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by

Hidekazu Furusho
Yasushi Komori
Kohji Matsumoto
Hirofumi Tsumura

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Hidekazu Furusho
Yasushi Komori
Kohji Matsumoto
Hirofumi Tsumura

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Graduate School of Mathematics
Nagoya University
Chikusa-ku, Furo-cho
Nagoya 464-8602
Japan

Department of Mathematics
Rikkyo University
Nishi-Ikebukuro, Toshima-ku
Tokyo 171-8501
Japan

Department of Mathematics
and Information Sciences
Tokyo Metropolitan University
1-1, Minami-Ohsawa, Hachioji
Tokyo 192-0397
Japan

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DESINGULARIZATION OF COMPLEX MULTIPLE ZETA-FUNCTIONS, FUNDAMENTALS OF p-ADIC MULTIPLE L-FUNCTIONS, AND EVALUATION OF THEIR SPECIAL VALUES

HIDEKAZU FURUSHO, YASUSHI KOMORI, KOHJI MATSUMOTO, AND HIROFUMI TSUMURA

Abstract. This paper deals with a multiple version of zeta- and L-functions both in the complex case and in the p-adic case:

(I). Our motivation in the complex case is to find suitable rigorous meaning of the values of multivariable multiple zeta-functions at non-positive integer points.

(a). A desingularization of multiple zeta-functions (of the generalized Euler-Zagier type): We reveal that multiple zeta-functions (which are known to be meromorphic in the whole space with whose singularities lying on infinitely many hyperplanes) turn to be entire on the whole space after taking the desingularization. Further we show that the desingularized function is given by a suitable finite 'linear' combination of multiple zeta-functions with some arguments shifted. It is also shown that specific combinations of Bernoulli numbers attain the special values at their non-positive integers of the desingularized ones.

(b). Twisted multiple zeta-functions: Those can be continued to entire functions, and their special values at non-positive integer points can be explicitly calculated.

(II). Our work in the p-adic case is to develop the study on analytic side of the Kubota-Leopoldt p-adic L-functions into the multiple setting. We construct p-adic multiple L-functions, multivariable versions of their p-adic L-functions, by using a specific p-adic measure. We establish their various fundamental properties:

(a). Evaluation at non-positive integers: We establish their intimate connection with the above complex multiple zeta-functions by showing that the special values of the p-adic multiple L-functions at non-positive integers are expressed by the twisted multiple Bernoulli numbers, the special values of the complex multiple zeta-functions at non-positive integers.

(b). Multiple Kummer congruence: We extend Kummer congruence for Bernoulli numbers to congruences for the twisted multiple Bernoulli numbers.

(c). Functional relations with a parity condition: We extend the vanishing property of the Kubota-Leopoldt p-adic L-functions with odd characters to our p-adic multiple L-functions.

(d). Evaluation at positive integers: We establish their close relationship with the p-adic twisted multiple polylogarithms by showing that the special values of the p-adic multiple L-functions at positive integers are described by those of the p-adic twisted multiple polylogarithms at roots of unity, which generalizes the previous result of Coleman in the single variable case.

Contents

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0. Introduction

The aim of the present paper is to consider multiple zeta- and L-functions both in the complex case and in the p-adic case, and especially study their special values at integer points.

Let our story begin with the **multiple zeta-function of the generalized Euler-Zagier type** defined by

$$\zeta_r((s_j);(\gamma_j)) = \zeta_r(s_1,\ldots,s_r;\gamma_1,\ldots,\gamma_r) := \sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\prod_{j=1}^{r}(m_1\gamma_1+\cdots+m_j\gamma_j)^{-s_j} \tag{0.1}$$

for complex variables $s_1,\ldots,s_r$, where $\gamma_1,\ldots,\gamma_r$ are complex parameters whose real parts are all positive. Series (0.1) converges absolutely in the region

$$\mathcal{D}_r = \{(s_1,\ldots,s_r)\in\mathbb{C}^r \mid \Re(s_{r-k+1}+\cdots+s_r) > k \ (1\leqslant k\leqslant r)\}. \tag{0.2}$$

The first work which established the meromorphic continuation of (0.1) is Essouabri's thesis [18]. The third-named author [37, Theorem 1] showed that (0.1) can be continued meromorphically to the whole complex space with infinitely many (possible) singular hyperplanes.

A special case of (0.1) is the **multiple zeta-function of Euler-Zagier type** defined by

$$\zeta_r((s_j)) = \zeta_r(s_1,s_2,\ldots,s_r) = \sum_{m_1,\ldots,m_r=1}^{\infty}\prod_{j=1}^{r}(m_1+\cdots+m_j)^{-s_j}, \tag{0.3}$$

which is absolutely convergent in $\mathcal{D}_r$.

Note that $\zeta_r((s_j)) = \zeta_r((s_j);(1))$. Its special value $\zeta_r(n_1,\ldots,n_r)$ when $n_1,\ldots,n_r$ are positive integers makes sense when $n_r>1$. It is called the **multiple zeta value** (abbreviated as MZV), history of whose study goes back to the work of Euler [19] published in 1776 [1]. For a couple of these decades, it has been intensively studied in various fields including number theory, algebraic geometry, low dimensional topology and mathematical physics.

[1] You can find several literatures which cite the paper saying as if it were published in 1775. But according to Euler archive `http://eulerarchive.maa.org/`, it was written in 1771, presented in 1775 and published in 1776.

On the other hand, after the work of meromorphic continuation of (0.1) mentioned above, it is natural to ask how is the behavior of $\zeta_r(-n_1,\ldots,-n_r)$ when $n_1,\ldots,n_r$ are positive (or non-negative) integers. However, as we will mention in Section 1.1, in most cases these points are on singular loci, and hence they are points of indeterminacy. Therefore we can raise the following fundamental problem.

Problem 0.1. Are there any 'rigorous' ways to give a meaning of $\zeta_r(-n_1,\ldots,-n_r)$ for $n_1,\ldots,n_r \in \mathbb{Z}_{\geqslant 0}$?

Several approaches to this problem have been done so far. Guo and Zhang [25], and also Manchon and Paycha [36] discussed a kind of renormalization method. In the present paper we will develop yet another approach, called the *desingularization*, in Section 1.4. The Riemann zeta-function $\zeta(s)$ is a meromorphic function on the complex plane $\mathbb{C}$ with a simple and unique pole at $s=1$. Hence $(s-1)\zeta(s)$ is an entire function. This simple fact may be regarded as a technique to resolve a singularity of $\zeta(s)$ and yield an entire function. Our desingularization method is motivated by this simple observation. We will show that a suitable *finite* sum of multiple zeta-functions turns to be an entire function whose special values at non-positive integers are described explicitly in terms of Bernoulli numbers.

Another possible approach to the above Problem 0.1 is to consider the twisted multiple series. Let $\xi_1,\ldots,\xi_r \in \mathbb{C}$ be roots of unity. For $\gamma_1,\ldots,\gamma_r \in \mathbb{C}$ with $\Re\gamma_j > 0$ $(1 \leqslant j \leqslant r)$, define the **multiple zeta-function of the generalized Euler-Zagier-Lerch type** by

$$\zeta_r((s_j);(\xi_j);(\gamma_j)) := \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \prod_{j=1}^{r} \xi_j^{m_j} (m_1\gamma_1+\cdots+m_j\gamma_j)^{-s_j}, \tag{0.4}$$

which is absolutely convergent in the region $\mathcal{D}_r$ defined by (0.2). We note that the multiple zeta-function of the generalized Euler-Zagier type (0.1) is its special case, that is,

$$\zeta_r((s_j);(\gamma_j)) = \zeta_r((s_j);(1);(\gamma_j)).$$

Because of the existence of the twisting factor $\xi_1,\ldots,\xi_r$, we can see (in Theorem 1.13 below) that, if all ξ_j is not equal to 1, series (0.4) can be continued to an entire function, hence its values at non-positive integer points have a rigorous meaning. Moreover we will show that those values can be written explicitly in terms of twisted multiple Bernoulli numbers.

The above second approach naturally leads us to the theory of p-adic multiple L-functions. Our next main theme in the present paper is to search for a multiple analogue of Kubota-Leopoldt p-adic L-functions.

In the 1960s, Kubota and Leopoldt [35] gave the first construction of the p-adic analogue of the Dirichlet L-function $L(s,\chi)$ associated with the Dirichlet character χ, which is called the p-adic L-function denoted by $L_p(s;\chi)$. This can be regarded as a p-adic interpolation of values of the Dirichlet L-function at non-positive integers, based on Kummer's congruences for Bernoulli numbers.

Iwasawa [28] proposed a different way of constructing p-adic L-functions, based on the study of the arithmetic of Galois modules associated with certain towers of algebraic number fields called $\mathbb{Z}_p$-extensions (see also [29]). In particular, his method shows that the p-adic

L-function can be expressed by use of the power series. His study, called the Iwasawa theory, developed spectacularly with recognition of the importance of p-adic L-functions.

In the 1970s, other constructions of p-adic L-functions were given, for example, by Amice and Fresnel [3], Coates [13], Koblitz [31] and Serre [42]. More generally p-adic L-functions over totally real number fields were constructed by Barsky [5], Cassou-Noguès [12], Deligne and Ribet [17]. These works are based on the p-adic properties of values of abelian L-functions over totally real number fields at non-positive integers. In fact, the approach of Deligne and Ribet were built on the theory of p-adic Hilbert modular forms, while those of Barsky and of Cassou-Noguès were built on the pioneering work of Shintani [43]. Shintani's work also inspired the theory of multivariable p-adic L-functions. In fact, motivated by [43], Imai [27] constructed certain multivariable p-adic L-functions and Hida [26] constructed p-adic analogues of Shintani's L-functions.

Recently, the second, the third and the fourth named author [34] introduced a certain double analogue of the Kubota-Leopoldt p-adic L-function, which could also be seen as a p-adic analogue of the double ($r = 2$) zeta function of a specific case. Its evaluation at non-positive integers and a certain functional relation with the Kubota-Leopoldt p-adic L-function were shown.

In the present paper we generalize the method in [34] to construct the multivariable **p-adic multiple L-function**,

$$L_{p,r}(s_1,\ldots,s_r;\omega^{k_1},\ldots,\omega^{k_r};\gamma_1,\ldots,\gamma_r;c) \tag{0.5}$$

It is a p-adic valued function for p-adic variables $s_1,\ldots,s_r$, which is attached to each $\omega^{k_1},\ldots,\omega^{k_r}$ (ω: the Teichmüller character, $k_1,\ldots,k_r\in\mathbb{Z}$), $\gamma_1,\ldots,\gamma_r\in\mathbb{Z}_p$ and $c\in\mathbb{N}_{>1}$ with $(c,p)=1$.

It can be seen as a multiple analogue of the Kubota-Leopoldt p-adic L-function and the above mentioned p-adic double L-function and also regarded as a p-adic analogues of the complex multivariable multiple zeta-functions (0.4) in a sense. Actually it can be constructed by a multiple analogue of the p-adic gamma transform of a p-adic measure of Koblitz [31]. We investigate p-adic properties of (0.5), particularly its p-adic analyticity in the parameter $s_1,\ldots,s_r$ and its p-adic continuity in the parameter c.

Evaluation of (0.5) at integral points is one of our main themes of this paper: By explicitly describing its special values at all non-positive integer points in terms on twisted Bernoulli numbers, we show that our p-adic multiple L-function (0.5) is closely connected to the complex multiple zeta function (0.4) of the generalized Euler-Zagier-Lerch type via their special values at non-positive integers. Our evaluation yields two particular results; multiple Kummer congruence and functional relations. The multiple Kummer congruence is a certain p-adic congruence among the special values, which includes an ordinal Kummer congruence for Bernoulli numbers as a very special case and where we also discover a newly-looked (or not?) type of congruence for Bernoulli numbers. The functional relations are linear relations among p-adic multiple L-functions as a function, which reduce (0.5) as a linear sum of (0.5) with lower r. The relations is attached to a parity of $k_1+\cdots+k_r$. It extends the known fact in the single variable case that the Kubota-Leopoldt p-adic L-function with odd character is identically zero function. As the double variable case, we recover the result of [34] that a certain p-adic double L-function is equal to the Kubota-Leopoldt p-adic L-function up to a minor factor.

Whilst, as for evaluation of (0.5) at all positive integers, we show that the special value of p-adic multiple L-function (0.5) with $\gamma_1 = \cdots = \gamma_r = 1$ at any positive integer points $(n_1, \ldots, n_r)$ is given by the special values of **p-adic twisted multiple polylogarithms** (p-adic TMPL's, in short)

$$Li^{(p)}_{n_1,\ldots,n_r}(\xi_1, \ldots, \xi_r; z) \tag{0.6}$$

at unity. The above p-adic TMPL (0.6) is a p-adic valued function for p-adic variable z, which is attached to each positive integers $n_1, \ldots, n_r$ and certain p-adic parameters $\xi_1, \ldots, \xi_r$ (which are occasionally roots of unity.) We construct the function by using Coleman's p-adic iterated integration theory [15]. The construction generalizes that of p-adic polylogarithm by Coleman [15] and that of p-adic multiple polylogarithm and p-adic multiple zeta (L-)values by the first-named author [22, 23] and Yamashita [47]. Our result here shows that there is a close connection between the theory of p-adic multiple L-functions of the Kubota-Leopoldt type initiated in [34], and the theory of p-adic multiple polylogarithms developed by the first-named author [22, 23] which were introduced under a very different motivation. It also generalizes the previous result of Coleman [15] which connects the Kubota-Leopoldt p-adic L-function with his p-adic polylogarithm. In Remark 4.4 of [34] it is written that it is unclear whether there is some connection between these two theories or not. Our results in the present paper give an answer to this question.

We remark that our multivariable p-adic multiple L-functions are different from p-adic multiple zeta functions by Tangedal and Young [44], which are one variable p-adic functions stemming from the works of Shintani, Cassou-Noguès, Yoshida and Kashio, though pursuing relationship between our works and theirs might be a significant research. A relationship to non-commutative Iwasawa theory (Coates, Fukaya, Kato, Sujatha and Venjakob [14], etc) might be another direction. The theory is a generalization of Iwasawa theory but it looks lacking its analytic side. It is not clear whether our work in this paper lead any direction related to this or not, which might be worthy to discuss further.

Here is the plan of this paper: In Section 1, we will introduce the *twisted r-ple Bernoulli numbers* $\{\mathfrak{B}((n_j);(\xi_j);(\gamma_j))\}$ associated with integers $\{n_j\}_{1\leqslant j\leqslant r}$ such that $n_j \geqslant -1$, roots of unity $\{\xi_j\}_{1\leqslant j\leqslant r}$ and complex numbers γ_j $(1 \leqslant j \leqslant r)$(Definition 1.6) as generalizations of ordinary Bernoulli numbers. We will show that $\zeta_r((s_j);(\xi_j);(\gamma_j))$ is analytically continued to the whole space as an entire function and interpolates $\{\mathfrak{B}((n_j);(\xi_j);(\gamma_j))\}$ at non-positive integers when all ξ_j are not 1 (Theorem 1.13). On the other hand, when ξ_j are all equal to 1, then $\zeta_r((s_j);(1);(\gamma_j)) = \zeta_r((s_j);(\gamma_j))$ is meromorphically continued to the whole space with whose singularities lying on infinitely many hyperplanes. We will introduce and develop the method of *desingularization*, which is to resolve all singularities of $\zeta_r((s_j);(\gamma_j))$. We will construct a function $\zeta_r^{\rm des}((s_j);(\gamma_j))$, called *the desingularized multiple zeta-function* (Definition 1.16), which will be shown to be entire (Theorem 1.19). Furthermore we will show that

$\zeta_r^{\rm des}((s_j);(\gamma_j))$ can be expressed as a finite 'linear' combination of $\zeta_r((s_j+m_j);(\gamma_j))$, the multiple zeta-functions of the generalized Euler-Zagier type whose arguments are appropriately shifted by integers m_j $(1 \leqslant j \leqslant r)$ (Theorem 1.23). It is where we stress that summing up suitable *finite* combinations of $\zeta_r((s_j+m_j);(\gamma_j))$ causes that a marvelous cancellation of all of their *infinitely* many singular hyperplanes occurs and consequently $\zeta_r^{\rm des}((s_j);(\gamma_j))$ turns to be entire. We will also describe the special values of $\zeta_r^{\rm des}((s_j);(\gamma_j))$ at non-positive integers in terms of ordinary Bernoulli numbers (Theorem 1.22).

In Section 2, we will construct the *p-adic r-ple L-function* $L_{p,r}((s_j);(\omega^{k_j});(\gamma_j);c)$ associated with $\gamma_j \in \mathbb{Z}_p$, $k_j \in \mathbb{Z}$ $(1 \leqslant j \leqslant r)$ and a positive integer $c(\geqslant 2)$ with $(c,p)=1$ (Definition 2.9). The case $r=1$ essentially coincides with the Kubota-Leopoldt p-adic L-function (Example 2.12). Also the case $(r,c)=(2,2)$ coincides with the p-adic double L-function introduced in [34] as mentioned above (Example 2.13). Our main technique of construction is due to Koblitz [31]. By using a specific p-adic measure, we will define $L_{p,r}((s_j);(\omega^{k_j});(\gamma_j);c)$ as a multiple version of the p-adic Γ-transform that provides a p-adic analyticity in (s_j) (Theorem 2.14). We will further show a non-trivial fact that the map $c \mapsto L_{p,r}((s_j);(\omega^{k_j});(\gamma_j);c)$ can be continuously extended to any p-adic integer c as a p-adic continuous function (Theorem 2.17).

In Section 3, we will describe the values of $L_{p,r}((s_j);(\omega^{k_j});(\gamma_j);c)$ at non-positive integers as a sum of $\{\mathfrak{B}((n_j);(\xi_j);(\gamma_j))\}$ (Theorem 3.1). We will see that our $L_{p,r}((s_j);(\omega^{k_j});(\gamma_j);c)$ is a p-adic interpolation of a certain sum (3.2) of the complex multiple zeta-functions $\zeta_r((s_j);(\xi_j);(\gamma_j))$ (Remark 3.2). As an application of the theorem, we will obtain a multiple version of the Kummer congruences (Theorem 3.10). In fact, the case $r=1$ coincides with the ordinary Kummer congruences for Bernoulli numbers. Also we will give some functional relations with a parity condition among p-adic multiple L-functions (Theorem 3.15). The functional relations can be regarded as multiple versions of the well-known fact that $L_p(s;\omega^{2k+1})$ is the zero-function (see Example 3.19). We will see that they also recover the functional relations shown in [34] as a special case $(r,c)=(2,2)$ (Example 3.20).

In Section 4, we will describe the special values of $L_{p,r}((s_j);(\omega^{k_j});(1);c)$ at positive integers. In Theorem 4.41 we will establish their close relation to those of *p-adic TMPL's* $Li^{(p)}_{n_1,\ldots,n_r}(\xi_1,\ldots,\xi_r;z)$ (cf. Definition 4.29) at roots of unity, which is an extension of the previous result of Coleman [15]. For this aim, we will introduce *p-adic rigid TMPL's* $\ell^{(p)}_{n_1,\ldots,n_r}(\xi_1,\ldots,\xi_{r-1},z)$ and *p-adic partial TMPL's* $\ell^{\equiv(\alpha_1,\ldots,\alpha_r),(p)}_{n_1,\ldots,n_r}(\xi_1,\ldots,\xi_{r-1},z)$ (Definition 4.4 and 4.16 respectively) as in-between functions. In Subsection 4.1 the above special values at positive integers will be shown to be connected with the special values of p-adic rigid TMPL's at roots unity (Theorem 4.9). Basic properties of these in-between functions will be presented in Subsection 4.2. We will show an explicit relationship between p-adic rigid TMPL's and p-adic TMPL's (Theorem 4.35) by transmitting through their connections with p-adic partial TMPL's to obtain Theorem 4.41 in Subsection 4.3.

1. Complex multiple zeta-functions

In this section, we will first recall the analytic properties of complex multiple zeta-functions of Euler-Zagier type (0.3) and of the zeta-function of Lerch type (1.10) which interpolates the twisted Bernoulli numbers (1.4). Next we will introduce multiple twisted Bernoulli numbers (Definition 1.6) which are connected with multiple zeta-functions of the generalized Euler-Zagier-Lerch type (0.4). It will be shown that the functions in the non-unity case (1.20) are analytically continued to the whole space as entire functions and interpolate these numbers at non-positive integers (Theorem 1.13). In the final subsection, we will develop our method of desingularization (Definition 1.16) of multiple zeta-functions of the generalized Euler-Zagier type (0.1). They are meromorphically continued to the whole space with whose singularities lying on infinitely many hyperplanes. Our desingularization is a method to reduce them into entire functions (Theorem 1.19). We will further show that the desingularized functions are given by a suitable finite 'linear' combination of multiple zeta-functions (0.1) with some arguments shifted (Theorem 1.23). It is where we see a miraculous cancellation of all of their *infinitely* many singular hyperplanes occurring there by taking a suitable *finite* combination of these functions. We will prove that certain combinations of Bernoulli numbers attain the special values at their non-positive integers of the desingularized functions (Theorem 1.22). Our method might be said as a multiple series analogue of the procedure reducing the Riemann zeta function $\zeta(s)$ into the entire function $(s-1)\zeta(s)$ (Example 1.18). These observations lead to the construction of p-adic multiple L-functions which will be discussed in the next section.

1.1. **Basic facts.** Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ be the set of natural numbers, non-negative integers, rational integers, rational numbers, real numbers and complex numbers, respectively. Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$. For $s \in \mathbb{C}$, denote by $\Re s$ and $\Im s$ the real and the imaginary parts of s, respectively.

Let χ be a primitive Dirichlet character and denote the conductor of χ by f_χ. The Dirichlet L-function associated with χ is defined by

$$L(s,\chi)=\sum_{m=1}^{\infty}\frac{\chi(m)}{m^s}.$$

In the case $\chi=\chi_0$, namely the trivial character with $f_{\chi_0}=1$, $L(s,\chi_0)$ is equal to $\zeta(s)$.

It is well-known that $L(s,\chi)$ is an entire function when $\chi\neq\chi_0$, and $\zeta(s)$ is a meromorphic function on $\mathbb{C}$ with a simple pole at $s=1$, and satisfies

$$\begin{aligned}
&\zeta(1-k)=\begin{cases}-\frac{B_k}{k} & (k\in\mathbb{N}_{>1})\\ -\frac{1}{2} & (k=1),\end{cases}\\
&L(1-k,\chi)=-\frac{B_{k,\chi}}{k}\quad (k\in\mathbb{N};\ \chi\neq\chi_0),
\end{aligned}\tag{1.1}$$

where $\{B_n\}$ and $\{B_{n,\chi}\}$ are the Bernoulli numbers [2] and the generalized Bernoulli numbers associated with χ defined by

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!},$$
$$\sum_{a=1}^{f}\frac{\chi(a)te^{at}}{e^{ft}-1}=\sum_{n=0}^{\infty}B_{n,\chi}\frac{t^n}{n!}\qquad (f=f_\chi),$$

respectively (see [46, Theorem 4.2]). Note that $B_{n,\chi_0}=B_n$ $(n\in\mathbb{N}_0)$ except for $B_{1,\chi_0}=-B_1=\frac{1}{2}$.

The multiple zeta-function of Euler-Zagier type is defined by (0.3). As was mentioned in the Introduction, the research of (0.3) goes back to a paper of Euler. In the late 1990s, several authors investigated its analytic properties, though their results have not been published (for the details, see the survey article [38]). In the early 2000s, Zhao [48] and Akiyama, Egami and Tanigawa [1] independently showed that the multiple zeta-function (0.3) can be meromorphically continued to $\mathbb{C}^r$. Furthermore, the *exact* locations of singularities of (0.3) were explicitly determined as follows.

Theorem 1.1 ([1, Theorem 1]). *The multiple zeta-function* (0.3) *can be meromorphically continued to* $\mathbb{C}^r$ *with infinitely many singular hyperplanes*

$$\begin{aligned}&s_r=1,\quad s_{r-1}+s_r=2,1,0,-2,-4,-6,\ldots,\\ &s_{r-k+1}+s_{r-k+2}+\cdots+s_r=k-n\quad (3\leqslant k\leqslant r,\ n\in\mathbb{N}_0).\end{aligned}\tag{1.2}$$

Multiple zeta values, namely the special values of (0.3) at positive integers, are equal to the special values of multiple polylogarithm

$$Li_{n_1,\ldots,n_r}(z_1,\ldots,z_r):=\sum_{0<k_1<\cdots<k_r}\frac{z_1^{k_1}\cdots z_r^{k_r}}{k_1^{n_1}\cdots k_r^{n_r}}$$

(which is an r-variable complex analytic function converging on polyunit disk) at unity, namely

$$\zeta_r(n_1,\ldots,n_r)=Li_{n_1,\ldots,n_r}(1,\ldots,1)\tag{1.3}$$

for $n_1,\ldots,n_r\in\mathbb{N}$ with $n_r>1$.

In the last section of this paper a p-adic analogue of the equality (1.3) will be attained.

As stated above, the multiple zeta-function (0.3) but for the single variable case $(r=1)$ has infinitely many singular hyperplanes and almost all non-positive integer points lie there. It causes an indeterminacy of its special values at non-positive integers. For example, according to [1, 2],

$$\lim_{\varepsilon_1\to0}\lim_{\varepsilon_2\to0}\zeta_2(\varepsilon_1,\varepsilon_2)=\frac{1}{3},$$
$$\lim_{\varepsilon_2\to0}\lim_{\varepsilon_1\to0}\zeta_2(\varepsilon_1,\varepsilon_2)=\frac{5}{12},$$

[2]or better to be called Seki-Bernoulli numbers, because K. Seki published the work on these numbers, independently, before J. Bernoulli.

$$\lim_{\varepsilon\to 0}\zeta_2(\varepsilon,\varepsilon)=\frac{3}{8}.$$

There are some other explicit formulas for the values at those non-positive integer points as limit values when the way of approaching those points are fixed ([1], [33], [41] et al.)

1.2. **Twisted multiple Bernoulli numbers.** We will review Koblitz' definition of twisted Bernoulli numbers. Then we will introduce twisted multiple Bernoulli numbers, their multiple analogue, in Definition 1.6 and investigate their expression as combinations of twisted Bernoulli numbers in Proposition 1.7.

Definition 1.2 ([31, p. 456])**.** For any root of unity ξ, we define the **twisted Bernoulli numbers** $\{\mathfrak{B}_n(\xi)\}$ by

$$\mathfrak{H}(t;\xi)=\frac{1}{1-\xi e^t}=\sum_{n=-1}^{\infty}\mathfrak{B}_n(\xi)\frac{t^n}{n!}. \tag{1.4}$$

Remark 1.3. Koblitz [31] generally defined the twisted Bernoulli numbers associated with primitive Dirichlet characters. The above $\{\mathfrak{B}_n(\xi)\}$ correspond to χ_0.

In the case $\xi=1$, we have

$$\mathfrak{B}_{-1}(1)=-1,\qquad \mathfrak{B}_n(1)=-\frac{B_{n+1}}{n+1}\quad (n\in\mathbb{N}_0). \tag{1.5}$$

In the case $\xi\neq 1$, we have $\mathfrak{B}_{-1}(\xi)=0$ and $\mathfrak{B}_n(\xi)=\frac{1}{1-\xi}H_n\left(\xi^{-1}\right)$ $(n\in\mathbb{N}_0)$, where $\{H_n(\lambda)\}_{n\geqslant 0}$ are what is called the Frobenius-Euler numbers associated with λ defined by

$$\frac{1-\lambda}{e^t-\lambda}=\sum_{n=0}^{\infty}H_n(\lambda)\frac{t^n}{n!}$$

(see Frobenius [21]). We obtain from (1.4) that $\mathfrak{B}_n(\xi)\in\mathbb{Q}(\xi)$. For example,

$$\begin{aligned}
&\mathfrak{B}_0(\xi)=\frac{1}{1-\xi},\quad \mathfrak{B}_1(\xi)=\frac{\xi}{(1-\xi)^2},\quad \mathfrak{B}_2(\xi)=\frac{\xi(\xi+1)}{(1-\xi)^3},\\
&\mathfrak{B}_3(\xi)=\frac{\xi(\xi^2+4\xi+1)}{(1-\xi)^4},\quad \mathfrak{B}_4(\xi)=\frac{\xi(\xi^3+11\xi^2+11\xi+1)}{(1-\xi)^5},\ldots
\end{aligned} \tag{1.6}$$

Let μ_k be the group of kth roots of unity. Using the relation

$$\frac{1}{X-1}-\frac{k}{X^k-1}=\sum_{\substack{\xi\in\mu_k\\ \xi\neq 1}}\frac{1}{1-\xi X}\qquad (k\in\mathbb{N}_{>1}) \tag{1.7}$$

for an indeterminate X, we obtain the following.

Proposition 1.4. *Let $c\in\mathbb{N}_{>1}$. For $n\in\mathbb{N}_0$,*

$$\left(1-c^{n+1}\right)\frac{B_{n+1}}{n+1}=\sum_{\substack{\xi^c=1\\ \xi\neq 1}}\mathfrak{B}_n(\xi). \tag{1.8}$$

Remark 1.5. Let ξ be a root of unity. As an analogue of (1.1), it holds that

$$\phi(-k;\xi)=\mathfrak{B}_k(\xi)\quad (k\in\mathbb{N}_0), \tag{1.9}$$

where $\phi(s;\xi)$ is the **zeta-function of Lerch type** defined by the meromorphic continuation of the series

$$\phi(s;\xi)=\sum_{m\geqslant 1}\xi^m m^{-s}\qquad (\Re s>1) \tag{1.10}$$

(cf. [32, Chapter 2, Section 1]).

We see that (1.8) can also be given from the relation

$$\left(c^{1-s}-1\right)\zeta(s)=\sum_{\substack{\xi^c=1\\ \xi\neq 1}}\phi(s;\xi). \tag{1.11}$$

Now we define certain multiple analogues of twisted Bernoulli numbers.

Definition 1.6. Let $r\in\mathbb{N}$, $\gamma_1,\ldots,\gamma_r\in\mathbb{C}$ and let $\xi_1,\ldots,\xi_r\in\mathbb{C}$ be roots of unity. Set

$$\mathfrak{H}_r((t_j);(\xi_j);(\gamma_j)):=\prod_{j=1}^r\mathfrak{H}(\gamma_j(\sum_{k=j}^r t_k);\xi_j)=\prod_{j=1}^r\frac{1}{1-\xi_j\exp\left(\gamma_j\sum_{k=j}^r t_k\right)} \tag{1.12}$$

and define **twisted multiple Bernoulli numbers** [3] $\{\mathfrak{B}(n_1,\ldots,n_r;(\xi_j);(\gamma_j))\}$ by

$$\mathfrak{H}_r((t_j);(\xi_j);(\gamma_j))=\sum_{n_1=-1}^\infty\cdots\sum_{n_r=-1}^\infty\mathfrak{B}(n_1,\ldots,n_r;(\xi_j);(\gamma_j))\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!}. \tag{1.13}$$

In the case $r=1$, we have $\mathfrak{B}_n(\xi_1)=\mathfrak{B}(n;\xi_1;1)$. Note that if $\xi_j\neq 1$ $(1\leqslant j\leqslant r)$ then $\mathfrak{H}_r((t_j);(\xi_j);(\gamma_j))$ is holomorphic around the origin with respect to the parameters $t_1,\ldots,t_r$, hence the singular part does not appear on the right-hand side of (1.13).

We immediately obtain the following from (1.4), (1.12) and (1.13).

Proposition 1.7. *Let $\gamma_1,\ldots,\gamma_r\in\mathbb{C}$ and $\xi_1,\ldots,\xi_r\in\mathbb{C}$ be roots of unity. Then $\mathfrak{B}(n_1,\ldots,n_r;(\xi_j);(\gamma_j))$ can be expressed as a polynomial in $\{\mathfrak{B}_n(\xi_j)\,|\,1\leqslant j\leqslant r,\ n\geqslant 0\}$ and $\{\gamma_1,\ldots,\gamma_r\}$ with $\mathbb{Q}$-coefficients, that is, a rational function in $\{\xi_j\}$ and $\{\gamma_j\}$ with $\mathbb{Q}$-coefficients.*

Example 1.8. We consider the case $r=2$ and $\xi_j\neq 1$ $(j=1,2)$. Substituting (1.4) into (1.12) in the case $r=2$, we have

$$\begin{aligned}\mathfrak{H}_2(t_1,t_2;\xi_1,\xi_2;\gamma_1,\gamma_2)&=\frac{1}{1-\xi_1\exp\left(\gamma_1(t_1+t_2)\right)}\frac{1}{1-\xi_2\exp\left(\gamma_2 t_2\right)}\\ &=\left(\sum_{m=0}^\infty\mathfrak{B}_m(\xi_1)\frac{\gamma_1^m(t_1+t_2)^m}{m!}\right)\left(\sum_{n=0}^\infty\mathfrak{B}_n(\xi_2)\frac{\gamma_2^n t_2^n}{n!}\right)\\ &=\sum_{m=0}^\infty\sum_{n=0}^\infty\mathfrak{B}_m(\xi_1)\mathfrak{B}_n(\xi_2)\left(\sum_{\substack{k,j\geqslant 0\\ k+j=m}}\frac{t_1^k t_2^j}{k!j!}\right)\gamma_1^m\gamma_2^n\frac{t_2^n}{n!}.\end{aligned}$$

Putting $l:=n+j$, we have

$$\mathfrak{H}_2(t_1,t_2;\xi_1,\xi_2;\gamma_1,\gamma_2)=\sum_{k=0}^\infty\sum_{l=0}^\infty\sum_{j=0}^l\binom{l}{j}\mathfrak{B}_{k+j}(\xi_1)\mathfrak{B}_{l-j}(\xi_2)\gamma_1^{k+j}\gamma_2^{l-j}\frac{t_1^k}{k!}\frac{t_2^l}{l!},$$

---

[3] We are not sure which is better, "twisted multiple", or "multiple twisted". But we will skip this problem because it looks that these two adjectives are "commutative" here.

which gives

$$\mathfrak{B}(k,l;\xi_1,\xi_2;\gamma_1,\gamma_2)=\sum_{j=0}^{l}\binom{l}{j}\mathfrak{B}_{k+j}(\xi_1)\mathfrak{B}_{l-j}(\xi_2)\gamma_1^{k+j}\gamma_2^{l-j}\quad (k,l\in\mathbb{N}_0). \tag{1.14}$$

For example, we can obtain from (1.6) that

$$\mathfrak{B}(0,0;\xi_1,\xi_2;\gamma_1,\gamma_2)=\frac{1}{(1-\xi_1)(1-\xi_2)},\quad \mathfrak{B}(1,0;\xi_1,\xi_2;\gamma_1,\gamma_2)=\frac{\xi_1\gamma_1}{(1-\xi_1)^2(1-\xi_2)},$$
$$\mathfrak{B}(0,1;\xi_1,\xi_2;\gamma_1,\gamma_2)=\frac{\xi_1\gamma_1+\xi_2\gamma_2-\xi_1\xi_2(\gamma_1+\gamma_2)}{(1-\xi_1)^2(1-\xi_2)^2},$$
$$\mathfrak{B}(1,1;\xi_1,\xi_2;\gamma_1,\gamma_2)=\frac{\xi_1^2\gamma_1(\gamma_1-\xi_2(\gamma_1+\gamma_2))+\xi_1\gamma_1(\gamma_1-\xi_2(\gamma_1-\gamma_2))}{(1-\xi_1)^3(1-\xi_2)^2},\ldots$$

The following series will be treated in our desingularization method in subsection 1.4.

Definition 1.9. For $c\in\mathbb{R}$ and $\gamma_1,\ldots,\gamma_r\in\mathbb{C}$ with $\Re\gamma_j>0$ $(1\leqslant j\leqslant r)$, define

$$\begin{aligned}\widetilde{\mathfrak{H}}_r((t_j);(\gamma_j);c)&=\prod_{j=1}^{r}\left(\frac{1}{\exp\left(\gamma_j\sum_{k=j}^{r}t_k\right)-1}-\frac{c}{\exp\left(c\gamma_j\sum_{k=j}^{r}t_k\right)-1}\right)\\&=\prod_{j=1}^{r}\left(\sum_{m=1}^{\infty}(1-c^m)B_m\frac{\left(\gamma_j\sum_{k=j}^{r}t_k\right)^{m-1}}{m!}\right).\end{aligned}\tag{1.15}$$

In particular when $c\in\mathbb{N}_{>1}$, by use of (1.7), we have

$$\begin{aligned}\widetilde{\mathfrak{H}}_r((t_j);(\gamma_j);c)&=\prod_{j=1}^{r}\sum_{\substack{\xi_j^c=1\\ \xi_j\neq1}}\frac{1}{1-\xi_j\exp\left(\gamma_j\sum_{k=j}^{r}t_k\right)}\\&=\sum_{\substack{\xi_1^c=1\\ \xi_1\neq1}}\cdots\sum_{\substack{\xi_r^c=1\\ \xi_r\neq1}}\mathfrak{H}_r((t_j);(\xi_j);(\gamma_j)).\end{aligned}\tag{1.16}$$

Remark 1.10. We note that $\widetilde{\mathfrak{H}}_r((t_j);(\gamma_j);c)$ is holomorphic around the origin with respect to the parameters (t_j), and tends to 0 as $c\to1$. We also note that the Bernoulli numbers appear in the Maclaurin expansion of the limit

$$\lim_{c\to1}\frac{1}{(c-1)^r}\widetilde{\mathfrak{H}}_r((t_j);(\gamma_j);c).$$

These are important points in our arguments on desingularization methods developed in Subsection 1.4.

Example 1.11. Similarly to Example 1.8, we obtain from (1.15) with any $c\in\mathbb{R}$ that

$$\begin{aligned}&\widetilde{\mathfrak{H}}_2(t_1,t_2;\gamma_1,\gamma_2;c)\\&=\sum_{k,l=0}^{\infty}\left\{\sum_{j=0}^{l}\binom{l}{j}\left(1-c^{k+j+1}\right)\left(1-c^{l-j+1}\right)\frac{B_{k+j+1}}{k+j+1}\frac{B_{l-j+1}}{l-j+1}\gamma_1^{k+j}\gamma_2^{l-j}\right\}\frac{t_1^kt_2^l}{k!l!}.\end{aligned}\tag{1.17}$$

Therefore it follows from (1.13) and (1.16) that

$$\begin{aligned}
&\sum_{\substack{\xi_1\in\mu_c\\ \xi_1\neq 1}}\sum_{\substack{\xi_2\in\mu_c\\ \xi_2\neq 1}}\mathfrak{B}(k,l;\xi_1,\xi_2;\gamma_1,\gamma_2)\\
&\quad=\sum_{j=0}^{l}\binom{l}{j}\left(1-c^{k+j+1}\right)\left(1-c^{l-j+1}\right)\frac{B_{k+j+1}}{k+j+1}\frac{B_{l-j+1}}{l-j+1}\gamma_1^{k+j}\gamma_2^{l-j}\quad (k,l\in\mathbb{N}_0)
\end{aligned}\tag{1.18}$$

for $c\in\mathbb{N}_{>1}$.

Remark 1.12. Kaneko [30] defined the poly-Bernoulli numbers $\{B_n^{(k)}\}_{n\in\mathbb{N}_0}$ $(k\in\mathbb{Z})$ by use of the polylogarithm of order k. Explicit relations between twisted multiple Bernoulli numbers and poly-Bernoulli numbers are not clearly known. It is noted that, for example,

$$B_l^{(2)}=\sum_{j=0}^{l}\binom{l}{j}\frac{B_{l-j}B_j}{j+1}\quad (l\in\mathbb{N}_0),$$

which resembles (1.14) and (1.18).

1.3. **Multiple zeta-functions.** Corresponding to the twisted multiple Bernoulli numbers $\{\mathfrak{B}((n_j);(\xi_j);(\gamma_j))\}$ is the multiple zeta-function of the generalized Euler-Zagier-Lerch type (0.4) defined in the Introduction, which is a multiple analogue of $\phi(s;\xi)$. This function can be continued analytically to the whole space and interpolates $\mathfrak{B}((n_j);(\xi_j);(\gamma_j))$ at non-positive integers (Theorem 1.13).

Assume $\xi_j\neq 1$ $(1\leqslant j\leqslant r)$. Using the well-known relation

$$u^{-s}=\frac{1}{\Gamma(s)}\int_0^\infty e^{-ut}t^{s-1}dt,$$

we obtain

$$\begin{aligned}
&\zeta_r((s_j);(\xi_j);(\gamma_j))\\
&=\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\left(\prod_{j=1}^{r}\xi_j^{m_j}\right)\left(\prod_{k=1}^{r}\frac{1}{\Gamma(s_k)}\right)\int_{[0,\infty)^r}\prod_{k=1}^{r}\exp(-t_k(\sum_{j\leqslant k}m_j\gamma_j))\prod_{k=1}^{r}t_k^{s_k-1}dt_k\\
&=\left(\prod_{k=1}^{r}\frac{1}{\Gamma(s_k)}\right)\int_{[0,\infty)^r}\prod_{j=1}^{r}\frac{\xi_j\exp(-\gamma_j(\sum_{k=j}^{r}t_k))}{1-\xi_j\exp(-\gamma_j(\sum_{k=j}^{r}t_k))}\prod_{k=1}^{r}t_k^{s_k-1}dt_k\\
&=\left(\prod_{k=1}^{r}\frac{1}{(e^{2\pi i s_k}-1)\Gamma(s_k)}\right)\int_{\mathcal{C}^r}\prod_{j=1}^{r}\frac{\xi_j\exp(-\gamma_j(\sum_{k=j}^{r}t_k))}{1-\xi_j\exp(-\gamma_j(\sum_{k=j}^{r}t_k))}\prod_{k=1}^{r}t_k^{s_k-1}dt_k\\
&=(-1)^r\left(\prod_{k=1}^{r}\frac{1}{(e^{2\pi i s_k}-1)\Gamma(s_k)}\right)\int_{\mathcal{C}^r}\mathfrak{H}_r((t_j),(\xi_j^{-1}),(\gamma_j))\prod_{k=1}^{r}t_k^{s_k-1}dt_k,
\end{aligned}\tag{1.19}$$

where $\mathcal{C}$ is the Hankel contour, that is, the path consisting of the positive real axis (top side), a circle around the origin of radius ε (sufficiently small), and the positive real axis (bottom side) (see Figure 1). Note that the third equality holds because we can let $\varepsilon\to 0$ on the fourth member of (1.19). In fact, the integrand of the fourth member is holomorphic around the origin with respect to the parameters (t_j) because of $\xi_j\neq 1$ $(1\leqslant j\leqslant r)$. Here we can

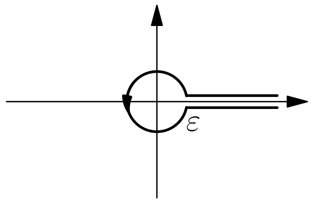

FIGURE 1. The Hankel contour $\mathcal{C}$

easily show that the integral on the last member of (1.19) is absolutely convergent in a usual manner with respect to the Hankel contour. Hence we obtain the following.

Theorem 1.13. *Let $\xi_1,\ldots,\xi_r\in\mathbb{C}$ be roots of unity and $\gamma_1,\ldots,\gamma_r\in\mathbb{C}$ with $\Re\gamma_j>0$ $(1\leqslant j\leqslant r)$. Assume that*

$$\xi_j\neq 1 \quad \text{for all } j\ (1\leqslant j\leqslant r). \tag{1.20}$$

Then, with the above notation, $\zeta_r((s_j);(\xi_j);(\gamma_j))$ can be analytically continued to $\mathbb{C}^r$ as an entire function in (s_j). For $n_1,\ldots,n_r\in\mathbb{N}_0$,

$$\zeta_r((-n_j);(\xi_j);(\gamma_j))=(-1)^{r+n_1+\cdots+n_r}\mathfrak{B}((n_j);(\xi_j^{-1});(\gamma_j)). \tag{1.21}$$

Proof. Since the contour integral on the right-hand side of (1.19) is holomorphic for all $(s_k)\in\mathbb{C}^r$, we see that $\zeta_r((s_j);(\xi_j);(\gamma_j))$ can be meromorphically continued to $\mathbb{C}^r$ and its possible singularities are located on hyperplanes $s_k=l_k\in\mathbb{N}$ $(1\leqslant k\leqslant r)$ outside of the region of convergence because $(e^{2\pi is_k}-1)\Gamma(s_k)$ does not vanish at $s_k\in\mathbb{Z}_{\leqslant 0}$. Furthermore, for $s_k=l_k\in\mathbb{N}$, the integrand of the contour integral with respect to t_k on the last member of (1.19) is holomorphic around $t_k=0$. Therefore, for $l_k\in\mathbb{N}$, we see that

$$\lim_{s_k\to l_k}\int_{\mathcal{C}}\mathfrak{H}_r((t_j),(\xi_j^{-1}),(\gamma_j))t_k^{s_k-1}dt_k=\int_{C_\varepsilon}\mathfrak{H}_r((t_j),(\xi_j^{-1}),(\gamma_j))t_k^{l_k-1}dt_k=0,$$

because of the residue theorem, where $C_\varepsilon=\{\varepsilon e^{i\theta}\,|\,0\leqslant\theta\leqslant 2\pi\}$ for any sufficiently small ε. Consequently this implies that $\zeta_r((s_j);(\xi_j);(\gamma_j))$ has no singularity on $s_k=l_k$, namely $\zeta_r((s_j);(\xi_j);(\gamma_j))$ is entire. Finally, substituting (1.13) into (1.19), setting $(s_j)=(-n_j)$ and using

$$\lim_{s\to -n}\frac{1}{(e^{2\pi is}-1)\Gamma(s)}=\frac{(-1)^n n!}{2\pi i}\quad (n\in\mathbb{N}_0),$$

we obtain (1.21). Thus we complete the proof of Theorem 1.13. □

The partial cases of Theorem 1.13 can be recovered by the special cases of the results in Matsumoto-Tanigawa [39], Matsumoto-Tsumura [40], and de Crisenoy [16].

In [37, Theorem 1], it is shown that the multiple zeta-function $\zeta_r((s_j);(\xi_j);(\gamma_j))$ of the generalized Euler-Zagier-Lerch type (0.4) with all $\xi_j=1$ is meromorphically continued to the whole space $\mathbb{C}^r$ with *possible* singularities. A more general type of multiple zeta-function is treated in [33], where equation (1.21) without the assertion of being an entire function is shown in the case of $\xi_j\neq 1$ for all j and the meromorphic continuation of $\zeta_r((s_j);(\xi_j);(\gamma_j))$ is also given.

Remark 1.14. Without the assumption (1.20), it should be noted that (1.19) does not hold generally, more strictly the third equality on the right-hand side does not hold because the Hankel contours necessarily cross the singularities of the integrand.

In the forthcoming paper [24], we will show the necessary and sufficient condition that $\zeta_r((s_j);(\xi_j);(\gamma_j))$ is entire, and will determine the exact locations of singularities when it is not entire:

Theorem 1.15. *Let $\xi_1,\ldots,\xi_r \in \mathbb{C}$ be roots of unity and $\gamma_1,\ldots,\gamma_r \in \mathbb{C}$ with $\Re\gamma_j > 0$ $(1 \leqslant j \leqslant r)$. Then $\zeta_r((s_j);(\xi_j);(\gamma_j))$ can be entire if and only if the condition* (1.20) *holds. When it is not entire, one the following cases occurs:*

(i) *The function $\zeta_r((s_j);(\xi_j);(\gamma_j))$ has infinitely many simple singular hyperplanes when $\xi_j = 1$ for some j $(1 \leqslant j \leqslant r-1)$.*
(ii) *The function $\zeta_r((s_j);(\xi_j);(\gamma_j))$ has a unique simple singular hyperplane $s_r = 1$ when $\xi_j \neq 1$ for all j $(1 \leqslant j \leqslant r-1)$ and $\xi_r = 1$.*

1.4. **Desingularization of multiple zeta-functions.** In this subsection we introduce and develop our method of desingularization. In our previous subsection we saw that the multiple zeta-function $\zeta_r((s_j);(\gamma_j))$ of the generalized Euler-Zagier type (0.1) is meromorphically continued to the whole space with 'true' singularities whilst the multiple zeta function $\zeta_r((s_j);(\xi_j);(\gamma_j))$ of the generalized Euler-Zagier-Lerch type (0.4) under the non-unity assumption (1.20) is analytically continued to $\mathbb{C}^r$ as an entire function. Our desingularization is a technique to resolve all singularities of $\zeta_r((s_j);(\gamma_j))$ by making use of the holomorphicity of $\zeta_r((s_j);(\xi_j);(\gamma_j))$ and to produce an entire function $\zeta_r^{\rm des}((s_j);(\gamma_j))$. Consider the following expression:

$$\zeta_r^{\rm des}((s_j);(\gamma_j)) := \lim_{c\to 1}\frac{1}{(c-1)^r}\sum_{\substack{\xi_1^c=1\\ \xi_1\neq 1}}\cdots\sum_{\substack{\xi_r^c=1\\ \xi_r\neq 1}}\zeta_r((s_j);(\xi_j);(\gamma_j)) \tag{1.22}$$

This is surely nonsense, because $c \in \mathbb{N}_{>1}$ on the right-hand side. However our fundamental idea is symbolized in this primitive expression. Our idea is motivated from a very simple observation

$$(1-s)\zeta(s) = \lim_{c\to 1}\frac{1}{c-1}\left(c^{1-s}-1\right)\zeta(s).$$

Here on the left-hand side we find an entire function $(1-s)\zeta(s)$, which is merely a product of $(1-s)$ and the meromorphic function $\zeta(s)$ with a simple pole at $s = 1$. While on the right hand-side, when $c \in \mathbb{N}_{>1}$, we may associate a decomposition

$$\frac{1}{c-1}\left(c^{1-s}-1\right)\zeta(s) = \frac{1}{c-1}\sum_{\substack{\xi^c=1\\ \xi\neq 1}}\phi(s;\xi)$$

into a sum of entire functions $\phi(s;\xi) = \zeta_1(s;\xi;1)$.

Our desingularization method, a rigorous mathematical formulation to give a meaning of (1.22) will be settled in Definition 1.16. An application of desingularization to the Riemann zeta function $\zeta(s)$ is given in Example 1.18. We will see in Theorem 1.19 that our $\zeta_r^{\rm des}((s_j);(\gamma_j))$ is entire on the whole space $\mathbb{C}^r$. We stress that $\zeta_r^{\rm des}((s_j);(\gamma_j))$ is worthy of an important object from the viewpoint of the analytic theory of multiple zeta-functions. In fact, its values at not only all positive or all non-positive integer points but also arbitrary integer points are fully determined (see Example 1.33).

Theorem 1.22 will prove that suitable combinations of Bernoulli numbers attain the special values at non-positive integers of $\zeta_r^{\rm des}((s_j);(\gamma_j))$. Theorem 1.23, which is the most important theorem in this subsection, will reveal that our desingularized multiple zeta-function $\zeta_r^{\rm des}((s_j);(\gamma_j))$ is actually given by a finite 'linear' combination of the multiple zeta-function $\zeta_r((s_j+m_j);(\gamma_j))$ with some arguments appropriately shifted by $m_j \in \mathbb{Z}$ $(1 \leqslant j \leqslant r)$. We will see there that *infinitely* many singular hyperplanes of $\zeta_r((s_j);(\gamma_j))$ are miraculously canceled by taking only the *finite* 'linear' combination of appropriately shifted ones. Example 1.26 and Remark 1.27 are our specific observations for double variable case.

Definition 1.16. For $\gamma_1, \dots, \gamma_r \in \mathbb{C}$ with $\Re\gamma_j > 0$ $\quad(1 \leqslant j \leqslant r)$, the **desingularized multiple zeta-function**, which we also call the **desingularization of** $\zeta_r((s_j);(\gamma_j))$, is defined by

$$\begin{aligned}&\zeta_r^{\rm des}((s_j);(\gamma_j))\\ &:= \lim_{\substack{c\to 1\\ c\in\mathbb{R}\setminus\{1\}}} \frac{(-1)^r}{(c-1)^r} \prod_{k=1}^r \frac{1}{(e^{2\pi i s_k}-1)\Gamma(s_k)} \int_{\mathcal{C}^r} \widetilde{\mathfrak{H}}_r((t_j);(\gamma_j);c) \prod_{k=1}^r t_k^{s_k-1} dt_k \end{aligned} \tag{1.23}$$

for $(s_j) \in \mathbb{C}^r$, where $\mathcal{C}$ is the Hankel contour used in (1.19). Note that (1.23) is well-defined because the convergence of the contour integral and of the limit with respect to $c \to 1$ can be justified from Theorem 1.19 (see below).

Remark 1.17. By (1.19) and (1.16), we may say that equation (1.23) is a rigorous way to make sense of the nonsense equation (1.22). We will discuss equation (1.22) again in the p-adic case (Subsection 2.2).

Example 1.18. In the case $r = 1$, set $(r, \gamma_1) = (1, 1)$ in (1.23). Similarly to [46, Theorem 4.2], we can easily see that

$$\begin{aligned}\zeta_1^{\rm des}(s;1) &= \lim_{c\to 1} \frac{(-1)}{c-1} \cdot \frac{1}{(e^{2\pi i s}-1)\Gamma(s)} \int_{\mathcal{C}} \left(\frac{1}{e^t-1} - \frac{c}{e^{ct}-1} \right) t^{s-1} dt \\ &= \lim_{c\to 1} \frac{(-1)}{c-1} \left(\zeta(s) - c \sum_{m=1}^{\infty} \frac{1}{(cm)^s} \right) \\ &= \lim_{c\to 1} \frac{(-1)}{c-1} \left(1 - c^{1-s}\right) \zeta(s) = (1-s)\zeta(s).\end{aligned} \tag{1.24}$$

Hence $\zeta_1^{\rm des}(s;1)$ can be analytically continued to $\mathbb{C}$.

More generally we can prove the following theorem.

Theorem 1.19. *For $\gamma_1, \dots, \gamma_r \in \mathbb{C}$ with $\Re\gamma_j > 0$ $\quad(1 \leqslant j \leqslant r)$,*

$$\begin{aligned}&\zeta_r^{\rm des}((s_j);(\gamma_j))\\ &= \prod_{k=1}^r \frac{1}{(e^{2\pi i s_k}-1)\Gamma(s_k)} \int_{\mathcal{C}^r} \lim_{c\to 1} \frac{(-1)^r}{(c-1)^r} \widetilde{\mathfrak{H}}_r((t_j);(\gamma_j);c) \prod_{k=1}^r t_k^{s_k-1} dt_k\end{aligned}$$

$$= \prod_{k=1}^{r} \frac{1}{(e^{2\pi i s_k}-1)\Gamma(s_k)}$$
$$\times \int_{\mathcal{C}^r} \prod_{j=1}^{r} \lim_{c\to 1} \frac{(-1)}{c-1} \left(\frac{1}{\exp\left(\gamma_j \sum_{k=j}^{r} t_k\right)-1} - \frac{c}{\exp\left(c\gamma_j \sum_{k=j}^{r} t_k\right)-1} \right) \prod_{k=1}^{r} t_k^{s_k-1} dt_k, \quad (1.25)$$

which can be analytically continued to $\mathbb{C}^r$ *as an entire function in* (s_j).

For the proof of (1.25), it is enough to prove that if $|c-1|$ is sufficiently small, then there exists a function $F:\mathcal{C}^r \to \mathbb{R}_{>0}$ independent of c such that

$$|(c-1)^{-r}\widetilde{\mathfrak{H}}_r((t_j);(\gamma_j);c)| \leqslant F((t_j)) \qquad ((t_j)\in\mathcal{C}^r), \tag{1.26}$$

$$\int_{\mathcal{C}^r} F((t_j)) \prod_{k=1}^{r} |t_k^{s_k-1} dt_k| < \infty. \tag{1.27}$$

Now we aim to construct $F((t_j))$ which satisfies these conditions. Let $\mathcal{N}(\varepsilon)=\{z\in\mathbb{C}\mid |z|\leqslant\varepsilon\}$ and $\mathcal{S}(\theta)=\{z\in\mathbb{C}\mid |\arg z|\leqslant\theta\}$.

Let $\gamma_1,\ldots,\gamma_r\in\mathbb{C}$ with $\Re\gamma_j>0 \quad (1\leqslant j\leqslant r)$. Then the following lemma is obvious.

Lemma 1.20. *There exist* $\varepsilon>0$ *and* $0<\theta<\pi/2$ *such that*

$$\gamma_j \sum_{k=j}^{r} t_k \in \mathcal{N}(1)\cup\mathcal{S}(\theta) \tag{1.28}$$

for any $(t_j)\in\mathcal{C}^r$*, where* $\mathcal{C}$ *is the Hankel contour involving a circle around the origin of radius* ε *(see* (1.19)*).*

Further we prove the following lemma.

Lemma 1.21. *Let* $c\in\mathbb{R}\setminus\{1\}$ *satisfying that* $|c-1|$ *is sufficiently small. Then there exists a constant* $A>0$ *independent of* c *such that*

$$|c-1|^{-1}\left|\frac{1}{e^y-1}-\frac{c}{e^{cy}-1}\right| < Ae^{-\Re y/2} \tag{1.29}$$

for any $y\in\mathcal{N}(1)\cup\mathcal{S}(\theta)$.

Proof. It is noted that there exists a constant $C>0$ such that

$$|c-1|^{-1}\left|\frac{1}{e^y-1}-\frac{c}{e^{cy}-1}\right| < C \quad (y\in\mathcal{N}(1)),$$

where we interpret this inequality for $y=0$ as that for $y\to 0$. Also, for any $y\in\mathcal{S}(\theta)\setminus\mathcal{N}(1)$, we have

$$\begin{aligned} |c-1|^{-1}\left|\frac{1}{e^y-1}-\frac{c}{e^{cy}-1}\right| &= |c-1|^{-1}\left|\frac{e^{cy}-ce^y+c-1}{(e^y-1)(e^{cy}-1)}\right| \\ &= |c-1|^{-1}\left|\frac{e^{cy}-e^y+(1-c)(e^y-1)}{(e^y-1)(e^{cy}-1)}\right| \\ &\leqslant |c-1|^{-1}\frac{|e^{cy}-e^y|}{|e^y-1||e^{cy}-1|}+\frac{1}{|e^{cy}-1|}. \end{aligned}$$

Hence it is necessary to estimate

$$|c-1|^{-1}\frac{|e^{cy}-e^y|}{|e^y-1||e^{cy}-1|}.$$

We note that

$$\Big|\frac{e^{ay}-1}{a}\Big| = \Big|\sum_{j=1}^{\infty}\frac{a^{j-1}y^j}{j!}\Big| \leqslant\ |y|\sum_{l=0}^{\infty}\frac{|ay|^l}{l!} \leqslant\ |y|e^{|ay|}.$$

Since $|y| \leqslant \Re y/\cos\theta$, we have

$$\begin{aligned}|c-1|^{-1}\frac{|e^{cy}-e^{y}|}{|e^{y}-1||e^{cy}-1|} &= \frac{1}{|1-e^{-y}||e^{cy}-1|}\frac{|e^{(c-1)y}-1|}{|c-1|}\\ &\leqslant \frac{1}{|1-e^{-y}||e^{cy}-1|}|y|e^{|(c-1)y|}\\ &\leqslant \frac{|y|e^{\Re y(|c-1|/\cos\theta)}}{|1-e^{-y}||e^{cy}-1|}.\end{aligned}$$

Therefore, if $|c-1|$ is sufficiently small, then there exists a constant $A>0$ such that

$$|c-1|^{-1}\frac{|e^{cy}-e^{y}|}{|e^{y}-1||e^{cy}-1|} \leqslant Ae^{-\Re y/2}.$$

This completes the proof. $\square$

Proof of Theorem 1.19. With the notation provided in Lemmas 1.20 and 1.21, we set

$$\begin{aligned}F((t_j)) &= A^r\prod_{j=1}^{r}\exp\left(-\Re(\gamma_j\sum_{k=j}^{r}t_k/2)\right) = A^r\exp\left(-\sum_{j=1}^{r}\Re(\gamma_j\sum_{k=j}^{r}t_k/2)\right)\\ &= A^r\exp\left(-\sum_{k=1}^{r}\Re(t_k(\sum_{j=1}^{k}\gamma_j/2))\right) = A^r\prod_{k=1}^{r}\exp\left(-\Re(t_k(\sum_{j=1}^{k}\gamma_j/2))\right).\end{aligned}$$

Then it is clear that $F((t_j))$ satisfies (1.26) and (1.27). Hence, by Lebesgue's convergence theorem we see that (1.25) holds.

Similarly to the proof of Theorem 1.13, since the contour integral on the right-hand side of (1.25) is holomorphic for all $(s_k)\in\mathbb{C}^r$, we see that $\zeta_r^{\mathrm{des}}((s_j);(\gamma_j))$ can be meromorphically continued to $\mathbb{C}^r$ and its possible singularities are located on hyperplanes $s_k = l_k\in\mathbb{N}$ $(1\leqslant k\leqslant r)$ outside of the region of convergence because $(e^{2\pi i s_k}-1)\Gamma(s_k)$ does not vanish at $s_k\in\mathbb{Z}_{\leqslant 0}$. Furthermore, for $s_k = l_k\in\mathbb{N}$, the integrand of the contour integral with respect to t_k on the right-hand side of (1.25) is holomorphic around $t_k=0$. Therefore, for $l_k\in\mathbb{N}$, we see that

$$\begin{aligned}&\lim_{s_k\to l_k}\int_{\mathcal{C}}\lim_{c\to 1}\frac{(-1)}{c-1}\left(\frac{1}{\exp\left(\gamma_j\sum_{\nu=j}^{r}t_\nu\right)-1}-\frac{c}{\exp\left(c\gamma_j\sum_{\nu=j}^{r}t_\nu\right)-1}\right)t_k^{s_k-1}dt_k\\ &= -\int_{C_\varepsilon}\lim_{c\to 1}\frac{1}{c-1}\left(\frac{1}{\exp\left(\gamma_j\sum_{\nu=j}^{r}t_\nu\right)-1}-\frac{c}{\exp\left(c\gamma_j\sum_{\nu=j}^{r}t_\nu\right)-1}\right)t_k^{l_k-1}dt_k\\ &= 0,\end{aligned}$$

because of the residue theorem, where $C_\varepsilon = \{\varepsilon e^{i\theta}\,|\,0\leqslant\theta\leqslant 2\pi\}$ for any sufficiently small ε. Consequently this implies that $\zeta_r^{\mathrm{des}}((s_j);(\gamma_j))$ has no singularity on $s_k = l_k$, namely $\zeta_r^{\mathrm{des}}((s_j);(\gamma_j))$ is entire. Thus we complete the proof of Theorem 1.19. $\square$

Theorem 1.22. *For $\gamma_1,\ldots,\gamma_r \in \mathbb{C}$ with $\Re\gamma_j > 0 \quad (1 \leqslant j \leqslant r)$,*

$$\begin{aligned}&\prod_{j=1}^{r} \frac{\left(1-\gamma_j\sum_{k=j}^r t_k\right)\exp\left(\gamma_j\sum_{k=j}^r t_k\right)-1}{\left(\exp\left(\gamma_j\sum_{k=j}^r t_k\right)-1\right)^2} \\ &\quad = \sum_{m_1,\ldots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r}\zeta_r^{\mathrm{des}}((-m_j);(\gamma_j))\prod_{j=1}^{r}\frac{t_j^{m_j}}{m_j!}.\end{aligned} \tag{1.30}$$

Hence, for $(k_j) \in \mathbb{N}_0^r$,

$$\begin{aligned}\zeta_r^{\mathrm{des}}((-k_j);(\gamma_j)) &= \prod_{l=1}^{r}(-1)^{k_l}k_l! \\ &\quad \times \sum_{\substack{\nu_{11}\geqslant 0 \\ \nu_{12},\nu_{22}\geqslant 0 \\ \cdots \\ \nu_{1r},\ldots,\nu_{rr}\geqslant 0 \\ \sum_{d=1}^{j}\nu_{dj}=k_j \\ (1\leqslant j\leqslant r)}} \prod_{j=1}^{r}\left(B_{1+\sum_{l=j}^r \nu_{jl}}\gamma_j^{\sum_{l=j}^r\nu_{jl}}\frac{1}{\prod_{d=1}^j \nu_{dj}!}\right).\end{aligned} \tag{1.31}$$

Proof. By (1.15), we have

$$\begin{aligned}&\lim_{c\to 1}\frac{(-1)^r}{(c-1)^r}\widetilde{\mathfrak{H}}_r((t_j)_{j=1}^r;(\gamma_j)_{j=1}^r;c) \\ &= \lim_{c\to 1}\prod_{j=1}^{r}\frac{(-1)}{c-1}\left(\frac{1}{\exp\left(\gamma_j\sum_{k=j}^r t_k\right)-1}-\frac{c}{\exp\left(c\gamma_j\sum_{k=j}^r t_k\right)-1}\right) \\ &= \prod_{j=1}^{r}\frac{\left(1-\gamma_j\sum_{k=j}^r t_k\right)\exp\left(\gamma_j\sum_{k=j}^r t_k\right)-1}{\left(\exp\left(\gamma_j\sum_{k=j}^r t_k\right)-1\right)^2}.\end{aligned}$$

Hence we obtain (1.30) from (1.25). Also, by (1.15), we have

$$\begin{aligned}&\lim_{c\to 1}\frac{(-1)^r}{(c-1)^r}\widetilde{\mathfrak{H}}_r((t_j)_{j=1}^r;(\gamma_j)_{j=1}^r;c) \\ &= \lim_{c\to 1}\prod_{j=1}^{r}\left(\sum_{m_j=1}^{\infty}\frac{c^{m_j}-1}{c-1}B_{m_j}\frac{\left(\gamma_j\sum_{l=j}^r t_l\right)^{m_j-1}}{m_j!}\right) \\ &= \prod_{j=1}^{r}\left(\sum_{m_j=1}^{\infty}B_{m_j}\frac{\left(\gamma_j\sum_{l=j}^r t_l\right)^{m_j-1}}{(m_j-1)!}\right) \\ &= \prod_{j=1}^{r}\left(\sum_{n_j=0}^{\infty}B_{n_j+1}\gamma_j^{n_j}\sum_{\substack{\nu_{jj},\ldots,\nu_{jr}\geqslant 0 \\ \sum_{l=j}^r \nu_{jl}=n_j}}\frac{t_j^{\nu_{jj}}}{\nu_{jj}!}\cdots\frac{t_r^{\nu_{jr}}}{\nu_{jr}!}\right) \\ &= \sum_{\substack{\nu_{11}\geqslant 0 \\ \nu_{12},\nu_{22}\geqslant 0 \\ \cdots \\ \nu_{1r},\nu_{2r},\ldots,\nu_{rr}\geqslant 0}}\prod_{j=1}^{r}\left(B_{1+\sum_{l=j}^r\nu_{jl}}\gamma_j^{\sum_{l=j}^r\nu_{jl}}\frac{t_j^{\sum_{d=1}^j\nu_{dj}}}{\prod_{d=1}^j\nu_{dj}!}\right).\end{aligned}$$

Hence, substituting the above relation into (1.25) and using the residue theorem with

$$\lim_{s\to -k}\left(e^{2\pi is}-1\right)\Gamma(s)=\frac{(2\pi i)(-1)^k}{k!}\qquad(k\in\mathbb{N}_0),$$

we have

$$\begin{aligned}\zeta_r^{\mathrm{des}}((-k_j);(\gamma_j))&=\prod_{l=1}^{r}\frac{(-1)^{k_l}k_l!}{2\pi i}\\&\quad\times(2\pi i)^r\sum_{\substack{\nu_{11}\geqslant 0\\ \nu_{12},\nu_{22}\geqslant 0\\ \cdots\\ \nu_{1r},\dots,\nu_{rr}\geqslant 0\\ \sum_{d=1}^{j}\nu_{dj}=k_j\\ (1\leqslant j\leqslant r)}}\prod_{j=1}^{r}\left(B_{1+\sum_{l=j}^{r}\nu_{jl}}\gamma_j^{\sum_{l=j}^{r}\nu_{jl}}\frac{1}{\prod_{d=1}^{j}\nu_{dj}!}\right).\end{aligned}$$

Thus we obtain the assertion. $\square$

Now we give a certain expression of $\zeta_r^{\mathrm{des}}((s_j);(\gamma_j))$ in terms of $\zeta_r((s_j);(1);(\gamma_j))$ (Theorem 1.23), which can be regarded as multiple versions of $\zeta_1^{\mathrm{des}}(s;1)=(1-s)\zeta(s)$ in the case $r=1$ (see Examples 1.18 and 1.25).

For $s_j\in\mathbb{C}$ with $\Re s_j>1$ $(1\leqslant j\leqslant r)$ and $\gamma_1,\dots,\gamma_r\in\mathbb{C}$ with $\Re\gamma_j>0$ $(1\leqslant j\leqslant r)$, we set

$$\begin{aligned}I_{c,r}(s_1,\dots,s_r;\gamma_1,\dots,\gamma_r)&:=\frac{1}{\prod_{j=1}^{r}\Gamma(s_j)}\int_{[0,\infty)^r}\prod_{j=1}^{r}dt_j\prod_{j=1}^{r}t_j^{s_j-1}\\&\quad\times\prod_{j=1}^{r}\left(\frac{1}{\exp\left(\gamma_j\sum_{k=j}^{r}t_k\right)-1}-\frac{c}{\exp\left(c\gamma_j\sum_{k=j}^{r}t_k\right)-1}\right).\end{aligned}\tag{1.32}$$

From Definition 1.16, we see that

$$\zeta_r^{\mathrm{des}}((s_j);(\gamma_j))=\lim_{c\to 1}\frac{(-1)^r}{(c-1)^r}I_{c,r}((s_j);(\gamma_j)).$$

For indeterminates u_j,v_j $(1\leqslant j\leqslant r)$, we set

$$\mathcal{G}((u_j),(v_j)):=\prod_{j=1}^{r}\Big(1-(u_jv_j+\cdots+u_rv_r)(v_j^{-1}-v_{j-1}^{-1})\Big)\tag{1.33}$$

with the convention $v_0^{-1}=0$, and also define the set of integers $\{a_{\boldsymbol{l},\boldsymbol{m}}\}$ by

$$\mathcal{G}((u_j),(v_j))=\sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r\\ \boldsymbol{m}=(m_j)\in\mathbb{Z}^r\\ \sum_{j=1}^{r}m_j=0}}a_{\boldsymbol{l},\boldsymbol{m}}\prod_{j=1}^{r}u_j^{l_j}v_j^{m_j},\tag{1.34}$$

where the sum on the right-hand side is obviously a finite sum. Note that the condition $\sum_{j=1}^{r}m_j=0$ for the summation indices $\boldsymbol{m}=(m_j)$ can be deduced from the fact that the right-hand side of (1.33) is a homogeneous polynomial of degree 0 in (v_j), namely so is that of (1.34).

Theorem 1.23. *For $\gamma_1,\ldots,\gamma_r\in\mathbb{C}$ with $\Re\gamma_j>0$ $(1\leqslant j\leqslant r)$,*

$$\zeta_r^{\rm des}((s_j);(\gamma_j))=\sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r\\ \boldsymbol{m}=(m_j)\in\mathbb{Z}^r\\ \sum_{j=1}^r m_j=0}} a_{\boldsymbol{l},\boldsymbol{m}}\Big(\prod_{j=1}^r (s_j)_{l_j}\Big)\zeta_r(s_1+m_1,\ldots,s_r+m_r;(1);(\gamma_j)) \tag{1.35}$$

holds for all $(s_j)\in\mathbb{C}^r$, where $(s)_0=1$ and $(s)_k=s(s+1)\cdots(s+k-1)$ $(k\in\mathbb{N})$ are the Pochhammer symbols.

We emphasize here that each term of the right-hand side of (1.35) is meromorphic with infinitely many singularities but taking the above *finite* sum of the shifted functions causes 'miraculous' cancellations all of the *infinitely* many singularities to conclude an entire function.

Remark 1.24. In (1.35), the condition $\sum_{j=1}^r m_j=0$ implies that all zeta-functions appearing on the both sides have the same *weight* $s_1+\cdots+s_r$.

Proof of Theorem 1.23. First we assume that $\Re s_j$ is sufficiently large for $1\leqslant j\leqslant r$. From (1.19) with $(\xi_j)=(1)$, we have

$$\zeta_r((s_j);(1);(\gamma_j))=\frac{1}{\prod_{j=1}^r\Gamma(s_j)}\int_{[0,\infty)^r}\prod_{j=1}^r\frac{t_j^{s_j-1}}{\exp\Big(\gamma_j\sum_{k=j}^r t_k\Big)-1}\prod_{j=1}^r dt_j. \tag{1.36}$$

Using the relation

$$\begin{aligned}&\lim_{c\to1}\frac{(-1)}{c-1}\Big(\frac{1}{e^y-1}-\frac{c}{e^{cy}-1}\Big)\\ &\quad=\frac{-1+e^y-ye^y}{(e^y-1)^2}=\frac{1}{e^y-1}-\frac{ye^y}{(e^y-1)^2}\ =E(y)\ (say),\end{aligned}$$

we have

$$\begin{aligned}\zeta_r^{\rm des}((s_j);(\gamma_j))&=\lim_{c\to1}\frac{(-1)^r}{(c-1)^r}I_{c,r}((s_j);(\gamma_j))\\ &=\lim_{c\to1}\frac{1}{\prod_{j=1}^r\Gamma(s_j)}\int_{[0,\infty)^r}\prod_{j=1}^r dt_j\prod_{j=1}^r t_j^{s_j-1}\\ &\qquad\times\prod_{j=1}^r\frac{(-1)}{c-1}\left(\frac{1}{\exp\Big(\gamma_j\sum_{k=j}^r t_k\Big)-1}-\frac{c}{\exp\Big(c\gamma_j\sum_{k=j}^r t_k\Big)-1}\right)\\ &=\frac{1}{\prod_{j=1}^r\Gamma(s_j)}\int_{[0,\infty)^r}\prod_{j=1}^r dt_j\prod_{j=1}^r t_j^{s_j-1}\prod_{j=1}^r E\Big(\gamma_j\sum_{k=j}^r t_k\Big).\end{aligned} \tag{1.37}$$

We calculate the last product of (1.37). Using the relations

$$\frac{1}{e^y-1}=\sum_{n=1}^\infty e^{-ny},\qquad \frac{e^y}{(e^y-1)^2}=\sum_{n=1}^\infty ne^{-ny},$$

we have, for $J\subset\{1,\ldots,r\}$,

$$\int_{[0,\infty)^r}\prod_{j=1}^r dt_j\prod_{j=1}^r t_j^{s_j-1}\prod_{j\notin J}\frac{1}{\exp\Big(\gamma_j\sum_{k=j}^r t_k\Big)-1}\prod_{j\in J}\frac{\Big(\gamma_j\sum_{k=j}^r t_k\Big)\exp\Big(\gamma_j\sum_{k=j}^r t_k\Big)}{\Big(\exp\Big(\gamma_j\sum_{k=j}^r t_k\Big)-1\Big)^2}$$

$$
\begin{aligned}
&= \int_{[0,\infty)^r} \prod_{j=1}^r dt_j \prod_{j=1}^r t_j^{s_j-1} \prod_{j\notin J} \sum_{n_j=1}^\infty \exp\Bigl(-n_j\gamma_j \sum_{k=j}^r t_k\Bigr) \\
&\qquad \times \prod_{j\in J} \sum_{n_j=1}^\infty n_j \exp\Bigl(-n_j\gamma_j \sum_{k=j}^r t_k\Bigr) \prod_{j\in J}\Bigl(\gamma_j \sum_{k=j}^r t_k\Bigr) \\
&= \sum_{n_1,\ldots,n_r\geq 1} \Bigl(\prod_{j\in J} n_j\gamma_j\Bigr) \int_{[0,\infty)^r} \prod_{j=1}^r dt_j \prod_{j=1}^r t_j^{s_j-1} \prod_{j=1}^r \exp\Bigl(-t_j \sum_{k=1}^j n_k\gamma_k\Bigr) \prod_{j\in J}\Bigl(\sum_{k=j}^r t_k\Bigr) \\
&= \sum_{n_1,\ldots,n_r\geq 1} \Bigl(\prod_{j\in J}\Bigl(\sum_{k=1}^j n_k\gamma_k - \sum_{k=1}^{j-1} n_k\gamma_k\Bigr)\Bigr) \\
&\qquad \times \int_{[0,\infty)^r} \prod_{j=1}^r dt_j \prod_{j=1}^r t_j^{s_j-1} \prod_{j=1}^r \exp\Bigl(-t_j \sum_{k=1}^j n_k\gamma_k\Bigr) \prod_{j\in J}\Bigl(\sum_{k=j}^r t_k\Bigr) \\
&= \sum_{\boldsymbol{l}\in\mathbb{N}_0^r} b_{J,\boldsymbol{l}} \sum_{n_1,\ldots,n_r\geq 1} \Bigl(\prod_{j\in J}\Bigl(\sum_{k=1}^j n_k\gamma_k - \sum_{k=1}^{j-1} n_k\gamma_k\Bigr)\Bigr) \\
&\qquad \times \int_{[0,\infty)^r} \prod_{j=1}^r dt_j \prod_{j=1}^r t_j^{s_j+l_j-1} \prod_{j=1}^r \exp\Bigl(-t_j \sum_{k=1}^j n_k\gamma_k\Bigr) \\
&= \sum_{\boldsymbol{l}\in\mathbb{N}_0^r} b_{J,\boldsymbol{l}} \sum_{n_1,\ldots,n_r\geq 1} \Bigl(\prod_{j\in J}\Bigl(\sum_{k=1}^j n_k\gamma_k - \sum_{k=1}^{j-1} n_k\gamma_k\Bigr)\Bigr) \prod_{j=1}^r \Gamma(s_j+l_j) \frac{1}{\bigl(\sum_{k=1}^j n_k\gamma_k\bigr)^{s_j+l_j}} \\
&= \sum_{\boldsymbol{l}\in\mathbb{N}_0^r} b_{J,\boldsymbol{l}} \prod_{j=1}^r \Gamma(s_j+l_j) \sum_{n_1,\ldots,n_r\geq 1} \sum_{K\subset J\setminus\{1\}} (-1)^{|K|} \prod_{j=1}^r \frac{1}{\bigl(\sum_{k=1}^j n_k\gamma_k\bigr)^{s_j+l_j-\delta_{j\in J\setminus K}-\delta_{j+1\in K}}} \\
&= \sum_{\boldsymbol{l}\in\mathbb{N}_0^r} b_{J,\boldsymbol{l}} \sum_{K\subset J\setminus\{1\}} (-1)^{|K|} \Bigl(\prod_{j=1}^r \Gamma(s_j+l_j)\Bigr) \zeta_r((s_j+l_j-\delta_{j\in J\setminus K}-\delta_{j+1\in K});(1);(\gamma_j)), \qquad (1.38)
\end{aligned}
$$

where $|K|$ implies the number of elements of K,

$$
\delta_{i\in I} = \begin{cases} 1 & (i\in I) \\ 0 & (i\notin I) \end{cases}
$$

for $I\subset J$, and

$$
\prod_{j\in J}\Bigl(\sum_{k=j}^r t_k\Bigr) = \sum_{\boldsymbol{l}\in\mathbb{N}_0^r} b_{J,\boldsymbol{l}} \prod_{j=1}^r t_j^{l_j}. \qquad (1.39)
$$

Hence, by (1.37) we have

$$
\begin{aligned}
&\zeta_r^{\mathrm{des}}((s_j);(\gamma_j)) \\
&= \sum_{J\subset\{1,\ldots,r\}} (-1)^{|J|} \sum_{\boldsymbol{l}\in\mathbb{N}_0^r} b_{J,\boldsymbol{l}} \sum_{K\subset J\setminus\{1\}} (-1)^{|K|} \Bigl(\prod_{j=1}^r \frac{\Gamma(s_j+l_j)}{\Gamma(s_j)}\Bigr) \zeta_r((s_j+l_j-\delta_{j\in J\setminus K}-\delta_{j+1\in K});(1);(\gamma_j)) \\
&= \sum_{J\subset\{1,\ldots,r\}} \sum_{K\subset J\setminus\{1\}} (-1)^{|J\setminus K|} \sum_{\boldsymbol{l}\in\mathbb{N}_0^r} b_{J,\boldsymbol{l}} \Bigl(\prod_{j=1}^r (s_j)_{l_j}\Bigr) \zeta_r((s_j+l_j-\delta_{j\in J\setminus K}-\delta_{j+1\in K});(1);(\gamma_j)).
\end{aligned}
\qquad (1.40)
$$

Finally we set

$$H((u_j),(v_j)) := \sum_{J\subset\{1,\ldots,r\}} \sum_{K\subset J\setminus\{1\}} (-1)^{|J\setminus K|} \sum_{\boldsymbol{l}\in\mathbb{N}_0^r} b_{J,\boldsymbol{l}} \prod_{j=1}^r u_j^{l_j} v_j^{l_j-\delta_{j\in J\setminus K}-\delta_{j+1\in K}}$$

and aim to prove that

$$\mathcal{G}((u_j),(v_j)) = H((u_j),(v_j)). \tag{1.41}$$

It follows from (1.39) that

$$\begin{aligned} H((u_j),(v_j)) &= \sum_{J\subset\{1,\ldots,r\}} \sum_{K\subset J\setminus\{1\}} (-1)^{|J\setminus K|} \Bigl(\prod_{j\in J}\sum_{k=j}^r u_k v_k\Bigr) \prod_{j=1}^r v_j^{-\delta_{j\in J\setminus K}-\delta_{j+1\in K}} \\ &= \sum_{J\subset\{1,\ldots,r\}} \Bigl(\prod_{j\in J}\sum_{k=j}^r u_k v_k\Bigr) \sum_{K\subset J\setminus\{1\}} \prod_{j\in J\setminus K} (-v_j^{-1}) \prod_{j\in K} v_{j-1}^{-1}. \end{aligned}$$

Since $v_0^{-1}=0$, we have

$$\sum_{K\subset J\setminus\{1\}} \prod_{j\in J\setminus K} (-v_j^{-1}) \prod_{j\in K} v_{j-1}^{-1} = \prod_{j\in J}(-v_j^{-1}+v_{j-1}^{-1}).$$

Hence we obtain

$$\begin{aligned} H((u_j),(v_j)) &= \sum_{J\subset\{1,\ldots,r\}} \prod_{j\in J} \Bigl(\sum_{k=j}^r u_k v_k\Bigr)(-v_j^{-1}+v_{j-1}^{-1}) \\ &= \prod_{j=1}^r \Bigl(\Bigl(\sum_{k=j}^r u_k v_k\Bigr)(-v_j^{-1}+v_{j-1}^{-1})+1\Bigr) \\ &= \prod_{j=1}^r \Bigl(1-\Bigl(\sum_{k=j}^r u_k v_k\Bigr)(v_j^{-1}-v_{j-1}^{-1})\Bigr) = \mathcal{G}((u_j),(v_j)). \end{aligned} \tag{1.42}$$

Combining (1.34), (1.40) and (1.42), and regarding $(s_j)_{l_j}$ and $\zeta_r((s_j+l_j-\delta_{j\in J\setminus K}-\delta_{j+1\in K});(1);(\gamma_j))$ as indeterminates $u_j^{l_j}$ and $v_j^{l_j}$, we see that (1.35) holds when $\Re s_j$ is sufficiently large for $1\leqslant j\leqslant r$. It is known that each function on the right-hand side can be continued meromorphically to $\mathbb{C}^r$ (see [37, Theorem 1]). Since $\zeta_r^{\rm des}((s_j);(\gamma_j))$ is entire, we see that (1.35) holds for all $(s_j)\in\mathbb{C}^r$. Thus we complete the proof of Theorem 1.23. $\square$

Example 1.25. In the case $r=1$ and $\gamma_1=1$, we have

$$\mathcal{G}(u_1,v_1) = 1-u_1v_1v_1^{-1} = 1-u_1,$$

namely, $a_{0,0}(1)=1$ and $a_{1,0}(1)=-1$. Hence we have

$$\zeta_1^{\rm des}(s;1) = \lim_{c\to1}\frac{(-1)}{c-1}I_{c,1}(s) = (s)_0\zeta_1(s;1;1)-(s)_1\zeta_1(s;1;1) = (1-s)\zeta(s),$$

which coincides with (1.24). Note that $\zeta_1^{\rm des}(-k;1)=(-1)^kB_{k+1}$ $(k\in\mathbb{N}_0)$.

Example 1.26. In the case $r=2$, we can easily check that

$$\begin{aligned} \mathcal{G}((u_j),(v_j)) &= (1-(u_1v_1+u_2v_2)v_1^{-1})(1-u_2v_2(v_2^{-1}-v_1^{-1})) \\ &= (1-u_1)(1-u_2)+(u_2^2-u_1u_2)v_1^{-1}v_2-u_2^2v_1^{-2}v_2^2. \end{aligned}$$

Then (1.35) implies that

$$\begin{aligned}\zeta_2^{\mathrm{des}}(s_1,s_2;\gamma_1,\gamma_2)&=(s_1-1)(s_2-1)\zeta_2(s_1,s_2;(1);\gamma_1,\gamma_2)\\&\quad+s_2(s_2+1-s_1)\zeta_2(s_1-1,s_2+1;(1);\gamma_1,\gamma_2)\\&\quad-s_2(s_2+1)\zeta_2(s_1-2,s_2+2;(1);\gamma_1,\gamma_2).\end{aligned}\tag{1.43}$$

Let $k,l\in\mathbb{N}_0$. By (1.31), we obtain

$$\zeta_2^{\mathrm{des}}(-k,-l;\gamma_1,\gamma_2)=(-1)^{k+l}\sum_{\nu=0}^{l}\binom{l}{\nu}B_{k+\nu+1}B_{l-\nu+1}\gamma_1^{k+\nu}\gamma_2^{l-\nu}.\tag{1.44}$$

Remark 1.27. Setting $(\gamma_1,\gamma_2)=(1,1)$ in (1.43), we obtain

$$\begin{aligned}\zeta_2^{\mathrm{des}}(s_1,s_2;1,1)&=(s_1-1)(s_2-1)\zeta_2(s_1,s_2)\\&\quad+s_2(s_2+1-s_1)\zeta_2(s_1-1,s_2+1)-s_2(s_2+1)\zeta_2(s_1-2,s_2+2).\end{aligned}\tag{1.45}$$

From Theorem 1.19, we see that $\zeta_2^{\mathrm{des}}(s_1,s_2;1,1)$ on the left-hand side of (1.45) is entire, though each double zeta-function (defined by (0.3)) on the right-hand side of (1.45) has infinitely many singularities (see Theorem 1.1). In fact, we can explicitly write $\zeta_2^{\mathrm{des}}(-m,-n;1,1)$ in terms of Bernoulli numbers by (1.44), though the values of $\zeta_2(s_1,s_2)$ at non-positive integers (except for regular points) cannot be determined uniquely because they are irregular singularities (see [1]).

Example 1.28. In the case $r=3$, we can see that

$$\begin{aligned}\mathcal{G}((u_j),(v_j))&=(1-(u_1v_1+u_2v_2+u_3v_3)v_1^{-1})(1-(u_2v_2+u_3v_3)(v_2^{-1}-v_1^{-1}))\\&\qquad\times(1-u_3v_3(v_3^{-1}-v_2^{-1}))\\&=-(u_1-1)(u_2-1)(u_3-1)+(u_1-1)(u_2-u_3)u_3v_2^{-1}v_3\\&\quad+(u_1-1)u_3^2v_2^{-2}v_3^2+(u_1-u_2)u_2(u_3-1)v_1^{-1}v_2\\&\quad+u_3(-u_1+2u_2-u_1u_2+u_2^2+u_1u_3-2u_2u_3)v_1^{-1}v_3\\&\quad-u_3^2(-1+u_1-2u_2+u_3)v_1^{-1}v_2^{-1}v_3^2+u_3^3v_1^{-1}v_2^{-2}v_3^3\\&\quad+u_2^2(u_3-1)v_1^{-2}v_2^2-u_2(2+u_2-2u_3)u_3v_1^{-2}v_2v_3\\&\quad+u_3^2(-1-2u_2+u_3)v_1^{-2}v_3^2-u_3^3v_1^{-2}v_2^{-1}v_3^3.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}&\zeta_3^{\mathrm{des}}(s_1,s_2,s_3;\gamma_1,\gamma_2,\gamma_3)\\&\quad=-(s_1-1)(s_2-1)(s_3-1)\zeta_3(s_1,s_2,s_3;(1);\gamma_1,\gamma_2,\gamma_3)\\&\qquad+(s_1-1)(-1+s_2-s_3)s_3\zeta_3(s_1,s_2-1,s_3+1;(1);\gamma_1,\gamma_2,\gamma_3)\\&\qquad+(s_1-1)s_3(s_3+1)\zeta_3(s_1,s_2-2,s_3+2;(1);\gamma_1,\gamma_2,\gamma_3)\\&\qquad+(-1+s_1-s_2)s_2(s_3-1)\zeta_3(s_1-1,s_2+1,s_3;(1);\gamma_1,\gamma_2,\gamma_3)\\&\qquad+s_3(s_2-s_1s_2+s_2^2+s_1s_3-2s_2s_3)\zeta_3(s_1-1,s_2,s_3+1;(1);\gamma_1,\gamma_2,\gamma_3)\\&\qquad-s_3(s_3+1)(1+s_1-2s_2+s_3)\zeta_3(s_1-1,s_2-1,s_3+2;(1);\gamma_1,\gamma_2,\gamma_3)\\&\qquad+s_3(s_3+1)(s_3+2)\zeta_3(s_1-1,s_2-2,s_3+3;(1);\gamma_1,\gamma_2,\gamma_3)\end{aligned}$$

$$
\begin{aligned}
&+ s_2(s_2+1)(s_3-1)\zeta_3(s_1-2, s_2+2, s_3; (1); \gamma_1, \gamma_2, \gamma_3)\\
&- s_2(1+s_2-2s_3)s_3\zeta_3(s_1-2, s_2+1, s_3+1; (1); \gamma_1, \gamma_2, \gamma_3)\\
&+ s_3(s_3+1)(1-2s_2+s_3)\zeta_3(s_1-2, s_2, s_3+2; (1); \gamma_1, \gamma_2, \gamma_3)\\
&- s_3(s_3+1)(s_3+2)\zeta_3(s_1-2, s_2-1, s_3+3; (1); \gamma_1, \gamma_2, \gamma_3).
\end{aligned}
$$

Let $k, l, m \in \mathbb{N}_0$. By (1.31), we have

$$
\begin{aligned}
\zeta_3^{\mathrm{des}}(-k, -l, -m; \gamma_1, \gamma_2, \gamma_3) = (-1)^{k+l+m} \sum_{\nu=0}^{m} \sum_{\rho=0}^{m-\nu} \sum_{\kappa=0}^{l} \binom{l}{\kappa} \binom{m}{\nu\ \rho}&\\
\times B_{k+\nu+\kappa+1} B_{l-\kappa+\rho+1} B_{m-\nu-\rho+1} \gamma_1^{k+\nu+\kappa+1} \gamma_2^{l-\kappa+\rho+1} \gamma_3^{m-\nu-\rho+1}&,
\end{aligned}
$$

where $\binom{m}{\nu\ \rho} = \frac{m!}{\nu!\,\rho!\,(m-\nu-\rho)!}$.

Remark 1.29. Our desingularization method in this paper is for multiple zeta-functions of the generalized Euler-Zagier type (0.1). In our forthcoming paper [24], we will extend our desingularization method into more general multiple series.

Remark 1.30. Arakawa and Kaneko [4] defined an entire function $\xi_k(s)$ associated with poly-Bernoulli numbers $\{B_n^{(k)}\}$ mentioned in Remark 1.12. It is known that, for example,

$$
\begin{aligned}
\xi_1(s) &= s\zeta(s+1),\\
\xi_2(s) &= -\zeta_2(2, s) + s\zeta_2(1, s+1) + \zeta(2)\zeta(s).
\end{aligned}
$$

Hence, from the fact $\xi_1(s) = -\zeta_1^{\mathrm{des}}(s+1; 1)$ (see (1.24)), it seems quite interesting if we acquire explicit relations between $\xi_k(s)$ and $\zeta_k^{\mathrm{des}}((s_j); (1))$.

Related to Connes-Kreimer's renormalization procedure in quantum field theory, Guo and Zhang [25] and Manchon and Paycha [36] introduced methods using certain Hopf algebras to give well-defined special values of the multiple zeta-functions at non-positive integers.

Example 1.31. According to their computation table (in loc.cit.), Guo-Zhang's renormalized value $\zeta_2^{\mathrm{GZ}}(0, -2)$ of $\zeta_2(s_1, s_2)$ at its singularity $(s_1, s_2) = (0, -2)$ is

$$
\zeta_2^{\mathrm{GZ}}(0, -2) = \frac{1}{120},
$$

while Manchon-Paycha's value $\zeta_2^{\mathrm{MP}}(0, -2)$ is

$$
\zeta_2^{\mathrm{MP}}(0, -2) = \frac{7}{720}.
$$

On the other hand, our desingularized method gives

$$
\zeta_2^{\mathrm{des}}(0, -2) = \frac{1}{18}.
$$

Several methods yield several different values.

Question 1.32. Are there any relationships between our desingularization method and their renormalization methods?

We emphasize that since our $\zeta_r^{\mathrm{des}}((s_j); (1))$ is entire, their special values at integers points which are neither all positive nor all non-positive are well-determined. These values might be also worthy to study.

Example 1.33. As explicit examples which we mentioned above, we can give

$$\begin{aligned}
\zeta_2^{\rm des}(-1,1) &= \frac{1}{8},\\
\zeta_2^{\rm des}(-1,4) &= \zeta(3)-\zeta(4),\\
\zeta_2^{\rm des}(3,-3) &= \frac{3}{4}-\frac{1}{15}\zeta(3),\\
\zeta_2^{\rm des}(4,-3) &= \frac{1}{2}+\frac{1}{2}\zeta(2)-\frac{1}{10}\zeta(4).
\end{aligned}$$

Also we can give the following examples for non-admissible indices:

$$\begin{aligned}
\zeta_2^{\rm des}(2,1) &= -\zeta(2)+2\zeta(3),\\
\zeta_2^{\rm des}(3,1) &= 2\zeta(3)-\frac{5}{4}\zeta(4).
\end{aligned}$$

For the details, see our forthcoming paper [24].

2. p-adic multiple L-functions

In this section, based on the consideration in the previous section, we will construct certain p-adic multiple L-functions in Definition 2.9 which can be regarded as multiple analogues of the Kubota-Leopoldt p-adic L-functions and also p-adic analogues of complex multiple L-(zeta-)functions. We will investigate several p-adic properties of them, in particular, a p-adic analyticity on certain variables in Theorem 2.14 and a p-adic continuity on other variables in Theorem 2.17.

2.1. Kubota-Leopoldt p-adic L-functions.

Here we will review of the Kubota-Leopoldt p-adic L-functions.

First we prepare ordinary notation. For a prime number p, let $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\overline{\mathbb{Q}}_p$, $\mathbb{C}_p$, $\mathcal{O}_{\mathbb{C}_p}$ and $\mathfrak{M}_{\mathbb{C}_p}$ be the set of p-adic integers, p-adic numbers, the algebraic closure of $\mathbb{Q}_p$, the p-adic completion of $\overline{\mathbb{Q}}_p$, the ring of integers of $\mathbb{C}_p$ and its maximal ideal. Fixing embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$, we identify $\overline{\mathbb{Q}}$ simultaneously as a subset of $\mathbb{C}$ and $\mathbb{C}_p$. Denote by $|\cdot|_p$ the p-adic absolute value, and by μ_c the group of cth roots of unity in $\mathbb{C}_p$ for $c \in \mathbb{N}$. Let $\mathbf{P}^1(\mathbb{C}_p) = \mathbb{C}_p \cup \{\infty\}$ with $|\infty|_p = \infty$.

Let $\mathbb{Z}_p^\times$ be the unit group in $\mathbb{Z}_p$ and

$$W := \begin{cases} \{\pm 1\} & (\text{if } p=2) \\ \{\xi \in \mathbb{Z}_p^\times \mid \xi^{p-1}=1\} & (\text{if } p \geqslant 3), \end{cases}$$

$$q := \begin{cases} 4 & (\text{if } p=2) \\ p & (\text{if } p \geqslant 3). \end{cases} \tag{2.1}$$

Then it is well-known that W forms a set of complete representatives of $(\mathbb{Z}_p/q\mathbb{Z}_p)^\times$, and

$$\mathbb{Z}_p^\times = W \times (1+q\mathbb{Z}_p),$$

where we denote this correspondence by $x = \omega(x)\cdot\langle x\rangle$ for $x \in \mathbb{Z}_p^\times$ (see [29, Section 3]). Noting $\omega(m) \equiv m \pmod{q\mathbb{Z}_p}$ for $(m,q)=1$, we see that

$$\omega \,:\, (\mathbb{Z}/q\mathbb{Z})^\times \to W \tag{2.2}$$

is the primitive Dirichlet character of conductor q, which is called the *Teichmüller character*.

Now we recall the Kubota-Leopoldt p-adic L-function which is defined by use of p-adic integral with respect to p-adic measure (for the details, see [32, Chapter 2]; also [31], [46, Chapter 5]). Following Koblitz [32, Chapter 2] we consider a p-adic measure $\mathfrak{m}_z$ on $\mathbb{Z}_p$ for $z \in \mathbf{P}^1(\mathbb{C}_p)$ with $|z-1|_p \geqslant 1$ by

$$\mathfrak{m}_z\left(j+p^N\mathbb{Z}_p\right) := \frac{z^j}{1-z^{p^N}} \quad \in \mathcal{O}_{\mathbb{C}_p} \quad (0 \leqslant j \leqslant p^N-1). \tag{2.3}$$

Corresponding to this measure we can define the p-adic integral

$$\int_{\mathbb{Z}_p} f(x)d\mathfrak{m}_z(x) := \lim_{N\to\infty}\sum_{a=0}^{p^N-1} f(a)\mathfrak{m}_z\left(a+p^N\mathbb{Z}_p\right) \quad \in \mathbb{C}_p$$

for any continuous function $f \,:\, \mathbb{Z}_p \to \mathbb{C}_p$. The p-adic integral over a compact open subset U of $\mathbb{Z}_p$ is defined by

$$\int_U f(x)d\mathfrak{m}_z(x) := \int_{\mathbb{Z}_p} f(x)\chi_U(x)d\mathfrak{m}_z(x) \quad \in \mathbb{C}_p$$

where $\chi_U(x)$ is the characteristic function of U.

For an arbitrary $t \in \mathbb{C}_p$ with $|t|_p < p^{-1/(p-1)}$, we see that e^{tx} is a continuous function in $x \in \mathbb{Z}_p$. We obtain the following proposition by regarding (1.4) as the Maclaurin expansion of $\mathfrak{H}(t;\xi)$ in $\mathbb{C}_p$ and using Koblitz' result [32, Chapter 2, Equations (1.4), (2.4) and (2.5)].

Proposition 2.1. *For any primitive root of unity $\xi \neq 1 \in \mathbb{C}_p$ of order prime to p,*

$$\mathfrak{H}(t;\xi) = \int_{\mathbb{Z}_p} e^{tx}d\mathfrak{m}_\xi(x) \quad \in \mathbb{C}_p[[t]]. \tag{2.4}$$

Therefore

$$\int_{\mathbb{Z}_p} x^n d\mathfrak{m}_\xi(x) = \mathfrak{B}_n(\xi) \qquad (n \in \mathbb{N}_0). \tag{2.5}$$

For any $c \in \mathbb{N}_{>1}$ with $(c,p)=1$, we consider

$$\widetilde{\mathfrak{m}}_c\left(j+p^N\mathbb{Z}_p\right) := \sum_{\substack{\xi\in\mathbb{C}_p\setminus\{1\}\\ \xi^c=1}} \mathfrak{m}_\xi\left(j+p^N\mathbb{Z}_p\right), \tag{2.6}$$

which is a p-adic measure on $\mathbb{Z}_p$ introduced by Mazur (see [32, Chapter 2]). Considering the Galois action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, we see that $\widetilde{\mathfrak{m}}_c$ is a $\mathbb{Q}_p$-valued measure, therefore a $\mathbb{Z}_p$-valued measure.

Note that

$$\begin{aligned}\widetilde{\mathfrak{m}}_c\left(pj+p^N\mathbb{Z}_p\right) &= \sum_{\substack{\xi\in\mathbb{C}_p\setminus\{1\}\\ \xi^c=1}} \frac{\xi^{pj}}{1-\xi^{p^N}} \\ &= \sum_{\substack{\rho\in\mathbb{C}_p\setminus\{1\}\\ \rho^c=1}} \frac{\rho^j}{1-\rho^{p^{N-1}}} = \widetilde{\mathfrak{m}}_c\left(j+p^{N-1}\mathbb{Z}_p\right),\end{aligned} \tag{2.7}$$

because $(c,p)=1$. From (1.8) and (2.5), we have the following.

Lemma 2.2 (cf. [32, Chapter 2, (2.6)]). *For any $c \in \mathbb{N}_{>1}$ with $(c,p)=1$,*

$$\int_{\mathbb{Z}_p} x^n d\widetilde{\mathfrak{m}}_c(x) = \left(c^{n+1}-1\right)\zeta(-n) = \left(1-c^{n+1}\right)\frac{B_{n+1}}{n+1} \in \mathbb{Q} \quad (n \in \mathbb{N}), \tag{2.8}$$

and

$$\int_{\mathbb{Z}_p} 1\, d\widetilde{\mathfrak{m}}_c(x) = \frac{c-1}{2} \in \mathbb{Z}_p. \tag{2.9}$$

Note that c is odd when $p=2$.

Here we define the **Kubota-Leopoldt p-adic L-function**.

Definition 2.3 ([32, Chapter 2, (4.5)]). For $k \in \mathbb{Z}$ and $c \in \mathbb{N}_{>1}$ with $(c,p)=1$,

$$L_p(s;\omega^k) := \frac{1}{\langle c\rangle^{1-s}\omega^k(c)-1}\int_{\mathbb{Z}_p^\times} \langle x\rangle^{-s}\omega^{k-1}(x)d\widetilde{\mathfrak{m}}_c(x) \quad \in \mathbb{C}_p$$

for $s \in \mathbb{C}_p$ with $|s|_p < qp^{-1/(p-1)}$.

Remark 2.4. We note that $\langle x\rangle^{-s}$ can be defined as an $\mathcal{O}_{\mathbb{C}_p}$-valued rigid-analytic function (cf. Notation 4.1 below) in $s \in \mathbb{C}_p$ with $|s|_p < qp^{-1/(p-1)}$ (see [46, p. 54]).

Proposition 2.5 ([32, Chapter 2, Theorem], [46, Theorem 5.11 and Corollary 12.5]). *For $k \in \mathbb{Z}$, $L_p(s;\omega^k)$ is rigid-analytic in the sense of Notation 4.1 below (except for a pole at $s=1$ when $k \equiv 0$ mod $p-1$) and satisfies that*

$$\begin{aligned} L_p(1-m;\omega^k) &= \left(1-\omega^{k-m}(p)p^{m-1}\right) L\left(1-m,\omega^{k-m}\right) \\ &= -\left(1-\omega^{k-m}(p)p^{m-1}\right)\frac{B_{m,\omega^{k-m}}}{m} \quad (m \in \mathbb{N}). \end{aligned} \tag{2.10}$$

We know that $B_{m,\omega^{k-m}}=0$ $(m \in \mathbb{N})$ when k is odd. Hence we obtain the following.

Proposition 2.6 ([46, p.57, Remark]). *When k is any odd integer, $L_p(s;\omega^k)$ is the zero-function.*

Remark 2.7. The above fact will be generalized to the multiple case as functional relations between the p-adic multiple L-functions defined in the following section (see Theorem 3.15 and Example 3.19).

2.2. **Construction and properties of p-adic multiple L-functions.** We defined a certain p-adic double L-function associated with the Teichmüller character for any odd prime p in [34]. In this section, as its generalization with slight modification, we will introduce a p-adic multiple L-functions $L_{p,r}(s_1,\ldots,s_r;\omega^{k_1},\ldots,\omega^{k_r};\gamma_1,\ldots,\gamma_r;c)$ in Definition 2.9 and investigate their basic properties: the p-adic rigid analyticity with respect to the parameter $s_1,\ldots,s_r$ (Theorem 2.14) and the p-adic continuity with respect to the parameter c (Theorem 2.17) which yields a non-trivial p-adic property of their special values (Corollary 2.23). At the end of this subsection we will give a discussion toward a construction of a p-adic analogue of the entire zeta-function $\zeta_r^{\rm des}(s_1,\ldots,s_r;\gamma_1,\ldots,\gamma_r)$ previously constructed in Subsection 1.4.

Let $r \in \mathbb{N}$ and $\gamma_1, \ldots, \gamma_r \in \mathbb{Z}_p$. Let $\xi_1, \ldots, \xi_r \in \mathbb{C}_p$ be roots of unity other than 1 whose orders are prime to p.

Then, combining (2.4) and (1.12), we have

$$\begin{aligned}
\mathfrak{H}_r((t_j),(\xi_j),(\gamma_j)) &= \prod_{j=1}^{r} \int_{\mathbb{Z}_p} e^{\gamma_j(\sum_{k=j}^{r} t_k)x_j} d\mathfrak{m}_{\xi_j}(x_j) \\
&= \int_{\mathbb{Z}_p^r} \exp\left(\sum_{k=1}^{r}\left(\sum_{j=1}^{k} x_j\gamma_j\right) t_k\right) \prod_{j=1}^{r} d\mathfrak{m}_{\xi_j}(x_j) \\
&= \sum_{n_1,\ldots,n_r=0}^{\infty} \int_{\mathbb{Z}_p^r} \prod_{k=1}^{r}\left(\sum_{j=1}^{k} x_j\gamma_j\right)^{n_k} \prod_{j=1}^{r} d\mathfrak{m}_{\xi_j}(x_j) \frac{t_1^{n_1}\cdots t_r^{n_r}}{n_1!\cdots n_r!}.
\end{aligned}$$

Hence as a multiple analogue of Proposition 2.1 we have the following.

Proposition 2.8. *For $n_1, \ldots, n_r \in \mathbb{N}_0$, $\gamma_1, \ldots, \gamma_r \in \mathbb{Z}_p$ and roots $\xi_1, \ldots, \xi_r \in \mathbb{C}_p^\times \setminus \{1\}$ of unity whose orders are prime to p,*

$$\int_{\mathbb{Z}_p^r} \prod_{k=1}^{r}(\sum_{j=1}^{k} x_j\gamma_j)^{n_k} \prod_{j=1}^{r} d\mathfrak{m}_{\xi_j}(x_j) = \mathfrak{B}((n_j);(\xi_j);(\gamma_j)). \tag{2.11}$$

We set

$$\left(\mathbb{Z}_p^r\right)'_{\{\gamma_j\}} := \left\{ (x_j) \in \mathbb{Z}_p^r \;\middle|\; p \nmid x_1\gamma_1,\ p \nmid (x_1\gamma_1 + x_2\gamma_2), \ldots,\ p \nmid \sum_{j=1}^{r} x_j\gamma_j \right\}, \tag{2.12}$$

and

$$\mathfrak{X}_r(d) := \left\{ (s_1, \ldots, s_r) \in \mathbb{C}_p^r \;\middle|\; |s_j|_p < d^{-1}p^{-1/(p-1)}\ (1 \leqslant j \leqslant r) \right\} \tag{2.13}$$

for $d \in \mathbb{R}_{>0}$.

Definition 2.9. For $r \in \mathbb{N}$, $k_1, \ldots, k_r, \in \mathbb{Z}$, $\gamma_1, \ldots, \gamma_r \in \mathbb{Z}_p$ and $c \in \mathbb{N}_{>1}$ with $(c,p) = 1$, the **p-adic multiple L-function** of depth r is the $\mathbb{C}_p$-valued function on $(s_j) \in \mathfrak{X}_r\left(q^{-1}\right)$ defined by

$$\begin{aligned}
&L_{p,r}(s_1, \ldots, s_r; \omega^{k_1}, \ldots, \omega^{k_r}; \gamma_1, \ldots, \gamma_r; c) \\
&:= \int_{\left(\mathbb{Z}_p^r\right)'_{\{\gamma_j\}}} \langle x_1\gamma_1\rangle^{-s_1}\langle x_1\gamma_1 + x_2\gamma_2\rangle^{-s_2}\cdots\langle\sum_{j=1}^{r} x_j\gamma_j\rangle^{-s_r} \\
&\qquad \times \omega^{k_1}(x_1\gamma_1)\cdots\omega^{k_r}(\sum_{j=1}^{r} x_j\gamma_j)\prod_{j=1}^{r} d\widetilde{\mathfrak{m}}_c(x_j).
\end{aligned} \tag{2.14}$$

Remark 2.10. In the case $\gamma_1 \in p\mathbb{Z}_p$, we see that $\left(\mathbb{Z}_p^r\right)'_{\{\gamma_j\}}$ is an empty set, hence we regard $L_{p,r}((s_j);(\omega^{k_j});(\gamma_j);c)$ as the zero-function.

Remark 2.11. Note that we can define, more generally,

$$L_{p,r}(s_1, \ldots, s_r; (\omega^{k_j}); (\gamma_j); (c_j)) := \int_{\left(\mathbb{Z}_p^r\right)'_{\{\gamma_j\}}} \langle x_1\gamma_1\rangle^{-s_1}\langle x_1\gamma_1 + x_2\gamma_2\rangle^{-s_2}\cdots\langle\sum_{j=1}^{r} x_j\gamma_j\rangle^{-s_r}$$

$$\times\,\omega^{k_1}(x_1\gamma_1)\cdots\omega^{k_r}(\sum_{j=1}^{r}x_j\gamma_j)\prod_{j=1}^{r}d\widetilde{\mathfrak{m}}_{c_j}(x_j)$$

for $c_j \in \mathbb{N}_{>1}$ with $(c_j,p)=1$ $(1\leqslant j\leqslant r)$. Then we can naturally generalize the following arguments in the remaining sections.

In the next section, we will see that the above p-adic multiple L-function can be seen as a p-adic interpolation of a certain finite sum (3.2) of complex multiple zeta functions (cf. Theorem 3.1 and Remark 3.2).

The following example shows that the Kubota-Leopoldt p-adic L-function is essentially a special case of our p-adic multiple L-function with $r=1$.

Example 2.12. For $s\in\mathbb{C}_p$ with $|s|_p<qp^{-1/(p-1)}$, $\gamma_1\in\mathbb{Z}_p^\times$, $k\in\mathbb{Z}$ and $c\in\mathbb{N}_{>1}$ with $(c,p)=1$, we have

$$\begin{aligned}L_{p,1}(s;\omega^{k-1};\gamma_1;c)&=\int_{\mathbb{Z}_p^\times}\langle x\gamma_1\rangle^{-s}\omega^{k-1}(x\gamma_1)d\widetilde{\mathfrak{m}}_c(x)\\&=\langle\gamma_1\rangle^{-s}\omega^{k-1}(\gamma_1)(\langle c\rangle^{1-s}\omega^{k}(c)-1)\cdot L_p(s;\omega^k).\end{aligned}\tag{2.15}$$

The next example shows that when $r=2$, we recover the notion of the p-adic double L-function which has been studied by the second-, the third- and the fourth-named authors.

Example 2.13. Let p be an odd prime, $r=2$, $c=2$ and $\eta\in p\mathbb{Z}_p$. Then, for $s_1,s_2\in\mathbb{C}_p$ with $|s_j|_p<p^{1-1/(p-1)}$ $(j=1,2)$ and $k_1,k_2\in\mathbb{Z}$, our p-adic L-function is given by

$$\begin{aligned}&L_{p,2}(s_1,s_2;\omega^{k_1},\omega^{k_2};1,\eta;2)\\&\quad=\int_{\mathbb{Z}_p^\times\times\mathbb{Z}_p}\langle x_1\rangle^{-s_1}\langle x_1+x_2\eta\rangle^{-s_2}\omega^{k_1}(x_1)\omega^{k_2}(x_1+x_2\eta)d\widetilde{\mathfrak{m}}_2(x_1)d\widetilde{\mathfrak{m}}_2(x_2),\end{aligned}\tag{2.16}$$

which is the p-adic double L-function introduced in [34].

The next theorem claims that our p-adic multiple L-function is rigid-analytic (cf. Notation 4.1 below) with respect to the parameters $s_1,\ldots,s_r$:

Theorem 2.14. *Let $k_1,\ldots,k_r\in\mathbb{Z}$, $\gamma_1,\ldots,\gamma_r\in\mathbb{Z}_p$ and $c\in\mathbb{N}_{>1}$ with $(c,p)=1$. Then $L_{p,r}((s_j);(\omega^{k_j});(\gamma_j);c)$ has the following expansion*

$$\begin{aligned}&L_{p,r}(s_1,\ldots,s_r;\omega^{k_1},\ldots,\omega^{k_r};\gamma_1,\ldots,\gamma_r;c)\\&\quad=\sum_{n_1,\ldots,n_r=0}^{\infty}C\left(n_1,\ldots,n_r;(\omega^{k_j});(\gamma_j);c\right)s_1^{n_1}\cdots s_r^{n_r}\end{aligned}$$

for $(s_j)\in\mathfrak{X}_r\left(q^{-1}\right)$, where $C\left((n_j);(\omega^{k_j});(\gamma_j);c\right)\in\mathbb{Z}_p$ satisfies

$$\left|C\left(n_1,\ldots,n_r;(\omega^{k_j});(\gamma_j);c\right)\right|_p\leqslant\left(qp^{-1/(p-1)}\right)^{-n_1-\cdots-n_r}.\tag{2.17}$$

In order to prove this theorem, we prepare the following two lemmas which are generalizations of [46, Proposition 5.8 and its associated lemma].

Lemma 2.15. *Let f be a continuous function from $\mathbb{Z}_p^r$ to $\mathbb{Q}_p$ defined by*

$$f(X_1,\ldots,X_r)=\sum_{n_1,\ldots,n_r=0}^{\infty}a(n_1,\ldots,n_r)\prod_{j=1}^{r}\binom{X_j}{n_j},$$

where $a(n_1,\ldots,n_r)\in\mathbb{Z}_p$. Suppose there exist $d,M\in\mathbb{R}_{>0}$ with $d<p^{-1/(p-1)}<1$ such that $|a(n_1,\ldots,n_r)|_p\leqslant Md^{\sum_{j=1}^{r}n_j}$ for any $(n_j)\in\mathbb{N}_0^r$. Then $f(X_1,\ldots,X_r)$ may be expressed as

$$f(X_1,\ldots,X_r)=\sum_{n_1,\ldots,n_r=0}^{\infty}C(n_1,\ldots,n_r)X_1^{n_1}\cdots X_r^{n_r}\in\mathbb{Q}_p[[X_1,\ldots,X_r]]$$

which converges absolutely in $\mathfrak{X}_r(d)$, where

$$|C(n_1,\ldots,n_r)|_p\leqslant M(d^{-1}p^{-1/(p-1)})^{-n_1-\cdots-n_r}.$$

Lemma 2.16. *Let*

$$P_l(X_1,\ldots,X_r)=\sum_{n_1,\ldots,n_r\geqslant 0}a(n_1,\ldots,n_r;l)\prod_{j=1}^{r}X_j^{n_j}\quad(l\in\mathbb{N}_0)$$

be a sequence of power series with $\mathbb{C}_p$-coefficients which converges in a fixed subset D of $\mathbb{C}_p^r$ and suppose

(i) *when $l\to\infty$, $a(n_1,\ldots,n_r;l)\to a(n_1,\ldots,n_r;0)$ for each $(n_j)\in\mathbb{N}_0^r$,*
(ii) *for each $(X_j)\in D$ and any $\varepsilon>0$, there exists an $N_0=N_0((X_j),\varepsilon)$ such that*

$$\left|\sum_{\substack{(n_j)\in\mathbb{N}_0^r\\ \sum n_j>N_0}}a(n_1,\ldots,n_r;l)\prod_{j=1}^{r}X_j^{n_j}\right|_p<\varepsilon$$

uniformly in $l\in\mathbb{N}$.

Then $P_l((X_j))\to P_0((X_j))$ as $l\to\infty$ for any $(X_j)\in D$.

Proof of Lemma 2.16. For any ε and (X_j), we can choose N_0 as above. Then

$$|P_l((X_j))-P_0((X_j))|_p\leqslant\max_{\sum n_j\leqslant N_0}\left\{\varepsilon,|a(n_1,\ldots,n_r;0)-a(n_1,\ldots,n_r;l)|_p\cdot\prod_{j=1}^{r}|X_j|_p^{n_j}\right\}\leqslant\varepsilon$$

for any sufficiently large l. □

Proof of Lemma 2.15. We can write

$$\binom{X}{n}=\frac{1}{n!}\sum_{m=0}^{n}b(n,m)X^m\qquad(n\in\mathbb{N}_0),$$

where $b(n,m)\in\mathbb{Z}$. For any $l\in\mathbb{N}$, let

$$\begin{aligned}\mathcal{P}_l(X_1,\ldots,X_r)&=\sum_{\substack{n_1,\ldots,n_r\geqslant 0\\ n_1+\cdots+n_r\leqslant l}}a(n_1,\ldots,n_r)\prod_{j=1}^{r}\binom{X_j}{n_j}\\&=\sum_{\substack{0\leqslant m_1,\ldots,m_r\leqslant l\\ m_1+\cdots+m_r\leqslant l}}C(m_1,\ldots,m_r;l)\prod_{j=1}^{r}X_j^{m_j},\end{aligned}$$

say. We aim to show that $\mathcal{P}_l(X_1,\ldots,X_r)$ satisfies the conditions (i) and (ii) for $P_l(X_1,\ldots,X_r)$ in Lemma 2.16. In fact, using the notation

$$\lambda_n(m)=\begin{cases}1 & (m\leqslant n)\\ 0 & (m>n),\end{cases}$$

we have

$$\begin{aligned}\mathcal{P}_l(X_1,\ldots,X_r)&=\sum_{n_1+\cdots+n_r\leqslant l}\frac{a(n_1,\ldots,n_r)}{n_1!\cdots n_r!}\sum_{m_1\leqslant n_1}\cdots\sum_{m_r\leqslant n_r}\prod_{j=1}^r b(n_j,m_j)X_j^{m_j}\\ &=\sum_{n_1+\cdots+n_r\leqslant l}\frac{a(n_1,\ldots,n_r)}{n_1!\cdots n_r!}\sum_{m_1\leqslant l}\cdots\sum_{\substack{m_r\leqslant l\\ m_1+\cdots+m_r\leqslant l}}\prod_{j=1}^r b(n_j,m_j)\lambda_{n_j}(m_j)X_j^{m_j}\\ &=\sum_{\substack{m_1,\ldots,m_r\leqslant l\\ m_1+\cdots+m_r\leqslant l}}\left(\sum_{\substack{m_1\leqslant n_1,\ldots,m_r\leqslant n_r\\ n_1+\cdots+n_r\leqslant l}}\frac{a(n_1,\ldots,n_r)}{n_1!\cdots n_r!}\prod_{j=1}^r b(n_j,m_j)\right)\prod_{j=1}^r X_j^{m_j}.\end{aligned}$$

Hence we have

$$C(m_1,\ldots,m_r;l)=\sum_{\substack{m_1\leqslant n_1,\ldots,m_r\leqslant n_r\\ n_1+\cdots+n_r\leqslant l}}\frac{a(n_1,\ldots,n_r)}{n_1!\cdots n_r!}\prod_{j=1}^r b(n_j,m_j).$$

Therefore, noting $b(n_j,m_j)\in\mathbb{Z}$ and using $|n!|_p>p^{-n/(p-1)}$, we obtain

$$\begin{aligned}|C(m_1,\ldots,m_r;l)|_p&\leqslant\max_{\substack{m_1\leqslant n_1,\ldots,m_r\leqslant n_r\\ n_1+\cdots+n_r\leqslant l}}\frac{Md^{\sum n_j}}{|n_1!\cdots n_r!|_p}\\ &\leqslant\max_{\substack{m_1\leqslant n_1,\ldots,m_r\leqslant n_r\\ n_1+\cdots+n_r\leqslant l}}M\left(dp^{1/(p-1)}\right)^{\sum n_j}\leqslant M\left(d^{-1}p^{-1/(p-1)}\right)^{-m_1-\cdots-m_r}.\end{aligned}\tag{2.18}$$

Furthermore, we have

$$\begin{aligned}&|C(m_1,\ldots,m_r;l+k)-C(m_1,\ldots,m_r;l)|_p\\ &\leqslant\left|\sum_{\substack{m_1\leqslant n_1,\ldots,m_r\leqslant n_r\\ l<n_1+\cdots+n_r\leqslant l+k}}\frac{a(n_1,\ldots,n_r)}{n_1!\cdots n_r!}\prod_{j=1}^r b(n_j,m_j)\right|_p\\ &\leqslant\max_{\substack{m_1\leqslant n_1,\ldots,m_r\leqslant n_r\\ l<n_1+\cdots+n_r\leqslant l+k}}\frac{Md^{\sum n_j}}{|n_1!\cdots n_r!|_p}\\ &\leqslant\max_{\substack{m_1\leqslant n_1,\ldots,m_r\leqslant n_r\\ l<n_1+\cdots+n_r\leqslant l+k}}M\left(dp^{1/(p-1)}\right)^{\sum n_j}\leqslant M\left(d^{-1}p^{-1/(p-1)}\right)^{-l-1}\end{aligned}$$

for any $l\in\mathbb{N}$. Hence $\{C(m_1,\ldots,m_r;l)\}$ is a Cauchy sequence in l. Therefore there exists

$$C(m_1,\ldots,m_r;0)=\lim_{l\to\infty}C(m_1,\ldots,m_r;l)\in\mathbb{Q}_p\quad((m_j)\in\mathbb{N}_0)$$

with $|C(m_1,\ldots,m_r;0)|_p\leqslant M(d^{-1}p^{-1/(p-1)})^{-m_1-\cdots-m_r}$. For $(X_j)\in\mathfrak{X}_r(d)$ defined by (2.13), let

$$\mathcal{P}_0(X_1,\ldots,X_r)=\sum_{n_1,\ldots,n_r\geqslant 0}C(n_1,\ldots,n_r;0)X_1^{n_1}\cdots X_r^{n_r}.$$

Then $\mathcal{P}_0(X_1,\ldots,X_r)$ converges absolutely in $\mathfrak{X}_r(d)$. Moreover, by (2.18), we have

$$\left|\sum_{\substack{n_1,\ldots,n_r\geqslant 0\\ n_1+\cdots+n_r\geqslant N_0}} C(n_1,\ldots,n_r;l)X_1^{n_1}\cdots X_r^{n_r}\right|_p \leqslant \max_{n_1+\cdots+n_r\geqslant N_0}\left\{M\prod_{j=1}^r\left(d^{-1}p^{-1/(p-1)}\right)^{-n_j}|X_j|_p^{n_j}\right\}\to 0 \quad (n_1,\ldots,n_r\to\infty)$$

uniformly in l, for $(X_j)\in\mathfrak{X}_r(d)$. Therefore, by Lemma 2.16, we obtain

$$f(X_1,\ldots,X_r)=\lim_{l\to\infty}\mathcal{P}_l(X_1,\ldots,X_r)=\mathcal{P}_0(X_1,\ldots,X_r)$$

for $(X_j)\in\mathfrak{X}_r(d)$. Thus we complete the proof of Lemma 2.15. □

Proof of Theorem 2.14. Considering the binomial expansion in (2.14), we have, for $(s_j)\in\mathfrak{X}_r=\mathfrak{X}_r\left(q^{-1}\right)$,

$$\begin{aligned}
&L_{p,r}(s_1,\ldots,s_r;\omega^{k_1},\ldots,\omega^{k_r};\gamma_1,\ldots,\gamma_r;c)\\
&=\sum_{n_1,\ldots,n_r=0}^{\infty}\left\{\int_{(\mathbb{Z}_p^r)'_{\{\gamma_j\}}}\prod_{\nu=1}^r\left(\langle\sum_{j=1}^{\nu}x_j\gamma_j\rangle-1\right)^{n_\nu}\right.\\
&\qquad\left.\times\,\omega^{k_1}(x_1\gamma_1)\cdots\omega^{k_r}(\sum_{j=1}^r x_j\gamma_j)\prod_{j=1}^r d\widetilde{\mathfrak{m}}_c(x_j)\right\}\prod_{j=1}^r\binom{-s_j}{n_j}\\
&=\sum_{n_1,\ldots,n_r=0}^{\infty}a(n_1,\ldots,n_r)\prod_{j=1}^r\binom{-s_j}{n_j},
\end{aligned}$$

say. Since $\langle\sum_{j=1}^{\nu}x_j\gamma_j\rangle\equiv 1 \bmod q$, we have $|a(n_1,\ldots,n_r)|_p\leqslant q^{-\sum n_j}$. Applying Lemma 2.15 with $d=q^{-1}$ and $M=1$, we obtain the proof of Theorem 2.14. □

Next, we discuss a p-adic continuity of our p-adic multiple L-function (2.14) with respect to the parameter c.

Theorem 2.17. *Let $s_1,\ldots,s_r\in\mathbb{Z}_p$, $k_1,\ldots,k_r\in\mathbb{Z}$, $\gamma_1,\ldots,\gamma_r\in\mathbb{Z}_p$ and $c\in\mathbb{N}_{>1}$ with $(c,p)=1$. Then the map*

$$c\mapsto L_{p,r}(s_1,\ldots,s_r;\omega^{k_1},\ldots,\omega^{k_r};\gamma_1,\ldots,\gamma_r;c)$$

is continuously extended to $c\in\mathbb{Z}_p^\times$ as a p-adic continuous function.

Moreover the extension is uniformly continuous with respect to both parameters c and (s_j). Namely, for any given $\varepsilon>0$, there always exists $\delta>0$ such that

$$|L_{p,r}(s_1,\ldots,s_r;\omega^{k_1},\ldots,\omega^{k_r};\gamma_1,\ldots,\gamma_r;c)-L_{p,r}(s_1',\ldots,s_r';\omega^{k_1},\ldots,\omega^{k_r};\gamma_1,\ldots,\gamma_r;c')|_p<\varepsilon$$

holds for any $c,c'\in\mathbb{Z}_p^\times$ with $|c-c'|_p<\delta$ and $s_j,s_j'\in\mathbb{Z}_p$ with $|s_j-s_j'|_p<\delta$ ($1\leqslant j\leqslant r$).

By [46, Subsection 12.2], the space $\mathrm{Meas}_{\mathbb{Z}_p}(\mathbb{Z}_p)$ of $\mathbb{Z}_p$-valued measures on $\mathbb{Z}_p$ is identified with the completed group algebra $\mathbb{Z}_p[[\mathbb{Z}_p]]$;

$$\mathrm{Meas}_{\mathbb{Z}_p}(\mathbb{Z}_p)\simeq\mathbb{Z}_p[[\mathbb{Z}_p]]. \tag{2.19}$$

Again by loc. cit. Theorem 7.1, it is identified with the one parameter formal power series ring $\mathbb{Z}_p[[T]]$ by sending $1 \in \mathbb{Z}_p[[\mathbb{Z}_p]]$ to $1+T \in \mathbb{Z}_p[[T]]$;

$$\mathbb{Z}_p[[\mathbb{Z}_p]] \simeq \mathbb{Z}_p[[T]]. \tag{2.20}$$

By loc. cit. Subsection 12.2, we obtain the following.

Lemma 2.18. *Let $c \in \mathbb{N}_{>1}$ with $(c,p)=1$. By the above correspondences, $\widetilde{\mathfrak{m}}_c \in \mathrm{Meas}_{\mathbb{Z}_p}(\mathbb{Z}_p)$ corresponds to*

$$g_c(T) = \frac{1}{(1+T)-1} - \frac{c}{(1+T)^c-1} \in \mathbb{Z}_p[[T]]. \tag{2.21}$$

Lemma 2.19. *The map $c \mapsto g_c(T)$ is uniquely extended into a p-adic continuous function $g : \mathbb{Z}_p^\times \to \mathbb{Z}_p[[T]]$.*

Proof. The series $(1+T)^c$ makes sense in $\mathbb{Z}_p[[T]]$ for $c \in \mathbb{Z}_p$ and moreover continuous with respect to $c \in \mathbb{Z}_p$ because we have

$$\mathbb{Z}_p[[T]] \simeq \varprojlim \mathbb{Z}_p[T]/\big((1+T)^{p^N}-1\big)$$

(cf. [46, Theorem 7.1]). When $c \in \mathbb{Z}_p^\times$, $\frac{c}{(1+T)^c-1}$ belongs to $\frac{1}{T}+\mathbb{Z}_p[[T]]$. Hence $g_c(T)$ belongs to $\mathbb{Z}_p[[T]]$. $\square$

For $c \in \mathbb{Z}_p^\times$ we denote $\widetilde{\mathfrak{m}}_c$ to be the $\mathbb{Z}_p$-valued measure on $\mathbb{Z}_p$ which corresponds to $g_c(T)$ by (2.19) and (2.20). We note that it coincides with (2.6) when $c \in \mathbb{N}_{>1}$ with $(c,p)=1$.

Remark 2.20. For the parameters $s_1, \ldots, s_r \in \mathbb{C}_p$ with $|s_j|_p < qp^{-1/(p-1)}$ $(1 \leqslant j \leqslant r)$, $k_1, \ldots, k_r \in \mathbb{Z}$, $\gamma_1, \ldots, \gamma_r \in \mathbb{Z}_p$ and $c \in \mathbb{Z}_p^\times$, the p-adic multiple L-function

$$L_{p,r}(s_1, \ldots, s_r; \omega^{k_1}, \ldots, \omega^{k_r}; \gamma_1, \ldots, \gamma_r; c)$$

is defined by (2.14) with the above measure $\widetilde{\mathfrak{m}}_c$.

Let $C(\mathbb{Z}_p^r, \mathbb{C}_p)$ be the $\mathbb{C}_p$-Banach space of continuous $\mathbb{C}_p$-valued functions on $\mathbb{Z}_p^r$ with $||f|| := \sup_{x \in \mathbb{Z}_p^r} |f(x)|_p$ and $\mathrm{Step}(\mathbb{Z}_p^r)$ be the set of $\mathbb{C}_p$-valued locally constant functions on $\mathbb{Z}_p^r$. The subspace $\mathrm{Step}(\mathbb{Z}_p^r)$ is dense in $C(\mathbb{Z}_p^r, \mathbb{C}_p)$ (cf. [46, Section 12.1]).

Proposition 2.21. *For each function $f \in C(\mathbb{Z}_p, \mathbb{C}_p)$, the map*

$$\Phi_f : \mathbb{Z}_p^\times \to \mathbb{C}_p$$

sending $c \mapsto \int_{\mathbb{Z}_p} f(z) d\widetilde{\mathfrak{m}}_c(z)$ is p-adically continuous.

Proof. First assume that $f = \chi_{j,N}$ where $\chi_{j,N}$ $(0 \leqslant j < p^N,\ N \in \mathbb{N})$ is the characteristic function of the set $j + p^N\mathbb{Z}_p$. Then $\Phi_f(c)$ is calculated to be $a_j(g_c)$ (cf. [46, Section 12.2]) which is defined by

$$g_c(T) \equiv \sum_{i=0}^{p^N-1} a_i(g_c)(1+T)^i \pmod{(1+T)^{p^N}-1}.$$

Since the map $c \mapsto g_c$ is continuous, $c \mapsto \Phi_f(c) = a_j(g_c)$ is continuous in this case. This implies the continuity of Φ_f in c when $f \in \mathrm{Step}(\mathbb{Z}_p)$.

Next assume $f \in C(\mathbb{Z}_p, \mathbb{C}_p)$ and $c \in \mathbb{Z}_p^\times$. Then for any $g \in C(\mathbb{Z}_p, \mathbb{C}_p)$ and $c' \in \mathbb{Z}_p^\times$, we have

$$\begin{aligned}\Phi_f(c) - \Phi_f(c') &= \int_{\mathbb{Z}_p} f(z) d\widetilde{\mathfrak{m}}_c(z) - \int_{\mathbb{Z}_p} f(z) d\widetilde{\mathfrak{m}}_{c'}(z) \\ &= \int_{\mathbb{Z}_p} (f(z) - g(z)) \cdot d(\widetilde{\mathfrak{m}}_c(z) - \widetilde{\mathfrak{m}}_{c'}(z)) + \int_{\mathbb{Z}_p} g(z) \cdot d(\widetilde{\mathfrak{m}}_c(z) - \widetilde{\mathfrak{m}}_{c'}(z)) \\ &= (\Phi_{f-g}(c) - \Phi_{f-g}(c')) + (\Phi_g(c) - \Phi_g(c')).\end{aligned}$$

Since $\mathrm{Step}(\mathbb{Z}_p)$ is dense in $C(\mathbb{Z}_p, \mathbb{C}_p)$, there exists $g_0 \in \mathrm{Step}(\mathbb{Z}_p)$ with $||f - g_0|| < \varepsilon$ for any given $\varepsilon > 0$. So for any $c \in \mathbb{Z}_p^\times$ we have

$$|\Phi_{f-g_0}(c)|_p = \Big| \int_{\mathbb{Z}_p} (f(z) - g_0(z)) \cdot d\widetilde{\mathfrak{m}}_c(z) \Big|_p \leqslant ||f - g_0|| < \varepsilon$$

because $\widetilde{\mathfrak{m}}_c$ is a $\mathbb{Z}_p$-valued measure.

On the other hand, since $g_0 \in \mathrm{Step}(\mathbb{Z}_p)$, there exists a $\delta > 0$ such that

$$|\Phi_{g_0}(c) - \Phi_{g_0}(c')|_p < \varepsilon$$

holds for any $c' \in \mathbb{Z}_p^\times$ with $|c - c'|_p < \delta$.

Therefore for $f \in C(\mathbb{Z}_p, \mathbb{C}_p)$, $c \in \mathbb{Z}_p^\times$ and any $\varepsilon > 0$, there always exists a $\delta > 0$ such that

$$|\Phi_f(c) - \Phi_f(c')|_p \leqslant \max\left\{ |\Phi_{f-g_0}(c)|_p, |\Phi_{f-g_0}(c')|_p, |\Phi_{g_0}(c) - \Phi_{g_0}(c')|_p \right\} < \varepsilon$$

holds for any $c' \in \mathbb{Z}_p^\times$ with $|c - c'|_p < \delta$. $\square$

By generalizing our arguments above, we obtain the following.

Proposition 2.22. *For each function $f \in C(\mathbb{Z}_p^r, \mathbb{C}_p)$, the map*

$$\Phi_f^r : \mathbb{Z}_p^\times \to \mathbb{C}_p$$

sending $c \mapsto \int_{\mathbb{Z}_p^r} f(x_1, \ldots, x_r) \prod_{j=1}^r d\widetilde{\mathfrak{m}}_c(x_j)$ is p-adically continuous.

A proof of Theorem 2.17 is attained by the above proposition. This is the reason why we restrict Theorem 2.17 to the case for $s_1, \ldots, s_r \in \mathbb{Z}_p$.

Proof of Theorem 2.17. Let us fix notation: Put

$W(s_1, \ldots s_r; x_1, \ldots, x_r)$

$$:= \langle x_1\gamma_1 \rangle^{-s_1} \langle x_1\gamma_1 + x_2\gamma_2 \rangle^{-s_2} \cdots \langle \sum_{j=1}^r x_j\gamma_j \rangle^{-s_r} \cdot \omega^{k_1}(x_1\gamma_1) \cdots \omega^{k_r}(\sum_{j=1}^r x_j\gamma_j) \cdot \chi_{(\mathbb{Z}_p^r)'_{\{\gamma_j\}}}(x_1, \ldots, x_r)$$

where $\chi_{(\mathbb{Z}_p^r)'_{\{\gamma_j\}}}(x_1, \ldots, x_r)$ is the characteristic function of $\left(\mathbb{Z}_p^r\right)'_{\{\gamma_j\}}$ (cf. Definition 2.9). Then we have

$$L_{p,r}(s_1, \ldots, s_r; \omega^{k_1}, \ldots, \omega^{k_r}; \gamma_1, \ldots, \gamma_r; c) := \int_{\mathbb{Z}_p^r} W(s_1, \ldots, s_r; x_1, \ldots, x_r) \cdot \prod_{j=1}^r d\widetilde{\mathfrak{m}}_c(x_j). \tag{2.22}$$

Below we will prove that the map

$$\Psi : (s_1, \ldots, s_r, c) \in \mathbb{Z}_p^r \times \mathbb{Z}_p^\times \mapsto L_{p,r}(s_1, \ldots, s_r; \omega^{k_1}, \ldots, \omega^{k_r}; \gamma_1, \ldots, \gamma_r; c) \in \mathbb{Z}_p$$

is continuous:

First, by Proposition 2.22 we see that the function is continuous with respect to $c \in \mathbb{Z}_p^\times$ for each fixed $(s_1, \ldots, s_r) \in \mathbb{Z}_p^r$. Namely, for each fixed $(s_1, \ldots, s_r) \in \mathbb{Z}_p^r$ and $c \in \mathbb{Z}_p^\times$ and for any given $\varepsilon > 0$, there always exists $\delta > 0$ such that

$$|L_{p,r}(s_1, \ldots, s_r; \omega^{k_1}, \ldots, \omega^{k_r}; \gamma_1, \ldots, \gamma_r; c) - L_{p,r}(s_1, \ldots, s_r; \omega^{k_1}, \ldots, \omega^{k_r}; \gamma_1, \ldots, \gamma_r; c')|_p < \varepsilon,$$

equivalently

$$|\Psi(s_1, \ldots, s_r; c) - \Psi(s_1, \ldots, s_r; c')|_p < \varepsilon \tag{2.23}$$

for all $c' \in \mathbb{Z}_p^\times$ with $|c - c'|_p < \delta$.

Next, take $M \in \mathbb{N}$ such that $\varepsilon > p^{-M} > 0$. Since it is easy to see that there exists $\delta' > 0$ such that

$$(1 + u)^d \equiv 1 \pmod{p^M}$$

for any $u \in p\mathbb{Z}_p$ and $d \in \mathbb{Z}_p$ with $|d|_p < \delta'$ (actually you may take δ' for $\delta' < p^{-M}$), we have

$$x^s \equiv x^{s'} \pmod{p^M}$$

for all $x \in (1 + p\mathbb{Z}_p)$ and $s, s' \in \mathbb{Z}_p$ with $|s - s'| < \delta'$ for such δ'. Therefore

$$W(s_1, \ldots, s_r; x_1, \ldots, x_r) \equiv W(s_1', \ldots, s_r'; x_1, \ldots, x_r) \pmod{p^M}$$

holds for all $(x_1, \ldots, x_r) \in \mathbb{Z}_p^r$ when $|s_i - s_i'|_p < \delta'$ $(1 \leqslant i \leqslant r)$. So, in that case, the inequality

$$\begin{aligned}|L_{p,r}(s_1, \ldots, s_r; \omega^{k_1}, \ldots, \omega^{k_r}; \gamma_1, \ldots, \gamma_r; c) - L_{p,r}(s_1', \ldots, s_r'; \omega^{k_1}, \ldots, \omega^{k_r}; \gamma_1, \ldots, \gamma_r; c)|_p \\ \leqslant p^{-M} < \varepsilon\end{aligned}$$

holds for any $c \in \mathbb{Z}_p^\times$ because $\widetilde{\mathfrak{m}}_c$ is a $\mathbb{Z}_p$-valued measure. Therefore, for any given $\varepsilon > 0$, there always exists $\delta' > 0$ such that when $|s_i - s_i'|_p < \delta'$ $(1 \leqslant i \leqslant r)$,

$$|\Psi(s_1, \ldots, s_r; c) - \Psi(s_1', \ldots, s_r'; c)|_p < \varepsilon \tag{2.24}$$

holds for all $c \in \mathbb{Z}_p^\times$.

By (2.23) and (2.24), we see that for each fixed $(s_1, \ldots, s_r) \in \mathbb{Z}_p^r$ and $c \in \mathbb{Z}_p^\times$, and for any given $\varepsilon > 0$, there always exist $\delta, \delta' > 0$ such that

$$\begin{aligned}&\bigl|\Psi(s_1, \ldots, s_r; c) - \Psi(s_1', \ldots, s_r'; c')\bigr|_p \\ &\quad = \bigl|\Psi(s_1, \ldots, s_r; c) - \Psi(s_1, \ldots, s_r; c') + \Psi(s_1, \ldots, s_r; c') - \Psi(s_1', \ldots, s_r'; c')\bigr|_p \\ &\quad \leqslant \max\Bigl\{\bigl|\Psi(s_1, \ldots, s_r; c) - \Psi(s_1, \ldots, s_r; c')\bigr|_p, \bigl|\Psi(s_1, \ldots, s_r; c') - \Psi(s_1', \ldots, s_r'; c')\bigr|_p\Bigr\} < \varepsilon\end{aligned}$$

for $|s_i - s_i'|_p < \delta'$ $(1 \leqslant i \leqslant r)$ and $|c - c'|_p < \delta$. Thus we get the desired continuity of $\Psi(s_1, \ldots, s_r; c)$.

The uniform continuity of $\Psi(s_1, \ldots, s_r; c)$ is almost obvious because now we know that the function is continuous and the source set $\mathbb{Z}_p^r \times \mathbb{Z}_p^\times$ is compact. □

As a corollary of Theorem 2.17, the following non-trivial property of special values of p-adic multiple L-functions at non-positive integer points is obtained.

Corollary 2.23. *The right-hand side of equation* (3.1) *of Theorem 3.1 (in the next section) is p-adically continuous with respect to c.*

Our final discussion here is towards a construction of a p-adic analogue of the entire function $Z(s_1,\ldots,s_r;\gamma_1,\ldots,\gamma_r)$ which was constructed as a multiple analogue of the entire function $(1-s)\zeta(s)$ by equation (1.23): As was explained in Remark 1.17, an idea of the equation is conceptually (but not mathematically) expressed as in (1.22). By Theorem 3.1 and Remark 3.2 (both in the next section), our $L_{p,r}(s_1,\ldots,s_r;\omega^{n_1},\ldots,\omega^{n_r};\gamma_1,\ldots,\gamma_r;c)$ is a p-adic interpolation of (3.2), which also has appeared on the right-hand side of (1.22).

By $\lim_{c\to 1} g_c(T)=0$, we have, for $s_1,\ldots,s_r\in\mathbb{Z}_p$, $k_1,\ldots,k_r\in\mathbb{Z}$, $\gamma_1,\ldots,\gamma_r\in\mathbb{Z}_p$,

$$\lim_{\substack{c\to 1\\ c\in\mathbb{Z}_p^\times}} L_{p,r}(s_1,\ldots,s_r;\omega^{k_1},\ldots,\omega^{k_r};\gamma_1,\ldots,\gamma_r;c)=0.$$

While, by

$$\lim_{c\to 1}\frac{g_c(T)}{c-1}\in\mathbb{Q}_p[[T]]\setminus\mathbb{Z}_p[[T]],$$

we can not say that the limit $\lim_{c\to 1}\frac{1}{c-1}\widetilde{\mathfrak{m}}_c$ converges to a measure. Thus the following is unclear.

Problem 2.24. For $s_1,\ldots,s_r\in\mathbb{Z}_p$, $k_1,\ldots,k_r\in\mathbb{Z}$ and $\gamma_1,\ldots,\gamma_r\in\mathbb{Z}_p$, does

$$\lim_{\substack{c\to 1\\ c\in\mathbb{Z}_p^\times\setminus\{1\}}}\frac{1}{(c-1)^r}L_{p,r}(s_1,\ldots,s_r;\omega^{k_1},\ldots,\omega^{k_r};\gamma_1,\ldots,\gamma_r;c) \tag{2.25}$$

converge?

If the limit (2.25) exists and happens to be a rigid analytic function, we may call it a p-adic analogue of the desingularized zeta function $\zeta_r^{\mathrm{des}}(s_1,\ldots,s_r;\gamma_1,\ldots,\gamma_r)$. We remind that the problem is affirmative in the case when $r=1$ and $\gamma=1$. Actually we have

$$\lim_{c\to 1}(c-1)^{-1}L_{p,1}(s;\omega^k;1;c)=(1-s)\cdot L_p(s;\omega^{k+1}),$$

which was already mentioned in Example 2.12. We also note that the limit converges when $(s_1,\ldots,s_r)=(-n_1,\ldots,-n_r)$ for $n_1,\ldots,n_r\in\mathbb{N}_0$ by Theorem 3.7 in the next section.

3. Special values of p-adic multiple L-functions at non-positive integers

We will consider the special values of our p-adic multiple L-functions (Definition 2.9) at non-positive integers. We will express them in terms of twisted multiple Bernoulli numbers (Definition 1.6) in Theorems 3.1 and 3.7. We will see that our p-adic multiple L-function is a p-adic interpolation of a certain sum (3.2) of the entire complex multiple zeta-functions of generalized Euler-Zagier-Lerch type in Remark 3.2. Based on the evaluations, we will generalize the well-known Kummer congruence for ordinary Bernoulli numbers to the multiple Kummer congruence for twisted multiple Bernoulli numbers in Theorem 3.10. We will show certain functional relations with a parity condition among p-adic multiple L-functions in Theorem 3.15. These functional relations will be seen as multiple generalizations of the vanishing property of the Kubota-Leopoldt p-adic L-function with odd characters (cf. Proposition 2.6) in the single variable case and of the functional relations for the p-adic double L-function (shown in [34]) in the double variable case under the special condition $c = 2$. Many examples will be investigated in this section.

3.1. **Evaluation of p-adic multiple L-functions at non-positive integers.** Based on the consideration in the previous sections, we determine values of p-adic multiple L-functions at non-positive integers as follows.

Theorem 3.1. *For $n_1, \ldots, n_r \in \mathbb{N}_0$, $\gamma_1, \ldots, \gamma_r \in \mathbb{Z}_p$, and $c \in \mathbb{N}_{>1}$ with $(c,p) = 1$.*

$$\begin{aligned} & L_{p,r}(-n_1, \ldots, -n_r; \omega^{n_1}, \ldots, \omega^{n_r}; \gamma_1, \ldots, \gamma_r; c) \\ &= \sum_{\substack{\xi_1^c=1 \\ \xi_1\neq 1}} \cdots \sum_{\substack{\xi_r^c=1 \\ \xi_r\neq 1}} \mathfrak{B}((n_j);(\xi_j);(\gamma_j)) \\ &+ \sum_{d=1}^{r} \left(-\frac{1}{p}\right)^d \sum_{1\leqslant i_1<\cdots<i_d\leqslant r} \sum_{\rho_{i_1}^p=1} \cdots \sum_{\rho_{i_d}^p=1} \sum_{\substack{\xi_1^c=1 \\ \xi_1\neq 1}} \cdots \sum_{\substack{\xi_r^c=1 \\ \xi_r\neq 1}} \mathfrak{B}((n_j);((\prod_{j\leqslant i_l} \rho_{i_l})^{\gamma_j}\xi_j);(\gamma_j)), \end{aligned} \tag{3.1}$$

where the empty product is interpreted as 1.

In Definition 1.6, the twisted multiple Bernoulli number $\mathfrak{B}((n_j);(\xi_j);(\gamma_j))$ was defined for $\gamma_1, \ldots, \gamma_r \in \mathbb{C}$ and roots of unity $\xi_1, \ldots, \xi_r \in \mathbb{C}$. Here we define $\mathfrak{B}((n_j);(\xi_j);(\gamma_j))$ for $\gamma, \ldots, \gamma_r \in \mathbb{Z}_p$ and roots of unity $\xi_1, \ldots, \xi_r \in \mathbb{C}_p$ by (1.12) and (1.13).

Remark 3.2. When $\gamma_1, \ldots, \gamma_r \in \mathbb{Z}_p \cap \overline{\mathbb{Q}}$ satisfy $\Re\gamma_j > 0 \quad (1 \leqslant j \leqslant r)$, we obtain from the above theorem and Theorem 1.13 that

$$\begin{aligned} L_{p,r}(-n_1, \ldots, -n_r; \omega^{n_1}, \ldots, \omega^{n_r}; \gamma_1, \ldots, \gamma_r; c) = (-1)^{r+n_1+\cdots+n_r} \Big\{ \sum_{\substack{\xi_1^c=1 \\ \xi_1\neq 1}} \cdots \sum_{\substack{\xi_r^c=1 \\ \xi_r\neq 1}} \zeta_r((-n_j);(\xi_j);(\gamma_j)) \\ + \sum_{d=1}^{r} \left(-\frac{1}{p}\right)^d \sum_{1\leqslant i_1<\cdots<i_d\leqslant r} \sum_{\rho_{i_1}^p=1} \cdots \sum_{\rho_{i_d}^p=1} \sum_{\substack{\xi_1^c=1 \\ \xi_1\neq 1}} \cdots \sum_{\substack{\xi_r^c=1 \\ \xi_r\neq 1}} \zeta_r((-n_j);((\prod_{j\leqslant i_l} \rho_{i_l})^{\gamma_j}\xi_j);(\gamma_j)) \Big\}. \end{aligned}$$

We may say that the p-adic multiple L-function $L_{p,r}((s_j);(\omega^{k_1});(\gamma_j);c)$ is a p-adic interpolation of the following finite sum of multiple zeta-functions $\zeta_r((s_j);(\xi_j);(\gamma_j))$ which are entire

$$\sum_{\substack{\xi_1^c=1 \\ \xi_1\neq 1}} \cdots \sum_{\substack{\xi_r^c=1 \\ \xi_r\neq 1}} \zeta_r((s_j);(\xi_j);(\gamma_j)). \tag{3.2}$$

Proof of Theorem 3.1. We see that for any $\rho \in \mu_p$ and $\xi \in \mu_c \setminus \{1\}$,

$$\mathfrak{m}_{\rho\xi}(j + p^N\mathbb{Z}_p) = \frac{(\rho\xi)^j}{1-(\rho\xi)^{p^N}} = \rho^j \mathfrak{m}_\xi(j + p^N\mathbb{Z}_p) \qquad (N \geqslant 1).$$

Hence, using

$$\sum_{\rho^p=1} \rho^n = \begin{cases} 0 & (p \nmid n) \\ p & (p \mid n), \end{cases}$$

we have

$$L_{p,r}(-n_1, \ldots, -n_r; \omega^{n_1}, \ldots, \omega^{n_r}; \gamma_1, \ldots, \gamma_r; c)$$

$$= \int_{\mathbb{Z}_p^r} (x_1\gamma_1)^{n_1} \cdots (\sum_{j=1}^r x_j\gamma_j)^{n_r} \prod_{i=1}^r \left(1 - \frac{1}{p} \sum_{\rho_i^p=1} \rho_i^{\sum_{\nu=1}^i x_\nu\gamma_\nu}\right) \prod_{j=1}^r \sum_{\substack{\xi_j^c=1 \\ \xi_j\neq 1}} d\mathfrak{m}_{\xi_j}(x_j)$$

$$= \int_{\mathbb{Z}_p^r} (x_1\gamma_1)^{n_1} \cdots (\sum_{j=1}^r x_j\gamma_j)^{n_r} \sum_{\substack{\xi_1^c=1 \\ \xi_1\neq 1}} \cdots \sum_{\substack{\xi_r^c=1 \\ \xi_r\neq 1}} \prod_{j=1}^r d\mathfrak{m}_{\xi_j}(x_j)$$

$$+ \sum_{d=1}^r \left(-\frac{1}{p}\right)^d \sum_{1\leqslant i_1<\cdots<i_d\leqslant r} \sum_{\rho_{i_1}^p=1} \cdots \sum_{\rho_{i_d}^p=1} \sum_{\substack{\xi_1^c=1 \\ \xi_1\neq 1}} \cdots \sum_{\substack{\xi_r^c=1 \\ \xi_r\neq 1}}$$

$$\times \int_{\mathbb{Z}_p^r} \prod_{l=1}^r (\sum_{j=1}^l x_j\gamma_j)^{n_l} \prod_{l=1}^d \rho_{i_l}^{\sum_{\nu=1}^{i_l} x_\nu\gamma_\nu} \prod_{j=1}^r d\mathfrak{m}_{\xi_j}(x_j)$$

$$= \int_{\mathbb{Z}_p^r} (x_1\gamma_1)^{n_1} \cdots (\sum_{j=1}^r x_j\gamma_j)^{n_r} \sum_{\substack{\xi_1^c=1 \\ \xi_1\neq 1}} \cdots \sum_{\substack{\xi_r^c=1 \\ \xi_r\neq 1}} \prod_{j=1}^r d\mathfrak{m}_{\xi_j}(x_j)$$

$$+ \sum_{d=1}^r \left(-\frac{1}{p}\right)^d \sum_{1\leqslant i_1<\cdots<i_d\leqslant r} \sum_{\rho_{i_1}^p=1} \cdots \sum_{\rho_{i_d}^p=1} \sum_{\substack{\xi_1^c=1 \\ \xi_1\neq 1}} \cdots \sum_{\substack{\xi_r^c=1 \\ \xi_r\neq 1}}$$

$$\times \int_{\mathbb{Z}_p^r} \prod_{l=1}^r (\sum_{j=1}^l x_j\gamma_j)^{n_l} \prod_{j=1}^r d\mathfrak{m}_{\{(\prod_{j\leqslant i_l} \rho_{i_l})^{\gamma_j}\xi_j\}}(x_j).$$

Thus, by Proposition 2.8, we complete the proof. □

Remark 3.3. It is worthy to restate Corollary 2.23 saying that the right-hand side of the above equation (3.1) is p-adically continuous not only with respect to $n_1, \ldots, n_r$ but also with respect to c.

Considering the Galois action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, we obtain the following result from Theorems 2.14 and 2.17.

Corollary 3.4. *For $n_1, \ldots, n_r \in \mathbb{N}_0$, $\gamma_1, \ldots, \gamma_r \in \mathbb{Z}_p$ and $c \in \mathbb{Z}_p^\times$,*

$$L_{p,r}(-n_1, \ldots, -n_r; \omega^{n_1}, \ldots, \omega^{n_r}; \gamma_1, \ldots, \gamma_r; c) \in \mathbb{Z}_p,$$

hence the right-hand side of (3.1) *is in $\mathbb{Z}_p$, though it includes the terms like $\left(-\frac{1}{p}\right)^d$. In particular when $\gamma_1, \ldots, \gamma_r \in \mathbb{Z}_{(p)} := \{\frac{a}{b} \in \mathbb{Q} \,|\, a, b \in \mathbb{Z},\ (b,p) = 1\} = \mathbb{Z}_p \cap \mathbb{Q}$,*

$$L_{p,r}(-n_1, \ldots, -n_r; \omega^{n_1}, \ldots, \omega^{n_r}; \gamma_1, \ldots, \gamma_r; c) \in \mathbb{Z}_{(p)}.$$

The following examples are special cases ($r = 1$ and $r = 2$) of Theorem 3.1.

Example 3.5. For $n \in \mathbb{N}_0$ and $c \in \mathbb{N}_{>1}$ with $(c,p) = 1$,

$$L_{p,1}(-n; \omega^n; 1; c) = (1 - p^n) \sum_{\substack{\xi^c=1 \\ \xi\neq 1}} \mathfrak{B}_n(\xi), \tag{3.3}$$

which recovers the known fact

$$L_p(-n;\omega^{n+1}) = -(1-p^n)\frac{B_{n+1}}{n+1}$$

thanks to (1.8) and (2.10).

Example 3.6. For $n_1,n_2\in\mathbb{N}_0$, $\gamma_1,\gamma_2\in\mathbb{Z}_p$, and $c\in\mathbb{N}_{>1}$ with $(c,p)=1$,

$$\begin{aligned}
&L_{p,2}(-n_1,-n_2;\omega^{n_1},\omega^{n_2};\gamma_1,\gamma_2;c)\\
&\quad=\sum_{\substack{\xi_1^c=1\\ \xi_1\neq 1}}\sum_{\substack{\xi_2^c=1\\ \xi_2\neq 1}}\mathfrak{B}(n_1,n_2;\xi_1,\xi_2;\gamma_1,\gamma_2)\\
&\qquad-\frac{1}{p}\sum_{\rho_1^p=1}\sum_{\substack{\xi_1^c=1\\ \xi_1\neq 1}}\sum_{\substack{\xi_2^c=1\\ \xi_2\neq 1}}\mathfrak{B}(n_1,n_2;\rho_1^{\gamma_1}\xi_1,\xi_2;\gamma_1,\gamma_2)\\
&\qquad-\frac{1}{p}\sum_{\rho_2^p=1}\sum_{\substack{\xi_1^c=1\\ \xi_1\neq 1}}\sum_{\substack{\xi_2^c=1\\ \xi_2\neq 1}}\mathfrak{B}(n_1,n_2;\rho_2^{\gamma_1}\xi_1,\rho_2^{\gamma_2}\xi_2;\gamma_1,\gamma_2)\\
&\qquad+\frac{1}{p^2}\sum_{\rho_1^p=1}\sum_{\rho_2^p=1}\sum_{\substack{\xi_1^c=1\\ \xi_1\neq 1}}\sum_{\substack{\xi_2^c=1\\ \xi_2\neq 1}}\mathfrak{B}(n_1,n_2;(\rho_1\rho_2)^{\gamma_1}\xi_1,\rho_2^{\gamma_2}\xi_2;\gamma_1,\gamma_2).
\end{aligned}\tag{3.4}$$

More generally, we consider the generating function of $\{L_{p,r}((-n_j);(\omega^{n_j});(\gamma_j);c)\}$, that is,

$$F_{p,r}(t_1,\ldots,t_r;(\gamma_j);c)=\sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}L_{p,r}((-n_j);(\omega^{n_j});(\gamma_j);c)\prod_{j=1}^{r}\frac{t_j^{n_j}}{n_j!}.$$

Then we have the following.

Theorem 3.7. *Let $c\in\mathbb{Z}_p^{\times}$. For $\gamma_1,\ldots,\gamma_r\in\mathbb{Z}_p$,*

$$\begin{aligned}
&F_{p,r}(t_1,\ldots,t_r;(\gamma_j);c)\\
&=\prod_{j=1}^{r}\left(\frac{1}{\exp\left(\gamma_j\sum_{k=j}^{r}t_k\right)-1}-\frac{c}{\exp\left(c\gamma_j\sum_{k=j}^{r}t_k\right)-1}\right)\\
&+\sum_{d=1}^{r}\left(-\frac{1}{p}\right)^{d}\sum_{1\leqslant i_1<\cdots<i_d\leqslant r}\sum_{\rho_{i_1}^p=1}\cdots\sum_{\rho_{i_d}^p=1}\\
&\quad\times\prod_{j=1}^{r}\left(\frac{1}{(\prod_{j\leqslant i_l}\rho_{i_l})^{\gamma_j}\exp\left(\gamma_j\sum_{k=j}^{r}t_k\right)-1}-\frac{c}{(\prod_{j\leqslant i_l}\rho_{i_l})^{c\gamma_j}\exp\left(c\gamma_j\sum_{k=j}^{r}t_k\right)-1}\right)\\
&=\prod_{j=1}^{r}\left(\sum_{m_j=0}^{\infty}\left(1-c^{m_j+1}\right)\frac{B_{m_j+1}}{m_j+1}\frac{\left(\gamma_j\sum_{k=j}^{r}t_k\right)^{m_j}}{m_j!}\right)\\
&\quad+(-1)^r\sum_{d=1}^{r}\left(-\frac{1}{p}\right)^{d}\sum_{1\leqslant i_1<\cdots<i_d\leqslant r}\sum_{\rho_{i_1}^p=1}\cdots\sum_{\rho_{i_d}^p=1}\\
&\quad\times\prod_{j=1}^{r}\left(\sum_{m_j=0}^{\infty}(1-c^{m_j+1})\mathfrak{B}_{m_j}((\prod_{j\leqslant i_l}\rho_{i_l})^{\gamma_j})\frac{\left(\gamma_j\sum_{k=j}^{r}t_k\right)^{m_j}}{m_j!}\right).
\end{aligned}$$

In particular when $\gamma_1 \in \mathbb{Z}_p^\times$ and $\gamma_j \in p\mathbb{Z}_p$ $(2 \leqslant j \leqslant r)$,

$$\begin{aligned}
F_{p,r}(t_1,\ldots,t_r;(\gamma_j);c) &= \Biggl(\frac{1}{\exp(\gamma_1\sum_{k=1}^r t_k)-1} - \frac{c}{\exp(c\gamma_1\sum_{k=1}^r t_k)-1} \\
&\qquad - \frac{1}{\exp(p\gamma_1\sum_{k=1}^r t_k)-1} + \frac{c}{\exp(cp\gamma_1\sum_{k=1}^r t_k)-1}\Biggr) \\
&\qquad \times \prod_{j=2}^r \left(\frac{1}{\exp\left(\gamma_j\sum_{k=j}^r t_k\right)-1} - \frac{c}{\exp\left(c\gamma_j\sum_{k=j}^r t_k\right)-1}\right) \\
&= \left(\sum_{n_1=0}^\infty (1-p^{n_1})\left(1-c^{n_1+1}\right)\frac{B_{n_1+1}}{n_1+1}\frac{(\gamma_1\sum_{k=1}^r t_k)^{n_1}}{n_1!}\right) \\
&\qquad \times \prod_{j=2}^r \left(\sum_{n_j=0}^\infty \left(1-c^{n_j+1}\right)\frac{B_{n_j+1}}{n_j+1}\frac{\left(\gamma_j\sum_{k=j}^r t_k\right)^{n_j}}{n_j!}\right). \qquad (3.5)
\end{aligned}$$

We stress here that Theorem 3.1 holds for $c \in \mathbb{N}_{>1}$ with $(c,p)=1$, in contrast, Theorem 3.7 holds, more generally, for $c \in \mathbb{Z}_p^\times$,

Proof. For the moment assume that $c \in \mathbb{N}_{>1}$ with $(c,p)=1$. Combining (1.13) and (3.1), we have

$$\begin{aligned}
F_{p,r}((t_j);(\gamma_j);c) &= \sum_{\substack{\xi_1^c=1 \\ \xi_1\neq 1}} \cdots \sum_{\substack{\xi_r^c=1 \\ \xi_r\neq 1}} \prod_{j=1}^r \frac{1}{1-\xi_j\exp\left(\gamma_j\sum_{k=j}^r t_k\right)} \\
&\quad + \sum_{d=1}^r \left(-\frac{1}{p}\right)^d \sum_{1\leqslant i_1<\cdots<i_d\leqslant r} \sum_{\rho_{i_1}^p=1} \cdots \sum_{\rho_{i_d}^p=1} \sum_{\substack{\xi_1^c=1 \\ \xi_1\neq 1}} \cdots \sum_{\substack{\xi_r^c=1 \\ \xi_r\neq 1}} \\
&\quad \times \prod_{j=1}^r \frac{1}{1-(\prod_{j\leqslant i_l}\rho_{i_l})^{\gamma_j}\xi_j\exp\left(\gamma_j\sum_{k=j}^r t_k\right)} \\
&= \prod_{j=1}^r \sum_{\substack{\xi_j^c=1 \\ \xi_j\neq 1}} \frac{1}{1-\xi_j\exp\left(\gamma_j\sum_{k=j}^r t_k\right)} \\
&\quad + \sum_{d=1}^r \left(-\frac{1}{p}\right)^d \sum_{1\leqslant i_1<\cdots<i_d\leqslant r} \sum_{\rho_{i_1}^p=1} \cdots \sum_{\rho_{i_d}^p=1} \\
&\quad \times \prod_{j=1}^r \left(\sum_{\substack{\xi_j^c=1 \\ \xi_j\neq 1}} \frac{1}{1-(\prod_{j\leqslant i_l}\rho_{i_l})^{\gamma_j}\xi_j\exp\left(\gamma_j\sum_{k=j}^r t_k\right)}\right).
\end{aligned}$$

Hence, by (1.7), we obtain the first assertion. Next we assume $\gamma_1 \in \mathbb{Z}_p^\times$ and $\gamma_j \in p\mathbb{Z}_p$ $(2 \leqslant j \leqslant r)$. Then

$$(\prod_{j\leqslant i_l}\rho_{i_l})^{\gamma_j} = 1 \quad (2 \leqslant j \leqslant r).$$

Hence we have

$$\begin{aligned}F_{p,r}((t_j);(\gamma_j);c) &= \prod_{j=1}^{r}\left(\frac{1}{\exp\left(\gamma_j\sum_{k=j}^{r}t_k\right)-1}-\frac{c}{\exp\left(c\gamma_j\sum_{k=j}^{r}t_k\right)-1}\right)\\ &\quad+\sum_{d=1}^{r}\left(-\frac{1}{p}\right)^{d}\sum_{1\leqslant i_1<\cdots<i_d\leqslant r}\sum_{\rho_{i_1}^{p}=1}\cdots\sum_{\rho_{i_d}^{p}=1}\\ &\quad\times\left(\frac{1}{(\prod_{i_l}\rho_{i_l})\exp\left(\gamma_1\sum_{k=1}^{r}t_k\right)-1}-\frac{c}{(\prod_{i_l}\rho_{i_l})^{c}\exp\left(c\gamma_1\sum_{k=1}^{r}t_k\right)-1}\right)\\ &\quad\times\prod_{j=2}^{r}\left(\frac{1}{\exp\left(\gamma_j\sum_{k=j}^{r}t_k\right)-1}-\frac{c}{\exp\left(c\gamma_j\sum_{k=j}^{r}t_k\right)-1}\right).\end{aligned}$$

Using (1.7), we can rewrite the second term on the right-hand side as

$$\begin{aligned}&\prod_{j=2}^{r}\left(\frac{1}{\exp\left(\gamma_j\sum_{k=j}^{r}t_k\right)-1}-\frac{c}{\exp\left(c\gamma_j\sum_{k=j}^{r}t_k\right)-1}\right)\\ &\quad\times\sum_{d=1}^{r}\left(-\frac{1}{p}\right)^{d}\sum_{1\leqslant i_1<\cdots<i_d\leqslant r}p^{d}\left(\frac{1}{\exp\left(p\gamma_1\sum_{k=1}^{r}t_k\right)-1}-\frac{c}{\exp\left(cp\gamma_1\sum_{k=1}^{r}t_k\right)-1}\right).\end{aligned}$$

Noting

$$\sum_{d=1}^{r}\left(-\frac{1}{p}\right)^{d}\sum_{1\leqslant i_1<\cdots<i_d\leqslant r}p^{d}=\sum_{d=1}^{r}\binom{r}{d}(-1)^{d}=(1-1)^{r}-1=-1,$$

we obtain the second assertion.

Whence we get the formulas in the theorem for $c\in\mathbb{N}_{>1}$ with $(c,p)=1$. It should be noted that each coefficient of the left-hand sides of the formulas is expressed as polynomials on c. On the other hand each coefficient of the right-hand sides of the formulas is continuous on c by Theorem 2.17. Therefore the validity of the formulas is extended into $c\in\mathbb{Z}_p^{\times}$. $\square$

We consider the case $r=2$ with $(\gamma_1,\gamma_2)=(1,\eta)$ for $\eta\in\mathbb{Z}_p$ and $c\in\mathbb{N}_{>1}$ with $(c,p)=1$. From (1.14) and Example 3.6, we have

$$\begin{aligned}&L_{p,2}(-k_1,-k_2;\omega^{k_1},\omega^{k_2};1,\eta;c)\\ &=\sum_{\substack{\xi_1^{c}=1\\ \xi_1\neq1}}\sum_{\substack{\xi_2^{c}=1\\ \xi_2\neq1}}\sum_{\nu=0}^{k_2}\binom{k_2}{\nu}\mathfrak{B}_{k_1+\nu}(\xi_1)\mathfrak{B}_{k_2-\nu}(\xi_2)\eta^{k_2-\nu}\\ &-\frac{1}{p}\sum_{\rho_1^{p}=1}\sum_{\substack{\xi_1^{c}=1\\ \xi_1\neq1}}\sum_{\substack{\xi_2^{c}=1\\ \xi_2\neq1}}\sum_{\nu=0}^{k_2}\binom{k_2}{\nu}\mathfrak{B}_{k_1+\nu}(\rho_1\xi_1)\mathfrak{B}_{k_2-\nu}(\xi_2)\eta^{k_2-\nu}\\ &-\frac{1}{p}\sum_{\rho_2^{p}=1}\sum_{\substack{\xi_1^{c}=1\\ \xi_1\neq1}}\sum_{\substack{\xi_2^{c}=1\\ \xi_2\neq1}}\sum_{\nu=0}^{k_2}\binom{k_2}{\nu}\mathfrak{B}_{k_1+\nu}(\rho_2\xi_1)\mathfrak{B}_{k_2-\nu}(\rho_2^{\eta}\xi_2)\eta^{k_2-\nu}\end{aligned}$$

$$+\frac{1}{p^2}\sum_{\rho_1^p=1}\sum_{\rho_2^p=1}\sum_{\substack{\xi_1^c=1\\ \xi_1\neq 1}}\sum_{\substack{\xi_2^c=1\\ \xi_2\neq 1}}\sum_{\nu=0}^{k_2}\binom{k_2}{\nu}\mathfrak{B}_{k_1+\nu}(\rho_1\rho_2\xi_1)\mathfrak{B}_{k_2-\nu}(\rho_2^\eta\xi_2)\eta^{k_2-\nu}$$

for $k_1, k_2 \in \mathbb{N}_0$. Similarly to (1.8), we have

$$\sum_{\substack{\xi^c=1\\ \xi\neq 1}}\mathfrak{B}_n(\alpha\xi)=\begin{cases}\left(1-c^{n+1}\right)\frac{B_{n+1}}{n+1} & (\alpha=1)\\ c^{n+1}\mathfrak{B}_n(\alpha^c)-\mathfrak{B}_n(\alpha) & (\alpha\neq 1)\end{cases}\tag{3.6}$$

for $n \in \mathbb{N}_0$. Also, using (1.7) with $k = p$, we have

$$\sum_{\rho^p=1}\sum_{\substack{\xi^c=1\\ \xi\neq 1}}\mathfrak{B}_n(\rho\xi)=\sum_{\rho^p=1}\left(c^{n+1}\mathfrak{B}_n(\rho^c)-\mathfrak{B}_n(\rho)\right)=p^{n+1}\left(1-c^{n+1}\right)\frac{B_{n+1}}{n+1}\tag{3.7}$$

for $n \in \mathbb{N}_0$. Hence, by the continuity with respect to c, we have the following.

Example 3.8. For $k_1, k_2 \in \mathbb{N}_0$, $\eta \in \mathbb{Z}_p$ and $c \in \mathbb{Z}_p^\times$,

$$\begin{aligned}&L_{p,2}(-k_1,-k_2;\omega^{k_1},\omega^{k_2};1,\eta;c)\\ &=\sum_{\nu=0}^{k_2}\binom{k_2}{\nu}\left(1-p^{k_1+\nu}\right)\left(1-c^{k_1+\nu+1}\right)\frac{B_{k_1+\nu+1}}{k_1+\nu+1}\left(1-c^{k_2-\nu+1}\right)\frac{B_{k_2-\nu+1}}{k_2-\nu+1}\eta^{k_2-\nu}\\ &-\frac{1}{p}\sum_{\rho_2^p=1}\sum_{\nu=0}^{k_2}\binom{k_2}{\nu}\left(c^{k_1+\nu+1}\mathfrak{B}_{k_1+\nu}(\rho_2^c)-\mathfrak{B}_{k_1+\nu}(\rho_2)\right)\left(c^{k_2-\nu+1}\mathfrak{B}_{k_2-\nu}(\rho_2^{c\eta})-\mathfrak{B}_{k_2-\nu}(\rho_2^\eta)\right)\eta^{k_2-\nu}\\ &+\frac{1}{p}\sum_{\rho_2^p=1}\sum_{\nu=0}^{k_2}\binom{k_2}{\nu}p^{k_1+\nu}\left(1-c^{k_1+\nu+1}\right)\frac{B_{k_1+\nu+1}}{k_1+\nu+1}\left(c^{k_2-\nu+1}\mathfrak{B}_{k_2-\nu}(\rho_2^{c\eta})-\mathfrak{B}_{k_2-\nu}(\rho_2^\eta)\right)\eta^{k_2-\nu}.\end{aligned}\tag{3.8}$$

In particular when $\eta \in p\mathbb{Z}_p$,

$$\begin{aligned}&L_{p,2}(-k_1,-k_2;\omega^{k_1},\omega^{k_2};1,\eta;c)\\ &=\sum_{\nu=0}^{k_2}\binom{k_2}{\nu}\left(1-p^{k_1+\nu}\right)\left(1-c^{k_1+\nu+1}\right)\left(1-c^{k_2-\nu+1}\right)\frac{B_{k_1+\nu+1}B_{k_2-\nu+1}}{(k_1+\nu+1)(k_2-\nu+1)}\eta^{k_2-\nu},\end{aligned}\tag{3.9}$$

given from (1.5) and (3.7), where the case $c = 2$ was already obtained in our previous paper [34, Section 4]. In the special case when $k_1 \in \mathbb{N}$, $k_2 \in \mathbb{N}_0$ and $k_1 + k_2$ is odd,

$$L_{p,2}(-k_1,-k_2;\omega^{k_1},\omega^{k_2};1,\eta;c)=\frac{c-1}{2}\left(1-c^{k_1+k_2+1}\right)\left(1-p^{k_1+k_2}\right)\frac{B_{k_1+k_2+1}}{k_1+k_2+1}.\tag{3.10}$$

Furthermore, computing the coefficient of $t_1^{n_1}t_2^{n_2}t_3^{n_3}$ in (3.5) with $r = 3$, we obtain the following.

Example 3.9. For $k_1, k_2, k_3 \in \mathbb{N}_0$, $\eta_1, \eta_2 \in p\mathbb{Z}_p$ and $c \in \mathbb{Z}_p^\times$,

$$\begin{aligned}&L_{p,3}(-k_1,-k_2,-k_3;\omega^{k_1},\omega^{k_2},\omega^{k_3};1,\eta_1,\eta_2;c)\\ &=\sum_{\nu_1=0}^{k_3}\sum_{\nu_2=0}^{k_3-\nu_1}\sum_{\kappa=0}^{k_2}\binom{k_2}{\kappa}\binom{k_3}{\nu_1\ \nu_2}\left(1-p^{k_1+\nu_2+\kappa}\right)\left(1-c^{k_1+\nu_2+\kappa+1}\right)\left(1-c^{k_2+\nu_1-\kappa+1}\right)\\ &\quad\times\left(1-c^{k_3-\nu_1-\nu_2+1}\right)\frac{B_{k_1+\nu_2+\kappa+1}}{k_1+\nu_2+\kappa+1}\frac{B_{k_2+\nu_1-\kappa+1}}{k_2+\nu_1-\kappa+1}\frac{B_{k_3-\nu_1-\nu_2+1}}{k_3-\nu_1-\nu_2+1}\end{aligned}$$

$$\times\, \eta_1^{k_2+\nu_1-\kappa}\eta_2^{k_3-\nu_1-\nu_2}, \tag{3.11}$$

where $\binom{N}{\nu_1\ \nu_2}=\frac{N!}{\nu_1!\ \nu_2!\ (N-\nu_1-\nu_2)!}$.

3.2. **Multiple Kummer congruences.** We will formulate multiple Kummer congruences for twisted multiple Bernoulli numbers, certain generalization of the Kummer congruences for Bernoulli numbers (see (3.14) below), from the viewpoint of p-adic multiple L-functions (Theorem 3.10). Then we will extract specific congruences for only ordinary Bernoulli numbers in depth 2 case (Example 3.13).

First we recall the ordinary Kummer congruences. By (2.15), we know from [32, p. 31] that

$$L_{p,1}(1-m;\omega^{m-1};1;c)\equiv L_{p,1}(1-n;\omega^{n-1};1;c) \pmod{p^l} \tag{3.12}$$

for $m,n\in\mathbb{N}_0$ and $l\in\mathbb{N}$ with $m\equiv n \pmod{(p-1)p^{l-1}}$ and $c\in\mathbb{N}_{>1}$. Therefore, from (3.3), we see that

$$(1-c^m)(1-p^{m-1})\frac{B_m}{m}\equiv(1-c^n)(1-p^{n-1})\frac{B_n}{n} \pmod{p^l}. \tag{3.13}$$

Note that $c\in\mathbb{N}_{>1}$ can be chosen arbitrarily under the condition $(c,p)=1$. In particular when $p>2$, for even positive integers m and n satisfying $m\equiv n \pmod{(p-1)p^{l-1}}$ and $n\not\equiv 0 \pmod{p-1}$, we can choose c satisfying $1-c^m\equiv 1-c^n \pmod{p^l}$ and $(1-c^m,p)=(1-c^n,p)=1$, and consequently obtain the ordinary Kummer congruences (see [32, p. 32]):

$$(1-p^{m-1})\frac{B_m}{m}\equiv(1-p^{n-1})\frac{B_n}{n} \pmod{p^l}. \tag{3.14}$$

As a multiple analogue of (3.13), we have the following.

Theorem 3.10 (**Multiple Kummer congruence).** *Let $m_1,\ldots,m_r,n_1,\ldots,n_r\in\mathbb{N}_0$ with $m_j\equiv n_j \bmod (p-1)p^{l_j-1}$ for $l_j\in\mathbb{N}$ $(1\leqslant j\leqslant r)$. Then, for $\gamma_1,\ldots,\gamma_r\in\mathbb{Z}_p$ and $c\in\mathbb{N}_{>1}$ with $(c,p)=1$,*

$$\begin{aligned}&L_{p,r}(-m_1,\ldots,-m_r;\omega^{m_1},\ldots,\omega^{m_r};\gamma_1,\ldots,\gamma_r;c)\\&\equiv L_{p,r}(-n_1,\ldots,-n_r;\omega^{n_1},\ldots,\omega^{n_r};\gamma_1,\ldots,\gamma_r;c) \quad \left(\bmod\ p^{\min\{l_j\,|\,1\leqslant j\leqslant r\}}\right).\end{aligned} \tag{3.15}$$

In other words,

$$\begin{aligned}&\sum_{\substack{\xi_1^c=1\\ \xi_1\neq1}}\cdots\sum_{\substack{\xi_r^c=1\\ \xi_r\neq1}}\mathfrak{B}((m_j);(\xi_j);(\gamma_j))\\&+\sum_{d=1}^{r}\left(-\frac{1}{p}\right)^d\sum_{1\leqslant i_1<\cdots<i_d\leqslant r}\ \sum_{\rho_{i_1}^p=1}\cdots\sum_{\rho_{i_d}^p=1}\ \sum_{\substack{\xi_1^c=1\\ \xi_1\neq1}}\cdots\sum_{\substack{\xi_r^c=1\\ \xi_r\neq1}}\mathfrak{B}((m_j);((\prod_{j\leqslant i_l}\rho_{i_l})^{\gamma_j}\xi_j);(\gamma_j))\\&\equiv\sum_{\substack{\xi_1^c=1\\ \xi_1\neq1}}\cdots\sum_{\substack{\xi_r^c=1\\ \xi_r\neq1}}\mathfrak{B}((n_j);(\xi_j);(\gamma_j))\\&\quad+\sum_{d=1}^{r}\left(-\frac{1}{p}\right)^d\sum_{1\leqslant i_1<\cdots<i_d\leqslant r}\ \sum_{\rho_{i_1}^p=1}\cdots\sum_{\rho_{i_d}^p=1}\ \sum_{\substack{\xi_1^c=1\\ \xi_1\neq1}}\cdots\sum_{\substack{\xi_r^c=1\\ \xi_r\neq1}}\mathfrak{B}((n_j);((\prod_{j\leqslant i_l}\rho_{i_l})^{\gamma_j}\xi_j);(\gamma_j))\\&\qquad\quad\left(\bmod\ p^{\min\{l_j\,|\,1\leqslant j\leqslant r\}}\right).\end{aligned} \tag{3.16}$$

Proof. The proof works in the same way to the case of $L_p(s;\chi)$ stated above (for the details, see [32, Chapter 2, Section 3]). As for the integrand of (2.22), we have

$$\left(\sum_{\nu=1}^{j} x_\nu\gamma_\nu\right)^{m_j} \equiv \left(\sum_{\nu=1}^{j} x_\nu\gamma_\nu\right)^{n_j} \quad \left(\bmod p^{l_j}\right) \quad (1\leqslant j\leqslant r) \tag{3.17}$$

for $(x_j)\in\left(\mathbb{Z}_p^r\right)'_{\{\gamma_j\}}$. Hence (3.17) directly yields (3.15). It follows from Theorem 3.1 that (3.16) holds. □

Remark 3.11. In the case $p=2$, since

$$\left(\mathbb{Z}/2^l\mathbb{Z}\right)^\times \simeq \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2^{l-2}\mathbb{Z} \quad (l\in\mathbb{N}_{>2}),$$

(3.17) holds under the condition

$$m_j\equiv n_j \bmod 2^{l_j-2} \quad (l_j\in\mathbb{N}_{>2};\ 1\leqslant j\leqslant r), \tag{3.18}$$

so does (3.15) under (3.18). Hence the following examples also hold under (3.18) in the case $p=2$.

In the case $r=1$, Theorem 3.10 is nothing but (3.12), a reformulation of (3.13).

In the case $r=2$, we obtain the following.

Example 3.12. For $m_1,m_2,n_1,n_2\in\mathbb{N}_0$ with $m_j\equiv n_j \bmod (p-1)p^{l_j-1}$ $(l_1,l_2\in\mathbb{N})$, $\eta\in\mathbb{Z}_p$ and $c\in\mathbb{Z}_p^\times$,

$$\begin{aligned}
&\sum_{\nu=0}^{m_2}\binom{m_2}{\nu}\left(1-p^{m_1+\nu}\right)\left(1-c^{m_1+\nu+1}\right)\frac{B_{m_1+\nu+1}}{m_1+\nu+1}\left(1-c^{m_2-\nu+1}\right)\frac{B_{m_2-\nu+1}}{m_2-\nu+1}\eta^{m_2-\nu}\\
&-\frac{1}{p}\sum_{\rho_2^p=1}\sum_{\nu=0}^{m_2}\binom{m_2}{\nu}\left(c^{m_1+\nu+1}\mathfrak{B}_{m_1+\nu}(\rho_2^c)-\mathfrak{B}_{m_1+\nu}(\rho_2)\right)\left(c^{m_2-\nu+1}\mathfrak{B}_{m_2-\nu}(\rho_2^{c\eta})-\mathfrak{B}_{m_2-\nu}(\rho_2^{\eta})\right)\eta^{m_2-\nu}\\
&+\frac{1}{p}\sum_{\rho_2^p=1}\sum_{\nu=0}^{m_2}\binom{m_2}{\nu}p^{m_1+\nu}\left(1-c^{m_1+\nu+1}\right)\frac{B_{m_1+\nu+1}}{m_1+\nu+1}\left(c^{m_2-\nu+1}\mathfrak{B}_{m_2-\nu}(\rho_2^{c\eta})-\mathfrak{B}_{m_2-\nu}(\rho_2^{\eta})\right)\eta^{m_2-\nu}\\
&\equiv\sum_{\nu=0}^{n_2}\binom{n_2}{\nu}\left(1-p^{n_1+\nu}\right)\left(1-c^{n_1+\nu+1}\right)\frac{B_{n_1+\nu+1}}{n_1+\nu+1}\left(1-c^{n_2-\nu+1}\right)\frac{B_{n_2-\nu+1}}{n_2-\nu+1}\eta^{n_2-\nu}\\
&-\frac{1}{p}\sum_{\rho_2^p=1}\sum_{\nu=0}^{n_2}\binom{n_2}{\nu}\left(c^{n_1+\nu+1}\mathfrak{B}_{n_1+\nu}(\rho_2^c)-\mathfrak{B}_{n_1+\nu}(\rho_2)\right)\left(c^{n_2-\nu+1}\mathfrak{B}_{n_2-\nu}(\rho_2^{c\eta})-\mathfrak{B}_{n_2-\nu}(\rho_2^{\eta})\right)\eta^{n_2-\nu}\\
&+\frac{1}{p}\sum_{\rho_2^p=1}\sum_{\nu=0}^{n_2}\binom{n_2}{\nu}p^{n_1+\nu}\left(1-c^{n_1+\nu+1}\right)\frac{B_{n_1+\nu+1}}{n_1+\nu+1}\left(c^{n_2-\nu+1}\mathfrak{B}_{n_2-\nu}(\rho_2^{c\eta})-\mathfrak{B}_{n_2-\nu}(\rho_2^{\eta})\right)\eta^{n_2-\nu}\\
&\qquad\left(\bmod p^{\min\{l_1,l_2\}}\right).
\end{aligned}\tag{3.19}$$

Proof. We obtain the claim by setting $(\gamma_1,\gamma_2)=(1,\eta)$ for $\eta\in\mathbb{Z}_p$ in (3.15) and using Example 3.8. □

Next we consider certain congruences between ordinary Bernoulli numbers.

Example 3.13. We set $\eta = p$ and $c \in \mathbb{N}_{>1}$ with $(c,p)=1$ in (3.19). Note that this case was already calculated in (3.9) with $\eta = p$. Hence we similarly obtain the following **double Kummer congruence** for ordinary Bernoulli numbers:

$$\begin{aligned}
&\sum_{\nu=0}^{m_2} \binom{m_2}{\nu} \left(1-p^{m_1+\nu}\right)\left(1-c^{m_1+\nu+1}\right)\left(1-c^{m_2-\nu+1}\right) \frac{B_{m_1+\nu+1}B_{m_2-\nu+1}}{(m_1+\nu+1)(m_2-\nu+1)} p^{m_2-\nu} \\
&\equiv \sum_{\nu=0}^{n_2} \binom{n_2}{\nu} \left(1-p^{n_1+\nu}\right)\left(1-c^{n_1+\nu+1}\right)\left(1-c^{n_2-\nu+1}\right) \frac{B_{n_1+\nu+1}B_{n_2-\nu+1}}{(n_1+\nu+1)(n_2-\nu+1)} p^{n_2-\nu} \\
&\qquad \left(\bmod\ p^{\min\{l_1,l_2\}}\right) \qquad (3.20)
\end{aligned}$$

for $m_1,m_2,n_1,n_2 \in \mathbb{N}_0$ with $m_j \equiv n_j \bmod (p-1)p^{l_j-1}$ $(l_j \in \mathbb{N};\ j=1,2)$. In particular when m_1 and m_2 are of different parity, we have from (3.10) that

$$\begin{aligned}
&\frac{c-1}{2}\left(1-c^{m_1+m_2+1}\right)\left(1-p^{m_1+m_2}\right)\frac{B_{m_1+m_2+1}}{m_1+m_2+1} \\
&\equiv \frac{c-1}{2}\left(1-c^{n_1+n_2+1}\right)\left(1-p^{n_1+n_2}\right)\frac{B_{n_1+n_2+1}}{n_1+n_2+1} \quad \left(\bmod\ p^{\min\{l_1,l_2\}}\right), \qquad (3.21)
\end{aligned}$$

which is equivalent to (3.13) in the case $(c,p)=1$ with $p^2 \nmid (c-1)$. Therefore, choosing c suitably, we obtain the ordinary Kummer congruence (3.14). From this observation, we can regard (3.20) as a certain general form of the Kummer congruence including (3.14). We do not know whether our (3.20) is a kind of new congruence of Bernoulli numbers or it follows from the ordinary Kummer congruence (3.13).

In the case $r=3$, we obtain the following.

Example 3.14. Using (3.11) with $(\gamma_1,\gamma_2,\gamma_3)=(1,p,p)$, we can similarly obtain the following **triple Kummer congruence** for ordinary Bernoulli numbers:

$$\begin{aligned}
&\sum_{\nu_1=0}^{m_3}\sum_{\nu_2=0}^{m_3-\nu_1}\sum_{\kappa=0}^{m_2}\binom{m_2}{\kappa}\binom{m_3}{\nu_1\ \nu_2}\left(1-p^{m_1+\nu_2+\kappa}\right)\left(1-c^{m_1+\nu_2+\kappa+1}\right)\left(1-c^{m_2+\nu_1-\kappa+1}\right) \\
&\quad \times\left(1-c^{m_3-\nu_1-\nu_2+1}\right)\frac{B_{m_1+\nu_2+\kappa+1}}{m_1+\nu_2+\kappa+1}\frac{B_{m_2+\nu_1-\kappa+1}}{m_2+\nu_1-\kappa+1}\frac{B_{m_3-\nu_1-\nu_2+1}}{m_3-\nu_1-\nu_2+1}p^{m_2+m_3-\kappa-\nu_2} \\
&\equiv \sum_{\nu_1=0}^{n_3}\sum_{\nu_2=0}^{n_3-\nu_1}\sum_{\kappa=0}^{n_2}\binom{n_2}{\kappa}\binom{n_3}{\nu_1\ \nu_2}\left(1-p^{n_1+\nu_2+\kappa}\right)\left(1-c^{n_1+\nu_2+\kappa+1}\right)\left(1-c^{n_2+\nu_1-\kappa+1}\right) \\
&\quad \times\left(1-c^{n_3-\nu_1-\nu_2+1}\right)\frac{B_{n_1+\nu_2+\kappa+1}}{n_1+\nu_2+\kappa+1}\frac{B_{n_2+\nu_1-\kappa+1}}{n_2+\nu_1-\kappa+1}\frac{B_{n_3-\nu_1-\nu_2+1}}{n_3-\nu_1-\nu_2+1}p^{n_2+n_3-\kappa-\nu_2} \\
&\quad \left(\bmod\ p^{\min\{l_1,l_2,l_3\}}\right) \qquad (3.22)
\end{aligned}$$

for $m_1,m_2,m_3,n_1,n_2,n_3 \in \mathbb{N}_0$ with $m_j \equiv n_j \bmod (p-1)p^{l_j-1}$ $(l_j \in \mathbb{N};\ j=1,2,3)$. It is also unclear whether this is new or follows from (3.13).

3.3. **Functional relations for p-adic multiple L-functions.** In this subsection, we will prove certain functional relations with a parity condition among p-adic multiple L-functions

(Theorem 3.15). They extend the vanishing property of the Kubota-Leopoldt p-adic L-function with odd characters in the single variable case (cf. Proposition 2.6 and Example 3.19) and the functional relations among p-adic double L-functions proved by [34] in the double variable case under the condition $c = 2$ (cf. Example 3.20). Also this can be regarded as a certain p-adic analogue of the parity result for MZVs (see Remark 3.18). It should be noted that the following functional relations are peculiar to the p-adic case, which are derived from a relation (3.31) among Bernoulli numbers.

Let $r \geqslant 2$. For each $J \subset \{1, \ldots, r\}$ with $1 \in J$, we write

$$J = \{j_1(J), j_2(J), \ldots, j_{|J|}(J)\} \quad (1 = j_1(J) < \cdots < j_{|J|}(J)),$$

where $|J|$ implies the number of elements of J. In addition, we formally let $j_{|J|+1}(J) = r+1$.

Theorem 3.15. *For $r \in \mathbb{N}_{>1}$, let $k_1, \ldots, k_r \in \mathbb{Z}$ with $k_1 + \cdots + k_r \not\equiv r \pmod 2$, $\gamma_1 \in \mathbb{Z}_p$, $\gamma_2, \ldots, \gamma_r \in p\mathbb{Z}_p$ and $c \in \mathbb{Z}_p^\times$. Then, for $s_1, \ldots, s_r \in \mathbb{Z}_p$,*

$$\begin{aligned}
&L_{p,r}(s_1, \ldots, s_r; \omega^{k_1}, \ldots, \omega^{k_r}; \gamma_1, \gamma_2, \ldots, \gamma_r; c) \\
&= - \sum_{\substack{J \subsetneq \{1,\ldots,r\} \\ 1 \in J}} \left(\frac{1-c}{2}\right)^{r-|J|} \\
&\quad \times L_{p,|J|}\Big(\Big(\sum_{j_\mu(J) \leqslant l < j_{\mu+1}(J)} s_l\Big)_{\mu=1}^{|J|}; \big(\omega^{\sum_{j_\mu(J) \leqslant l < j_{\mu+1}(J)} k_l}\big)_{\mu=1}^{|J|}; (\gamma_j)_{j \in J}; c\Big). \qquad (3.23)
\end{aligned}$$

For the proof of this theorem, we prepare some notation. Similarly to Definition 1.9, we consider $\widetilde{\mathfrak{H}}_1(t;\gamma;c)$ in $\mathbb{C}_p$, that is,

$$\begin{aligned}
\widetilde{\mathfrak{H}}_1(t;\gamma;c) &= \frac{1}{e^{\gamma t} - 1} - \frac{c}{e^{c\gamma t} - 1} \\
&= \sum_{m=0}^{\infty} \left(1 - c^{m+1}\right) B_{m+1} \frac{(\gamma t)^m}{(m+1)!} \qquad (3.24) \\
&= \frac{c-1}{2} + \sum_{m=1}^{\infty} \left(c^{m+1} - 1\right) \zeta(-m) \frac{(\gamma t)^m}{m!}
\end{aligned}$$

for $c \in \mathbb{Z}_p^\times$ and $\gamma \in \mathbb{Z}_p$. For simplicity, we set

$$\mathcal{B}(n;\gamma;c) := \left(1 - c^{n+1}\right) \frac{B_{n+1}}{n+1} \gamma^n \quad (n \in \mathbb{N}). \qquad (3.25)$$

It follows from (2.8) and (2.9) that

$$\widetilde{\mathfrak{H}}_1(t;\gamma;c) = \sum_{m=0}^{\infty} \int_{\mathbb{Z}_p} x^m d\widetilde{\mathfrak{m}}_c(x) \frac{(\gamma t)^m}{m!} = \int_{\mathbb{Z}_p} e^{x\gamma t} d\widetilde{\mathfrak{m}}_c(x), \qquad (3.26)$$

$$\begin{aligned}
\mathcal{H}_1(t;\gamma;c) :&= \widetilde{\mathfrak{H}}_1(t;\gamma;c) - \frac{c-1}{2} = \sum_{m=1}^{\infty} \mathcal{B}(m;\gamma;c) \frac{t^m}{m!} \\
&= \int_{\mathbb{Z}_p} (e^{x\gamma t} - 1) d\widetilde{\mathfrak{m}}_c(x). \qquad (3.27)
\end{aligned}$$

We know that $B_{2m+1} = 0$, namely $\mathcal{B}(2m;\gamma;c) = 0$ for $m \in \mathbb{N}$. Hence we have

$$\mathcal{H}_1(-t;\gamma;c) = -\mathcal{H}_1(t;\gamma;c). \qquad (3.28)$$

Now we define certain multiple analogues of Bernoulli numbers as follows. For $r \in \mathbb{N}$ and $\gamma_1, \ldots, \gamma_r \in \mathbb{Z}_p$, we set

$$\begin{aligned}\mathcal{H}_r(t_1, \ldots, t_r; \gamma_1, \ldots, \gamma_r; c) &:= \prod_{j=1}^{r} \mathcal{H}_1(\sum_{k=j}^{r} t_k; \gamma_j; c) \\ &\quad \left(= \prod_{j=1}^{r} \left\{ \widetilde{\mathfrak{H}}_1(\sum_{k=j}^{r} t_k; \gamma_j; c) - \frac{c-1}{2} \right\} \right) \\ &\quad = \sum_{n_1=1}^{\infty} \cdots \sum_{n_r=1}^{\infty} \mathcal{B}(n_1, \ldots, n_r; \gamma_1, \ldots, \gamma_r; c) \frac{t_1^{n_1}}{n_1!} \cdots \frac{t_r^{n_r}}{n_r!}.\end{aligned} \tag{3.29}$$

By (3.28), we have

$$\mathcal{H}_r(-t_1, \ldots, -t_r; \gamma_1, \ldots, \gamma_r; c) = (-1)^r \mathcal{H}_r(t_1, \ldots, t_r; \gamma_1, \ldots, \gamma_r; c). \tag{3.30}$$

Hence, when $n_1 + \cdots + n_r \not\equiv r \pmod 2$, we obtain

$$\mathcal{B}(n_1, \ldots, n_r; \gamma_1, \ldots, \gamma_r; c) = 0. \tag{3.31}$$

Remark 3.16. From (1.15), we have

$$\widetilde{\mathfrak{H}}_r((t_j); (\gamma_j); c) = \prod_{j=1}^{r} \widetilde{\mathfrak{H}}_1(\sum_{k=j}^{r} t_k; \gamma_j; c). \tag{3.32}$$

Comparing (3.29) with (3.32), we see that $\mathcal{H}_r((t_j); (\gamma_j); c)$ is defined as a slight modification of $\widetilde{\mathfrak{H}}_r((t_j); (\gamma_j); c)$ so that (3.30) holds.

It follows from (3.25), (3.27) and (3.29) that $\mathcal{B}(n_1, \ldots, n_r; \gamma_1, \ldots, \gamma_r; c)$ can be expressed as a polynomial in $\{B_n\}_{n \geqslant 1}$ and $\{\gamma_1, \ldots, \gamma_r\}$ with $\mathbb{Q}$-coefficients.

Lemma 3.17. *For $n_1, \ldots, n_r \in \mathbb{N}$, $\gamma_1, \ldots, \gamma_r \in \mathbb{Z}_p$ and $c \in \mathbb{Z}_p^{\times}$,*

$$\mathcal{B}(n_1, \ldots, n_r; \gamma_1, \ldots, \gamma_r; c) = \sum_{\substack{J \subset \{1, \ldots, r\} \\ 1 \in J}} \left(\frac{1-c}{2} \right)^{r-|J|} \int_{\mathbb{Z}_p^{|J|}} \prod_{l=1}^{r} (\sum_{\substack{j \in J \\ j \leqslant l}} x_j \gamma_j)^{n_l} \prod_{j \in J} d\widetilde{\mathfrak{m}}_c(x_j), \tag{3.33}$$

where the empty product is interpreted as 1.

Proof. From (3.27) and (3.29), we have

$$\begin{aligned}&\mathcal{H}_r(t_1, \ldots, t_r; \gamma_1, \ldots, \gamma_r; c) \\ &\quad = \int_{\mathbb{Z}_p^r} \prod_{j=1}^{r} (e^{x_j \gamma_j (\sum_{l=j}^{r} t_l)} - 1) \prod_{j=1}^{r} d\widetilde{\mathfrak{m}}_c(x_j) \\ &\quad = \int_{\mathbb{Z}_p^r} \sum_{J \subset \{1, \ldots, r\}} (-1)^{r-|J|} \exp\Big(\sum_{j \in J} x_j \gamma_j (\sum_{l=j}^{r} t_l) \Big) \prod_{j=1}^{r} d\widetilde{\mathfrak{m}}_c(x_j) \\ &\quad = \int_{\mathbb{Z}_p^r} \sum_{J \subset \{1, \ldots, r\}} (-1)^{r-|J|} \exp\Big(\sum_{l=1}^{r} t_l (\sum_{\substack{j \in J \\ j \leqslant l}} x_j \gamma_j) \Big) \prod_{j=1}^{r} d\widetilde{\mathfrak{m}}_c(x_j)\end{aligned}$$

$$= \sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} \sum_{J\subset\{1,\ldots,r\}} (-1)^{r-|J|} \int_{\mathbb{Z}_p^r} \prod_{l=1}^{r} \frac{\left(t_l(\sum_{\substack{j\in J\\ j\leqslant l}} x_j\gamma_j)\right)^{n_l}}{n_l!} \prod_{j=1}^{r} d\widetilde{\mathfrak{m}}_c(x_j)$$
$$= \sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} \sum_{J\subset\{1,\ldots,r\}} (-1)^{r-|J|} \int_{\mathbb{Z}_p^r} \prod_{l=1}^{r} (\sum_{\substack{j\in J\\ j\leqslant l}} x_j\gamma_j)^{n_l} \prod_{j=1}^{r} d\widetilde{\mathfrak{m}}_c(x_j) \frac{t_1^{n_1}}{n_1!} \cdots \frac{t_r^{n_r}}{n_r!}.$$

Here we consider each coefficient of $\frac{t_1^{n_1}}{n_1!} \cdots \frac{t_r^{n_r}}{n_r!}$ for $n_1, \ldots, n_r \in \mathbb{N}$. If $1 \notin J$ then

$$\sum_{\substack{j\in J\\ j\leqslant 1}} x_j\gamma_j$$

is an empty sum which implies 0. Hence we obtain from (2.9) that

$$\begin{aligned}
&\mathcal{B}(n_1, \ldots, n_r; \gamma_1, \ldots, \gamma_r; c)\\
&\quad = \sum_{J\subset\{1,\ldots,r\}} (-1)^{r-|J|} \int_{\mathbb{Z}_p^r} \prod_{l=1}^{r} (\sum_{\substack{j\in J\\ j\leqslant l}} x_j\gamma_j)^{n_l} \prod_{j=1}^{r} d\widetilde{\mathfrak{m}}_c(x_j)\\
&\quad = \sum_{\substack{J\subset\{1,\ldots,r\}\\ 1\in J}} (-1)^{r-|J|} \int_{\mathbb{Z}_p^r} \prod_{l=1}^{r} (\sum_{\substack{j\in J\\ j\leqslant l}} x_j\gamma_j)^{n_l} \prod_{j=1}^{r} d\widetilde{\mathfrak{m}}_c(x_j)\\
&\quad = \sum_{\substack{J\subset\{1,\ldots,r\}\\ 1\in J}} \Big(\frac{1-c}{2}\Big)^{r-|J|} \int_{\mathbb{Z}_p^{|J|}} \prod_{l=1}^{r} (\sum_{\substack{j\in J\\ j\leqslant l}} x_j\gamma_j)^{n_l} \prod_{j\in J} d\widetilde{\mathfrak{m}}_c(x_j)
\end{aligned}$$

for $n_1, \ldots, n_r \in \mathbb{N}$. Thus we complete the proof. □

Proof of Theorem 3.15. If $\gamma_1 \in p\mathbb{Z}_p$, then it follows from Remark 2.10 that all functions on the both sides of (3.23) are zero-functions. Hence (3.23) holds trivially. Therefore we only consider the case $\gamma_1 \in \mathbb{Z}_p^\times,\ \gamma_2, \ldots, \gamma_r \in p\mathbb{Z}_p$ in (3.33).

First we consider the case $\gamma_1 = 1$. Setting $\gamma_1 = 1$ and $\gamma_2, \ldots, \gamma_r \in p\mathbb{Z}_p$ in (3.33), we have

$$\begin{aligned}
&\mathcal{B}(n_1, \ldots, n_r; \gamma_1, \gamma_2, \ldots, \gamma_r; c)\\
&\quad + \sum_{\substack{J\subset\{1,\ldots,r\}\\ 1\in J}} \Big(\frac{1-c}{2}\Big)^{r-|J|} \int_{\mathbb{Z}_p^{|J|}} \prod_{l=1}^{r} (\sum_{\substack{j\in J\\ j\leqslant l}} x_j\gamma_j)^{n_l} \prod_{j\in J} d\widetilde{\mathfrak{m}}_c(x_j)
\end{aligned}$$

for $n_1, \ldots, n_r \in \mathbb{N}$. From the condition $\gamma_1 = 1$ and $\gamma_2, \ldots, \gamma_r \in p\mathbb{Z}_p$, we can see that

$$(\mathbb{Z}_p^{|J|})'_{\{\gamma_j\}_{j\in J}} = \mathbb{Z}_p^\times \times \mathbb{Z}_p^{|J|-1} = \mathbb{Z}_p^{|J|} \setminus \left(p\mathbb{Z}_p \times \mathbb{Z}_p^{|J|-1}\right)$$

for any $J \subset \{1, \ldots, r\}$ with $1 \in J$. Therefore we have

$$\begin{aligned}
&\sum_{\substack{J\subset\{1,\ldots,r\}\\ 1\in J}} \Big(\frac{1-c}{2}\Big)^{r-|J|}\\
&\quad \times L_{p,|J|}\Big(\Big(-\sum_{j_\mu(J)\leqslant l<j_{\mu+1}(J)} n_l\Big)_{\mu=1}^{|J|}; \big(\omega^{\sum_{j_\mu(J)\leqslant l<j_{\mu+1}(J)} n_l}\big)_{\mu=1}^{|J|}; (\gamma_j)_{j\in J}; c\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\substack{J\subset\{1,\ldots,r\}\\ 1\in J}} \Big(\frac{1-c}{2}\Big)^{r-|J|}\int_{\mathbb{Z}_p^\times\times\mathbb{Z}_p^{|J|-1}} \prod_{l=1}^{r}(\sum_{\substack{j\in J\\ j\leqslant l}} x_j\gamma_j)^{n_l}\prod_{j\in J} d\widetilde{\mathfrak{m}}_c(x_j)\\
&= \sum_{\substack{J\subset\{1,\ldots,r\}\\ 1\in J}} \Big(\frac{1-c}{2}\Big)^{r-|J|}\int_{\mathbb{Z}_p^{|J|}} \prod_{l=1}^{r}(\sum_{\substack{j\in J\\ i\leqslant l}} x_j\gamma_j)^{n_l}\prod_{j\in J} d\widetilde{\mathfrak{m}}_c(x_j)\\
&\quad - p^{\sum_{l=1}^r n_l}\sum_{\substack{J\subset\{1,\ldots,r\}\\ 1\in J}} \Big(\frac{1-c}{2}\Big)^{r-|J|}\int_{\mathbb{Z}_p^{|J|}} \prod_{l=1}^{r}(x_1+\sum_{\substack{j\in J\\ 1<j\leqslant l}} x_j\gamma_j/p)^{n_l}\prod_{j\in J} d\widetilde{\mathfrak{m}}_c(x_j)\\
&= \mathcal{B}(n_1,\ldots,n_r;1,\gamma_2,\ldots,\gamma_r;c) - p^{\sum_{l=1}^r n_l}\mathcal{B}(n_1,\ldots,n_r;1,\gamma_2/p,\ldots,\gamma_r/p;c),
\end{aligned}
\tag{3.34}
$$

which is equal to 0 when $n_1+\cdots+n_r\not\equiv r \pmod 2$ because of (3.31). Let $k_1,\ldots,k_r\in\mathbb{Z}$ with $k_1+\cdots+k_r\not\equiv r \pmod 2$. Then the above consideration gives that

$$
\begin{aligned}
&\sum_{\substack{J\subsetneq\{1,\ldots,r\}\\ 1\in J}} \Big(\frac{1-c}{2}\Big)^{r-|J|}\\
&\times L_{p,|J|}\Big(\Big(-\sum_{j_\mu(J)\leqslant l<j_{\mu+1}(J)} n_l\Big)_{\mu=1}^{|J|};\big(\omega^{\sum_{j_\mu(J)\leqslant l<j_{\mu+1}(J)}k_l}\big)_{\mu=1}^{|J|};(\gamma_j)_{j\in J};c\Big)\\
&\quad + L_{p,r}(-n_1,\ldots,-n_r;\omega^{k_1},\ldots,\omega^{k_r};\gamma_1,\gamma_2,\ldots,\gamma_r;c)=0
\end{aligned}
\tag{3.35}
$$

holds on

$$S_{\{k_j\}}=\{(n_1,\ldots,n_r)\in\mathbb{N}^r \mid n_j\in k_j+(p-1)\mathbb{Z}\ (1\leqslant j\leqslant r)\}.$$

Since $S_{\{k_j\}}$ is dense in $\mathbb{Z}_p^r$, we obtain (3.23) in the case $\gamma_1=1$.

Next we consider the case $\gamma_1\in\mathbb{Z}_p^\times$. We can easily check that if $\gamma_1\in\mathbb{Z}_p^\times$,

$$
\begin{aligned}
&L_{p,r}((s_j);(\omega^{k_j});(\gamma_j);c)\\
&\quad=\langle\gamma_1\rangle^{-s_1-\cdots-s_r}\omega^{k_1+\cdots+k_r}(\gamma_1)\ L_{p,r}((s_j);(\omega^{k_j});(\gamma_j/\gamma_1);c),\\
&L_{p,|J|}\Big(\Big(\sum_{j_\mu(J)\leqslant l<j_{\mu+1}(J)} s_l\Big)_{\mu=1}^{|J|};\big(\omega^{\sum_{j_\mu(J)\leqslant l<j_{\mu+1}(J)}k_l}\big)_{\mu=1}^{|J|};(\gamma_j)_{j\in J};c\Big)\\
&\quad=\langle\gamma_1\rangle^{-s_1-\cdots-s_r}\omega^{k_1+\cdots+k_r}(\gamma_1)\\
&\qquad\times L_{p,|J|}\Big(\Big(\sum_{j_\mu(J)\leqslant l<j_{\mu+1}(J)} s_l\Big)_{\mu=1}^{|J|};\big(\omega^{\sum_{j_\mu(J)\leqslant l<j_{\mu+1}(J)}k_l}\big)_{\mu=1}^{|J|};(\gamma_j/\gamma_1)_{j\in J};c\Big)
\end{aligned}
$$

for $J\subsetneq\{1,\ldots,r\}$ with $1\in J$. Since we already proved (3.23) corresponding to the case $\{\gamma_j/\gamma_1\}_{j=1}^r$ in the above argument, we consequently obtain (3.23) in the case $\gamma_1\in\mathbb{Z}_p^\times$ by multiplying

$$\langle\gamma_1\rangle^{-s_1-\cdots-s_r}\omega^{k_1+\cdots+k_r}(\gamma_1)$$

on the both sides. Thus we obtain the proof of Theorem 3.15. □

Remark 3.18. Note that each p-adic multiple L-function appearing on the right-hand side of (3.23) is of lower depth than r. The relation (3.23) reminds us of the *parity result* for MZVs which implies that the MZV whose depth and weight are of different parity can be expressed as a polynomial in MZVs of lower depth with $\mathbb{Q}$-coefficients. In fact, (3.35) shows that $L_{p,r}((-n_j);(\omega^{k_j});(\gamma_j);c)$ can be expressed as a polynomial in p-adic multiple L-values of lower depth than r at non-positive integers with $\mathbb{Q}$-coefficients, when $n_1+\cdots+n_r \not\equiv r \pmod 2$. This can be regarded as a p-adic version of the parity result for p-adic multiple L-values. By the density property, we obtain its continuous version, that is, the functional relation (3.23).

The following result corresponds to the case $r=1$ in Theorem 3.15.

Example 3.19. Setting $r=1$ in (3.34), we obtain from Definition 2.3 that

$$\begin{aligned}&\left(\frac{1-c}{2}\right)^{r-1}\int_{\mathbb{Z}_p^\times} x_1^{n_1}d\widetilde{\mathfrak{m}}_c(x_1)\left(=\left(\frac{1-c}{2}\right)^{r-1}\left(c^{n_1+1}-1\right)L_p(-n_1;\omega^{n_1+1})\right)\\ &=\mathcal{B}(n_1;1;c)-p^{n_1}\mathcal{B}(n_1;1;c)=0\end{aligned}$$

for $n_1\in\mathbb{N}$ satisfying $n_1\not\equiv 1 \pmod 2$ namely n_1+1 is odd, because of (3.31). Hence, if k is odd then $L_p(-n_1;\omega^k)=0$ for $n_1\in\mathbb{N}$ satisfying $n_1+1\equiv k \pmod{p-1}$. This implies

$$L_p(s,\omega^k)\equiv 0$$

when k is odd, the statement of Proposition 2.6. Thus we can regard Theorem 3.15 as a multiple version of Proposition 2.6.

Next we consider the case $r=2$.

Example 3.20. Let $k,l\in\mathbb{N}_0$ with $2\nmid(k+l)$, $\gamma_1\in\mathbb{Z}_p^\times$, $\gamma_2\in p\mathbb{Z}_p$ and $c\in\mathbb{Z}_p^\times$. Then, for $s_1,s_2\in\mathbb{Z}_p$, we obtain from (3.23) and (2.15) that

$$\begin{aligned}&L_{p,2}(s_1,s_2;\omega^k,\omega^l;\gamma_1,\gamma_2;c)\\ &=\frac{c-1}{2}\langle\gamma_1\rangle^{-s_1-s_2}\omega^{k+l}(\gamma_1)\left(\langle c\rangle^{1-s_1-s_2}\omega^{k+l+1}(c)-1\right)L_p(s_1+s_2;\omega^{k+l+1}).\end{aligned}\tag{3.36}$$

Note that this functional relation in the case $p>2$, $c=2$ and $\gamma_1=1$ was already proved in [34] by a different method.

Remark 3.21. As we noted above, when $k+l$ is even, $L_p(s;\omega^{k+l+1})$ is the zero-function, so is the right-hand of (3.36). On the other hand, even if $k+l$ is even, the left-hand of (3.36) is not necessarily the zero-function. In fact, it follows from (3.9) that

$$L_{p,2}(-1,-1;\omega,\omega;1,\gamma_2;c)=\frac{(1-c^2)^2(1-p)}{4}B_2^2\gamma_2$$

for $\gamma_2\in p\mathbb{Z}_p$, which does not vanish if $\gamma_2\neq 0$. Therefore this case implies that $L_{p,2}(s_1,s_2;\omega,\omega;1,\gamma_2;c)$ is not the zero-function. Therefore $L_{p,2}(s_1,s_2;\omega^k,\omega^l;1,\gamma_2;c)$ seems to have more information beyond the Kubota-Leopoldt p-adic L-function.

Further we consider the case $r=3$.

Example 3.22. Let $k_1,k_2,k_3\in\mathbb{Z}$ with $2\mid(k_1+k_2+k_3)$, $\gamma_1\in\mathbb{Z}_p^\times$, $\gamma_2,\gamma_3\in p\mathbb{Z}_p$ and $c\in\mathbb{Z}_p^\times$. Then, for $s_1,s_2,s_3\in\mathbb{Z}_p$, we obtain from (3.23) that

$$\begin{aligned}&L_{p,3}(s_1,s_2,s_3;\omega^{k_1},\omega^{k_2},\omega^{k_3};\gamma_1,\gamma_2,\gamma_3;c)\\ &=\frac{1-c}{2}L_{p,2}(s_1,s_2+s_3;\omega^{k_1},\omega^{k_2+k_3};\gamma_1,\gamma_2;c)+\frac{1-c}{2}L_{p,2}(s_1+s_2,s_3;\omega^{k_1+k_2},\omega^{k_3};\gamma_1,\gamma_3;c)\end{aligned}$$

$$
\begin{aligned}
&- \left(\frac{1-c}{2}\right)^2 \langle \gamma_1 \rangle^{-s_1-s_2-s_3} \omega^{k_1+k_2+k_3}(\gamma_1) \\
&\quad \times \left(\langle c \rangle^{1-s_1-s_2-s_3} \omega^{k_1+k_2+k_3+1}(c) - 1\right) L_p(s_1+s_2+s_3; \omega^{k_1+k_2+k_3+1}).
\end{aligned}
$$

Note that from (2.15) the third term on the right-hand side vanishes because $L_p(s;\omega^k)$ is the zero function when k is odd.

Using Theorem 3.15 and Examples 3.20 and 3.22, we can immediately obtain the following result by induction on $r \geqslant 2$.

Corollary 3.23. *Let $r \in \mathbb{N}_{\geqslant 2}$, $k_1, \ldots, k_r \in \mathbb{Z}$ with $k_1 + \cdots + k_r \not\equiv r \pmod 2$, $\gamma_1 \in \mathbb{Z}_p$, $\gamma_2, \ldots, \gamma_r \in p\mathbb{Z}_p$ and $c \in \mathbb{Z}_p^\times$. Then*

$$L_{p,r}(s_1, \ldots, s_r; \omega^{k_1}, \ldots, \omega^{k_r}; \gamma_1, \ldots, \gamma_r; c)$$

can be expressed as a polynomial in p-adic j-ple L-functions for $j \in \{1, 2, \ldots, r-1\}$ satisfying $j \not\equiv r \pmod 2$, with $\mathbb{Q}$-coefficients.

4. Special values of p-adic multiple L-functions at positive integers

In our previous section, particularly in Theorem 3.1, we saw that the special values of our p-adic multiple L-functions at non-positive integers are expressed in terms of the twisted multiple Bernoulli numbers (Definition 1.6), which are the special values of the complex multiple zeta-functions of generalized Euler-Zagier-Lerch type at non-positive integers (cf. Theorem 1.13). In contrast, in this section we will discuss their special values at positive integers. For this purpose, we will introduce a specific p-adic function in each subsection. In the complex case, the special values of multiple zeta function (cf. (0.3)) at positive integers are given by the special values of multiple polylogarithms at unity (see (1.3)). Our main result is Theorem 4.41 where we will show a p-adic analogue of the equality (1.3): We will establish a close relationship of our p-adic multiple L-functions with the p-adic TMPL's [4], generalizations of p-adic multiple polylogarithm introduced by the first-named author [22, 23] and Yamashita [47] for a study of the p-adic realization of certain motivic fundamental group. It is achieved by showing that the special values of p-adic multiple L-function at positive integers are described by p-adic twisted multiple L-values; the special values at unity of the p-adic TMPL's. (Definition 4.29, see also [22, 47]). It generalizes a previous work of Coleman [15]. To connect p-adic multiple L-functions with p-adic TMPL's, we will introduce p-adic rigid TMPL's (Definition 4.4) and their partial versions (Definition 4.16) as intermediate objects and investigate their several basic properties mainly in Subsections 4.1 and 4.2.

4.1. p-adic rigid twisted multiple polylogarithms.

This subsection is to introduce p-adic rigid TMPL's (Definition 4.4) and to give a description of special values of p-adic multiple L-functions at positive integers by special values of p-adic rigid TMPL's at roots unity (Theorem 4.9) which extends Coleman's result (4.7).

[4] In this paper, TMPL stands for twisted multiple polylogarithm.

First, we briefly review the minimum basics of rigid analysis in our specific case.

Notation 4.1 (cf. [10] etc.)**.** Let $\alpha_1,\dots,\alpha_n \in \mathbb{C}_p$ and $\delta_1,\dots,\delta_n \in \mathbb{Q}_{>0}$ and $\delta_0 \in \mathbb{Q}_{\geqslant 0}$. The space

$$X = \left\{ z \in \mathbf{P}^1(\mathbb{C}_p) \;\middle|\; |z-\alpha_i|_p \geqslant \delta_i \ (i=1,\dots,n), |z|_p \leqslant 1/\delta_0 \right\} \tag{4.1}$$

is equipped with a structure of *affinoid*, a special type of rigid analytic space. A *rigid analytic function* on X is a functions $f(z)$ on X which admits the convergent expansion

$$f(z) = \sum_{m\geqslant 0} a_m(\infty;f) z^m + \sum_{i=1}^{n}\sum_{m>0} \frac{a_m(\alpha_i;f)}{(z-\alpha_i)^m}$$

with $\mathbb{C}_p$-coefficients. The expressions are actually unique (the Mittag-Leffler decompositions), which can be shown from, for example [20, I.1.3]. We denote the algebra of rigid analytic functions on X by $A^{\mathrm{rig}}(X)$.

Notation 4.2. For a in $\mathbf{P}^1(\mathbb{C}_p)$, $\bar{a}$ means the image $\mathrm{red}(a)$ by the reduction map

$$\mathrm{red}: \mathbf{P}^1(\mathbb{C}_p) \to \mathbf{P}^1(\overline{\mathbb{F}}_p) \left(= \overline{\mathbb{F}}_p \cup \{\bar{\infty}\}\right),$$

where $\overline{\mathbb{F}}_p$ is the algebraic closure of $\mathbb{F}_p$. For a finite subset $D \subset \mathbf{P}^1(\mathbb{C}_p)$, we define $\bar{D} = \mathrm{red}(D) \subset \mathbf{P}^1(\overline{\mathbb{F}}_p)$. For $a_0 \in \mathbf{P}^1(\overline{\mathbb{F}}_p)$, its tubular neighborhood $]a_0[$ means the inverse image $\mathrm{red}^{-1}(a_0)$ of the reduction map. Namely, $]\bar{a}[= \{x \in \mathbf{P}^1(\mathbb{C}_p) \,|\, |x-a|_p < 1\}$ for $a \in \mathbb{C}_p$, $]\bar{0}[= \mathfrak{M}_{\mathbb{C}_p}$ and $]\bar{\infty}[= \mathbf{P}^1(\mathbb{C}_p) \backslash \mathcal{O}_{\mathbb{C}_p}$. For a finite subset $S \subset \mathbf{P}^1(\overline{\mathbb{F}}_p)$, we define $]S[:= \mathrm{red}^{-1}(S) \subset \mathbf{P}^1(\mathbb{C}_p)$. By abuse of notation, we denote $A^{\mathrm{rig}}(\mathbf{P}^1(\mathbb{C}_p) -]S[)$ by $A^{\mathrm{rig}}(\mathbf{P}^1 \setminus S)$.

We remind two fundamental properties of rigid analytic functions:

Proposition 4.3 ([10, Chapter 6], etc.)**.** *Let X be as in* (4.1) *Then the following holds.*

(i) *Coincidence principle: If two rigid analytic functions $f(z)$ and $g(z)$ coincide in a sub-affinoid of X, then they coincide on the whole of X.*

(ii) *The algebra $A^{rig}(X)$ forms a Banach algebra with the supremum norm.*

The following function plays a main role in this subsection.

Definition 4.4. Let $n_1,\dots,n_r \in \mathbb{N}$ and $\xi_1,\dots,\xi_{r-1} \in \mathbb{C}_p$ with $|\xi_j|_p \leqslant 1$ $(1 \leqslant j \leqslant r-1)$. The **$p$-adic rigid TMPL** $\ell^{(p)}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_{r-1},z)$ is defined by the following p-adic power series:

$$\ell^{(p)}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_{r-1},z) := \sum_{\substack{0<k_1<\cdots<k_r \\ (k_1,p)=\cdots=(k_r,p)=1}} \frac{\xi_1^{k_1}\cdots\xi_{r-1}^{k_{r-1}} z^{k_r}}{k_1^{n_1}\cdots k_r^{n_r}} \tag{4.2}$$

which converges for $z \in]\bar{0}[= \{x \in \mathbb{C}_p \mid |x|_p < 1\}$ by $|\xi_j|_p \leqslant 1$ for $1 \leqslant j \leqslant r-1$.

It will be proved that it is rigid analytic in Proposition 4.7 and is furthermore overconvergent in Theorem 4.22. We remark that when $r=1$, $\ell^{(p)}_n(z)$ is equal to the p-adic polylogarithm $\ell^{(p)}_n(z)$ in [15, p.196]. The following integral expressions are generalization of [15, Lemma 7.2].

Theorem 4.5. *Let $n_1,\ldots,n_r\in\mathbb{N}$ and $\xi_1,\ldots,\xi_{r-1}\in\mathbb{C}_p$ with $|\xi_j|_p\leqslant 1$ $(1\leqslant j\leqslant r-1)$. Set a finite subset S of $\boldsymbol{P}^1(\overline{\mathbb{F}}_p)$ by* [5]

$$S=\{\overline{1},\overline{\xi_{r-1}^{-1}},\ldots,\overline{(\xi_1\cdots\xi_{r-1})^{-1}}\}. \tag{4.3}$$

Then the p-adic rigid TMPL $\ell^{(p)}_{n_1,\ldots,n_r}(\xi_1,\ldots,\xi_{r-1},z)$ is extended into $\boldsymbol{P}^1(\mathbb{C}_p)-]S[$ as a function on z by the following p-adic integral expression:

$$\ell^{(p)}_{n_1,\ldots,n_r}(\xi_1,\ldots,\xi_{r-1},z)=$$
$$\int_{(\mathbb{Z}_p^r)'_{\{1\}}}\langle x_1\rangle^{-n_1}\langle x_1+x_2\rangle^{-n_2}\cdots\langle x_1+\cdots+x_r\rangle^{-n_r}\cdot\omega(x_1)^{-n_1}\omega(x_1+x_2)^{-n_2}\cdots\omega(x_1+\cdots+x_r)^{-n_r}$$
$$d\mathfrak{m}_{\xi_1\cdots\xi_{r-1}z}(x_1)\cdots d\mathfrak{m}_{\xi_{r-1}z}(x_{r-1})d\mathfrak{m}_z(x_r), \tag{4.4}$$

where $(\mathbb{Z}_p^r)'_{\{1\}}=\Big\{(x_1,\ldots,x_r)\in\mathbb{Z}_p^r\ \Big|\ p\nmid x_1,p\nmid(x_1+x_2),\ldots,p\nmid(x_1+\cdots+x_r)\Big\}$ (cf. (2.12)).

Proof. Since $\langle x\rangle\cdot\omega(x)=x\neq 0$ for $x\in\mathbb{Z}_p^\times$, the right-hand side of (4.4) is

$$\int_{(\mathbb{Z}_p^r)'_{\{1\}}}x_1^{-n_1}(x_1+x_2)^{-n_2}\cdots(x_1+\cdots+x_r)^{-n_r}d\mathfrak{m}_{\xi_1\cdots\xi_{r-1}z}(x_1)\cdots d\mathfrak{m}_{\xi_{r-1}z}(x_{r-1})d\mathfrak{m}_z(x_r)$$
$$=\lim_{M\to\infty}\sum_{\substack{0<l_1,\ldots,l_r<p^M\\(l_1,p)=\cdots=(l_1+\cdots+l_r,p)=1}}\frac{\xi_1^{l_1}\xi_2^{l_1+l_2}\cdots\xi_{r-1}^{l_1+\cdots+l_{r-1}}z^{l_1+\cdots+l_r}}{l_1^{n_1}(l_1+l_2)^{n_2}\cdots(l_1+\cdots+l_r)^{n_r}}$$
$$\cdot\frac{1}{1-(\xi_1\cdots\xi_{r-1}z)^{p^M}}\cdots\frac{1}{1-(\xi_{r-1}z)^{p^M}}\cdot\frac{1}{1-z^{p^M}}. \tag{4.5}$$
$$=\lim_{M\to\infty}\quad g^M_{n_1,\ldots,n_r}(\xi_1,\ldots,\xi_{r-1};z)\qquad\text{(say)}.$$

By a direct calculation it can be shown that it is equal to the right-hand side of (4.2) when $|z|_p<1$. □

As for $g^M_{n_1,\ldots,n_r}(\xi_1,\ldots,\xi_{r-1};z)$ defined in the above proof , we have

Lemma 4.6. *Fix $n_1,\ldots,n_r,M\in\mathbb{N}$ and $\xi_1,\ldots,\xi_{r-1}\in\mathbb{C}_p$ with $|\xi_j|_p=1$ $(1\leqslant j\leqslant r-1)$. Then, for $z_0\in\mathbf{P}^1(\mathbb{C}_p)-]\overline{1},\overline{\xi_{r-1}^{-1}},\ldots,\overline{(\xi_1\cdots\xi_{r-1})^{-1}}[$, we have*

$$g^M_{n_1,\ldots,n_r}(\xi_1,\ldots,\xi_{r-1};z_0)\in\mathcal{O}_{\mathbb{C}_p}$$

and

$$g^{M+1}_{n_1,\ldots,n_r}(\xi_1,\ldots,\xi_{r-1};z_0)\equiv g^M_{n_1,\ldots,n_r}(\xi_1,\ldots,\xi_{r-1};z_0)\pmod{p^M}.$$

Proof. When $z_0\in\mathbf{P}^1(\mathbb{C}_p)-]\overline{1},\overline{\xi_{r-1}^{-1}},\ldots,\overline{(\xi_1\cdots\xi_{r-1})^{-1}},\overline{\infty}[$,
namely when $z_0\in\mathcal{O}_{\mathbb{C}_p}-]\overline{1},\overline{\xi_{r-1}^{-1}},\ldots,\overline{(\xi_1\cdots\xi_{r-1})^{-1}}[$, it is clear that $g^M_{n_1,\ldots,n_r}(\xi_1,\ldots,\xi_{r-1};z_0)\in\mathcal{O}_{\mathbb{C}_p}$. In the definition of $g^{M+1}_{n_1,\ldots,n_r}(\xi_1,\ldots,\xi_{r-1};z)$, l_j $(1\leqslant j\leqslant r)$ is running in the interval $(0,p^{M+1})$. Writing $l_j=l'_j+kp^M$ $(0<l'_j<p^M$ and $1\leqslant k\leqslant p-1)$, we have

$$g^{M+1}_{n_1,\ldots,n_r}(\xi_1,\ldots,\xi_{r-1};z_0)\equiv\sum_{\substack{0<l'_1,\ldots,l'_r<p^M\\(l'_1,p)=\cdots=(l'_1+\cdots+l'_r,p)=1}}\frac{\xi_1^{l'_1}\xi_2^{l'_1+l'_2}\cdots\xi_{r-1}^{l'_1+\cdots+l'_{r-1}}z_0^{l'_1+\cdots+l'_r}}{l_1'^{n_1}(l'_1+l'_2)^{n_2}\cdots(l'_1+\cdots+l'_r)^{n_r}}$$

[5] Here we ignore the multiplicity.

$$
\begin{aligned}
&\cdot \{1+(\xi_1\cdots\xi_{r-1}z_0)^{p^M}+(\xi_1\cdots\xi_{r-1}z_0)^{2p^M}+\cdots+(\xi_1\cdots\xi_{r-1}z_0)^{(p-1)p^M}\}\\
&\qquad\cdots\\
&\cdot \{1+(\xi_{r-1}z_0)^{p^M}+(\xi_{r-1}z_0)^{2p^M}+\cdots+(\xi_{r-1}z_0)^{(p-1)p^M}\}\\
&\cdot \{1+z_0^{p^M}+z_0^{2p^M}+\cdots+z_0^{(p-1)p^M}\}\\
&\cdot \frac{1}{1-(\xi_1\cdots\xi_{r-1}z)^{p^{M+1}}}\cdots\frac{1}{1-(\xi_{r-1}z)^{p^{M+1}}}\cdot\frac{1}{1-z^{p^{M+1}}} \pmod{p^M}\\
&\qquad = g^M_{n_1,\dots,n_r}(\xi_1,\dots,\xi_{r-1};z).
\end{aligned}
$$

When $z_0\in]\bar\infty[$, put $\varepsilon=\frac{1}{z_0}\in]\bar 0[$. By a direct calculation, $g^M_{n_1,\dots,n_r}(\xi_1,\dots,\xi_{r-1};z_0)\in\mathcal{O}_{\mathbb{C}_p}$. We have

$$
\begin{aligned}
g^M_{n_1,\dots,n_r}(\xi_1,\dots,\xi_{r-1};z_0) &= \sum_{\substack{0<l_1,\dots,l_r<p^M\\ (l_1,p)=\cdots=(l_1+\cdots+l_r,p)=1}} \frac{(\frac{\xi_1\xi_2\cdots\xi_{r-1}}{\varepsilon})^{l_1}(\frac{\xi_2\cdots\xi_{r-1}}{\varepsilon})^{l_2}\cdots(\frac{\xi_{r-1}}{\varepsilon})^{l_{r-1}}(\frac{1}{\varepsilon})^{l_r}}{l_1^{n_1}(l_1+l_2)^{n_2}\cdots(l_1+\cdots+l_r)^{n_r}}\\
&\quad\cdot\frac{1}{1-(\frac{\xi_1\cdots\xi_{r-1}}{\varepsilon})^{p^M}}\cdots\frac{1}{1-(\frac{\xi_{r-1}}{\varepsilon})^{p^M}}\cdot\frac{1}{1-(\frac{1}{\varepsilon})^{p^M}}\\
&=(-1)^r \sum_{\substack{0<l_1,\dots,l_r<p^M\\ (l_1,p)=\cdots=(l_1+\cdots+l_r,p)=1}} \frac{(\frac{\varepsilon}{\xi_1\xi_2\cdots\xi_{r-1}})^{p^M-l_1}(\frac{\varepsilon}{\xi_2\cdots\xi_{r-1}})^{p^M-l_2}\cdots(\frac{\varepsilon}{\xi_{r-1}})^{p^M-l_{r-1}}\varepsilon^{p^M-l_r}}{l_1^{n_1}(l_1+l_2)^{n_2}\cdots(l_1+\cdots+l_r)^{n_r}}\\
&\quad\cdot\frac{1}{1-(\frac{\varepsilon}{\xi_1\cdots\xi_{r-1}})^{p^M}}\cdots\frac{1}{1-(\frac{\varepsilon}{\xi_{r-1}})^{p^M}}\cdot\frac{1}{1-\varepsilon^{p^M}}\\
&=(-1)^r \sum_{\substack{0<l_1,\dots,l_r<p^M\\ (l_1,p)=\cdots=(l_1+\cdots+l_r,p)=1}} \frac{(\frac{\varepsilon}{\xi_1\xi_2\cdots\xi_{r-1}})^{l_1}(\frac{\varepsilon}{\xi_2\cdots\xi_{r-1}})^{l_2}\cdots(\frac{\varepsilon}{\xi_{r-1}})^{l_{r-1}}\varepsilon^{l_r}}{(p^M-l_1)^{n_1}(2p^M-l_1-l_2)^{n_2}\cdots(rp^M-l_1-\cdots-l_r)^{n_r}}\\
&\quad\cdot\frac{1}{1-(\frac{\varepsilon}{\xi_1\cdots\xi_{r-1}})^{p^M}}\cdots\frac{1}{1-(\frac{\varepsilon}{\xi_{r-1}})^{p^M}}\cdot\frac{1}{1-\varepsilon^{p^M}}\\
&\equiv(-1)^{r+n_1+\cdots+n_r} \sum_{\substack{0<l_1,\dots,l_r<p^M\\ (l_1,p)=\cdots=(l_1+\cdots+l_r,p)=1}} \frac{(\frac{\varepsilon}{\xi_1\xi_2\cdots\xi_{r-1}})^{l_1}(\frac{\varepsilon}{\xi_2\cdots\xi_{r-1}})^{l_2}\cdots(\frac{\varepsilon}{\xi_{r-1}})^{l_{r-1}}\varepsilon^{l_r}}{{l_1}^{n_1}(l_1+l_2)^{n_2}\cdots(l_1+\cdots+l_r)^{n_r}}\\
&\quad\cdot\frac{1}{1-(\frac{\varepsilon}{\xi_1\cdots\xi_{r-1}})^{p^M}}\cdots\frac{1}{1-(\frac{\varepsilon}{\xi_{r-1}})^{p^M}}\cdot\frac{1}{1-\varepsilon^{p^M}} \pmod{p^M}\\
&=(-1)^{r+n_1+\cdots+n_r}g^M_{n_1,\dots,n_r}(\xi_1^{-1},\dots,\xi_{r-1}^{-1};\varepsilon).
\end{aligned}
$$

Therefore by our previous argument and by $|\xi_j|_p=|\xi_j^{-1}|_p=1$, it follows that

$$
\begin{aligned}
g^{M+1}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_{r-1};z_0)&\equiv(-1)^{r+n_1+\cdots+n_r}g^{M+1}_{n_1,\dots,n_r}(\xi_1^{-1},\dots,\xi_{r-1}^{-1};\varepsilon)\\
&\equiv(-1)^{r+n_1+\cdots+n_r}g^{M}_{n_1,\dots,n_r}(\xi_1^{-1},\dots,\xi_{r-1}^{-1};\varepsilon)\equiv g^M_{n_1,\dots,n_r}(\xi_1,\dots,\xi_{r-1};z_0) \pmod{p^M}.
\end{aligned}
$$

□

Theorem 4.5 and Lemma 4.6 imply the following:

Proposition 4.7. *Fix $n_1,\dots,n_r\in\mathbb{N}$ and $\xi_1,\dots,\xi_{r-1}\in\mathbb{C}_p$ with $|\xi_j|_p=1$ ($1\leqslant j\leqslant r-1$). By our integral formula* (4.4)*, the p-adic rigid TMPL $\ell^{(p)}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_{r-1},z)$ is a rigid analytic* *function on* $\mathbf{P}^1(\mathbb{C}_p)-]S[$. *Namely,*

$$\ell^{(p)}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_{r-1},z)\in A^{\mathrm{rig}}(\mathbf{P}^1\setminus S).$$

Proof. Since the space $\mathbf{P}^1(\mathbb{C}_p)-]S[$ is an affinoid and the algebra $A^{\mathrm{rig}}(\mathbf{P}^1\setminus S)$ forms a Banach algebra by the supremum norm (cf. Notation 4.1), by Lemma 4.6, the rational functions

$$g^M_{n_1,\dots,n_r}(\xi_1,\dots,\xi_{r-1};z)\in A^{\mathrm{rig}}(\mathbf{P}^1\setminus S)$$

uniformly converge to a rigid analytic function $\ell(z)\in A^{\mathrm{rig}}(\mathbf{P}^1\setminus S)$ when M goes to ∞ thanks to Proposition 4.3. It is easy to see that the restriction of $\ell(z)$ into $\mathbf{P}^1(\mathbb{C}_p)-]S[$ coincides with (4.5), hence with $\ell^{(p)}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_{r-1},z)$. Therefore the analytic continuation of $\ell^{(p)}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_{r-1},z)$ is given by $\ell(z)$. $\square$

From now on we will employ the same symbol $\ell^{(p)}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_{r-1},z)$ to denote its analytic continuation.

We note that, by (4.5),

Lemma 4.8. *For* $n_1,\dots,n_r\in\mathbb{N}$ *and* $\xi_1,\dots,\xi_{r-1}\in\mathbb{C}_p$ *with* $|\xi_j|_p=1$ $(1\leqslant j\leqslant r-1)$, *we have*

$$\ell^{(p)}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_{r-1},\infty)=0.$$

Proof. By a direct computation, $g^M_{n_1,\dots,n_r}(\xi_1,\dots,\xi_{r-1};\infty)=0$. Then the claim is obtained because $\ell^{(p)}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_{r-1},z)$ is defined to be the limit of $g^M_{n_1,\dots,n_r}(\xi_1,\dots,\xi_{r-1};z)$. $\square$

The special values of the p-adic multiple L-function at positive integer points are described in terms of the special values of p-adic rigid TMPL at roots of unity:

Theorem 4.9. *For* $n_1,\dots,n_r\in\mathbb{N}$ *and* $c\in\mathbb{N}_{>1}$ *with* $(c,p)=1$,

$$L_{p,r}(n_1,\dots,n_r;\omega^{-n_1},\dots,\omega^{-n_r};1,\dots,1;c)=\sum_{\substack{\xi_1^c=\cdots=\xi_r^c=1\\ \xi_1\cdots\xi_r\neq1,\ \dots,\ \xi_{r-1}\xi_r\neq1,\ \xi_r\neq1}}\ell^{(p)}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_r). \tag{4.6}$$

Proof. By definition,

$$\begin{aligned}L_{p,r}(n_1,\dots,n_r;\omega^{-n_1},\dots,\omega^{-n_r};1,\dots,1;c)=&\\ \int_{(\mathbb{Z}_p^r)'_{\{1\}}}\langle x_1\rangle^{-n_1}\langle x_1+x_2\rangle^{-n_2}\cdots\langle x_1+\cdots+x_r\rangle^{-n_r}&\\ \cdot\,\omega(x_1)^{-n_1}\omega(x_1+x_2)^{-n_2}\cdots\omega(x_1+\cdots+x_r)^{-n_r}d\widetilde{\mathfrak{m}}_c(x_1)\cdots d\widetilde{\mathfrak{m}}_c(x_r)&\end{aligned}$$

where $\widetilde{\mathfrak{m}}_c=\sum_{\substack{\xi^c=1\\ \xi\neq1}}\mathfrak{m}_\xi$. By (4.4) and

$$\begin{aligned}(\widetilde{\mathfrak{m}}_c)^r&=\Big\{\sum_{\substack{\xi_1'^c=1\\ \xi_1'\neq1}}\mathfrak{m}_{\xi_1'}\Big\}\times\Big\{\sum_{\substack{\xi_2'^c=1\\ \xi_2'\neq1}}\mathfrak{m}_{\xi_2'}\Big\}\times\cdots\times\Big\{\sum_{\substack{\xi_r'^c=1\\ \xi_r'\neq1}}\mathfrak{m}_{\xi_r'}\Big\}\\ &=\sum_{\substack{\xi_1^c=\cdots=\xi_r^c=1\\ \xi_1\cdots\xi_r\neq1,\dots,\xi_{r-1}\xi_r\neq1,\xi_r\neq1}}\mathfrak{m}_{\xi_1\cdots\xi_r}\times\mathfrak{m}_{\xi_2\cdots\xi_r}\times\cdots\times\mathfrak{m}_{\xi_{r-1}\xi_r}\times\mathfrak{m}_{\xi_r}\end{aligned}$$

we get our formula. □

Remark 4.10. It is worthy to note that the right-hand side of (4.6) is p-adically continuous not only with respect to $n_1, \ldots, n_r$ but also with respect to c by Theorem 2.17.

As a special case when $r = 1$ of Theorem 4.9, we recover Coleman's formula in [15, p.203] below by Example 2.12.

Example 4.11. For $n \in \mathbb{N}_{>1}$ and $c \in \mathbb{N}_{>1}$ with $(c, p) = 1$,

$$(c^{1-n} - 1) \cdot L_p(n; \omega^{1-n}) = \sum_{\substack{\xi^c=1 \\ \xi \neq 1}} \ell_n^{(p)}(\xi). \tag{4.7}$$

When $r = 2$, we have

Example 4.12. For $n_1, n_2 \in \mathbb{N}$ and $c \in \mathbb{N}_{>1}$ with $(c, p) = 1$,

$$L_{p,2}(n_1, n_2; \omega^{-n_1}, \omega^{-n_2}; 1, 1; c) = \sum_{\substack{\xi_1^c=\xi_2^c=1 \\ \xi_1\xi_2 \neq 1,\ \xi_2 \neq 1}} \ell_{n_1,n_2}^{(p)}(\xi_1, \xi_2).$$

4.2. **p-adic partial twisted multiple polylogarithms.** We will prove that p-adic rigid TMPL's (Definition 4.4) are overconvergent in Theorem 4.21. In order to do that, p-adic partial TMPL's will be introduced in Definition 4.16 and their properties will be investigated.

First,we recall overconvergent functions and rigid cohomologies in our particular case (consult [6] for a general theory)

Notation 4.13. Let $S = \{s_0, \ldots, s_d\}$ (all s_i's are distinct) be a finite subset of $\mathbf{P}^1(\overline{\mathbb{F}}_p)$. An *overconvergent function* of $\mathbf{P}^1 \setminus S$ is a function belonging to the $\mathbb{C}_p$-algebra

$$A^\dagger(\mathbf{P}^1 \setminus S) := \underset{\lambda \to 1^-}{\text{ind-lim}}\ A^{\text{rig}}(U_\lambda)$$

where U_λ is the affinoid "obtained by removing all closed discs of radius λ around $\hat{s}_i$ (: a lift of s_i) from $\mathbf{P}^1(\mathbb{C}_p)$", i.e.

$$U_\lambda := \mathbf{P}^1(\mathbb{C}_p) \setminus \bigcup_{0 \leqslant i \leqslant d} z_i^{-1}\left(\{\alpha \in \mathbb{C}_p \mid |\alpha|_p < \lambda\}\right) \tag{4.8}$$

and z_i is a local parameter

$$z_i :]s_i[\overset{\sim}{\to}]0[. \tag{4.9}$$

(It is noted that the above $\hat{s}_i$ is equal to $z_i^{-1}(0)$.) An overconvergent function of $\mathbf{P}^1 - S$ is, in short, a function which can be analytically extended into an affinoid which is bigger than $\mathbf{P}^1(\mathbb{C}_p) -]s_0, s_1, \ldots, s_d[$. We note that the definition of the space of overconvergent functions does not depend on the choice of local parameter z_i.

The following lemmas are quite useful.

Lemma 4.14. *Assume $s_0 = \bar{\infty}$ and take $\hat{s_0} = \infty$. Then we have a description:*

$$A^\dagger(\mathbf{P}^1 \setminus S) \simeq \left\{ f(z) = \sum_{r \geqslant 0} a_r(\hat{s}_0; f) z^r + \sum_{l=1}^{d} \sum_{m>0} \frac{a_m(\hat{s}_l; f)}{(z-\hat{s}_l)^m} \in \mathbb{C}_p[[z, \frac{1}{z-\hat{s}_l}]] \;\middle|\; \frac{|a_m(\hat{s}_l; f)|_p}{\lambda^m} \to 0 \ (m \to \infty) \ \textit{for some } 0<\lambda<1 \quad (0 \leqslant l \leqslant d) \right\}. \tag{4.10}$$

While we have

$$A^{rig}(\mathbf{P}^1 \setminus S) \simeq \left\{ f(z) = \sum_{m \geqslant 0} a_m(\hat{s}_0; f) z^m + \sum_{l=1}^{d} \sum_{m>0} \frac{a_m(\hat{s}_l; f)}{(z-\hat{s}_l)^m} \in \mathbb{C}_p[[z, \frac{1}{z-\hat{s}_l}]] \;\middle|\; |a_m(\hat{s}_l; f)|_p \to 0 \ (m \to \infty) \ \textit{for } 0 \leqslant l \leqslant d \right\}. \tag{4.11}$$

Proof. They follow from the definitions (cf. Notation 4.1). □

We note that

$$A^\dagger(\mathbf{P}^1 \setminus S) \subset A^{\mathrm{rig}}(\mathbf{P}^1 \setminus S).$$

The following is one of the most important properties of overconvergent functions.

Lemma 4.15. *Let $f(z) \in A^\dagger(\mathbf{P}^1 \setminus S)$. Under the above assumption, there exists a (unique up to constant) solution $F(z) \in A^\dagger(\mathbf{P}^1 \setminus S)$ of the differential equation*

$$dF(z) = f(z)dz$$

if and only if the residues of the differential 1-form $f(z)dz$, i.e. $a_1(\hat{s}_l; f)$ $(1 \leqslant l \leqslant d)$ are all 0.

Proof. When $a_1(\hat{s}_l; f)$ $(1 \leqslant l \leqslant d)$ are all 0, integrations of $f(z)$ in (4.10) are formally given by the following power series

$$\sum_{r \geqslant 1} \frac{a_{r-1}(\hat{s}_0; f)}{r} \cdot z^r + \sum_{l=1}^{d} \sum_{m>0} \frac{a_{m+1}(\hat{s}_l; f)}{-m} \cdot \frac{1}{(z-\hat{s}_l)^m} + \text{constant}.$$

Then by replacing the λ by λ' such that $\lambda < \lambda' < 1$ and using $\mathrm{ord}_p(n) = O(\log n / \log p)$, we get

$$\frac{|a_{m+1}(\hat{s}_l; f)|_p}{|m|_p} \cdot \frac{1}{\lambda'^m} \to 0 \quad (m \to \infty).$$

Whence $F(z)$ belongs to $A^\dagger(\mathbf{P}^1 \setminus S)$. The 'if'-part is obtained. The 'only if'-part is easy. □

The lemma actually is a consequence of the fact that $\dim H^1_\dagger(\mathbf{P}^1 \setminus S) = d$.

The following function is a main object in this subsection.

Definition 4.16. Let $n_1, \dots, n_r \in \mathbb{N}$ and $\xi_1, \dots, \xi_{r-1} \in \mathbb{C}_p$ with $|\xi_j|_p \leqslant 1$ $(1 \leqslant j \leqslant r-1)$. Let $\alpha_1, \dots, \alpha_r \in \mathbb{N}$ with $0 < \alpha_j < p$ $(1 \leqslant j \leqslant r)$. The **$p$-adic partial TMPL** $\ell^{\equiv(\alpha_1,\dots,\alpha_r),(p)}_{n_1,\dots,n_r}(\xi_1, \dots \xi_{r-1}, z)$ is defined by the following p-adic power series:

$$\ell^{\equiv(\alpha_1,\dots,\alpha_r),(p)}_{n_1,\dots,n_r}(\xi_1, \dots, \xi_{r-1}, z) := \sum_{\substack{0<k_1<\cdots<k_r \\ k_1 \equiv \alpha_1, \dots, k_r \equiv \alpha_r \bmod p}} \frac{\xi_1^{k_1} \cdots \xi_{r-1}^{k_{r-1}} z^{k_r}}{k_1^{n_1} \cdots k_r^{n_r}} \tag{4.12}$$

which converges for $z \in]\bar{0}[$.

Similarly to (4.5), we have a limit expression of them.

Proposition 4.17. *Let $n_1,\ldots,n_r \in \mathbb{N}$, $\xi_1,\ldots,\xi_{r-1} \in \mathbb{C}_p$ with $|\xi_j|_p \leqslant 1$ $(1 \leqslant j \leqslant r-1)$ and $\alpha_1,\ldots,\alpha_r \in \mathbb{N}$ with $0<\alpha_j<p$ $(1 \leqslant j \leqslant r)$. When $z \in]\bar{0}[$, the function $\ell^{\equiv(\alpha_1,\ldots,\alpha_r),(p)}_{n_1,\ldots,n_r}(\xi_1,\ldots,\xi_{r-1},z)$ is expressed as*

$$\ell^{\equiv(\alpha_1,\ldots,\alpha_r),(p)}_{n_1,\ldots,n_r}(\xi_1,\ldots,\xi_{r-1},z) = \lim_{M\to\infty} \sum_{\substack{0<l_1,\ldots,l_r<p^M \\ l_1\equiv\alpha_1, l_1+l_2\equiv\alpha_2,\ldots,l_1+\cdots+l_r\equiv\alpha_r \bmod p}} \frac{\xi_1^{l_1}\xi_2^{l_1+l_2}\cdots\xi_{r-1}^{l_1+\cdots+l_{r-1}} z^{l_1+\cdots+l_r}}{l_1^{n_1}(l_1+l_2)^{n_2}\cdots(l_1+\cdots+l_r)^{n_r}}$$
$$\cdot \frac{1}{1-(\xi_1\cdots\xi_{r-1}z)^{p^M}}\cdots\frac{1}{1-(\xi_{r-1}z)^{p^M}}\cdot\frac{1}{1-z^{p^M}}. \tag{4.13}$$

Proof. It can be proved by a direct calculation. □

It is worthy to note that here the condition $\alpha_j \neq 0$ $(1 \leqslant j \leqslant r)$ is necessary to make the limit convergent.

Proposition 4.18. *Let $n_1,\ldots,n_r \in \mathbb{N}$, $\xi_1,\ldots,\xi_{r-1} \in \mathbb{C}_p$ with $|\xi_j|_p = 1$ $(1 \leqslant j \leqslant r-1)$ and $\alpha_1,\ldots,\alpha_r \in \mathbb{N}$ with $0<\alpha_j<p$ $(1 \leqslant j \leqslant r)$. Set S as in (4.3). Then the function $\ell^{\equiv(\alpha_1,\ldots,\alpha_r),(p)}_{n_1,\ldots,n_r}(\xi_1,\ldots,\xi_{r-1},z)$ is analytically extended into $\mathbf{P}^1(\mathbb{C}_p)-]S[$ as a rigid analytic function. Namely,*

$$\ell^{\equiv(\alpha_1,\ldots,\alpha_r),(p)}_{n_1,\ldots,n_r}(\xi_1,\ldots,\xi_{r-1},z) \in A^{rig}(\mathbf{P}^1 \setminus S).$$

Proof. The relation

$$\ell^{\equiv(\alpha_1,\ldots,\alpha_r),(p)}_{n_1,\ldots,n_r}(\xi_1,\ldots,\xi_{r-1},z) = \frac{1}{p^r}\sum_{\rho_1^p=\cdots=\rho_r^p=1} \rho_1^{-\alpha_1}\cdots\rho_r^{-\alpha_r}\ell^{(p)}_{n_1,\ldots,n_r}(\rho_1\xi_1,\ldots,\rho_{r-1}\xi_{r-1},\rho_r z) \tag{4.14}$$

holds on $]\bar{0}[$. Since $\ell^{(p)}_{n_1,\ldots,n_r}(\rho_1\xi_1,\ldots,\rho_{r-1}\xi_{r-1},\rho_r z)$'s are all rigid analytic on the above space by Proposition 4.7, the function $\ell^{\equiv(\alpha_1,\ldots,\alpha_r),(p)}_{n_1,\ldots,n_r}(\xi_1,\ldots,\xi_{r-1},z)$ can be extended there as a rigid analytic function to keep the equation. □

From now on we will employ the same symbol $\ell^{\equiv(\alpha_1,\ldots,\alpha_r),(p)}_{n_1,\ldots,n_r}(\xi_1,\ldots,\xi_{r-1},z)$ to denote its analytic continuation.

The following formulas are necessary to prove our Theorem 4.21.

Lemma 4.19. *Let $n_1,\ldots,n_r \in \mathbb{N}$, $\xi_1,\ldots,\xi_{r-1} \in \mathbb{C}_p$ with $|\xi_j|_p = 1$ $(1 \leqslant j \leqslant r-1)$ and $\alpha_1,\ldots,\alpha_r \in \mathbb{N}$ with $0<\alpha_j<p$ $(1 \leqslant j \leqslant r)$.*

(i) *For $n_r > 1$,*

$$\frac{d}{dz}\ell^{\equiv(\alpha_1,\ldots,\alpha_r),(p)}_{n_1,\ldots,n_r}(\xi_1,\ldots,\xi_{r-1},z) = \frac{1}{z}\ell^{\equiv(\alpha_1,\ldots,\alpha_r),(p)}_{n_1,\ldots,n_{r-1},n_r-1}(\xi_1,\ldots,\xi_{r-1},z).$$

(ii) *For $n_r = 1$ and $r \neq 1$,*

$$\frac{d}{dz}\ell^{\equiv(\alpha_1,\ldots,\alpha_r),(p)}_{n_1,\ldots,n_r}(\xi_1,\ldots,\xi_{r-1},z) = \begin{cases} \frac{z^{\alpha_r-\alpha_{r-1}-1}}{1-z^p}\ell^{\equiv(\alpha_1,\ldots,\alpha_{r-1}),(p)}_{n_1,\ldots,n_{r-1}}(\xi_1,\ldots,\xi_{r-2},\xi_{r-1}z) \\ \qquad\qquad \text{if} \quad \alpha_r > \alpha_{r-1}, \\ \frac{z^{\alpha_r-\alpha_{r-1}+p-1}}{1-z^p}\ell^{\equiv(\alpha_1,\ldots,\alpha_{r-1}),(p)}_{n_1,\ldots,n_{r-1}}(\xi_1,\ldots,\xi_{r-2},\xi_{r-1}z) \\ \qquad\qquad \text{if} \quad \alpha_r \leqslant \alpha_{r-1}. \end{cases}$$

(iii) For $n_r = 1$ and $r = 1$ with $\alpha_r = \alpha$,

$$\frac{d}{dz}\ell_1^{\equiv\alpha,(p)}(z) = \frac{z^{\alpha-1}}{1-z^p}.$$

Proof. They can be proved by a direct computation. □

By Lemma 4.8 and (4.14),

Remark 4.20. For $n_1,\ldots,n_r \in \mathbb{N}$, $\xi_1,\ldots,\xi_{r-1} \in \mathbb{C}_p$ with $|\xi_j|_p = 1$ $(1 \leqslant j \leqslant r-1)$ and $\alpha_1,\ldots,\alpha_r \in \mathbb{N}$ with $0 < \alpha_j < p$ $(1 \leqslant j \leqslant r)$, the following hold:

(i) $\ell_{n_1,\ldots,n_r}^{\equiv(\alpha_1,\ldots,\alpha_r),(p)}(\xi_1,\ldots,\xi_{r-1},0) = \ell_{n_1,\ldots,n_r}^{\equiv(\alpha_1,\ldots,\alpha_r),(p)}(\xi_1,\ldots,\xi_{r-1},\infty) = 0$.

(ii) $\ell_{n_1,\ldots,n_r}^{(p)}(\xi_1,\ldots,\xi_{r-1},z) = \sum_{0<\alpha_1,\ldots,\alpha_r<p} \ell_{n_1,\ldots,n_r}^{\equiv(\alpha_1,\ldots,\alpha_r),(p)}(\xi_1,\ldots,\xi_{r-1},z)$.

Next we discuss a new property of our functions.

Theorem 4.21. *Let $n_1,\ldots,n_r \in \mathbb{N}$, $\xi_1,\ldots,\xi_{r-1} \in \mathbb{C}_p$ with $|\xi_j|_p = 1$ $(1 \leqslant j \leqslant r-1)$ and $\alpha_1,\ldots,\alpha_r \in \mathbb{N}$ with $0 < \alpha_j < p$ $(1 \leqslant j \leqslant r)$. Set S as in* (4.3). *The function $\ell_{n_1,\ldots,n_r}^{\equiv(\alpha_1,\ldots,\alpha_r),(p)}(\xi_1,\ldots,\xi_{r-1},z)$ is an overconvergent function of $\mathbf{P}^1 \setminus S$. Namely,*

$$\ell_{n_1,\ldots,n_r}^{\equiv(\alpha_1,\ldots,\alpha_r),(p)}(\xi_1,\ldots,\xi_{r-1},z) \in A^\dagger(\mathbf{P}^1 \setminus S).$$

Proof. It is achieved by induction on weight $n_1 + \cdots + n_r$.

(i). Assume that the weight is equal to 1, i.e. $r = 1$ and $n_1 = 1$. By changing variable $w(z) = \frac{1}{z-1}$, we see that $\mathbf{P}^1 \setminus \{\bar{1}\}$ is identified with $\mathbf{P}^1 \setminus \{\bar{\infty}\}$. By a direct calculation, it can be checked that $\frac{z^{\alpha-1}}{1-z^p} \cdot \frac{dz}{dw}$ as a function on w belongs to $A^\dagger(\mathbf{P}^1 \setminus \{\bar{\infty}\})$. (We note that it also follows from the fact that that it is a rational function on w whose poles are all of the form $w = \frac{1}{\zeta_p - 1}$ with $\zeta_p \in \mu_p$.) So by Lemma 4.15 with $d = 0$, there exists a unique (modulo constant) function $F(w)$ in $A^\dagger(\mathbf{P}^1 \setminus \{\bar{\infty}\})$ such that

$$\frac{dF}{dw} = \frac{z^{\alpha-1}}{1-z^p} \cdot \frac{dz}{dw}.$$

Therefore, there exists a unique function $F(z)$ in $A^\dagger(\mathbf{P}^1 \setminus \{\bar{1}\})$ such that

$$F(0) = 0 \quad \text{and} \quad \frac{dF(z)}{dz} = \frac{z^{\alpha-1}}{1-z^p}.$$

By Proposition 4.18 and Lemma 4.19 (iii), $\ell_1^{\equiv\alpha,(p)}(z)$ is also a unique function in $A^{\mathrm{rig}}(\mathbf{P}^1 \setminus \{\bar{1}\})$ satisfying the above properties and $F(z)|_{\mathbf{P}^1(\mathbb{C}_p) -]\bar{1}[}$ belongs to $A^{\mathrm{rig}}(\mathbf{P}^1 \setminus \{\bar{1}\})$. Thus we have

$$F(z)|_{\mathbf{P}^1(\mathbb{C}_p) -]\bar{1}[} \equiv \ell_1^{\equiv\alpha,(p)}(z).$$

So by the coincidence principle of rigid analytic functions we can say that $\ell_1^{\equiv\alpha,(p)}(z)$ can be uniquely extended into a rigid analytic space bigger than $\mathbf{P}^1(\mathbb{C}_p) -]\bar{1}[$ by $F(z) \in A^\dagger(\mathbf{P}^1 \setminus \{\bar{1}\})$.

(ii). Assume that $n_r > 1$. We put

$$S_\infty = S \cup \{\bar{\infty}\} \qquad \text{and} \qquad S_{\infty,0} = S \cup \{\bar{\infty}\} \cup \{\bar{0}\}$$

and take a lift $\{\hat{s_0}, \hat{s_1}, \ldots, \hat{s_d}\}$ of $S_{\infty,0}$ with

$$\hat{s_0} = \infty \qquad \text{and} \qquad \hat{s_1} = 0.$$

By our assumption

$$\ell^{\equiv(\alpha_1,\dots,\alpha_r),(p)}_{n_1,\dots,n_{r-1},n_r-1}(\xi_1,\dots,\xi_{r-1},z)\in A^\dagger(\mathbf{P}^1\setminus S)$$

and by $\frac{dz}{z}\in\Omega^{\dagger,1}(\mathbf{P}^1\setminus\{\overline{\infty},\overline{0}\})$, we have

$$\ell^{\equiv(\alpha_1,\dots,\alpha_r),(p)}_{n_1,\dots,n_{r-1},n_r-1}(\xi_1,\dots,\xi_{r-1},z)\frac{dz}{z}\in\Omega^{\dagger,1}(\mathbf{P}^1\setminus S_{\infty,0}).$$

Here $\Omega^{\dagger,1}(\mathbf{P}^1\setminus\tilde{S})$ ($\tilde{S}$: a finite subset of $\mathbf{P}^1(\overline{\mathbb{F}}_p)$) denotes the space of rigid differential 1-forms there, i.e. $\Omega^{\dagger,1}(\mathbf{P}^1\setminus\tilde{S})=A^{\dagger,1}(\mathbf{P}^1\setminus\tilde{S})dz$.

Put

$$f(z):=\frac{1}{z}\ell^{\equiv(\alpha_1,\dots,\alpha_r),(p)}_{n_1,\dots,n_{r-1},n_r-1}(\xi_1,\dots,\xi_{r-1},z)\in A^\dagger(\mathbf{P}^1\setminus S_{\infty,0}). \tag{4.15}$$

Since $\ell^{\equiv(\alpha_1,\dots,\alpha_r),(p)}_{n_1,\dots,n_{r-1},n_r}(\xi_1,\dots,\xi_{r-1},z)$ belongs to $A^{\mathrm{rig}}(\mathbf{P}^1\setminus S)\Big(\subset A^{\mathrm{rig}}(\mathbf{P}^1\setminus S_{\infty,0})\Big)$ by Proposition 4.18 and it satisfies the differential equation in Lemma 4.19.(i), i.e. its differential is equal to $f(z)$, we have particularly

$$a_m(\hat{s}_1;f)=0\qquad(m>0) \tag{4.16}$$

(recall $\hat{s_1}=0$) and

$$a_1(\hat{s}_l;f)=0\qquad(2\leqslant l\leqslant d) \tag{4.17}$$

by (4.10) and (4.11).

By (4.15) and (4.16),

$$f(z)\in A^\dagger(\mathbf{P}^1\setminus S_\infty).$$

By (4.17) and Lemma 4.15, there exists a unique function $F(z)$ in $A^\dagger(\mathbf{P}^1\setminus S_\infty)$, i.e. a function $F(z)$ which is rigid analytic on an affinoid V of

$$\mathbf{P}^1(\mathbb{C}_p)-]S_\infty[\quad=\quad\mathbf{P}^1(\mathbb{C}_p)-]\bar{\infty},S[$$

such that

$$F(0)=0\quad\text{and}\quad dF(z)=f(z)dz. \tag{4.18}$$

Since $\ell^{\equiv(\alpha_1,\dots,\alpha_r),(p)}_{n_1,\dots,n_{r-1},n_r}(\xi_1,\dots,\xi_{r-1},z)$ is also a unique function in $A^{\mathrm{rig}}(\mathbf{P}^1\setminus S)$ satisfying (4.18), the restrictions of both $F(z)$ and $\ell^{\equiv(\alpha_1,\dots,\alpha_r),(p)}_{n_1,\dots,n_{r-1},n_r}(\xi_1,\dots,\xi_{r-1},z)$ into the subspace $\mathbf{P}^1(\mathbb{C}_p)-]S_\infty[$ must coincide, i.e.

$$F(z)\Big|_{\mathbf{P}^1(\mathbb{C}_p)-]S_\infty[}\equiv\ell^{\equiv(\alpha_1,\dots,\alpha_r),(p)}_{n_1,\dots,n_{r-1},n_r}(\xi_1,\dots,\xi_{r-1},z)\Big|_{\mathbf{P}^1(\mathbb{C}_p)-]S_\infty[}.$$

Hence by the coincidence principle of rigid analytic functions, there is a rigid analytic function $G(z)$ on the union of V and $\mathbf{P}^1(\mathbb{C}_p)-]S[$ whose restriction to V is equal to $F(z)$ and whose restriction to $\mathbf{P}^1(\mathbb{C}_p)-]S[$ is equal to $\ell^{\equiv(\alpha_1,\dots,\alpha_r),(p)}_{n_1,\dots,n_{r-1},n_r}(\xi_1,\dots,\xi_{r-1},z)$. So we can say that

$$\ell^{\equiv(\alpha_1,\dots,\alpha_r),(p)}_{n_1,\dots,n_{r-1},n_r}(\xi_1,\dots,\xi_{r-1},z)\in A^{\mathrm{rig}}(\mathbf{P}^1\setminus S)$$

can be rigid analytically extended into a bigger rigid analytic space by $G(z)$. Namely,

$$\ell^{\equiv(\alpha_1,\dots,\alpha_r),(p)}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_{r-1},z)\in A^\dagger(\mathbf{P}^1\setminus S).$$

(iii). Assume that $n_r=1$ ($r\geqslant 2$). Put

$$\beta(z):=\begin{cases}\frac{z^{\alpha_r-\alpha_{r-1}-1}}{1-z^p} & \text{if}\quad \alpha_r>\alpha_{r-1},\\ \frac{z^{\alpha_r-\alpha_{r-1}+p-1}}{1-z^p} & \text{if}\quad \alpha_r\leqslant\alpha_{r-1}.\end{cases}$$

By our assumption

$$\ell^{\equiv(\alpha_1,\dots,\alpha_{r-1}),(p)}_{n_1,\dots,n_{r-1}}(\xi_1,\dots,\xi_{r-2},\xi_{r-1}z) \in A^\dagger(\mathbf{P}^1 \setminus \{\overline{\xi_{r-1}^{-1}},\dots,\overline{(\xi_1\cdots\xi_{r-1})^{-1}}\})$$

and by $\beta(z)dz \in \Omega^{\dagger,1}(\mathbf{P}^1 \setminus \{\overline{\infty},\overline{1}\})$, we have

$$\ell^{\equiv(\alpha_1,\dots,\alpha_{r-1}),(p)}_{n_1,\dots,n_{r-1}}(\xi_1,\dots,\xi_{r-2},\xi_{r-1}z)\cdot\beta(z)dz \in \Omega^{\dagger,1}(\mathbf{P}^1 \setminus S_\infty).$$

Put

$$f(z) := \ell^{\equiv(\alpha_1,\dots,\alpha_{r-1}),(p)}_{n_1,\dots,n_{r-1}}(\xi_1,\dots,\xi_{r-2},\xi_{r-1}z)\cdot\beta(z) \in A^\dagger(\mathbf{P}^1 \setminus S_\infty).$$

Then it follows from the same arguments as the ones in (ii) that

$$\ell^{\equiv(\alpha_1,\dots,\alpha_r),(p)}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_{r-1},z) \in A^\dagger(\mathbf{P}^1 \setminus S).$$

□

We saw in Proposition 4.7 that $\ell^{(p)}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_{r-1},z)$ is a rigid analytic function, namely $\ell^{(p)}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_{r-1},z) \in A^{\mathrm{rig}}(\mathbf{P}^1 \setminus S)$. But actually we can say more:

Theorem 4.22. *Let $n_1,\dots,n_r \in \mathbb{N}$, $\xi_1,\dots,\xi_{r-1} \in \mathbb{C}_p$ with $|\xi_j|_p = 1$ ($1 \leqslant j \leqslant r-1$). Set S as in* (4.3)*. The function $\ell^{(p)}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_{r-1},z)$ is an overconvergent function of $\mathbf{P}^1 \setminus S$. Namely,*

$$\ell^{(p)}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_{r-1},z) \in A^\dagger(\mathbf{P}^1 \setminus S).$$

Proof. By our previous proposition, we have

$$\ell^{\equiv(\alpha_1,\dots,\alpha_r),(p)}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_{r-1},z) \in A^\dagger(\mathbf{P}^1 \setminus S).$$

Then by Remark 4.20 (ii), we have $\ell^{(p)}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_{r-1},z) \in A^\dagger(\mathbf{P}^1 \setminus S)$ because $A^\dagger(\mathbf{P}^1 \setminus S)$ forms an algebra. □

Example 4.23. Especially when $r = 1$ we have

$$\ell^{(p)}_n(z) \in A^\dagger(\mathbf{P}^1 \setminus \{\bar{1}\}).$$

Actually in [15, Proposition 6.2], Coleman showed that

$$\ell^{(p)}_n(z) \in A^{\mathrm{rig}}(\tilde{X})$$

with $\tilde{X} = \left\{ z \in \mathbf{P}^1(\mathbb{C}_p) \;\middle|\; |z-1|_p > p^{\frac{-1}{p-1}} \right\}$.

Remark 4.24. In [23, Definition 2.13], the first named author introduced the overconvergent p-adic MPL $Li^\dagger_{n_1,\dots,n_r}(z)$. Asking a relationship between this $Li^\dagger_{n_1,\dots,n_r}(z)$ and our $\ell^{(p)}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_{r-1},z)$ would be one of the questions which we are interested in.

4.3. **p-adic twisted multiple polylogarithms.** In [22, 47] p-adic TMPL's (see Definition 4.29) were introduced as a multiple generalization of Coleman's p-adic polylogarithm [15] and p-adic multiple L-value is introduced as its special value at 1. The aim of this subsection is to clarify a relationship between p-adic rigid TMPL's (see Definition 4.4) and p-adic TMPL's in Theorem 4.35 and to establish a relationship between special values at positive integers of our p-adic multiple L-functions (see Definition 2.9) and the p-adic twisted multiple L-values, the special values of p-adic TMPL's at 1, in Theorem 4.40. The theorem extends the previous result (4.30) of Coleman in depth 1 case shown in [15].

First we recall Coleman's p-adic iterated integration theory [15] in our particular case. For other nice expositions of his theory, see [8, Section 5], [11, Subsection 2.2.1] and [22, Subsection 2.1]. The integration here is different from the integration using a certain p-adic measure which is explained in Subsection 2.1.

Notation 4.25. Fix $\varpi \in \mathbb{C}_p$. The *p-adic logarithm* $\log^{\varpi}$ associated to the *branch* ϖ means the locally rigid analytic group homomorphism $\mathbb{C}_p^{\times} \to \mathbb{C}_p$ with the usual Taylor expansion for log on $]\bar{1}[= 1 + \mathfrak{M}_{\mathbb{C}_p}$. It is uniquely characterized by $log^{\varpi}(p) = \varpi$ because $\mathbb{C}_p^{\times} \simeq p^{\mathbb{Q}} \times \mu_\infty \times (1 + \mathfrak{M}_{\mathbb{C}_p})$. We call this $\varpi \in \mathbb{C}_p$ the *branch parameter* of the p-adic logarithm.

Let $S = \{s_0, \dots, s_d\}$ (all s_i's are distinct) be a finite subset of $\mathbf{P}^1(\overline{\mathbb{F}}_p)$. Define

$$A^{\varpi}_{\mathrm{loc}}(\mathbf{P}^1 \setminus S) := \prod_{x \in \mathbf{P}^1(\overline{\mathbb{F}}_p)} A^{\varpi}_{log}(U_x), \qquad \Omega^{\varpi}_{\mathrm{loc}}(\mathbf{P}^1 \setminus S) := \prod_{x \in \mathbf{P}^1(\overline{\mathbb{F}}_p)} \Omega^{\varpi}_{log}(U_x)$$

where

$$A^{\varpi}_{log}(U_x) := \begin{cases} A^{\mathrm{rig}}(]x[) & x \notin S, \\ \lim\limits_{\lambda \to 1^-} A^{\mathrm{rig}}\Big(]x[\cap U_\lambda\Big)\Big[log^{\varpi}(z_i)\Big] & x = s_i \quad (0 \leqslant i \leqslant d), \end{cases}$$

$$\Omega^{\varpi}_{log}(U_x) := \begin{cases} \Omega^{\mathrm{rig},1}(]x[) & x \notin S, \\ A^{\varpi}_{log}(U_x) dz_i & x = s_i \quad (0 \leqslant i \leqslant d). \end{cases}$$

Here U_λ is the affinoid in (4.8) and z_i is a local parameter (4.9). We note that $log^{\varpi}(z_i)$ is a locally analytic function defined on $]s_i[-z_i^{-1}(0)$ whose differential is $\frac{dz_i}{z_i}$ and it is transcendental over $A^{\mathrm{rig}}\Big(]s_i[\cap U_\lambda\Big)$ and

$$\lim_{\lambda \to 1^-} A^{\mathrm{rig}}\Big(]s_i[\cap U_\lambda\Big) \cong \Big\{ f(z_i) = \sum_{n=-\infty}^{\infty} a_n z_i^n \ (a_n \in \mathbb{C}_p) \text{ converging on } \lambda < |z_i|_p < 1 \text{ for some } 0 \leqslant \lambda < 1 \Big\}$$

(see [7]). We remark that these definitions of $A^{\varpi}_{log}(U_x)$ and $\Omega^{\varpi}_{log}(U_x)$ are independent of any choice of local parameters z_i modulo standard isomorphisms. By taking a component-wise derivative, we obtain a $\mathbb{C}_p$-linear map $d : A^{\varpi}_{\mathrm{loc}}(\mathbf{P}^1 \setminus S) \to \Omega^{\varpi}_{\mathrm{loc}}(\mathbf{P}^1 \setminus S)$. We may regard $A^{\dagger}(\mathbf{P}^1 \setminus S)$ and $\Omega^{\dagger,1}(\mathbf{P}^1 \setminus S)$ in Notation 4.13 to be a subspace of $A^{\varpi}_{\mathrm{loc}}(\mathbf{P}^1 \setminus S)$ and $\Omega^{\varpi}_{\mathrm{loc}}(\mathbf{P}^1 \setminus S)$ respectively.

By the work of Coleman [15], we have an $A^{\dagger}(\mathbf{P}^1 \setminus S)$-subalgebra

$$A^{\varpi}_{\mathrm{Col}}(\mathbf{P}^1 \setminus S)$$

of $A^{\varpi}_{\mathrm{loc}}(\mathbf{P}^1 \setminus S)$, which we call *the ring of Coleman functions* attached to a branch parameter $\varpi \in \mathbb{C}_p$, and a $\mathbb{C}_p$-linear map

$$\int_{(\varpi)} : A^{\varpi}_{\mathrm{Col}}(\mathbf{P}^1 \setminus S) \underset{A^{\dagger}(\mathbf{P}^1 \setminus S)}{\otimes} \Omega^{\dagger,1}(\mathbf{P}^1 \setminus S) \to A^{\varpi}_{\mathrm{Col}}(\mathbf{P}^1 \setminus S) \Big/ \mathbb{C}_p \cdot 1$$

satisfying $d\, \big|_{A^{\varpi}_{\mathrm{Col}}(\mathbf{P}^1 \setminus S)} \circ \int_{(\varpi)} = id_{A^{\varpi}_{\mathrm{Col}}(\mathbf{P}^1 \setminus S) \otimes \Omega^{\dagger,1}(\mathbf{P}^1 \setminus S)}$, which we call *the p-adic (Coleman) integration* attached to a branch parameter $\varpi \in \mathbb{C}_p$. We often drop the subscript $_{(\varpi)}$.

Since $A^{\varpi}_{\mathrm{Col}}(\mathbf{P}^1 \setminus S)$ is a subspace of $A^{\varpi}_{\mathrm{loc}}(\mathbf{P}^1 \setminus S)$, each element f of $A^{\varpi}_{\mathrm{Col}}(\mathbf{P}^1 \setminus S)$ can be seen as a map $f : U \to \mathbb{C}_p$ by $f|_{U\cap]x[} \in A^{\varpi}_{\log}(U_x)$ for each residue $]x[$, where U is an affinoid bigger than $\mathbf{P}^1(\mathbb{C}_p) -]S[$. This is how we regard f as a function.

Here we recall two important properties of Coleman functions below.

Proposition 4.26 (Branch Independency Principle [22, Proposition 2.3])**.** *For $\varpi_1, \varpi_2 \in \mathbb{C}_p$ define the isomorphisms*

$$\iota_{\varpi_1,\varpi_2} : A^{\varpi_1}_{\mathrm{loc}}(\mathbf{P}^1 \setminus S) \xrightarrow{\sim} A^{\varpi_2}_{\mathrm{loc}}(\mathbf{P}^1 \setminus S) \quad \text{and} \quad \tau_{\varpi_1,\varpi_2} : \Omega^{\varpi_1}_{\mathrm{loc}}(\mathbf{P}^1 \setminus S) \xrightarrow{\sim} \Omega^{\varpi_2}_{\mathrm{loc}}(\mathbf{P}^1 \setminus S)$$

which is obtained by replacing each $\log^{\varpi_1}(z_{s_i})$ by $\log^{\varpi_2}(z_{s_i})$ for $1 \leqslant i \leqslant d$. Then

$$\iota_{\varpi_1,\varpi_2}(A^{\varpi_1}_{\mathrm{Col}}(\mathbf{P}^1 \setminus S)) = A^{\varpi_2}_{\mathrm{Col}}(\mathbf{P}^1 \setminus S),$$
$$\tau_{\varpi_1,\varpi_2}(A^{\varpi_1}_{\mathrm{Col}}(\mathbf{P}^1 \setminus S) \otimes \Omega^{\dagger,1}(\mathbf{P}^1 \setminus S)) = A^{\varpi_2}_{\mathrm{Col}}(\mathbf{P}^1 \setminus S) \otimes \Omega^{\dagger,1}(\mathbf{P}^1 \setminus S)$$

and

$$\iota_{\varpi_1,\varpi_2} \circ \int_{(\varpi_1)} = \int_{(\varpi_2)} \circ \tau_{\varpi_1,\varpi_2} \mod \mathbb{C}_p \cdot 1.$$

Namely the following diagram is commutative:

$$\begin{array}{ccc}
A^{\varpi_1}_{\mathrm{Col}}(\mathbf{P}^1 \setminus S) \underset{A^{\dagger}(\mathbf{P}^1\setminus S)}{\otimes} \Omega^{\dagger,1}(\mathbf{P}^1 \setminus S) & \xrightarrow{\tau_{\varpi_1,\varpi_2}} & A^{\varpi_2}_{\mathrm{Col}}(\mathbf{P}^1 \setminus S) \underset{A^{\dagger}(\mathbf{P}^1\setminus S)}{\otimes} \Omega^{\dagger,1}(\mathbf{P}^1 \setminus S) \\
{\scriptstyle \int_{(\varpi_1)}}\big\downarrow & & \big\downarrow{\scriptstyle \int_{(\varpi_2)}} \\
A^{\varpi_1}_{\mathrm{Col}}(\mathbf{P}^1 \setminus S) \Big/ \mathbb{C}_p \cdot 1 & \xrightarrow{\iota_{\varpi_1,\varpi_2}} & A^{\varpi_2}_{\mathrm{Col}}(\mathbf{P}^1 \setminus S) \Big/ \mathbb{C}_p \cdot 1.
\end{array}$$

Proposition 4.27 (Coincidence Principle [15, Chapter IV])**.** *Put $\varpi \in \mathbb{C}_p$. Let $f \in A^{\varpi}_{\mathrm{Col}}(\mathbf{P}^1 \setminus S)$. Suppose that $f|_U \equiv 0$ for an admissible open subset U (cf.[8, 15]) of $\mathbf{P}^1(\mathbb{C}_p)$, (that is, U is of the form $\left\{z \in \mathbf{P}^1(\mathbb{C}_p) \;\middle|\; |z-\alpha_i|_p > \delta_i \ (i = 1, \ldots, n), |z|_p < 1/\delta_0\right\}$ for some $\alpha_1, \ldots, \alpha_n \in \mathbb{C}_p$ and $\delta_1, \ldots, \delta_n \in \mathbb{Q}_{>0}$ and $\delta_0 \in \mathbb{Q}_{\geqslant 0}$). Then $f = 0$.*

It follows from this proposition that a locally constant Coleman function is globally constant.

Notation 4.28. Fix a branch $\varpi \in \mathbb{C}_p$ and let $\omega \in A^{\varpi}_{\mathrm{Col}}(\mathbf{P}^1 \setminus S) \underset{A^{\dagger}(\mathbf{P}^1\setminus S)}{\otimes} \Omega^{\dagger,1}(\mathbf{P}^1 \setminus S)$. Then by Coleman's integration theory, there exists (uniquely modulo constant) a Coleman function $F_\omega \in A^{\varpi}_{\mathrm{Col}}(\mathbf{P}^1 \setminus S)$ such that $\int \omega \equiv F_\omega$ (modulo constant). For $x, y \in]\mathbf{P}^1 \setminus S[$, we define

$$\int_x^y \omega := F_\omega(y) - F_\omega(x).$$

It is clear that $\int_x^y \omega$ does not depend on any choice of F_ω (although it may depend on the choice of a branch $\varpi \in \mathbb{C}_p$). If both $F_\omega(x)$ and $F_\omega(y)$ make sense when x or y belongs to $]S_0[$, we also denote $F_\omega(y) - F_\omega(x)$ by $\int_x^y \omega$. By letting x fixed and y vary, we regard $\int_x^y \omega$ as the Coleman function which is characterized by $dF_\omega = \omega$ and $F_\omega(x) = 0$. We note that

$$\iota_{\varpi_1,\varpi_2}\left(\int_{(\varpi_1),x}^y \omega\right) = \int_{(\varpi_2),x}^y \tau_{\varpi_1,\varpi_2}(\omega). \tag{4.19}$$

We will apply Coleman's theory to the function defined below, which is a main object in this subsection.

Definition 4.29. Let $n_1, \ldots, n_r \in \mathbb{N}$, $\xi_1, \ldots, \xi_r \in \mathbb{C}_p$ with $|\xi_j|_p \leqslant 1$ $(1 \leqslant j \leqslant r)$. The **$p$-adic TMPL** $Li^{(p)}_{n_1,\ldots,n_r}(\xi_1, \ldots, \xi_r; z)$ is defined by the following p-adic power series:

$$Li^{(p)}_{n_1,\ldots,n_r}(\xi_1, \ldots, \xi_r; z) := \sum_{0<k_1<\cdots<k_r} \frac{\xi_1^{k_1} \cdots \xi_r^{k_r} z^{k_r}}{k_1^{n_1} \cdots k_r^{n_r}} \tag{4.20}$$

which converges for $z \in]\bar{0}[$ by $|\xi_j|_p \leqslant 1$ for $1 \leqslant j \leqslant r$.

Remark 4.30.

(i) We note that when $r = 1$ and $\xi = 1$, $Li^{(p)}_n(1; z)$ is equal to Coleman's p-adic polylogarithm $\ell_n(z)$ in [15, p. 195].

(ii) The special case when $\xi_1 = \cdots = \xi_r = 1$ is investigated by the first-named author in [23] where $Li^{(p)}_{n_1,\ldots,n_r}(1, \ldots, 1; z)$ is introduced as p-adic multiple polylogarithm.

(iii) Yamashita [47] treats the case when all ξ_j are roots of unity whose orders are prime to p.

The following can be proved by a direct computation.

Lemma 4.31. *Let $n_1, \ldots, n_r \in \mathbb{N}$, $\xi_1, \ldots, \xi_r \in \mathbb{C}_p$ with $|\xi_j|_p \leqslant 1$ $(1 \leqslant j \leqslant r)$.*

(i) $\dfrac{d}{dz} Li^{(p)}_1(\xi_1; z) = \dfrac{\xi_1}{1 - \xi_1 z}$.

(ii) $$\frac{d}{dz} Li^{(p)}_{n_1,\ldots,n_r}(\xi_1, \ldots, \xi_r; z) = \begin{cases} \frac{1}{z} Li^{(p)}_{n_1,\ldots,n_{r-1},n_r-1}(\xi_1, \ldots, \xi_r; z) & \textit{if} \quad n_r > 1, \\ \frac{\xi_r}{1-\xi_r z} Li^{(p)}_{n_1,\ldots,n_{r-1}}(\xi_1, \ldots, \xi_{r-2}, \xi_{r-1}\xi_r; z) & \textit{if} \quad n_r = 1. \end{cases}$$

The definition below is suggested by the differential equations above.

Theorem-Definition 4.32. *Fix a branch of the p-adic logarithm by $\varpi \in \mathbb{C}_p$. Let $n_1, \ldots, n_r \in \mathbb{N}$, $\xi_1, \ldots, \xi_r \in \mathbb{C}_p$ with $|\xi_j|_p \leqslant 1$ $(1 \leqslant j \leqslant r)$. Put*

$$S_r := \{\bar{0}, \bar{\infty}, \overline{(\xi_r)^{-1}}, \overline{(\xi_{r-1}\xi_r)^{-1}}, \ldots, \overline{(\xi_1 \cdots \xi_r)^{-1}}\} \subset \mathbf{P}^1(\bar{\mathbb{F}}_p).$$

Then we have the Coleman function attached to $\varpi \in \mathbb{C}_p$

$$Li^{(p),\varpi}_{n_1,\ldots,n_r}(\xi_1, \ldots, \xi_r; z) \in A^{\varpi}_{Col}(\mathbf{P}^1 \setminus S_r)$$

which is constructed by the following iterated integrals:

$$Li^{(p),\varpi}_1(\xi_1; z) = -\log^{\varpi}(1 - \xi_1 z) = \int_0^z \frac{\xi_1}{1 - \xi_1 t} dt, \tag{4.21}$$

$$Li^{(p),\varpi}_{n_1,\ldots,n_r}(\xi_1, \ldots, \xi_r; z) = \begin{cases} \int_0^z Li^{(p),\varpi}_{n_1,\ldots,n_{r-1},n_r-1}(\xi_1, \ldots, \xi_r; t) \frac{dt}{t} & \textit{if} \quad n_r > 1, \\ \int_0^z Li^{(p),\varpi}_{n_1,\ldots,n_{r-1}}(\xi_1, \ldots, \xi_{r-2}, \xi_{r-1}\xi_r; t) \frac{\xi_r dt}{1-\xi_r t} & \textit{if} \quad n_r = 1. \end{cases} \tag{4.22}$$

Proof. It is achieved by the induction on weight $w := n_1 + \cdots + n_r$.

When $w = 1$, the integration starting from 0 makes sense because the differential form $\frac{\xi_1}{1-\xi_1 t} dt$ has no pole at $t = 0$. Hence we have (4.21).

When $w > 1$ and $n_r > 1$, by our induction assumption, $Li^{(p),\varpi}_{n_1,\ldots,n_r-1}(\xi_1, \ldots, \xi_r; 0)$ is equal to 0 because it is an integration from 0 to 0. So $Li^{(p),\varpi}_{n_1,\ldots,n_r-1}(\xi_1, \ldots, \xi_r; t)$ has a zero of order at least 1. So the integrand of the right-hand side of (4.22) has no pole at $t = 0$. The integration (4.22) starting from 0 makes sense. The case when $n_r = 1$ can be proved in a same (or an easier) way. □

Remark 4.33. It is easy to see that $Li^{(p),\varpi}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_r;z) = Li^{(p)}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_r;z)$ when $|z|_p<1$ for all branch $\varpi\in\mathbb{C}_p$. Thus the Coleman function $Li^{(p),\varpi}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_r;z)\in A^{\varpi}_{\mathrm{Col}}(\mathbf{P}^1\setminus S_r)$ is defined on an affinoid bigger than $\mathbf{P}^1(\mathbb{C}_p) -]S_r\setminus\{\bar{0}\}[$.

Proposition 4.34. *Fix a branch $\varpi\in\mathbb{C}_p$. Let $n_1,\dots,n_r\in\mathbb{N}$, $\xi_1,\dots,\xi_r\in\mathbb{C}_p$ with $|\xi_j|_p\leqslant 1$ $(1\leqslant j\leqslant r)$. Then the restriction of the p-adic TMPL $Li^{(p),\varpi}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_{r-1},\xi_r;z)$ into $\mathbf{P}^1(\mathbb{C}_p) -]S_r\setminus\{\bar{0}\}[$ does not depend on any choice of the branch $\varpi\in\mathbb{C}_p$.*

Proof. It is achieved by the induction on weight $w:=n_1+\cdots+n_r$. Take $z\in\mathbf{P}^1(\mathbb{C}_p) -]S_r\setminus\{\bar{0}\}[$.

When $w=1$, it is easy to see that $Li^{(p),\varpi}_1(\xi_1;z) = -\log^{\varpi}(1-\xi_1 z)$ is free from any choice of the branch $\varpi\in\mathbb{C}_p$.

Consider the case when $w>1$ and $n_r>1$. Due to the existence of a pole of $\frac{dt}{t}$ at $t=0$, the integration (4.22) might have a 'log-term' on its restriction into $]\bar{0}[$. However it is easy to see that the restriction of $Li^{(p),\varpi}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_r;t)$ into $]\bar{0}[$ has no log-term because it is given by the series expansion (4.20) by Remark 4.33. By our induction assumption, the restriction of $Li^{(p),\varpi}_{n_1,\dots,n_r-1}(\xi_1,\dots,\xi_r;t)$ into $\mathbf{P}^1(\mathbb{C}_p) -]S_r\setminus\{\bar{0}\}[$ is independent of any choice of the branch $\varpi\in\mathbb{C}_p$. Therefore $Li^{(p),\varpi}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_r;t)$ has no log-term and does not depend on any choice of the branch.

The above proof also works for the case when $w>1$ and $n_r=1$. It is noted that the branch independency follows since two poles of $\frac{\xi_r dt}{1-\xi_r t}$ are outside of the region $\mathbf{P}^1(\mathbb{C}_p) -]S_r\setminus\{\bar{0}\}[$. ☐

The p-adic rigid TMPL $\ell^{(p)}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_{r-1},\xi_r z)$ in Subsection 4.1 is described by our p-adic TMPL $Li^{(p),\varpi}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_{r-1},\xi_r;z)$ as follows:

Theorem 4.35. *Fix a branch $\varpi\in\mathbb{C}_p$. Let $n_1,\dots,n_r\in\mathbb{N}$, $\xi_1,\dots,\xi_r\in\mathbb{C}_p$ with $|\xi_j|_p=1$ $(1\leqslant j\leqslant r)$. The equality*

$$\begin{aligned}&\ell^{(p)}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_{r-1},\xi_r z)\\ &\quad=\frac{1}{p^r}\sum_{0<\alpha_1,\dots,\alpha_r<p}\ \sum_{\rho_1^p=\cdots=\rho_r^p=1}\rho_1^{-\alpha_1}\cdots\rho_r^{-\alpha_r}Li^{(p),\varpi}_{n_1,\dots,n_r}(\rho_1\xi_1,\dots,\rho_r\xi_r;z)\end{aligned}\tag{4.23}$$

holds for $z\in\mathbf{P}^1(\mathbb{C}_p) -]S_r\setminus\{\bar{0}\}[$.

Proof. By the power series expansion (4.12) and (4.20) and Remark 4.20.(ii), it is easy to see that the equality holds on $]\bar{0}[$. By Theorem 4.22, the left-hand side belongs to $A^{\dagger}(\mathbf{P}^1\setminus S_r)$ $(\subset A^{\varpi}_{\mathrm{Col}}(\mathbf{P}^1\setminus S_r))$ and by Theorem-Definition 4.32, the right-hand side belongs $A^{\varpi}_{\mathrm{Col}}(\mathbf{P}^1\setminus S_r)$. Therefore by the coincidence principle (Proposition 4.27), the equality actually holds on the whole space of $\mathbf{P}^1(\mathbb{C}_p) -]S_r\setminus\{\bar{0}\}[$, actually on an affinoid bigger than the space. ☐

The following is a reformulation of the equation in Theorem 4.35.

Theorem 4.36. *Fix a branch $\varpi\in\mathbb{C}_p$. Let $n_1,\dots,n_r\in\mathbb{N}$, $\xi_1,\dots,\xi_r\in\mathbb{C}_p$ with $|\xi_j|_p=1$ $(1\leqslant j\leqslant r)$. The equality*

$$\ell^{(p)}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_{r-1},\xi_r z)$$

$$= Li^{(p),\varpi}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_r;z)+\sum_{d=1}^{r}\left(-\frac{1}{p}\right)^d\sum_{1\leqslant i_1<\cdots<i_d\leqslant r}\sum_{\rho_{i_1}^p=1}\cdots\sum_{\rho_{i_d}^p=1}Li^{(p),\varpi}_{n_1,\dots,n_r}\Big(((\prod_{l=1}^{d}\rho_{i_l}^{\delta_{i_l j}})\xi_j);z\Big) \tag{4.24}$$

holds for $z\in\mathbf{P}^1(\mathbb{C}_p)-]S_r\setminus\{\bar{0}\}[$, where δ_{ij} is the Kronecker delta.

Proof. It is a consequence of the following direct computation:

$$\ell^{(p)}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_{r-1},\xi_r z)=\sum_{\substack{0<k_1<\cdots<k_r\\(k_1,p)=\cdots=(k_r,p)=1}}\frac{\xi_1^{k_1}\cdots\xi_r^{k_r}z^{k_r}}{k_1^{n_1}\cdots k_r^{n_r}}=\sum_{0<k_1<\cdots<k_r}\left\{\prod_{i=1}^{r}\left(1-\frac{1}{p}\sum_{\rho_i^p=1}\rho_i^{k_i}\right)\frac{\xi_i^{k_i}}{k_i^{n_i}}\right\}z^{k_r}$$

$$=Li^{(p)}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_r;z)+\sum_{d=1}^{r}\left(-\frac{1}{p}\right)^d\sum_{1\leqslant i_1<\cdots<i_d\leqslant r}\sum_{\rho_{i_1}^p=1}\cdots\sum_{\rho_{i_d}^p=1}Li^{(p)}_{n_1,\dots,n_r}\Big(((\prod_{l=1}^{d}\rho_{i_l}^{\delta_{i_l j}})\xi_j);z\Big).$$

□

Example 4.37. In the case when $r=1$, (4.24) gives

$$\ell^{(p)}_n(z)=Li^{(p)}_n(1;z)-\frac{1}{p}\sum_{\rho^p=1}Li^{(p)}_n(\rho;z).$$

But by the distribution relation (cf.[15])

$$\frac{1}{r}\sum_{\eta\in\mu_r}Li^{(p)}_n(\eta;z)=\frac{1}{r^n}Li^{(p)}_n(1;z^r) \tag{4.25}$$

for $r\geqslant 1$, we recover Coleman's formula

$$\ell^{(p)}_n(z)=Li^{(p)}_n(1;z)-\frac{1}{p^n}Li^{(p)}_n(1;z^p)$$

shown in [15, Section VI].

We define the p-adic twisted multiple L-values by the special values of the p-adic TMPL's at unity under a certain condition below.

Theorem-Definition 4.38. *Fix a branch $\varpi\in\mathbb{C}_p$. Let $n_1,\dots,n_r\in\mathbb{N}$, $\rho_1,\dots,\rho_r\in\mu_p$ and $\xi_1,\dots,\xi_r\in\mu_c$ with $(c,p)=1$. Assume that*

$$\xi_1\cdots\xi_r\neq 1,\quad \xi_2\cdots\xi_r\neq 1,\quad \dots,\quad \xi_{r-1}\xi_r\neq 1,\quad \xi_r\neq 1. \tag{4.26}$$

Then the special value of

$$Li^{(p),\varpi}_{n_1,\dots,n_r}(\rho_1\xi_1,\dots,\rho_r\xi_r;z) \tag{4.27}$$

*at $z=1$ is well-defined. Furthermore it is free from any choice of the branch $\varpi\in\mathbb{C}_p$ and it belongs to $\mathbb{Q}_p(\mu_{cp})$. We call the value by **p-adic twisted multiple L-value** and denote it by*

$$Li^{(p)}_{n_1,\dots,n_r}(\rho_1\xi_1,\dots,\rho_r\xi_r). \tag{4.28}$$

Proof. By our assumption (4.26),

$$1\in\mathbf{P}^1(\mathbb{C}_p)-]S_r\setminus\{\bar{0}\}[.$$

Hence the special value of (4.27) at $z=1$ makes sense by Remark 4.33.

The branch independency of the special value follows from Proposition 4.34.

The points 0 and 1 and all the differential 1-forms $\frac{\xi_r dt}{1-\xi_r t}$, $\frac{(\xi_{r-1}\xi_r)dt}{1-(\xi_{r-1}\xi_r)t}$, $\ldots$, $\frac{(\xi_1\cdots\xi_r)dt}{1-(\xi_1\cdots\xi_r)t}$ are defined over $\mathbb{Q}_p(\mu_{cp})$. Then by the Galois equivalency stated in [9] Remark 2.3, the special value of (4.27) for $z\in\mathbb{Q}_p(\mu_{cp})$ is invariant under the action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p(\mu_{cp}))$ if we take $\varpi\in\mathbb{Q}_p(\mu_{cp})$. Since we have shown that the special values at $z=1$ is free from the choice of ϖ, the value (4.28) must belong to $\mathbb{Q}_p(\mu_{cp})$. □

Remark 4.39.

(i) The p-adic multiple zeta values are introduced as the special values at $z=1$ of p-adic MPL $Li^{(p),\varpi}_{n_1,\ldots,n_r}(1,\ldots,1;z)$, the function (4.27) with all $\rho_i=1$ and $\xi_i=1$, and their basic properties are investigated by the first-named author in [22].

(ii) The p-adic multiple L-values introduced by Yamashita [47] are special values at $z=1$ of the function (4.27) with all $\rho_i=1$ and $(\xi_r,n_r)\neq(1,1)$. Unver's [45] cyclotomic p-adic multi-zeta values might be closely related to his values.

By Theorems 4.9 and Theorem 4.35, we have the following. A very nice point here is that our assumption (4.26) appeared as the condition of the summation in equation (4.6).

Theorem 4.40. *For $n_1,\ldots,n_r\in\mathbb{N}$ and $c\in\mathbb{N}_{>1}$ with $(c,p)=1$,*

$$L_{p,r}(n_1,\ldots,n_r;\omega^{-n_1},\ldots,\omega^{-n_r};1,\ldots,1;c)=\\ \frac{1}{p^r}\sum_{0<\alpha_1,\ldots,\alpha_r<p}\ \sum_{\rho_1^p=\cdots=\rho_r^p=1}\ \sum_{\substack{\xi_1^c=\cdots=\xi_r^c=1\\ \xi_1\cdots\xi_r\neq1,\ldots,\xi_{r-1}\xi_r\neq1,\xi_r\neq1}}\rho_1^{-\alpha_1}\cdots\rho_r^{-\alpha_r}\cdot Li^{(p)}_{n_1,\ldots,n_r}(\rho_1\xi_1,\ldots,\rho_r\xi_r).$$

The above theorem is reformulated to the following, which is comparable to Theorem 3.1.

Theorem 4.41. *For $n_1,\ldots,n_r\in\mathbb{N}$ and $c\in\mathbb{N}_{>1}$ with $(c,p)=1$,*

$$\begin{aligned}
&L_{p,r}(n_1,\ldots,n_r;\omega^{-n_1},\ldots,\omega^{-n_r};1,\ldots,1;c)\\
&\quad=\sum_{\substack{\xi_1^c=1\\ \xi_1\neq1}}\cdots\sum_{\substack{\xi_r^c=1\\ \xi_r\neq1}}Li^{(p)}_{n_1,\ldots,n_r}\left(\frac{\xi_1}{\xi_2},\frac{\xi_2}{\xi_3},\ldots,\frac{\xi_r}{\xi_{r+1}}\right)\\
&\quad+\sum_{d=1}^{r}\left(-\frac{1}{p}\right)^d\sum_{1\leqslant i_1<\cdots<i_d\leqslant r}\ \sum_{\rho_{i_1}^p=1}\cdots\sum_{\rho_{i_d}^p=1}\ \sum_{\substack{\xi_1^c=1\\ \xi_1\neq1}}\cdots\sum_{\substack{\xi_r^c=1\\ \xi_r\neq1}}Li^{(p)}_{n_1,\ldots,n_r}\Big(\big(\frac{\prod_{l=1}^d\rho_{i_l}^{\delta_{i_lj}}\xi_j}{\xi_{j+1}}\big)\Big),
\end{aligned}\tag{4.29}$$

where we put $\xi_{r+1}=1$.

Proof. It follows from Theorem 4.9 and Theorem 4.36. □

Thus the positive integer values of the p-adic multiple L-function are described as linear combinations of the special values of the p-adic TMPL (4.27) at roots of unity. This might be regarded as a p-adic analogue of the equality (1.3).

As a special case of Theorem 4.40 we have the following. When $r=1$, we have

Example 4.42. For $n \in \mathbb{N}$ and $c \in \mathbb{N}_{>1}$ with $(c,p)=1$, we have

$$(c^{1-n}-1)\cdot L_p(n;\omega^{1-n}) = \frac{1}{p}\sum_{0<\alpha<p}\sum_{\rho^p=1}\sum_{\substack{\xi^c=1\\ \xi\neq 1}}\rho^{-\alpha}Li_n^{(p)}(\rho\xi)$$

by Example 2.12. This formula and (4.25) recover Coleman's equation [15, (4)]

$$L_p(n;\omega^{1-n}) = (1-\frac{1}{p^n})Li_n^{(p)}(1). \tag{4.30}$$

When $r=2$, we have

Example 4.43. For $a,b \in \mathbb{N}$ and $c \in \mathbb{N}_{>1}$ with $(c,p)=1$,

$$\begin{aligned}
&L_{p,2}(a,b;\omega^{-a},\omega^{-b};1,1;c)\\
&=\frac{1}{p^2}\sum_{0<\alpha,\beta<p}\ \sum_{\rho_1,\rho_2\in\mu_p}\ \sum_{\substack{\xi_1,\xi_2\in\mu_c\\ \xi_1\xi_2\neq1,\xi_2\neq1}}\rho_1^{-\alpha}\rho_2^{-\beta}Li_{a,b}(\rho_1\xi_1,\rho_2\xi_2)\\
&=\sum_{\substack{\xi_1^c=1\\ \xi_1\neq1}}\sum_{\substack{\xi_2^c=1\\ \xi_2\neq1}}Li_{a,b}^{(p)}\left(\frac{\xi_1}{\xi_2},\xi_2\right)-\frac{1}{p}\sum_{\rho^p=1}\sum_{\substack{\xi_1^c=1\\ \xi_1\neq1}}\sum_{\substack{\xi_2^c=1\\ \xi_2\neq1}}\left\{Li_{a,b}^{(p)}\left(\frac{\rho\xi_1}{\xi_2},\xi_2\right)+Li_{a,b}^{(p)}\left(\frac{\xi_1}{\xi_2},\rho\xi_2\right)\right\}\\
&\qquad+\frac{1}{p^2}\sum_{\rho_1^p=1}\sum_{\rho_2^p=1}\sum_{\substack{\xi_1^c=1\\ \xi_1\neq1}}\sum_{\substack{\xi_2^c=1\\ \xi_2\neq1}}Li_{a,b}^{(p)}\left(\frac{\rho_1\xi_1}{\xi_2},\rho_2\xi_2\right).
\end{aligned}$$

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H. Furusho
Graduate School of Mathematics
Nagoya University
Furo-cho, Chikusa-ku
Nagoya 464-8602, Japan
furusho@math.nagoya-u.ac.jp

Y. Komori
Department of Mathematics
Rikkyo University
Nishi-Ikebukuro, Toshima-ku
Tokyo 171-8501, Japan
komori@rikkyo.ac.jp

K. Matsumoto
Graduate School of Mathematics
Nagoya University
Furo-cho, Chikusa-ku
Nagoya 464-8602, Japan
kohjimat@math.nagoya-u.ac.jp

H. Tsumura
Department of Mathematics and Information Sciences
Tokyo Metropolitan University
1-1, Minami-Ohsawa, Hachioji
Tokyo 192-0397, Japan
tsumura@tmu.ac.jp