PUNCTURED PLANE PARTITIONS AND THE q-DEFORMED KNIZHNIK–ZAMOLODCHIKOV AND HIROTA EQUATIONS

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Abstract. We consider partial sum rules for the homogeneous limit of the solution of the q-deformed Knizhnik–Zamolodchikov equation with reflecting boundaries in the Dyck path representation. We show that these partial sums arise in a solution of the discrete Hirota equation, and prove that they are the generating functions of τ^2-weighted punctured cyclically symmetric transpose complement plane partitions where $\tau = -(q+q^{-1})$. In the cases of no or minimal punctures, we prove that these generating functions coincide with τ^2-enumerations of vertically symmetric alternating sign matrices and modifications thereof.

Contents

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1. Introduction

The discrete Hirota equation [11, 29] is a difference equation for a function f in three discrete variables:

$$f(n,i,j)f(n-2,i,j) = f(n-1,i-1,j)f(n-1,i+1,j) + \tau^2 f(n-1,i,j-1)f(n-1,i,j+1). \quad (1.1)$$

The factor τ^2 can be absorbed into f by rescaling $f(n,i,j) \to \tau^{-j(j+1)} f(n,i,j)$. Equation (1.1) is also known as the octahedron recurrence as the variables n,i,j are natural coordinates for the vertices of an octahedron, see e.g. [26].

In order to fix a particular solution to (1.1) we need to specify boundary conditions. An interesting choice is the following. Let $A = (a_{ij})_{1\le i,j\le n}$ be an $n \times n$ matrix and set

$$f(0,i,j) = 1, \qquad f(1,i,j) = a_{ij} \quad 1 \le i,j \le n. \quad (1.2)$$

With these boundary conditions, the discrete Hirota equation defines the τ^2-deformation of the determinant studied by Robbins and Rumsey [25]. In particular, the τ^2-determinant of A is defined by

$$|A|_{\tau^2} = f(n,1,1). \quad (1.3)$$

For $\tau = \mathrm{i}$ the τ^2-determinant reduces to the ordinary determinant which can be expanded as a sum over permutation matrices. In their famous work [25], Robbins and Rumsey showed that solutions to (1.1) with the boundary condition (1.2) can be written as a sum over alternating sign matrices (ASMs), see also [2].

It is well known that ASMs are equinumerous to totally symmetric self-complementary plane partitions (TSSCPPs). Surprisingly, generating functions of τ^2-enumerations of TSSCPPs and other symmetry classes of plane partitions, as studied by Robbins [24], appear as normalisations for homogeneous solutions of the q-deformed Knizhnik–Zamolodchikov (qKZ) equation recently obtained by Di Francesco and Zinn-Justin [7, 9]. This result will be generalised below, when we consider a class of punctured cyclically symmetric transpose complement plane partitions (PCSTCPPs), whose weighted enumerations also arise in the qKZ equation.

It was already observed by Robbins and Kuperberg [24, 13] that the τ^2-enumerations of CSTCPPs (without puncture) are closely related to τ^2 enumerations of vertically symmetric alternating-sign matrices (VSASMs) and other symmetry classes of ASMs. A precise statement will be proved below. Closing the circle, these enumerations comprise a particular solution of (1.1), albeit with a different boundary condition than (1.2). We hope that this paper will be a further step in resolving the Razumov–Stroganov conjectures and the discovery of a bijection between ASMs and TSSCPPs.

Throughout the following we will use the notation $[x]_q$ for the usual q-number

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}.$$

The notation $[x]$ will always refer to base q.

2. q-deformed Knizhnik–Zamolodchikov equation

Definition 1. *The* ***Temperley–Lieb algebra*** *of type A_L, denoted by $\mathcal{T}_L^{\rm A}(q)$, is the unital algebra defined in terms of generators e_i, $i=1,\ldots,L-1$, satisfying the relations*

$$\begin{aligned} &e_i^2 = -[2]e_i, \quad e_ie_j = e_je_i \quad \forall\, i,j:\, |i-j|>1,\\ &e_ie_{i\pm1}e_i = e_i. \end{aligned} \tag{2.1}$$

Every representation of the Temperley–Lieb algebra defines a solvable lattice model through the definition of so-called R-matrices. An R-matrix is a representation of the following Baxterised element of the Temperley–Lieb algebra:

$$R_i(u) = \frac{[1-u]-[u]\,e_i}{[1+u]}, \tag{2.2}$$

where $u\in\mathbb{C}\setminus\{-1\}$ is the spectral parameter.

Definition 2. *A* ***Dyck path*** *$\alpha=(\alpha_0,\alpha_1,\ldots,\alpha_L)$ is a sequence of integers α_i (heights) such that $\alpha_{i+1}=\alpha_i\pm1$, $\alpha_i\geq 0$ and $\alpha_0=\alpha_L=0$.*

The Temperley–Lieb algebra has a known action on Dyck paths, which is well-documented in the literature, see for example [16, 5] and references therein. The span of Dyck paths forms a module of $\mathcal{T}_L^{\rm A}(q)$, and we will denote its basis elements by $|\alpha\rangle$, where α runs over the set $\mathcal{D}_L$ of Dyck paths of length L. Let us now consider a linear combination $|\Psi\rangle$ of states $|\alpha\rangle$ with coefficients ψ_α taking values in the ring of formal series in L variables $q^{\pm x_i}$, $i=1,2,\ldots,L$:

$$|\Psi(x_1,\ldots,x_L)\rangle = \sum_{\alpha\in\mathcal{D}_L}\psi_\alpha(x_1,\ldots,x_L)|\alpha\rangle.$$

The q-deformed Knizhnik–Zamolodchikov equation on a segment (with reflecting boundaries) is a system of finite difference equations on the vector $|\Psi\rangle$. This equation reads (see [12, 6])

$$\begin{aligned} R_i(x_i-x_{i+1})|\Psi\rangle &= \pi_i|\Psi\rangle, \qquad \forall\, i=1,\ldots,L-1,\\ |\Psi\rangle &= \pi_0|\Psi\rangle,\\ |\Psi\rangle &= \pi_L|\Psi\rangle, \end{aligned} \tag{2.3}$$

where R_i are the Baxterised elements of the Temperley–Lieb algebra. The operators $R_i(x_i-x_{i+1})$ act on states $|\alpha\rangle$, whereas the operators π_i permute or reflect arguments of the coefficient functions:

$$\pi_i\psi_\alpha(\ldots,x_i,x_{i+1},\ldots) = \psi_\alpha(\ldots,x_{i+1},x_i,\ldots),$$

$$\pi_0\psi_\alpha(x_1,\ldots) = \psi_\alpha(-x_1,\ldots), \tag{2.4}$$

$$\pi_L\psi_\alpha(\ldots,x_L) = \psi_\alpha(\ldots,-\lambda-x_L). \tag{2.5}$$

The shift $\lambda\in\mathbb{C}$ is a parameter related to the level of the qKZ equation, see [10].

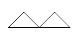
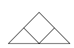
In [7] polynomial solutions of the qKZ equation were studied in the limit where $x_i\to 0$. Interestingly, in this limit the coefficients ψ_α turn out to be polynomials with positive coefficients in τ^2, where $\tau=-[2]=-q-q^{-1}$. Furthermore, based on observations on explicit solutions for small values of L, an intriguing connection between (generalised) sum rules and the enumeration of weighted CSTCPPs was conjectured. This was subsequently proved using multi-integral formulae [9]. Here we shall generalise this result.

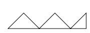
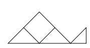
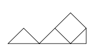
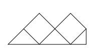
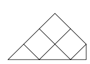
3. Explicit solutions

In [7] explicit solutions of the qKZ equation in the limit $x_i \to 0$ were obtained for $L \leq 8$. In order to illustrate our results we shall here list the first few solutions. We will write shorthand ψ_α for the limit $x_i \to 0$ of $\psi_\alpha(x_1, \ldots, x_L)$. The complete solution is determined up to an overall normalisation, and we will choose

$$\psi_\Omega = \tau^{\lfloor L/2 \rfloor (\lfloor L/2 \rfloor - 1)/2}$$

for the coefficient corresponding to the maximal Dyck path $\Omega \in \mathcal{D}_L : \Omega_i = \min\{i, L + \epsilon_L - i\}$, $\epsilon_L = L \bmod 2$. Together with the solution we list certain powers $\tau^{c_{\alpha,n}}$ whose meaning will become clear below.

$L=4$	α	ψ_α	$\tau^{\pm c_{\alpha,1}}$	
		$1+\tau^2$	1	
		τ	$\tau^{\pm 1}$	

$L=5$	α	ψ_α	$\tau^{\pm c_{\alpha,2}}$	$\tau^{\pm c_{\alpha,1}}$
		$\tau^2(2+\tau^2)$	1	
		τ^3	$\tau^{\pm 1}$	
		$\tau(2+\tau^2)$	$\tau^{\pm 1}$	
		$1+2\tau^2$	$\tau^{\pm 2}$	1
		τ	$\tau^{\pm 1}$	$\tau^{\pm 1}$

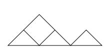
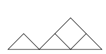
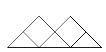
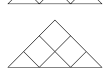
$L=6$	α	ψ_α	$\tau^{\pm c_{\alpha,2}}$	$\tau^{\pm c_{\alpha,1}}$
		$1+5\tau^2+4\tau^4+\tau^6$	1	
		$\tau(2+2\tau^2+\tau^4)$	$\tau^{\pm 1}$	
		$\tau(1+3\tau^2+\tau^4)$	$\tau^{\pm 1}$	
		$2\tau^2(1+\tau^2)$	$\tau^{\pm 2}$	1
		τ^3	$\tau^{\pm 1}$	$\tau^{\pm 1}$

Table 1. Explicit solutions of the qKZ equation in the homogeneous limit

An immediate observation about these solutions was already noted in [7]:

Conjecture 1. *The components $\psi_\alpha(x_1, \ldots, x_L)$ of the polynomial solution of the qKZ equation of type A in the limit $x_i \to 0$, $i = 1, \ldots, L$ are, up to an overall factor which is a power of τ, polynomials in τ^2 with* ***positive*** *integer coefficients. Here $\tau = -[2]$.*

Polynomiality and integrality of the coefficients is proved in [9].

Based on these explicit solutions, and solutions obtained for $L = 9$ and $L = 10$ in [5], we discovered underlying discrete bilinear relations. In order to uncover these relations we have to introduce certain partial sums over the components ψ_α of the solution. Let us first define the paths $\Omega(L,p) \in \mathcal{D}_L$ whose local minima lie at height $\tilde{p}$, where

$$\tilde{p} = \lfloor (L-1)/2 \rfloor - p, \qquad p = 0, \dots, \lfloor (L-1)/2 \rfloor.$$

Figure 1 illustrates the path $\Omega(12,3)$. We further define the subset $\mathcal{D}_{L,p}$ of Dyck paths of length L which lie above $\Omega(L,p)$, i.e. whose local minima lie on or above height $\tilde{p}$. Formally, this subset is described as

$$\mathcal{D}_{L,p} = \{\alpha \in \mathcal{D}_L |\ \alpha_i \geq \Omega_i(L,p) = \min(\Omega_i, \tilde{p})\},$$

where Ω_i are integer heights of the maximal Dyck path $\Omega = \Omega(L,0)$.

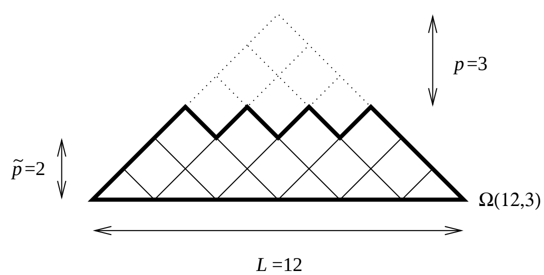


FIGURE 1. The path $\Omega(12,3) \in \mathcal{D}_{12,3}$.

To each Dyck path we associate an integer $c_{\alpha,p}$. Let $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_L) \in \mathcal{D}_{L,p}$ be a Dyck path of length L whose minima lie on or above height $\tilde{p}$. Then $c_{\alpha,p}$ is defined as the signed sum of boxes between α and $\Omega(L,p)$, where the boxes at height $\tilde{p} + h$ are assigned $(-1)^{h-1}$ for $h \geq 1$. An example is given in Figure 2, and an explicit expression for $c_{\alpha,p}$ is given by

$$c_{\alpha,p} = \frac{(-1)^{\tilde{p}}}{2} \sum_{i=2}^{L-1} (-1)^i (\alpha_i - \Omega_i(L,p)). \tag{3.1}$$

In the next section we will consider certain properties of the partial weighted sums

$$S_\pm(L,p) = \sum_{\alpha \in \mathcal{D}_{L,p}} \tau^{\pm c_{\alpha,p}}\, \psi_\alpha. \tag{3.2}$$

The partial sums corresponding to the solutions in Table 1 are given in Table 2.

4. Plane partitions and the discrete Hirota equation

It was observed in [21] that for $\tau = 1$ ($q = \mathrm{e}^{2\pi \mathrm{i}/3}$), the partial sums in Table 2 satisfy a discrete bilinear relation called Pascal's hexagon, or the discrete Boussinesq equation which, in turn, is a two dimensional reduction of the discrete Hirota equation (see [29] and references therein). Here we generalise this result to arbitrary τ, and show that the partial

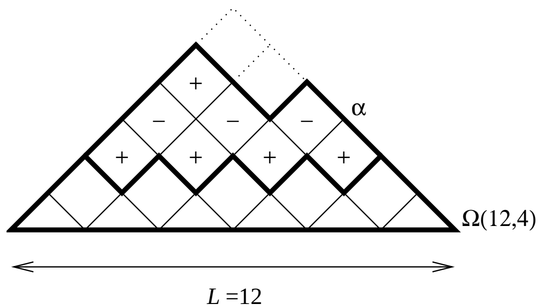


FIGURE 2. Definition of the number $c_{\alpha,p}$ as the signed sum of boxes between the α and the path $\widetilde{\Omega}(12,4)$. In this figure $L = 12$ and $p = 4$ and $c_{\alpha,4} = 4 - 3 + 1 = 2$.

$L = 4$	$S_-(4,0) = \tau$	$S_+(4,0) = \tau$
	$S_-(4,1) = 2 + \tau^2$	$S_+(4,1) = 1 + 2\tau^2$
$L = 5$	$S_-(5,0) = \tau$	$S_+(5,0) = \tau$
	$S_-(5,1) = 2(1 + \tau^2)$	$S_+(5,1) = 1 + 3\tau^2$
	$S_-(5,2) = \tau^{-2}(1 + 5\tau^2 + 4\tau^4 + \tau^6)$	$S_+(5,2) = \tau^2(6 + 5\tau^2)$
$L = 6$	$S_-(6,0) = \tau^3$	$S_+(6,0) = \tau^3$
	$S_-(6,1) = \tau^2(3 + 2\tau^2)$	$S_+(6,1) = \tau^2(2 + 3\tau^2)$
	$S_-(6,2) = 6 + 13\tau^2 + 6\tau^4 + \tau^6$	$S_+(6,2) = 1 + 8\tau^2 + 12\tau^4 + 5\tau^6$

TABLE 2. Partial sums corresponding to the solutions in Table 1.

sums satisfy the discrete Hirota equation. Based on experimental data for solutions up to $L = 10$ [5], we were led to introduce the following polynomials: [1]

$$T(L,p,k) = \det_{1\le \ell,m\le p}\left(\sum_{r=0}^{2p}\binom{\ell+k-1}{r-\ell}\binom{m+L-2p-k}{2m-r}\tau^{2(2m-r)}\right). \tag{4.1}$$

We then have the following result for the functional form of the partial sums for arbitrary system size L:

Proposition 1.

$$S_+(L,p) = \tau^{\nu_{L,p}}\, T(L,p,\lfloor L/2\rfloor - p), \tag{4.2}$$
$$S_-(L,p) = \tau^{\nu_{L,p}}\, T(L,p,\lfloor L/2\rfloor - p + 1), \tag{4.3}$$

[1] The idea to consider such type determinants comes from the observation that some of the polynomials given in Table 2 coincide with the conjectural generating functions of VSASMs and VHSASMs $T_n(\tau^2,\mu)|_{\mu=0,1}$ (see [24], Table 4.3). The latter functions are given by similar determinants.

where

$$\nu_{L,p} = \frac{1}{2}\big(\lfloor L/2\rfloor(\lfloor L/2\rfloor - 1) - p(p+1)\big).$$

This proposition will be proved using multiple integral equations in Section 6.

4.1. **Punctured symmetric plane partitions.** Using the standard presentation of the determinant of $p\times p$ matrix A in terms of the minors of $p \times 2p$ matrices B and C: $A = BC^t$, the polynomials $T(L,p,k)$ can alternatively be written as

$$T(L,p,k) = \sum_{1\le r_1<r_2<\ldots<r_p} \det_{1\le \ell,m\le p}\left(\binom{\ell+k-1}{r_m-\ell}\right) \times \det_{1\le \ell,m\le p}\left(\binom{\ell+L-2p-k}{2\ell-r_m}\tau^{2(2\ell-r_m)}\right). \quad (4.4)$$

By the use of Lindström-Gessel-Viennot formula [14] the last expression can be interpreted as the generating function of two sets of paths with the same end-point, as in Figure 3. One set of paths starts at positions $(\ell,\ell+k-1)$, $(\ell=1,\ldots,p)$ and has diagonal NW-SE and vertical steps; the other set starts at $(\ell-L+2p+k,-\ell-L+2p+k)$ with diagonal NE-SW and vertical steps. Each vertical step below the horizontal line (green online) is assigned a weight τ^2. All other steps are assigned weight 1. Up to an overall factor $\tau^{(k-1)p}$ this is also the generating function of pairs of paths where all vertical steps are assigned a weight τ.

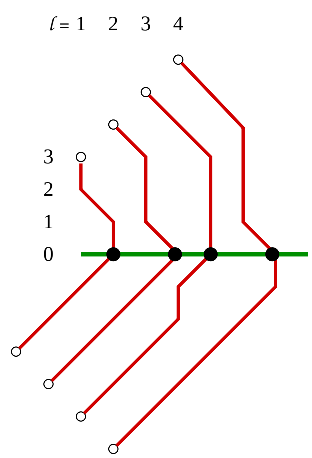


FIGURE 3. An example of two sets of paths with common endpoints enumerated by $T(L,p,k)$ for $L=13$, $p=4$ and $k=3$.

Consider stacking a large cube with smaller cubes, starting in a corner of the large cube, in such a way that the smaller cubes can only be stacked on top or next to each other. Such stackings are called plane partitions, which are well known to be equivalent to rhombus tilings of a hexagon. The paths in Figure 3 arise naturally when one considers cyclically symmetric transpose complement plane partitions (CSTCPPs) and generalisations thereof, as was done by Ciucu and Krattenthaler [3]. CSTCPPs correspond to rhombus tilings of a hexagon which are invariant under rotations over $2\pi/3$ as well as under reflections across a symmetry axis not passing through the hexagon corners. In [3] the more general problem was considered of cyclically symmetric transpose complement tilings of a hexagon with a triangular hole, as in Figure 4. We will call such tilings punctured CSTCPPs, or PCSTCPPs, the size of the triangular puncture being determined by the difference of the lengths of the sides of the hexagon.

If we weight PCSTCPPs by assigning a weight τ^2 to each vertical step below the bisecting line (green online) in the South-East region of Figure 4, the weighted enumeration of tilings of the fundamental domain of PCSTCPPs is equivalent to the weighted enumeration of paths in Figure 3. We have sketched the bijection in Figure 4, full details may be found in [3] where the enumeration ($\tau = 1$) is considered, which is shown to factorise completely, see also (5.1). For general τ but p restricted to $p = \lfloor (L-1)/2 \rfloor$ PCSTCPPs were considered in the context of the qKZ equation by Di Francesco [7].

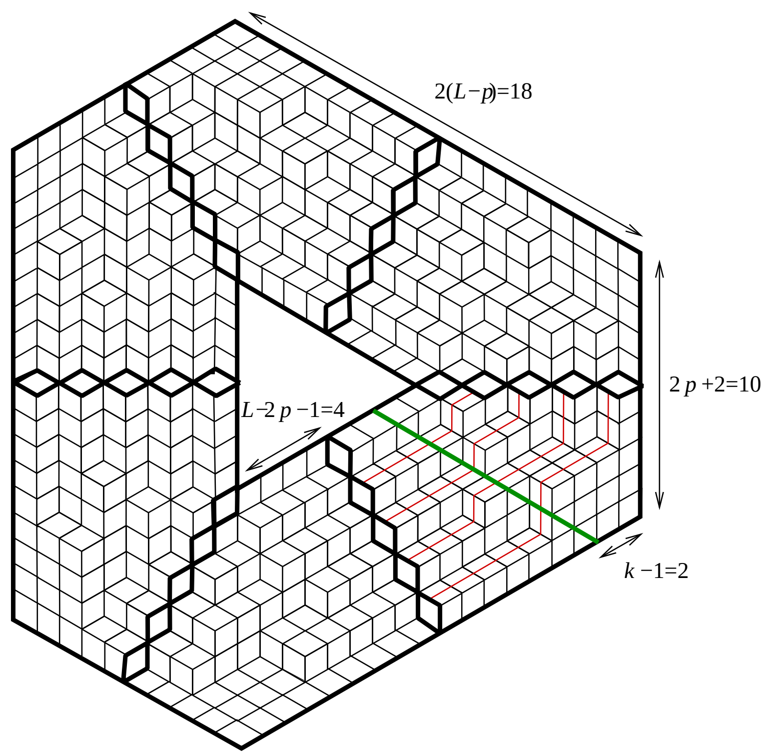


Figure 4. A punctured cyclically symmetric transpose complement plane partition (PCSTCPP) and its fundamental region. The linear size of the puncture is given by the difference of the lengths of the sides of the hexagon. The position of the bisecting line (green online) determines a particular weighting of the PCSTCPP: vertical steps below this line are assigned a weight τ^2.

Interestingly, there is another way to interpret the paths in Figure 3 in terms of punctured cyclically symmetric plane partitions. When we remove a central cubic region from the large cube, and consider special sixfold rotational symmetric rhombus tilings of the corresponding punctured hexagon by defining a fundamental region as in the upper right corner of Figure 5, we obtain a subset of punctured cyclically symmetric self complement plane partitions (PCSSCPPs).

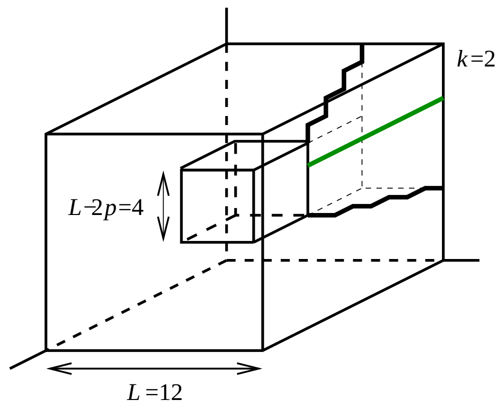


FIGURE 5. Fundamental region for a subclass of punctured cyclically symmetric self complement plane partitions (PCSSCPPs). The position of the bisecting line (green online) determines a particular weighting of a PCSSCPP.

PCSSCPPs are enumerated by giving a special weight to the non-intersecting lattice paths in Figure 3, see [3]. Here we will not discuss this, but will only consider those PCSSCPPs as defined in Figure 6. The paths in this picture correspond to those in Figure 3. In the East region of Figure 6, diagonal steps below the bisecting line (green online) are assigned a weight τ^2.

4.2. **Discrete Hirota equation.** We end this section with the following important observation for which we do not know a good interpretation. The polynomials $T(L,p,k)$ defined in (4.1) satisfy a discrete bilinear relation:

Proposition 2. *The polynomials* (4.1) *satisfy the recurrence*

$$\begin{aligned} T(L,p,k)T(L-2,p-2,k+2) &= T(L-1,p-2,k+2)T(L-1,p,k) \\ &\quad + \tau^2 T(L-2,p-1,k)T(L,p-1,k+2). \end{aligned} \tag{4.5}$$

This proposition will be proved in Section 7.

The recurrence (4.5) is equivalent to the equation with which we began this paper: (4.5) becomes the discrete Hirota equation (1.1) or octahedron recurrence, when we change variables to $n = L-p-k$, $i = 2p+k-L$, $j = p+k$, and $T(L,p,k) = f(n,i,j)$:

$$\begin{aligned} f(n,i,j)f(n-2,i,j) &= f(n-1,i-1,j)f(n-1,i+1,j) \\ &\quad + \tau^2 f(n-1,i,j-1)f(n-1,i,j+1). \end{aligned}$$

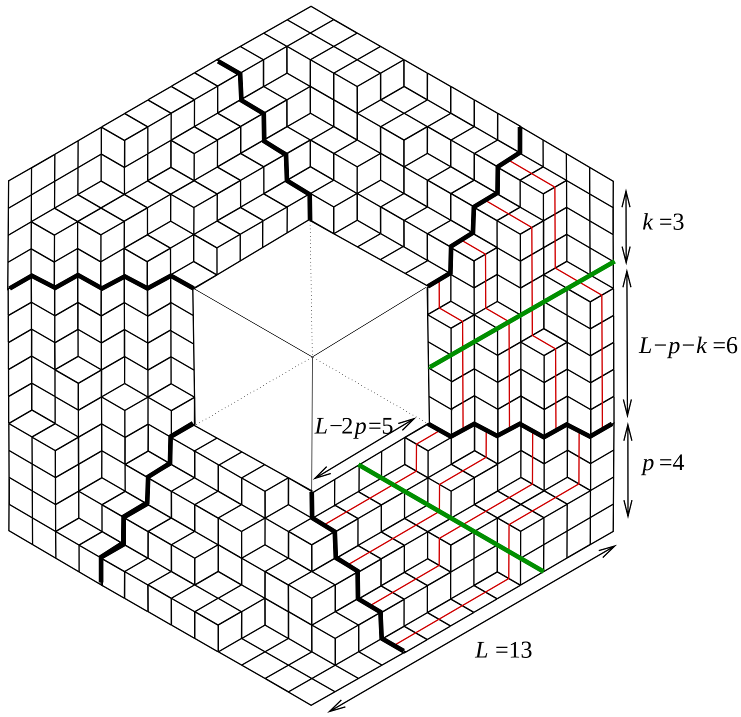


FIGURE 6. A PCSSCPP of size $L = 13$ with a puncture of size $L-2p = 5$. PCSSCPPs are enumerated by non-intersecting lattice paths, as indicated by the red lines. The paths in this picture define a natural subset of all PCSSCPPs, and correspond to those in Figure 3. In the East region, diagonal steps below the bisecting line (green online) are assigned a weight τ^2. All other steps are assigned weight 1. The value of k, $k = 3$ in this figure, determines the position of the bisecting green line.

5. Fully packed loop diagrams and the Razumov–Stroganov conjecture

In this section we will discuss another combinatorial interpretation of the solutions of the qKZ equation. At $\tau = 1$ the qKZ equation is equivalent to an eigenvalue equation for the transfer matrix of the O(1) loop model. In this context a relation was conjectured in [19] (as a variant of a similar conjecture by Razumov and Stroganov [22, 1, 23]) between the solutions in Section 3 for L even and refined enumerations of vertically symmetric alternating-sign matrices (VSASMs). For L odd there is a connection to related objects, see below. The qKZ equation is a generalisation of the O(1) eigenvalue equation to $\tau \neq 1$, as found by Pasquier [18], and Di Francesco and Zinn-Justin [8].

The Razumov–Stroganov conjecture can be roughly described as follows. First a simple bijection of VSASMs to fully packed loop diagrams is made, see e.g. [20]. To each such fully packed loop diagram is associated a Dyck path; this will be described below. For a given Dyck path α there correspond many FPL diagrams, and their number is precisely ψ_α at $\tau = 1$. For general τ there is as yet no interpretation of ψ_α as a weighted enumeration of FPL diagrams or VSASMs. However, as we will show in Proposition 3 at the end of this section, there is such an interpretation for the total τ-normalisations $S_-(2n, n-1)$ of the qKZ solution. We will see that $S_+(2n, n-1)$ and $S_-(2n-1, n-1)$ are also related to ASM generating functions.

As shown in the previous section, the normalisation $S_-(2n, n-1)$ is also equal to a generating function of τ^2-weighted PCSTCPPs with a small puncture of size 2, establishing a connection between such plane partitions and weighted VSASMs. A similar connection was considered in [7]. Other equalities between generating functions of weighted cyclically symmetric plane partitions and generating functions related to weighted enumerations of symmetric ASMs were conjectured by Robbins and Kuperberg [24, 13]. We will prove some of these in Section 8.

Finding the τ-statistic on FPL diagrams that gives rise to an interpretation of ψ_α for $\tau \neq 1$ should provide clues for explicit bijections between (P)CSTCPPs and symmetry classes of ASMs, as well as for an explicit bijection between TSSCPPs and unrestricted ASMs. In fact, in this section we will make another small step towards such a bijection by elucidating the role played by the parameter p which determines the size of the puncture in PCSTCPPs.

5.1. **Fully packed loop diagrams.** For L even, consider the set of vertices of a piece of square lattice of size $L/2 \times L$. In the case of L odd, take a piece of size $(L-1)/2 \times (L+1)$. On this set of points, draw bonds such that each internal vertex has exactly two drawn bonds. On the boundary we impose the condition that every other ingoing bond on the left hand side, bottom and right hand side is drawn, starting at the top left vertex. [2] In this way, the drawn bonds form closed polygons or connect outgoing bonds to each other, see for example Figures 7 and 8. From Kuperberg's work [13] it follows that FPL diagrams of even size are equinumerous with VSASMs. Those of odd size are conjectured[3] to be equinumerous with CSTCPPs [19].

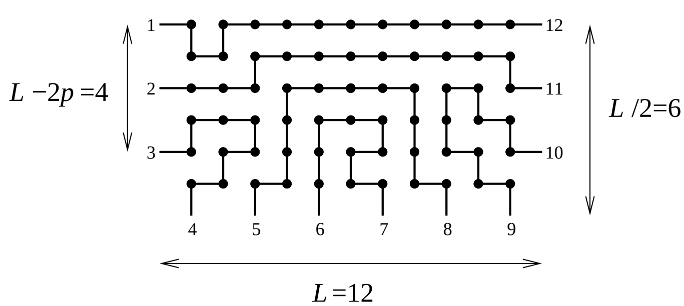


FIGURE 7. An FPL diagram of size $L = 12$ with $L/2 - p = 2$ loop lines connecting loop terminals $1, 2$ with $L, L-1$ respectively.

We will label the outgoing bonds by successive integers, as in Figures 7 and 8. As each outgoing bond is connected to another outgoing bond, FPL diagrams can be naturally

[2] These boundary conditions and the shape of the lattice are of particular relevance in this paper, but more general FPL diagrams can be considered, see e.g. [4].

[3] To our knowledge no proof of this assertion exists.

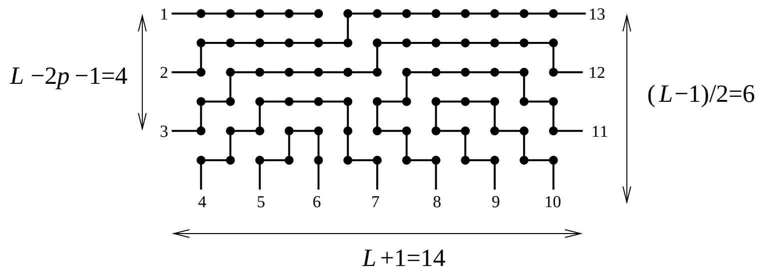


FIGURE 8. An FPL diagram of size $L = 13$ with $\lfloor L/2 \rfloor - p = 2$ loop lines connecting external loop terminals $2, 3$ with $L, L-1$ respectively.

labeled by link patterns, or equivalently, two-row Young tableaux or Dyck paths. For example, the diagram in Figure 8 has link pattern ((((())(()))) which is short hand for saying that 1 is connected to the top of the diagram, 2 is connected to 13, 3 to 12 etc. In general, to each link pattern correspond many FPL diagrams. This information can also be coded in two-row standard Young tableaux. The entries of the first row of the Young tableau correspond to the positions of opening parentheses '(' in a link pattern, and the entries of the second row to the positions of the closing parentheses ')'. The FPL diagram of Figure 8 carries as a label the standard Young tableau given in Figure 9.

1	2	3	4	5	8	9
6	7	10	11	12	13	

FIGURE 9. Standard Young tableau corresponding to the FPL diagram in Figure 8.

Yet another way of coding the same information is by using Dyck paths. This will be useful when making a connection to the results of Section 3. Each entry in the first row of the standard Young tableau represents an up step, while those in the second row represent down steps. The Dyck path corresponding to Figure 9 is given in Figure 10.

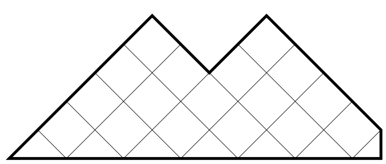


FIGURE 10. Dyck path corresponding to the FPL diagram in Figure 8 and the standard Young tableau in Figure 8.

5.2. **Subsets of FPL diagrams and punctured plane partitions.** As discussed above, FPL diagrams of even size are equinumerous with VSASMs, and it follows from the results of Section 4 that they are also equinumerous to PCSTCPPs with a small puncture of size 2. Here we shall formulate a refined correspondence between subsets of FPL diagrams and PCSTCPPs.

Recall the subset $\mathcal{D}_{L,p}$ of Dyck paths whose local minima lie on or above height $\tilde{p}$ where $\tilde{p} = \lfloor (L-1)/2 \rfloor - p$, see Section 3. Each path in $\mathcal{D}_{L,p}$ is a label for FPL diagrams whose loop terminals $1, \dots \tilde{p}$ (in the case of L even) are connected to terminals $L - \tilde{p} + 1, \dots, L$ respectively, see Figure 7. In the case of L odd, the Dyck paths in $\mathcal{D}_{L,p}$ label FPL diagrams with loop connections between terminals $2, \dots, \tilde{p}+1$ and $L - \tilde{p} + 1, \dots, L$ respectively, see Figure 8.

Definition 3. *A **p-restricted FPL diagram** is an FPL diagram whose corresponding Dyck path belongs to $\mathcal{D}_{L,p}$.*

In other words, in a p-restricted FPL diagram of even size, the first $L/2 - p$ loop terminals are connected to the last $L/2 - p$ loop terminals. This classification of FPL diagrams allows us to formulate a refined correspondence between FPL diagrams and symmetric plane partitions:

Conjecture 2. *The total number of p-restricted FPL diagrams of size $L = 2n$ is equal to the total number $S_{\pm}(2n,p)|_{\tau=1}$ of PCSTCPPs having sides of length $2(L-p)$ and $2(p+1)$ with a triangular puncture of size $2(L-2p-1)$, see Figure 4.*

As a side remark we note that $S_{\pm}(L,p)|_{\tau=1}$ at $\tau = 1$ factorises (this was proved in [15, 3] in the context of PCSTCPPs) and takes the form

$$S_{\pm}(L,p)|_{\tau=1} = 2^{(p+1)(L-p)} \prod_{j=1}^{p} \frac{\Gamma(L-j+1)\Gamma((2L+2j+3)/6)\Gamma((L-2j+3)/3)}{\Gamma(L-2j+1)\Gamma(j+1/2)\Gamma((2L-j+3)/3)}. \quad (5.1)$$

This expression is equivalent to earlier conjectured expressions for the total number of p-restricted FPL diagrams in the context of the O(1) loop model and the Razumov-Stroganov conjecture [17, 21].

One may ask if a generalisation of Conjecture 2 holds for $\tau \neq 1$. As we have seen in Section 4.1, the parameter τ^2 is a natural weight for (punctured) symmetric plane partitions. One can therefore ask whether it also describes a natural statistic on ASMs or FPL diagrams. In the case of $p = \lfloor (L-1)/2 \rfloor$ ($\tilde{p} = 0$) this question can be answered in the affirmative. It was already observed by Robbins and Kuperberg [24, 13] that some τ^2-generating functions for ordinary CSTCPPs, i.e. without puncture, are the same as certain polynomials arising in τ^2-enumerations of symmetric ASMs where each -1 is assigned a weight τ^2. In the case of FPL diagram this amounts to giving a weight τ to every two consecutive vertical or horizontal steps (such consecutive steps correspond to either a $+1$ or -1 in ASM language). Our expressions for the partial sums in Table 2 can be directly compared with Kuperberg's Table 4. Using the notations of [13], see also Section 8, we collect this observation with two others in the following intriguing proposition:

Proposition 3.

$$\begin{aligned} T(2n,n-1,2) &= S_-(2n,n-1) = A_{\mathrm{V}}(2n+1;\tau^2), \\ T(2n,n-1,1) &= S_+(2n,n-1) = A^{(2)}_{\mathrm{VHP}}(4n+2;\tau^2), \\ T(2n-1,n-1,1) &= \tau^{n-1} S_-(2n-1,n-1) = \tilde{A}^{(2)}_{\mathrm{UU}}(4n;\tau^2). \end{aligned}$$

This proposition will be proved in Section 8.

It is an open problem to find a generalisation of Conjecture 2 to arbitrary weight τ, or a generalisation of Proposition 3 to arbitrary p, i.e. to find a correspondence between τ^2-weighted p-restricted FPL diagrams and τ^2-enumerations of PCSTCPPs.

6. Proof of the determinant formula for partial sums

In this section we prove Proposition 1 using the formalism developed in [9, 32] and which consists of writing integral formulae for solutions of the qKZ equation. For the sake of simplicity, we shall work out separately the two possible parities of the size L.

6.1. **Even size.** Assume $L = 2n$. The following set of integrals was introduced in [9]:

$$\psi_{a_1,\dots,a_n} = \prod_{1\le i<j\le L}[1+x_i-x_j][1-x_i-x_j]\oint\cdots\oint\prod_{\ell=1}^{n}\frac{\log q}{q-q^{-1}}\frac{\mathrm{d}y_\ell}{\pi\mathrm{i}}$$
$$\frac{\prod_{1\le\ell<m\le n}[y_\ell-y_m][1+y_\ell-y_m][y_\ell+y_m]\prod_{1\le\ell\le m\le n}[1+y_\ell+y_m]}{\prod_{\ell=1}^{n}\prod_{i=1}^{L}[y_\ell+x_i]\prod_{i=1}^{a_\ell}[y_\ell-x_i]\prod_{i=a_\ell+1}^{L}[1+y_\ell-x_i]} \tag{6.1}$$

where the contour integrals surround the poles at $x_i - 1$. Here $a_1, \dots, a_n$ form a non-decreasing sequence of integers between 1 and $L-1$.

The relation to the solution of the qKZ system (2.3) is as follows: up to normalization by a symmetric factor of the parameters x_i, the $\psi_{a_1,\dots,a_n}$ are linear combinations of the components of the solution:

$$\psi_{a_1,\dots,a_n} = \sum_\alpha C_{a_1,\dots,a_n;\alpha}\psi_\alpha \tag{6.2}$$

with coefficients $C_{a_1,\dots,a_n;\alpha}$ that are described explicitly in appendix A of [32], and which we shall define here by recurrence. First, $C_{\emptyset;\emptyset} = 1$. Next, for a pair $(a_1, \dots, a_n; \alpha)$ of length L, consider any local maximum i of the path α, and the new path α' obtained by removing the two steps before and after i. Call k the number of ℓ such that $a_\ell = i$. If $k = 0$, $C_{a_1,\dots,a_n;\alpha} = 0$. If $k > 0$, consider the new sequence $a'_1, \dots, a'_{n-1}$ obtained from $a_1, \dots, a_n$ by removing one "i" and replacing each remaining a_ℓ with: itself if $a_\ell < i$; $i-1$ if $a_\ell = i$; $a_\ell - 2$ if $a_\ell > i$. Then $C_{a_1,\dots,a_n;\alpha} = [k]C_{a'_1,\dots,a'_{n-1};\alpha'}$ (and this definition is independent of the choice of local maximum).

Since we are interested in the values of the ψ_α at $x_i = 0$, let us set $x_i = 0$ in (6.1) and perform the change of variables $u_\ell = [1+y_{n+1-\ell}]/[y_{n+1-\ell}]$. At this stage it is convenient to reindex the integers as $b_\ell = L - a_{n+1-\ell}$, $\ell = 1, \dots, n$, and to define $\bar\psi_{b_1,\dots,b_n} := \psi_{a_1,\dots,a_n}$, so that

$$\psi_{a_1,\dots,a_n} = \bar\psi_{b_1,\dots,b_n} = \oint\cdots\oint\prod_{\ell=1}^{n}\frac{\mathrm{d}u_\ell}{2\pi\mathrm{i}\,u_\ell^{b_\ell}}\Bigg[\prod_{1\le\ell\le m\le n}(1-u_\ell u_m)$$
$$\times\prod_{1\le\ell<m\le n}(u_m-u_\ell)(1+\tau u_m+u_\ell u_m)(\tau+u_\ell+u_m)\Bigg] \tag{6.3}$$

Here the normalization is chosen in such a way that $\psi_{1,2,\dots,n} = \psi_\Omega = \tau^{n(n-1)/2}$ (note that in this case the integrals are trivial and can be performed by simply setting $u_\ell = 0$ in the numerator of the integrand).

We now consider specific $\bar\psi_{b_1,\dots,b_n}$ which will reproduce our sums $S_\pm(L,p)$. Fix a non-negative integer p, and let $\tilde p = n-1-p$. Consider sequences $(b_1,\dots,b_n)$ of the form

$$b_\ell = \begin{cases} \ell & 1 \le \ell \le \tilde p + 1 \\ 2\ell - \tilde p - 1 - \epsilon_{\ell-\tilde p-1} & \tilde p + 2 \le \ell \le n \end{cases}$$

where $\epsilon_1,\dots,\epsilon_p \in \{0,1\}$. We have the following

Lemma 1.

$$\bar\psi_{b_1,\dots,b_n} = \bar\psi_{1,\dots,\tilde p+1,\tilde p+3-\epsilon_1,\dots,2n-\tilde p-1-\epsilon_p} = \sum_{\substack{\alpha\in\mathcal{D}_{L,p} \\ \forall \ell, \alpha_{L-\tilde p-2\ell-1}<\alpha_{L-\tilde p-2\ell} \text{ iff } \epsilon_\ell=1}} \psi_\alpha$$

Proof. We shall proceed by induction. Fix a sequence of integers b_ℓ as in the lemma, that is in terms of the mirror-symmetric sequence a_ℓ,

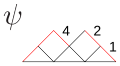
$$a_\ell = \begin{cases} 2\ell + \tilde p - 1 + \epsilon_{p+1-\ell} & 1 \le \ell \le p \\ \ell + n - 1 & p+1 \le \ell \le n \end{cases}$$

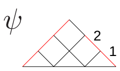
Note that $a_\ell \ge \tilde p + 1$. Let α be a Dyck path. Consider a local maximum i of α. One can always assume $i \le n$. There are two cases:

1. $i \le \tilde p$. In this case, $\alpha \notin \mathcal{D}_{L,p}$. We find immediately that there are zero $a_\ell = i$, so that the coefficient is zero.

2. $\tilde p < i \le n < L - \tilde p$. Call $\ell = \lfloor (i - \tilde p + 1)/2 \rfloor$. There are four cases depending on the parity of i and the value of $\epsilon_{p+1-\ell}$. If $\epsilon_{p+1-\ell} = 0$ and $i = 2\ell + \tilde p \neq a_\ell$, there are no a_ℓ equal to i so that the coefficient is zero. Since i is a local maximum, we have $\alpha_{L-\tilde p-2(p+1-\ell)-1=i-1} < \alpha_{L-\tilde p-2(p+1-\ell)=i}$ satisfying the inequality in the summation of lemma 1 (despite $\epsilon_{p+1-\ell} = 0$). Similarly, if $\epsilon_{p+1-\ell} = 1$ and $i = 2\ell + \tilde p - 1 \neq a_\ell$, there are no a_ℓ equal to i so that the coefficient is zero, and $\alpha_{L-\tilde p-2(p+1-\ell)-1=i} > \alpha_{L-\tilde p-2(p+1-\ell)=i+1}$ violating the inequality in the summation of lemma 1 (despite $\epsilon_{p+1-\ell} = 1$). In the other two cases, we have $a_\ell = i$ and the equivalence in the summation of lemma 1 is valid. We can then apply the definition by recurrence of the coefficient $C_{a_1,\dots,a_n;\alpha}$; the new sequence $a'_1,\dots,a'_{n-1}$ is exactly the same type as $a_1,\dots,a_n$, that is defined by the same ϵ_i with $\epsilon_{p+1-\ell}$ skipped. On the other hand it is clear that the other conditions on α in the summation of lemma 1 are equivalent to the conditions on α' (α with the local maximum removed) with the new sequence $a'_1,\dots,a'_{n-1}$. One then uses the induction hypothesis to conclude. $\square$

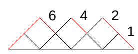
Example 1. *Consider the two sequences with $L = 6$, $p = 1$ ($\tilde p = 1$). We find*

$\bar\psi_{1,2,4} = \psi$  $= 2\tau^2(1+\tau^2)$

$\bar\psi_{1,2,3} = \psi$  $= \tau^3$

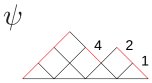
In size $L=8$, with $p=2$ ($\tilde p=1$),

$\bar\psi_{1,2,4,6}=\psi$

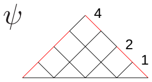


$=6\tau^3+21\tau^5+18\tau^7+5\tau^9$

$\bar\psi_{1,2,4,5}=\psi$

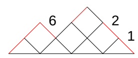


$+\psi$



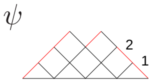
$=5\tau^4+7\tau^6+3\tau^8$

$\bar\psi_{1,2,3,6}=\psi$



$=3\tau^4+8\tau^6+3\tau^8$

$\bar\psi_{1,2,3,5}=\psi$



$=3\tau^5+3\tau^7$

The edges in red are those whose labels (counted from right to left) appear in the sequence of integers iff the edge is a down step (in fact the first and last $\tilde p+1$ edges are fixed by the fact that the paths are in $\mathcal{D}_{L,p}$).

Note that taken together, the various sequences for a given L and p reproduce the full set of paths of $\mathcal{D}_{L,p}$. Furthermore, by direct computation using formula (3.1) (grouping together pairs $\alpha_{L-\tilde p-2i-1}$ and $\alpha_{L-\tilde p-2i}$ in the sum and using $\alpha_{L-\tilde p-2i}-\alpha_{L-\tilde p-2i-1}=2\epsilon_i-1$), it is easy to check that all ψ_α that contribute to a given $\bar\psi_{1,\dots,\tilde p+1,\tilde p+3-\epsilon_1,\dots,2n-\tilde p-1-\epsilon_p}$ have the same integer $c_{\alpha,p}=\sum_{i=1}^p\epsilon_i$.

We thus define

$$S(L,p|t)=\sum_{\epsilon_1,\dots,\epsilon_p\in\{0,1\}} t^{\sum_{i=1}^p\epsilon_i}\bar\psi_{1,\dots,\tilde p+1,\tilde p+3-\epsilon_1,\dots,2n-\tilde p-1-\epsilon_p} \tag{6.4}$$

and claim that $S_\pm(L,p)=S(L,p|\tau^{\pm1})$.

Using (6.3), we now obtain the following integral representation for $S(L,p|t)$:

$$\begin{aligned}S(L,p|t)=\oint\cdots\oint\left(\prod_{\ell=1}^{\tilde p+1}\frac{\mathrm{d}u_\ell}{2\pi\mathrm{i}\,u_\ell^\ell}\right)\left(\prod_{\ell=\tilde p+2}^{n}\frac{\mathrm{d}u_\ell(1+tu_\ell)}{2\pi\mathrm{i}\,u_\ell^{2\ell-\tilde p-1}}\right)\Bigg[\prod_{1\le\ell\le m\le n}(1-u_\ell u_m)\\ \times\prod_{1\le\ell<m\le n}(u_m-u_\ell)(1+\tau u_m+u_\ell u_m)(\tau+u_\ell+u_m)\Bigg]\end{aligned} \tag{6.5}$$

The first integrals over $u_1,\dots,u_{\tilde p+1}$ can be performed successively by simply setting the corresponding variables to zero. The result, after shifting the indices of the variables, is:

$$\begin{aligned}S(L,p|t)=\oint\cdots\oint\prod_{\ell=1}^{p}\frac{\mathrm{d}u_\ell(1+tu_\ell)(1+\tau u_\ell)^{\tilde p+1}(\tau+u_\ell)^{\tilde p+1}}{2\pi\mathrm{i}\,u_\ell^{2\ell}}\Bigg[\prod_{1\le\ell\le m\le p}(1-u_\ell u_m)\\ \times\prod_{1\le\ell<m\le p}(u_m-u_\ell)(1+\tau u_m+u_\ell u_m)(\tau+u_\ell+u_m)\Bigg]\end{aligned} \tag{6.6}$$

Next we use the following lemma:

Lemma 2. *If AS designates antisymmetrisation:* $AS(f(u_1,\dots,u_p)) = \sum_{\sigma\in\mathcal{S}_p}(-1)^\sigma f(u_{\sigma(1)},\dots,u_{\sigma(p)})$, *and* $(\cdots)_{\le 0}$ *means keeping only non-positive powers of a Laurent polynomial in the variables* u_ℓ, *then the following equality holds:*

$$\left\{\prod_{1\le\ell\le m\le p}(1-u_\ell u_m)\ \mathrm{AS}\left(\prod_{\ell=1}^p u_\ell^{-2\ell+1}\prod_{1\le\ell<m\le p}(1+u_\ell u_m+\tau u_m)\right)\right\}_{\le 0}$$
$$= \mathrm{AS}\left(\prod_{\ell=1}^p\left(u_\ell^{-\ell}(\tau+u_\ell^{-1})^{\ell-1}\right)\right) = \prod_{\ell=1}^p u_\ell^{-1}\prod_{1\le\ell<m\le p}(u_m^{-1}-u_\ell^{-1})(\tau+u_\ell^{-1}+u_m^{-1}) \quad (6.7)$$

This is equivalent to formula (4.5) of [32]. It is a slightly stronger version of the proposition of [30], and can be proved along the same lines. We use it to symmetrize the integrand:

$$S(L,p|t) = \frac{1}{p!}\oint\cdots\oint\prod_{\ell=1}^p\frac{\mathrm{d}u_\ell(1+tu_\ell)(1+\tau u_\ell)^{\tilde p+1}(\tau+u_\ell)^{\tilde p+1}}{2\pi\mathrm{i}\,u_\ell}$$
$$\prod_{1\le\ell<m\le p}(u_m-u_\ell)(\tau+u_\ell+u_m)(u_m^{-1}-u_\ell^{-1})(\tau+u_\ell^{-1}+u_m^{-1}) \quad (6.8)$$

Noting that $\prod_{1\le\ell<m\le p}(u_m-u_\ell)(\tau+u_\ell+u_m)$ is just the Vandermonde determinant of the $u_\ell(\tau+u_\ell)$ and similarly for the other factors, we can finally pull the determinants out of the integral, resulting in:

$$S(L,p|t) = \det_{1\le\ell,m\le p}\left[\oint\frac{\mathrm{d}u}{2\pi iu}(1+tu)u^{\ell-m+\tilde p}(\tau+u)^{\ell+\tilde p}(\tau+u^{-1})^{m+\tilde p}\right] \quad (6.9)$$

For general t, by using the binomial formula we can evaluate this to be

$$S(L,p|t) = \det_{1\le\ell,m\le p}\left[\sum_r\tau^{\tilde p+2m+2\ell-2r-1}\binom{\ell+\tilde p}{r-\ell}\left(\tau\binom{m+\tilde p}{2m-r}+t\binom{m+\tilde p}{2m-r-1}\right)\right] \quad (6.10)$$

where we recall that $\tilde p = L/2-1-p$.

At $t=\tau$ this expression simplifies slightly:

$$S_+(L,p) = \det_{1\le\ell,m\le p}\left[\sum_r\tau^{\tilde p+2m+2\ell-2r}\binom{\ell+\tilde p}{r-\ell}\binom{m+\tilde p+1}{2m-r}\right] \quad (6.11)$$

as well as at $t=\tau^{-1}$:

$$S_-(L,p) = \det_{1\le\ell,m\le p}\left[\sum_r\tau^{\tilde p+2m+2\ell-2r}\binom{\ell+\tilde p+1}{r-\ell}\binom{m+\tilde p}{2m-r}\right] \quad (6.12)$$

The summation over r is such that only a finite number of terms is non-zero; in practice a possible range is $0\le r\le 2p$. One can check that formulae (6.11,6.12) match the expressions given in proposition 1 for L even.

6.2. **Odd size.** Assume $L = 2n+1$. The reasoning being exactly identical to the case L even, we only provide the key formulae. The starting point is formally the same integral formula as previously:

$$\psi_{a_1,\dots,a_n} = \prod_{1\le i<j\le L} [1+x_i-x_j][1-x_i-x_j] \oint\cdots\oint \prod_{\ell=1}^{n} \frac{\log q}{q-q^{-1}} \frac{\mathrm{d}y_\ell}{\pi \mathrm{i}} \frac{\prod_{1\le \ell<m\le n}[y_\ell-y_m][1+y_\ell-y_m][y_\ell+y_m]\prod_{1\le\ell\le m\le n}[1+y_\ell+y_m]}{\prod_{\ell=1}^{n}\prod_{i=1}^{L}[y_\ell+x_i]\prod_{i=1}^{a_\ell}[y_\ell-x_i]\prod_{i=a_\ell+1}^{L}[1+y_\ell-x_i]} \tag{6.13}$$

but with an odd number of parameters x_i, so that it produces a slightly different expression when the x_i are set to zero:

$$\psi_{a_1,\dots,a_n} = \bar\psi_{b_1,\dots,b_n} = \oint\cdots\oint \prod_{\ell=1}^{n} \frac{\mathrm{d}u_\ell}{2\pi\mathrm{i}\, u_\ell^{b_\ell}} \Bigg[\prod_{1\le\ell\le m\le n} (1-u_\ell u_m)(1+\tau u_m+u_\ell u_m) \times \prod_{1\le \ell<m\le n} (u_m-u_\ell)(\tau+u_\ell+u_m)\Bigg] \tag{6.14}$$

Define, for p a non-negative integer, $\tilde p = n-p$ and

$$S(L,p|t) = \sum_{\epsilon_1,\dots,\epsilon_p\in\{0,1\}} t^{\sum_{i=1}^{p}\epsilon_i}\bar\psi_{1,\dots,\tilde p,\tilde p+2-\epsilon_1,\dots,2n-\tilde p-\epsilon_p} \tag{6.15}$$

so that $S_\pm(L,p) = S(L,p|\tau^{\pm 1})$. We obtain the following integral expression for $S(L,p|t)$:

$$S(L,p|t) = \oint\cdots\oint \left(\prod_{\ell=1}^{\tilde p}\frac{\mathrm{d}u_\ell}{2\pi\mathrm{i}\, u_\ell^{\ell}}\right)\left(\prod_{\ell=\tilde p+1}^{n}\frac{\mathrm{d}u_\ell(1+tu_\ell)}{2\pi i u_\ell^{2\ell-\tilde p}}\right)\Bigg[\prod_{1\le\ell\le m\le n}(1-u_\ell u_m) \times \prod_{\ell=1}^{n}(1+\tau u_\ell+u_\ell^2)\prod_{1\le\ell<m\le n}(u_m-u_\ell)(1+\tau u_m+u_\ell u_m)(\tau+u_\ell+u_m)\Bigg] \tag{6.16}$$

We integrate over the first $\tilde p$ variables and reindex the remaining ones:

$$S(L,p|t) = \oint\cdots\oint \prod_{\ell=1}^{p}\frac{\mathrm{d}u_\ell(1+tu_\ell)(1+\tau u_\ell)^{\tilde p}(\tau+u_\ell)^{\tilde p}}{2\pi\mathrm{i}\, u_\ell^{2\ell}}\Bigg[\prod_{1\le\ell\le m\le p}(1-u_\ell u_m) \times \prod_{\ell=1}^{n}(1+\tau u_\ell+u_\ell^2)\prod_{1\le\ell<m\le p}(u_m-u_\ell)(1+\tau u_m+u_\ell u_m)(\tau+u_\ell+u_m)\Bigg] \tag{6.17}$$

We use lemma 2 and pull the determinants out of the integral as before:

$$\begin{aligned}
S(L,p|t) &= \frac{1}{p!}\oint\cdots\oint\prod_{\ell=1}^{p}\frac{\mathrm{d}u_\ell(1+tu_\ell)(1+\tau u_\ell)^{\tilde p}(\tau+u_\ell)^{\tilde p}(1+\tau u_\ell+u_\ell^2)}{2\pi\mathrm{i}\,u_\ell^2}\\
&\qquad\prod_{1\le\ell<m\le p}(u_m-u_\ell)(\tau+u_\ell+u_m)(u_m^{-1}-u_\ell^{-1})(\tau+u_\ell^{-1}+u_m^{-1})\\
&= \det_{1\le\ell,m\le p}\left[\oint\frac{\mathrm{d}u}{2\pi\mathrm{i}\,u}(1+tu)(1+\tau u_\ell+u_\ell^2)u^{\ell-m+\tilde p-1}(\tau+u)^{\ell+\tilde p-1}(\tau+u^{-1})^{m+\tilde p-1}\right]\\
&= \det_{1\le\ell,m\le p}\left[\oint\frac{\mathrm{d}u}{2\pi\mathrm{i}\,u}(1+tu)u^{\ell-m+\tilde p}(\tau+u)^{\ell+\tilde p-1}(\tau+u^{-1})^{m+\tilde p}\right] \qquad (6.18)
\end{aligned}$$

In the last line we wrote $(1+\tau u+u^2)/u=(\tau+u^{-1})+u$ and noted that the second term reproduces the column $m-1$ of the matrix and thus can be subtracted without changing the determinant.

One finally obtains

$$S(L,p|t)=\det_{1\le\ell,m\le p}\left[\sum_r\tau^{\tilde p+2m+2\ell-2r-3}\binom{\ell+\tilde p-1}{r-\ell}\left(\tau\binom{m+\tilde p}{2m-r}+t\binom{m+\tilde p}{2m-r-1}\right)\right] \qquad (6.19)$$

where we recall that $\tilde p=(L-1)/2-p$.

At $t=\tau$ this simplifies to

$$S_+(L,p)=\det_{1\le\ell,m\le p}\left[\sum_r\tau^{\tilde p+2m+2\ell-2r-3}\binom{\ell+\tilde p-1}{r-\ell}\binom{m+\tilde p+1}{2m-r}\right] \qquad (6.20)$$

whereas at $t=\tau^{-1}$:

$$S_-(L,p)=\det_{1\le\ell,m\le p}\left[\sum_r\tau^{\tilde p+2m+2\ell-2r-1}\binom{\ell+\tilde p}{r-\ell}\binom{m+\tilde p}{2m-r}\right] \qquad (6.21)$$

thus reproducing the expressions of proposition 1 for L odd.

7. Proof of the bilinear recurrence relations for $T(L,p,k)$

In this section we prove Proposition 2. To simplify the formulae of this section we introduce $k'=L-2p-k$, so that definition (4.1) reads

$$T(L,p,k)\;=\;\det_{1\le\ell,m\le p}T_{\ell m}\,,\qquad T_{\ell m}(k,k')=\sum_r\binom{\ell+k-1}{2m-\ell-r}\binom{m+k'}{r}\tau^{2r}\,. \qquad (7.1)$$

Here the limits of summation in r are automatically fixed by the conditions $\binom{n}{r}=0$ $\forall\, r<0$ and $\forall\, r>n$. We begin with a derivation of yet another determinant formula for $T(L,p,k)$.

Lemma 3.

$$T(L,p,k)\;=\;\det_{1\le\ell,m\le p+1}U_{\ell m}\,, \qquad (7.2)$$

where

$$U_{\ell,1}(p)\;=\;(-1)^{\ell-1}\,\tau^{2(p+1-\ell)}\,,\qquad U_{\ell,m+1}(k,k')\;=\;T_{\ell m}(k,k'-1)\,. \qquad (7.3)$$

Proof. To check the identity $\det U = \det T$ we shall perform linear transformations of the matrix U not affecting its determinant. First, we combine adjacent rows of U with the aim to set to zero all components of the first column, except the last element $U_{p+1,1} = (-1)^p$:

$$V_{\ell m} = U_{\ell,m+1} + \tau^2 U_{\ell+1,m+1}.$$

Then, by decomposing the determinant of the resulting matrix along the first column we find

$$\det_{1\le \ell,m\le p+1} U_{\ell m} \;=\; \det_{1\le \ell,m\le p} V_{\ell m}\,.$$

The lemma now follows by noticing that the rows of T are linear combinations of those of V,

$$\begin{aligned}
V_{\ell m} &= \sum_r \left(\binom{\ell+k-1}{2m-\ell-r}\binom{m+k'-1}{r} + \binom{\ell+k}{2m-\ell-r}\binom{m+k'-1}{r-1}\right)\tau^{2r} \\
&= \sum_r \left(\binom{\ell+k-1}{2m-\ell-r}\binom{m+k'}{r} + \binom{\ell+k-1}{2m-\ell-r-1}\binom{m+k'-1}{r-1}\right)\tau^{2r} \\
&= T_{\ell m}(k,k') + \tau^2 T_{\ell,m-1}(k,k'),
\end{aligned}$$

where we have used Pascal's rule $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$ to go from the second to the third line. Hence we find

$$\sum_{j=0}^{m-1} (-\tau^2)^j\, V_{\ell,m-j} \;=\; T_{\ell m},$$

from which we conclude that $\det V = \det T$. □

Our derivation of formula (4.5) is based on the use of the following particular example of a Plücker relation for determinants (for the general case see [27]). Consider a pair of $n\times n$ matrices $A_{\ell m}$ and $B_{\ell m}$. Denote by A_ℓ the ℓ-th row of the matrix A and introduce notation

$$\det A = |A|\,, \qquad A \;=\; [A_1,\ldots,A_n]\,,$$

In this notation the Plücker relation reads

$$|A|\,|B| = \sum_{j=1}^{n} |A_1,\ldots,A_{n-1},B_j| \times |B_1,\ldots,B_{j-1},A_n,B_{j+1},\ldots,B_n|\,, \tag{7.4}$$

where the sum is taken over permutations of the last row of A with each row of B.

We now take $n = p+1$ and, recalling the definition of the matrix U in (7.3), substitute for A and B the following matrices:

$$A(k,k') = U(k,k') = [U_1,\ldots,U_{p+1}]\,, \tag{7.5}$$
$$B(k,k') = [\delta_1, U_2,\ldots,U_p,\delta_{p+1}]\,, \tag{7.6}$$

where $\delta_{\ell m}$ is Kronecker's delta.

By Lemma 3 we have $|A| = T(L,p,k)$. To calculate the determinant of B we expand along its top and bottom rows and then notice that the first column of the resulting

$(p-1)\times(p-1)$ matrix is also of δ-type:

$$U_{\ell,2}(k,k') = T_{\ell,1}(k,k'-1) = \sum_r \binom{\ell+k-1}{2-\ell-r}\binom{k'}{r}\tau^{2r} = \delta_{\ell,2} \quad \text{for} \quad \ell \geq 2.$$

So we calculate

$$\begin{aligned} |B| \;=\; \det_{1\leq \ell,m\leq p-2} U_{\ell+2,m+2}(k,k') \;&=\; \det_{1\leq \ell,m\leq p-2} T_{\ell+2,m+1}(k,k'-1) \\ &=\; \det_{1\leq \ell,m\leq p-2} T_{\ell m}(k+2,k') \;=\; T(L-2,p-2,k+2). \end{aligned}$$

Now let us consider the right hand side of the relation (7.4). Here only permutations with the top and the bottom rows of the matrix B give non-vanishing contributions. Permuting the last row in A and the first row in B gives the following factors:

$$\begin{aligned} |U_1,\dots U_p,\delta_1| \;&=\; (-1)^p \det_{1\leq \ell,m\leq p} U_{\ell,m+1}(k,k') \\ &=\; (-1)^p \det_{1\leq \ell,m\leq p} T_{\ell m}(k,k'-1) \;=\; (-1)^p\, T(L-1,p,k)\,, \end{aligned}$$

$$\begin{aligned} |U_{p+1},U_2,\dots U_p,\delta_{p+1}| \;&=\; (-1)^{p-1}\,|U_2,\dots,U_p,U_{p+1}| \\ &=\; (-1)^p \det_{1\leq \ell,m\leq p-1} U_{\ell m}(k+2,k'+1) \;=\; (-1)^p\, T(L-1,p-2,k+2)\,, \end{aligned}$$

where in the last calculation when passing to the second line one i) expands the determinant along the first column noticing that $U_{\ell+1,2}=\delta_{\ell,1}$ and ii) redefines matrix indices using the identity $U_{\ell+2,m+1}(k,k') = U_{\ell m}(k+2,k'+1)\ \ \forall\, m>1$.

Permuting the last row in A and the last row in B gives the following factors:

$$|U_1,\dots,U_p,\delta_{p+1}| \;=\; |U_1,\dots,U_p| = \tau^2 \det_{1\leq \ell,m\leq p} U_{\ell m}(k,k') \;=\; \tau^2\, T(L-2,p-1,k)\,.$$

The factor τ^2 is extracted from the p-dependent first column of the matrix U: $U_{\ell,1}(p+1) = \tau^2 U_{\ell,1}(p)$;

$$\begin{aligned} |\delta_1,U_2,\dots U_{p+1}| \;&=\; \det_{1\leq \ell,m\leq p} U_{\ell+1,m+1}(k,k') \\ &=\; \det_{1\leq \ell,m\leq p-1} U_{\ell+2,m+2}(k,k') \;=\; \det_{1\leq \ell,m\leq p-1} T_{\ell m}(k+2,k') \;=\; T(L,p-1,k+2)\,. \end{aligned}$$

Here when passing to the second line of the calculation we first expand $\det U_{\ell+1,m+1}$ along the first column $U_{\ell+1,1}=\delta_{\ell,1}$ and then redefines indices of the matrix T: $T_{\ell+2,m+1}(k,k'-1) = T_{\ell m}(k+2,k')$.

Thus, the Plücker relation (7.4) for the matrices A and B defined in (7.5) and (7.6) produces the equality (4.5).

8. Proof of τ^2-enumeration of ASMs

In this section, we prove Proposition 3. We start from the determinant formulae of [13] for the enumeration of various symmetry classes of Alternating Sign Matrices, and reduce them to our own determinant formulae for $S_\pm$. In what follows, we keep Kuperberg's notations (even though they are non-standard), to ease the comparison of formulae with [13]. In each case, one starts with configurations of the six-vertex model, which for certain particular boundary conditions are identified with Alternating Sign Matrices in various symmetry classes. There are three distinct Boltzmann weights for the six-vertex model, taking into account $\mathbb{Z}_2$ symmetry, and they are parametrised as $w_{\mathrm{a}} = ax^{-1}y - a^{-1}xy^{-1}$, $w_{\mathrm{b}} = axy^{-1} - a^{-1}x^{-1}y$, $w_{\mathrm{c}} = a^2 - a^{-2}$, x and y being row/column spectral parameter and a a global parameter. In the ASM language, a weight w_{c} is assigned to a ± 1, and weights w_{b} and w_{a} are assigned to zeroes. In the end we must take the homogeneous limit where $w_{\mathrm{a}} = w_{\mathrm{b}}$ and $w_{\mathrm{c}}/w_{\mathrm{a}} = \tau = -q - q^{-1}$: this ensures that adding a -1 to an ASM, that is *two* extra vertices of type c (adding a -1 also increases the number of $+1$ by 1), produces a weight τ^2. This is achieved by setting all spectral parameters to 1 and $a = -q$. Similar parameters, called b and c, which are related to boundary weights will be used below.

To be self-contained, we will give the matrices of Kuperberg relevant to this paper. It is useful to first define the functions σ and α by

$$\sigma(x) = x - x^{-1}, \qquad \alpha(x) = \sigma(ax)\sigma(a/x).$$

The relevant matrices then are

$$M_{\mathrm{U}}(n;\vec{x},\vec{y})_{ij} = \frac{1}{\alpha(x_i/y_j)} - \frac{1}{\alpha(x_i y_j)}, \tag{8.1}$$

$$\begin{aligned} M_{\mathrm{UU}}(n;\vec{x},\vec{y})_{ij} = {} & \frac{\sigma(b/y_j)\sigma(cx_i)}{\alpha(ax_i/y_j)} - \frac{\sigma(b/y_j)\sigma(c/x_i)}{\alpha(a/x_i y_j)} - \frac{\sigma(by_j)\sigma(cx_i)}{\alpha(ax_i y_j)} \\ & + \frac{\sigma(by_j)\sigma(c/x_i)}{\alpha(ay_j/x_i)}, \end{aligned} \tag{8.2}$$

$$M^{+}_{\mathrm{HT}}(n;\vec{x},\vec{y})_{ij} = \frac{1}{\sigma(ay_j/x_i)} + \frac{1}{\sigma(ax_i/y_j)}. \tag{8.3}$$

Since the equalities of Proposition 3 are known to be true at $\tau = 1$ (they are consequences of the various relations between enumerations of ASMs found in [13], as well the relations between ASMs and PPs discussed in [9]), and since both sides are easily checked to be polynomials in τ of the same degree, we can safely drop various trivial factors in the calculation, keeping only the determinant itself as well as factors that become singular in the homogeneous limit.

8.1. **First formula.** In [13], it is explained how VSASMs are a special case of UASMs; more precisely, the partition function of VSASMs can be obtained from the more general one of UASMs by tuning a certain boundary parameter. It is also noted there that the boundary parameter only enters the formula for the partition function in prefactors, and not in the determinant itself. In particular the enumeration of VSASMs and UASMs are essentially the same. We shall therefore write directly the partition function for UASMs of size $2n$ without all the regular prefactors; using (8.1) it takes the form:

$$Z_{\mathrm{UASM}} \propto \frac{1}{\Delta(x^2)\Delta(y^2)\Delta^*(x^2)\Delta^*(y^2)} \times \det_{1\le i,j\le n} \left(\frac{1}{(a^2x_i^2 - y_j^2)(a^2y_j^2 - x_i^2)(a^2 - x_i^2y_j^2)(1 - a^2x_i^2y_j^2)} \right)$$

where Δ stands for the Vandermonde determinant, e.g. $\Delta(x^2) = \prod_{i<j}(x_j^2 - x_i^2)$; and $\Delta^*(x^2) = \prod_{i<j}(1 - x_i^2 x_j^2)$. The x_i, y_j are spectral parameters which will eventually be set to one.

We use the following integral representation:

$$\frac{1}{(a^2x^2-y^2)(a^2y^2-x^2)(a^2-x^2y^2)(1-a^2x^2y^2)} = \\ \frac{a^3(a^2-1)^2}{(a^2+1)\,x^2y^4(x^2-a^2)(a^2x^2-1)} \oint \frac{\mathrm{d}u}{2\pi\mathrm{i}} \frac{u(u+a+a^{-1})}{\mu(u,x)\mu(u,1/x)\mu(u,a/y)\mu(u,ay)}, \tag{8.4}$$

where

$$\mu(u,x) = a(a+u) - (1+au)x^2. \tag{8.5}$$

The contour of integration in (8.4) surrounds the x-dependent poles but not the y-dependent ones. This identity can be checked directly by residues. The various prefactors, as well as the integral sign, can be pulled out of the determinant and we thus find

$$Z_{\mathrm{UASM}} \propto \frac{1}{\Delta(x^2)\Delta(y^2)\Delta^*(x^2)\Delta^*(y^2)} \oint \frac{\mathrm{d}u_1(u_1+a+a^{-1})u_1}{2\pi\mathrm{i}} \cdots \oint \frac{\mathrm{d}u_n(u_n+a+a^{-1})u_n}{2\pi\mathrm{i}} \\ \det\left(\frac{1}{\mu(u_i,x_j)\mu(u_i,1/x_j)}\right) \det\left(\frac{1}{\mu(u_i,a/y_j)\mu(u_i,ay_j)}\right)$$

In order to compute these determinants, we perform the following change of variables:

$$X = -\frac{(1-x^2)^2}{(1-a^2x^2)(1-a^{-2}x^2)} \qquad Y = -\frac{(1-y^2)^2}{(1-a^2y^2)(1-a^{-2}y^2)}$$

and use the factorizations

$$\mu(u,x)\mu(u,1/x) = (a^2-x^2)(a^2-x^{-2})(1-Xu(a+a^{-1}+u)), \\ \mu(u,ay)\mu(u,a/y) = a^2u^2(a^2-y^2)(a^2-y^{-2})(1-Yu^{-1}(a+a^{-1}+u^{-1})).$$

Note that $a + a^{-1} = \tau$. Again one can get rid of the trivial factors and obtain

$$Z_{\mathrm{UASM}} \propto \frac{1}{\Delta(x^2)\Delta(y^2)\Delta^*(x^2)\Delta^*(y^2)} \oint \frac{\mathrm{d}u_1(u_1+\tau)}{2\pi\mathrm{i}\,u_1} \cdots \oint \frac{\mathrm{d}u_n(u_n+\tau)}{2\pi\mathrm{i}\,u_n} \\ \det\left(\frac{1}{1-u_i(\tau+u_i)X_j}\right) \det\left(\frac{1}{1-u_i^{-1}(\tau+u_i^{-1})Y_j}\right),$$

where the contours of integration surround the X_i-dependent poles. The determinants are now of Cauchy type and can be evaluated exactly:

$$Z_{\mathrm{UASM}} \propto \frac{\Delta(X)\Delta(Y)}{\Delta(x^2)\Delta(y^2)\Delta^*(x^2)\Delta^*(y^2)} \oint \frac{\mathrm{d}u_1(u_1+\tau)}{2\pi\mathrm{i}\,u_1} \cdots \oint \frac{\mathrm{d}u_n(u_n+\tau)}{2\pi\mathrm{i}\,u_n} \\ \Delta(u(1+\tau u))\Delta(u^{-1}(1+\tau u^{-1})) \prod_{i,j} \frac{1}{(1-u_i(\tau+u_i)X_j)(1-u_i^{-1}(\tau+u_i^{-1})Y_j)}.$$

The Vandermonde determinants outside the integral cancel each other, leaving only a regular part, due to

$$X_i - X_j = \frac{(1-a^2)^2(x_i^2-x_j^2)(1-x_i^2x_j^2)}{a^2(1-a^2x_i^2)(1-a^{-2}x_i^2)(1-a^2x_j^2)(1-a^{-2}x_j^2)}.$$

At this stage one can take the homogeneous limit, that is set $x_i = y_i = 1$, or $X_i = Y_i = 0$, which results in the simple expression

$$A_{\mathrm{V}}(2n+1;\tau^2) \propto \oint \frac{\mathrm{d}u_1(\tau+u_1)}{2\pi\mathrm{i}\,u_1}\cdots\oint \frac{\mathrm{d}u_n(\tau+u_n)}{2\pi\mathrm{i}\,u_n}\Delta(u(\tau+u))\Delta(u^{-1}(\tau+u^{-1})),$$

where the integrals are performed around zero. We can once again exchange determinant and integral sign and find

$$A_{\mathrm{V}}(2n+1;\tau^2) \propto \det_{0\le \ell,m\le n-1} \oint \frac{\mathrm{d}u}{2\pi\mathrm{i}\,u} u^{\ell-m}(\tau+u)^{\ell+1}(\tau+u^{-1})^m$$

The first column of this matrix being $(1,0,\ldots)$, we can restrict the range of the indices to

$$A_{\mathrm{V}}(2n+1;\tau^2) \propto \det_{1\le \ell,m\le n-1} \oint \frac{\mathrm{d}u}{2\pi\mathrm{i}\,u} u^{\ell-m}(\tau+u)^{\ell+1}(\tau+u^{-1})^m$$

which is identical to (6.9) with $t=\tau^{-1}$, $p=n-1$, $\tilde{p}=0$.

8.2. **Second formula.** The reasoning is exactly the same; only the matrix elements of the determinant are slightly modified. In the case of $A^{(2)}_{\mathrm{VHPASM}}$, we must use the matrix M_{UU} defined in (8.2) with parameters $b=1/a$, $c=a$. In this case we use the following identity to represent the matrix elements in factorised form:

$$\frac{-a^2y^2(1+x^4)+\Big(\big(a^4+a^2+1\big)\,(1+y^4)-\big(a^2+1\big)^2\,y^2\Big)\,x^2}{xy(a^2x^2-y^2)(a^2y^2-x^2)(a^2-x^2y^2)(1-a^2x^2y^2)}a^2$$
$$= (1-a^2)^2\oint \frac{\mathrm{d}u}{2\pi\mathrm{i}}\frac{u(1+u(a+a^{-1}))}{\mu(u,x)(u,1/x)\mu(u,a/y)\mu(u,ay)}.$$

We thus find the following expression for the "partition function" $Z^{(2)}_{\mathrm{VHPASM}}$ (which is really a ratio of the partition function of UUASMs by the partition function of UASMs)

$$Z^{(2)}_{\mathrm{VHPASM}} \propto \frac{\Delta(X)\Delta(Y)}{\Delta(x^2)\Delta(y^2)\Delta^*(x^2)\Delta^*(y^2)}\oint\frac{\mathrm{d}u_1(1+\tau u_1)}{2\pi\mathrm{i}\,u_1}\cdots\oint\frac{\mathrm{d}u_n(1+\tau u_n)}{2\pi\mathrm{i}\,u_n}$$
$$\Delta(u(1+\tau u))\Delta(u^{-1}(1+\tau u^{-1}))\prod_{i,j}\frac{1}{(1-u_i(\tau+u_i)X_j)(1-u_i^{-1}(\tau+u_i^{-1})Y_j)},$$

and in the homogeneous limit,

$$A^{(2)}_{\mathrm{VHPASM}}(2n+1;\tau^2) \propto \oint\frac{\mathrm{d}u_1(1+\tau u_1)}{2\pi\mathrm{i}\,u_1}\cdots\oint\frac{\mathrm{d}u_n(1+\tau u_n)}{2\pi\mathrm{i}\,u_n}\Delta(u(\tau+u))\Delta(u^{-1}(\tau+u^{-1}))$$
$$\propto \det_{0\le \ell,m\le n-1}\oint\frac{\mathrm{d}u}{2\pi\mathrm{i}}u^{\ell-m}(\tau+u)^{\ell}(\tau+u^{-1})^{m+1}.$$

Once again one can remove the first line and column, and we recover (6.9) with $t=\tau$, $p=n-1$, $\tilde{p}=0$.

8.3. **Third formula.** The generating function $\tilde{A}^{(2)}_{\mathrm{UU}}(n;\tau^2)$ is defined as one of the factors of $A^{(2)}_{\mathrm{HT}}(2n;\tau^2,1)$, the other being $A^{(2)}_{\mathrm{UU}}(n;\tau^2,1,1)$. To compute this generating function we need $Z^{+,(2)}_{\mathrm{HT}}(2n;(\vec{x},\vec{x}^{-1}),(\vec{y},\vec{y}^{-1}))$ which is defined in terms of the matrix M^{+}_{HT}, see (8.3). The matrix $M^{+}_{\mathrm{HT}}\left(2n;(\vec{x},\vec{x}^{-1}),(\vec{y},\vec{y}^{-1})\right)$ commutes with

$$P=\begin{pmatrix}0 & I_n\\ I_n & 0\end{pmatrix},$$

and thus can be brought to block-diagonal form with blocks

$$M^{+,\pm}_{\mathrm{HT}}(n;\vec{x},\vec{y})_{ij}=M^{+}_{\mathrm{HT}}(n;\vec{x},\vec{y})_{ij}\pm M^{+}_{\mathrm{HT}}(n;\vec{x},\vec{y}^{-1})_{ij}.$$

These matrices are readily identified with the matrix M_{UU} with $b=c=\mathrm{i}$ and $b=c=1$ respectively:

$$\begin{aligned}M^{+,-}_{\mathrm{HT}}(n;\vec{x},\vec{y})_{ij}&=\frac{1}{\sigma(\mathrm{i}x)\sigma(\mathrm{i}y)}M_{\mathrm{UU}}(n;\vec{x},\vec{y})_{ij}\qquad(b=c=\mathrm{i}),\\ M^{+,+}_{\mathrm{HT}}(n;\vec{x},\vec{y})_{ij}&=-\frac{1}{\sigma(x)\sigma(y)}M_{\mathrm{UU}}(n;\vec{x},\vec{y})_{ij}\qquad(b=c=1).\end{aligned}$$

The further reasoning is again analogous to that in Section 8.1, and in order to compute the homogeneous limit of $\det M^{(2)}_{\mathrm{HT}}(n;\vec{x},\vec{y})$ we now use the identity

$$\begin{aligned}&\frac{(a^2+1)^2x^2y^2-a^2(x^2+y^2)(1+x^2y^2)}{xy(a^2x^2-y^2)(a^2y^2-x^2)(a^2-x^2y^2)(1-a^2x^2y^2)}a^2\\ &\qquad=(1-a^2)^2\oint\frac{\mathrm{d}u}{2\pi\mathrm{i}}\frac{u}{\mu(u,x)(u,1/x)\mu(u,a/y)\mu(u,ay)}.\end{aligned}\tag{8.6}$$

Thus we find

$$\begin{aligned}\tilde{Z}^{(2)}_{\mathrm{UU}}\propto\ &\frac{\Delta(X)\Delta(Y)}{\Delta(x^2)\Delta(y^2)\Delta^*(x^2)\Delta^*(y^2)}\oint\frac{\mathrm{d}u_1}{2\pi\mathrm{i}\,u_1}\cdots\oint\frac{\mathrm{d}u_n}{2\pi\mathrm{i}\,u_n}\\ &\Delta(u(1+\tau u))\Delta(u^{-1}(1+\tau u^{-1}))\prod_{i,j}\frac{1}{(1-u_i(\tau+u_i)X_j)(1-u_i^{-1}(\tau+u_i^{-1})Y_j)},\end{aligned}$$

and

$$\begin{aligned}\tilde{A}^{(2)}_{\mathrm{UU}}(4n;\tau^2)&\propto\oint\frac{\mathrm{d}u_1}{2\pi\mathrm{i}\,u_1}\cdots\oint\frac{\mathrm{d}u_n}{2\pi\mathrm{i}\,u_n}\Delta(u(\tau+u))\Delta(u^{-1}(\tau+u^{-1}))\\ &\propto\det_{0\le\ell,m\le n-1}\oint\frac{\mathrm{d}u}{2\pi\mathrm{i}}u^{\ell-m}(\tau+u)^{\ell}(\tau+u^{-1})^{m}.\end{aligned}$$

Removing the first line and column, we recover (6.18) with $t=\tau^{-1}$, $p=n-1$, $\tilde{p}=0$.

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