FLOER HOMOLOGY, NIELSEN THEORY AND SYMPLECTIC ZETA FUNCTIONS

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Abstract. We describe connection between symplectic Floer homology for surfaces and Nielsen fixed point theory. A new zeta functions of symplectic origin are defined. We show that special values of symplectic zeta functions are Reidemeister torsions.

Contents

1. Introduction

In two dimensions a diffeomorphism is symplectic if it preserves volumes. As a consequence, the symplectic geometry of surfaces lacks many of the interesting phenomena which are encountered in higher dimensions. For example, two symplectic automorphisms of a closed surface are symplectically isotopic iff they are homotopic, by a theorem of Moser[15]. On other hand symplectic fixed point theory is very nontrivial in dimension 2, as shown by the Poincare-Birkhoff theorem. It is known that symplectic Floer homology on surface is a simple model for the instanton Floer homology of the mapping torus of the surface diffeomorphism [17].

In this article we define new zeta functions related to symplectic Floer homology groups and investigate their analytical properties. We show that special values of these zeta functions are Reidemeister torsions. We also prove the following result: if ϕ is a nontrivial orientation preserving periodic diffeomorphism of a compact connected surface M of Euler characteristic $\chi(M) < 0$, then ϕ is monotone with respect to some ϕ-invariant area form and $HF_*(\phi) \cong \mathbb{Z}_2^{N(\phi)}$, $\dim HF_*(\phi) = N(\phi)$ where, $N(\phi)$ denotes the Nielsen number of ϕ and $HF_*(\phi)$ denotes symplectic Floer homology group. Due to P. Seidel [18] $\dim HF_*(\phi)$ is a new symplectic invariant of a four-dimensional symplectic manifold with nonzero first Betti number. This 4-manifold produced from symplectomorphism ϕ by a

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surgery construction which is a variation of earlier constructions due to McMullen-Taubes, Fintushel-Stern and J. Smith.

The author came to the idea that Nilsen numbers are connected with Floer homology of surface diffeomorphisms at the Autumn 2000, after conversations with Joel Robbin. Last years I try to find a cohomological theory which lies behind of the Nielsen zeta function [4]. This was my motivation when I studied the symplectic Floer homology of surface diffeomorphisms. The results of this paper were reported in the author talk on the International Conference Topological Methods in Nonlinear Analysis, June 2001 in Bendlewo, Poland. I am very gratefull to Dietmar Salomon who send me handwritten notes of Wu- Chung Hsiang paper "A speculation on Floer theory and Nielsen theory" and to Dan Burghelea, Stefan Haller, Wu- Chung Hsiang, Jarek Kedra, Wilhelm Klingenberg, Francois Laudenbach, Kaoru Ono, Yuli Rudjak, Andrei Tyurin and Vladimir Turaev for very useful discussions of the results of this paper. The delay in publication was connected with the absence of the important notion of monotonicity, introduced by P. Seidel[17] in the Spring 2001. It is interesting to compare our Question 14 and Theorem 1 of the recent important preprint of Ralf Gautschi[9].

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2. The monotonicity condition

In this section we discuss the notion of monotonicity as defined in [17, 9]. Monotonicity plays important role for Floer homology in two dimensions. For a more detailed account we refer to the original articles of P. Seidel and R. Gautschi.

Throughout this article, M denotes a closed connected and oriented 2-manifold of genus ≥ 2. Pick an everywhere positive two-form ω on M.

Let $\phi \in \mathrm{Symp}(M, \omega)$, the group of symplectic automorphisms of the two-dimensional symplectic manifold M, ω. The mapping torus of ϕ, $T_\phi = \mathbb{R} \times M/(t+1, x) \sim (t, \phi(x))$, is a 3-manifold fibered over $S^1 = \mathbb{R}/\mathbb{Z}$. There are two natural second cohomology classes on T_ϕ, denoted by $[\omega_\phi]$ and c_ϕ. The first one is represented by the closed two-form ω_ϕ which is induced from the pullback of ω to $\mathbb{R} \times M$. The second is the Euler class of the vector bundle

$$V_\phi = \mathbb{R} \times TM/(t+1, \xi_x) \sim (t, \mathrm{d}\phi_x \xi_x),$$

which is of rank 2 and inherits an orientation from TM.

$\phi \in \mathrm{Symp}(M, \omega)$ is called **monotone**, if

$$[\omega_\phi] = (\mathrm{area}_\omega(M)/\chi(M)) \cdot c_\phi$$

in $H^2(;\mathbb{R})$; throughout this article $\mathrm{Symp}^m(M, \omega)$ denotes the set of monotone symplectomorphisms.

Now $H^2(T_\phi; \mathbb{R})$ fits into the following short exact sequence [17, 9]

$$0 \longrightarrow \frac{H^1(M;\mathbb{R})}{\mathrm{im}(\mathrm{id} - \phi^*)} \xrightarrow{d} H^2(T_\phi; \mathbb{R}) \xrightarrow{r^*} H^2(M; \mathbb{R}), \longrightarrow 0. \tag{1}$$

where the map r^* is restriction to the fiber. The map d is defined as follows. Let $\rho : I \to \mathbb{R}$ be a smooth function which vanishes near 0 and 1 and satisfies $\int_0^1 \rho \, \mathrm{d}t = 1$. If θ is a closed 1-form on M, then $\rho \cdot \theta \wedge \mathrm{d}t$ defines a closed 2-form on T_ϕ; indeed $d[\theta] = [\rho \cdot \theta \wedge \mathrm{d}t]$. The map $r : M \hookrightarrow T_\phi$ assigns to each $x \in M$ the equivalence class of $(1/2, x)$. Note, that

$r^*\omega_\phi = \omega$ and r^*c_ϕ is the Euler class of TM. Hence, by (1), there exists a unique class $m(\phi) \in H^1(M;\mathbb{R})/\operatorname{im}(\mathrm{id} - \phi^*)$ satisfying

$$d\, m(\phi) = [\omega_\phi] - (\mathrm{area}_\omega(M)/\chi(M)) \cdot c_\phi,$$

where χ denotes the Euler characteristic. Therefore, ϕ is monotone if and only if $m(\phi) = 0$.

We recall the fundamental properties of $\mathrm{Symp}^m(M, \omega)$ from [17, 9]. Let $\mathrm{Diff}^+(M)$ denotes the group of orientation preserving diffeomorphisms of M.

(Naturality) If $\phi \in \mathrm{Symp}^m(M, \omega), \psi \in \mathrm{Diff}^+(M)$, then $\psi^{-1}\phi\psi \in \mathrm{Symp}^m(M, \psi^*\omega)$.

(Isotopy) Let $(\psi_t)_{t\in I}$ be an isotopy in $\mathrm{Symp}(M, \omega)$, i.e. a smooth path with $\psi_0 = \mathrm{id}$. Then $m(\phi \circ \psi_1) = m(\phi) + [\mathrm{Flux}(\psi_t)_{t\in I}]$ in $H^1(M;\mathbb{R})/\operatorname{im}(\mathrm{id} - \phi^*)$; see [17, Lemma 6]. For the definition of the flux homomorphism see [14].

(Inclusion) The inclusion $\mathrm{Symp}^m(M, \omega) \hookrightarrow \mathrm{Diff}^+(M)$ is a homotopy equivalence. This follows from the isotopy property, surjectivity of the flux homomorphism and Moser's isotopy theorem [15] which says that each element of the mapping class group admits representatives which preserve ω. Furthermore, the Earl-Eells Theorem [3] implies that every connected component of $\mathrm{Symp}^m(M, \omega)$ is contractible.

(Floer homology) To every $\phi \in \mathrm{Symp}^m(M, \omega)$ symplectic Floer homology theory assigns a $\mathbb{Z}_2$-graded vector space $HF_*(\phi)$ over $\mathbb{Z}_2$, with an additional multiplicative structure, called the quantum cap product, $H^*(M;\mathbb{Z}_2) \otimes HF_*(\phi) \longrightarrow HF_*(\phi)$. Each $\psi \in \mathrm{Diff}^+(M)$ induces an isomorphism $HF_*(\phi) \cong HF_*(\psi^{-1}\phi\psi)$ of $H^*(M;\mathbb{Z}_2)$-modules.

(Invariance) If $\phi, \phi' \in \mathrm{Symp}^m(M, \omega)$ are isotopic, then $HF_*(\phi)$ and $HF_*(\phi')$ are naturally isomorphic as $H^*(M;\mathbb{Z}_2)$-modules. This is proven in [17, Page 7]. Note that every Hamiltonian perturbation of ϕ (see [2]) is also in $\mathrm{Symp}^m(M, \omega)$.

Now let g be a mapping class of M, i.e. an isotopy class of $\mathrm{Diff}^+(M)$. Pick an area form ω and a representative $\phi \in \mathrm{Symp}^m(M, \omega)$ of g. Then $HF_*(\phi)$ is an invariant of g, which is denoted by $HF_*(g)$. Note that $HF_*(g)$ is independent of the choice of an area form ω by Moser's isotopy theorem [15] and naturality of Floer homology.

3. Symplectic Floer homology

Let $\phi \in \mathrm{Symp}(M, \omega)$.There are two ways of constructing Floer homology detecting its fixed points, $\mathrm{Fix}(\phi)$. Firstly, the graph of ϕ is a Lagrangian submanifold of $M \times M, (-\omega) \times \omega)$ and its fixed points correspond to the intersection points of graph(ϕ) with the diagonal $\Delta = \{(x, x) \in M \times M\}$. Thus we have the Floer homology of the Lagrangian intersection $HF_*(M \times M, \Delta, graph(\phi))$. This intersection is transversal if the fixed points of ϕ are nondegenerate, i.e. if 1 is not an eigenvalue of $d\phi(x)$, for $x \in \mathrm{Fix}(\phi)$. The second approach was mentioned by Floer in [6] and presented with details by Dostoglou and Salomon in [2].We follow here Seidel's approach [17] which, comparable with [2], uses a larger class of perturbations, but such that the perturbed action form is still cohomologous to the unperturbed. As a consequence, the usual invariance of Floer homology under Hamiltonian isotopies is extended to the stronger property stated above. Let now $\phi \in \mathrm{Symp}^m(M, \omega)$, i.e ϕ is monotone. Firstly, we give the definition of $HF_*(\phi)$ in the special case where all the fixed points of ϕ are non-degenerate, i.e. for all $y \in \mathrm{Fix}(\phi)$, $\det(\mathrm{id} - \mathrm{d}\phi_y) \neq 0$, and then following Seidels approach [17] we consider general case when

ϕ has degenerate fixed points. Let $\Omega_\phi = \{y \in C^\infty(\mathbb{R}, M) \mid y(t) = \phi(y(t+1))\}$ be the twisted free loop space, which is also the space of sections of $T_\phi \to S^1$. The action form is the closed one-form α_ϕ on Ω_ϕ defined by

$$\alpha_\phi(y)Y = \int_0^1 \omega(dy/dt, Y(t))\, dt.$$

where $y \in \Omega_\phi$ and $Y \in T_y\Omega_\phi$, i.e. $Y(t) \in T_{y(t)}M$ and $Y(t) = \mathrm{d}\phi_{y(t+1)}Y(t+1)$ for all $t \in \mathbb{R}$.

The tangent bundle of any symplectic manifold admits an almost complex structure $J : TM \longrightarrow TM$ which is compatible with ω in sense that $(v, w) = \omega(v, Jw)$ defines a Riemannian metric. Let $J = (J_t)_{t\in\mathbb{R}}$ be a smooth path of ω-compatible almost complex structures on M such that $J_{t+1} = \phi^* J_t$. If $Y, Y' \in T_y\Omega_\phi$, then $\int_0^1 \omega(Y'(t), J_tY(t))\mathrm{d}t$ defines a metric on the loop space Ω_ϕ. So the critical points of α_ω are the constant paths in Ω_ϕ and hence the fixed points of ϕ. The negative gradient lines of α_ω with respect to the metric above are solutions of the partial differential equations with boundary conditions

$$\begin{cases} u(s,t) = \phi(u(s,t+1)), \\ \partial_s u + J_t(u)\partial_t u = 0, \\ \lim_{s\to\pm\infty} u(s,t) \in \mathrm{Fix}(\phi) \end{cases} \tag{2}$$

These are exactly Gromov's pseudoholomorphic curves [10].

For $y^\pm \in \mathrm{Fix}(\phi)$, let $\mathcal{M}(y^-, y^+; J, \phi)$ denote the space of smooth maps $u : \mathbb{R}^2 \to M$ which satisfy the equations (2). Now to every $u \in \mathcal{M}(y^-, y^+; J, \phi)$ is associated a Fredholm operator D_u which linearizes (2) in suitable Sobolev spaces. The index of this operator is given by the so called Maslov index $\mu(u)$, which satisfies $\mu(u) = \deg(y^+) - \deg(y^-) \bmod 2$, where $(-1)^{\deg y} = \mathrm{sign}(\det(\mathrm{id} - \mathrm{d}\phi_y))$. We have no bubling, since for surface $\pi_2(M) = 0$. For a generic J, every $u \in \mathcal{M}(y^-, y^+; J, \phi)$ is regular, meaning that D_u is onto. Hence, by the implicit function theorem, $\mathcal{M}_k(y^-, y^+; J, \phi)$ is a smooth k-dimensional manifold, where $\mathcal{M}_k(y^-, y^+; J, \phi)$ denotes the subset of those $u \in \mathcal{M}(y^-, y^+; J, \phi)$ with $\mu(u) = k \in \mathbb{Z}$. Translation of the s-variable defines a free $\mathbb{R}$-action on 1-dimensional manifold $\mathcal{M}_1(y^-, y^+; J, \phi)$ and hence the quotient is a discrete set of points. The energy of a map $u : \mathbb{R}^2 \to M$ is given by $E(u) = \int_{\mathbb{R}} \int_0^1 \omega\big(\partial_t u(s,t), J_t\partial_t u(s,t)\big)\, \mathrm{d}t\mathrm{d}s$ for all $y \in \mathrm{Fix}(\phi)$. P.Seidel has proved in [17] that if ϕ is monotone, then the energy is constant on each $\mathcal{M}_k(y^-, y^+; J, \phi)$. Since all fixed points of ϕ are nondegenerate the set $\mathrm{Fix}(\phi)$ is a finite set and the $\mathbb{Z}_2$-vector space $CF_*(\phi) := \mathbb{Z}_2^{\#\mathrm{Fix}(\phi)}$ admits a $\mathbb{Z}_2$-grading with $(-1)^{\deg y} = \mathrm{sign}(\det(\mathrm{id} - \mathrm{d}\phi_y))$, for all $y \in \mathrm{Fix}(\phi)$. The boundness of the energy $E(u)$ for monotone ϕ implies that the 0-dimensional quotients $\mathcal{M}_1(y_-, y_+, J, \phi)/\mathbb{R}$ are actually finite sets. Denoting by $n(y_-, y_+)$ the number of points mod 2 in each of them, one defines a differential $\partial_J : CF_*(\phi) \to CF_{*+1}(\phi)$ by $\partial_J y_- = \sum_{y_+} n(y_-, y_+)y_+$. Due to gluing theorem this Floer boundary operator satisfies $\partial_J \circ \partial_J = 0$. For gluing theorem to hold one needs again the boundness of the energy $E(u)$. It follows that $(CF_*(\phi), \partial_J)$ is a chain complex and its homology is by definition the Floer homology of ϕ denoted $HF_*(\phi)$. It is independent of J and is an invariant of ϕ.

If ϕ has degenerate fixed points one needs to perturb equations (2) in order to define the Floer homology. Equivalently, one could say that the action form needs to be perturbed. The necessary analysis is given in [17] is essentially the same as in the slightly different situations considered in [2]. But Seidel's approach also differs from the usual one in [2].

He uses a larger class of perturbations, but such that the perturbed action form is still cohomologous to the unperturbed.

4. Nielsen numbers and Floer homology

Before discussing the main results of the paper, we briefly describe the few basic notions of Nielsen fixed point theory which will be used. We assume X to be a connected, compact polyhedron and $f : X \to X$ to be a continuous map. Let $p : \tilde{X} \to X$ be the universal cover of X and $\tilde{f} : \tilde{X} \to \tilde{X}$ a lifting of f, i.e. $p \circ \tilde{f} = f \circ p$. Two liftings $\tilde{f}$ and $\tilde{f}'$ are called *conjugate* if there is a $\gamma \in \Gamma \cong \pi_1(X)$ such that $\tilde{f}' = \gamma \circ \tilde{f} \circ \gamma^{-1}$. The subset $p(Fix(\tilde{f})) \subset Fix(f)$ is called *the fixed point class of f determined by the lifting class* $[\tilde{f}]$.Two fixed points x_0 and x_1 of f belong to the same fixed point class iff there is a path c from x_0 to x_1 such that $c \cong f \circ c$ (homotopy relative endpoints). This fact can be considered as an equivalent definition of a non-empty fixed point class. Every map f has only finitely many non-empty fixed point classes, each a compact subset of X. A fixed point class is called *essential* if its index is nonzero. The number of essential fixed point classes is called the *Nielsen number* of f, denoted by $N(f)$.The Nielsen number is always finite. $R(f)$ and $N(f)$ are homotopy invariants. In the category of compact, connected polyhedra, the Nielsen number of a map is, apart from certain exceptional cases, equal to the least number of fixed points of maps with the same homotopy type as f.

The following Lemma was first proven in [11]. We repeat here the arguments from [9].

Lemma 1. [11] *Let ϕ a non-trivial orientation preserving periodic diffeomorphism of a compact connected surface M of Euler characteristic $\chi(M) \leq 0$. Then each fixed point class of ϕ consists of a single point.*

Proof. First assume that M is closed. The uniformization theorem states that in every conformal class of metrics on M, there is a unique metric of constant curvature -1 if $\chi(M) < 0$ or 0 if $\chi(M) = 0$. This implies that the unique representative of a ϕ-invariant conformal class of metrics (such a class exists since ϕ is finite order) is itself ϕ-invariant. Hence we can pick a ϕ-invariant metric of constant curvature -1 or 0 on M and lift ϕ to an isometry $\tilde{\phi}$ of the universal cover $\tilde{M}$ of M. $\tilde{M}$ is either isometric to the hyperbolic plane H^2 or the Euclidean plane $\mathbb{R}^2$.

Let $x \in \mathrm{Fix}(\phi)$ and let $\tilde{\phi}, \tilde{x}$ be lifts of ϕ, x to $\tilde{M}$, such that $\tilde{\phi}(\tilde{x}) = \tilde{x}$. Note, that a fixed point of ϕ is in the same class as x if and only if it can be lifted to a fixed point of $\tilde{\phi}$. Assume by contradiction that $\tilde{y} \neq \tilde{x}$ is a fixed point of $\tilde{\phi}$. It follows that the unique geodesic going through $\tilde{x}$ and $\tilde{y}$ is pointwise fixed by $\tilde{\phi}$. In particular, since $\tilde{\phi}$ preserves orientation, $\mathrm{d}\tilde{\phi}_{\tilde{x}} = \mathrm{id}$. This implies that $\tilde{\phi} = \mathrm{id}$, because an isometry of H^2 or $\mathbb{R}^2$ is determined by its value and differential at one point. This proves claim 3 in the case that M is closed.

The case $\partial M \neq \emptyset$ is reduced to the above case by gluing two copies of M together along a ϕ-invariant tubular neighborhood of ∂M. The glued manifold is closed and of Euler characteristic ≤ 0; φ extends to a non-trivial diffeomorphism ϕ', which is orientation preserving and of finite order. Hence, every fixed point class of ϕ' is a single point. The same therefore holds for ϕ. This ends the proof. ☐

R. Gautschi gave two criteria for monotonicity which we use later on.

Let ω be an area form on M and $\phi \in \mathrm{Symp}(\Sigma, \omega)$.

Lemma 2. [9] *Assume that every class $\alpha \in \ker(\mathrm{id} - \phi_*) \subset H_1(M;\mathbb{Z})$ is represented by a map $\gamma : S \to \mathrm{Fix}(\phi)$, where S is a compact oriented 1-manifold. Then ϕ is monotone.*

Lemma 3. [9] *If ϕ^k is monotone for some $k > 0$, then ϕ is monotone. If ϕ is monotone, then ϕ^k is monotone for all $k > 0$.*

Proof. We repeat Gautschi arguments from [9] here. Recall that T_ϕ is the orbit space of the $\mathbb{Z}$-action $n \cdot (t, x) = (t + n, \phi^{-n}(x))$, where $n \in \mathbb{Z}$ and $(t, x) \in \mathbb{R} \times \Sigma$. If we only divide out by the subgroup $k\mathbb{Z}$, for $k \in \mathbb{N}_{>0}$, we naturally get the mapping torus of ϕ^k. Further dividing by $\mathbb{Z}/k\mathbb{Z}$ defines the k-fold covering map $p_k : T_{\phi^k} \to T_\phi$. It is straight forward to check that

$$p_k^*[\omega_\phi] = [\omega_{\phi^k}] \quad \text{and} \quad p_k^* c_\phi = c_{\phi^k}. \tag{3}$$

The first equality follows immediately from the definitions. To prove the second, note that $p_k^*\big((TM \times \mathbb{R})/\mathbb{Z}\big) \cong (TM \times \mathbb{R})/k\mathbb{Z} \cong V_{\phi^k}$, where the $\mathbb{Z}$-action on $\mathbb{R} \times T\Sigma$ is given by $n \cdot (t, \xi_x) = (t + n, \mathrm{d}\phi_x^{-n}\xi_x)$, for $n \in \mathbb{Z}$ and $\xi_x \in T_xM$. The lemma follows from (3) and the fact that p_k^* is injective. To prove injectivity, define the map $a_*^k : H^2(T_{\phi^k};\mathbb{R}) \to H^2(T_\phi;\mathbb{R})$ by averaging differential forms; a_*^k is a left inverse of p_k^*, i.e. $a_*^k \circ p_k^* = \mathrm{id}$. This ends the proof of the lemma. □

We shall say that $\phi : M \to M$ is a periodic map of period m, if ϕ^m is the identity map $id_M : M \to M$.

Theorem 4. *If ϕ is a non-trivial orientation preserving periodic diffeomorphism of a compact connected surface M of Euler characteristic $\chi(M) < 0$, then ϕ is monotone with respect to some ϕ-invariant area form and*

$$HF_*(\phi) \cong \mathbb{Z}_2^{N(\phi)}, \ \ \dim HF_*(\phi) = N(\phi),$$

where $N(\phi)$ denotes the Nielsen number of ϕ.

Proof. Let ϕ be a periodic diffeomorphism of least period l. First note that if $\tilde\omega$ is an area form on M, then area form $\omega := \sum_{i=1}^{\ell} (\phi^i)^*\tilde\omega$ is ϕ-invariant, i.e. $\phi \in \mathrm{Symp}(M, \omega)$. By periodicity of ϕ, ϕ^l is the identity map $id_M : M \to M$. Then from Lemmas 2 and 3 it follows that ω can be chosen such that $\phi \in \mathrm{Symp}^m(M, \omega)$.

Lemma 1 implies that every $y \in \mathrm{Fix}(\phi)$ forms a different fixed point class of ϕ, so $\#\,\mathrm{Fix}(\phi) = N(\phi)$. This has an immediate consequence for the Floer complex $(CF_*(\phi), \partial_J)$ with respect to a generic $J = (J_t)_{t\in\mathbb{R}}$. If $y^\pm \in \mathrm{Fix}(\phi)$ are in different fixed point classes, then $\mathcal{M}(y^-, y^+; J, \phi) = \emptyset$. This follows from the first equation in (2). Then the boundary map in the Floer complex is zero $\partial_J = 0$ and $\mathbb{Z}_2$-vector space $CF_*(\phi) := \mathbb{Z}_2^{\#\,\mathrm{Fix}(\phi)} = \mathbb{Z}_2^{N(\phi)}$. This immediately implies $HF_*(\phi) \cong \mathbb{Z}_2^{N(\phi)}$ and $\dim HF_*(\phi) = N(\phi)$.

□

Remark 5. Since every fixed point of ϕ has fixed point index 1 [11, 9], the Lefschetz fixed point formula implies that $\#\,\mathrm{Fix}(\phi) = N(\phi) = L(\phi)$, where $L(\phi)$ is the Lefschetz number(see section 5). So, theorem 4 agrees with the more general theorem of R. Gautschi for a diffeomorphism of finite type [9].

Theorem 6. *If ϕ is a hyperbolic diffeomorphism of a 2-dimensional torus T^2, then ϕ is symplectic and*

$$HF_*(\phi) \cong \mathbb{Z}_2^{N(\phi)}, \ \ \dim HF_*(\phi) = N(\phi)$$

where $N(\phi) = |\det(E - \phi_*)|$ *denotes the Nielsen number of* ϕ *and* ϕ_* *is an induced homomorphism on the fundamental group of* T^2.

Proof. Hyperbolicity of ϕ means that the covering linear map $\tilde{\phi}: R^2 \to R^2$, $\det \tilde{\phi} = 1$ has no eigenvalue of modulus one. The hyperbolic diffeomorphism of a 2-dimensional torus T^2 is area preserving so symplectic. In fact, the covering map $\tilde{\phi}$ has a unique fixed point, which is the origin; hence, by the covering homotopy theorem , the fixed points of ϕ are pairwise Nielsen nonequivalent.The index of each Nielsen fixed point class, consisting of one fixed point, coincides with its Lefschetz index, and by the hyperbolicity of ϕ, the later is not equal to zero.Thus the Nielsen number $N(\phi) = \#\,\mathrm{Fix}(\phi)$. If $y^{\pm} \in \mathrm{Fix}(\phi)$ are in different Nielsen fixed point classes, then $\mathcal{M}(y^-, y^+; J, \phi) = \emptyset$. This follows from the first equation in (2). Then the boundary map in the Floer complex is zero $\partial_J = 0$ and $\mathbb{Z}_2$-vector space $CF_*(\phi) := \mathbb{Z}_2^{\#\mathrm{Fix}(\phi)} = \mathbb{Z}_2^{N(\phi)}$. This immediately implies $HF_*(\phi) \cong \mathbb{Z}_2^{N(\phi)}$ and $\dim HF_*(\phi) = N(\phi)$.

□

Remark 7. It is interesting to compare this result with the first computation by Marcin Poźniak of the Floer homology of linear symplectomorphisms in case of torus[16].

5. Symplectic zeta functions

Let $\Gamma = \pi_0(Diff^+(M))$ be the mapping class group of a closed connected oriented surface M of genus ≥ 2. Pick an everywhere positive two-form ω on M. A isotopy theorem of Moser [15] says that each mapping class of $g \in \Gamma$, i.e. an isotopy class of $Diff^+(M)$, admits representatives which preserve ω. Due to Seidel[17] we can pick a monotone representative $\phi \in \mathrm{Symp}^m(M, \omega)$ of g. Then $HF_*(\phi)$ is an invariant of g, which is denoted by $HF_*(g)$. Note that $HF_*(g)$ is independent of the choice of an area form ω by Moser's theorem and naturality of Floer homology. By Gautschi lemma 3 symplectomorphisms ϕ^n are also monotone for all $n > 0$. Taking a dynamical point of view, we consider the iterates of monotone symplectomorphism ϕ and define the first symplectic zeta function of ϕ as the following power series:

$$\chi_\phi(z) = \exp\left(\sum_{n=1}^{\infty} \frac{\chi(HF_*(\phi^n))}{n} z^n\right),$$

where $\chi(HF_*(\phi^n))$ is the Euler characteristic of Floer homology group of ϕ^n. Then $\chi_\phi(z)$ is an invariant of g, which we denote by $\chi_g(z)$.

Let us consider the Lefschetz zeta function

$$L_\phi(z) := \exp\left(\sum_{n=1}^{\infty} \frac{L(\phi^n)}{n} z^n\right),$$

where $L(\phi^n) := \sum_{k=0}^{\dim X} (-1)^k \operatorname{tr}\left[\phi_{*k}^n : H_k(M;\mathbb{Q}) \to H_k(M;\mathbb{Q})\right]$ is the Lefschetz number of ϕ^n. The Lefschetz zeta function is always a rational function of z and is given by the formula:

$$L_\phi(z) = \prod_{k=0}^{\dim X} \det\left(E - \phi_{*k}.z\right)^{(-1)^{k+1}}.$$

Theorem 8. *Symplectic zeta function $\chi_\phi(z)$ is a rational function of z and*

$$\chi_\phi(z) = L_\phi(z) = \prod_{k=0}^{\dim X} \det\big(E - \phi_{*k}.z\big)^{(-1)^{k+1}}.$$

Proof. If for every n all the fixed points of ϕ^n are non-degenerate, i.e. for all $x \in \mathrm{Fix}(\phi^n)$, $\det(\mathrm{id} - \mathrm{d}\phi^n(x)) \neq 0$, then we have(see section 3):

$$\chi(HF_*(\phi^n)) = \sum_{x=\phi^n(x)} \mathrm{sign}(\det(\mathrm{id} - \mathrm{d}\phi^n(x))) = L(\phi^n).$$

If we have degenerate fixed points one needs to perturb equations (2) in order to define the Floer homology. The necessary analysis is given in [17] is essentially the same as in the slightly different situations considered in [2], where the above connection between the Euler characteristic and the Lefschetz number was firstly established. □

Remark 9. Theorem 8 shows that symplectic zeta function $\chi_\phi(z)$ counts symplectic periodic points of ϕ algebraically- in the Lefschetz way.

The next issue is a relation of the symplectic zeta function with the Reidemeister torsion, a very important topological invariant. We will show that special value of symplectic zeta function $\chi_\phi(z)$ is a Reidemeister torsion. Let $T_\phi := (X \times I)/(x,0) \sim (\phi(x),1)$ be the mapping tori of ϕ. We shall consider the bundle $p : T_\phi \to S^1$ over the circle S^1 with fibers M. We assume here that E is a flat, complex vector bundle with finite dimensional fibre and base S^1. We form its pullback p^*E over T_ϕ. Note that the vector spaces $H^i(p^{-1}(b), \mathbb{C})$ with $b \in S^1$ form a flat vector bundle over S^1, which we denote H^iM. The integral lattice in $H^i(p^{-1}(b), \mathbb{R})$ determines a flat density by the condition that the covolume of the lattice is 1. We suppose that the bundle $E \otimes H^iM$ is acyclic for all i. Under these conditions D. Fried [8] has shown that the bundle p^*E is acyclic, and we have

$$\tau(T_\phi; p^*E) = \prod_i \tau(S^1; E \otimes H^iM)^{(-1)^i}. \tag{4}$$

Let g be the prefered generator of the group $\pi_1(S^1)$ and let $A = \rho(g)$ where $\rho : \pi_1(S^1) \to GL(V)$. Then the holonomy around g of the bundle $E \otimes H^iM$ is $A \otimes \phi_i^*$.

Since $\tau(S^1; E) = \mid \det(I - A) \mid$ it follows from (4) that

$$\tau(T_\phi; p^*E) = \prod_i \mid \det(I - A \otimes \phi_i^*) \mid^{(-1)^i}. \tag{5}$$

We now consider the special case in which E is one-dimensional, so A is just a complex scalar λ of modulus one. Then in terms of the rational function $L_\phi(z)$ we have [8]:

$$\tau(T_\phi; p^*E) = \prod_i \mid \det(I - \lambda \cdot \phi_i^*) \mid^{(-1)^i} = \mid L_\phi(\lambda) \mid^{-1} \tag{6}$$

This proves the following

Theorem 10. *The Reidemeister torsion is the special value of the symplectic zeta function:*

$$\tau\,(T_\phi; p^*E) = \mid \chi_\phi(\lambda) \mid^{-1},$$

where λ is the holonomy of the one-dimensional flat complex bundle E over S^1.

Now we define the second symplectic zeta function for monotone symplectomorphism ϕ as the following power series:

$$F_\phi(z) \quad = \quad \exp\left(\sum_{n=1}^{\infty} \frac{\dim HF_*(\phi^n)}{n} z^n\right).$$

Motivation for this definition is the theorem 4 and nice analytical properties of the Nielsen zeta function $N_\phi(z) = \exp\left(\sum_{n=1}^{\infty} \frac{N(\phi^n)}{n} z^n\right)$, see [4, 5]. We denote the numbers $\dim HF_*(\phi^n)$ by N_n. Let $\mu(d), d \in N$, be the Möbius function from number theory. As is known, it is given by the following equations: $\mu(d) = 0$ if d is divisible by a square different from one ; $\mu(d) = (-1)^k$ if d is not divisible by a square different from one , where k denotes the number of prime divisors of d; $\mu(1) = 1$.

Theorem 11. *Let ϕ be a non-trivial orientation preserving periodic diffeomorphism of least period m of a compact connected surface M of Euler characteristic $\chi(M) < 0$. Then the zeta function $F_\phi(z)$ is a radical of a rational function and*

$$F_\phi(z) = \prod_{d|m} \sqrt[d]{(1 - z^d)^{-P(d)}},$$

where the product is taken over all divisors d of the period m, and $P(d)$ is the integer

$$P(d) = \sum_{d_1|d} \mu(d_1) N_{d|d_1}.$$

Proof. Since $\phi^m = id$,then for each $j, N_j = N_{m+j}$. If $(k, m) = 1$, then there exist positive integers t and q such that $kt = mq + 1$. So $(\phi^k)^t = \phi^{kt} = \phi^{mq+1} = \phi^{mq}\phi = (\phi^m)^q\phi = \phi$. Consequently, $\mathrm{Fix}((\phi^k)^t) = \mathrm{Fix}(\phi)$. We have $\mathrm{Fix}(\phi) \subset \mathrm{Fix}((\phi^k)$ and $\mathrm{Fix}((\phi^k) \subset \mathrm{Fix}((\phi^k)^t) = \mathrm{Fix}(\phi)$. Then $\mathrm{Fix}(\phi) = \mathrm{Fix}((\phi^k)$ and $N_1 = N_k$. One can prove completely analogously that $N_d = N_{di}$, if $(i, m/d) = 1$, where d is a divisor of m. Using these sequences of equal numbers, one can regroup the terms of the series in the exponential of the zeta function so as to get logarithmic functions by adding and subtracting missing terms with necessary coefficients. We show how to do this first for period $m = p^l$, where p is a prime number. We have the following sequence of equal numbers:

$$N_1 = N_k, (k, p^l) = 1 (\text{i.e., no}\, N_{ip}, N_{ip^2}, \dots, N_{ip^l}, i = 1, 2, 3, \dots),$$

$$N_p = N_{2p} = N_{3p} = \dots = N_{(p-1)p} = N_{(p+1)p} = \dots \ (\text{no}\, N_{ip^2}, N_{ip^3}, \dots, N_{ip^l})$$

etc.; finally,

$$N_{p^{l-1}} = N_{2p^{l-1}} = \dots (\text{no}\ \ N_{ip^l})$$

and separately the number N_{p^l}.
Further,

$$\begin{aligned}
\sum_{i=1}^{\infty}\frac{N_i}{i}z^i &= \sum_{i=1}^{\infty}\frac{N_1}{i}z^i+\sum_{i=1}^{\infty}\frac{(N_p-N_1)}{p}\frac{z^{pi}}{i}+\\
&+\sum_{i=1}^{\infty}\frac{(N_{p^2}-(N_p-N_1)-N_1)}{p^2}\frac{z^{p^2i}}{i}+...\\
&+\sum_{i=1}^{\infty}\frac{(N_{p^l}-...-(N_p-N_1)-N_1)}{p^l}\frac{z^{p^li}}{i}\\
&= -N_1\cdot\log(1-z)+\frac{N_1-N_p}{p}\cdot\log(1-z^p)+\\
&+\frac{N_p-N_{p^2}}{p^2}\cdot\log(1-z^{p^2})+...\\
&+\frac{N_{p^{l-1}}-N_{p^l}}{p^l}\cdot\log(1-z^{p^l}).
\end{aligned}$$

For an arbitrary period m , we get completely analogously,

$$\begin{aligned}
F_f(z) &= \exp\left(\sum_{i=1}^{\infty}\frac{N_i}{i}z^i\right)\\
&= \exp\left(\sum_{d\mid m}\sum_{i=1}^{\infty}\frac{P(d)}{d}\cdot\frac{z^{di}}{i}\right)\\
&= \exp\left(\sum_{d\mid m}\frac{P(d)}{d}\cdot\log(1-z^d)\right)\\
&= \prod_{d\mid m}\sqrt[d]{(1-z^d)^{-P(d)}},
\end{aligned}$$

where the integers $P(d)$ are calculated recursively by the formula

$$P(d)=N_d-\sum_{d_1\mid d;d_1\neq d}P(d_1).$$

Moreover, if the last formula is rewritten in the form

$$N_d=\sum_{d_1\mid d}\mu(d_1)\cdot P(d_1)$$

and one uses the Möbius Inversion law for real functions in number theory, then

$$P(d)=\sum_{d_1\mid d}\mu(d_1)\cdot N_{d/d_1},$$

where $\mu(d_1)$ is the Möbius function in number theory. The lemma is proved. $\square$

Corollary 12. *If in Theorem 2 the period m is a prime number, then*

$$F_f(z) = \frac{1}{(1-z)^{N_1}} \cdot \sqrt[m]{(1-z^m)^{N_1-N_m}}.$$

We denote by $\zeta_\phi(z)$ the Artin-Mazur zeta function

$$\zeta_\phi(z) := \exp\left(\sum_{n=1}^{\infty} \frac{I(\phi^n)}{n} z^n\right),$$

where $I(\phi^n)$ is the number of isolated fixed points of ϕ^n.

Theorem 13. *If ϕ is a hyperbolic diffeomorphism of a 2-dimensional torus T^2, then the symplectic zeta function $F_\phi(z)$ is a rational function and $F_\phi(z) = N_\phi(z) = \zeta_\phi(z) = (L_\phi(\sigma \cdot z))^{(-1)^r}$, where r is equal to the number of $\lambda_i \in Spec(\tilde{\phi})$ such that $\mid \lambda_i \mid > 1$, p is equal to the number of $\mu_i \in Spec(\tilde{\phi})$ such that $\mu_i < -1$ and $\sigma = (-1)^p$.*

Proof. From theorem 6 and [1] it follows that $\dim HF_*(\phi^n) = \#\operatorname{Fix}(\phi^n) = I(\phi^n) = N(\phi^n) = \mid \det(E - \tilde{\phi^n}) \mid = \mid L(\phi^n) \mid$ [1]. Thus $\dim HF_*(\phi^n) = \#\operatorname{Fix}(\phi^n) = I(\phi^n) = N(\phi^n) = (-1)^{r+pn} \cdot \det(E - \tilde{\phi^n}) = (-1)^{r+pn} \cdot L(\phi^n)$. Now a direct computation ends the proof of the theorem.

□

6. Further remarks and questions

6.1. **Open questions.**

Question 14. *For which mapping classes of $g \in \Gamma = \pi_0(Diff^+(M))$ we have*

$$HF_*(g) = \mathbb{Z}_2^{N(g)}, \ \dim HF_*(g) = N(g),$$

where $N(g)$ denotes the Nielsen number of g, i.e when there is a monotone representative $\phi \in \operatorname{Symp}^m(M,\omega)$ of g such that

$$HF_*(\phi) = \mathbb{Z}_2^{N(\phi)}, \ \dim HF_*(\phi) = N(\phi)?$$

Remark 15. (i) We think that the arguments of R. Gautschi in [9] can be used to answer the 14 for a algebraically finite mapping class [19].

(ii) The most interesting case is a case of a pseudo-Anosov mapping class [19]. For pseudo-Anosov "diffeomorphism" we also have, as in theorems 4 and 6, a topological separation of fixed points [19, 12, 13], i.e the Nielsen number of pseudo-Anosov diffeomorphism equals to the number of fixed points and there are no connecting orbits between them. But pseudo-Anosov diffeomorphism is a symplectic automorphism only on the complement of his fixed points set and this fact creates a problem.

(iii) From results of R. Gautschi it follows that the answer on our question is positive for the geometric monodromy of an isolated curve singularity. In this case the Nielsen number is zero and Floer homology groups are trivial(see the proof of the theorem 4 in [9]).

(iv) Due to P. Seidel [18] $\dim HF_*(\phi)$ is a new symplectic invariant of a four-dimensional symplectic manifold with nonzero first Betti number. This 4-manifold produced from symplectomorphism ϕ by a surgery construction which is a variation of earlier constructions due to McMullen-Taubes, Fintushel-Stern and J. Smith.

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