THE STRUCTURE OF THE NILPOTENT CONE, THE KAZHDAN–LUSZTIG MAP AND ALGEBRAIC GROUP ANALOGUES OF THE SLODOWY SLICES

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ABSTRACT. We define algebraic group analogues of the Slodowy transversal slices to adjoint orbits in a complex semisimple Lie algebra $\mathfrak{g}$. The new slices are transversal to the conjugacy classes in an algebraic group with Lie algebra $\mathfrak{g}$. These slices are associated to the pairs $(\mathfrak{p}, s)$, where $\mathfrak{p}$ is a parabolic subalgebra in $\mathfrak{g}$ and s is an element of the Weyl group W of $\mathfrak{g}$. For such slices we prove an analogue of the Kostant cross–section theorem for the action of a unipotent group. To each element $s \in W$ we also naturally associate a parabolic subalgebra $\mathfrak{p}$ and construct the corresponding slice. In case of classical Lie algebras we consider some other examples of the new slices for which the parabolic subalgebra $\mathfrak{p}$ is constructed with the help of the grading associated to a nilpotent element $e \in \mathfrak{g}$, and the element $s \in W$ is naturally associated to e via the Kazhdan–Lusztig map. We also realize simple Kleinian singularities as the singularities of the fibers of the restriction of the conjugation quotient map to the slices associated to pairs $(\mathfrak{b}, s)$, where $\mathfrak{b}$ is a Borel subalgebra in $\mathfrak{g}$ and s is an element of W whose representative in G is subregular.

1. INTRODUCTION

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, G the adjoint group of $\mathfrak{g}$, $e \in \mathfrak{g}$ a nonzero nilpotent element in $\mathfrak{g}$. By the Jacobson–Morozov theorem there is an $\mathfrak{sl}_2$–triple (e, h, f) associated to e, i.e. elements $f, h \in \mathfrak{g}$ such that $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. Fix such an $\mathfrak{sl}_2$–triple.

Let $z(f)$ be the centralizer of f in $\mathfrak{g}$. The affine space $s(e) = e + z(f)$ is called the Slodowy slice to the adjoint orbit of e at point e. Slodowy slices were introduced in [17] as a technical tool for the study of the singularities of the adjoint quotient of $\mathfrak{g}$. We recall that if $\mathfrak{h}$ is a Cartan subalgebra in $\mathfrak{g}$ and W is the Weyl group of $\mathfrak{g}$ then, after identification $\mathfrak{g} \simeq \mathfrak{g}^*$ with the help of the Killing form, the adjoint quotient can be defined as the morphism $\delta_\mathfrak{g} : \mathfrak{g} \to \mathfrak{h}/W$ induced by the inclusion $\mathbb{C}[\mathfrak{h}]^W \simeq \mathbb{C}[\mathfrak{g}]^G \hookrightarrow \mathbb{C}[\mathfrak{g}]$. The fibers of $\delta_\mathfrak{g}$ are unions of adjoint orbits in $\mathfrak{g}$. Each fiber of $\delta_\mathfrak{g}$ contains a single orbit which consists of regular elements. The singularities of the fibers correspond to irregular elements.

Slodowy studied the singularities of the adjoint quotient by restricting the morphism $\delta_\mathfrak{g}$ to the slices $s(e)$ which turn out to be transversal to the adjoint orbits in $\mathfrak{g}$. In particular, for regular nilpotent e the restriction $\delta_\mathfrak{g} : s(e) \to \mathfrak{h}/W$ is an isomorphism, and $s(e)$ is a cross–section for the set of the adjoint orbits of regular elements in $\mathfrak{g}$. For subregular e the fiber $\delta_\mathfrak{g}^{-1}(0)$ has one singular point which is a simple singularity, and $s(e)$ can be regarded as a deformation of this singularity.

In this paper we are going to outline a similar construction for algebraic groups. Let G be a complex simple algebraic group with Lie algebra $\mathfrak{g}$. In case of algebraic groups, instead of the adjoint quotient map, one should consider the conjugation quotient map $\delta_G : G \to H/W$ generated by the inclusion $\mathbb{C}[H]^W \simeq \mathbb{C}[G]^G \hookrightarrow \mathbb{C}[G]$, where H is the maximal torus of G corresponding to the Cartan subalgebra $\mathfrak{h}$ and W is the Weyl group of the pair (G, H). Some fibers of this map are singular and one can study these singularities by restricting δ_G to certain transversal slices to conjugacy classes in G. We are going to define such slices in this paper.

Note that the fibers of the adjoint quotient map and those of the conjugation quotient map are generally not isomorphic. More precisely, by Theorem 3.15 in [17] there are open neighborhoods U of 1 in H/W and U' of 0 in $\mathfrak{h}/W$ and a surjective morphism $\gamma : U \to U'$ such that the fibers $\delta_G^{-1}(u)$ and $\delta_{\mathfrak{g}}^{-1}(\gamma(u))$ are isomorphic for $u \in U$ as G–spaces. But globally such isomorphisms do not exist. In fact any fiber of the adjoint quotient map can be translated by a contracting $\mathbb{C}^*$–action to a fiber over any small neighborhood of 0 in $\mathfrak{h}/W$. But there is no similar statement for the fibers of the conjugation quotient map, and the problem of the study of singularities of the fibers of the conjugation quotient map and of their resolutions and deformations is more difficult than the same problem in case of the adjoint quotient map.

The other construction where the Slodowy slices or, more precisely, noncommutative deformations of algebras of regular functions on these slices, play an important role is the Whittaker or, more generally, generalized Gelfand–Graev representations of the Lie algebra $\mathfrak{g}$ (see [8, 11]). Namely, to each nilpotent element in $\mathfrak{g}$ one can associate the corresponding category of Gelfand–Graev representations. The category of generalized Gelfand–Graev representations associated to a nilpotent element $e \in \mathfrak{g}$ is equivalent to the category of finitely generated left modules over the W–algebra associated to e. This remarkable result was proved by Kostant in case of regular nilpotent $e \in \mathfrak{g}$ (see [11]) and by Skryabin in the general case (see Appendix to [12]). A more direct proof of Skryabin's theorem was obtained in [6]. This proof, as well as the original Kostant's proof, is based on the study of the commutative graded algebra associated to the algebra W–algebra.

The main observation of [6, 11] is that this commutative algebra is isomorphic to the algebra of regular functions on the Slodowy slice $s(e)$. This isomorphism is established with the help of a cross–section theorem proved in [11] in case of regular nilpotent e and in [6] in the general case. We briefly recall the main statement of this theorem.

Let χ be the element of $\mathfrak{g}^*$ which corresponds to e under the isomorphism $\mathfrak{g} \simeq \mathfrak{g}^*$ induced by the Killing form. Under the action of ad h we have a decomposition

$$(1.1) \qquad \mathfrak{g} = \oplus_{i \in \mathbb{Z}} \mathfrak{g}(i), \text{ where } \mathfrak{g}(i) = \{x \in \mathfrak{g} \mid [h, x] = ix\}.$$

The skew–symmetric bilinear form ω on $\mathfrak{g}(-1)$ defined by $\omega(x, y) = \chi([x, y])$ is nondegenerate. Fix an isotropic subspace l of $\mathfrak{g}(-1)$ with respect to ω and denote by $l^{\perp_\omega}$ the annihilator of l with respect to ω.

Let

$$(1.2) \qquad \mathfrak{m}_l = l \oplus \bigoplus_{i \leq -2} \mathfrak{g}(i), \ \mathfrak{n}_l = l^{\perp_\omega} \oplus \bigoplus_{i \leq -2} \mathfrak{g}(i).$$

Note that $\mathfrak{m}_l \subset \mathfrak{n}_l$, both $\mathfrak{m}_l$ and $\mathfrak{n}_l$ are nilpotent Lie subalgebras of $\mathfrak{g}$.

The Kostant cross–section theorem asserts that the adjoint action map $N_l \times s(e) \to e + \mathfrak{m}_l^{\perp_\mathfrak{g}}$ is an isomorphism of varieties, where N_l is the Lie subgroup of G corresponding to the Lie subalgebra $\mathfrak{n}_l \subset \mathfrak{g}$, and $\mathfrak{m}_l^{\perp_\mathfrak{g}}$ is the annihilator of $\mathfrak{m}_l$ in $\mathfrak{g}$ with respect to the Killing form.

Now a natural question is: are there any analogues of the Slodowy slices for algebraic groups? According to a general theorem proved in [17], if G is an algebraic group one can construct a transversal slice to the set of G–orbits at each point of any variety V equipped with a G–action. In particular, such slices exist for $V = G$ equipped with conjugation action. But we are rather interested in special transversal slices which can be used for the study of singularities of the conjugation quotient map and for which an analogue of the Kostant cross–section theorem holds.

In paper [23] R. Steinberg introduced a natural analogue of the slice $s(e)$ for regular nilpotent e. We recall that $s(e)$ is a cross–section for the set of adjoint orbits of regular elements in $\mathfrak{g}$. In paper [23] a cross–section for the set of conjugacy classes of regular elements in the connected simply connected group G' with Lie algebra $\mathfrak{g}$ is constructed. We briefly recall Steinberg's construction.

If e is regular nilpotent then, in the notation introduced above, $\mathfrak{g}(-1) = 0$, and $\mathfrak{n}_l = \mathfrak{m}_l = \mathfrak{n}$, where $\mathfrak{n}$ is a maximal nilpotent subalgebra of $\mathfrak{g}$. Let $\mathfrak{p} = \bigoplus_{i \leq 0} \mathfrak{g}(i)$ be the Borel subalgebra containing $\mathfrak{n}$, $\mathfrak{h} = \mathfrak{g}(0)$ the Cartan subalgebra of $\mathfrak{g}$, and W the Weyl group of the pair $(\mathfrak{g}, \mathfrak{h})$. Fix a system of positive simple roots associated to the pair $(\mathfrak{h}, \mathfrak{p})$. Let $s \in W$ be a Coxeter element, i.e. a product of the reflections corresponding to the simple roots. Fix a representative for s in G'. We denote this representative by the same letter. Let N be the unipotent subgroup in G' corresponding to the Lie algebra $\mathfrak{n}$, and $\overline{P}$ the opposite Borel subgroup with Lie algebra $\overline{\mathfrak{p}} = \bigoplus_{i \geq 0} \mathfrak{g}(i)$.

Steinberg introduced a subgroup $N_s \subset N$, $N_s = \{n \in N \mid sns^{-1} \in \overline{P}\}$, and proved that the set $N_s s^{-1}$ is a cross–section for the set of conjugacy classes of regular elements in the connected simply connected algebraic group G' associated to the Lie algebra $\mathfrak{g}$. Moreover, in [13] it is shown that the conjugation map $N \times N_s s^{-1} \to N s^{-1} N$ is an isomorphism of varieties. The last statement is an algebraic group analogue of the Kostant cross–section theorem.

As it was observed in [15, 16] the analogue of the Kostant cross–section theorem for the slice $N_s s^{-1}$ is the main ingredient of the construction of the Whittaker model of the center of the quantum group and of the Whittaker representations for quantum groups. The purpose of this paper is to construct other special transversal slices to conjugacy classes in a complex semisimple algebraic group G and to find an analogue of the Kostant cross–section theorem for these slices. We expect that these results can be applied to define deformed W–algebras and the generalized Gelfand–Graev representations for quantum groups. An initial step in this programme was realized in preprint [14] where the Poisson deformed W–algebras are defined with the help of Poisson reduction in algebraic Poisson–Lie groups.

As we shall see in Section 2 transversal slices in G similar to the Steinberg slice appear in a quite general setting. They are associated to pairs $(\mathfrak{p}, s)$, where $\mathfrak{p}$ is a parabolic subalgebra in the Lie algebra $\mathfrak{g}$ of the group G, and $s \in G$ is a representative of an element of the Weyl group W such that the operator Ads has no nonzero fixed points in $\mathfrak{n}$ and $\overline{\mathfrak{n}}$, where $\mathfrak{n}$ is the nilradical of $\mathfrak{p}$, and $\overline{\mathfrak{n}}$ the nilradical of the opposite parabolic subalgebra. Such a slice always contains the element $s^{-1} \in G$. Note that since any Weyl group element $s \in W$ has finite order

its representative $s \in G$ is semisimple, and hence the slice associated to a pair $(\mathfrak{p}, s)$ is transversal to the set of conjugacy classes in G at the semisimple element $s^{-1} \in G$ while the Slodowy slice associated to a nilpotent element e is always transversal to the set of adjoint orbits at the nilpotent element e.

To any element $s \in W$ we also naturally associate some parabolic subalgebras $\mathfrak{p} \subset \mathfrak{g}$ and construct the corresponding slices. In the set of those parabolic subalgebras one can distinguish generic parabolic subalgebras for which the semisimple parts of their Levi factors only contain fixpoints of the operator $\mathrm{Ad}\,s$. The generic parabolic subalgebras associated to elements of the Weyl group play the role of parabolic subalgebras $\mathfrak{p} = \bigoplus_{i \leq 0} \mathfrak{g}(i)$ associated to nilpotent elements of $\mathfrak{g}$ with the help of grading (1.1).

In case of classical simple Lie algebras other types of slices in G are discussed in Section 3. In those examples the parabolic subalgebra $\mathfrak{p}$ is constructed with the help of grading (1.1) associated to a nilpotent element $e \in \mathfrak{g}$, $\mathfrak{p} = \bigoplus_{i \leq 0} \mathfrak{g}(i)$. The element $s \in W$ is also naturally associated to e via the Kazhdan–Lusztig map which sends nilpotent conjugacy classes in $\mathfrak{g}$ to conjugacy classes in W (see [10]). This map can be explicitly described in case of the classical Lie algebras (see [18, 19]), and we explicitly construct the corresponding examples of slices in G in Section 3. In case of $\mathfrak{g} = \mathfrak{sl}_n$ the dimension of such a slice associated to a nilpotent element $e \in \mathfrak{g}$ coincides with the dimension of the Slodowy slice $s(e)$. This establishes a correspondence between Slodowy slices in the Lie algebra $\mathfrak{g}$ and some special transversal slices in the Lie group G. In the general case a similar correspondence is difficult to establish since at the moment there is no explicit description of the Kazhdan–Lusztig map for exceptional Lie algebras (see, however, paper [20] where the values of this map for some nilpotent elements were calculated). But such a correspondence presumably exists.

In Section 4 we apply the results of Section 3 to describe simple singularities in terms of transversal slices in algebraic groups. In our construction we use elements of the Weyl group associated to subregular nilpotent elements via the Kazhdan–Lusztig map. Representatives of these Weyl group elements in G are also subregular.

2. TRANSVERSAL SLICES TO CONJUGACY CLASSES IN ALGEBRAIC GROUPS

In this section we introduce algebraic group counterparts of the Slodowy slices and prove an analogue of the Kostant cross–section theorem for them.

Let G be a complex semisimple algebraic group, $\mathfrak{g}$ its Lie algebra. Let P be a parabolic subgroup of G, L its Levi factor, and N the unipotent radical of P. Denote by $\mathfrak{p}$, $\mathfrak{l}$ and $\mathfrak{n}$ the Lie algebras of P, L and N, respectively. Let $\overline{P}$ be the opposite parabolic subgroup and $\overline{N}$ the unipotent radical of $\overline{P}$. Denote by $\overline{\mathfrak{p}}$ and $\overline{\mathfrak{n}}$ the Lie algebras of $\overline{P}$ and $\overline{N}$, respectively.

Let $s \in W$ be an element of the Weyl group W of G. Fix a representative of s in G. We denote this representative by the same letter, $s \in G$. The element $s \in G$ naturally acts on G by conjugations. Let Z be the set of s-fixed points in L,

$$Z = \{z \in L \mid szs^{-1} = z\},$$

and
$$N_s = \{n \in N \mid sns^{-1} \in \overline{P}\}.$$

Clearly, Z and N_s are subgroups in G, and Z normalizes both N and N_s. Denote by $\mathfrak{n}_s$ and $\mathfrak{z}$ the Lie algebras of N_s and Z, respectively.

Now consider the subvariety $N_s Z s^{-1} \subset G$.

Proposition 2.1. *Assume that for any $x \in \mathfrak{n}$ and any $y \in \overline{\mathfrak{n}}$, $x, y \neq 0$ there exist $k, k' \in \mathbb{N}$ such that $\mathrm{Ad}(s^k)(x) \notin \mathfrak{p}$ and $\mathrm{Ad}(s^{k'})(y) \notin \overline{\mathfrak{p}}$. Then the conjugation map $G \times N_s Z s^{-1} \to G$ is smooth, and the variety $N_s Z s^{-1} \subset G$ is a transversal slice to the set of conjugacy classes in G.*

Proof. We have to show that the conjugation map

$$(2.1) \qquad \gamma : G \times N_s Z s^{-1} \to G$$

has the surjective differential.

Note that the set of smooth points of map (2.1) is stable under the G–action by left translations on the first factor of $G \times N_s Z s^{-1}$. Therefore it suffices to show that the differential of map (2.1) is surjective at points $(1, n_s z s^{-1})$, $n_s \in N_s$, $z \in Z$.

In terms of the right trivialization of the tangent bundle TG and the induced trivialization of $T(N_s Z s^{-1})$ the differential of map (2.1) at points $(1, n_s z s^{-1})$ takes the form

$$(2.2) \qquad d\gamma_{(1, n_s z s^{-1})} : (x, (n, w)) \to (Id - \mathrm{Ad}(n_s z s^{-1}))x + n + w,$$
$$x \in \mathfrak{g} \simeq T_1(G), (n, w) \in \mathfrak{n}_s + \mathfrak{z} \simeq T_{n_s z s^{-1}}(N_s Z s^{-1}).$$

Now we need a convenient description of the parabolic subalgebras $\mathfrak{p}$, $\overline{\mathfrak{p}}$ and of their nilradicals $\mathfrak{n}$, $\overline{\mathfrak{n}}$. Recall that one can fix a semisimple element $h \in \mathfrak{g}$ such that if we introduce the linear subspaces $\mathfrak{g}_k \subset \mathfrak{g}, k \in \mathbb{C}$ by $\mathfrak{g}_k = \{x \in \mathfrak{g} \mid [h, x] = kx\}$ then we have the following linear space isomorphisms $\mathfrak{n} = \bigoplus_{k<0} \mathfrak{g}_k$, $\mathfrak{p} = \bigoplus_{k \leq 0} \mathfrak{g}_k$, $\mathfrak{l} = \{x \in \mathfrak{g} \mid [h, x] = 0\}$, $\overline{\mathfrak{p}} = \bigoplus_{k \geq 0} \mathfrak{g}_k$. Let $\mathfrak{h}$ be the Cartan subalgebra in $\mathfrak{g}$ containing h and normalizing $\mathfrak{p}$. Denote by H the subgroup in G corresponding to $\mathfrak{h}$. Clearly, H normalizes Z, N and N_s.

In order to show that map (2.2) is surjective we consider the following map

$$(2.3) \qquad \sigma : G \times N_s Z H s^{-1} \to G$$

induced by the action of the group G on itself by conjugations. We shall identify map (2.2) with the restriction of the differential of map (2.3) to a certain subspace in the tangent space to $G \times N_s Z H s^{-1}$ at point $(1, n_s z s^{-1}) \in G \times N_s Z H s^{-1}$, $n_s \in N_s, z \in Z$.

First observe that in terms of the right trivialization of the tangent bundle TG the tangent bundle $T(N_s Z H s^{-1})$ can be identified with $(\mathfrak{n}_s + \mathfrak{z} + \mathfrak{h}) \times N_s Z H s^{-1}$. Consider the subbundle $T'(N_s Z H s^{-1})$ in $T(N_s Z H s^{-1})$ which is isomorphic to $(\mathfrak{n}_s + \mathfrak{z}) \times N_s Z H s^{-1}$ in terms of the right trivialization of TG. Clearly, the tangent bundle $T(N_s Z s^{-1})$ is identified with $(\mathfrak{n}_s + \mathfrak{z}) \times N_s Z s^{-1}$, and hence the restriction of $T'(N_s Z H s^{-1})$ to $N_s Z s^{-1} \subset N_s Z H s^{-1}$ coincides with $T(N_s Z s^{-1})$.

Now straightforward calculation shows that for any $n_s \in N_s$ and $z \in Z$ the restriction of the differential $d\sigma_{(1, n_s z s^{-1})}$ of map (2.3) to the subspace

$$T_1 G + T_{n_s z s^{-1}}(N_s Z s^{-1}) \simeq T_1 G + T'_{n_s z s^{-1}}(N_s Z H s^{-1}) \subset T_1 G + T_{n_s z s^{-1}}(N_s Z H s^{-1})$$

coincides with differential (2.2) of map (2.1). Therefore in order to prove that map (2.2) is surjective it suffices to show that the map

$$d\sigma_{(1,n_s z s^{-1})} : T_1 G + T'_{n_s z s^{-1}}(N_s Z H s^{-1}) \to T_{n_s z s^{-1}} G$$

is surjective.

More generally we shall prove that the image of the map

$$(2.4) \qquad d\sigma_{(1,n_s z c s^{-1})} : T_1 G + T'_{n_s z c s^{-1}}(N_s Z H s^{-1}) \to T_{n_s z c s^{-1}} G,$$

$$n_s \in N_s, \ z \in Z, c \in H$$

coincides with $\mathfrak{g} \simeq T_{n_s z c s^{-1}} G$.

Define a $\mathbb{C}^*$–action on the group G as follows

$$(2.5) \qquad t \cdot g = e^{-th} g e^{th}, t \in \mathbb{C}^*, g \in G.$$

The subset $N_s Z H s^{-1}$ of G is invariant under action (2.5). Moreover, action (2.3) is $\mathbb{C}^*$–equivariant, and the set of points $(1, n_s z c s^{-1})$, $n_s \in N_s$, $z \in Z$, $c \in H$ and the subbundle $T'(N_s Z H s^{-1})$ of $T(N_s Z H s^{-1})$ are stable under this action. Since N is the Lie group of $\mathfrak{n} = \bigoplus_{k<0} \mathfrak{g}_k$, $\mathfrak{g}_k = \{x \in \mathfrak{g} \mid [h, x] = kx\}$ and Z, H are the Lie groups of $\mathfrak{z}, \mathfrak{h} \subset \mathfrak{l} = \{x \in \mathfrak{g} \mid [h, x] = 0\}$ the variety $N_s Z H s^{-1}$ can be contracted by the $\mathbb{C}^*$–action to $Z H s^{-1}$. Therefore it suffices to prove that map (2.4) is surjective at points $(1, z c s^{-1})$, $z \in Z, c \in H$.

In terms of the right trivialization of the tangent bundle TG we have

$$(2.6) \, d\sigma_{(1,z c s^{-1})} : (x, (n, w)) \to (Id - \mathrm{Ad}(z c s^{-1}))x + n + w, \ x \in \mathfrak{g} \simeq T_1(G),$$

$$(n, w) \in \mathfrak{n}_s + \mathfrak{z} \simeq T'_{z c s^{-1}}(N_s Z H s^{-1}).$$

In order to show that the image of map (2.6) coincides with $T_{z c s^{-1}} G \simeq \mathfrak{g}$ we shall need a direct vector space decomposition of the Lie algebra $\mathfrak{g}$,

$$(2.7) \qquad \mathfrak{g} = \mathfrak{n}_s + \mathfrak{n}' + \mathfrak{z} + \mathfrak{z}' + \overline{\mathfrak{n}},$$

where $\mathfrak{n}' = \mathrm{Ad} s^{-1}(\mathfrak{n}) \cap \mathfrak{n}$ and $\mathfrak{z}'$ is a complementary subspace to $\mathfrak{z}$ in $\mathfrak{l}$. For any element $y \in \mathfrak{g}$ we denote by $y = y_{\mathfrak{n}_s} + y_{\mathfrak{n}'} + y_{\mathfrak{z}} + y_{\mathfrak{z}'} + y_{\overline{\mathfrak{n}}}$ its decomposition corresponding to direct vector space decomposition (2.7).

We shall show that for any $y \in \mathfrak{g} \simeq T_{z c s^{-1}} G$ one can find $x = x_{\mathfrak{n}'} + x_{\mathfrak{z}'} + x_{\overline{\mathfrak{n}}} \in \mathfrak{g} \simeq T_1(G)$ and $(n, w) \in \mathfrak{n}_s + \mathfrak{z} \simeq T'_{z c s^{-1}}(N_s Z H s^{-1})$ such that

$$(2.8) \qquad d\sigma_{(1,z c s^{-1})}(x, (n, w)) = y$$

Using explicit formula (2.6) for the differential of map (2.3) we deduce from (2.8) the following equation for x, n and w

$$(2.9) \qquad (Id - \mathrm{Ad}(z c s^{-1}))(x_{\mathfrak{n}'} + x_{\mathfrak{z}'} + x_{\overline{\mathfrak{n}}}) + n + w = y_{\mathfrak{n}_s} + y_{\mathfrak{n}'} + y_{\mathfrak{z}} + y_{\mathfrak{z}'} + y_{\overline{\mathfrak{n}}}.$$

Taking the $\mathfrak{n}_s, \mathfrak{n}', \mathfrak{z}, \mathfrak{z}'$ and $\overline{\mathfrak{n}}$–components of the l.h.s. and of the r.h.s. of the last equation we reduce (2.9) to a system of linear equations,

$$(2.10) \qquad x_{\mathfrak{n}'} - (\mathrm{Ad}(z c s^{-1}) x_{\mathfrak{n}'})_{\mathfrak{n}'} = y_{\mathfrak{n}'},$$

$$(2.11) \quad x_{\overline{\mathfrak{n}}} + x_{\mathfrak{z}'} - (\mathrm{Ad}(z c s^{-1})(x_{\overline{\mathfrak{n}}} + x_{\mathfrak{z}'}))_{\mathfrak{z}'+\overline{\mathfrak{n}}} - (\mathrm{Ad}(z c s^{-1}) x_{\mathfrak{n}'})_{\mathfrak{z}'+\overline{\mathfrak{n}}} = y_{\overline{\mathfrak{n}}} + y_{\mathfrak{z}'},$$

$$(2.12) \qquad -(\mathrm{Ad}(z c s^{-1})(x_{\overline{\mathfrak{n}}} + x_{\mathfrak{z}'}))_{\mathfrak{z}} - (\mathrm{Ad}(z c s^{-1}) x_{\mathfrak{n}'})_{\mathfrak{z}} + w = y_{\mathfrak{z}},$$

$$(2.13) \qquad -(\mathrm{Ad}(z c s^{-1})(x_{\overline{\mathfrak{n}}} + x_{\mathfrak{z}'}))_{\mathfrak{n}_s} + n = y_{\mathfrak{n}_s}.$$

Now recall that for each $x \in \mathfrak{n}(\overline{\mathfrak{n}})$, $x \neq 0$ there exists $k \in \mathbb{N}$ such that $\mathrm{Ad}(s^k)(x) \notin \mathfrak{p}(\overline{\mathfrak{p}})$. Observe also that the element s normalizes both Z and H, and Z and H

normalize $\mathfrak{n}$, $\overline{\mathfrak{n}}$ and $\mathfrak{l}$. Therefore the operator $\mathrm{Ad}(zcs^{-1})$ has no fixed points in the subspaces $\mathfrak{n}'$ and $\mathfrak{z}' + \overline{\mathfrak{n}}$, and the operators $\mathfrak{n}' \to \mathfrak{n}'$, $x_{\mathfrak{n}'} \mapsto x_{\mathfrak{n}'} - (\mathrm{Ad}(zcs^{-1})x_{\mathfrak{n}'})_{\mathfrak{n}'}$ and $\mathfrak{z}' + \overline{\mathfrak{n}} \to \mathfrak{z}' + \overline{\mathfrak{n}}$, $x_{\overline{\mathfrak{n}}} + x_{\mathfrak{z}'} \mapsto x_{\overline{\mathfrak{n}}} + x_{\mathfrak{z}'} - (\mathrm{Ad}(zcs^{-1})(x_{\overline{\mathfrak{n}}} + x_{\mathfrak{z}'}))_{\mathfrak{z}' + \overline{\mathfrak{n}}}$ are invertible. Now from equation (2.10) one can find $x_{\mathfrak{n}'}$ in a unique way, and after that equation (2.11) uniquely determines $x_{\overline{\mathfrak{n}}} + x_{\mathfrak{z}'}$. Finally from equations (2.12) and (2.13) we obtain w and n, respectively. This completes the proof. $\qquad\square$

We call the variety $N_s Z s^{-1}$ the transversal slice in G associated to the pair $(\mathfrak{p}, s)$. The following statement is an analogue of the Kostant cross–section theorem for the slice $N_s Z s^{-1} \subset G$.

Proposition 2.2. *Assume that for each $x \in \mathfrak{n}$, $x \neq 0$ there exists $k \in \mathbb{N}$ such that $\mathrm{Ad}(s^k)(x) \notin \mathfrak{n}$. Then the conjugation map*

$$(2.14) \qquad\qquad \alpha : N \times N_s Z s^{-1} \to N Z s^{-1} N$$

is an isomorphism of varieties.

Proof. First observe that the variety $N Z s^{-1} N$ is isomorphic to $N_s Z s^{-1} N$ and hence the domain of map (2.14) is isomorphic to $N Z s^{-1} N \simeq N_s Z s^{-1} N$ as a variety. Therefore in order to prove that map (2.14) is an isomorphism it suffices to show that the centralizer of each point in $N_s Z s^{-1}$ for the action of N by conjugations is trivial. Since the group N is unipotent the exponential map establishes a diffeomorphism of this centralizer and of its Lie algebra. Therefore the centralizer of each point in $N_s Z s^{-1}$ for the action of N by conjugations is trivial if and only if the differential of map (2.14) is surjective at points $(1, n_s z s^{-1})$, $n_s \in N_s$, $z \in Z$.

In terms of the right trivialization of the tangent bundle TG and the induced trivializations of $T(N_s Z s^{-1})$ and $T(N_s Z s^{-1} N)$ the differential of map (2.14) at points $(1, n_s z s^{-1})$ takes the form

$$(2.15) \qquad d\alpha_{(1, n_s z s^{-1})} : (x, (n, w)) \to (Id - \mathrm{Ad}(n_s z s^{-1}))x + n + w,$$

$$x \in \mathfrak{n} \simeq T_1(N), (n, w) \in \mathfrak{n}_s + \mathfrak{z} \simeq T_{n_s z s^{-1}}(N_s Z s^{-1}).$$

Now in order to show that map (2.15) is surjective we are going to apply the trick used in the proof of the previous proposition. First we consider the conjugation map,

$$(2.16) \qquad\qquad \beta : N \times N_s Z H s^{-1} \to N Z H s^{-1} N,$$

where H is the Cartan subgroup in G introduced in the proof of Proposition 2.1. Map (2.16) is the restriction of map (2.3) to the subset $N \times N_s Z H s^{-1} \subset G \times N_s Z H s^{-1}$.

As in the previous proposition we identify map (2.15) with the restriction of the differential of map (2.16) to the subspace

$$T_1 N + T_{n_s z s^{-1}}(N_s Z s^{-1}) \simeq T_1 N + T'_{n_s z s^{-1}}(N_s Z H s^{-1}) \subset T_1 N + T_{n_s z s^{-1}}(N_s Z H s^{-1})$$

at points $(1, n_s z s^{-1}) \in N \times N_s Z H s^{-1}$, $n_s \in N_s, z \in Z$.

Beside of the subbundle $T'(N_s Z H s^{-1})$ of $T(N_s Z H s^{-1})$ which is isomorphic to $(\mathfrak{n}_s + \mathfrak{z}) \times N_s Z H s^{-1}$ in terms of the right trivialization of TG we introduce a similar subbundle $T'(N_s Z H s^{-1} N)$ of $T(N_s Z H s^{-1} N)$ in the following way.

Using the fact that $N Z H s^{-1} N \simeq N_s Z H s^{-1} N$, one can show that in terms of the right trivialization of the tangent bundle TG the tangent space $T_{n_s z c s^{-1} q}(N_s Z H s^{-1} N)$, $n_s \in N_s, z \in Z, c \in H, q \in N$ is isomorphic to $\mathfrak{n}_s + \mathfrak{z} + \mathfrak{h} + \mathrm{Ad}(n_s z c s^{-1})(\mathfrak{n})$. We define the subbundle $T'(N_s Z H s^{-1} N)$ in $T(N_s Z H s^{-1} N)$ the fiber $T'_{n_s z c s^{-1} q}(N_s Z H s^{-1} N)$

of which at each point $n_s z c s^{-1} q \in N_s Z H s^{-1} N$ is identified with $\mathfrak{n}_s + \mathfrak{z} + \mathrm{Ad}(n_s z c s^{-1})(\mathfrak{n})$ in terms of the right trivialization of TG. The fiber $T_{n_s z s^{-1} q}(N_s Z s^{-1} N)$ of the tangent bundle $T(N_s Z s^{-1} N)$ at each point $n_s z s^{-1} q \in N_s Z s^{-1} N$ is also identified with $\mathfrak{n}_s + \mathfrak{z} + \mathrm{Ad}(n_s z s^{-1})(\mathfrak{n})$. Therefore the restriction of the bundle $T'(N_s Z H s^{-1} N)$ to $N_s Z s^{-1} N \subset N_s Z H s^{-1} N$ coincides with $T(N_s Z s^{-1} N)$.

Now in order to prove that map (2.15) is surjective it suffices to show that the map

$$d\beta_{(1, n_s z s^{-1})} : T_1 N + T'_{n_s z s^{-1}}(N_s Z H s^{-1}) \to T_{n_s z s^{-1}}(N_s Z s^{-1} N) \simeq$$
$$\simeq T'_{n_s z s^{-1}}(N_s Z H s^{-1} N) \subset T_{n_s z s^{-1}}(N_s Z H s^{-1} N)$$

is surjective.

More generally we shall prove that the image of the map

$$d\beta_{(1, n_s z c s^{-1})} : T_1 N + T'_{n_s z c s^{-1}}(N_s Z H s^{-1}) \to T_{n_s z c s^{-1}}(N_s Z H s^{-1} N),$$
$$n_s \in N_s, \ z \in Z, c \in H$$

belongs to the subspace $T'_{n_s z c s^{-1}}(N_s Z H s^{-1} N) \subset T_{n_s z c s^{-1}}(N_s Z H s^{-1} N)$, and the map

$$(2.17) \qquad d\beta_{(1, n_s z c s^{-1})} : T_1 N + T'_{n_s z c s^{-1}}(N_s Z H s^{-1}) \to T'_{n_s z c s^{-1}}(N_s Z H s^{-1} N),$$
$$n_s \in N_s, \ z \in Z, c \in H$$

is an isomorphism.

Observe that the subsets N, $N_s Z H s^{-1}$ and $N_s Z H s^{-1} N$ of G are invariant under $\mathbb{C}^*$–action (2.5) on the group G. Moreover, action (2.16) is $\mathbb{C}^*$–equivariant, and the set of points $(1, n_s z c s^{-1})$, $n_s \in N_s$, $z \in Z$, $c \in H$ and the subbundles $T'(N_s Z H s^{-1})$ of $T(N_s Z H s^{-1})$ and $T'(N_s Z H s^{-1} N)$ of $T(N_s Z H s^{-1} N)$ are stable under this action. Since N is the Lie group of $\mathfrak{n} = \bigoplus_{k<0} \mathfrak{g}_k$, $\mathfrak{g}_k = \{x \in \mathfrak{g} \mid [h, x] = kx\}$ and Z, H are the Lie groups of $\mathfrak{z}, \mathfrak{h} \subset \mathfrak{l} = \{x \in \mathfrak{g} \mid [h, x] = 0\}$ the variety $N_s Z H s^{-1}$ can be contracted by the $\mathbb{C}^*$–action to $Z H s^{-1}$. Therefore it suffices to prove that map (2.17) is an isomorphism at points $(1, z c s^{-1})$, $z \in Z, c \in H$.

In terms of the right trivialization of the tangent bundle TG we have

$$(2.18) \qquad d\beta_{(1, z d s^{-1})} : (x, (n, w)) \to (Id - \mathrm{Ad}(z c s^{-1}))x + n + w,$$
$$x \in \mathfrak{n} \simeq T_1(N), \ (n, w) \in \mathfrak{n}_s + \mathfrak{z} \simeq T'_{z c s^{-1}}(N_s Z H s^{-1}).$$

Now by dimensional count it suffices to show that the image of map (2.18) coincides with $T'_{z c s^{-1}}(N_s Z H s^{-1} N) \simeq \mathfrak{n}_s + \mathfrak{z} + \mathrm{Ad}(z c s^{-1})(\mathfrak{n})$.

Indeed, introducing the subspace $\mathfrak{n}' = \mathrm{Ad} s^{-1}(\mathfrak{n}) \cap \mathfrak{n}$ we get a direct vector space decomposition $\mathfrak{n} = \mathfrak{n}_s + \mathfrak{n}'$. For any $x \in \mathfrak{n}$ we denote by $x_{\mathfrak{n}_s}$ and $x_{\mathfrak{n}'}$ the components of x in this decomposition, $x = x_{\mathfrak{n}_s} + x_{\mathfrak{n}'}$. For any $x \in \mathfrak{n} \simeq T_1(N)$ and $(n, w) \in \mathfrak{n}_s + \mathfrak{z} \simeq T'_{z c s^{-1}}(N_s Z H s^{-1})$ we obviously have

$$(2.19) \quad d\beta_{(1, z c s^{-1})}(x, (n, w)) = (Id - \mathrm{Ad}(z c s^{-1}))x + n + w =$$
$$= \mathrm{Ad}(z c s^{-1})(\mathrm{Ad}(s c^{-1} z^{-1}) x_{\mathfrak{n}'} - x) + x_{\mathfrak{n}_s} + n + w \in \mathfrak{n}_s + \mathfrak{z} + \mathrm{Ad}(z c s^{-1})(\mathfrak{n}),$$

and hence the image of the map

$$d\beta_{(1, z c s^{-1})} : T_1 N + T'_{z c s^{-1}}(N_s Z H s^{-1}) \to T_{z c s^{-1}}(N_s Z H s^{-1} N)$$

is a subspace of $T'_{z c s^{-1}}(N_s Z H s^{-1} N) \simeq \mathfrak{n}_s + \mathfrak{z} + \mathrm{Ad}(z c s^{-1})(\mathfrak{n})$.

Now let $y_{\mathfrak{n}_s} + y_{\mathfrak{z}} + \mathrm{Ad}(z c s^{-1}) v \in \mathfrak{n}_s + \mathfrak{z} + \mathrm{Ad}(z c s^{-1})(\mathfrak{n})$, $y_{\mathfrak{n}_s} \in \mathfrak{n}_s, y_{\mathfrak{z}} \in \mathfrak{z}, v \in \mathfrak{n}$ be an arbitrary element of $\mathfrak{n}_s + \mathfrak{z} + \mathrm{Ad}(z c s^{-1})(\mathfrak{n}) \simeq T'_{z c s^{-1}}(N_s Z H s^{-1} N)$. We prove

that there exist elements $x \in \mathfrak{n} \simeq T_1(N)$ and $(n, w) \in \mathfrak{n}_s + \mathfrak{z} \simeq T'_{zcs^{-1}}(N_s ZH s^{-1})$ such that

$$(2.20) \qquad d\beta_{(1, zcs^{-1})}(x, (n, w)) = (Id - \mathrm{Ad}(zcs^{-1}))x + n + w =$$
$$= y_{\mathfrak{n}_s} + y_{\mathfrak{z}} + \mathrm{Ad}(zcs^{-1})v \in \mathfrak{n}_s + \mathfrak{z} + \mathrm{Ad}(zcs^{-1})(\mathfrak{n}).$$

First by (2.19) equation (2.20) implies that

$$\mathrm{Ad}(zcs^{-1})(\mathrm{Ad}(sc^{-1}z^{-1})x_{\mathfrak{n}'} - x) + x_{\mathfrak{n}_s} + n + w = y_{\mathfrak{n}_s} + y_{\mathfrak{z}} + \mathrm{Ad}(zcs^{-1})v$$

which is equivalent to a system of linear equations,

$$(2.21) \qquad\qquad\qquad \mathrm{Ad}(sc^{-1}z^{-1})x_{\mathfrak{n}'} - x = v,$$

$$(2.22) \qquad\qquad\qquad\qquad x_{\mathfrak{n}_s} + n = y_{\mathfrak{n}_s},$$

$$(2.23) \qquad\qquad\qquad\qquad\qquad w = y_{\mathfrak{z}}.$$

Now recall that for each $x \in \mathfrak{n}$, $x \neq 0$ there exists $k \in \mathbb{N}$ such that $\mathrm{Ad}(s^k)(x) \notin \mathfrak{n}$. Observe also that the element s normalizes both Z and H, and Z and H normalize $\mathfrak{n}$. Therefore the operator $\mathfrak{n} \to \mathfrak{n}$, $x \mapsto \mathrm{Ad}(sc^{-1}z^{-1})x_{\mathfrak{n}'} - x$ is invertible, and hence equation (2.21) has a unique solution. From equations (2.22) and (2.23) we immediately find n and w in a unique way. Therefore map (2.17) is an isomorphism, and hence map (2.14) is an isomorphism of varieties as well. This completes the proof. $\qquad\square$

Note that the conditions of Proposition 2.2 imposed on s are weaker than the conditions of Proposition 2.1. Actually Proposition 2.2 is essentially a statement about unipotent groups only, and if the conditions of Proposition 2.2 are satisfied then the corresponding variety $N_s Z s^{-1}$ is not necessarily a transversal slice to the set of conjugacy classes in the group G.

Now for each Weyl group element we construct some natural parabolic subalgebras which satisfy the conditions of Propositions 2.1 and 2.2. First recall that in the classification theory of conjugacy classes in Weyl group W of complex simple Lie algebra $\mathfrak{g}$ the so-called primitive (or semi–Coxeter in another terminology) elements play a primary role. The primitive elements $w \in W$ are characterized by the property $\det(1 - w) = \det a$, where a is the Cartan matrix of $\mathfrak{g}$. According to the results of [2] each element s of the Weyl group of the pair $(\mathfrak{g}, \mathfrak{h})$ is a primitive element in the Weyl group W' of a regular semisimple Lie subalgebra $\mathfrak{g}' \subset \mathfrak{g}$ of the form

$$\mathfrak{g}' = \mathfrak{h}' + \sum_{\alpha \in \Delta'} X_\alpha,$$

where Δ' is a root subsystem of the root system Δ of $\mathfrak{g}$, X_α is the root subspace of $\mathfrak{g}$ corresponding to root α, and $\mathfrak{h}'$ is a Lie subalgebra of $\mathfrak{h}$.

Proposition 2.3. *Let s be an element of the Weyl group of the pair $(\mathfrak{g}, \mathfrak{h})$. Denote a representative of s in G by the same letter. Assume that s is primitive in the Weyl group W' of a pair $(\mathfrak{g}', \mathfrak{h}')$, where $\mathfrak{g}'$ is a regular subalgebra of $\mathfrak{g}$ and $\mathfrak{h}' \subset \mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}'$. Let $\mathfrak{p}$ be a parabolic subalgebra in $\mathfrak{g}$ defined by $\mathfrak{p} = \bigoplus_{n \geq 0} \mathfrak{g}_n$, where $\mathfrak{g}_n = \{x \in \mathfrak{g} \mid [h, x] = nx\}$ for some $h \in \mathfrak{h}'$. Denote by $\mathfrak{n}$ the nilradical of $\mathfrak{p}$, $\mathfrak{n} = \bigoplus_{n > 0} \mathfrak{g}_n$. Then for any $x \in \mathfrak{n}$ and $y \in \bar{\mathfrak{n}}$, $x, y \neq 0$ there exist $k, k' \in \mathbb{N}$ such that $\mathrm{Ad}(s^k)(x) \notin \mathfrak{p}$ and $\mathrm{Ad}(s^{k'})(y) \notin \bar{\mathfrak{p}}$.*

Proof. Assume that there exists $x \in \mathfrak{n}$ such that for each $k \in \mathbb{N}$ $\mathrm{Ad}(s^k)(x) \in \mathfrak{p}$. Since s has finite order as an operator acting on the root system Δ and $\mathrm{Ad}s$ sends each root subspace X_α to $X_{s(\alpha)}$ our assumption implies that there is an element $\eta \in \mathbb{Z}\Delta$ such that $s\eta = \eta$ and $\eta(h) > 0$; one can take η to be the sum of different roots corresponding to root vectors in the $\mathrm{Ad}s$–orbit of x.

Now recall that s is primitive in the Weyl subgroup W' and that any primitive element of W' is a product of rank $\mathfrak{g}'$ reflections with respect to linearly independent roots (see [2]). Therefore η must be fixed not only by s but also by all those reflections (see [22]), and hence η must belong to the orthocomplement of the subspace $\mathfrak{h}' \subset \mathfrak{h}$. In particular, since $h \in \mathfrak{h}'$ we deduce that $\eta(h) = 0$. Thus we arrive at a contradiction. This completes the proof. $\square$

Summarizing the discussion above and recalling Propositions 2.1 and 2.2 one can formulate the following statement.

Proposition 2.4. *Let s be an element of the Weyl group of the pair $(\mathfrak{g}, \mathfrak{h})$. Denote by the same letter a representative of s in G. Assume that s is primitive in the Weyl group W' of a pair $(\mathfrak{g}', \mathfrak{h}')$, where $\mathfrak{g}'$ is a regular subalgebra of $\mathfrak{g}$ and $\mathfrak{h}' \subset \mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}'$. Let $\mathfrak{p}$ be the parabolic subalgebra in $\mathfrak{g}$ defined by $\mathfrak{p} = \bigoplus_{n \geq 0} \mathfrak{g}_n$, where $\mathfrak{g}_n = \{x \in \mathfrak{g} \mid [h, x] = nx\}$ for some $h \in \mathfrak{h}'$. Let $\mathfrak{l}$ and $\mathfrak{n}$ be the Levi factor and the nilradical of $\mathfrak{p}$. Denote by P, L and N the Lie subgroups of G corresponding to Lie subalgebras $\mathfrak{p}$, $\mathfrak{l}$ and $\mathfrak{n}$, respectively. Let*

$$Z = \{z \in L \mid szs^{-1} = z\}$$

be the centralizer of s in L, and

$$N_s = \{n \in N \mid sns^{-1} \in \overline{P}\},$$

where $\overline{P}$ is the parabolic subalgebra opposite to P. Then the conjugation map $G \times N_sZs^{-1} \to G$ is smooth, the variety $N_sZs^{-1} \subset G$ is a transversal slice to the set of conjugacy classes in G, and the conjugation map

$$(2.24) \qquad\qquad N \times N_sZs^{-1} \to NZs^{-1}N$$

is an isomorphism of varieties.

In conclusion we note that one can naturally associate a generic parabolic subalgebra $\mathfrak{p}$ to any element $s \in W$. Namely, choosing a generic element $h \in \mathfrak{h}'$ used in the previous proposition one can construct a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ with the help of h in such a way that the semisimple part of the Levi factor $\mathfrak{l}$ of $\mathfrak{p}$ is contained in the centralizer of s in $\mathfrak{g}$.

Indeed, assume again that s is primitive in the Weyl group W' of a pair $(\mathfrak{g}', \mathfrak{h}')$, where $\mathfrak{g}'$ is a regular subalgebra of $\mathfrak{g}$ and $\mathfrak{h}' \subset \mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}'$. Let $\Delta^\perp$ be the subset of the root system Δ which consists of the roots orthogonal to $\mathfrak{h}'$,

$$\Delta^\perp = \{\alpha \in \Delta : \alpha \perp \mathfrak{h}'\}.$$

Obviously, $\Delta^\perp$ is a root subsystem of Δ. Since s is contained in the Weyl subgroup W', and elements of W' act trivially on the orthocomplement of $\mathfrak{h}'$ in $\mathfrak{h}$, $\Delta^\perp$ consists of the roots which are fixed by the action of s. Denote by $\widetilde{\Delta}$ the complementary subset of $\Delta^\perp$ in Δ, $\widetilde{\Delta} = \Delta \setminus \Delta^\perp$ and by $\widetilde{\Delta}'$ the orthogonal projection of $\widetilde{\Delta}$ onto $\mathfrak{h}'$. The set $\widetilde{\Delta}'$ is a finite subset of the real form $\mathfrak{h}'_{\mathbb{R}}$ of $\mathfrak{h}'$, the real span of simple

coroots in $\mathfrak{h}'$. Let Π be the union of hyperplanes in $\mathfrak{h}'_{\mathbb{R}}$ which are orthogonal to the elements of $\widetilde{\Delta}'$,

$$\Pi = \cup_{\alpha \in \widetilde{\Delta}'} V_\alpha, V_\alpha = \{x \in \mathfrak{h}'_{\mathbb{R}} : \alpha(x) = 0\}.$$

Let h be an arbitrary element of $\mathfrak{h}'_{\mathbb{R}}$ which belongs to the complement of Π in $\mathfrak{h}'_{\mathbb{R}}$, $h \in \mathfrak{h}'_{\mathbb{R}} \setminus \Pi$. Denote by $\mathfrak{p}$ the parabolic subalgebra of $\mathfrak{g}$ associated to h, $\mathfrak{p} = \bigoplus_{n \geq 0} \mathfrak{g}_n$, where $\mathfrak{g}_n = \{x \in \mathfrak{g} \mid [h,x] = nx\}$. By the choice of h the semisimple part $\mathfrak{m}$ of the Levi factor $\mathfrak{l}$ of $\mathfrak{p}$ is the semisimple subalgebra of $\mathfrak{g}$ with the root system $\Delta^\perp$. Therefore $\mathfrak{m}$ is fixed by the action of $\mathrm{Ad}s$, where we denote a representative of the Weyl group element s in G by the same letter. Thus $\mathfrak{m}$ is contained in the set of fixpoints of the operator $\mathrm{Ad}s$. In fact in this case $\mathfrak{z} = \mathfrak{m} \oplus \mathfrak{h}_z$ and $\mathfrak{z} \cap \mathfrak{h} = \mathfrak{h}'^\perp$, where $\mathfrak{h}_z$ is a Lie subalgebra of the center of $\mathfrak{l}$ and $\mathfrak{h}'^\perp$ is the orthogonal complement of $\mathfrak{h}'$ in $\mathfrak{h}$ with respect to the Killing form

Moreover, if we denote by N and Z the subgroups of G corresponding to the Lie subalgebras $\mathfrak{n}$ and $\mathfrak{z}$ and by N_s the subgroup of N defined by $N_s = \{n \in N \mid sns^{-1} \in \overline{N}\}$, where $\overline{N}$ is the opposite unipotent subgroup of G then, according to Proposition 2.4, the corresponding variety $N_s Z s^{-1} \subset G$ is a transversal slice to the set of conjugacy classes in G, and the conjugation map

$$(2.25) \qquad\qquad N \times N_s Z s^{-1} \to N Z s^{-1} N$$

is an isomorphism of varieties.

Note that according to the definition of N_s the dimension of the slice $N_s Z s^{-1}$ is equal to $l(s) + F + \dim \mathfrak{h}'^\perp$, where $l(s) = \dim N_s$ is the length of s in W with respect to the system of simple positive roots associated to the Borel subalgebra contained in $\mathfrak{p}$, i.e. the number of the corresponding simple reflections entering a reduced decomposition of s in W, and F is the number of fixed points of s in the root system Δ.

3. Transversal slices in algebraic groups associated to nilpotent elements in their Lie algebras

In this section we consider other examples of the slices introduced in the previous section. In these examples the parabolic subalgebras and the elements of the Weyl group which enter the definition of the slices are associated to nilpotent elements in the underlying Lie algebra via gradings (1.1) and the Kazhdan–Lusztig map, respectively.

First we recall the definition of the Kazhdan–Lusztig map [10]. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, G the adjoint group of $\mathfrak{g}$. Let $\mathbb{C}[[\varepsilon]]$ be the ring of complex power series in a formal variable ε, $\mathfrak{m} = \varepsilon\mathbb{C}[[\varepsilon]]$ its maximal ideal, $\mathbb{C}((\varepsilon))$ its field of fractions, and $\mathfrak{g}((\varepsilon)) = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((\varepsilon))$.

For every nilpotent element $e \in \mathfrak{g}$ there exists a dense open subset U of $e + \mathfrak{m}\mathfrak{g}$ such that all elements of U are regular semisimple and their centralizers in $\mathfrak{g}((\varepsilon))$ are all $G(\mathbb{C}((\varepsilon)))$–conjugate ([10], Proposition 6.1).

The conjugacy classes of Cartan subalgebras in $\mathfrak{g}((\varepsilon))$ under the action of $G(\mathbb{C}((\varepsilon)))$ are parametrized by the conjugacy classes in W, the Weyl group of $\mathfrak{g}$ ([10], Lemma 1.1). Thus one can associate to the G-orbit $\mathbb{O}_e$ of a nilpotent element $e \in \mathfrak{g}$ a $G(\mathbb{C}((\varepsilon)))$–conjugacy class of Cartan subalgebras of $\mathfrak{g}((\varepsilon))$, and hence a conjugacy class s_e in W. The map $\mathbb{O}_e \mapsto s_e$ introduced in [10] is called the Kazhdan-Lusztig map.

We recall that in case of classical Lie algebras the nilpotent conjugacy classes and the Kazhdan–Lusztig map can be explicitly described. In particular, the description of the nilpotent conjugacy classes in the Lie algebra $\mathfrak{sl}_n$ is given in terms of the Jordan normal form.

Proposition 3.1. *Let $\mathfrak{g} = \mathfrak{sl}_n$, V the fundamental representation of $\mathfrak{g}$ and $e \in \mathfrak{g}$ a nilpotent element of $\mathfrak{g}$. Then there exist a partition $d = \{d_1, \ldots, d_r\}$ of n, $\sum_{i=1}^r d_i = n$, and vectors $v_1, \ldots, v_r \in V$ such that all $e^j v_i$ with $1 \leq i \leq r$ and $0 \leq j < d_i$ are a basis for V and such that $e^{d_i} v_i = 0$ for all i. Moreover, two nilpotent elements of $\mathfrak{g}$ belong to the same adjoint orbit if and only if they have the same partition.*

The Weyl group W of $\mathfrak{sl}_n$ is the group of permutations of n elements, and the conjugacy classes in W are also parametrized by partitions of n. Therefore there is a natural one–to–one correspondence between nilpotent orbits in $\mathfrak{sl}_n$ and conjugacy classes in the Weyl group of $\mathfrak{sl}_n$. It turns out that this correspondence coincides with that given by the Kazhdan–Lusztig map, and we have the following proposition.

Proposition 3.2. *([10], Proposition 9.3) Let $e \in \mathfrak{sl}_n$ be a nilpotent element the adjoint orbit of which corresponds to the partition $d = \{d_1, \ldots, d_r\}$ of n. Then the conjugacy class s_e in W that corresponds to the adjoint orbit of e has the same partition as e.*

One can choose a representative $s \in \mathrm{SL}(n)$ for an element from s_e in such a way that using the notation of the previous proposition we have for all $i = 1, \ldots, r$

$$s : e^j v_i \to e^{j-1} v_i, \; j = 1, \ldots, d_i - 1$$

and

$$s : v_i \to (-1)^{d_i - 1} e^{d_i - 1} v_i.$$

In the symplectic and orthogonal cases the nilpotent orbits and the Kazhdan–Lusztig map can also be described in terms of partitions.

Proposition 3.3. *([7], Section 1.11, Theorem 2) Let $\mathfrak{g} = \mathfrak{sp}_{2n}$, V the fundamental representation of $\mathfrak{g}$, and φ the skew–symmetric nondegenerate bilinear form on V preserved by the action of $\mathfrak{g}$. For every nilpotent element $e \in \mathfrak{g}$ there exist a partition $d = \{d_1, \ldots, d_r\}$ of $2n$, $\sum_{i=1}^r d_i = 2n$, and vectors $v_1, \ldots, v_r \in V$ such that:*

(a) $e^j v_i$ with $1 \leq i \leq r$ and $0 \leq j < d_i$ are a basis for V.
(b) $e^{d_i} v_i = 0$ for all i.
(c) If d_i is even then

$$\varphi(e^j v_i, e^h v_i) = \left\{ \begin{array}{ll} (-1)^j, & \text{if } j + h = d_i - 1 \\ 0, & \text{otherwise} \end{array} \right. ,$$

and $\varphi(e^j v_i, e^h v_k) = 0$ for all $k \neq i$ and all j, h.
(d) If d_i is odd then there exists an integer $\delta_i \in \{\pm 1\}$ and an index $i^* \neq i$, $1 \leq i^* \leq r$ with $d_{i^*} = d_i$ such that

$$\varphi(e^j v_i, e^h v_{i^*}) = \left\{ \begin{array}{ll} (-1)^j, & \text{if } j + h = d_i - 1 \\ 0, & \text{otherwise} \end{array} \right. ,$$

and $\varphi(e^j v_i, e^h v_k) = 0$ for all $k \neq i^*$ and all j, h.

Moreover, two nilpotent elements of $\mathfrak{g}$ belong to the same adjoint orbit if and only if they have the same partition.

Let $E = \{1, -1, \ldots, n, -n\}$ and let W_0 be the permutation group of E. For $\mathfrak{sp}_{2n}$ the Weyl group W can be identified with $\{w \in W_0 \mid w(-i) = -w(i) \text{ for each } i\}$. One can attach to each element $w \in W$ two partitions a and b as follows. Let X be a w–orbit in E. Then $-X$ is also a w–orbit. If $X \neq -X$ then a gets one part $a_i = \mid X \mid$ for the pair of orbits $X, -X$. If $X = -X$ then $\mid X \mid$ is even, and b gets one part $b_i = \mid X \mid /2$ for the orbit X. The pair of partitions (a, b) characterizes completely the conjugacy class of w in W, and the partitions (a, b) which arise in this way are exactly those for which $\sum_i a_i + \sum_i b_i = n$.

Proposition 3.4. ([18], **Theorem A**) *Let $e \in \mathfrak{sp}_{2n}$ be a nilpotent element the adjoint orbit of which corresponds to the partition $d = \{d_1, \ldots, d_r\}$ of $2n$. Let a be the partition which has one part d_i for each pair d_i, d_{i^*} of equal odd parts of d, and let b be the partition which has one part $d_i/2$ for each even part d_i of d. Then the conjugacy class s_e in W that corresponds to the adjoint orbit of e has the partition (a, b).*

One can choose a representative $s \in \mathrm{Sp}(2n)$ for an element from s_e in such a way that using the notation of the previous proposition we have

(a) If d_i is odd then

$$s : e^j v_i \rightarrow \delta_i e^{j+\delta_i} v_i \text{ if } j, j + \delta_i = 0, \ldots, d_i - 1$$

and

$$s : v_i \rightarrow e^{d_i - 1} v_i \text{ if } \delta_i = -1$$

or

$$s : e^{d_i - 1} v_i \rightarrow v_i \text{ if } \delta_i = 1.$$

(b) If d_i is even then

$$s : v_i \rightarrow (-1)^{d_i/2} e^{d_i/2} v_i,$$
$$s : e^j v_i \rightarrow e^{j-1} v_i \text{ if } j = 1, \ldots, d_i/2 - 1,$$
$$s : e^j v_i \rightarrow -e^{j+1} v_i \text{ if } j = d_i/2, \ldots, d_i - 2,$$
$$s : e^{d_i - 1} v_i \rightarrow e^{d_i/2 - 1} v_i.$$

Proposition 3.5. ([7], **Section 1.11, Theorem 1**) *Let $\mathfrak{g} = \mathfrak{so}_n$, V the fundamental representation of $\mathfrak{g}$, and φ the symmetric nondegenerate bilinear form on V preserved by the action of $\mathfrak{g}$. For every nilpotent element $e \in \mathfrak{g}$ there exist a partition $d = \{d_1, \ldots, d_r\}$ of n, $\sum_{i=1}^r d_i = n$, and vectors $v_1, \ldots, v_r \in V$ such that:*

(a) $e^j v_i$ with $1 \leq i \leq r$ and $0 \leq j < d_i$ are a basis for V.

(b) $e^{d_i} v_i = 0$ for all i.

(c) If d_i is odd then

$$\varphi(e^j v_i, e^h v_i) = \begin{cases} (-1)^j, & \text{if } j + h = d_i - 1 \\ 0, & \text{otherwise} \end{cases},$$

and $\varphi(e^j v_i, e^h v_k) = 0$ for all $k \neq i$ and all j, h.

(d) If d_i is even then there exists an integer $\delta_i \in \{\pm 1\}$ and an index $i^* \neq i$, $1 \leq i^* \leq r$ with $d_{i^*} = d_i$ such that

$$\varphi(e^j v_i, e^h v_{i^*}) = \begin{cases} (-1)^j, & \text{if } j + h = d_i - 1 \\ 0, & \text{otherwise} \end{cases},$$

and $\varphi(e^j v_i, e^h v_k) = 0$ for all $k \neq i^*$ and all j, h.

Moreover, two nilpotent elements of $\mathfrak{g}$ belong to the same adjoint orbit if and only if they have the same partition.

If n is odd then the Weyl group W of $\mathfrak{g} = \mathfrak{so}_n$ is the same as that of $\mathfrak{g} = \mathfrak{sp}_{n-1}$ and the conjugacy classes in W are therefore also parametrized by pairs of partitions. For n even one can embed W in a Weyl group $\tilde{W}$ of type B_N. The conjugacy class of $\tilde{W}$ corresponding to the pair of partitions (a, b) is contained in W if b has an even number of parts and is disjoint from W otherwise. Moreover, if it is contained in W then it is a single conjugacy class in W, unless b is the empty partition and all the parts of a are even, in which case it splits into two classes.

Proposition 3.6. ([18], **Theorem B**) *Let $e \in \mathfrak{so}_n$ be a nilpotent element the adjoint orbit of which corresponds to the partition $d = \{d_1, \ldots, d_r\}$ of n. Let d^{even} and d^{odd} be the partitions which consist respectively of the even parts and of the odd parts of d, written as decreasing sequences. Define partitions a and b as follows.*

(a) For each even i such that $d_i^{even} \neq 0$, if the number of odd parts of d larger than d_i^{even} is even, then a has one part equal to d_i^{even}, and otherwise b has two parts equal to $d_i^{even}/2$.

(b) For each odd i such that $d_i^{odd} = d_{i+1}^{odd} \neq 0$, a has one part equal to d_i^{odd}.

(c) For each odd i such that $d_i^{odd} \neq d_{i+1}^{odd} \neq 0$, b has one part equal to $(d_i^{odd} - 1)/2$ and one part equal to $(d_{i+1}^{odd} + 1)/2$.

(d) For each odd i such that $d_i^{odd} \neq d_{i+1}^{odd} = 0$, b has one part equal to $(d_i^{odd} - 1)/2$.

Then (a, b) is the pair of partitions that corresponds to the conjugacy class s_e in W.

One can choose a representative $s \in \mathrm{SO}(n)$ for an element from s_e in such a way that using the notation introduced above we have

(a) For each even i such that $d_i^{even} \neq 0$, if the number of odd parts of d larger than d_i^{even} is even, then $d_{i-1}^{even} = d_{i*}^{even} = d_i^{even}$ and

$$s : e^j v_{i(i^*)} \to \delta_i e^{j+\delta_i} v_{i(i^*)} \text{ if } j, j + \delta_i = 0, \ldots, d_i^{even} - 1$$

and

$$s : v_{i(i^*)} \to e^{d_i^{even}-1} v_{i(i^*)} \text{ if } \delta_{i(i^*)} = -1$$

or

$$s : e^{d_i^{even}-1} v_{i(i^*)} \to v_{i(i^*)} \text{ if } \delta_{i(i^*)} = 1.$$

For each even i such that $d_i^{even} \neq 0$, if the number of odd parts of d larger than d_i^{even} is odd, then $d_{i-1}^{even} = d_{i*}^{even} = d_i^{even}$, and

$$s : v_{i(i^*)} \to (-1)^{d_i^{even}/2} e^{d_i^{even}/2} v_{i(i^*)},$$

$$s : e^j v_{i(i^*)} \to e^{j-1} v_{i(i^*)} \text{ if } j = 1, \ldots, d_i^{even}/2 - 1,$$

$$s : e^j v_{i(i^*)} \to -e^{j+1} v_{i(i^*)} \text{ if } j =, d_i^{even}/2 \ldots, d_i^{even}/2 - 2,$$

$$s : e^{d_i^{even}-1} v_{i(i^*)} \to e^{d_i^{even}/2-1} v_{i(i^*)}.$$

(b) For each odd i such that $d_i^{odd} = d_{i+1}^{odd} \neq 0$,

$$s : v_i \to (-1)^{(d_i^{odd}-1)/2} w_i, \text{ where } w_i = 1/2 e^{(d_i^{odd}-1)/2} (v_i + \sqrt{-1} v_{i+1}),$$

$$s : v_{i+1} \to (-1)^{(d_i^{odd}+1)/2} e^{(d_i^{odd}+1)/2} v_i,$$

$$s : e^j v_{i(i+1)} \to e^{j-1} v_{i(i+1)} \text{ if } j = 1, \ldots, (d_i^{odd} - 3)/2,$$

$$s : e^j v_{i(i+1)} \to -e^{j+1} v_{i(i+1)} \text{ if } j = (d_i^{odd} + 1)/2, \ldots, d_i^{odd} - 2,$$

$$s : e^{d_i^{odd}-1} v_i \to w_{i+1}, \text{ where } w_{i+1} = 1/2 e^{(d_i^{odd}-1)/2} (v_i - \sqrt{-1} v_{i+1}),$$

$$s : e^{d_i^{odd}-1} v_{i+1} \to e^{(d_i^{odd}-3)/2} v_i,$$

$$s : w_i \rightarrow -e^{(d_i^{odd}+1)/2} v_{i+1},$$

$$s : w_{i+1} \rightarrow e^{(d_i^{odd}-3)/2} v_{i+1}.$$

(c) For each odd i such that $d_i^{odd} \neq d_{i+1}^{odd} \neq 0$,

$$s : v_i \rightarrow (-1)^{(d_i^{odd}+1)/2} e^{(d_i^{odd}+1)/2} v_i,$$

$$s : v_{i+1} \rightarrow w_i,$$

where $w_i = 1/2((-1)^{(d_i^{odd}-1)/4} e^{(d_i^{odd}-1)/2} v_i + \sqrt{-1}(-1)^{(d_{i+1}^{odd}-1)/4} e^{(d_{i+1}^{odd}-1)/2} v_{i+1})$,

$$s : e^j v_{i(i+1)} \rightarrow e^{j-1} v_{i(i+1)} \text{ if } j = 1, \ldots, (d_{i(i+1)}^{odd} - 3)/2,$$

$$s : e^j v_{i(i+1)} \rightarrow -e^{j+1} v_{i(i+1)} \text{ if } j = (d_{i(i+1)}^{odd} + 1)/2, \ldots, d_{i(i+1)}^{odd} - 2,$$

$$s : e^{d_i^{odd}-1} v_i \rightarrow e^{(d_i^{odd}-3)/2} v_i,$$

$$s : e^{d_{i+1}^{odd}-1} v_{i+1} \rightarrow w_{i+1},$$

where $w_{i+1} = 1/2((-1)^{(d_i^{odd}-1)/4} e^{(d_i^{odd}-1)/2} v_i - \sqrt{-1}(-1)^{(d_{i+1}^{odd}-1)/4} e^{(d_{i+1}^{odd}-1)/2} v_{i+1})$,

$$s : w_i \rightarrow (-1)^{(d_{i+1}^{odd}+1)/2} e^{(d_{i+1}^{odd}+1)/2} v_{i+1},$$

$$s : w_{i+1} \rightarrow e^{(d_{i+1}^{odd}-3)/2} v_{i+1}.$$

(d) For each odd i such that $d_i^{odd} \neq d_{i+1}^{odd} = 0$,

$$s : v_i \rightarrow (-1)^{(d_i^{odd}+1)/2} e^{(d_i^{odd}+1)/2} v_i,$$

$$s : e^j v_i \rightarrow e^{j-1} v_i \text{ if } j = 1, \ldots, (d_i^{odd} - 3)/2,$$

$$s : e^j v_i \rightarrow -e^{j+1} v_i \text{ if } j = (d_i^{odd} + 1)/2, \ldots, d_i^{odd} - 2,$$

$$s : e^{d_i^{odd}-1} v_i \rightarrow e^{(d_i^{odd}-3)/2} v_i,$$

$$s : e^{(d_i^{odd}-1)/2} v_i \rightarrow e^{(d_i^{odd}-1)/2} v_i.$$

In case of $\mathfrak{g} = \mathfrak{so}_{4n}$ the last proposition does not describe the Kazhdan–Lusztig map completely. If partition d has only even dimensional Jordan blocks then Proposition 3.6 tells only that s_e is one of the two conjugacy classes in W corresponding to the pair of partitions $(a, 0)$ with $a_i = d_{2i}$, $i \geq 1$. This ambiguity in the description of the Kazhdan–Lusztig map was removed in [19]. We shall not use the results of [19] in this paper. In case of exceptional Lie algebras some results on the Kazhdan–Lusztig map were obtained in [20].

Now let $\mathfrak{g}$ be a classical complex simple Lie algebra. We recall that to any nilpotent element $e \in \mathfrak{g}$ one can associate a grading in $\mathfrak{g}$. Indeed, by the Jacobson–Morozov theorem there is an $\mathfrak{sl}_2$–triple (e, h, f) associated to e, i.e. elements $f, h \in \mathfrak{g}$ such that $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. Fix such an $\mathfrak{sl}_2$–triple. Under the action of ad h we have a decomposition

(3.1) $$\mathfrak{g} = \oplus_{i \in \mathbb{Z}} \mathfrak{g}(i), \text{ where } \mathfrak{g}(i) = \{x \in \mathfrak{g} \mid [h, x] = ix\}.$$

Following [7], Sections 3.3, 3.4 we explicitly describe grading (3.1) in terms of the bases of the fundamental representation V introduced in Propositions 3.1, 3.3 and 3.5. Namely, let $V(m)$ be the span of all $e^j v_i$ with $2j + 1 - d_i = m$. Clearly,

(3.2) $$V = \oplus_{m \in \mathbb{Z}} V(m),$$

and hence the Lie algebra $\mathfrak{gl}(V)$ acquires a $\mathbb{Z}$–grading,

(3.3) $$\mathfrak{gl}(V)(k) = \{x \in \mathfrak{gl}(V) \mid x(V(m)) \subset V(m+k) \text{ for all } m \in \mathbb{Z}\}.$$

Moreover in the orthogonal and symplectic cases the grading $V = \oplus_{m \in \mathbb{Z}} V(m)$ is compatible with the bilinear form φ on V in the sense that

$$\text{If } \varphi(V(m), V(n)) \neq 0 \text{ then } m + n = 0.$$

One can show that any classical Lie algebra $\mathfrak{g}$ inherits a $\mathbb{Z}$–grading from $\mathfrak{gl}(V)$, i.e.

$$(3.4) \qquad \mathfrak{g} = \oplus_{k \in \mathbb{Z}} \mathfrak{g}(k), \text{ where } \mathfrak{g}(k) = \mathfrak{g} \cap \mathfrak{gl}(V)(k).$$

Now let G be an algebraic Lie group such that $\mathfrak{g}$ is the Lie algebra of G. Fix a nilpotent element $e \in \mathfrak{g}$ and the corresponding grading (3.1). Let $\mathfrak{p} = \oplus_{i \leq 0} \mathfrak{g}(i)$ be the parabolic subalgebra defined with the help of grading (3.1). We shall associate a transversal slice in G to the pair $(\mathfrak{p}, s)$, where s is the representative of an element from the conjugacy class in the Weyl group W defined in Propositions 3.2, 3.4, 3.6 with the help of the Kazhdan–Lusztig map.

Denote by $P \subset G$ the parabolic subgroup corresponding to the parabolic Lie subalgebra $\mathfrak{p}$, and by N the unipotent radical of P. Then the Levi factor L of P is the subgroup of G corresponding to the Lie subalgebra $\mathfrak{g}(0)$, N is the subgroup of G corresponding to the Lie subalgebra $\mathfrak{n} = \oplus_{i < 0} \mathfrak{g}(i)$, and the opposite parabolic subgroup $\overline{P}$ corresponds to the Lie subalgebra $\overline{\mathfrak{p}} = \oplus_{i \geq 0} \mathfrak{g}(i)$. The unipotent radical $\overline{N}$ of $\overline{P}$ has the Lie algebra $\overline{\mathfrak{n}} = \oplus_{i > 0} \mathfrak{g}(i)$.

Proposition 3.7. *Let G be a classical complex simple Lie group, $\mathfrak{g}$ its Lie algebra, $e \in \mathfrak{g}$ a nilpotent element and $s \in G$ the representative of an element from the conjugacy class s_e in W defined in Propositions 3.2, 3.4, 3.6 for classical Lie algebras. Let Z be the set of s-fixed points in L,*

$$Z = \{z \in L \mid szs^{-1} = z\},$$

and

$$N_s = \{n \in N \mid sns^{-1} \in \overline{P}\}.$$

Then the variety $N_s Z s^{-1} \subset G$ is a transversal slice to the set of conjugacy classes in G, and the conjugation map

$$(3.5) \qquad \alpha : N \times N_s Z s^{-1} \to N Z s^{-1} N$$

is an isomorphism of varieties.

The proof of this proposition is based on the following proposition in the formulation of which we use the notation introduced above.

Proposition 3.8. *Let $\mathfrak{g}$ be a classical complex simple Lie algebra, G an algebraic Lie group with Lie algebra $\mathfrak{g}$, $s \in W$ an element of the Weyl group of $\mathfrak{g}$, and $e \in \mathfrak{g}$ a nilpotent element in $\mathfrak{g}$. We denote a representative for s in G by the same letter. Assume that there is a homogeneous basis F in the fundamental representation V of $\mathfrak{g}$, with respect to grading (3.2) associated to e, and that F is also selfdual in the orthogonal and symplectic cases in the sense that for each element v of the basis F the corresponding element v^* of the dual basis, with respect to the bilinear form φ on V, is a multiple of an element of the basis F.*

Assume also that s maps elements of the basis F of V to nonzero multiples of elements of F, and in the s–orbit of each element of the basis F all elements of maximal degree k, $k \in \mathbb{Z}$, $k \geq 0$ are proportional to a single element v_{max}, the minimal degree of elements in the same orbit is $-k$, and all elements of the minimal degree are proportional to a single element v_{min}.

Assume, moreover, that if for two s–orbits containing elements v_{max} and w_{max}, respectively, of equal maximal degree k there exists $n \in \mathbb{N}$ such that all elements of maximal degree k in those orbits are of the form $s^{mn}v_{max}$ and $s^{mn}w_{max}$, $m \in \mathbb{N}$, respectively, then one of the following alternatives is true:

either there exists $m \in \mathbb{N}$ such that the degree of the element $s^m v_{max}$ is strictly less than the degree of the element $s^m w_{max}$;

or the degree of $s^m v_{max}$ equals to the degree of $s^m w_{max}$, for any $m \in \mathbb{Z}$.

Then the variety $N_s Z s^{-1} \subset G$ is a transversal slice to the set of conjugacy classes in G, and the conjugation map

$$(3.6) \qquad \alpha : N \times N_s Z s^{-1} \rightarrow N Z s^{-1} N$$

is an isomorphism of varieties.

Proof. We prove that the conditions of Propositions 2.1 and 2.2 are satisfied, i.e. for any $x \in \mathfrak{n}$ and any $y \in \overline{\mathfrak{n}}$, $x, y \neq 0$ there exist $k, k' \in \mathbb{N}$ such that $\mathrm{Ad}(s^k)(x) \notin \mathfrak{p}$ and $\mathrm{Ad}(s^{k'})(y) \notin \overline{\mathfrak{p}})$.

First we show that it suffices to prove this statement for a class of simple elements of $\mathfrak{g}$. We call an element $x \in \mathfrak{g}$ simple if it maps each element of the basis F to a multiple of an element of F.

Since the basis F is selfdual the Lie algebra $\mathfrak{g}$ is generated, as a linear space, by simple elements. Since s maps elements of F to multiples of elements of F the operator $\mathrm{Ad}s$ also maps simple elements to simple elements. Therefore it suffices to prove that for each simple element $x \in \mathfrak{n}(\overline{\mathfrak{n}})$ there exists $k \in \mathbb{N}$ such that $\mathrm{Ad}(s^k)(x) \notin \mathfrak{p}(\overline{\mathfrak{p}})$.

Let $x \in \mathfrak{n}$ be a simple element mapping an element v of F to an element w of higher degree, and w is also a multiple of an element of F. Suppose that the s–orbit of v contains an element v_{max} of maximal degree k and an element v_{min} of minimal degree $-k$, $k \in \mathbb{Z}$, $k \geq 0$, the s–orbit of w contains an element w_{max} of maximal degree l and an element w_{min} of minimal degree $-l$, $l \in \mathbb{Z}$, $l \geq 0$.

If $k > l$ then there is an element x' in the s–orbit of x which maps the element v_{min} of degree $-k$ to an element w' of a higher degree, w' being an element from the orbit of w. Therefore $x' \notin \mathfrak{p} = \oplus_{i \leq 0}\mathfrak{g}(i)$.

If $k < l$ then there is an element x'' in the s–orbit of x which maps an element from the orbit of v to the element w_{max} of higher degree l. Therefore $x'' \notin \mathfrak{p} = \oplus_{i \leq 0}\mathfrak{g}(i)$.

If $k = l$ then there is an element x' in the s–orbit of x which maps the element v_{min} of degree $-k$ to an element w' of a higher or equal degree, w' being an element from the orbit of w. Therefore $x' \notin \mathfrak{n} = \oplus_{i < 0}\mathfrak{g}(i)$. We have to show that there exists such an x' which is also not an element of $\mathfrak{p} = \oplus_{i \leq 0}\mathfrak{g}(i)$.

Suppose that $\mathrm{Ad}(s^m)x \in \mathfrak{p}$ for any $m \in \mathbb{N}$. Then for any $m \in \mathbb{N}$ all elements in the orbit of v which are mapped by $\mathrm{Ad}(s^m)x$ to w_{max} must be proportional to v_{max}, and the intersection of the image of v_{min} under the action of $\mathrm{Ad}(s^m)x$ with the orbit of w is either zero or belongs to the one–dimensional subspace of V generated by w_{min}. Without loss of generality we assume that there is an element $x' = \mathrm{Ad}(s^r)x$, $r \in \mathbb{N}$ in the s–orbit of x which maps $v_{max} = s^r v$ to w_{max}, $x'v_{max} = w_{max}$.

Recall that by assumption s maps elements of the basis F to multiples of elements of F and that the number of elements of F is finite. Therefore there exist a minimal positive integer number n such that $s^n w_{min} = a w_{min}$, $a \neq 0$ and a minimal positive integer number n' such that $s^{n'} v_{max} = b v_{max}$, $b \neq 0$.

Since in the s–orbit of each element of the basis F all elements of the maximal degree are proportional to each other and all elements of the minimal degree are proportional to each other n is also the minimal positive integer number such that $s^n w_{max} = a' w_{max}$ and n' is also the minimal positive integer number such that $s^{n'} v_{min} = b' v_{min}$.

Therefore we have $\mathrm{Ad}(s^n)(x') s^n v_{max} = s^n x' v_{max} = s^n w_{max} = a' w_{max}$, and hence $s^n v_{max} = b'' v_{max}$. Since n' is the minimal positive integer number such that $s^{n'} v_{max} = b v_{max}$ and all the elements proportional to v_{max} in the s–orbit of v are of the form $s^{mn'} v_{max}$, $m \in \mathbb{Z}$ we must have $n = n't$, $t \in \mathbb{N}$. Similarly we obtain that $n' = nt'$, $t' \in \mathbb{N}$. Therefore $n = n'$, and all elements of maximal degree k in the orbits of v_{max} and w_{max} are of the form $s^{mn} v_{max}$ and $s^{mn} w_{max}$, $m \in \mathbb{N}$.

Now by assumption either there exists $m \in \mathbb{N}$ such that the degree of the element $s^m v_{max}$ is strictly less than the degree of the element $s^m w_{max}$ or the degree of $s^m v_{max}$ equals to the degree of $s^m w_{max}$, for any $m \in \mathbb{Z}$.

In the first case the degree of the element $\mathrm{Ad}(s^m)(x') s^m v_{max} = s^m w_{max}$ is strictly greater than the degree of the element $s^m v_{max}$, and hence $\mathrm{Ad}(s^m)(x') \notin \mathfrak{p}$.

In the second case the degree of the element $\mathrm{Ad}(s^m)(x') s^m v_{max} = s^m w_{max}$ is equal to the degree of the element $s^m v_{max}$ for any $m \in \mathbb{Z}$. Now recall that there exists $r \in \mathbb{N}$ such that $s^{-r} v_{max} = v$, $\mathrm{Ad} s^{-r} x' = x$ and the degree of the element $w = xv = \mathrm{Ad} s^{-r} x' s^{-r} v_{max} = s^{-r} w_{max}$ is higher than the degree of $v = s^{-r} v_{max}$.

Thus in both cases we come to a contradiction. Therefore there exists $m \in \mathbb{N}$ such that $\mathrm{Ad}(s^m) x \notin \mathfrak{p}$.

The case of simple elements from $\overline{\mathfrak{n}}$ can be considered in a similar way. This completes the proof. $\qquad\square$

Proof of Proposition 3.7. One checks straightforwardly that the conditions of the previous proposition are satisfied for the basis of the fundamental representation V in terms of which the action of s on V is defined in Propositions 3.2, 3.4, 3.6. Now the statement of Proposition 3.7 follows from Proposition 3.8. This completes the proof. $\qquad\square$

In case of $\mathfrak{g} = \mathfrak{sl}_n$ the dimension of the slice $N_s Z s^{-1}$ associated to a nilpotent element $e \in \mathfrak{g}$ in Proposition 3.7 coincides with the dimension of the Slodowy slice $s(e)$. More precisely we have the following proposition.

Proposition 3.9. *Let $\mathfrak{g} = \mathfrak{sl}_n$, $e \in \mathfrak{g}$ a nilpotent element, and $Z(e)$ the centralizer of e in $G = SL(n)$, $\mathfrak{z}(e)$ the Lie algebra of $Z(e)$. Let $s \in \mathrm{SL}(n)$ be the representative for an element from s_e defined in Proposition 3.2. Then $\dim(N_s) = \dim(Z(e) \cap \mathfrak{n})$, $\mathfrak{z} = \mathfrak{z}(e) \cap \mathfrak{l}$, and $\dim(N_s Z s^{-1}) = \dim(Z(e)) = \dim(s(e))$.*

Proof. We shall use the notation introduced in Propositions 3.1 and 3.2 and the description (3.4) of grading (3.1) for classical Lie algebras.

First one checks straightforwardly that the Lie algebra $\mathfrak{n}_s$ of N_s has the linear basis which consists of the elements

$$x : v_{i_1} \rightarrow \begin{cases} e^{j_2} v_{i_2} & \text{if } 0 \le j_2 \le d_{i_2} - 1 \text{ and } 1 - d_{i_1} < 2j_2 + 1 - d_{i_2} \le d_{i_1} + 1 \\ 0 & \text{otherwise} \end{cases}.$$

An elementary calculation shows that the number of such x's, i.e. $\dim(N_s)$, is equal to

$$D = \sum_{i,j=1}^{r} \min(d_i, d_j) - \sum_{i,j,d_i=d_j} 1.$$

According to [7], Sections 3.1 and 3.7 the number D is also equal to $\dim(z(e) \cap \mathfrak{n})$, i.e. $\dim(N_s) = \dim(z(e) \cap \mathfrak{n})$.

Let $x \in \mathfrak{z} \subset \mathfrak{g}(0)$ be an element of $\mathfrak{z}$. Since $x \in \mathfrak{g}(0)$ it is completely determined by the values

$$(3.7) \qquad x(e^{j_1} v_{i_1}) = \sum_{i_2, d_{i_2} - d_{i_1} + j_1 \geq 0} c_{i_1, j_1, i_2} e^{d_{i_2} - d_{i_1} + j_1} v_{i_2}.$$

The element x is also s–invariant, and hence from the definition of s given in Proposition 3.2 it follows that the only nonzero coefficients c_{i_1, j_1, i_2} in formula (3.7) correspond to the terms with $d_{i_1} = d_{i_2}$ and $j_1 = j_2$. Otherwise for some $k > 0$ the degree of the element $\mathrm{Ad}(s^k) x$ is not equal to zero. We deduce that

$$(3.8) \qquad x(e^{j_1} v_{i_1}) = \sum_{i_2, d_{i_2} = d_{i_1}} c_{i_1, j_1, i_2} e^{j_1} v_{i_2}.$$

The fact that x is s–invariant also implies that the coefficients c_{i_1, j_1, i_2} in formula (3.9) do not depend on j_1, i.e.

$$(3.9) \qquad x(e^{j_1} v_{i_1}) = \sum_{i_2, d_{i_2} = d_{i_1}} c_{i_1, i_2} e^{j_1} v_{i_2}.$$

Now one can check directly that the elements of the form (3.9) belong to $\mathfrak{z}(e) \cap \mathfrak{l}$.

Conversely, the results of [7], Section 3.7 imply that any element of $\mathfrak{z}(e) \cap \mathfrak{l}$ is of the form (3.9), and, as we just proved, all such elements are s–invariant. Therefore $\mathfrak{z} = \mathfrak{z}(e) \cap \mathfrak{l}$, and we also obviously have $\dim(N_s Z s^{-1}) = \dim(Z(e)) = \dim(s(e))$. This completes the proof. $\qquad \square$

4. Subregular elements and simple singularities

Recall that the definition of transversal slices to adjoint orbits in a complex simple Lie algebra $\mathfrak{g}$ and to conjugacy classes in a complex simple algebraic group G given in [17] was motivated by the study of simple singularities. Simple singularities appear in algebraic group theory as some singularities of the fibers of the conjugation quotient map $\delta_G : G \to H/W$ generated by the inclusion $\mathbb{C}[H]^W \simeq \mathbb{C}[G]^G \hookrightarrow \mathbb{C}[G]$, where H is a maximal torus of G and W is the Weyl group of the pair (G, H). Some fibers of this map are singular, the singularities correspond to irregular elements of G, and one can study these singularities by restricting δ_G to certain transversal slices to conjugacy classes in G. Simple singularities can be identified with the help of the following proposition proved in [17].

Proposition 4.1. ([17], Section 6.5) *Let S be a transversal slice for the conjugation action of G on itself. Assume that S has dimension $r + 2$, where r is the rank of G. Then the fibers of the restriction of the adjoint quotient map to S, $\delta_G : S \to H/W$, are normal surfaces with isolated singularities. A point $x \in S$ is an isolated singularity of such a fiber iff x is subregular in G, and S can be regarded as a deformation of this singularity.*

Moreover, if $t \in H$, and $x \in S$ is a singular point of the fiber $\delta_G^{-1}(t)$, then x is a rational double point of type ${}_h\Delta_i$ for a suitable $i \in \{1, \ldots, m\}$, where Δ_i are the components in the decomposition of the Dynkin diagram $\Delta(t)$ of the centralizer $Z_G(t)$ of t in G, $\Delta(t) = \Delta_1 \cup \ldots \cup \Delta_m$. If Δ_i is of type A, D or E then ${}_h\Delta_i = \Delta_i$; otherwise ${}_h\Delta_i$ is the homogeneous diagram of type A, D or E associated to Δ_i by the rule ${}_h B_n = A_{2n-1}$, ${}_h C_n = D_{n+1}$, ${}_h F_4 = E_6$, ${}_h G_2 = D_4$.

Note that all simple singularities of types A, D and E were explicitly constructed in [17] using certain special transversal slices in complex simple Lie algebras $\mathfrak{g}$ and the adjoint quotient map $\delta_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{h}/W$ generated by the inclusion $\mathbb{C}[\mathfrak{h}]^W \simeq \mathbb{C}[\mathfrak{g}]^G \hookrightarrow \mathbb{C}[\mathfrak{g}]$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and W is the Weyl group of the pair $(\mathfrak{g}, \mathfrak{h})$. Below we give an alternative description of simple singularities in terms of transversal slices in algebraic groups. In our construction we shall use the transversal slices defined in Section 2. The corresponding elements $s \in W$ will be associated to subregular nilpotent elements in $\mathfrak{g}$ via the Kazhdan–Lusztig map. In case of classical simple Lie algebras we shall use the description of the nilpotent elements and of the Kazhdan–Lusztig map in terms of partitions (see Propositions 3.1–3.6 and the notation used there). We start with the description of the values of this map on the subregular adjoint orbits.

Lemma 4.2. *Let $\mathfrak{g}$ be a complex simple Lie algebra, $\mathfrak{b}$ a Borel subalgebra of $\mathfrak{g}$ containing Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$. Let W be the Weyl group of the pair $(\mathfrak{g}, \mathfrak{h})$. Denote by $\Gamma = \{\alpha_1, \ldots, \alpha_r\}$, $r = \operatorname{rank} \mathfrak{g}$ the corresponding system of simple positive roots of $\mathfrak{g}$ and by Δ the root system of $\mathfrak{g}$. Fix a system of root vectors $e_\alpha \in \mathfrak{g}$, $\alpha \in \Delta$. One can choose a representative e in the unique subregular nilpotent adjoint orbit of $\mathfrak{g}$ and a representative s_e of length $l(s_e)$ in the conjugacy class in W, which corresponds to e under the Kazhdan–Lusztig map, as follows (below we use the convention of [5] for the numbering of simple roots, and for the classical Lie algebras we indicate the sizes d_i of the corresponding blocks of e and the lengths of the cycles in the partition corresponding to s_e, in the notation of Propositions 3.1–3.6; for the exceptional lie algebras we give the type of e according to classification [1] and the type of s_e according to classification [2]):*

- A_{n-1}, $\mathfrak{g} = \mathfrak{sl}_n$, $d_1 = n - 1$, $d_2 = 1$,

$$e = e_{\alpha_1} + \ldots + e_{\alpha_{l-1}} + e_{\alpha_l + \alpha_{l+1}} + e_{\alpha_{l+2}} + \ldots + e_{\alpha_{n-1}} \ (l = 1, \ldots n - 2),$$

  s_e has one cycle of length n-1 and one cycle of length 1,

$$s_e = s_{\alpha_1} \ldots s_{\alpha_{l-1}} s_{\alpha_l + \alpha_{l+1}} s_{\alpha_{l+2}} \ldots s_{\alpha_{n-1}},$$

  $l(s_e) = n$;
- B_n, $\mathfrak{g} = \mathfrak{so}_{2n+1}$, $d_1 = 2n - 1$, $d_2 = 1$, $d_3 = 1$,

$$e = e_{\alpha_1} + \ldots + e_{\alpha_{n-2}} + e_{\alpha_{n-1} + \alpha_n} + e_{\alpha_n},$$

  the partition b corresponding to s_e has one cycle of length n-1 and one cycle of length 1,

$$s_e = s_{\alpha_1} \ldots s_{\alpha_{n-2}} s_{\alpha_{n-1} + \alpha_n} s_{\alpha_n},$$

  $l(s_e) = n + 2$;
- C_n, $\mathfrak{g} = \mathfrak{sp}_{2n}$, $d_1 = 2n - 2$, $d_2 = 2$,

$$e = e_{\alpha_1} + \ldots + e_{\alpha_{n-2}} + e_{2\alpha_{n-1} + \alpha_n} + e_{\alpha_n},$$

  the partition b corresponding to s_e has one cycle of length n-1 and one cycle of length 1,

$$s_e = s_{\alpha_1} \ldots s_{\alpha_{n-2}} s_{2\alpha_{n-1} + \alpha_n} s_{\alpha_n},$$

  $l(s_e) = n + 2$;
- D_n, $\mathfrak{g} = \mathfrak{so}_{2n}$, $d_1 = 2n - 3$, $d_2 = 3$,

$$e = e_{\alpha_1} + \ldots + e_{\alpha_{n-4}} + e_{\alpha_{n-3} + \alpha_{n-2}} + e_{\alpha_{n-2} + \alpha_{n-1}} + e_{\alpha_{n-1}} e_{\alpha_n},$$

the partition b corresponding to s_e has one cycle of length n-2 and one cycle of length 2,

$$s_e = s_{\alpha_1} \ldots s_{\alpha_{n-4}} s_{\alpha_{n-3}+\alpha_{n-2}} s_{\alpha_{n-2}+\alpha_{n-1}} s_{\alpha_{n-1}} s_{\alpha_n},$$

$l(s_e) = n + 2;$

- E_6, the type of e is $E_6(a_1)$,

$$e = e_{\alpha_1} + e_{\alpha_2+\alpha_3} + e_{\alpha_4} + e_{\alpha_5} + e_{\alpha_3+\alpha_6} + e_{\alpha_6},$$

s_e has type $E_6(a_1)$,

$$s_e = s_{\alpha_1} s_{\alpha_2+\alpha_3} s_{\alpha_4} s_{\alpha_5} s_{\alpha_3+\alpha_6} s_{\alpha_6},$$

$l(s_e) = 8;$

- E_7, the type of e is $E_7(a_1)$,

$$e = e_{\alpha_1} + e_{\alpha_2} + e_{\alpha_3+\alpha_4} + e_{\alpha_5} + e_{\alpha_6} + e_{\alpha_7} + e_{\alpha_4+\alpha_7},$$

s_e has type $E_7(a_1)$,

$$s_e = s_{\alpha_1} s_{\alpha_2} s_{\alpha_3+\alpha_4} s_{\alpha_5} s_{\alpha_6} s_{\alpha_7} s_{\alpha_4+\alpha_7},$$

$l(s_e) = 9;$

- E_8, the type of e is $E_8(a_1)$,

$$e = e_{\alpha_1} + e_{\alpha_2} + e_{\alpha_3} + e_{\alpha_4+\alpha_5} + e_{\alpha_5+\alpha_8} + e_{\alpha_6} + e_{\alpha_7} + e_{\alpha_8},$$

s_e has type $E_8(a_1)$,

$$s_e = s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4+\alpha_5} s_{\alpha_5+\alpha_8} s_{\alpha_6} s_{\alpha_7} s_{\alpha_8},$$

$l(s_e) = 10;$

- F_4, the type of e is $F_4(a_1)$,

$$e = e_{\alpha_1} + e_{\alpha_2} + e_{\alpha_2+2\alpha_3} + e_{\alpha_3+\alpha_4},$$

s_e has type B_4,

$$s_e = s_{\alpha_1} s_{\alpha_2} s_{\alpha_2+2\alpha_3} s_{\alpha_3+\alpha_4},$$

$l(s_e) = 6;$

- G_2, the type of e is $G_2(a_1)$,

$$e = e_{2\alpha_1+\alpha_2} + e_{\alpha_2},$$

s_e has type A_2,

$$s_e = s_{3\alpha_1+\alpha_2} s_{\alpha_2},$$

$l(s_e) = 4.$

Proof. For the classical Lie algebras this lemma follows from Propositions 3.1–3.6 and the formula for the dimension of the centralizer $\mathfrak{z}(e)$ of a nilpotent element e with partition $d_1 \geq d_2 \geq \ldots \geq d_q > 0$ (see [7], Sect. 3.1–3.2),

$$\dim \mathfrak{z}(e) = n - 1 + 2 \sum_{i=1}^{q} (i-1)d_i, \ (\mathfrak{g} = \mathfrak{sl}_n),$$

$$\dim \mathfrak{z}(e) = \frac{n}{2} + \sum_{i=1}^{q} (i-1)d_i - \sum_{i;d_i \text{ odd}} \frac{\varepsilon}{2}, \ (\varepsilon = 1 \text{ for } \mathfrak{g} = \mathfrak{so}_n, \text{ and } \varepsilon = -1 \text{ for } \mathfrak{g} = \mathfrak{sp}_n).$$

For the exceptional Lie algebras of types E_6, E_7 and E_8 the subregular nilpotent elements are also semiregular, and they are described explicitly in [5]. For the exceptional Lie algebras of types F_4 and G_2 the subregular nilpotent elements can

be easily found using the tables of the dimensions of the centralizers of nilpotent elements given in [5] and the classification of nilpotent elements [1]. The values of the Kazhdan–Lusztig map for exceptional Lie algebras were calculated in [20]. The table of values given in [20] is not complete. But in all cases one can find the value of the Kazhdan–Lusztig map on the subregular nilpotent orbit. Note that for the semiregular elements the Kazhdan–Lusztig map has a very simple form, $\sum_{\alpha \in \Delta} e_\alpha \mapsto \prod_{\alpha \in \Delta} s_\alpha$ (see [2, 4]).   □

Now using elements s_e defined in Lemma 4.2 and Proposition 2.1 we construct transversal slices to conjugacy classes in the algebraic group G.

Proposition 4.3. *Let G be a complex simple algebraic group with Lie algebra $\mathfrak{g}$, $\mathfrak{b}$ a Borel subalgebra of $\mathfrak{g}$, H is the maximal torus of the Borel subgroup B corresponding to the Borel subalgebra $\mathfrak{b}$, N is the unipotent radical of B, $\overline{B}$ is the opposite Borel subgroup in G. Let e be the subregular nilpotent element in $\mathfrak{g}$ defined in the previous lemma and s_e the element of the Weyl group of $\mathfrak{g}$ associated to e in Lemma 4.2. Denote by $s \in G$ a representative of s_e in G. Let $\mathfrak{p} = \oplus_{i \leq 0} \mathfrak{g}(i)$ be the parabolic subalgebra of $\mathfrak{g}$ defined with the help of grading (1.1) associated to e.*

Then s is subregular in G and the the conditions of Propositions 2.1 and 2.2 are satisfied for the pairs $(\mathfrak{b}, s)$ and $(\mathfrak{p}, s)$, the subgroups N_s and Z being the same in both cases, and hence the variety $N_s Z s^{-1}$, where $Z = \{z \in H \mid szs^{-1} = z\}$, $N_s = \{n \in N \mid sns^{-1} \in \overline{B}\}$, is a transversal slice to the set of conjugacy classes in G and the maps $N \times N_s Z s^{-1} \to N Z s^{-1} N$ and $N' \times N_s Z s^{-1} \to N' Z s^{-1} N'$ are isomorphisms of varieties. Here N' is the unipotent radical of the parabolic Lie subgroup of G corresponding to the Lie subalgebra $\mathfrak{p}$. The slice $N_s Z s^{-1}$ has dimension $r + 2$, $r = $ rank $\mathfrak{g}$.

Proof. Consider first the case of the pair $(\mathfrak{b}, s)$. If $\mathfrak{g}$ is not of type A then for the pair $(\mathfrak{b}, s)$ the conditions of Propositions 2.1 and 2.2 are satisfied since the action of the element s_e on the root lattice has no fixed points in the sublattice $\bigoplus_{\alpha \in \Gamma} \mathbb{N}\alpha$. Indeed if $\mathfrak{g}$ is not of type A then s_e is the product of rank $\mathfrak{g}$ reflections with respect to linearly independent roots, and any fixed point for the action of s_e on the Cartan subalgebra $\mathfrak{h}$ must be fixed by all those reflections (see, for instance, [2]), i.e. any such fixed point must be orthogonal to rank $\mathfrak{g}$ linearly independent roots in $\mathfrak{h}$. Therefore zero is the only fixed point for the action of s_e on $\mathfrak{h}$ if $\mathfrak{g}$ is not of type A. This implies that the centralizer $\mathfrak{z}$ of s in the Levi factor $\mathfrak{h}$ of the Borel subalgebra $\mathfrak{b}$ is zero, $\mathfrak{z} = 0$.

If $\mathfrak{g}$ is of type A one can verify straightforwardly that for the pair $(\mathfrak{b}, s)$ the conditions of Propositions 2.1 and 2.2 are satisfied. Note that in this case $\mathfrak{z} = \mathbb{C}(\omega_l - \omega_{l+1})$, where ω_l and ω_{l+1} are the fundamental weights dual to α_l and α_{l+1}, respectively (here we keep the notation of Lemma 4.2).

If $\mathfrak{g}$ is not of type A then $\dim N_s = l(s_e) = $ rank $\mathfrak{g} + 2$ and $Z = 1$. If $\mathfrak{g}$ is of type A then $\dim N_s = l(s_e) = $ rank $\mathfrak{g} + 1$ and $\dim Z = 1$. Therefore $\dim N_s Z s^{-1} = $ rank $\mathfrak{g} + 2$ in all cases.

Obviously the dimension of the centralizer of s in G is equal to the dimension of the orbit space for the adjoint action of s on $\mathfrak{g}$. One verifies straightforwardly that this dimension is equal to rank $\mathfrak{g} + 2$ in all cases.

Since the nilradical of $\mathfrak{p}$ is contained in the nilradical of $\mathfrak{b}$ the conditions of Propositions 2.1 and 2.2 are also satisfied for the pair $(\mathfrak{p}, s)$. For the classical Lie algebras and the pair $(\mathfrak{p}, s)$ this is also a particular case of Proposition 3.7.

The fact that the subgroups N_s and Z are the same for both pairs $(\mathfrak{b}, s)$ and $(\mathfrak{p}, s)$ easily follows from the observation that the semisimple part of the Levi factor of $\mathfrak{p}$ is an $\mathfrak{sl}_2$–triple in $\mathfrak{g}$ with the nilpotent elements being simple root vectors in $\mathfrak{g}$, and this $\mathfrak{sl}_2$–triple is not invariant under the action of Ad s. $\square$

An immediate corollary of the last proposition and of Proposition 4.1 is the following construction of simple singularities in terms of the conjugation quotient map.

Proposition 4.4. *Let $\mathfrak{g}$ be a complex simple Lie algebra, G a complex simple algebraic group with Lie algebra $\mathfrak{g}$, $\mathfrak{b}$ a Borel subalgebra of $\mathfrak{g}$, H is the maximal torus of the Borel subgroup $B \subset G$ corresponding to the Borel subalgebra $\mathfrak{b}$, N is the unipotent radical of B, $\overline{B}$ is the opposite Borel subgroup in G, and W the Weyl group of the pair (G, H). Let e be the subregular nilpotent element in $\mathfrak{g}$ defined in Lemma 4.2 and s_e the element of the Weyl group of $\mathfrak{g}$ associated to e in Lemma 4.2. Denote by $s \in G$ a representative of s_e in G. Let $Z = \{z \in H \mid szs^{-1} = z\}$ and $N_s = \{n \in N \mid sns^{-1} \in \overline{B}\}$.*

Then the fibers of the restriction of the adjoint quotient map to the transversal slice $N_s Z s^{-1}$ to the conjugacy classes in G, $\delta_G : N_s Z s^{-1} \to H/W$, are normal surfaces with isolated singularities. A point $x \in N_s Z s^{-1}$ is an isolated singularity of such a fiber iff x is subregular in G, and $N_s Z s^{-1}$ can be regarded as a deformation of this singularity.

Moreover, if $t \in H$, and $x \in N_s Z s^{-1}$ is a singular point of the fiber $\delta_G^{-1}(t)$, then x is a rational double point of type ${}_h\Delta_i$ for a suitable $i \in \{1, \ldots, m\}$, where Δ_i are the components in the decomposition of the Dynkin diagram $\Delta(t)$ of the centralizer $Z_G(t)$ of t in G, $\Delta(t) = \Delta_1 \cup \ldots \cup \Delta_m$. If Δ_i is of type A, D or E then ${}_h\Delta_i = \Delta_i$; otherwise ${}_h\Delta_i$ is the homogeneous diagram of type A, D or E associated to Δ_i by the rule ${}_hB_n = A_{2n-1}$, ${}_hC_n = D_{n+1}$, ${}_hF_4 = E_6$, ${}_hG_2 = D_4$.

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