Green Pseudodifferential Operators on Manifolds with Edges

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Abstract

The Green operators occurring in the pseudodifferential calculus on manifolds with edges have been defined via their mapping properties. We give a transparent equivalent description of them in terms of their Schwartz kernels.

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Contents

1 *Introduction*

Let M be a compact manifold with singularity S, i.e., M is a compact Hausdorff topological space, $S \subset M$ and $M \setminus S$ is a smooth manifold without boundary.

For any smooth symbol $a(z,\zeta)$ of order m on $T^*(M \setminus S)$, we may consider a pseudodifferential operator $A = \mathrm{op}\,(a)$, defined first on functions of compact support in $M \setminus S$.

Since $M \setminus S$ is non-compact, it is not to be expected that A is determined uniquely modulo "small" operators, even if $a(z,\zeta)$ is "smooth" up to the singularity. To make use of the compactness of M, we have to extend the operator A to functions defined up to S.

For this purpose, the idea suggested by Schulze [10, 13] is to find a proper reformulation of the operator A close to the singularity. More precisely, one looks for an operator $A^{(R)}$ defined on functions which are supported close to S, such that $A^{(R)} = A$ modulo smoothing operators on $M \setminus S$. Such a reformulation certainly depends on the structure of M near the singularity.

Each singularity S, when blown up, produces a specific class of degenerated differential operators, called *totally characteristic*. In this way we guess a class of smooth symbols on $T^*(M \setminus S)$ to be reformulated. Then, one arrives at various subalgebras of the algebra of pseudodifferential operators on $M \setminus S$. These are of the form

$$\varphi_0 A^{(R)} \psi_0 + \varphi_\infty A \psi_\infty, \tag{1.1}$$

where $(\varphi_0, \varphi_\infty)$ is a C^∞ partition of unity on $M \setminus S$ with

$$\begin{cases} \varphi_0 \equiv 1 & \text{close to } S, \\ \varphi_0 \equiv 0 & \text{away from a small neighborhood of } S, \end{cases}$$

and ψ_ν "covers" φ_ν.

Such operators are referred to as pseudodifferential operators on manifolds with singularities and act in relevant Sobolev spaces on M.

Because of the compatibility of A and $A^{(R)}$ away from the singularity, each operator of the form (1.1) possesses an *interior symbol*, which is perhaps suppressed to control its behavior up to the singularity. Moreover, when realized as a pseudodifferential operator along S, the operator $A^{(R)}$ has an operator-valued symbol acting in the fibers of the "normal bundle" of S, perhaps, after blowing up the singularity in an appropriate way. The latter operators are sometimes called "transversal operators" and admit additional symbols homogeneous with respect to a group action in the fibers (so-called "singular symbols").

While the invertibility of the singular symbol (*Lopatinskii condition*) is a necessary condition for the Fredholm property of the operator in question, such is not usually the case. The invertibility of the interior symbol implies only that the singular symbol is a Fredholm operator. Thus, to achieve the invertibility of the singular symbol, one needs to border $A^{(R)}$ with a number of smoothing operators in

$M \setminus S$, that however affect the singular symbol of $A^{(R)}$. These are pseudodifferential operators along S with classical operator-valued symbols, called the *Green symbols*.

In this way we obtain what looks like the operators in Boutet de Monvel's theory of boundary value problems and encompasses this theory, namely

$$A^{(B)} = \phi_0 \begin{pmatrix} A^{(R)} & P \\ T & E \end{pmatrix} \psi_0 + \phi_\infty \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \psi_\infty, \tag{1.2}$$

where the operator P has the meaning of a *corestriction* or *potential operator* with respect to S, T of a *trace operator*, and E is nothing else than a matrix of pseudodifferential operators along S with scalar symbols.

A familiar argument, based on pseudolocality of the operators in question, shows that if $\begin{pmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{pmatrix}$ is a parametrix for $\begin{pmatrix} A^{(R)} & P \\ T & E \end{pmatrix}$ and Π a parametrix for A, then

$$\Pi^{(B)} = \phi_0 \begin{pmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{pmatrix} \psi_0 + \phi_\infty \begin{pmatrix} \Pi & 0 \\ 0 & 0 \end{pmatrix} \psi_\infty,$$

is a parametrix for (1.2).

When carried out on the symbolic level, the parametrix construction for the operator $\begin{pmatrix} A^{(R)} & P \\ T & E \end{pmatrix}$ invests the class of admissible operator-valued symbols of $A^{(R)}$. It is therefore adequate to have all the "inverse symbols" from the very beginning in the class.

When seeking an analytical expression for the index of an operator $A^{(B)}$, one has to construct a very exact parametrix inverting this operator not as usual up to a compact error but up to a *trace class* error. For the operator A this construction is usually made by means of interior symbolic calculus. But for the operator-valued symbols in question differentiation in the covariable does not improve their differential properties. So we need to find another way to indicate that such differentiation improves the properties of the symbols.

The present paper gives a new characterization of the Green pseudodifferential operators, generated in the parametrix construction for elliptic operators on a manifold with edge singularities. This will be performed locally on the level of specific operator-valued symbols in a wedge of the form $X^\Delta \times U$, where $X^\Delta = (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$ is the model cone with base X which is a compact C^∞ manifold, and $U \subset \mathbb{R}^q$ is an open set, the edge. An allowed case is $\dim X = 0$; then the wedge equals $\overline{\mathbb{R}}_+ \times U$ that may be regarded as the local form of a C^∞ manifold with boundary, where the model cone corresponds to the inner normal $\overline{\mathbb{R}}_+$ and the edge to a patch U of the boundary.

The role of the Green operators for solving elliptic (say differential) equations within a pseudodifferential algebra on a given configuration with edge singularities is analogous to that of the contribution from the boundary to classical Green's function in elliptic boundary value problems.

From the pseudodifferential calculus of boundary value problems with the transmission property it is known, cf. Boutet de Monvel [1], that Green's function equals a sum $\Pi + G$, where Π is a parametrix of the given elliptic operator and G a singular Green operator (in the notations of [1]) [1]. Locally G is (up to a smoothing Green operator) a pseudodifferential operator with an operator-valued symbol of special kind, cf. Schulze [13, 2.2.3]. This is the motivation to call the analogous objects for elliptic operators on more general manifolds with edges Green operators.

The structure of Green's function in the standard situation for elliptic boundary value problems is based on the elliptic regularity of solutions with smoothness up to the boundary when the right-hand sides and boundary data are smooth. This is a consequence of the transmission property of the involved interior symbols. Using the concept of discrete asymptotic types, the meaning and significance of the transmission property is now elucidated: roughly speaking, a pseudodifferential operator has the transmission property if Taylor asymptotics close to the boundary are preserved under its action.

There is an enormous gap between this comparatively simple special case and the general Green operators in edge problems (even when $\dim X = 0$, which corresponds to the case of violated transmission property). The reason is the complexity of phenomena occurring in the elliptic regularity of solutions near the edges. In contrast to the mentioned smoothness (Taylor asymptotics with respect to the distance variable t) for C^∞ solutions, it consists of asymptotics of the form

$$u(t,x,y) \sim \sum_{\nu=0}^{\infty} \sum_{j=0}^{m_\nu} f_{\nu,j}(x,y)\, t^{-p_\nu} (\log t)^k \quad \text{as} \quad t \to 0, \tag{1.3}$$

for every $y \in U$ with certain $p_\nu = p_\nu(y) \in \mathbb{C}$ satisfying $\Re p_\nu \to -\infty$ as $j\nu \to \infty$, $m_\nu = m_\nu(y) \in \mathbb{Z}_+$, and coefficients $f_{\nu,j}$ in $C^\infty(X)$, where the dependence on y is jumping in general, even chaotic, with no possibility to choose a numeration of the p_ν, m_ν in a unified manner under varying y.

This behavior of C^∞ solutions can be observed already in simplest cases of edge-degenerate differential operators. Solutions in weighted edge Sobolev spaces of finite smoothness are more difficult to characterize, cf. Chapter 9 in Egorov and Schulze [3]. The general strategy for understanding the solutions under the aspect of their behavior near the edges is to construct a parametrix $\Pi^{(B)}$ of the given operator $A^{(B)}$ and to compose this from the left to the equation $A^{(B)}u = f$. On the left-hand side we obtain $\Pi^{(B)} A^{(B)} = 1 + G^{(B)}$ for a smoothing operator in the calculus, Green in our terminology. Then the regularity for u will follow from the continuity both of $\Pi^{(B)}$ and $G^{(B)}$ between the distribution spaces with asymptotics. So the symbolic structures behind $\Pi^{(B)}$ and $G^{(B)}$ have to be rich enough to reflect the mentioned regularity of solutions with asymptotics. In addition the operators are required to act continuously within such distribution spaces and in addition to form an algebra.

The latter one, briefly called the edge pseudodifferential algebra (cf. Schulze [9]), has all these properties, and, in particular, there occur the Green pseudodifferential operators. They have operator-valued symbols, cf. the explicit definition below, in which possible asymptotic data are involved as parameters. Our characterization here will consist of a description of the kernels in terms of spaces with continuous

[1] By Green's function, one means the left upper corner of $\Pi^{(B)}$.

asymptotics. This covers all the special possibilities of positions of exponents in (1.3) as well as of the jumping m_ν and $f_{\nu,j}$ under varying $y \in U$, described by C^∞ functions on U with values in the analytic functionals in the plane of the complex Mellin covariable, that are point-wise discrete and finite orders. This aspects was studied in detail in Schulze [8].

The edge pseudo-differential calculus also contains trace and potential operators with respect to the edge, analogously to the boundary (trace and potential) conditions in boundary value problems, where the ellipticity is an analogue of the classical Lopatinskii condition. On the symbolic level they correspond to the trace and potential operator-valued symbols, also carrying asymptotic information inherited from the final elliptic regularity, here with continuous asymptotics, again. Similarly to boundary value problems, say with the transmission property, the composition between a potential and a trace symbol will be a Green symbol. Conversely our tensor product description in general can be understood as convergent sums of such compositions.

This relation shows intuitively how the Green symbols organize the interaction between the interior effects of the calculus outside the edge singularity and the specific contributions due to the edge. Green, trace and potential symbols can be understood in a unified way as block matrix-valued symbols, where the entry in the left upper corner are the proper Green objects, whereas the entries outside the diagonal are interpreted as trace and potential symbols. The right lower corner is a matrix of scalar classical symbols along the edge.

This point of view also belongs to the axiomatic ideas for the pseudodifferential calculus on configurations with polyhedral singularities of higher order. In the past years this theory has seen much progress, and in particular, the role of the Green operators there is to formulate the specific interactions of properties of solutions relative to the lower-dimensional skeletons. Here there arise many types of Green, trace and potential operators, operating between distribution spaces (with asymptotics) on the faces of various dimensions, and the structure of the (block matrices of these) Green operators is so rich that the various reductions to components of the lower-dimensional skeleton generate at once the pseudodifferential calculus that is originally given there from the knowledge of the analysis for the lower singularity orders (cf. Schulze [11] for second order corners). Here for edge singularities the lower dimensional face is the edge alone, and the block matrix shaped Green operators contain (as right lower corners) the classical pseudodifferential calculus on that C^∞ manifold.

2 *Edge Sobolev spaces*

Fix a smoothed norm function $\eta \mapsto \langle\eta\rangle$ on $\mathbb{R}^q$, i.e., a positive C^∞ function on $\mathbb{R}^q$ with the property that $\langle\eta\rangle = |\eta|$ for all $|\eta| \geq c > 0$.

Let L be a Banach space with norm $\|\cdot\|_L$. Denote by $\mathcal{L}_\sigma(L)$ the space of all continuous linear operators in L, equipped with the strong operator topology.

In the sequel, $(\kappa_\lambda)_{\lambda>0}$ stands for a strongly continuous group of operators on

L, i.e., κ_λ is a continuous mapping of $\mathbb{R}_+ \to \mathcal{L}_\sigma(L)$ satisfying the composition rule $\kappa_\lambda \kappa_\rho = \kappa_{\lambda\rho}$ for all $\lambda, \rho \in \mathbb{R}_+$. In particular, κ_λ is an isomorphism of L, for each $\lambda \in \mathbb{R}_+$.

Definition 2.1 *Given any $s \in \mathbb{R}$, the "twisted" Sobolev space $W^s(\mathbb{R}^q, L)$ is defined to be the completion of $\mathcal{S}(\mathbb{R}^q, L)$ with respect to the norm*

$$\|u\|_{W^s(\mathbb{R}^q,L)} = \left(\int_{\mathbb{R}^q} \langle\eta\rangle^{2s} \|\kappa_{\langle\eta\rangle^{-1}} \mathcal{F}_{y\mapsto\eta} u\|_L^2 d\eta \right)^{\frac{1}{2}} .$$

This concept can also be extended to the case where L is a Fréchet space. For our purpose, it is sufficient to do this for those L which are projective limits of Banach spaces.

Namely, suppose (L_ν) is a sequence of Banach spaces with continuous embeddings $L_{\nu+1} \hookrightarrow L_\nu$, and suppose the group actions agree on all the spaces. Then, we set $W^s(\mathbb{R}^q, L) = \text{proj lim}_{\nu\to\infty} W^s(\mathbb{R}^q, L_\nu)$.

The main property of the spaces $W^s(\mathbb{R}^q, L)$ is that these are semilocal spaces. In other words, the multiplication operator $u \mapsto \phi u$ is continuous on $W^s(\mathbb{R}^q, L)$, for each $\phi \in C^\infty_{comp}(\mathbb{R}^q)$. This allows one to define the local versions $W^s_{loc}(U, L)$ and $W^s_{comp}(U, L)$ of Sobolev spaces on each open set $U \subset \mathbb{R}^q$.

In particular, if $L = \mathcal{K}^{s,\gamma}(X^\square)$ is the weighted Sobolev space on the stretched cone $X^\square = \overline{\mathbb{R}}_+ \times X$ over a compact manifold X of dimension n, and $\kappa_\lambda u\,(t, x) = \lambda^{\frac{1+n}{2}} u(\lambda t, x)$, then $W^s_{loc}(X^\square \times U) := W^s_{loc}(U, \mathcal{K}^{s,\gamma}(X^\square))$ is known as the (edge) Sobolev space over the stretched wedge $X^\square \times U$ whose edge is U.

3 *Cone spaces with continuous asymptotics*

The continuous asymptotic type will be described in terms of analytic functionals. We start, therefore, with a short description of those functionals.

Namely, if K is a compact set in the complex plane, then we denote by $\mathcal{A}'_K$ the space of all $f \in \text{Hol}\,(\mathbb{C})'$ having K as a *carrier*. We have $\mathcal{A}'_K = \text{Hol}\,(K)'$ provided that the complement of K is connected.

The space $\mathcal{A}'_K$ is known to be a nuclear Fréchet space. Therefore, for any Fréchet space L, we may consider the space $\mathcal{A}'_K(\mathbb{C}, L)$ of L-valued analytic functionals carried by K.

Pick a cut-off function $\omega \in C^\infty_{comp}(\overline{\mathbb{R}}_+)$, i.e., any function with $\omega(t) \equiv 1$ near $t = 0$.

Suppose K lies on the left from the vertical line $\Re z = \frac{1+n}{2} - \gamma$, i.e., $\Re z < \frac{1+n}{2} - \gamma$ for all $z \in K$. Then, for any $f \in \mathcal{A}'_K(\mathbb{C}, C^\infty(X))$, the function $\omega(t)\,\langle f(x), t^{-z}\rangle$ is easily verified to lie in $\mathcal{K}^{\infty,\gamma}(X^\square)$.

We want to define subspaces of $\mathcal{K}^{s,\gamma}(X^\square)$ consisting of functions u such that the difference $u(t, x) - \omega(t)\,\langle f(x), t^{-z}\rangle$ has a gain in the weight. To this end, we need some preliminaries.

A weight datum $\mathfrak{d} = (\gamma, (-l, 0])$ consists of a number $\gamma \in \mathbb{R}$ and an interval $(-l, 0]$ with $0 < l \leq \infty$.

As described above, when working with the spaces $\mathcal{K}^{s,\gamma}(X^\square)$, we have to consider weight data $\mathfrak{d} = \left(\gamma - \frac{n}{2}, (-l, 0]\right)$.

A set $\sigma \subset \mathbb{C}$ is called a carrier of asymptotics if σ is closed, has connected complement and if the intersection of σ with each strip $\{c' \leq \Re z \leq c''\}$ is compact.

Given a weight datum $\mathfrak{d} = \left(\gamma - \frac{n}{2}, (-l, 0]\right)$, we are going to introduce asymptotic types related to $\mathfrak{d}$. For this purpose, we should distinguish the cases of finite and infinite weight intervals $(-l, 0]$.

If $l < \infty$, then by an *asymptotic type* related to $\mathfrak{d}$ is meant any pair $as = (\sigma, \Sigma)$, where

- σ is a carrier of asymptotics contained in the complex strip $\frac{1+n}{2} - \gamma - l \leq \Re z < \frac{1+n}{2} - \gamma$; and
- Σ is a closed subspace of $\mathcal{A}'_\sigma(\mathbb{C}, C^\infty(X))$.

If $l = \infty$, then by an asymptotic type related to $\mathfrak{d}$ is meant any collection $as = (\sigma, (\Sigma_\nu))$ where

- σ is a carrier of asymptotics contained in the complex strip $\Re z < \frac{1+n}{2} - \gamma$; and
- (Σ_ν) is a sequence of closed subspaces of $\mathcal{A}'_{K_\nu}(\mathbb{C}, C^\infty(X))$, with $K_\nu \subset\subset \sigma$ carriers of asymptotics and $\sup_{z \in K_\nu} \Re z \to -\infty$ as $\nu \to \infty$.

Let us denote by $\mathrm{As}\,(\mathfrak{d})$ the set of all asymptotic types related to $\mathfrak{d}$. We are now in a position to introduce our spaces with continuous asymptotics.

Definition 3.1

- *For asymptotic type $as = (\sigma, \Sigma)$ related to a weight datum with finite weight interval, the space $\mathcal{K}^{s,\gamma}_{as}(X^\square)$ is defined to consist of all $u \in \mathcal{K}^{s,\gamma}(X^\square)$ such that $u(t,x) - \omega(t)\,\langle f(x), t^{-z}\rangle \in \mathcal{K}^{s,\gamma+l-0}(X^\square)$.*
- *For asymptotic type $as = (\sigma, (\Sigma_\nu)$ related to a weight datum with infinite weight interval, the space $\mathcal{K}^{s,\gamma}_{as}(X^\square)$ is defined to consist of all $u \in \mathcal{K}^{s,\gamma}(X^\square)$ such that $u(t,x) - \omega(t)\sum_{\nu=1}^{N}\langle f_\nu(x), t^{-z}\rangle \in \mathcal{K}^{s,\gamma_N}(X^\square)$ for some sequence $f_\nu \in \Sigma_\nu$ and some sequence (γ_N) of real numbers convergent to $+\infty$.*

Notice that this definition is independent of the particular choice of the cut-off function ω.

To topologize the space $\mathcal{K}^{s,\gamma}_{as}(X^\square)$ in the case of finite weight intervals, denote by $\mathfrak{A}_{as}(X^\square)$ the space of functions $u \in \mathcal{K}^{\infty,\gamma}(X^\square)$ of the form $u(t,x) = \omega(t)\,\langle f(x), t^{-z}\rangle$, where $f \in \Sigma$.

By the Köthe-Grothendieck duality, $\mathcal{A}'_\sigma(\mathbb{C}, C^\infty(X)) \overset{top.}{\cong} \mathrm{Hol}\,(\hat{\mathbb{C}} \setminus \sigma, C^\infty(X))$ has a Fréchet topology which is nuclear. Then, we endow the subspace Σ by the induced topology. Moreover, the mapping of $\Sigma \to \mathfrak{A}_{as}(X^\square)$, given by $f \mapsto \omega(t)\,\langle f(x), t^{-z}\rangle$, is easily seen to be injective and surjective. Thus, we can give $\mathfrak{A}_{as}(X^\square)$ the topology induced by this algebraic isomorphism.

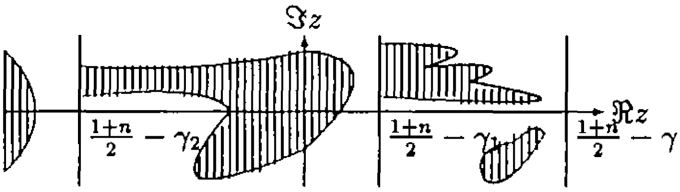

Fig. 1: A carrier of a continuous asymptotics with infinite weight interval.

Lemma 3.2 *For each asymptotic type 'as' related to a weight datum with finite weight interval, we have*

$$\mathcal{K}_{as}^{s,\gamma}(X^\square) = \mathcal{K}^{s,\gamma+l-0}(X^\square) + \mathfrak{A}_{as}(X^\square). \tag{3.1}$$

Proof. See Lemma 1.3.14 in Dorschfeldt [2].

□

We make $\mathcal{K}_{as}^{s,\gamma}(X^\square)$ a Fréchet space by endowing it with the topology of non-direct sum of Fréchet spaces. Moreover, if $\sigma \cap \{\Re z = \frac{1+n}{2} - \gamma - l\} = \emptyset$, then the sum (3.1) is direct.

We now turn to the case of infinite weight intervals. In this case, the space $\mathcal{K}_{as}^{s,\gamma}(X^\square)$ is given a projective limit topology. We shall not attempt to describe this topology in general, just looking at a model situation (cf. Figure 1).

Lemma 3.3 *Let 'as' be an asymptotic type associated with infinite weight interval. Suppose there is an increasing sequence $\gamma = \gamma_0 < \gamma_1 < \dots$ such that each K_ν is contained in the complex strip $\frac{1+n}{2} - \gamma_\nu \le \Re z < \frac{1+n}{2} - \gamma_{\nu-1}$. Then, for $as_N = \left(\cup_1^N K_\nu, \oplus_1^N\right)$, all the embeddings $\mathcal{K}_{as_{N+1}}^{s,\gamma}(X^\square) \hookrightarrow \mathcal{K}_{as_N}^{s,\gamma}(X^\square)$ are continuous and $\mathcal{K}_{as}^{s,\gamma}(X^\square) = \cap_{N=1}^\infty \mathcal{K}_{as_N}^{s,\gamma}(X^\square)$.*

Proof. *Ibid.*, Lemma 1.3.15.

□

When given the topology of the projective limit of the sequence $\left(\mathcal{K}_{as_N}^{s,\gamma}(X^\square)\right)$, the space $\mathcal{K}_{as}^{s,\gamma}(X^\square)$ becomes a Fréchet space.

Example 3.4 Let p be a fixed point in the strip $\frac{1+n}{2} - \gamma - l \le \Re z < \frac{1+n}{2} - \gamma$, and let $\sigma = \{p\}$. If $\tilde{\Sigma}$ is a subspace of $C^\infty(X)$, then the analytic functionals of the form

$$\langle f(x), u\rangle = \sum_{j=0}^{m} f_j(x)\,(\partial/\partial z)^j u\,(p), \quad u \in \mathrm{Hol}\,(\sigma),$$

with $f_j \in \tilde{\Sigma}$, form a subspace of $\mathcal{A}'_\sigma(\mathbb{C}, C^\infty(X))$. We have

$$\langle f(x), t^{-z}\rangle = \sum_{j=0}^{m} f_j(x)\, t^{-p}\,(\log t)^j, \quad t > 0. \tag{3.2}$$

What we obtain in this way, is known as discrete conormal asymptotics for $t \to 0+$ for solutions of elliptic equations on the stretched cone $\mathbb{R}_+ \times X$. The works of

Kondrat'ev [5], of Eskin [4], and, for pseudodifferential operators on the half-axis, of Schulze [10] show that one may expect a solution u of an elliptic equation on the half-axis $\mathbb{R}_+$ to have asymptotics of the form

$$u(t) \sim \sum_{\nu=0}^{\infty} \sum_{j=0}^{m_\nu} f_{\nu,j}(x)\, t^{-p_\nu} (\log t)^j, \quad t \to 0+, \tag{3.3}$$

with exponents $p_\nu \in \mathbb{C}$ satisfying $\Re p_\nu \to -\infty$ as $\nu \to \infty$. This motivates the definition of function spaces with continuous asymptotics.

□

In the local theory of boundary value problems, the analysis takes place in domains $\mathbb{R}_+ \times U$, where $U \subset \mathbb{R}^q$ is some open set of the boundary and $\mathbb{R}_+$ is the inner normal. In this context, solutions of elliptic equations depend on the variables $(t,x) \in \mathbb{R}_+ \times U$, and thus the poles p_ν and their multiplicities $m_\nu + 1$ may depend on the variable $y \in U$. It is the case, for example, in the well-known problem of Babushka (1986), i.e., the Dirichlet problem for the Laplace equation in an oblique cylinder, where the edge angle varies along the edge. This simple observation causes serious difficulties when one works in the framework of discrete asymptotic types, for the poles that appear may be spread over regions in $\mathbb{C}$ and the multiplicities of poles may vary with varying $y \in U$. In order to be able to cope with those difficulties, Schulze [8] introduced continuous asymptotic types. In recent years it turned out to be a very useful tool in wedge theory (cf. [13]).

If $u(t,x) = \omega(t)\,\langle f(x), t^{-z}\rangle$ is an element of $\mathfrak{A}_{as}(X^\square)$, with $as = (\sigma, \Sigma)$, then

$$\kappa_\lambda u\,(t,x) = \omega(\lambda t)\,\langle \lambda^{\frac{1+n}{2}-z} f(x), t^{-z}\rangle, \quad \lambda > 0.$$

For each fixed $\lambda > 0$, the function $\omega(\lambda t)$ is again of cut-off nature and so does not affect the property of being in $\mathfrak{A}_{as}(X^\square)$ modulo elements of $\mathcal{K}^{\infty,\gamma+l-0}(X^\square)$. On the other hand, $\lambda^{\frac{1+n}{2}-z} = \exp\left(\left(\frac{1+n}{2} - z\right)\right)\log\lambda\Big)$ is an entire function of z, hence the product $\lambda^{\frac{1+n}{2}-z} f(x)$ is again an analytic functional with values in $C^\infty(X)$ and carrier σ. Whether this functional belongs to Σ or not, would be a property of Σ itself. A kind of this property is that Σ is invariant under multiplication by entire functions. If such is the case, then $\kappa_\lambda u \in \mathfrak{A}_{as}(X^\square)$ modulo $\mathcal{K}^{\infty,\gamma+l-0}(X^\square)$, for all $\lambda > 0$.

Hence it follows that the space $\mathcal{K}^{s,\gamma}_{as}(X^\square)$ is invariant under the group action $(\kappa_\lambda)_{\lambda>0}$. Applying the general construction of Section 2 to the spaces $\mathcal{K}^{s,\gamma}_{as}(X^\square)$ yields (edge) Sobolev spaces with asymptotics $W^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_{as}(X^\square))$ over the stretched wedge $X^\square \times \mathbb{R}^q$.

4 *Pseudodifferential operators with operator-valued symbols*

The theory of pseudodifferential operator with operator-valued symbols is a natural extension of the scalar case.

Suppose that L and V are Banach spaces with fixed group actions $\left(\kappa_\lambda^{(L)}\right)_{\lambda\in\mathbb{R}_+}$ and $\left(\kappa_\lambda^{(V)}\right)_{\lambda\in\mathbb{R}_+}$ respectively.

To shorten notation, we suppress the indices (L) and (V) and write both $\kappa_\lambda^{(L)}$ and $\kappa_\lambda^{(V)}$ simply κ_λ when no confusion can arise.

Definition 4.1 *Given any open set $U \subset \mathbb{R}^N$ and number $m \in \mathbb{R}$, we let $S^m(U \times \mathbb{R}^q, \mathcal{L}(L \to V))$ consist of all $a \in C^\infty_{loc}(U \times \mathbb{R}^q, \mathcal{L}_b(L \to V))$ with the property that, for each compact set $K \subset U$ and each multi-indices $\alpha \in \mathbb{Z}^N_+$ and $\beta \in \mathbb{Z}^q_+$, there is a constant $c_{K,\alpha,\beta}$ such that*

$$\left\|\kappa^{-1}_{\langle\eta\rangle}\left(D^\alpha_y D^\beta_\eta a(y,\eta)\right)\kappa_{\langle\eta\rangle}\right\|_{\mathcal{L}_b(L\to V)} \le c_{K,\alpha,\beta}\,\langle\eta\rangle^{m-|\beta|} \quad \text{for all} \quad y\in K,\ \eta\in\mathbb{R}^q. \quad (4.1)$$

The best constants $c_{K,\alpha,\beta}$ in (4.1) form a system of seminorms on the space $S^m(U \times \mathbb{R}^q, \mathcal{L}(L\to V))$ under which it becomes a Fréchet space.

We mention that the asymptotic sums of symbols of decreasing orders can be carried cut within these symbol classes modulo

$$S^{-\infty}(U \times \mathbb{R}^q, \mathcal{L}(L\to V)) = \cap_m S^m(U \times \mathbb{R}^q, \mathcal{L}(L\to V)).$$

Classical symbols of order m are defined to be those $a \in S^m(U \times \mathbb{R}^q, \mathcal{L}(L\to V))$ which have asymptotic expansions into functions homogeneous in η of decreasing degrees $m-j$, $j \in \mathbb{Z}_+$. The homogeneity is understood in terms of the group actions.

More precisely, a function $a \in C^\infty_{loc}(U \times (\mathbb{R}^q \setminus 0), \mathcal{L}_b(L\to V))$ is called homogeneous of degree m in $\eta \ne 0$ if

$$a(y,\lambda\eta) = \lambda^m \kappa_\lambda a(y,\eta)\kappa_\lambda^{-1} \quad \text{for all} \quad \lambda\in\mathbb{R}_+,$$

provided $\eta \ne 0$. It is easy to see that if a is homogeneous of degree m in $\eta \ne 0$, then, for any excision function $\chi \in C^\infty_{loc}(\mathbb{R}^q$, the product $\chi(\eta)\,a(y,\eta)$ is in the symbol class $S^m(U \times \mathbb{R}^q, \mathcal{L}_b(L\to V))$. Now, a symbol $a \in S^m(U \times \mathbb{R}^q, \mathcal{L}_b(L\to V))$ is called *classical* if there are functions $a_{m-j} \in C^\infty_{loc}(U \times (\mathbb{R}^q \setminus 0), \mathcal{L}_b(L\to V))$ homogeneous of degree $m-j$ in $\eta \ne 0$, such that $a \sim \sum_{j=0}^\infty \chi a_{m-j}$ for some excision function χ. We denote by $S^m_{cl}(U \times \mathbb{R}^q, \mathcal{L}(L\to V))$ the space of all classical symbols.

For $a \in S^m_{cl}(U \times \mathbb{R}^q, \mathcal{L}(L\to V))$, the component $\sigma^m_{edge}(a)(y,\eta) = a_m(y,\eta)$ is called the *principal edge symbol* of a.

Lemma 4.2 *For each $a \in S^m_{cl}(U \times \mathbb{R}^q, \mathcal{L}(L\to V))$, we have*

$$\sigma_{edge}(a)(y,\eta) = \lim_{\lambda\to\infty} \lambda^{-m}\kappa_\lambda^{-1} a(y,\lambda\eta)\kappa_\lambda \quad (4.2)$$

in the operator norm of $\mathcal{L}_b(L\to V)$.

The symbols occurring in the edge algebra correspond to the particular choice of the spaces L and V, namely, $L = \mathcal{K}^{s,\gamma}(X^\square) \oplus \mathbb{C}^{d_-}$ and $V = \mathcal{K}^{s-m,\delta}(X^\square) \oplus \mathbb{C}^{d_+}$ or those with asymptotics, where both $\mathbb{C}^{d_-}$ and $\mathbb{C}^{d_+}$ are endowed with the identity group actions. They are not classical while the principal wedge symbol is yet defined by (4.2), the limit being in the strong operator topology of $\mathcal{L}_b(L \to V)$. We set

$$\mathrm{ES}^m(U \times \mathbb{R}^q, \mathfrak{d}) = \cap_{s\in\mathbb{R}} S^m(U \times \mathbb{R}^q, \mathcal{L}(\mathcal{K}^{s,\gamma}(X^\square) \oplus \mathbb{C}^{d-} \to \mathcal{K}^{s-m,\delta}(X^\square) \oplus \mathbb{C}^{d+})),$$

with $\mathfrak{d} = \left(-\gamma - \frac{n}{2}, \delta - \frac{n}{2}, (-l, 0]\right)$ a weight datum.

Given an operator-valued symbol $a(y, y', \eta) \in S^m(U \times U \times \mathbb{R}^q, \mathcal{L}(L \to V))$, with U an open subset of $\mathbb{R}^q$, we can form

$$\mathrm{op}(a)u\,(y) = \frac{1}{(2\pi)^q} \iint e^{\sqrt{-1}(y-y',\eta)} a(y, y', \eta) u(y') dy' d\eta, \quad y \in U, \tag{4.3}$$

defined first on $u \in C^\infty_{comp}(U, L)$. The double integral on the right-hand side of (4.3) is interpreted in the sense of oscillatory integrals, now for the corresponding vector- or operator-valued functions. In contrast to the scalar case, we have here the groups of isomorphisms $\left(\kappa^{(L)}_\lambda\right)_{\lambda\in\mathbb{R}_+}$ and $\left(\kappa^{(V)}_\lambda\right)_{\lambda\in\mathbb{R}_+}$ in the spaces L and V respectively. However, writing

$$\mathrm{op}(a)u\,(y) = \frac{1}{(2\pi)^q} \iint e^{\sqrt{-1}(y-y',\eta)} \kappa_{\langle\eta\rangle} \left(\kappa^{-1}_{\langle\eta\rangle} a(y, y', \eta) \kappa_{\langle\eta\rangle}\right) \left(\kappa^{-1}_{\langle\eta\rangle} u(y')\right) dy' d\eta$$

and using the estimates

$$\left|\kappa^{(\cdot)}_\lambda\right|_{\mathcal{L}(\cdot)} \le c^{(\cdot)} \left(\max\left(\lambda, \frac{1}{\lambda}\right)\right)^{R^{(\cdot)}}, \quad \lambda > 0, \tag{4.4}$$

with constants $c^{(\cdot)} > 0$ and $R^{(\cdot)} \ge 1$, allows us to perform all essential constructions for oscillatory integrals in the operator-valued set-up.

The operators $\mathrm{op}(a)$ given by (4.3), for $a \in S^m(U \times U \times \mathbb{R}^q, \mathcal{L}(L \to V))$, are called pseudodifferential operators of order m with operator-valued symbols. The space of such operators is denoted by $\mathcal{L}^m(U, \mathcal{L}(L \to V))$. We write $\mathcal{L}^m_{cl}(U, \mathcal{L}(L \to V))$ for the subspace consisting of those operators which correspond to the classical symbols.

5 *Kernels of pseudodifferential operators*

Every operator $A \in \mathcal{L}^m(U, \mathcal{L}(L \to V))$ is easily proved to induce a continuous mapping $A : C^\infty_{comp}(U, L) \to C^\infty_{loc}(U, V))$.

Moreover, the juxtaposition $u \mapsto Au$ yields a continuous mapping $C^\infty_{comp}(U) \to C^\infty_{loc}(U, \mathcal{L}(L \to V))$. Thus, setting

$$\langle K_A, g \otimes u\rangle = \langle g, Au\rangle_U, \quad \text{for} \quad g, u \in C^\infty_{comp}(U), \tag{5.1}$$

we obtain an operator-valued distribution K_A on the product $U \times U$. This distribution $K_A \in \mathcal{D}'(U \times U, \mathcal{L}_b(L \to V))$ is called the Schwartz kernel of the operator A [2].

[2]The kernel theorem of Schwartz is valid for operators in $\mathcal{L}_\epsilon(L'_\tau \to V)$, one of the spaces L, V being nuclear (see Theorem 1.4.2 in [15]).

Arguing formally, we may write

$$\langle g, Au\rangle_U = \iint_{U\times U} \left(\frac{1}{(2\pi)^q}\int_{\mathbb{R}^q} e^{\sqrt{-1}\langle y-y',\eta\rangle} a(y,y',\eta)d\eta\right) g(y)u(y')dydy',$$

whence $K_A(y',y) = \mathcal{F}^{-1}_{\eta\mapsto y-y'} a(y,y',\eta)$. Note that for every $s \in \mathbb{Z}_+$ there is an $m \in \mathbb{R}$ such that $\mathcal{F}^{-1}_{\eta\mapsto y-y'} a(y,y',\eta)$ is of class $C^s_{loc}(U \times U, \mathcal{L}_b(L \to V))$. In fact, it suffices to choose $m < -s - q - R^{(L)} - R^{(V)}$ for the constants $R^{(L)}$, $R^{(V)}$ from (4.4).

It follows that the operators in $\mathcal{L}^{-\infty}(U, \mathcal{L}(L \to V)) = \cap_{m\in\mathbb{R}} \mathcal{L}^m(U, \mathcal{L}(L \to U))$ have Schwartz kernels in $C^\infty_{loc}(U \times U, \mathcal{L}(L \to V))$. Conversely, if $K(y,y')$ is a kernel in $C^\infty_{loc}(U \times U, \mathcal{L}(L \to V))$, then the associated integral operator can be written as $A = \mathrm{op}(a)$ for an $a(y,y',\eta) \in S^{-\infty}(U \times U \times \mathbb{R}^q, \mathcal{L}(L \to V))$. It is sufficient to set $a(y,y',\eta) = (2\pi)^q e^{-\sqrt{-1}\langle y-y',\eta\rangle} K(y,y')\omega(\eta)$ with any $\omega \in C^\infty_{comp}(\mathbb{R}^q)$ satisfying $\int \omega(\eta)d\eta = 1$. We have thus proved the following result.

Lemma 5.1 *The space $\mathcal{L}^{-\infty}(U, \mathcal{L}(L \to V))$ coincides with the space of all integral operators of the form $Au = \int K(.,y')u(y')dy'$, where $K(y,y') \in C^\infty_{loc}(U \times U, \mathcal{L}(L \to V))$.*

We endow $\mathcal{L}^{-\infty}(U, \mathcal{L}(L \to V))$ with the Fréchet topology of the kernel space $C^\infty_{loc}(U \times U, \mathcal{L}_b(L \to V))$.

When interpreted in the sense of oscillatory integrals, the operator-valued function $\mathcal{F}^{-1}_{\eta\mapsto y-y'} a(y,y',\eta)$ is easily seen to be smooth away from the diagonal Δ of $U \times U$. It follows that sing supp $K_A \subset \Delta$.

An operator $A \in \mathcal{L}^m(U, \mathcal{L}(L \to V))$ is called *property supported* if, for arbitrary compact sets $K_1, K_2 \subset U$, both $(K_1 \times U) \cap \operatorname{supp} K_A$ and $(U \times K_2) \cap \operatorname{supp} K_A$ are compact sets in $U \times U$.

If $A \in \mathcal{L}^m(U, \mathcal{L}(L \to V))$ is property supported, then it extends by duality to continuous mappings

$$A: \mathcal{E}'(U,L) \to \mathcal{E}'(U,V),$$
$$A: \mathcal{D}'(U,L) \to \mathcal{D}'(U,V).$$

Lemma 5.2 *Every operator $A \in \mathcal{L}^m(U, \mathcal{L}(L \to V))$ can be written as $A = A' + A''$, where $A' \in \mathcal{L}^m(U, \mathcal{L}(L \to V))$ is property supported and $A'' \in L^{-\infty}(U, \mathcal{L}(L \to V))$ is smoothing.*

Proof. Indeed, pick a properly supported C^∞ function χ on $U \times U$, such that $\chi \equiv 1$ in a neighborhood of the diagonal Δ. Then

$$\begin{aligned} A' &= \mathrm{op}(\chi a), \\ A'' &= \mathrm{op}((1-\chi)a) \end{aligned}$$

fill the bill.

□

Thus, given any $m \in \mathbb{R}$, we have exact sequence

$$0 \to \frac{S^m(U \times U \times \mathbb{R}^n, \mathcal{L}(L \to V))}{S^{-\infty}(U \times U \times \mathbb{R}^n, \mathcal{L}(L \to V))} \to \frac{\mathcal{L}^m(U, \mathcal{L}(L \to V))}{\mathcal{L}^{-\infty}(U, \mathcal{L}(L \to V))} \to 0, \tag{5.2}$$

and each equivalence class in $\frac{\mathcal{L}^m(U,\mathcal{L}(L\to V))}{\mathcal{L}^{-\infty}(U,\mathcal{L}(L\to V))}$ has a property supported representative.

If $A \in \mathcal{L}^m(U,\mathcal{L}(L \to V))$ is property supported, then we may consider the operator-valued function $a(y,\eta) = e^{-\sqrt{-1}\langle y,\eta\rangle} A e^{\sqrt{-1}\langle \cdot,\eta\rangle}$, for $(y,\eta) \in T^*(U)$. It turns out that $a(y,\eta) \in S^m(T^*(U),\mathcal{L}(L\to V))$ and $A = \mathrm{op}(a)$ modulo $\mathcal{L}^{-\infty}(U,\mathcal{L}(L\to V))$. In other words, every equivalence class in $\frac{S^m(U\times U\times\mathbb{R}^n,\mathcal{L}(L\to V))}{S^{-\infty}(U\times U\times\mathbb{R}^n,\mathcal{L}(L\to V))}$ has a representative which is independent of y'.

We will say that $a(y,\eta) \in S^m(T^*(U),\mathcal{L}(L\to V))$ is a *complete symbol* of $A \in \mathcal{L}^m(U,\mathcal{L}(L\to V))$ if $A = \mathrm{op}(a)$ modulo $\mathcal{L}^{-\infty}(U,\mathcal{L}(L\to V))$. It follows from the above that every pseudodifferential operator A has a complete symbol. Moreover, every operator $A \in \mathcal{L}^m_{cl}(U,\mathcal{L}(L\to V))$ has a complete symbol in $S^m_{cl}(T^*(U),\mathcal{L}(L\to V))$.

The subspace $\mathcal{L}^m_\sigma(U,\mathcal{L}(L\to V))$ of $\mathcal{L}^m(U,\mathcal{L}(L\to V))$, consisting of operators supported in a proper closed set $\sigma \subset U\times U$ containing Δ, can be given the topology induced by the mapping $A \mapsto e^{-\sqrt{-1}\langle y,\eta\rangle} A e^{\sqrt{-1}\langle\cdot,\eta\rangle}$. Then, the whole space $\mathcal{L}^m(U,\mathcal{L}(L\to V))$ is endowed with the Fréchet topology of the non-direct sum $\mathcal{L}^m_\sigma(U,\mathcal{L}(L\to V)) + \mathcal{L}^{-\infty}(U,\mathcal{L}(L\to V))$.

6 *Green symbols*

The calculus of pseudodifferential operators on a manifold with edges (cf. Schulze [8]) is organized close to the edges in local coordinates $y \in U$, with U an open subset of $\mathbb{R}^q$, as the Fourier pseudodifferential calculus along U with operator-valued symbols acting in the weighted Sobolev spaces on the stretched model cone $X^\square = \overline{\mathbb{R}}_+ \times X$ of the wedge. These operator-valued symbols are thus the families $a(y,y',\eta) \in S^m(U\times U\times\mathbb{R}^q,\mathcal{L}(\mathcal{K}^{s,\gamma}(X^\square)\to\mathcal{K}^{s-m,\delta}(X^\square))$ which are pointwise elements of the cone algebra with asymptotics (cf. Schulze [10]). The cone operators are defined as sums like $a_\mathcal{M} + a_\mathcal{F} + s_\mathcal{M} + g$, with $a_\mathcal{M}$ a Mellin operator near $t=0$, $a_\mathcal{F}$ a Fourier operator away from $t=0$, $s_\mathcal{M}$ a smoothing Mellin operator, and g a Green operator. Now these items depend on $(y,y',\eta) \in U\times U\times\mathbb{R}^q$.

We discuss the parameter-dependent Green operators as particular classical operator-valued symbols to be called here Green edge symbols. As described above, this notation is motivated by the structure of Green's function of a classical elliptic boundary value problem. The Green function is, up to a fundamental solution of the elliptic operator, a pseudodifferential operator with a special Green symbol along the boundary. In this interpretation the boundary corresponds to the edge and the inner normal of the boundary to the model cone of the "wedge."

To this end, fix a weight datum $\mathfrak{d}' = \left(\delta - \frac{n}{2}, (-l,0]\right)$. Recall that, for an asymptotic type $as' \in \mathrm{As}(\mathfrak{d}')$, the space $\mathcal{K}^{s-m,\delta}_{as'}(X^\square)$ can be written as a projective limit of Banach spaces invariant under κ_λ. This is also the case for $\mathcal{K}^{\infty,\delta}_{as'}(X^\square)$, which gives us the symbol spaces $S^m(U\times\mathbb{R}^q,\mathcal{L}(\mathcal{K}^{s,\gamma}(X^\square)\to\mathcal{K}^{\infty,\delta}_{as'}(X^\square))$, with U an open set in $\mathbb{R}^N$.

Given any element $a \in \mathcal{L}(\mathcal{K}^{s,\gamma}(X^\square)\to\mathcal{K}^{t,\delta}(X^\square))$, we can define the transpose a' as an element of $\mathcal{L}(\mathcal{K}^{-t,-\delta}(X^\square)\to\mathcal{K}^{-s,-\gamma}(X^\square))$ via the non-degenerate pairings $\mathcal{K}^{s,\gamma}(X^\square)\times\mathcal{K}^{-s,-\gamma}(X^\square)\to\mathbb{C}$ induced by the inner product in $\mathcal{K}^{0,0}(X^\square)$. (Namely, we set $(a\,u,\overline{v})_{\mathcal{K}^{0,0}(X^\square)} = (u,\overline{a'v})_{\mathcal{K}^{0,0}(X^\square)}$ for all $u,v \in C^\infty_{comp}(X^\square)$. [3]) Thus, to each $a(y,\eta) \in S^m_{cl}(U \times \mathbb{R}^q, \mathcal{L}(\mathcal{K}^{s,\gamma}(X^\square) \to \mathcal{K}^{\infty,\delta}_{as'}(X^\square))$ there corresponds pointwise the transpose $a'(y,\eta)$, and we may demand that this operator-valued function belong to $S^m_{cl}(U \times \mathbb{R}^q, \mathcal{L}(\mathcal{K}^{s,-\delta}(X^\square) \to \mathcal{K}^{\infty,-\gamma}_{as''}(X^\square))$ for some other asymptotic type $as'' \in \mathrm{As}(\mathfrak{d}'')$, where $\mathfrak{d}'' = \left(-\gamma - \frac{n}{2}, (-l,0]\right)$.

Since we are again aimed at the analysis near $t=0$, we shall replace $\mathcal{K}^{\infty,\delta}_{as'}(X^\square)$ and $\mathcal{K}^{\infty,-\gamma}_{as''}(X^\square)$ by the subspaces

$$\begin{aligned} \mathcal{S}^{\delta}_{as'}(X^\square) &= \omega\, \mathcal{K}^{\infty,\delta}_{as'}(X^\square) + (1-\omega)\, \mathcal{S}(X^\square), \\ \mathcal{S}^{-\gamma}_{as''}(X^\square) &= \omega\, \mathcal{K}^{\infty,-\gamma}_{as''}(X^\square) + (1-\omega)\, \mathcal{S}(X^\square) \end{aligned}$$

respectively, where $\omega(t)$ is a cut-off function and $\mathcal{S}(X^\square) = \mathcal{S}(\overline{\mathbb{R}}_+, C^\infty(X))$. It is easily seen that $\mathcal{S}^{\delta}_{as'}(X^\wedge)$ and $\mathcal{S}^{-\gamma}_{as''}(X^\wedge)$ are independent of the concrete choice of ω.

Definition 6.1 *An operator-valued symbol* $\begin{pmatrix} g(y,\eta) & 0 \\ 0 & 0 \end{pmatrix}$ *in* $\mathrm{ES}^m(U \times \mathbb{R}^q, \mathfrak{d})$ *is called a Green edge symbol of order* m *with asymptotics if there are asymptotic types*

$$\begin{aligned} as' &\in \mathrm{As}(\delta - \tfrac{n}{2}, (-l,0]), \\ as'' &\in \mathrm{As}(-\gamma - \tfrac{n}{2}, (-l,0]) \end{aligned} \tag{6.1}$$

such that

$$\begin{aligned} g(y,\eta) &\in \cap_{s\in\mathbb{R}} S^m_{cl}(U \times \mathbb{R}^q, \mathcal{L}(\mathcal{K}^{s,\gamma}(X^\square) \to \mathcal{S}^{\delta}_{as'}(X^\square))), \\ g'(y,\eta) &\in \cap_{s\in\mathbb{R}} S^m_{cl}(U \times \mathbb{R}^q, \mathcal{L}(\mathcal{K}^{s,-\delta}(X^\square) \to \mathcal{S}^{-\gamma}_{as''}(X^\square))). \end{aligned}$$

Then, a *weight datum* $\mathfrak{d} = \left(-\gamma - \frac{n}{2}, \delta - \frac{n}{2}, (-l,0]\right)$ consists of real numbers $-\gamma - \frac{n}{2}$ and $\delta - \frac{n}{2}$ and an interval $(-l,0]$ with $0 < l \le \infty$.

For such a weight datum $\mathfrak{d}$, denote by $\mathrm{ES}^m_G(U \times \mathbb{R}^q, \mathfrak{d})$ the set of all Green edge symbols of order m and with asymptotic types satisfying (6.1).

It is worth pointing out here that the space $\mathrm{ES}^m_G(U \times \mathbb{R}^q, \mathfrak{d})$ is closed under multiplication from both the right and the left by functions in $C^\infty_{loc}(U, C^\infty_{comp}(X^\square))$ as well as by powers t^p, $p \in \mathbb{Z}_+$ (cf. Schulze [14, 3.3.1]).

Remark 6.2 *Smoothing Green operators are compact.*

7 *Edge conditions*

For studying elliptic pseudodifferential operators on manifolds with edges it will be necessary to formulate additional conditions of trace and potential type with respect to the edges. They will have the form of pseudodifferential operators with operator-valued symbols which are similar to the Green symbols from Definition 6.1.

To discuss this symbols let us agree to consider the finite-dimensional spaces $\mathbb{C}^{d_\pm}$ with the identity group actions.

[3]The space of Green edge symbols seems to depend on the particular choice of the scalar product in $\mathcal{K}^{0,0}(X^\square)$.

Definition 7.1 *An operator-valued symbol* $\begin{pmatrix} t(y,\eta) & 0 \\ 0 & 0 \end{pmatrix}$ *in* $\mathrm{ES}^m(U \times \mathbb{R}^q, \mathfrak{d})$ *is called a trace edge symbol of order* m *with asymptotics if there is an asymptotic type*

$$as'' \in As(-\gamma - \frac{n}{2}, (-l, 0])$$

such that

$$\begin{aligned} t(y,\eta) &\in \cap_{s\in\mathbb{R}} S^m_{cl}(U \times \mathbb{R}^q, \mathcal{L}(\mathcal{K}^{s,\gamma}(X^\square) \to \mathbb{C}^{d_+})), \\ t'(y,\eta) &\in S^m_{cl}(U \times \mathbb{R}^q, \mathcal{L}(\mathbb{C}^{d_+} \to \mathcal{S}^{-\gamma}_{as''}(X^\square))). \end{aligned}$$

For a weight datum $\mathfrak{d} = \left(-\gamma - \frac{n}{2}, \delta - \frac{n}{2}, (-l, 0]\right)$, denote by $\mathrm{ES}^m_T(U \times \mathbb{R}^q, \mathfrak{d})$ the subspace of $\mathrm{ES}^m(U \times \mathbb{R}^q, \mathfrak{d})$ consisting of all trace edge symbols with asymptotics.

Note that it is typical for the edge pseudodifferential calculus that the trace objects occur in integral form, in contrast to the case of standard boundary value problems. In other words, the traces which restrict the argument functions to the edges, possibly after differentiating them with respect to the t-variable (that is those of the form $t(y,\eta)u = c_j D_t^j u\,|_{t=0}$, with $j \in \mathbb{Z}_+$) do not belong to the trace operators here. This would be impossible any way, because in elliptic edge problems we cannot expect the solutions to have such traces on the edges. We will get in fact more general (e.g. discrete) asymptotics that are just the reason for our framework with arbitrary asymptotic types.

Definition 7.2 *An operator-valued symbol* $\begin{pmatrix} 0 & p(y,\eta) \\ 0 & 0 \end{pmatrix}$ *in* $\mathrm{ES}^m(U \times \mathbb{R}^q, \mathfrak{d})$ *is called a potential edge symbol of order* m *with asymptotics if there is an asymptotic type*

$$as' \in As(\delta - \frac{n}{2}, (-l, 0])$$

such that

$$\begin{aligned} p(y,\eta) &\in S^m_{cl}(U \times \mathbb{R}^q, \mathcal{L}(\mathbb{C}^{d_-} \to \mathcal{S}^{\delta}_{as'}(X^\square))), \\ p'(y,\eta) &\in \cap_{s\in\mathbb{R}} S^m_{cl}(U \times \mathbb{R}^q, \mathcal{L}(\mathcal{K}^{s,-\delta}(X^\square) \to \mathbb{C}^{d_-})). \end{aligned}$$

Given a weight datum $\mathfrak{d} = \left(-\gamma - \frac{n}{2}, \delta - \frac{n}{2}, (-l, 0]\right)$, denote by $\mathrm{ES}^m_P(U \times \mathbb{R}^q, \mathfrak{d})$ the subspace of $\mathrm{ES}^m(U \times \mathbb{R}^q, \mathfrak{d})$ consisting of all potential edge symbols with asymptotics.

It follows directly from these definitions that $\left(\begin{smallmatrix} 0 & p \\ 0 & 0 \end{smallmatrix}\right)$ is a potential edge symbol if and only if the transpose $\left(\begin{smallmatrix} 0 & 0 \\ p' & 0 \end{smallmatrix}\right)$ is a trace edge symbol. For this reason, potential edge symbols are sometimes called corestriction edge symbols.

The composition of a potential edge symbol and a trace edge symbol is a Green edge symbol.

For simplicity of notation, we use the same names 'Green,' 'trace' and 'potential' operators for the entries $g(y,\eta)$, $t(y,\eta)$ and $p(y,\eta)$, too, tacitly identifying them with the corresponding matrices.

We are now in a position to describe the edge conditions which are substitutes of boundary conditions for the case of edge problems. These are Fourier pseudodifferential operators along the edge with operator-valued symbols

$$a(y,\eta)=\begin{pmatrix} g(y,\eta) & p(y,\eta) \\ t(y,\eta) & e(y,\eta) \end{pmatrix}: \begin{matrix} \mathcal{K}^{s,\gamma}(X^\square) \\ \oplus \\ \mathbb{C}^{d_-} \end{matrix} \to \begin{matrix} \mathcal{K}^{s-m,\delta}(X^\square) \\ \oplus \\ \mathbb{C}^{d_+} \end{matrix}, \quad (y,\eta)\in U\times\mathbb{R}^q, \tag{7.1}$$

where $g(y,\eta)$ is a Green edge symbol, $t(y,\eta)$ a trace edge symbol, $p(y,\eta)$ a potential edge symbol, and $e(y,\eta) \in S^m_{cl}(U\times\mathbb{R}^q, \mathcal{L}(\mathbb{C}^{d_-}\to\mathbb{C}^{d_+}))$ is a $(d_+\times d_-)$-matrix of scalar classical symbols of order m along the edge.

Using edge conditions (7.1), one corrects the edge symbols originated with interior symbols of elliptic operators in the wedge, thus arriving at the isomorphism of the singular symbol.

8 *An example*

Before giving a kernel characterization of Green symbols, let us have look at the model situation.

Let $\gamma=\delta=0$ and $l=N+1>0$, and let $\dim X=0$ that corresponds to the case of boundary value problems.

Consider the asymptotic type $as=(\sigma,\Sigma)$, where

- $\sigma=(0,-1,-N)$;
- Σ is the space of analytic functionals of the form $\langle f,u\rangle=\sum_{\nu=0}^N f_\nu\, u(-\nu)$, for $u\in \mathrm{Hol}\,(\sigma)$, with f_ν a complex number.

If $f\in\Sigma$, then $\langle f,t^{-z}\rangle=\sum_{\nu=0}^N f_\nu\, t^\nu$. Since each function $u\in C^N_{loc}(\overline{\mathbb{R}}_+)$ can be written in the form

$$u(t)=\sum_{\nu=0}^N \frac{\partial^\nu u(0)}{\nu!}\,t^\nu+u'(t),$$

where $u'\in C^N_{loc}(\overline{\mathbb{R}}_+)$ vanishes up to order N at $t=0$ (in other words, u' is N-flat at $t=0$), such asymptotics are called the *Taylor asymptotics* of order N.

Hence it follows that $\mathfrak{A}_{as}(\overline{\mathbb{R}}_+)$ is the space of Taylor asymptotics of order N, and $\mathcal{S}^0_{as}(\overline{\mathbb{R}}_+)=\mathcal{S}(\overline{\mathbb{R}}_+)$ provided $N=\infty$.

This also suggests that the coefficients $f_{\nu,j}(x)$ in (3.3) play the role of traces at $t=0$ in general.

For the weight datum $\mathfrak{d}=(0,0,(-\infty,0])$, the space $\mathrm{ES}^m_G(U\times\mathbb{R}^q,\mathfrak{d})$ consists of all matrices $\begin{pmatrix} g(y,\eta) & 0 \\ 0 & 0\end{pmatrix}$ such that

$$\begin{aligned} g(y,\eta) &\in \cap_{s\in\mathbb{R}} S^m_{cl}(U\times\mathbb{R}^q,\mathcal{L}(\mathcal{K}^{s,0}(\overline{\mathbb{R}}_+)\to\mathcal{S}(\overline{\mathbb{R}}_+))), \\ g'(y,\eta) &\in \cap_{s\in\mathbb{R}} S^m_{cl}(U\times\mathbb{R}^q,\mathcal{L}(\mathcal{K}^{s,0}(\overline{\mathbb{R}}_+)\to\mathcal{S}(\overline{\mathbb{R}}_+))). \end{aligned}$$

The latter conditions are easily verified to amount to saying that

$$\begin{aligned} g(y,\eta) &\in S^m_{cl}(U\times\mathbb{R}^q,\mathcal{L}(L^2(\overline{\mathbb{R}}_+)\to\mathcal{S}(\overline{\mathbb{R}}_+))), \\ g'(y,\eta) &\in S^m_{cl}(U\times\mathbb{R}^q,\mathcal{L}(L^2(\overline{\mathbb{R}}_+)\to\mathcal{S}(\overline{\mathbb{R}}_+))). \end{aligned}$$

Note that these symbols just coincide with the singular Green symbols of type zero in the boundary symbolic algebra for pseudodifferential boundary value problems in Boutet de Monvel's class, cf. Schulze [12] and Schrohe [7].

More general singular Green symbols of type J in Boutet de Monvel's algebra have the form $\sum_{j=0}^{J} g_j(y,\eta)\, D_t^j$, with $g_j(y,\eta)$ a singular Green symbol of order $m-j$ and type 0. They are considered to act from $H^s(\overline{\mathbb{R}}_+)$, $s > J - \frac{1}{2}$, to $H^{s-J}(\overline{\mathbb{R}}_+)$. In the case of non-trivial model cone, a substitute for D_t is the totally characteristic derivative $(-t\partial_t)$. Thus, we might consider more general edge symbols $\sum_{j=0}^{J} g_j(y,\eta)\,(-t\partial_t)^j$, with $g_j(y,\eta)$ a Green edge symbol of order $m-j$ with asymptotics. However, these are not covered by Definition 6.1, though $(-t\partial_t)$ does not shift the weight.

We note that the space of all operators $g \in \mathcal{L}(L^2(\overline{\mathbb{R}}_+) \to \mathcal{S}(\overline{\mathbb{R}}_+))$, whose transposes are in $\mathcal{L}(L^2(\overline{\mathbb{R}}_+) \to \mathcal{S}(\overline{\mathbb{R}}_+))$, coincides with the set of all integral operators $(gu)(t) = \int_0^\infty K_g(t,t')u(t')dt'$, $u \in L^2(\overline{\mathbb{R}}_+)$, where $K_g \in \mathcal{S}(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+)$.

9 *Kernel characterization of Green symbols*

In the sequel, let

$$\begin{array}{rcl} as' & \in & \mathrm{As}(\delta - \frac{n}{2}, (-l, 0]), \\ as'' & \in & \mathrm{As}(-\gamma - \frac{n}{2}, (-l, 0]) \end{array}$$

stand for two fixed asymptotic types.

Lemma 9.1 *For each operator* $g : \mathcal{K}^{0,\gamma}(X^\square) \to \mathcal{K}^{-m,\delta}(X^\square)$, *the following conditions are equivalent:*

1) The mappings

$$\begin{array}{rcl} g & : & \mathcal{K}^{s,\gamma}(X^\square) \to \mathcal{S}^{\delta}_{as'}(X^\square), \\ g' & : & \mathcal{K}^{s,-\delta}(X^\square) \to \mathcal{S}^{-\gamma}_{as''}(X^\square) \end{array}$$

are continuous for all $s \in \mathbb{R}$.

2) The mappings

$$\begin{array}{rcl} g & : & \mathcal{K}^{0,\gamma}(X^\square) \to \mathcal{S}^{\delta}_{as'}(X^\square), \\ g' & : & \mathcal{K}^{0,-\delta}(X^\square) \to \mathcal{S}^{-\gamma}_{as''}(X^\square) \end{array}$$

are continuous.

Proof. Only the implication 2) $\Rightarrow$ 1) requires a proof. To do this we pick a g satisfying condition 2).

Since $\mathcal{K}^{s,\gamma}(X^\square)$ is a Hilbert space with dual $\mathcal{K}^{-s,-\gamma}(X^\square)$ and $\mathcal{S}^{\delta}_{as'}(X^\square)$ is a nuclear Fréchet space, it follows from $g \in \mathcal{L}(\mathcal{K}^{0,\gamma}(X^\square) \to \mathcal{S}^{\delta}_{as'}(X^\square))$ that g can be represented by a kernel

$$K_g \in \mathcal{S}^{\delta}_{as'}(X^\square) \otimes_\pi \mathcal{K}^{0,-\gamma}(X^\square)$$

(cf. Theorem 1.4.2 in [15]).

The same reasoning, when applied to $g' \in \mathcal{L}(\mathcal{K}^{0,-\delta}(X^\square) \to \mathcal{S}^{-\gamma}_{as''}(X^\square))$, yields

$$K_{g'} \in \mathcal{S}^{-\gamma}_{as''}(X^\square) \otimes_\pi \mathcal{K}^{0,\delta}(X^\square).$$

Since $K_g(t,x,t',x') = K_{g'}(t',x',t,x)$, we can assert that

$$K_g \in \left(\mathcal{S}^{\delta}_{as'}(X^\square) \otimes_\pi \mathcal{K}^{0,-\gamma}(X^\square)\right) \cap \left(\mathcal{K}^{0,\delta}(X^\square) \otimes_\pi \mathcal{S}^{-\gamma}_{as''}(X^\square)\right).$$

On the other hand, it is easy to see that, for each $s \in \mathbb{R}$, we have

$$\begin{aligned} &\left(\mathcal{S}^{\delta}_{as'}(X^\square) \otimes_\pi \mathcal{K}^{0,-\gamma}(X^\square)\right) \cap \left(\mathcal{K}^{0,\delta}(X^\square) \otimes_\pi \mathcal{S}^{-\gamma}_{as''}(X^\square)\right) \\ &= \left(\mathcal{S}^{\delta}_{as'}(X^\square) \otimes_\pi \mathcal{K}^{-s,-\gamma}(X^\square)\right) \cap \left(\mathcal{K}^{-s,\delta}(X^\square) \otimes_\pi \mathcal{S}^{-\gamma}_{as''}(X^\square)\right). \end{aligned} \quad (9.1)$$

Repeated application of the above arguments in reverse order shows now that the statement 1) holds, as required.

□

Since s is arbitrary in (9.1), we may take $-s$ as large as we like. Then, (9.1) implies the following result.

Corollary 9.2 *Under condition 2) of Lemma 9.1, we have* $K_g \in C^\infty_{loc}((\mathbb{R}_+ \times X) \times (X \times \mathbb{R}_+))$.

Now, we return to the equality (9.1). It is to be expected that the intersections in question are in fact equal to the tensor product $\mathcal{S}^{\delta}_{as'}(X^\square) \otimes_\pi \mathcal{S}^{-\gamma}_{as''}(X^\square)$. This question can be treated in the more general framework of topological tensor products. Let L, V be Fréchet spaces and let L_0, V_0 be nuclear Fréchet spaces with continuous embeddings $L_0 \hookrightarrow L$, $V_0 \hookrightarrow V$. Is it true that $(V_0 \otimes_\pi L) \cap (L_0 \otimes_\pi V) = V_0 \otimes_\pi L_0$? The equality would extend the well-known result (*Schwartz*'s *lemma*) that if a function $k(z,z')$ can be infinitely many times differentiated separately in z and z', then also the mixed derivatives exist. For this reason, the answer seems to be negative in general [4].

Lemma 9.3 *For each numbers* $\gamma, \delta \in \mathbb{R}$ *and asymptotic types* as' *and* as'', *it follows that*

$$\left(\mathcal{S}^{\delta}_{as'}(X^\square) \otimes_\pi \mathcal{K}^{0,-\gamma}(X^\square)\right) \cap \left(\mathcal{K}^{0,\delta}(X^\square) \otimes_\pi \mathcal{S}^{-\gamma}_{as''}(X^\square)\right) = \mathcal{S}^{\delta}_{as'}(X^\square) \otimes_\pi \mathcal{S}^{-\gamma}_{as''}(X^\square).$$

In the case of discrete asymptotic types, this result is proved by Rempel and Schulze (cf. Subsection 1.2.2 in [6]).

From Lemma 9.3 we deduce that each Green edge symbol $g(y,\eta)$ of order m and with asymptotic types as', as'' is represented by a unique kernel function $K_g(y,\eta;t,x,t',x')$ in $C^\infty_{loc}(U \times \mathbb{R}^q, \mathcal{S}^{\delta}_{as'}(X^\square) \otimes_\pi \mathcal{S}^{-\gamma}_{as''}(X^\square))$, such that

$$g(y,\eta)u\,(t,x) = \int_{X^\square} K_g(y,\eta;t,x,t',x')u(t',x')t'^n dt'dx' \quad \text{for} \quad u \in \mathcal{K}^{0,\gamma}(X^\square), \quad (9.2)$$

[4]Some suggestive evidence to the negative answer is that $(\mathcal{E} \otimes_\pi \mathcal{D}') \cap (\mathcal{D}' \otimes_\pi \mathcal{E}) \neq \mathcal{E} \otimes_\pi \mathcal{E}$, for the left side contains the kernels of all pseudodifferential operators on the manifold in question.

where dx is the Riemannian volume element on X.

Given a Green edge symbol $g(y,\eta)$ of order m, let us have a look at the sequence of homogeneous components of g. Denote by $g_j(y,\eta)$ the homogeneous component of $g(y,\eta)$ of degree j, $j \le m$. Then, $g_j(y,\eta)$ is uniquely determined by its restriction to the unit sphere in $\mathbb{R}^q$ via

$$g_j(y,\eta) = |\eta|^j\, \kappa_{|\eta|}\, g_j\left(y, \frac{\eta}{|\eta|}\right) \kappa_{|\eta|^{-1}}, \quad \eta \in \mathbb{R}^q \setminus \{0\}. \tag{9.3}$$

On the other hand, the above argument yields a kernel $K_{g_j}(y,\eta;t,x,t',x')$ in $C^\infty_{loc}(U \times S^{q-1}, \mathcal{S}^\delta_{as'}(X^\square) \otimes_\pi \mathcal{S}^{-\gamma}_{as''}(X^\square))$ for the restriction of $g_j(y,\eta)$ to the unit sphere $S^{q-1} = \{\eta \in \mathbb{R}^q : |\eta| = 1\}$. Hence it follows, by (9.3), that

$$\begin{aligned} g_j(y,\eta)u(t,x) &= |\eta|^j \int_{X^\square} K_{g_j}\left(y, \frac{\eta}{|\eta|}; t|\eta|, x, t', x'\right) u\left(\frac{t'}{|\eta|}, x'\right) t'^n dt' dx' \\ &= \int_{X^\square} |\eta|^{j+n+1} K_{g_j}\left(y, \frac{\eta}{|\eta|}; t|\eta|, x, t'|\eta|, x'\right) u(t',x') t'^n dt' dx'. \end{aligned}$$

Moreover, the kernel

$$|\eta|^{j+n+1}\, K_{g_j}\left(y, \frac{\eta}{|\eta|}; t, x, t', x'\right)$$

is homogeneous of order $j+n+1$ in η, and so, when multiplied by an excision function $\chi(\eta)$, belongs to

$$S^{j+n+1}_{cl}(U \times \mathbb{R}^q, \mathcal{S}^\delta_{as'}(X^\square) \otimes_\pi \mathcal{S}^{-\gamma}_{as''}(X^\square)) = S^{j+n+1}_{cl}(U \times \mathbb{R}^q) \otimes_\pi (\mathcal{S}^\delta_{as'}(X^\square) \otimes_\pi \mathcal{S}^{-\gamma}_{as''}(X^\square)),$$

the space $\mathcal{S}^\delta_{as'}(X^\square) \otimes_\pi \mathcal{S}^{-\gamma}_{as''}(X^\square))$ being endowed with the identity action. (We have used the fact that the space of classical symbols is nuclear.) Thus, we are allowed to take an asymptotic sum of these kernels within the symbol space $S^{m+n+1}_{cl}(U \times \mathbb{R}^q, \mathcal{S}^\delta_{as'}(X^\square) \otimes_\pi \mathcal{S}^{-\gamma}_{as''}(X^\square))$.

After this motivation, we are able to formulate our main result, which makes more precise the structure of kernels (9.2) of Green edge symbols with asymptotics.

Theorem 9.4 *For each* $k \in S^{m+n+1}_{cl}(U \times \mathbb{R}^q) \otimes_\pi \left(\mathcal{S}^\delta_{as'}(X^\square) \otimes_\pi \mathcal{S}^{-\gamma}_{as''}(X^\square)\right)$, *the operator family*

$$g_k(y,\eta) : u(t,x) \to \int_{X^\square} k(y,\eta; t\langle\eta\rangle, x, t'\langle\eta\rangle, x')\, u(t',x')\, t'^n dt' dx', \quad u \in \mathcal{K}^{0,\gamma}(X^\square), \tag{9.4}$$

is a Green edge symbol of order m with asymptotics as' and as''. Conversely, every Green edge symbol of order m with asymptotics as' and as'' has such a representation.

Proof. Let us first assume $g_k(y,\eta)$ to be of the form (9.4) with a kernel $k(y,\eta;t,x,t',x')$ in $S^{m+n+1}_{cl}(U \times \mathbb{R}^q) \otimes_\pi \left(\mathcal{S}^\delta_{as'}(X^\square) \otimes_\pi \mathcal{S}^{-\gamma}_{as''}(X^\square)\right)$. Then,

$$\begin{aligned} \kappa_{\langle\eta\rangle^{-1}} g_k(y,\eta) \kappa_{\langle\eta\rangle} u(t,x) &= \int_{X^\square} k(y,\eta; t, x, t'\langle\eta\rangle, x')\, u(t'\langle\eta\rangle, x')\, t'^n dt' dx' \\ &= \frac{1}{\langle\eta\rangle^{n+1}} \int_{X^\square} k(y,\eta;t,x,t',x')\, u(t',x')\, t'^n dt' dx'. \end{aligned}$$

By a familiar argument of topological tensor product, we may write the kernel $k(y,\eta;t,x,t',x')$ as a convergent sum

$$k(y,\eta;t,x,t',x') = \sum_{j=1}^{\infty} c_j\, a^{(j)}(y,\eta) \otimes k^{(j)}(t,x,t',x'),$$

where

$$\begin{array}{rclr} a^{(j)}(y,\eta) & \in & S_{cl}^{m+n+1}(U \times \mathbb{R}^q), & a^{(j)} \to 0 \ \text{ as } \ j \to \infty, \\ k^{(j)}(t,x,t',x') & \in & \mathcal{S}_{as'}^{\delta}(X^{\square}) \otimes_{\pi} \mathcal{S}_{as''}^{-\gamma}(X^{\square}), & k^{(j)} \to 0 \ \text{ as } \ j \to \infty, \end{array}$$

and

$$c_j \ \in \ \mathbb{C}, \qquad \textstyle\sum_{j=1}^{\infty} |c_j| < \infty.$$

By the above, the space $\mathcal{S}_{as'}^{\delta}(X^{\square})$ can be represented as the projective limit of a sequence of Banach spaces (V_ν). Then, for the operators

$$T_j u\,(t,x) = \int_{X^{\square}} k^{(j)}(t,x,t',x')\, u(t',x')\, t'^n dt'dx', \quad u \in \mathcal{K}^{0,\gamma}(X^{\square}),$$

we obtain

$$\|T_j\|_{\mathcal{L}(\mathcal{K}^{0,\gamma}(X^{\square}) \to V_\nu)} \to 0, \quad \text{as} \quad j \to \infty,$$

where $\nu = 1, 2, \ldots$. Moreover, using symbol estimates for $a^{(j)}(y,\eta)$, we can assert that

$$\|a^{(j)}(y,\eta)\, T_j\|_{\mathcal{L}(\mathcal{K}^{0,\gamma}(X^{\square}) \to V_\nu)} \le C_j\, \langle\eta\rangle^{m+n+1}\, \|T_j\|_{\mathcal{L}(\mathcal{K}^{0,\gamma}(X^{\square}) \to V_\nu)}$$

for $y \in K \subset\subset U$, with constants $C_j \to 0$ when $j \to \infty$. Hence it follows that

$$\|\kappa_{\langle\eta\rangle^{-1}}\, g_k(y,\eta)\, \kappa_{\langle\eta\rangle}\|_{\mathcal{L}(\mathcal{K}^{0,\gamma}(X^{\square}) \to V_\nu)} \le c\, \langle\eta\rangle^m$$

for all $y \in K$ and $\eta \in \mathbb{R}^q$, with c a constant depending on K, and for $\nu = 1, 2, \ldots$.

In an analogous manner we can argue for the derivatives $D_y^\alpha D_\eta^\beta g_k(y,\eta)$, where $\alpha \in \mathbb{Z}_+^N$ and $\beta \in \mathbb{Z}_+^q$, which yields the relevant estimates with $\langle\eta\rangle^{m-|\beta|}$ on the right.

For the transpose we can do the same. This finally shows that $g_k(y,\eta)$ is a Green edge symbol of order m with asymptotics as' and as'', as desired.

Conversely, let $g(y,\eta)$ be a Green edge symbol of order m with asymptotics as' and as''. Write

$$g(y,\eta) \sim \sum_{j=0}^{\infty} \chi(\eta)\, g_{m-j}(y,\eta),$$

where $g_{m-j}(y,\eta)$ is the homogeneous component of $g(y,\eta)$ of degree $m-j$ in $\eta \neq 0$ and $\chi(\eta)$ an excision function. The asymptotic sum can be carried out as a convergent sum

$$\tilde{g}(y,\eta) = \sum_{j=0}^{\infty} \chi(\varepsilon_j\, \eta)\, g_{m-j}(y,\eta),$$

with (ε_j) a sequence convergent to 0 fast enough. Then, the difference $g(y,\eta) - \tilde{g}(y,\eta)$ is a smoothing Green edge symbol with asymptotics.

As described above, each component $g_{m-j}(y,\eta)$ is defined by the unique kernel

$$|\eta|^{m-j+n+1}\, K_{g_{m-j}}\left(y,\frac{\eta}{|\eta|};t\langle\eta\rangle,x,t'\langle\eta\rangle,x'\right),$$

where $K_{g_{m-j}}(y,\eta;t,x,t',x') \in C^\infty_{loc}(U\times S^{q-1},\mathcal{S}^{\delta}_{as'}(X^\square)\otimes_\pi \mathcal{S}^{-\gamma}_{as''}(X^\square))$. We can assume, by decreasing ε_j if necessary, that $\langle\eta\rangle = |\eta|$ for $\chi(\varepsilon_j\eta)\neq 0$. Set

$$k_{m-j}(y,\eta;t,x,t',x') = \chi(\varepsilon_j\eta)\,|\eta|^{m-j+n+1}\, K_{g_{m-j}}\left(y,\frac{\eta}{|\eta|};t,x,t',x'\right);$$

then, $k_{m-j}\in S^{m-j+n+1}_{cl}(U\times\mathbb{R}^q)\otimes_\pi\left(\mathcal{S}^{\delta}_{as'}(X^\square)\otimes_\pi \mathcal{S}^{-\gamma}_{as''}(X^\square)\right)$ and

$$k_{m-j}(y,\eta;t\langle\eta\rangle,x,t'\langle\eta\rangle,x') = \chi(\varepsilon_j\eta)\,|\eta|^{m-j+n+1}\, K_{g_{m-j}}\left(y,\frac{\eta}{|\eta|};t|\eta|,x,t'|\eta|,x'\right). \tag{9.5}$$

We next claim that the series $\sum_{j=0}^\infty k_{m-j}(y,\eta;t,x,t',x')$ converges in the symbol space $S^{m+n+1}_{cl}(U\times\mathbb{R}^q)\otimes_\pi\left(\mathcal{S}^{\delta}_{as'}(X^\square)\otimes_\pi \mathcal{S}^{-\gamma}_{as''}(X^\square)\right)$ for a suitable choice of the constants ε_j. Here we may forget about the subscript '*cl*' since the summands are homogeneous of degree $m-j+n+1$ in η for $|\eta|\geq c$ and hence the convergence of the associated series of homogeneous components is trivial. Thus, letting

$$T_{m-j}(y,\eta)u\,(t,x) = \int_{X^\square} k_{m-j}(y,\eta;t,x,t',x')\,u(t',x')\,t'^n dt'dx', \quad u\in\mathcal{K}^{0,\gamma}(X^\square),$$

we see at once that the convergence of the series $\sum_{j=0}^\infty k_{m-j}(y,\eta;t,x,t',x')$ in the space $S^{m+n+1}_{cl}(U\times\mathbb{R}^q)\otimes_\pi\left(\mathcal{S}^{\delta}_{as'}(X^\square)\otimes_\pi \mathcal{S}^{-\gamma}_{as''}(X^\square)\right)$ to a limit $k(y,\eta;t,x,t',x')$ is equivalent to the convergence of the series $\sum_{j=0}^\infty T_{m-j}(y,\eta)$ in the symbol space $S^{m+n+1}(U\times\mathbb{R}^q,\mathcal{S}^{\delta}_{as'}(X^\square)\otimes_\pi \mathcal{S}^{-\gamma}_{as''}(X^\square))$ to a limit $T(y,\eta)$, where

$$T(y,\eta)u\,(t,x) = \int_{X^\square} k(y,\eta;t,x,t',x')\,u(t',x')\,t'^n dt'dx', \quad u\in\mathcal{K}^{0,\gamma}(X^\square).$$

If the spaces $\mathcal{S}^{\delta}_{as'}(X^\square)$ and $\mathcal{S}^{-\gamma}_{as''}(X^\square)$ are written as the projective limits of sequences of Banach spaces (V_ν) and (L_ν) respectively, then, by Lemma 9.3, the topology of $S^{m+n+1}(U\times\mathbb{R}^q),\mathcal{S}^{\delta}_{as'}(X^\square)\otimes \mathcal{S}^{-\gamma}_{as''}(X^\square))$ may be given by the family of seminorms

$$\begin{aligned} s'_{\alpha,\beta,K;\nu}(T) &= \sup_{(y,\eta)\in K\times\mathbb{R}^q}\langle\eta\rangle^{-m-n-1+|\beta|}\,\|D^\alpha_y D^\beta_\eta T(y,\eta)\|_{\mathcal{L}(\mathcal{K}^{0,\gamma}(X^\square)\to V_\nu)},\\ s''_{\alpha,\beta,K;\nu}(T) &= \sup_{(y,\eta)\in K\times\mathbb{R}^q}\langle\eta\rangle^{-m-n-1+|\beta|}\,\|D^\alpha_y D^\beta_\eta T(y,\eta)\|_{\mathcal{L}(\mathcal{K}^{0,-\delta}\to L_\nu)}, \end{aligned}$$

where α varies over $\mathbb{Z}^N_+$, β over $\mathbb{Z}^q_+$, K over compact subset of U, and $\nu=1,2,\ldots$. From the properties of the functions $k_{m-j}(y,\eta;t,x,t',x')$ we immediately see that the operator families $T_{m-j}(y,\eta)$ and $T'_{m-j}(y,\eta)$ are elements of

$$\begin{aligned} &S^{m-j+n+1}_{cl}\left(U\times\mathbb{R}^q,\mathcal{L}(\mathcal{K}^{0,\gamma}(X^\square)\to\mathcal{S}^{\delta}_{as'}(X^\square)),(I)\right)\\ &\quad = \text{proj }\lim_{\nu\to\infty} S^{m-j+n+1}_{cl}\left(U\times\mathbb{R}^q,\mathcal{L}(\mathcal{K}^{0,\gamma}(X^\square)\to V_\nu),(I)\right),\\ &S^{m-j+n+1}_{cl}\left(U\times\mathbb{R}^q,\mathcal{L}(\mathcal{K}^{0,-\delta}(X^\square)\to\mathcal{S}^{-\gamma}_{as''}(X^\square)),(I)\right)\\ &\quad = \text{proj }\lim_{\nu\to\infty} S^{m-j+n+1}_{cl}\left(U\times\mathbb{R}^q,\mathcal{L}(\mathcal{K}^{0,-\delta}(X^\square)\to L_\nu),(I)\right) \end{aligned}$$

respectively, all the symbol spaces being defined with respect to the identity group action in the involved spaces.

Now, the sequence $(T_{m-j}(y,\eta))$ has an asymptotic sum $T(y,\eta)$ in the space $S^{m+n+1}\left(U\times\mathbb{R}^q,\mathcal{L}(\mathcal{K}^{0,\gamma}(X^\square)\to\mathcal{S}^\delta_{as'}(X^\square)),(I)\right)$, such that at the same time $T'(y,\eta)$ is the asymptotic sum of the sequence of transposes $\left(T'_{m-j}(y,\eta)\right)$ in the space $S^{m+n+1}_{cl}\left(U\times\mathbb{R}^q,\mathcal{L}(\mathcal{K}^{0,-\delta}(X^\square)\to\mathcal{S}^{-\gamma}_{as''}(X^\square)),(I)\right)$. We obtain $T(y,\eta)$ as a convergent sum

$$T(y,\eta)=\sum_{j=0}^{\infty}\chi(\epsilon_j\eta)\,T_{m-j}(y,\eta)$$

for a suitable sequence of constants (ϵ_j) tending to 0 sufficiently fast. Moreover, when taking the asymptotic sum, we have uniqueness modulo smoothing symbols in $S^{-\infty}\left(U\times\mathbb{R}^q,\mathcal{L}(\mathcal{K}^{0,\gamma}(X^\square)\to\mathcal{S}^\delta_{as'}(X^\square)),(I)\right)$. Recall that the space of symbols of order $-\infty$ is independent of the group actions in the corresponding spaces, so that (I) is unnecessary in this case.

Taking $\epsilon_j\le\varepsilon_j$ for all j, which is an allowed choice, we then may modify the above ε_j, again, by taking them smaller, if necessary. Finally, it is possible to set $\epsilon_j=\varepsilon_j$. The convergence of the series $\sum_{j=0}^{\infty}\chi(\epsilon_j\eta)\,T_{m-j}(y,\eta)$, which relies on the systems of seminorms $s'_{\alpha,\beta,K;\nu}(\cdot)$, $s''_{\alpha,\beta,K;\nu}(\cdot)$, now corresponds exactly to the desired convergence of the series $\sum_{j=0}^{\infty}k_{m-j}(y,\eta;t,x,t',x')$ to a function $k(y,\eta;t,x,t',x')$ in the space $S^{m+n+1}_{cl}(U\times\mathbb{R}^q)\otimes_\pi\left(\mathcal{S}^\delta_{as'}(X^\square)\otimes_\pi\mathcal{S}^{-\gamma}_{as''}(X^\square)\right)$. We can now return to the original operator family $g(y,\eta)$, concluding that the difference

$$g_{-\infty}(y,\eta)=g(y,\eta)-\int_{X^\square}k(y,\eta;t\langle\eta\rangle,x,t'\langle\eta\rangle,x')\,(\cdot)\,t'^n dt'dx'$$

is a smoothing Green edge symbol with asymptotics as', as''. Indeed, by (9.5),

$$\begin{aligned}
&\int_{X^\square}k(y,\eta;t\langle\eta\rangle,x,t'\langle\eta\rangle,x')\,(\cdot)\,t'^n dt'dx'\\
&=\int_{X^\square}\sum_{j=0}^{\infty}k_{m-j}(y,\eta;t\langle\eta\rangle,x,t'\langle\eta\rangle,x')\,(\cdot)\,t'^n dt'dx'\\
&=\sum_{j=0}^{\infty}\chi(\epsilon_j\eta)\int_{X^\square}|\eta|^{m-j+n+1}K_{g_{j-m}}\left(y,\frac{\eta}{|\eta|};t|\eta|,x,t'|\eta|,x'\right)(\cdot)\,t'^n dt'dx'\\
&=\sum_{j=0}^{\infty}\chi(\epsilon_j\eta)\,g_{j-m}(y,\eta)\\
&\sim g(y,\eta),
\end{aligned}$$

as required.

To complete the proof, it remains to observe that each smoothing Green edge symbol with asymptotics as', as'' has a representation of the form

$$g_{-\infty}(y,\eta)u\,(t,x)=\int_{X^\square}k_{-\infty}(y,\eta;t\langle\eta\rangle,x,t'\langle\eta\rangle,x')\,u(t',x')\,t'^n dt'dx',\quad u\in\mathcal{K}^{0,\gamma}(X^\square),$$

with some $k_{-\infty}(y,\eta;t,x,t',x')\in S^{-\infty}(U\times\mathbb{R}^q)\otimes_\pi\left(\mathcal{S}^\delta_{as'}(X^\square)\otimes_\pi\mathcal{S}^{-\gamma}_{as''}(X^\square)\right)$.

□

10 *Kernel characterization of trace and potential symbols*

For every $k(y,\eta;t',x') \in S_{cl}^{m+\frac{n+1}{2}}(U \times \mathbb{R}^q) \otimes_\pi (\mathbb{C}^{d_+} \otimes \mathcal{S}_{as''}^{-\gamma}(X^\Box))$, the operator family

$$t_k(y,\eta):\ u(t,x) \to \int_{X^\Box} k(y,\eta;t'\langle\eta\rangle,x')\,u(t',x')\,t'^n dt'dx', \quad u \in \mathcal{K}^{0,\gamma}(X^\Box), \tag{10.1}$$

is a trace edge symbol of order m with asymptotics as''. Conversely, every trace edge symbol of order m with asymptotics as'' has such a representation.

Similarly, for every $k(y,\eta;t,x) \in S_{cl}^{m+\frac{n+1}{2}}(U \times \mathbb{R}^q) \otimes_\pi (\mathcal{S}_{as'}^{\delta}(X^\Box) \otimes \mathbb{C}^{d_-})$, the operator family

$$p_k(y,\eta):\ u \to k(y,\eta;t\langle\eta\rangle,x)\,u, \quad u \in \mathbb{C}^{d_-}, \tag{10.2}$$

is a potential edge symbol of order m with asymptotics as'. Conversely, every potential edge symbol of order m with asymptotics as' has such a representation.

The proofs of these assertions are quite analogous to the proof of Theorem 9.4 and are left to the reader.

In particular, we obtain from this characterization that, if p is a potential edge symbol of order m_1 with asymptotics as' and t is a trace edge symbol of order m_2 with asymptotics as'', then pt is a $(d_+ \times d_-)$-matrix of Green edge symbols of order $m_1 + m_2$ with asymptotics as' and as''.

Conversely, every matrix-valued Green edge symbol $g(y,\eta)$ of order m with asymptotics as', as'' can be written in the form

$$g(y,\eta) = \sum_j c_j\, p_j(y,\eta)\, t_j(y,\eta),$$

where $p_j \to 0$ are potential edge symbols of order $\frac{m}{2}$, $t_j \to 0$ trace edge symbols of order $\frac{m}{2}$, and $\sum_j |c_j| < \infty$ (cf. Egorov and Schulze [3, 7.2.1]).

11 *Some applications*

With the help of Theorem 9.4 it is easy to see that Green edge symbols with asymptotics are invariant under multiplication by powers of t both from the left and the right.

Indeed, for each asymptotics $u(t,x) = \omega(t)\langle f(x), t^{-z}\rangle$ of asymptotic type $as = (\sigma, \Sigma)$, we have

$$\begin{aligned} t^p\, u(t,x) &= \omega(t)\,\langle f(x), t^{-(z-p)}\rangle \\ &= \omega(t)\,\langle (z \mapsto z+p)^* f(x), t^{-z}\rangle, \end{aligned}$$

where $(z \mapsto z+p)^* f(x)$ is the pull-back of $f(x)$ under the biholomorphism $z \mapsto z+p$ of the complex plane. Thus, the product is of asymptotic type $(z \mapsto z+p)^* as =$

$(\sigma - p, (z \mapsto z+p)^*\Sigma)$. Hence it follows that

$$\begin{aligned} t^p \mathcal{S}^\gamma_{as}(X^\square) &= t^p\, \omega \left(\mathcal{K}^{\infty,\gamma+l-0}(X^\square) + \mathfrak{A}_{as}(X^\square) \right) + t^p\,(1-\omega)\,\mathcal{S}(X^\square) \\ &= \omega \left(\mathcal{K}^{\infty,\gamma+p+l-0}(X^\square) + \mathfrak{A}_{(z\mapsto z+p)^* as}(X^\square) \right) + (1-\omega)\,\mathcal{S}(X^\square) \\ &= \mathcal{S}^{\gamma+p}_{(z\mapsto z+p)^* as}(X^\square). \end{aligned} \tag{11.1}$$

Corollary 11.1 *Assume $g(y,\eta)$ is a Green edge symbol of order m with asymptotics as', as''. Then, for each $p_1, p_2 \in \mathbb{R}$, the composition $t^{p_2} g(y,\eta)\, t'^{p_1}$ is a Green edge symbol of order $m - p_1 - p_2$ with asymptotics $(z \mapsto z+p_2)^* as'$ and $(z \to z+p_1)^* as''$.*

Proof. By Theorem 9.4, we conclude that there exists a kernel $k(y,\eta;t,x,t',x')$ in $S^{m+n+1}_{cl}(U \times \mathbb{R}^q) \otimes_\pi (\mathcal{S}^\delta_{as'}(X^\square) \otimes_\pi \mathcal{S}^{-\gamma}_{as''}(X^\square))$ such that $g(y,\eta)$ is given by (9.4). Hence

$$\begin{aligned} & t^{p_2}\, g(y,\eta)\, t'^{p_1}\, u(t,x) \\ &= \int_{X^\square} \langle\eta\rangle^{-p_1-p_2} (t\langle\eta\rangle)^{p_2} k(y,\eta;t\langle\eta\rangle,x,t'\langle\eta\rangle,x')(t'\langle\eta\rangle)^{p_1}\, u(t',x')\, t'^n dt'dx' \\ &= \int_{X^\square} \tilde{k}(y,\eta;t\langle\eta\rangle,x,t'\langle\eta\rangle,x')\, u(t',x')\, t'^n dt'dx', \end{aligned}$$

where $\tilde{k}(y,\eta;t,x,t',x') = \langle\eta\rangle^{-p_1-p_2} t^{p_2}\, k(y,\eta;t,x,t',x')\, t^{p_1}$.

It follows from (11.1) that this new kernel $\tilde{k}(y,\eta;t,x,t',x')$ belongs to

$$S^{m-p_1-p_2+n+1}_{cl}(U \times \mathbb{R}^q) \otimes_\pi \left(\mathcal{S}^{\delta+p_2}_{(z\mapsto z+p_2)^* as'}(X^\square) \otimes_\pi \mathcal{S}^{-\gamma+p_1}_{(z\mapsto z+p_1)^* as''}(X^\square) \right).$$

Applying Theorem 9.4 again yields the desired assertion.

□

Yet another consequence of Theorem 9.4 is that the Green edge symbols with asymptotics are invariant under multiplication by functions of $C^\infty_{comp}(\overline{\mathbb{R}}_+, C^\infty_{loc}(X \times U))$ both from the left and the right. However, the asymptotic types change in general.

Corollary 11.2 *Assume $g(y,\eta)$ is a Green edge symbol of order m with asymptotics. Then, for each $\phi_1, \phi_2 \in C^\infty_{comp}(\overline{\mathbb{R}}_+, C^\infty_{loc}(X \times U))$, the composition $\phi_2\, g(y,\eta)\, \phi_1$ is a Green edge symbol of order m with asymptotics.*

Proof. We give the proof only for the composition $g(y,\eta)\,\phi_1$; similar arguments apply to the case of $\phi_2\, g(y,\eta)$.

Given any $\phi \in C^\infty_{comp}(\overline{\mathbb{R}}_+, C^\infty_{loc}(X \times U))$, we use the Taylor formula to write ϕ in the form

$$\phi(t,x,y) = \sum_{\nu=0}^{N} t^\nu\, \phi_\nu(x,y) + t^{N+1}\, \phi_{N+1}(t,x,y),$$

with arbitrary $N = 0,1,\ldots$ and $\phi_\nu(x,y) = \frac{\partial_t^\nu \phi(0,x,y)}{\nu!}$, $\nu = 0,1,\ldots,N$

Then, for each $N = 0, 1, \ldots$, we have

$$g(y,\eta)\,\phi(t',x',y) = \sum_{\nu=0}^{N} g(y,\eta)\,t'^{\nu}\,\phi_\nu(x',y) + g(y,\eta)\,t'^{N+1}\,\phi_{N+1}(t',x',y).$$

From Corollary 11.1 it follows that the composition $g(y,\eta)\,t'^{\nu}\,\phi_\nu(x',y)$ is a Green edge symbol of order $m-\nu$ with asymptotics as' and $(z \mapsto z+\nu)^* as''$. For $as'' = (\sigma'', \Sigma'')$, we consider a new asymptotic type $as = (\sigma, \Sigma)$, where

$$\begin{aligned} \sigma &= \cup_{\nu=0}^{\infty} (\sigma'' - \nu) \cap \{z \in \mathbb{C} : \Re z \geq \frac{1+n}{2} + \gamma - l\} \quad \text{(shadow condition)}, \\ \Sigma &= \oplus_{\nu=0}^{\infty} (z \mapsto z+\nu)^* \Sigma'' \,|_{(\sigma''-\nu)\cap\{z\in\mathbb{C}:\ \Re z \geq \frac{1+n}{2}+\gamma-l\}} \end{aligned}$$

[5].

Then, $g(y,\eta)\,t'^{\nu}\,\phi_\nu(x',y)$ is a Green edge symbol of order $m-\nu$ with asymptotics as' and as, and we may take an asymptotic sum of this symbols within the space. Since $g(y,\eta)\,t'^{N+1}\,\phi_{N+1}(t',x',y)$ is of order $m-N-1$ (while being not classical), we conclude that

$$g(y,\eta)\,\phi(t',x',y) \sim \sum_{\nu=0}^{\infty} g(y,\eta)\,t'^{\nu}\,\phi_\nu(x',y)$$

in the sense of asymptotic sums of Green edge symbols with asymptotics as' and as. This finishes the proof.

□

12 *Smoothing Mellin operators*

A standard asymptotic summation allows one to invert, up to smoothing Mellin edge symbols, the Mellin edge symbols with invertible conormal symbol. Smoothing Mellin edge symbols are used in explicit form only for a finite weight interval $(-l, 0]$, with $l = 1, 2, \ldots$. By a *smoothing Mellin symbol* of order $m \in \mathbb{R}$, is meant a family

$$s_{\mathcal{M}}(y,\eta) = \varphi_0(t\langle\eta\rangle)\,\frac{1}{t^m}\sum_{j=0}^{l-1}\sum_{|\alpha|\leq j} t^j\,\mathrm{op}_{\mathcal{M},\gamma_{j,\alpha}-\frac{n}{2}}(h_{j,\alpha})(y,\eta)\,\eta^\alpha\,\psi_0(t\langle\eta\rangle),$$

where φ_0, ψ_0 are cut-off functions close to $t=0$, and $h_{j,\alpha}$ are meromorphic functions on the complex plane, taking their values in the algebra of smoothing pseudodifferential operators on X.

The weights $\gamma_{j,\alpha}$ are chosen in such a way that the vertical line $\{z \in \mathbb{C} : \Re z = \frac{1+n}{2} - \gamma_{j,\alpha}\}$ does not meet any pole of $h_{j,\alpha}$ in the complex plane. Moreover, $s_{\mathcal{M}}(y,\eta)$ is considered to act from $\mathcal{K}^{s,\gamma}(X^\square)$ to $\mathcal{K}^{\infty,\delta}(X^\square)$, for any $s \in \mathbb{R}$. Therefore, the exponents m and j involved must satisfy

$$\begin{cases} \gamma_{j,\alpha} & \leq \gamma, \\ \gamma_{j,\alpha} - m + j & \geq \delta \end{cases}$$

[5]To define these restrictions, use the equality $\mathcal{A}'_{\sigma_1\cup\sigma_2} = \mathcal{A}'_{\sigma_1} + \mathcal{A}'_{\sigma_2}$ in the sense of non-direct sum of Fréchet spaces.

for all j and α.

Let us denote by $\mathrm{ES}^m_{SM}(U \times \mathbb{R}^q, \mathfrak{d})$ the space of all matrices $\begin{pmatrix} s_{\mathcal{M}}(y,\eta) & 0 \\ 0 & 0 \end{pmatrix}$, where $s_{\mathcal{M}}(y,\eta)$ is a smoothing Mellin edge symbols of order m and with respect to a weight datum $\mathfrak{d} = \left(-\gamma - \frac{n}{2}, \delta - \frac{n}{2}, (-l, 0]\right)$.

It is easy to see that $\mathrm{ES}^m_{SM}(U \times \mathbb{R}^q, \mathfrak{d}) \subset \mathrm{ES}^m_G(U \times \mathbb{R}^q, \mathfrak{d})$ if $l \leq \gamma - m - \delta$. As the operator of multiplication by t decreases the order m by 1, it follows that, for each smoothing Mellin edge symbol $s_{\mathcal{M}}(y,\eta)$ of order m, the composition $t^N s_{\mathcal{M}}(y,\eta)$ is a Green edge symbol of order $m - N$, provided N is large enough (precisely, $N \geq l - \gamma + m + \delta$). This is one of the motivations for introducing Green edge symbols.

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