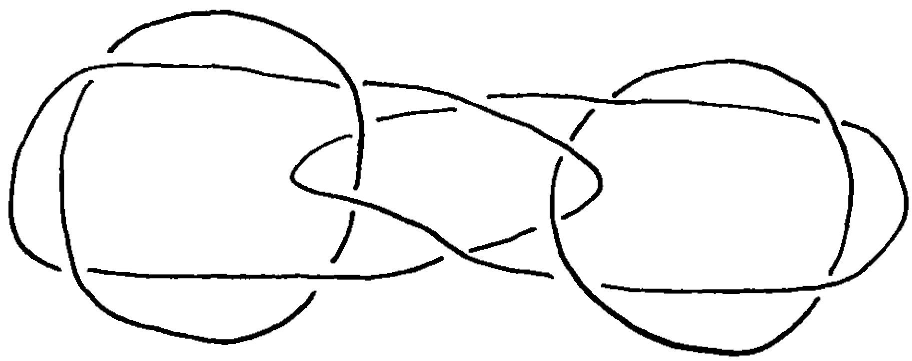
An obvious Dehn-twist gives a homeomorphism $S^3 - L \cong S^3 - L_1$ where L_1 is the following link:



The manifolds $\Gamma_{-7}(6,5)\backslash\mathbb{H}^3$ and $\Gamma_{-7}(6,6)\backslash\mathbb{H}^3$ are homeomorphic, [G,S,1].

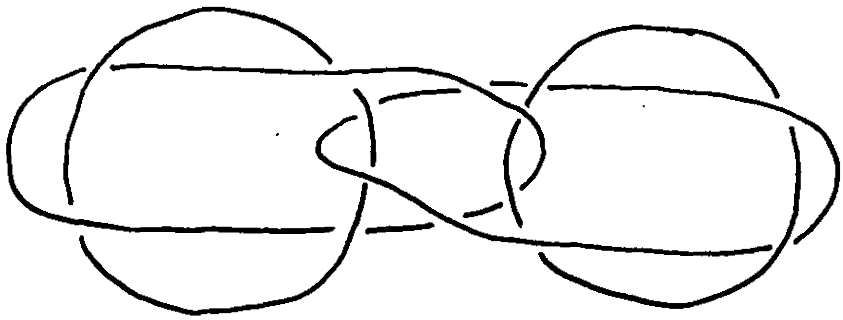
$\Gamma_{-7}(12,1)$:

The manifold $\Gamma_{-7}(12,1)\backslash\mathbb{H}^3$ is homeomorphic to $S^3 - L$ where L is the following link with 4 components:



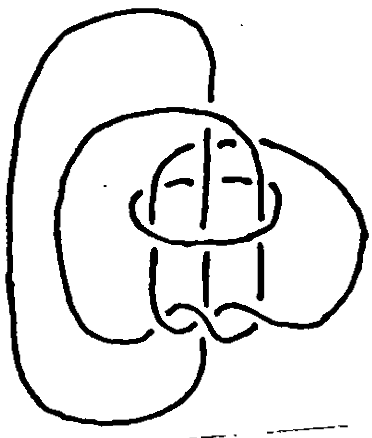
$\Gamma_{-7}(12,2)$:

The manifold $\Gamma_{-7}(12,2)\backslash\mathbb{H}^3$ is homeomorphic to $S^3 - L$ where L is the following link with 4 components:



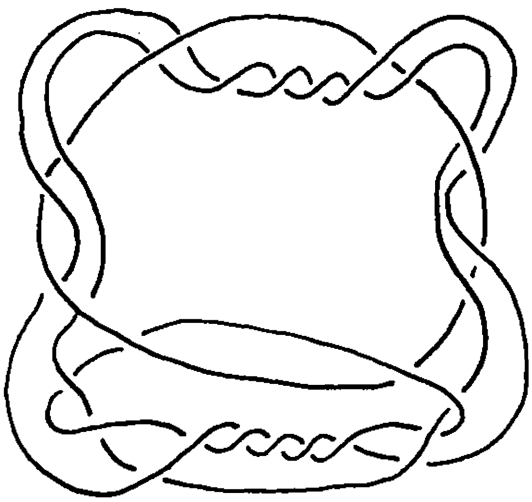
$$\Gamma_{-7}(12,3), \Gamma_{-7}(12,4), \Gamma_{-7}(12,5), \Gamma_{-7}(12,6):$$

These manifolds are homeomorphic to $S^3 - L$ where L is the following link with 4 components:



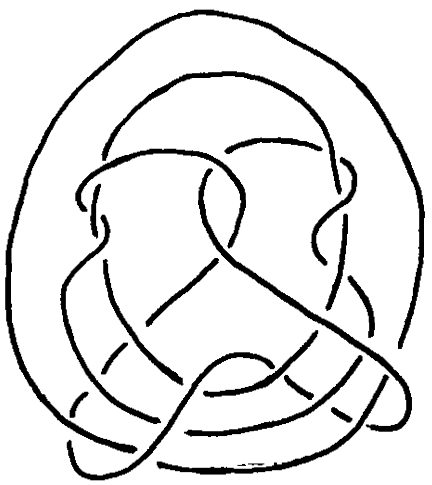
$$\Gamma_{-7}(12,7):$$

The manifold $\Gamma_{-7}(12,7)\backslash\mathbb{H}^3$ is homeomorphic to $S^3 - L$ where L is the following link with 3 components:



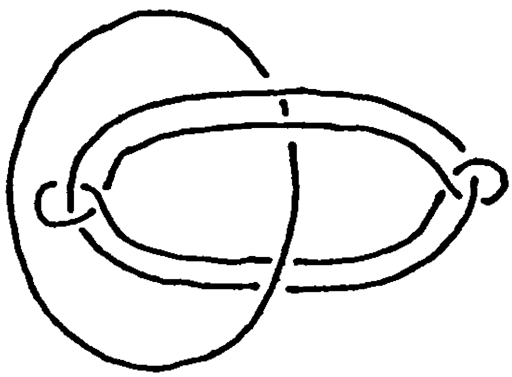
$\Gamma_{-7}(12,8)$:

The manifold $\Gamma_{-7}(12,8)\backslash\mathbb{H}^3$ is homeomorphic to $S^3 - L$ where L is the following link with 3 components:



$\Gamma_{-7}(12,9), \Gamma_{-7}(12,10), \Gamma_{-7}(12,12), \Gamma_{-7}(12,13), \Gamma_{-7}(12,15), \Gamma_{-7}(12,16)$:

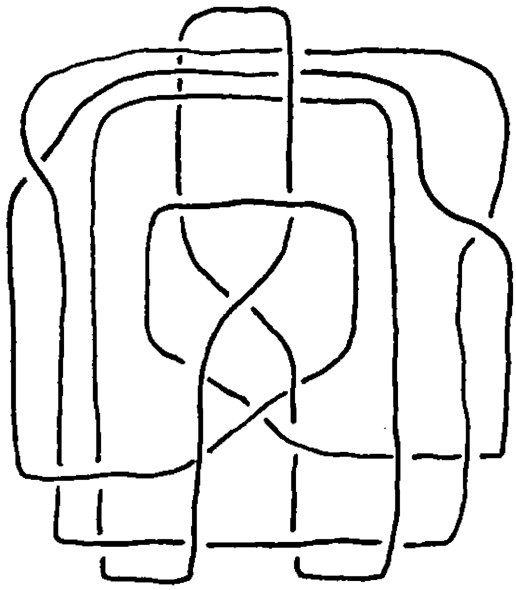
These corresponding manifolds are homeomorphic to $S^3 - L$ where L is the following link with 3 components:



For the proof see paragraph 3 and [G,S,1].

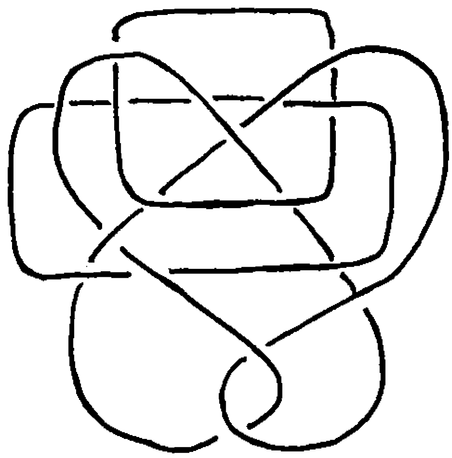
$\Gamma_{-7}(12, 11)$:

The manifold $\Gamma_{-7}(12, 11)\backslash\mathbb{H}^3$ is homeomorphic to $S^3 - L$ where L is the following link with 3 components:

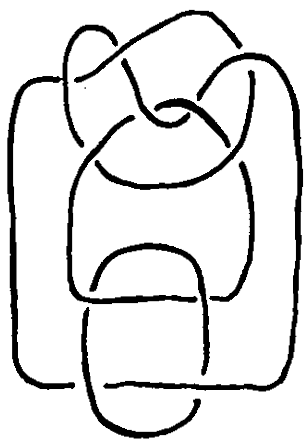


$\Gamma_{-7}(12, 14)$:

The manifold $\Gamma_{-7}(12, 14)\backslash\mathbb{H}^3$ is homeomorphic to $S^3 - L$ where L is the following link with 3 components:

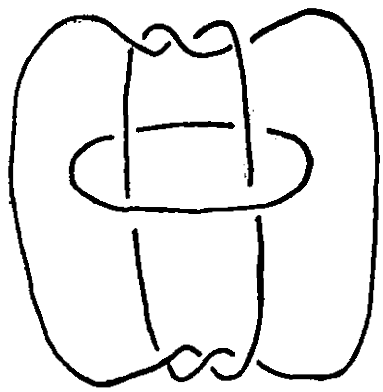


Under an obvious Dehn-twist $S^3 - L$ is homeomorphic to $S^3 - L_1$ where L_1 is the following slightly simpler link:



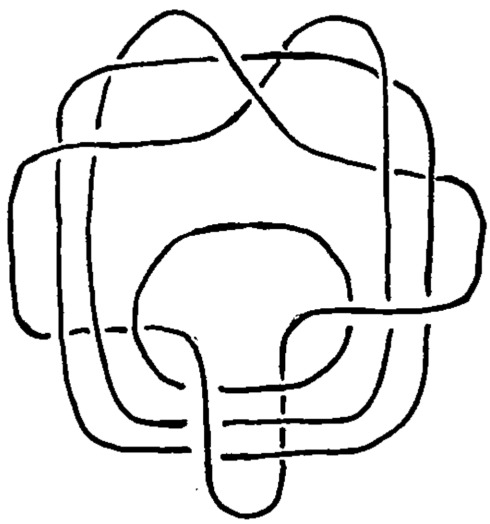
$\Gamma_{-7}(12,17)$:

The manifold $\Gamma_{-7}(12,17)\backslash\mathbb{H}^3$ is homeomorphic to $S^3 - L$ where L is the following link with 3 components:



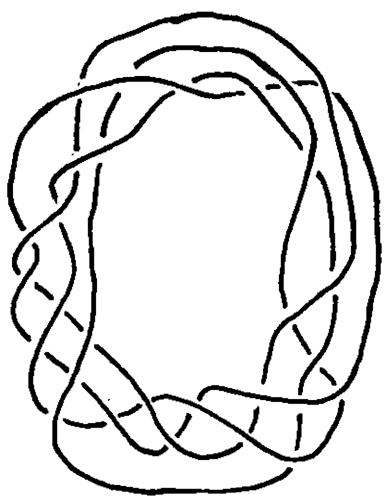
$\Gamma_{-7}(12,18):$

The manifold $\Gamma_{-7}(12,18)\backslash\mathbb{H}^3$ is homeomorphic to $S^3 - L$ where L is the following link with 3 components:



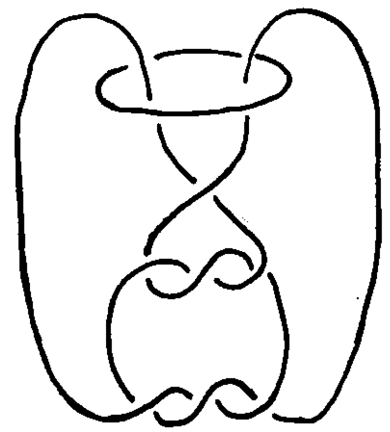
$\Gamma_{-7}(12,19):$

The manifold $\Gamma_{-7}(12,19)\backslash\mathbb{H}^3$ is homeomorphic to $S^3 - L$ where L is the following link with 3 components:



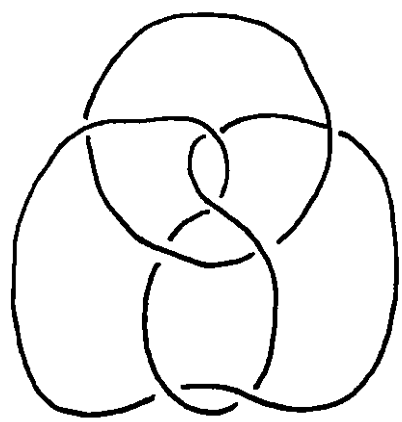
$\Gamma_{-7}(12, 20)$:

The manifold $\Gamma_{-7}(12,20)\backslash\mathbb{H}^3$ is homeomorphic to $S^3 - L$ where L is the following link with 2 components:

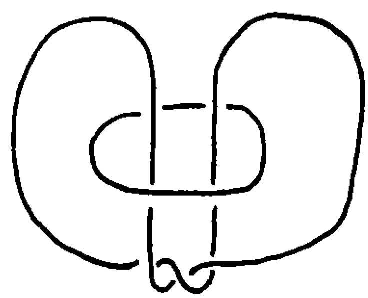


$\Gamma_{-7}(12, 21)$:

The manifold $\Gamma_{-7}(12,21)\backslash\mathbb{H}^3$ is homeomorphic to $S^3 - L$ where L is the following link with 2 components:

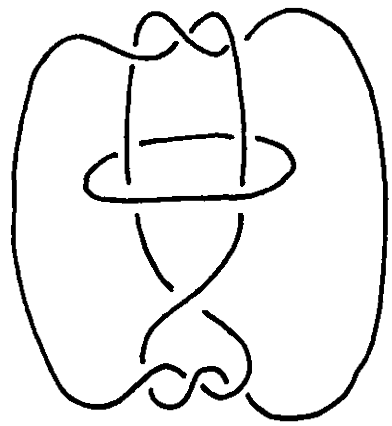


Under an obvious Dehn twist $S^3 - L$ is homeomorphic to the complement of the following link.



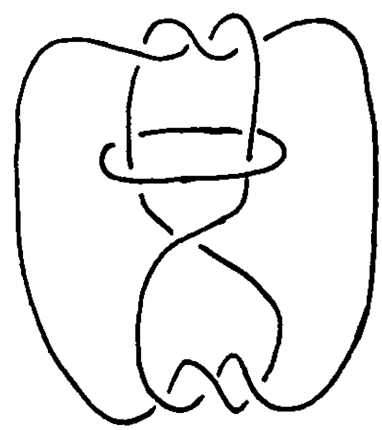
$\Gamma_{-7}(12,22)$:

The manifold $\Gamma_{-7}(12,22)\backslash\mathbb{H}^3$ is homeomorphic to $S^3 - L$ where L is the following link:



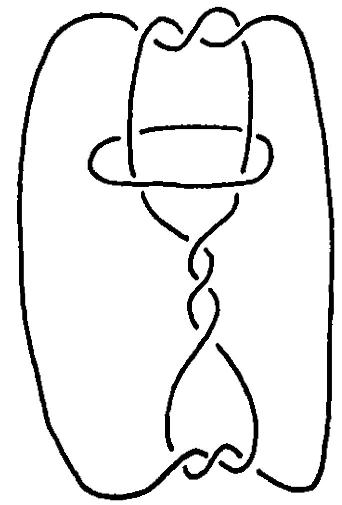
$\Gamma_{-7}(12,23)$:

The manifold $\Gamma_{-7}(12,23)\backslash\mathbb{H}^3$ is homeomorphic to $S^3 - L$ where L is the following link:



$\Gamma_{-7}(12,24)$:

The manifold $\Gamma_{-7}(12,24)\backslash\mathbb{H}^3$ is homeomorphic to $S^3 - L$ where L is the following link with 2 components:



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