NORMAL AFFINE SURFACES PROPERLY DOMINATED BY $\mathbb{C} \times \mathbb{C}^*$

by

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Introduction. In 1980, Nishino–Suzuki [11] had a beautiful theorem on a cluster set of a holomorphic mapping of a punctured disc in $\mathbb{C}$ into a compact non–singular surface, and Suzuki [12] applied it to the study of complex analytic compactifications of $\mathbb{C} \times \mathbb{C}^*$, $(\mathbb{C}^*)^2$.

Later, the author applied it to the linearization of a polynomial automorphism of $\mathbb{C}^2$ of finite order ([2]) and the determination of a normal affine surface properly dominated by $\mathbb{C}^2$ ([4]), (see also Miyanishi [8], Gurjar–Shastri [5]). An affine variety X is said to be properly dominated by an affine variety V if there is a proper morphism $f: V \longrightarrow X$ of V onto X.

Now, in this paper, we will apply it to the determination of a normal affine surface properly dominated by $\mathbb{C} \times \mathbb{C}^*$. Our main result is the following

Theorem. Let X be a normal affine surface properly dominated by $\mathbb{C} \times \mathbb{C}^*$. Then, (i) $X \cong \mathbb{C}^2$ or $\mathbb{C} \times \mathbb{C}^*$ if X is non–singular, (ii) $X \cong \mathbb{C}^2/G_a$ for a small finite subgroup G_a of $GL(2;\mathbb{C})$, or $X \cong \mathbb{C} \times \mathbb{C}^*/G_b$ for a small finite subgroup G_b of $\mathrm{Aut}(\mathbb{C} \times \mathbb{C}^*)$ with exactly two fixed points if X is singular.

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Notation

$b_i(M)$: i–th Betti number of M

$\chi(M)$: the Euler number of M

K_S : a canonical divisor on a normal Gorenstein surface S

$q(S) := \dim H^1(S,\mathcal{O}_S)$

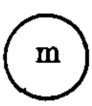
$P_g(S) := \dim H^2(S,\mathcal{O}_S)$

$P_m(S) := \dim H^0(S,\mathcal{O}(mK_S)$

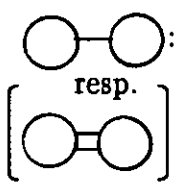
$\bar{\kappa}(X)$: the logarithmic Kodaira dimension of an open surface X

$c_1(\mathscr{L})$: the first Chern class of a line bundle $\mathscr{L}$

[D] : the line bundle defined by a divisor D



: a non–singular rational curve with the self–intersection number m



: transversal (resp. tangential) intersection of two non–singular rational curves corresponding to the vertices

$\mathrm{Aut}(\mathbb{C}^2)$: the group of algebraic automorphisms of $\mathbb{C}^2$

$\mathrm{Aut}(\mathbb{C} \times \mathbb{C}^*)$: the group of algebraic automorphisms of $\mathbb{C} \times \mathbb{C}^*$

§ 1. Theory of cluster sets of holomorphic mappings

1. Let S be a normal Gorenstein projective algebraic surface and $C = \bigcup_{i=1}^{n} C_i$ (C_i is irreducible) be an algebraic curve in S with Sing S $\cap$ C = ϕ . Assume that

(i) any singular point of $C = \bigcup_{i=1}^{n} C_i$ is an ordinary double point,

(ii) there is no non–singular rational irreducible component of C with the self–intersection number -1 which has at most two intersection points with other components of C .

We call such a pair (S,C) the minimal normal pair.

Let (S,C) be a minimal normal pair. Then we have

Lemma 1.1 (Nishino–Suzuki [11]). Assume that for each C_i , there is a holomorphic mapping $\phi_i : \Delta^* \longrightarrow S \setminus C$ of a punctured disc $\Delta^* := \{z \in \mathbb{C} \,;\, 0 < |z| < 1\}$ into $S \setminus C$ such that

$$C_i \subset \phi_i(0;S) \subset C ,$$

where $\phi_i(0;S) = \bigcap_{\rho > 0} \overline{\phi_i(\Delta_\rho^*)}$, $\Delta_\rho^* := \{z \in \mathbb{C};0 < |z| < \rho\}$, $\overline{\phi_i(\Delta_\rho^*)}$ is the closure of $\phi_i(\Delta_\rho^*)$ in S . Then the curve $C = \bigcup_{i=1}^{n} C_i$ must be one of the type from (α) to (ε) in Table I below, in which, for types (β_r) $(r \geq 2)$, (γ) , (γ') , (δ) , (ε) , each irreducible component of C is a non–singular rational curve and assigned Figures (1 – 5) represent the dual graph $\Gamma(C)$ of C.

2. Since S is Gorenstein, one can define a canonical divisor K_S on S.

Then we have

<u>Proposition 1.2</u>. Assume that $C = \bigcup_{i=1}^{n} C_i$ is one of the types (γ), (γ'), (δ), (ε) in the Table I. If K_S is written as follow:

$$K_S = \sum_{i=1}^{n} \lambda_i C_i \quad (\lambda_i \in \mathbb{Z}),$$

then the dual graph $\Gamma(C)$ is one of the type from F. 1 to F. 15 in Table II, in which, adjacent to the vertex representing the irreducible component C_i of C, we write the order λ_i of K_S on C_i.

For the proof, we need the following

<u>Lemma 1.3</u> (Lemma 6 in Suzuki [12]).

Let $A_1, \dots, A_m$ be irreducible non–singular rational curves on a smooth projective algebraic surface M such that $A := \bigcup_{i=1}^{m} A_i$ is simply connected. If there is a pair i,j $(1 \le i,j \le m)$ such that $(A_i^2) \ge 0$, $(A_j^2) \ge 0$ and $A_i \cap A_j = \phi$, then $(A_i^2) = (A_j^2) = 0$, and there is only one $A_k (k \ne i,j)$ which intersects $A_i \cup A_j$. Further, for this A_k, we have $(A_i \cdot A_k) = (A_j \cdot A_k) = \nu \ge 1$.

Table I.

Name of type	Explication of C
(α) $\alpha(n)$	an irreducible non–singular elliptic curve with the self–intersection number $(C^2) = n \geq 0$
(β) $\beta(n)$	an irreducible rational curve with only one ordinary double point and $(C^2) = n \geq 0$
(β_r) $\beta(n_1, \ldots, n_r)$ $(r \geq 2)$	Figure 1, all $n_i = -2$ or $\max\{n_i\} \geq 0$
(γ) $\gamma(n_1, \ldots, n_r)$ $(r \geq 1)$	Figure 2, all $n_i = -2$ or $\max\{n_1 + 1, n_2, \ldots, n_{r-1}, n_r + 1\} \geq 0$
(γ') $\gamma'(n_1, \ldots, n_r)(r \geq 1)$	Figure 3, $\max\{n_1 + 1, n_2, \ldots, n_r\} \geq 0$.
(δ) $\delta(n_0; \frac{q_1}{\ell_1}, \frac{q_2}{\ell_2}, \frac{q_3}{\ell_3})$	Figure 4, (i) $n_0 \geq -2$, (ii) $(\ell_1, \ell_2, \ell_3) = (3,3,3)$, $(2,4,4)$ or $(2,3,6-m)$ $(0 \leq m \leq 3)$, (iii) for each $i(1 \leq i \leq 3)$, (q_1, ℓ_i) is a pair of coprime integers such that $0 < q_i < \ell_i$ and that $\frac{\ell_i}{q_i} = n_{i_1} - \cfrac{1}{n_{i_2} - \cfrac{1}{\ddots \, n_{ir_i}}}$ (continued fraction expansion), where $n_{i,j} \geq 2$ are integers appearing in Figure 4
(ε) $\varepsilon(n_1, \ldots, n_r)$	Figure 5, $\max\{n_i\} \geq 0$.

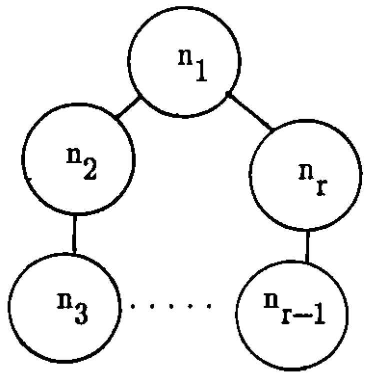


Figure 1

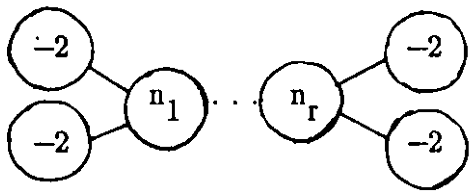


Figure 2

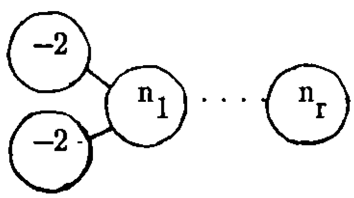


Figure 3

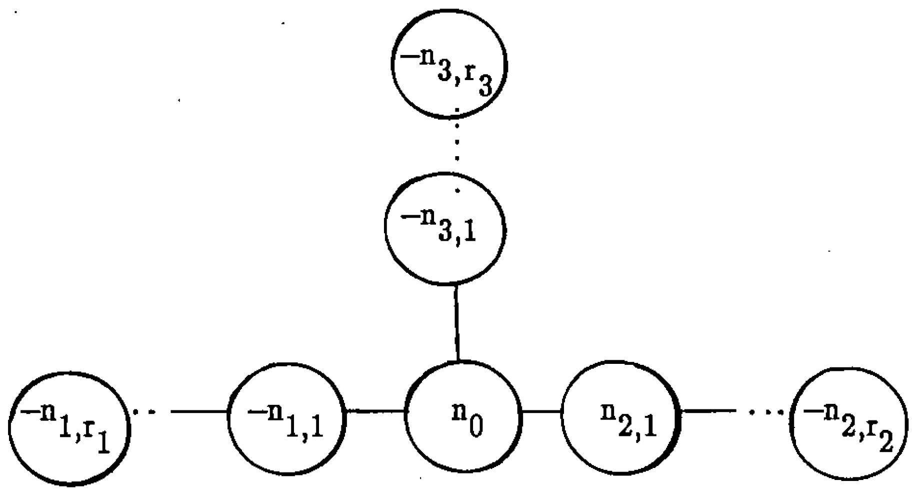


Figure 4

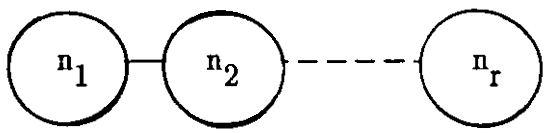


Figure 5

Let us denote, for simplicity, by $\langle n_1; \dots ;n_k\rangle$ the dual graph which looks like Figure 5. Let us consider the configuration $\langle m;0;n\rangle$ of non–singular rational curves in a smooth complex surface. Blowing up the point of intersection of the curves corresponding to the vertices with weights 0 and n, we have the configuration $\langle m;-1;-1;n-1\rangle$. Blowing down the (-1)–curve which is the proper transform of the rational curve corresponding to the vertex with weight 0, we have the new configuration $\langle m+1;0;n-1\rangle$. We call this transformation $\langle m;0;n\rangle \longmapsto \langle m+1;0;n-1\rangle$ the elementary transformation (with center 0). By a succession of N–elementary transformations, we have the configuration $\langle m+N;0;m-N\rangle$ of non–singular rational curves.

3. We will prove Proposition 1.2 below.

(1) The case where C is of the type (γ).

First, let us consider the case of $\max\{n_1+1, n_2, \dots, n_{k-1}, n_k+1\} \geq 0$.

Claim (1.1). $k \neq 1$.

Indeed, if $k = 1$, then we have the dual graph $\Gamma(C)$ which looks like Fig. 16.

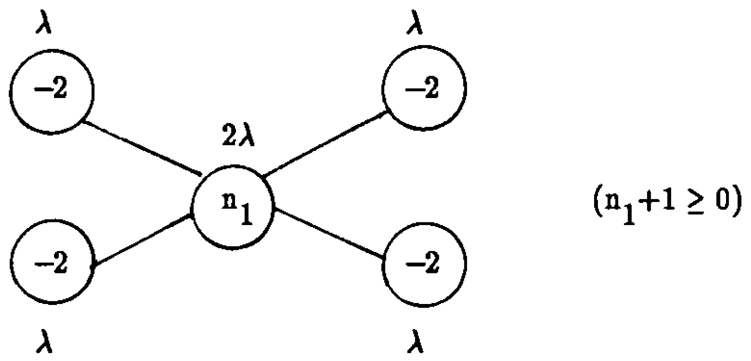


Fig. 16

By the adjunction formula, we have $2(2\lambda + 1) + (2\lambda + 1)n_1 = 0$, hence, $n_1 = -2$. Since $n_1 + 1 \geq 0$, this is a contradiction.

q.e.d.

Repeating blowing ups and elementary transformations on C, we have the dual graph which looks like Fig. 17.

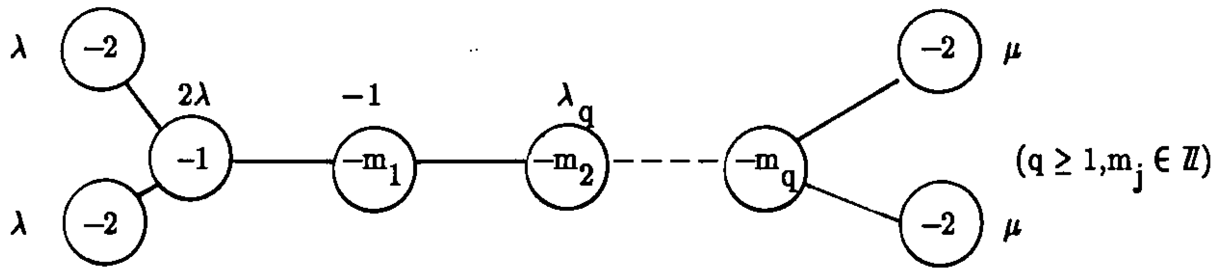
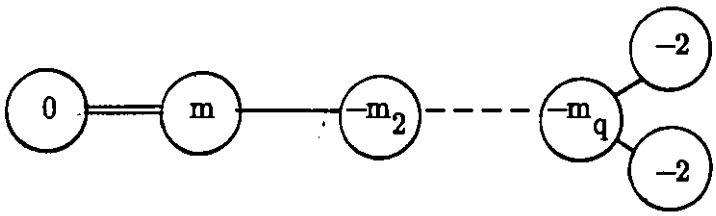


Fig. 17

Claim (1.2). $q = 2$ and $m_2 = 1$.

Indeed, if $q \geq 3$, then, contracting the (–1)–curve corresponding to the vertex with weight -1 in Fig. 17, we have the dual graph Fig. 17'.



$(m = 2 - m_1)$

Fig. 17'

By Lemma 1.3, we have $\max\{-m_2, \dots, -m_q\} < 0$. We may assume that $m_j \geq 2$ $(2 \leq j \leq q-1)$. If $m_q = 1$, then, by Lemma 1.3, we must have $q = 2$. This is a contradiction, since $q \geq 3$ by assumption. Thus $m_j \geq 2$ for $1 \leq j \leq q$.

By the adjunction formula, we have easily

$$(1.3) \qquad \begin{cases} 2\mu = [m_q, \dots, m_2]\,(\lambda_2+1) \\ 2\mu = [m_{q-1}, \dots, m_2]\,(\lambda_2+1) \end{cases} ,$$

where $[m_q, \dots, m_i]$ $(1 \leq i \leq q)$ represents an integer defined inductively by the following way:

$$(1.4)\qquad \begin{cases} [\phi] = 1, \quad [m_q] \ = m_q \\ [m_q, \ldots, m_i] \ = \ m_q \ [m_{q-1}, \cdots, m_i] - [m_{q-2}, \cdots, m_i] \ . \end{cases}$$

Since $[m_q, \cdots, m_i] > [m_{q-1}, \cdots, m_i]$ if $m_j \geq 2$ $(i \leq j \leq q)$, by (1.3), we have $\lambda_2 = -1$. Let C_1 be the irreducible component of C corresponding to the vertex with weight $-m_1$ (see Fig. 17). Then we have

$$m_1 - 2 = (K_S \cdot C_1) = 2\lambda + m_1 + \lambda_2 \ .$$

Since $\lambda_2 = -1$, we have $\lambda = -\frac{1}{2} \notin \mathbb{Z}$. This is a contradiction. Therefore we have $q \leq 2$. Using the adjunction formula, one can easily verify that $q \neq 1$. Thus, $q = 2$. By the adjunction formula, we have $(2\mu + 1)(m_1 - 1) = 0$, hence, $m_1 = 1$.

q.e.d.

By Claim (1.2), we have the dual graph F.2.

Next, if all $n_i = -2$, then we have the dual graph F.1.

(2) The case where C is of the type (γ').

Since $\max\{n_1 + 1, n_2, \cdots, n_k\} \geq 0$, repeating blowing ups and elementary transformations on C, we have the dual graph which looks like Fig. 18.

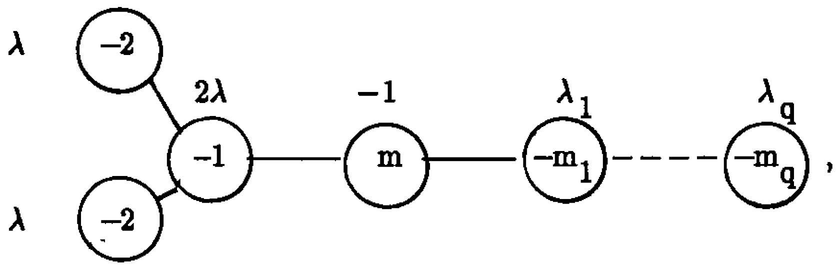


Fig. 18

where $q = 0$, or $q \geq 1$ and $m_j \geq 2$ $(1 \leq j \leq q)$ by Lemma 1.3. By the adjunction formula, we have $[m_q, \ldots, m_1](\lambda_1 + 1) = 1$. Thus we have $[m_q, \ldots, m_1] = 1$, namely, $q = 0$. Then we have $\lambda = -1$. Therefore we have the dual graph F.3.

(3) The case where C is of the type (δ).

Let C_0, C_{i,j_i} $(1 \leq i \leq 3,\ 1 \leq j_i \leq r_i)$ be the irreducible components of C corresponding to the vertices with weights $n_0, -n_{i,j_i}$, respectively (see Figure 4). Then we put

$$K_S = \lambda_0 C_0 + \sum_{i=1}^{3} \left(\sum_{j_i=1} \lambda_{i,j_i} \cdot C_{i,j_i}\right), \tag{1.5}$$

where $\lambda_0, \lambda_{i,j_i} \in \mathbb{Z}$. By the adjunction formula, we have

$$(1.6)\qquad \begin{cases} -2-n_0 = \sum_{i=1}^{3} \lambda_{i,1} + \lambda_0 \cdot n_0 \\ -2-n_{i,j_i-1} = \lambda_{i,j_i-2} - \lambda_{i,j_i-1} \cdot n_{i,j_i-1} + \lambda_{i,j_i} \\ -2-n_{i,r_i} = \lambda_{i,r_i-1} - \lambda_{i,r_i} \cdot n_{i,r_i} \end{cases}$$

We put $\ell_i = [n_{i,r_i}, \dots, n_{i,1}]$ and $q_i = [n_{i,r_i}, \dots, n_{i,2}]$. Since $n_{i,j_i} \geq 2$, we have $\ell_i \geq r_i + 1$, $\ell_i > q_i > 0$. By (1.6), we have easily

$$(1.7)\qquad \ell_i(\lambda_{i,1} + 1) - q_i(\lambda_0 + 1) = 1 \quad (1 \leq i \leq 3)$$

$$(1.8)\qquad (\lambda_0 + 1)n_0 + \Sigma(\lambda_{i,1} + 1) = 1$$

$$(1.9)\qquad (\lambda_0 + 1)(n_0 + \sum_{i=1}^{3} \frac{q_i}{\ell_i}) + \sum_{i=1}^{3} \frac{1}{\ell_i} = 1$$

Since $(\ell_1,\ell_2,\ell_3) = (3,3,3), (2,4,4), (2,3,6), (2,3,5), (2,3,4), (2,3,3)$, we have

(i) $\sum_{i=1}^{3} \frac{1}{\ell_i} = 1 \Leftrightarrow (\ell_1,\ell_2,\ell_3) = (3,3,3), (2,4,4), (2,3,6)$

(ii) $\sum_{i=1}^{3} \frac{1}{\ell_i} > 1 \Leftrightarrow (\ell_1,\ell_2,\ell_3) = (2,3,5), (2,3,4), (2,3,3)$

Case (i). $\sum_{i=1}^{3} \frac{1}{\ell_i} = 1$.

In this case, by (1.9), we have $(\lambda_0 + 1)(n_0 + \sum_{i=1}^{3} \frac{q_i}{\ell_i}) = 0$. By (1.7), we have $\lambda_0 + 1 \neq 0$. Hence, $\sum_{i=1}^{3} \frac{q_i}{\ell_i} + n_0 = 0$. Since $n_0 \geq -2$, we must have $n_0 = -1$ or -2 . If $n_0 = -1$, then we have $\left[\frac{q_1}{\ell_1}, \frac{q_2}{\ell_2}, \frac{q_3}{\ell_3}\right] = \left[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right], \left[\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right], \left[\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right]$. Thus we have the dual graphs F.4, F.5 and F.6, respectively. If $n_0 = -2$, then we have $\left[\frac{q_1}{\ell_1}, \frac{q_2}{\ell_2}, \frac{q_3}{\ell_3}\right] = \left[\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right], \left[\frac{1}{2}, \frac{3}{4}, \frac{3}{4}\right], \left[\frac{1}{2}, \frac{2}{3}, \frac{5}{6}\right]$. Thus we have the dual graphs F.7, F.8 and F.9, respectively.

Case (ii). $\sum_{i=1}^{3} \frac{1}{\ell_i} > 1$.

In this case, we have

$$-(\lambda_0 + 1) = \sum_{i=1}^{3} \frac{1}{\ell_i} - 1 \Big/ \sum_{i=1}^{3} \frac{q_i}{\ell_i} + n_0 \in \mathbb{Z} .$$

Hence, we have $n_0 = -1$ or -2 . If $n_0 = -1$, then we have $\left[\frac{q_1}{\ell_1}, \frac{q_2}{\ell_1}, \frac{q_3}{\ell_1}\right] = \left[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}\right], \left[\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right], \left[\frac{1}{2}, \frac{1}{3}, \frac{1}{3}\right]$ and $\lambda_0 = -2$. Thus we have the dual graphs F.10, F.11 and F.12, respectively. If $n_0 = -2$, then $-(\lambda_0 + 1) = \sum_{i=1}^{3} \frac{1}{\ell_i} - 1 \Big/ \sum_{i=1}^{3} \frac{q_i}{\ell_i} - 2$ is not integer. Hence $n_0 \neq -2$. Thus, we have finally the dual graphs F.4 – F.12.

(4) The case where C is of the type (ε) .

Since $\max\{n_i\} \geq 0$, repeating blowing ups and elementary transformations on C as before, we have the dual graphs F.13, F.14 and the following Fig. 19.

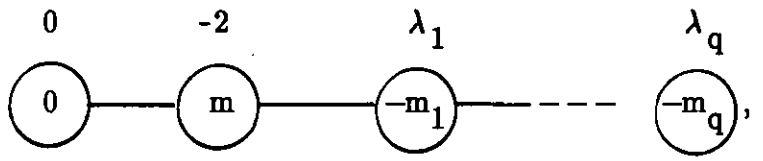


Fig. 19

where $q \geq 1$ and $m_j \geq 2\ (1 \leq j \leq q)$ by Lemma 1.3. By the adjunction formula, we have

$$(1.10) \qquad [m_q, \dots, m_1](\lambda_1 + 1) + [m_q, \dots, m_2] = 1$$

Thus we have

$$-(\lambda_1 + 1) = \frac{[m_q, \dots, m_2] - 1}{[m_q, \dots, m_1]} \in \mathbb{Z} .$$

Hence, $q = 1$ and $\lambda_1 = -1$. Thus we have the dual graph F.15. This completes the proof of Proposition 1.2.

<u>Corollary 1.4.</u> Let (S,C) be as in Proposition 1.2. Then we have Table III below.

Type	F.1	F.2	F.3	F.4	F.5
K_S^2	0	$m+4$	$n+4$	$-\lambda^2$	$-2\lambda^2$

Type	F.6	F.7	F.8	F.9	F.10
K_S^2	$-3\lambda^2$	0	0	0	-2

Type	F.11	F.12	F.13	F.14	F.15
K_S^2	-1	0	8	8	$8-m_1$

Table III

§ 2. A characterization of $\mathbb{C}^2$, $\mathbb{C} \times \mathbb{C}^*$

Let (S,C) be a minimal normal pair of a smooth projective algebraic surface S and an algebraic curve $C = \bigcup_{i=1}^{n} C_i$ on S. Assume that C is one of the type from (α) to (ε) in Table I. We put $X := S \setminus C$.

<u>Proposition 2.1</u>. If $H_i(X;\mathbb{Z}) = 0$ for $i > 0$, then $X \cong \mathbb{C}^2$.

<u>Proof</u>. Let us consider an exact sequence over $\mathbb{Z}$:

$$(2.1) \qquad \begin{array}{ccccccccc} \longrightarrow & H^i(S,C) & \longrightarrow & H^i(S) & \longrightarrow & H^i(C) & \longrightarrow & H^{i+1}(S,C) & \longrightarrow \\ & \wr\| & & & & & & \wr\| & \\ & H_{4-i}(X) & & & & & & H_{3-i}(X) & \end{array}$$

Since $H_i(X;\mathbb{Z}) = 0$, we have $H^i(S;\mathbb{Z}) \cong H^i(C;\mathbb{Z})$ for $i \geq 1$. In particular, $H^1(C;\mathbb{Z}) \cong H^1(S;\mathbb{Z}) = 0$. Thus C is one of the types (γ), (γ'), (δ), (ε). Since $b_1(S) = 0$, we have $q(S) = 0$. Further, $H^2(S;\mathbb{Z})$ is generated by $c_1([C_1]), \dots, c_1([C_n])$ over $\mathbb{Z}$. Thus we have

(i) $\quad K_S = \Sigma \lambda_i C_i \quad (\lambda_i \in \mathbb{Z})$

(ii) $\quad \det((C_i \cdot C_j)) \neq 0$

Let T be a tubular neighborhood of C in S and ∂T be the boundary of T. We may assume that $T \setminus C \approx \partial T$ (deformation retract). Let us consider the following diagram over $\mathbb{Z}$:

(2.2)
$$\begin{tikzcd}
0 \arrow[r] & H_2(\partial T) \arrow[r] & H_2(X) \oplus H_2(T) \arrow[r] \arrow[d, equal, "\wr"'] & H_2(S) \arrow[r] \arrow[d, equal] & H_1(\partial T) \arrow[r] & 0 \\
0 \arrow[r] & H_3(X) \arrow[r] \arrow[d, equal, "\wr"'] & H_2(C) \arrow[r] & H_2(S) \arrow[r] & H_2(X) \arrow[r] \arrow[u, "\wr"'] \arrow[d, equal, "\wr"'] & 0 \\
 & 0 & & & 0 &
\end{tikzcd}$$

Then, we have

(iii) $H_1(\partial T;\mathbb{Z}) = 0$

On the other hand, by the Noether formula, we have

(iv) $12P_g(S) = K_S^2 + b_2(C) - 10$,

since $q(S) = b_1(S) = 0$, $b_2(S) = b_2(C)$. By (i), we can apply Proposition 1.2. Thus, C is one of the type from F.1 to F.15. By (ii), C is one of the types F.3, F.10, F.11, F.12, F.13 and F.15 (c.f. Suzuki [p. 457, 12]). By (iii), we have the types F.10 and F.13. If C is of the type F.10, then, by (iv) and Table III, we have $12P_g(S) = -8 < 0$. This is a contradiction. Therefore C must be of the type F.13. In particular, $S \cong \Sigma_m$ (a Hirzebruch surface), since $b_2(S) = b_2(C) = 2$. One can easily show that $X = S \backslash C = \mathbb{C}^2$.

q.e.d.

<u>Proposition 2.2</u>. (c.f. Suzuki [12]). Let (S,C) and X be as above. If $H_1(X;\mathbb{Z}) = \mathbb{Z}$, $H_i(X;\mathbb{Z}) = 0$ for $i \geq 2$, then $X \cong \mathbb{C} \times \mathbb{C}^*$.

<u>Proof</u>. Since $H_2(X;\mathbb{Z}) = 0$, by (2.1), we have

$$(2.3) \qquad 0 \longrightarrow H^2(S;\mathbb{Z}) \longrightarrow H^2(C;\mathbb{Z}) \longrightarrow H_1(X;\mathbb{Z}) \longrightarrow H^3(S;\mathbb{Z}) = 0 .$$

Since $b_1(S) = b_3(S) \le b_1(X) = 1$ and S is projective, we have $b_1(C) = q(S)$, $b_1(S) = 0$, that is, C is one of the types (γ), (γ'), (δ), (ϵ). Further, we have

$$\text{(i)} \qquad K_S = \sum_{i=1}^{n} \lambda_i C_i \qquad (\lambda_i \in \mathbb{Z})$$

$$\text{(ii)} \qquad \det((C_i \cdot C_j)) = 0 .$$

By the Noether formula, we have

$$\text{(iii)} \qquad 12P_g(S) = K_S^2 + b_2(C) - 11 ,$$

since $b_2(S) = b_2(C) - 1$. Since $b^+(S) = 2P_g(S) + 1 \ge 1$ and $i^* : H^2(S;\mathbb{R}) \longrightarrow H^2(C;\mathbb{R})$ is injective, we have,

$$\text{(iv)} \qquad ((C_i \cdot C_j)) \text{ is not negative semi–definite.}$$

Now, by (i) and Proposition 1.2, we have the dual graph from F.1 to F.15. By (ii), we have the dual graphs F.1, F.2, F.4–F.9, F.14. By (iii), we have F.1, F.2, F.14 (see Table II). By (iv), we have F.2, F.14 (c.f. [p. 457, 12]). If C is of the type F.2., blowing down two (–1)–curves, we have a smooth projective surface S' and an algebraic curve C' with the dual graph $\Gamma(C')$ which looks like Fig. 20.

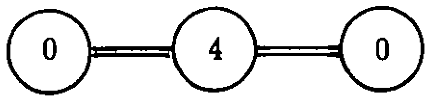


Fig. 20

Let C_i' $(i = 1,2)$, C_0' be the non-singular rational curves corresponding to the vertices with weights $0,4$, respectively. Then there is a ruling $\varphi : S' \longrightarrow \mathbb{P}^1$ which has C_i' as a smooth fiber and C_0' as a double section.

Thus S' is a rational ruled surface. Since $1 = b_2(C') - b_2(S) = b_2(C') - b_2(S') = 3 - b_2(S')$, we have $b_2(S') = 2$. Hence, we have $S' \cong \Sigma_n$ (in fact, $S' \cong \mathbb{P}^1 \times \mathbb{P}^1$ or Σ_1).

On the other hand, one can easily verify that $H_1(X;\mathbb{Z}) \cong H^2(C;\mathbb{Z})/H^2(S;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ (see Fig. 20). This cannot occur, since $H_1(X;\mathbb{Z}) \cong \mathbb{Z}$ by assumption. Therefore we have finally the dual graph F.14. Then one can verify that $S \cong \Sigma_m$ and $S \backslash C \cong \mathbb{C} \times \mathbb{C}^*$.

q.e.d.

§ 3. Affine surfaces properly dominated by $\mathbb{C} \times \mathbb{C}^*$

1. Let X be a normal affine surface over $\mathbb{C}$ and $f : \mathbb{C} \times \mathbb{C}^* \longrightarrow X$ be a proper morphism of $\mathbb{C} \times \mathbb{C}^*$ onto X. Let (S,C) be a minimal normal completion of X, namely, S is a normal projective algebraic surface and $C = \bigcup_{i=1}^{n} C_i$ is an algebraic curve on S such that $X \cong S \setminus C$. By a resolution of singularities, we may assume that S is non–singular at every point on C.

Lemma 3.1 C is one of the type from (α) to (ϵ) in Table I.

Proof (c.f. [2]). By the proof of Lemma 2 in Suzuki [12], we can find two regular points P_1, P_2 of C_i and a divergent sequence of points $\{(x_{kn}, y_{kn})\}_{n=1}^{\infty}$ in $\mathbb{C} \times \mathbb{C}^*$ satisfying (i) $\lim_{n \to \infty} f(x_{kn}, y_{kn}) = P_k$ $(k = 1,2)$, (ii) $\lim_{n \to \infty} x_{1n} = \lim_{n \to \infty} x_{2n} = \infty$, (iii) $x_{1n} \neq x_{2n}$. Further, we can find a holomorphic function with no zero on Δ_ρ^* such that $h(x_{kn}) = y_{kn} \neq 0$ for $k = 1,2$ and $n = 1,2, \ldots$. We put $\phi_i(x) := f(x, h(x))$. Then we have a holomorphic mapping $\phi_i : \Delta_\rho^* \longrightarrow S \setminus C$ such that $\phi_i(0;S)(\subset C)$ contains two regular points of C_i of C. Thus, by Proposition 3 of Nishino–Suzuki [11], we have $C_i \subset \phi_i(0;S) \subset C$. By Lemma 1.1, we have the claim.

q.e.d.

2. Since $\mathbb{C} \times \mathbb{C}^*$ is Stein and f is proper finite, X is also Stein, and further $H_i(X;\mathbb{Z}) = 0$ for $i > 2$, $H_2(X;\mathbb{Z})$ is a torsion free group (see Narasimhan [9], [10]).

Lemma 3.2. Assume that $H_1(X;\mathbb{Z})$ has a torsion. Then there is a finite unramified covering $\pi : \tilde{X} \longrightarrow X$ of X such that $H_1(\tilde{X};\mathbb{Z})$ is free.

Proof. Since X is Stein, we have $H^1(X,\mathcal{O}^*) \cong H^2(X;\mathbb{Z}) \cong H_2(X;\mathbb{Z}) \oplus \mathrm{Tor}\, H_1(X;\mathbb{Z})$ (the universal coefficient theorem). Then, there is a line bundle L on X with $L^{\otimes m} = 1$ for some $m > 1$ $(m \in \mathbb{Z})$. Let $s \not\equiv 0$ be a non–zero section of $L^{\otimes m} = 1$. Then we put $\tilde{X} := R(\sqrt[n]{s})$ (the Riemann domain over X). Then $\tilde{X}$ is a n–fold section of L, which is desired.

q.e.d.

Let V be the fiber product of $\mathbb{C} \times \mathbb{C}^*$ and $\tilde{X}$ over X. Then we have a commutative diagramm:

(3.1)
$$\begin{tikzcd} V \arrow[r, "f'"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\circlearrowleft"] & \tilde{X} \arrow[d, "\pi"] \\ \mathbb{C} \times \mathbb{C}^* \arrow[r, "f"'] & X \end{tikzcd},$$

where π', f' are the natural projections. Since $\pi : \tilde{X} \longrightarrow X$ is a finite unramified covering, so is $\pi' : V \longrightarrow \mathbb{C} \times \mathbb{C}^*$. Then we have $V \cong \mathbb{C} \times \mathbb{C}^*$ by Lemma 3.3 below.

Lemma 3.3. Let $v := \{v_1, \dots, v_k\}$ $(k \geq 0)$ be a set of k points in $\mathbb{C} \times \mathbb{C}^*$, where $v = \phi$ if $k = 0$. Let $\varphi' : M' \longrightarrow \mathbb{C} \times \mathbb{C}^* - v$ is a finite unramified algebraic covering. Then there is a finite unramified algebraic covering $\varphi : \mathbb{C} \times \mathbb{C}^* \longrightarrow \mathbb{C} \times \mathbb{C}^*$ such that $M' \cong \mathbb{C} \times \mathbb{C}^* - \varphi^{-1}(v)$. In particular, if $k = 0$, then $M' \cong \mathbb{C} \times \mathbb{C}^*$.

<u>Proof</u>. M' can be imbedded in a normal (affine) surface M such that φ' extends to a proper morphism $\varphi : M \longrightarrow \mathbb{C} \times \mathbb{C}^*$. Since $\mathbb{C} \times \mathbb{C}^*$ is smooth and φ' is unramified, M is smooth, and thus, $\varphi : M \longrightarrow \mathbb{C} \times \mathbb{C}^*$ is also unramified. Now, since $\pi_1(\mathbb{C} \times \mathbb{C}^*) \cong \mathbb{Z}$, we have $\pi_1(\mathbb{C} \times \mathbb{C}^*)/\pi_1(M) \cong \mathbb{Z}_m$ for some $m > 1$ $(m \in \mathbb{Z})$. Thus $\varphi : M \longrightarrow \mathbb{C} \times \mathbb{C}^*$ is equivalent to a covering $\phi : \mathbb{C} \times \mathbb{C}^* \longrightarrow \mathbb{C} \times \mathbb{C}^*$ with $\phi(z,w) = (z,w^m)$, where (z,w) is a coordinate system of $\mathbb{C} \times \mathbb{C}^*$.

q.e.d.

<u>Definition 3.4</u>. We call the normal affine surface $\tilde{X}$ the torsion free reduction of X.

§ 4. Proof of Theorem (Non–singular case)

Assume that X is non–singular. Since $f : \mathbb{C} \times \mathbb{C}^* \longrightarrow X$ is proper finite, we have $b_i(X) \leq b_i(\mathbb{C} \times \mathbb{C}^*)$ for $i \geq 1$ (c.f. Theorem (2.1) in Fujita [1]). Thus we have two cases:

(i) $b_i(X) = 0$ for $i > 0$,

(ii) $b_1(X) = 1$, $b_i(X) = 0$ for $i > 1$.

Since X is Stein, we have $H_i(X;\mathbb{Z}) = 0$ for $i \geq 2$ in any case.

Let (S,C) be the minimal normal completion of X. By Lemma 3.1, C is one of the type from (α) to (ε) in Table I. If $H_1(X;\mathbb{Z})$ is free, then, by Proposition 2.1 and Proposition 2.2, we have $X \cong \mathbb{C}^2$ or $\mathbb{C} \times \mathbb{C}^*$.

Lemma 4.1. $H_1(X;\mathbb{Z})$ has no torsion.

Proof. If $H_1(X;\mathbb{Z})$ has a torsion, then we take the torsion free reduction $\pi : \tilde{X} \longrightarrow X$. Then there is a proper morphism $f' : \mathbb{C} \times \mathbb{C}^* \longrightarrow \tilde{X}$ (see (3.1)). Since $H_1(\tilde{X};\mathbb{Z})$ is free, we have $\tilde{X} \cong \mathbb{C}^2$ or $\mathbb{C} \times \mathbb{C}^*$. In the case of $\tilde{X} = \mathbb{C}^2$, since $1 = \chi(\mathbb{C}^2) = (\deg \pi) \cdot \chi(X)$, we have $\deg \pi = 1$, namely, $\tilde{X} \cong X$. In the case of $\tilde{X} \cong \mathbb{C} \times \mathbb{C}^*$, by the same argument as in Proposition 1.2, we can see that the dual graph $\Gamma(C)$ looks like F.2 or F.14. If the dual graph looks like F.2, then one can easily see that $\bar{\kappa}(X) = 1$. On the other hand, since $\pi : \tilde{X} \longrightarrow X$ is a finite unramified covering, we have $-\infty = \bar{\kappa}(\mathbb{C} \times \mathbb{C}^*) = \bar{\kappa}(\tilde{X}) = \bar{\kappa}(X)$ (c.f. Iitaka [6]). This is a contradiction. Thus the dual graph must be F.14. Hence $X = S \setminus C = \mathbb{C} \times \mathbb{C}^*$. Therefore $H_1(X;\mathbb{Z})$ has no torsion.

q.e.d.

This completes the proof of the non–singular case.

§ 5. Proof of Theorem (Singular case)

Let $x = \{x_1, \dots, x_r\}$ be the singular points of X, and $\overline{U}_j$ be a small closed contractible neighborhood of x_j such that $\overline{U}_j \backslash x_j \approx \partial U_j$ (deformation retract) and $\overline{U}_j \cap \overline{U}_k = \phi$ $(j \neq k)$, where ∂U_j is the boundary of U_j. Take a point $v_j \in f^{-1}(x_j)$. Since $f : \mathbb{C} \times \mathbb{C}^* \longrightarrow X$ is a proper finite mapping, there is a small ball Δ_j with center v_j in $\mathbb{C} \times \mathbb{C}^*$ such that $f|\Delta_j : \Delta_j \longrightarrow U_j$ is a proper finite mapping with $\Delta_j \cap f^{-1}(x_j) = \{v_j\}$. Since $\pi_1(\Delta_j \backslash v_j) = 1$ and $\deg f|\Delta_j < +\infty$, $\pi_1(\overline{U}_j \backslash x_j) = \pi_1(\partial U_j)$ is a finite group.

Thus we have

<u>Lemma 5.1</u> Each x_j is a quotient singularity.

Now, let K be a subgroup of $\pi_1(X \backslash x)$ of finite index and $\sigma' : Y' \longrightarrow X \backslash x$ be a finite covering associated with the subgroup K. Then the finite quotient group $G := \pi(X \backslash x)/K$ acts on Y' freely. Let Z' be the fiber product of $\mathbb{C} \times \mathbb{C}^* - v$ and Y' over $X \backslash x$, where $v := f^{-1}(x)$. Then we have a commutative diagram:

(5.1)
$$\begin{tikzcd} Z' \arrow[r, "g'"] \arrow[d, "\tau'"'] \arrow[dr, phantom, "\circlearrowleft"] & X' \arrow[d, "\sigma'"] \\ \mathbb{C} \times \mathbb{C}^* - v \arrow[r, "f"'] & X \backslash x \end{tikzcd} ,$$

where τ', g' are the natural projections. One can easily show that $\tau' : Z' \longrightarrow \mathbb{C} \times \mathbb{C}^* - v$ is a finite unramified covering. Then Z' (resp. Y') can be imbedded in a normal affine surface Z (resp. Y) such that τ' (resp. σ') extends to a proper morphism $\tau : Z \longrightarrow \mathbb{C} \times \mathbb{C}^*$ (resp. $\sigma : Y \longrightarrow X$), since $\mathbb{C} \times \mathbb{C}^*$, X are normal and f, σ' are proper morphisms. Further, the morphism $g' : Z' \longrightarrow Y'$ also extends to a proper morphism $g : Z \longrightarrow Y$. Thus we have a commutative diagram:

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "\tau"'] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[d, "\sigma"] \\ \mathbb{C} \times \mathbb{C}^* \arrow[r, "f"'] & X \end{tikzcd} \tag{5.2}$$

In particular, G can be extended to algebraic automorphisms of Y (with isolated fixed points). Thus we have

<u>Lemma 5.2</u>. $X \cong Y/G$.

Next, since $\tau : Z - \tau^{-1}(v) \longrightarrow \mathbb{C} \times \mathbb{C}^* - v$ is a finite unramified covering, by Lemma 3.3, we have

<u>Lemma 5.3</u>. $Z \cong \mathbb{C} \times \mathbb{C}^*$.

We put $y := \sigma^{-1}(x)$ and $z = g^{-1}(y)$. Then we have $Y \backslash y = Y'$. Since $\pi_1(Z \backslash z) = \pi_1(\mathbb{C} \times \mathbb{C}^* - z) \cong \mathbb{Z}$, the image $\mathrm{Im}(\pi_1(Z \backslash z) \hookrightarrow \pi_1(Y \backslash y)$ is isomorphic to $\mathbb{Z}_m$ ($m \in \mathbb{N} \cup \{0\}$). Since $\mathrm{Im}(\pi_1(Z \backslash z)$ is a subgroup of $\pi_1(Y \backslash y)$ of finite index (c.f. Theorem (2.12) in Fujita [1]), $\pi_1(Y \backslash y)$ is a finite subgroup if $m \neq 0$.

Taking a finite covering associated with the subgroup $\mathrm{Im}(\pi_1(\mathbb{C} \times \mathbb{C}^* \backslash z))$ (or taking the universal covering if $\pi_1(Y \backslash y)$ is a finite group) if necessary, we may assume that $\pi_1(Y\backslash y) = 1$ or $\mathbb{Z}$. Further, taking the torsion free reduction of Y if necessary, we may assume that $H_1(Y;\mathbb{Z})$ is free.

Thus we have

<u>Lemma 5.4</u>. $\pi_1(Y\backslash y) = 1$ or $\mathbb{Z}$, and $H_1(Y;\mathbb{Z})$ is free.

Let W be a contractible neighborhood of y in Y with $W\backslash y \approx \partial W$ (deformation retract), where ∂W is the boundary of W. Then we have an exact sequence over $\mathbb{Z}$:

$$(5.3) \qquad \longrightarrow H_i(\partial W) \longrightarrow H_i(Y\backslash y) \oplus H_i(W) \longrightarrow H_i(Y) \longrightarrow H_{i-1}(\partial W) \longrightarrow .$$

We know that $y = \sigma^{-1}(x)$ consists of at worst quotient singularities (c.f. Lemma 5.1). Thus, both $\pi_1(\partial W)$ and $H_1(\partial W;\mathbb{Z})$ are finite groups.

On the other hand, since $g : \mathbb{C} \times \mathbb{C}^* - z \longrightarrow Y - y$ is proper finite, we have $b_i(Y\backslash y) \leq b_i(\mathbb{C} \times \mathbb{C}^* - z)$ namely, $b_1(Y\backslash y) \leq 1$ and $b_2(Y\backslash y) = 0$. Thus by Lemma 5.3 and (5.3) above, we have

<u>Lemma 5.5</u>.

(a) $\quad \pi_1(Y\backslash y) = 1 \Leftrightarrow \quad H_i(Y;\mathbb{Z}) = 0 \ (i > 0)$, $H_2(Y\backslash y;\mathbb{Z}) = 0$,
$\qquad\qquad\qquad\qquad\quad H_1(\partial W;\mathbb{Z}) = 0$,

(b) $\pi_1(Y\backslash y) = \mathbb{Z} \Leftrightarrow \quad H_1(Y;\mathbb{Z}) = \mathbb{Z}, \qquad H_i(Y;\mathbb{Z}) = 0 \quad (i > 1),$

$\qquad H_2(Y\backslash y;\mathbb{Z}) = 0, \qquad H_1(\partial W;\mathbb{Z}) = 0.$

Since $\pi_1(\partial W)$ is a finite group and $H_1(\partial W;\mathbb{Z}) = 0$, we have

Corollary 5.6 The set $y = \sigma^{-1}(x)$ consists of at worst rational double points of E_8 – type.

Let (S,C) be the minimal normal completion of X such that S is smooth in a neighborhood of $C = \bigcup_{i=1}^{n} C_i$ (see § 3). By Lemma 3.1, C is one of the type from (α) to (ε).

Let $\nu : \hat{S} \longrightarrow S$ be the minimal resolution of S and put $\nu^{-1}(y) = E = \bigcup_{j=1}^{m} E_j$, where $y = \sigma^{-1}(x) \hookrightarrow X \hookrightarrow S$. By Corollary 5.6, one can define a canonical divisor K_S on S, and further, we have

Lemma 5.7. Supp $K_{\hat{S}} \cap E = \phi$, $K_{\hat{S}} \cong K_S$.

Proof. We have $K_{\hat{S}} = \nu^* K_S + \Sigma n_j E_j \ (n_j \in \mathbb{Z})$. Since each E_j is (-2) – curve and the intersection matrix $((E_i \cdot E_j))$ is negative definite, we have all $n_j = 0$.

q.e.d.

Let us consider an exact sequence over $\mathbb{Z}$:

$$
(5.4)\qquad
\begin{array}{ccccccccc}
\longrightarrow & H^i(\hat{S}, C \cup E) & \longrightarrow & H^i(\hat{S}) & \longrightarrow & H^i(C \cup E) & \longrightarrow & H^{i+1}(\hat{S}, C \cup E) & \longrightarrow \\
& \wr\| & & & & & & \wr\| & \\
& H_{4-i}(Y\setminus y) & & & & & & H_{3-i}(Y\setminus y) &
\end{array}
$$

By Lemma 5.5, we have

$$
(5.5)\qquad 0 \longrightarrow H^2(\hat{S};\mathbb{Z}) \longrightarrow H^2(C \cup E;\mathbb{Z}) \longrightarrow H_1(Y\setminus y) \longrightarrow 0 .
$$

Indeed, by (5.4), we have $b_3(\hat{S}) \le b_1(Y\setminus y) \le 1$. Since $\hat{S}$ is projective, we have $0 = b_3(\hat{S}) = b_1(\hat{S}) = b_1(C) = q(\hat{S})$. By (5.5) and Lemma 5.6, we have

$$
\text{(i)}\qquad K_S = \sum_{i=1}^{n} \lambda_i C_i \qquad (\lambda_i \in \mathbb{Z}) ,
$$

$$
\text{(ii)}_a\qquad \det((C_i \cdot C_j)) \neq 0 \quad \text{if} \quad \pi_1(Y\setminus y) = 1 ,
$$

$$
\text{(ii)}_b\qquad \det((C_i \cdot C_j)) = 0 \quad \text{if} \quad \pi_1(Y\setminus y) \cong \mathbb{Z} .
$$

Since $b^+(\hat{S}) = 2P_g(\hat{S}) + 1 \ge 1$ and $((E_i \cdot E_j))$ is negative definite, we have

$$
\text{(iii)}\qquad ((C_i \cdot C_j)) \text{ is not negative semi-definite.}
$$

By the Noether formula, we have

$(\mathrm{iv})_a$ $\qquad 12P_g(\hat{S}) = K^2_{\hat{S}} + b_2(C) + 8t - 10$ if $\pi_1(Y\setminus y) = 1$,

$(\mathrm{iv})_b$ $\qquad 12P_g(\hat{S}) = K^2_{\hat{S}} + b_2(C) + 8t - 11$, if $\pi_1(Y/y) \cong \mathbb{Z}$,

where $t(\geq 0)$ is the number of singularities of E_8–type in Y .

Let ∂T be the boundary of a small tubular neighborhood T of C in $\hat{S}$. Replacing, in the diagram (2.2), C , S by $C \cup E$, $\hat{S}$, respectively, we have

$(\mathrm{v})_a$ $\qquad H_1(\partial T;\mathbb{Z}) = 0$ if $\pi_1(Y\setminus y) = 1$,

since $H_2(Y\setminus y;\mathbb{Z}) = 0$.

Proposition 5.8. $Y \cong \mathbb{C}^2$ or $\mathbb{C} \times \mathbb{C}^*$.

Proof. We have only to prove the smoothness of Y (see (5.2), Lemma 5.3, Lemma 5.5). By Lemma 5.5, we have two cases (a) and (b). First, let us consider the case (a). We have then (i), $(\mathrm{ii})_a$, (iii), $(\mathrm{iv})_a$, $(\mathrm{v})_a$ above. By the same argument as in Proposition 2.1, we have the dual graph $\Gamma(C)$ which looks like F.10 (with $12P_g(\hat{S}) = 8(t-1)$) , and F.13 (with $12P_g(\hat{S}) = 8t$) . Looking at the order of $K_{\hat{S}} = K_S$ in these graphs, one can easily see that $P_m(\hat{S}) = P_m(S) = 0$ for $m > 0$. Thus, if $\Gamma(C)$ looks like F.13, then we have $t = 0$, namely, Y is smooth. If $\Gamma(C)$ looks like F.10, then we have $t = 1$, namely, Sing Y consists of exactly one rational double point of E_8–type. Gurjar–Shastri [p. 481–482,5] proved, in this situation, that $\pi_1(Y\setminus y) \neq 1$. Therefore Y must be smooth if $\pi_1(Y\setminus y) = 1$.

Next, let us consider the case (b). We have then (i), $(ii)_b$, (iii), $(iv)_b$. By the same argument as in Proposition 2.2, we have the dual graph $\Gamma(C)$ which looks like F.2 (with $12P_g(\hat{S}) = m + 8t$), F.14 (with $12P_g(\hat{S}) = 8t$). Looking at these dual graphs (and Fig. 20), one can set that $\hat{S}$ has a structure of a ruled surface over a non–singular rational curve (c.f. Suzuki [p. 459,12]). Thus we have $P_g(\hat{S}) = 0$. If $\Gamma(C)$ looks like F.14, then $t = 0$, namely Y is smooth. If $\Gamma(C)$ looks like F.2, then we have $\bar{\kappa}(Y) = 1$ (c.f. [6]). On the other hand, since $g : Z \cong \mathbb{C} \times \mathbb{C}^* \longrightarrow Y$ is proper finite morphism, we have $\bar{\kappa}(Y) \leq \bar{\kappa}(\mathbb{C} \times \mathbb{C}^*) = -\infty$. This is a contradiction. Therefore Y is smooth if $\pi_1(Y \backslash y) \cong \mathbb{Z}$.

q.e.d.

By Lemma 5.2, we have

<u>Corollary 5.9</u>. $X \cong \mathbb{C}^2/G_a$ or $\mathbb{C} \times \mathbb{C}^*/G_b$, where G_a (resp. G_b) is a small finite subgroup of $\mathrm{Aut}(\mathbb{C}^*)$ (resp. $\mathrm{Aut}(\mathbb{C} \times \mathbb{C}^*)$).

<u>Lemma 5.10</u> (Miyanishi [7], Furushima [3]). For a finite subgroup of $\mathrm{Aut}(\mathbb{C}^2)$, there is an automorphism $\alpha = \alpha_G \in \mathrm{Aut}(\mathbb{C}^2)$ such that $\alpha \circ G \circ \alpha^{-1} := \{\alpha \circ g \circ \alpha^{-1} ; G \in G\} \subset GL(2,\mathbb{C})$.

Let $g(z,w) = (g_1(z,w)\ ,\ g_2(z,w))$ be an algebraic automorphism of $\mathbb{C} \times \mathbb{C}^*$, where (z,w) is a coordinate system of $\mathbb{C} \times \mathbb{C}^*$, and $g_j(z,w)$ $(j = 1,2)$ is a regular rational function on $\mathbb{C} \times \mathbb{C}^*$.

Let $\pi : \mathbb{C} \times \mathbb{C}^* \longrightarrow \mathbb{C}^*$ be the natural projection. For a point $w \in \mathbb{C}^*$, the restriction $\pi | g(\pi^{-1}(w)) : g(\pi^{-1}(w) \cong \mathbb{C} \longrightarrow \mathbb{C}^*$ is a non–zero regular rational function. Hence it must be non–zero constant. Thus we have $g(\pi^{-1}(w)) = \pi^{-1}(w')$ for some $w' \in \mathbb{C}^*$.

Therefore g induces an automorphism $\mu_g \in \mathrm{Aut}(\mathbb{C}^*)$ such that $\mu_g \circ \pi = \pi \circ g$, hence, $g_2(z,w) = \mu_g(w)$. Since $\left| \dfrac{\partial(g_1,g_2)}{\partial(z,w)} \right| \neq 0$ on $\mathbb{C} \times \mathbb{C}^*$, we have $g_1(z,w) = P_g(w) \cdot z + Q_g(w)$, where $P_g(w)$, $Q_g(w)$ are rational functions of w.

Now, since $\mu_g(w) = a_g \cdot w$, or $\frac{a_g}{w}$ $(a_g \in \mathbb{C}^*)$, the set $F(\mu_g)$ of fixed points of μ_g consists of two points if $F(\mu_g) \neq \phi$. Since any automorphism $h \neq \mathrm{id}$ of $\mathbb{C}$ has at most one fixed point, the number of fixed points of g is equal to two.

Thus we have

Lemma 5.11. $G_b \subset \mathrm{Aut}(\mathbb{C} \times \mathbb{C}^*)$ has exactly two (isolated) fixed points on $\mathbb{C} \times \mathbb{C}^*$.

By Corollary 5.9, Lemma 5.10, Lemma 5.11, we complete the proof of Theorem.

Remark. Similarly, one can also prove that a normal affine surfaces properly dominated by $(\mathbb{C}^*)^2$ is isomorphic to either $\mathbb{C}^2$, $\mathbb{C} \times \mathbb{C}^*$, $(\mathbb{C}^*)^2$ or $\mathbb{C}^2/G_a$, $\mathbb{C} \times \mathbb{C}^*/G_b$, $(\mathbb{C}^*)^2/G_c$ for a small finite subgroup G_a, G_b, G_c of $GL(2,\mathbb{C})$, $\mathrm{Aut}(\mathbb{C} \times \mathbb{C}^*)$, $\mathrm{Aut}(\mathbb{C}^*)^2$, respectively. The details will be discussed elsewhere.

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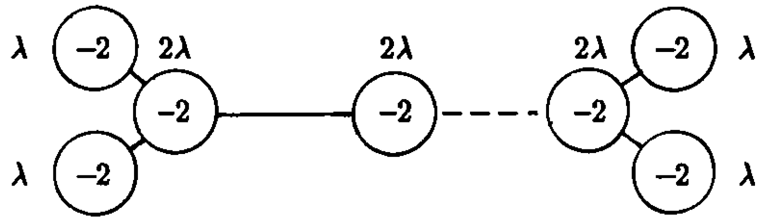
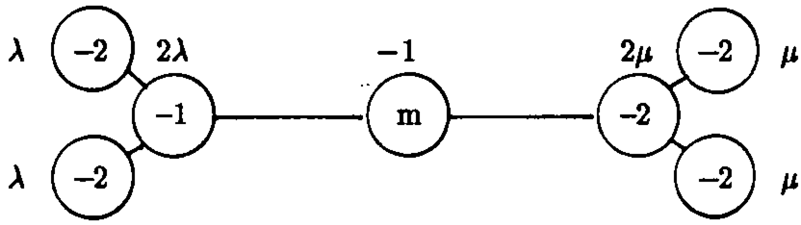
F.1

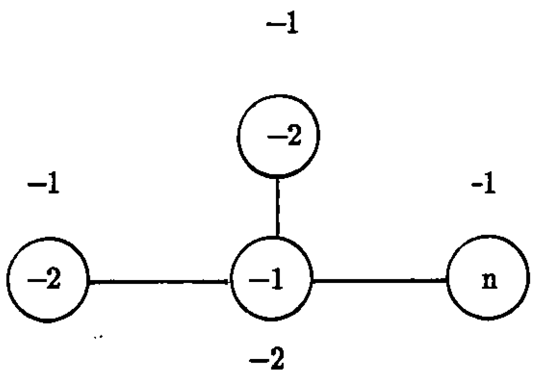


F.2



$(\lambda + \mu = -1)$

F.3



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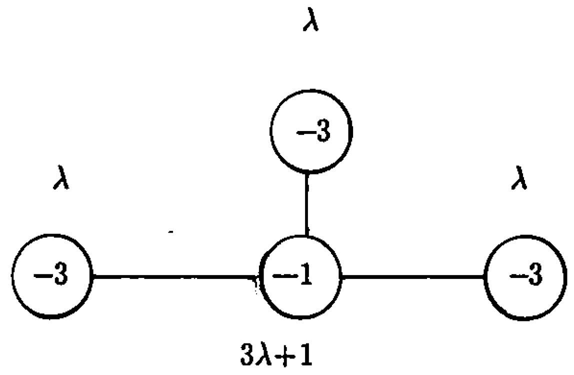

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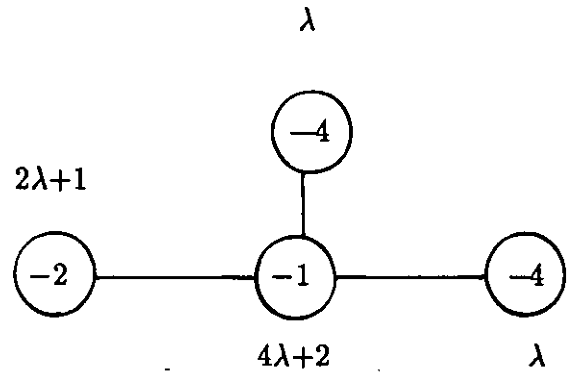

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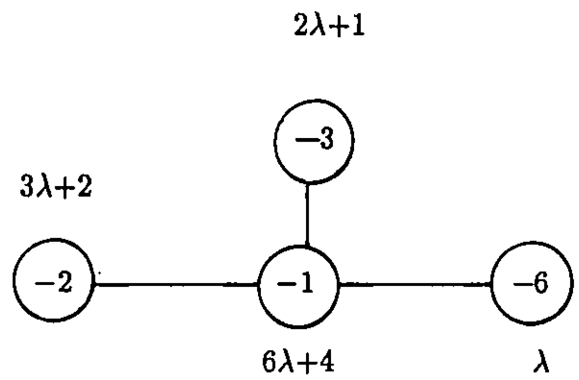

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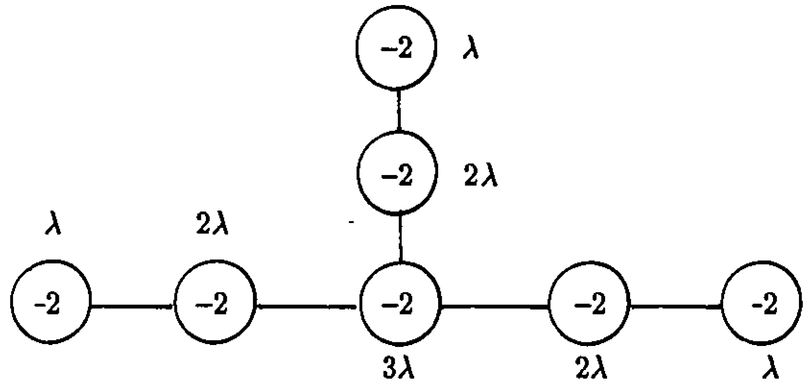
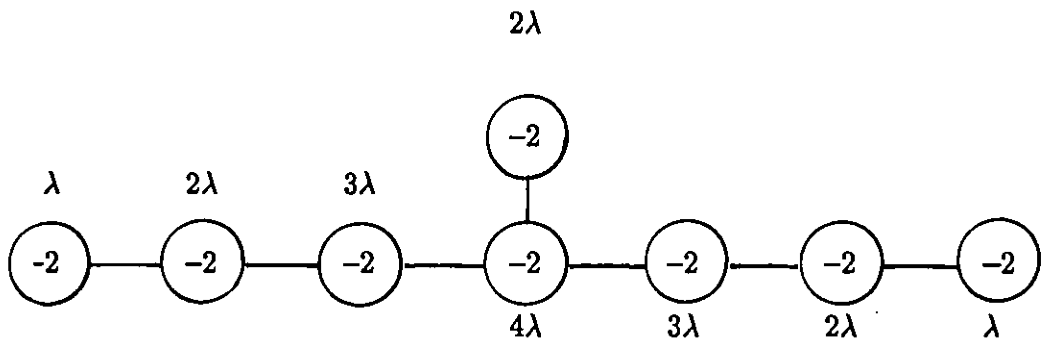
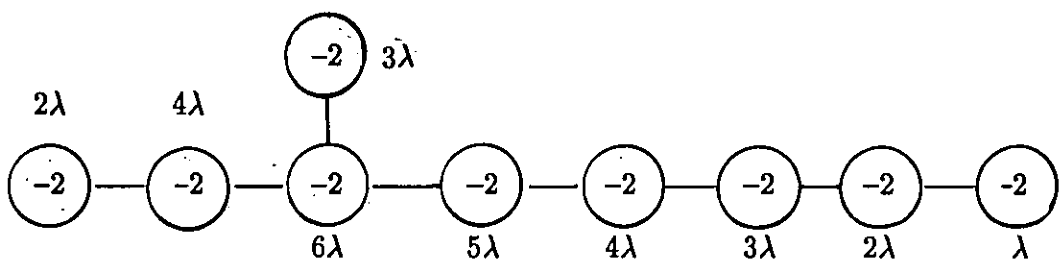

-2
λ
-2
2λ
λ
2λ
-2
-2
-2
-2
-2
3λ
2λ
λ

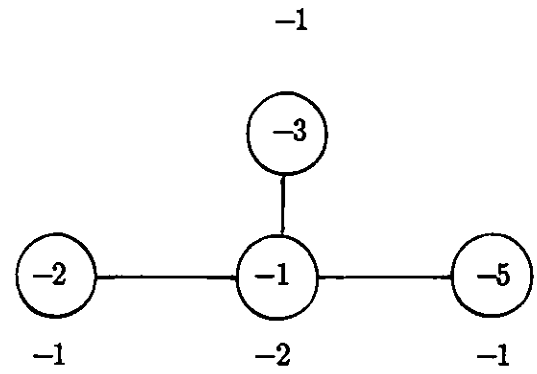
F.8


2λ
-2
λ
2λ
3λ
-2
-2
-2
-2
-2
-2
-2
4λ
3λ
2λ
λ

F.9


-2
3λ
2λ
4λ
-2
-2
-2
-2
-2
-2
-2
-2
6λ
5λ
4λ
3λ
2λ
λ

F.10



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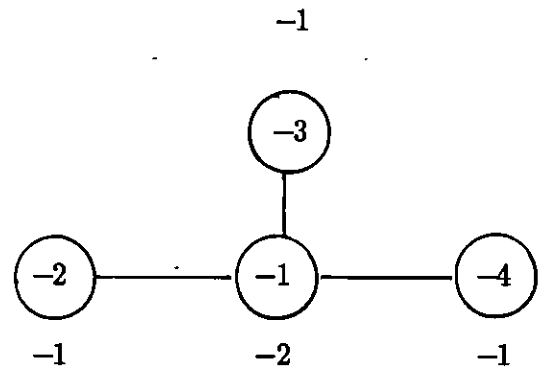

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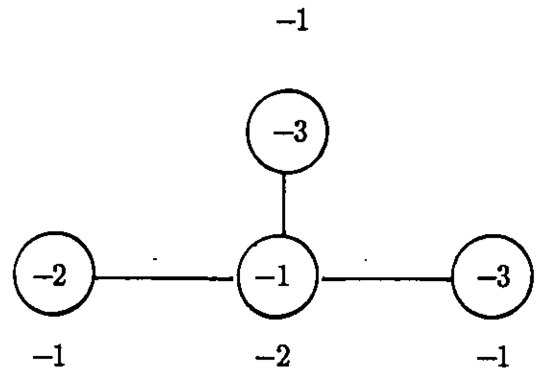

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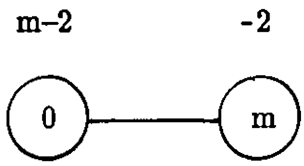
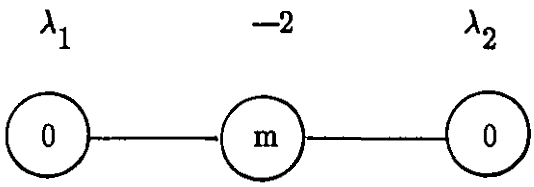
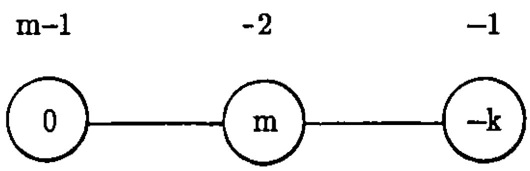


F.14



$(\lambda_1 + \lambda_2 = m - 2)$

F.15



$(k \geq 2)$

TABLE II