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Geometric adeles and the Riemann–Roch theorem for 1-cycles on surfaces

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Adelic proofs of the Riemann–Roch theorem for smooth proper irreducible curves over finite fields are well known, e.g. [I1], [T, 4.2]. There only characteristic zero case is treated, but it is easy to develop a similar proof in positive characteristic. That proof uses the translation invariant measure and integration, Fourier transform and harmonic analysis on adeles and their subquotients. However, adelic duality on its own is already sufficient to derive the Riemann–Roch theorem over an arbitrary field, as described in 1950/51 Artin's lectures in Princeton [A, Ch.XIV] for function fields over finite fields, [I2] for smooth proper irreducible curves over any field, [G] for proper irreducible curves over any field. There is also an essentially measure and integration free proof in [W, Ch.VI], which is however less lucid than the previous proofs.

Unlike dimension one, there are two different adelic structures in dimension two, one of more geometric nature, closely related to 1-cycles, and another of more analytic nature, so that one has measure, integration and zeta integral theory on them, this second structure is closely related to 0-cycles, [F3, Ch.2], [F4, Ch.1]. The former adelic structure on surfaces over finite fields was introduced yet in [P2], and it had been a long expectation that the geometric adeles would lead to a proof of the Riemann–Roch theorem for divisors on surfaces. An adelic proof of the Riemann–Roch theorem for smooth irreducible projective surfaces over finite fields was recently announced in [OP3]. The proof relies, in addition to foundational papers [P1], [B], [P2], on several technical developments in recent papers [O], [OP1], [OP2] whose full length is a three digital number of pages.

For further developments of adelic geometry, including an adelic interpretation of the intersection pairing and other applications in higher algebraic geometry, as well as applications of the two-dimensional zeta integral study to the study of the zeta function including the full BSD conjecture, concise and lucid proofs are needed.

The aim of this paper is to offer a new adelic interpretation of the intersection index on surfaces in terms of adelic Euler characteristic, without using K-theoretical constructions. The proof uses only foundational aspects of duality of geometric adeles $\mathbf{A}$, from parts of [P1], [B], [P2], and further develops their theories. The proof uses adelic self-duality and its corollaries, and can be viewed as another evidence of powerful applications of adelic geometry. The proof is very short and essentially easy. The main result implies the Riemann–Roch theorem on surfaces as soon as one uses the known relation between Zariski cohomologies and adelic complex cohomologies.

We also prove various aspects of self-duality of $\mathbf{A}$, which were stated without proof in several previous publications including [F4,§28] and [OP3], and establish a number of new properties in a form well suitable to several future generalizations.

We work with smooth proper irreducible surfaces over finite fields, but the proof is written in a such way that, similarly to [I2] and [G], it can be easily generalized to the case of proper irreducible surfaces over arbitrary fields.

An extension of the theory of this work to the case of arithmetic surfaces is expected, with an adelic interpretation of the Arakelov intersection, see Remarks at the end of the paper.

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In the introduction of [I2] Iwasawa wrote, "it has become clearer and clearer that the topological properties of [adeles] and of its related structures ..., have essential relations to the arithmetic of the [global field]". This text illustrates in a concise way a similar phenomenon in relation to algebraic geometry on surfaces.

I am grateful to A. Beilinson for discussions of adelic geometry and importance of the moving lemma. This work combines the two and the universal property of the intersection pairing. I am also grateful to A. Yekutieli, M. Kapranov and M. Morrow for a number of useful comments, and to D. Osipov for explaining some features of the very different approach in [OP3].

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0. Here is how the classical one-dimensional proof can be conducted. Let k be a global field and let $\mathbb{A}_{\rm k}$ be the adelic ring of k. Suppose k is of positive characteristic, i.e. the function field of a smooth proper irreducible curve C over a finite field $\mathbb{F}$, and assume that the latter is algebraically closed in k. For a divisor d on the curve we have an adelic complex

$$\mathcal{A}_{\rm k}(d): \qquad \mathrm{k} \oplus \mathbb{A}_{\rm k}(d) \longrightarrow \mathbb{A}_{\rm k}, \qquad (a,b) \mapsto a-b$$

where $\mathbb{A}_{\rm k}(d) = \{(\alpha_v) : v(\alpha_v) \geqslant -v(d)\}$, where v runs through all discrete valuations on k, i.e. all closed points of the curve, and $d = \sum v(d) d_v$ where d_v is the class of the valuation/closed point in the divisor group. This complex is quasi-isomorphic to the complexes $\mathbb{A}_{\rm k}(d) \longrightarrow \mathbb{A}_{\rm k}/\mathrm{k}$ and $\mathrm{k} \longrightarrow \mathbb{A}_{\rm k}/\mathbb{A}_{\rm k}(d)$. We have $H^0(\mathcal{A}_{\rm k}(d)) = \mathrm{k} \cap \mathbb{A}_{\rm k}(D)$, $H^1(\mathcal{A}_{\rm k}(d)) = \mathbb{A}_{\rm k}/(\mathrm{k} \cap \mathbb{A}_{\rm k}(D))$.

Take any non-zero differential form $\omega \in \Omega^1_{\mathrm{k}/\mathbb{F}}$. We use an associated pairing

$$\mathbb{A}_{\rm k} \times \mathbb{A}_{\rm k} \longrightarrow \mathbb{F}, \qquad (\alpha, \beta) \mapsto \sum_v \mathrm{Tr}_{k(v)/\mathbb{F}}\ \mathrm{res}_v(\alpha_v \beta_v \omega),$$

where $k(v)$ is the residue field of the local ring at v. Denote by $\mathfrak{c}$ the divisor of ω. Alternatively, take any nontrivial differential map (the terminology of [A, Ch.XIII §4]), i.e. continuous $\mathbb{F}$-linear map from $\mathbb{A}_{\rm k}$ to $\mathbb{F}$ (endowed with the discrete topology) which vanishes on k, and compose it with the multiplication $\mathbb{A}_{\rm k} \times \mathbb{A}_{\rm k} \longrightarrow \mathbb{A}_{\rm k}$, then the complement of $\mathbb{A}_{\rm k}(0)^\perp$ of $\mathbb{A}_{\rm k}(0)$ can be written as $\mathbb{A}_{\rm k}(\mathfrak{c})$ for an appropriate $\mathfrak{c}$.

Using (non-canonical) self-duality of the additive group of a local field it is easy to prove (non-canonical) self-duality of the additive group of $\mathbb{A}_{\rm k}$. It is also algebraically and topologically isomorphic to the group of continuous $\mathbb{F}$-linear maps from the adeles to $\mathbb{F}$. For a subgroup H of the adeles denote $H^\perp = \{\beta \in \mathbb{A}_{\rm k} : (H, \beta) = 0\}$. If C is a projective line, it is easy to see that one can find an open subgroup R (actually in the form $\mathbb{A}_{\rm k}(d)$) of $\mathbb{A}_{\rm k}$ such that the latter is the direct sum of R and k, hence k is a discrete subset of $\mathbb{A}_{\rm k}$, and the quotient $\mathbb{A}_{\rm k}/\mathrm{k}$ is compact, since R is compact. This property extends to the general case using the trace map from k down to $\mathbb{F}(t)$. The complement $\mathrm{k}^\perp$ is a k-space which contains k; $\mathrm{k}^\perp/\mathrm{k}$ is a closed subgroup of $\mathbb{A}_{\rm k}/\mathrm{k}$, hence compact. On the other hand, $\mathrm{k}^\perp$ is the group of continuous $\mathbb{F}$-linear maps from the compact $\mathbb{A}_{\rm k}/\mathrm{k}$ to $\mathbb{F}$, hence it is discrete. Then $\mathrm{k}^\perp/\mathrm{k}$ is discrete and compact, hence finite. Since k is an infinite field, we deduce $\mathrm{k}^\perp = \mathrm{k}$. Hence every differential map corresponds to a differential form ω.

Working with the pairing of the adeles with themselves, we get $\mathbb{A}_{\rm k}(d)^\perp = \mathbb{A}_{\rm k}(\mathfrak{c} - d)$, hence the group of continuous $\mathbb{F}$-linear maps from $H^0(\mathcal{A}_{\rm k}(d))$ to $\mathbb{F}$ is isomorphic to $\mathbb{A}_{\rm k}/H^0(\mathcal{A}_{\rm k}(d))^\perp = H^1(\mathcal{A}_{\rm k}(\mathfrak{c} - d))$. Now $\mathbb{A}_{\rm k}(d)$ is compact, so its intersection with k is discrete in the compact group, hence $H^0(\mathcal{A}_{\rm k}(d))$ is finite and so is $H^1(\mathcal{A}_{\rm k}(d))$, and we get $\dim_{\mathbb{F}} H^0(\mathcal{A}_{\rm k}(d)) = \dim_{\mathbb{F}} H^1(\mathcal{A}_{\rm k}(\mathfrak{c} - d))$. So for the Euler characteristic we obtain $\chi_{\mathcal{A}_{\rm k}}(d) = \dim_{\mathbb{F}} H^0(\mathcal{A}_{\rm k}(d)) - \dim_{\mathbb{F}} H^1(\mathcal{A}_{\rm k}(d)) = \chi_{\mathcal{A}_{\rm k}}(\mathfrak{c} - d)$.

Working with the virtual dimension of two $\mathbb{F}$-commensurable spaces G, H (which means $G\cap H$ is of finite $\mathbb{F}$-finite codimension in each of them), $\dim_{\mathbb{F}}(G:H) = \dim_{\mathbb{F}} G/(G\cap H) - \dim_{\mathbb{F}}(H/(G\cap H)$, and noting it is additive on short exact sequences related to $0 \to \mathrm{k} \to \mathbb{A}_{\mathrm{k}} \to \mathbb{A}_{\mathrm{k}}/\mathrm{k} \to 0$, we obtain $-\deg d = \dim_{\mathbb{F}}(\mathbb{A}_{\mathrm{k}}(0) : \mathbb{A}_{\mathrm{k}}(d)) = \chi_{\mathcal{A}_{\mathrm{k}}}(0) - \chi_{\mathcal{A}_{\mathrm{k}}}(d)$.

The two formulas for curves

$$\deg d = \chi_{\mathcal{A}_{\mathrm{k}}}(d) - \chi_{\mathcal{A}_{\mathrm{k}}}(0), \qquad \chi_{\mathcal{A}_{\mathrm{k}}}(d) = \chi_{\mathcal{A}_{\mathrm{k}}}(\mathfrak{c} - d)$$

are the ones from which everything else follows.

In particular, using them we get

$$-\deg d = \dim_{\mathbb{F}} H^0(\mathcal{A}_{\mathrm{k}}(0)) - \dim_{\mathbb{F}} H^0(\mathcal{A}_{\mathrm{k}}(\mathfrak{c})) - \dim_{\mathbb{F}} H^0(\mathcal{A}_{\mathrm{k}}(d)) + \dim_{\mathbb{F}} H^0(\mathcal{A}_{\mathrm{k}}(\mathfrak{c} - d)).$$

Since $\dim_{\mathbb{F}} H^0(\mathcal{A}_{\mathrm{k}}(0)) = 1$, we derive $\dim_{\mathbb{F}} H^0(\mathcal{A}_{\mathrm{k}}(\mathfrak{c})) = \dim_{\mathbb{F}} H^0(\mathcal{A}_{\mathrm{k}}(\mathfrak{c} - d)) + \deg d + 1 - g$, where $g = \dim_{\mathbb{F}} H^0(\mathcal{A}_{\mathrm{k}}(\mathfrak{c}))$. This is the *adelic Riemann–Roch theorem*. It is not difficult to deduce that the adelic cohomology groups $H^i(\mathcal{A}_{\mathrm{k}}(d))$ are isomorphic to the Zariski cohomology groups $H^i(C, \mathcal{O}_C(d))$ and then the previous equality gives the classical Riemann–Roch theorem.

This proof is presented in such a way that it immediately extends to smooth proper irreducible and even proper irreducible curves over any field F. In this more general proof one works with F-linear topology on the adeles for a function field of the curve over F and replaces compactness by F-linear compactness (see, e.g. [L,§6]), for more details see [I2], [G].

1. Let S be a smooth irreducible proper surface over a finite field $\mathbb{F}$, hence projective. Denote by K its function field and assume that no nontrivial finite extension of $\mathbb{F}$ is contained in K. For an irreducible proper curve y (we will call them just curves) on S let D_y be the divisor of y in Div(S) and let K_y be the fraction field of the completion $\mathcal{O}_y$ of the local ring of S at y. For a closed point $x \in y$ let $K_{x,y}$ be the product of all $K_{x,z}$ where z runs through all minimal prime ideals of $\mathcal{O}_x$ corresponding to y, i.e. through all formal branches $y(x)$ of y at x, and $K_{x,z}$ is the fraction field of the completion $\mathcal{O}_{x,z}$ of the localization of $\mathcal{O}_x$ at z. For a closed point x of S let K_x be the minimal subring of $K_{x,y}$ which contains $\mathcal{O}_x$ and K.

Put $\mathcal{O}_{x,y} = \prod_{z\in y(x)} \mathcal{O}_{x,z}$. Denote by $O_{x,z}$ the ring of integers of the two-dimensional local field $K_{x,z}$ with respect to any of its discrete valuations of rank 2. The residue field of $K_{x,z}$ with respect to its discrete valuation of rank 1 is one-dimensional local field $E_{x,z}$, its residue is finite field $k_z(x)$. See e.g. [F4, §24] and references therein for more information. In positive characteristic we can identify $E_{x,z}$ with a unique subfield of $K_{x,z}$ and then choosing a local parameter t of $K_{x,z}$ with respect to its discrete valuation of rank 1, for example a local parameter t_y of y on S, the field $E_{x,z}$ can be viewed as the formal power series field $E_{x,z}((t))$.

For every curve y on S similarly to [F4, §28] and [F3] define the two-dimensional geometric adelic space $\mathbf{A}_y = \cup \mathbb{A}_y^r$ as in [F4, §25]. The ring $\mathbf{A}_y$ is the two-dimensional adelic ring associated to the curve y on the surface S. It is a subring of $\prod_{x\in y} K_{x,y}$ and can be thought of as $\mathbb{A}_{k(y)}((t_y))$, the formal power series in t_y, a local parameter of y on S, over the adelic ring of the one-dimensional function field of y. It has a subring $\mathbf{OA}_y = \mathbb{A}_y^0 = \mathbf{A}_y \cap \prod_{x\in y} \mathcal{O}_{x,y}$, which can be viewed as the subring of integral power series in t_y.

Similarly to [F4, §28] define the geometric adelic ring $\mathbf{A}_S$ as the restricted product of $\mathbf{A}_y$ with respect to $\mathbf{OA}_y$. Equivalently, $\mathbf{A} = \mathbf{A}_S$ is the subring of all $\{(\alpha_{x,z}), \alpha_{x,z} \in K_{x,z})\}$ such that the following two restricted conditions are satisfied: for almost every y the element $\alpha_{x,z} \in \mathcal{O}_{x,z}$ for all $x \in y$ and there is r such that $(\alpha_{x,y})_{x\in y} \in t_y^r \mathbf{A}_y$ for all y. The adelic ring $\mathbf{A}_S$ equals A_{012} defined in [P2, §2.1]. The ring $\mathbf{A}_S$ is the union of subring $\mathbf{A}_S(D)$ where D runs through all 1-cycles (divisors) of S and $\mathbf{A}_S(D) = \{(\alpha_{x,y})_{x\in y} \in t_y^{r_y} \mathbf{A}_y$ for all $y\}$ where $r_y = -v_y(D)$, $D = \sum -r_y[y]$. The reason why we use the bold font for these (geometric) adelic objects is that the Bbb-notation is employed for analytic adelic objects $\mathbb{A}, \mathbb{B}$ on relative surfaces in [F4].

Denote

$$\mathbf{B} = \prod K_y \cap \mathbf{A}_S, \mathbf{C} = \prod K_x \cap \mathbf{A}_S, \mathbf{OA} = \prod \mathcal{O}_{x,y} \cap \mathbf{A}, \mathbf{OB} = \prod \mathcal{O}_y \cap \mathbf{A}_S, \mathbf{OC} = \prod \mathcal{O}_x \cap \mathbf{A}_S.$$

In line with [P1] (the complex there is the intersection of the following one with the product of copies of K), [B] and [P2] (there is a misprint in the definition of A_{02} there), in dimension two the (geometric) adelic complex is

$$\mathcal{A}_S = \mathcal{A}_S(0) : \qquad K \oplus \mathbf{OB} \oplus \mathbf{OC} \to \mathbf{B} \oplus \mathbf{C} \oplus \mathbf{OA} \to \mathbf{A},$$

or in more compatible with the underlying symplectic structure on flags of scheme points of S and more convenient for computations notation

$$\mathcal{A}_S = \mathcal{A}_S(0) : \qquad A_0 \oplus A_1 \oplus A_2 \longrightarrow A_{01} \oplus A_{02} \oplus A_{12} \longrightarrow A_{012}$$

and the adelic complexes for 1-cycles D

$$\mathcal{A}_S(D) : \qquad A_0 \oplus A_1(D) \oplus A_2(D) \longrightarrow A_{01} \oplus A_{02} \oplus A_{12}(D) \longrightarrow A_{012}$$

where $A_*(D) = A_{0*} \cap \mathbf{A}_S(D)$. The maps are $(a_0, a_1, a_2) \mapsto (a_0 - a_1, a_2 - a_0, a_1 - a_2)$ and $(a_{01}, a_{02}, a_{12}) \mapsto a_{01} + a_{02} + a_{12}$.

For a more general approach to geometric adeles see [B], [Y].

In dimension two, the best topologies to work with are sequentially saturated topologies, [F1], [C]. However, for the purposes of this paper we do not need them, since a subgroup is open in the topology we consider on $\mathbf{A}$ if and only if it is open in its sequential saturation. As appropriate for this paper, we will have more open subgroups of $\mathbf{A}$ than in [F4, Ch.1].

Define a translation invariant topology on $\mathbf{A}_y^r \simeq t_y^r \mathbb{A}_{k(y)}[[t_y]]$ which is isomorphic, looking at the coefficients of powers of t_y, to $\prod_{i=r}^{\infty} \mathbb{A}_{k(y)}$, as the product topology on the latter, where the topology on the one-dimensional adeles is the usual one. This topology does not depend on any of the related choices in positive characteristic. Moreover, in the general case it does not depend on the choices provided the ring embeddings and local parameters agree with the geometric structure. This topology coincides with the translation invariant topology in which $t_y^r(\prod_{x \in y} \prod_{z \in y(x)} W_{x,z} + t_y^m \mathbf{A}_y)$, $m \geqslant 1$, form an open base of 0, where $W_{x,z}$ are open subgroups of $\mathcal{O}_{x,z}$ and almost every $W_{x,z}$ is equal to $\sum_{i=0}^{i=m-1} t_y^i O(E_{x,z}) + t_y^m \mathcal{O}_{x,z}$, $O(E_{x,z})$ = the residue image of $O_{x,z}$, i.e. the ring of integers of $E_{x,z}$. Define a translation invariant topology on $\mathbf{A}_y$ as the inductive limit of the topologies on $\mathbf{A}_y^r$.

As usual, the topology of the restricted product $\prod' G_i$ of topological groups G_i with respect to their closed subgroups (H_i) is the translation invariant topology in which an open base of the identity element is formed by subsets $\prod_{i \in J} V_i \times \prod_{i \notin J} H_i$ of G_J, where J runs through finite subsets of I and V_i are in an open base of the identity element in G_i. Thus we get a translation invariant topology on $\mathbf{A}$ in which it is not a locally compact group. Each of $\mathbb{F}$-subspaces of $\mathcal{A}_S(D)$, including $A_*, A_*(D)$, is endowed with the induced topology. It is easy to see that the induced topology on $\mathbb{F}$ is discrete.

2. The group of characters of a topological group is a topological group with respect to the corresponding well known (compact to open) topology. Recall that every two-dimensional local field F is (non-canonically) self-dual, its groups of characters is $\{\beta \mapsto \chi_0(\alpha\beta) : \alpha \in F\}$ where χ_0 is a fixed nontrivial character, see e.g. [F2, Lemma 3].

There are two general constructions which extend the class of self-dual topological groups occurring in geometry. If the additive group of a ring R is endowed with a translation invariant topology with respect to which it is self-dual, then the additive group of the formal power series ring $R((t))$, endowed with the inductive limit topology of the direct product topology on $t^i R[[t]] \simeq \prod R$, is self-dual. If topological groups G_i are self-dual, then for a certain choice of their closed subgroups H_i satisfying natural conditions, the restricted product of G_i with respect to H_i is self-dual. In particular, the additive group of $\mathbf{A}_y$ is (non-canonically) self-dual, and as the next proposition shows, the additive group of $\mathbf{A}$ is self-dual.

Similarly to [A, Ch.XIII §4] we can call an $\mathbb{F}$-linear continuous homomorphism from $\mathbf{A}$ to $\mathbb{F}$ which vanishes on $A_{01} + A_{02}$ a differential map (this notion is related to a more general notion of locally differential operator in [Y, Def 3.1.8]). The following proposition includes a description of differential maps, each of which is of the form

$$d_\omega \colon \mathbf{A} \longrightarrow \mathbb{F}, \qquad (\alpha_{x,z}) \mapsto \sum_{x \in y} \sum_{z \in y(x)} \mathrm{Tr}_{x,z}\ \mathrm{res}_{x,z}(\alpha_{x,z}\omega)$$

where $\omega \in \Omega^2_{K/\mathbb{F}}$, y runs through all curves on S, x runs through all closed points of y, and $\mathrm{res}_{x,z} \colon K_{x,z} \to k_z(x)$ is the two-dimensional local residue which is the composite of the residue to the first residue field $E_{x,z}$ and then the residue to the second residue field $k_z(x)$, $\mathrm{Tr}_{x,z} \colon k_z(x) \to k(x)$ is the trace to the residue field $k(x)$ of $\mathcal{O}_x$. See [P1], [P2], [Y], [M1], [M2] for more detail about d_ω.

Let ω be a non-zero form. Denote by $\mathfrak{C}$ the divisor of ω, its class in Pic(S) is uniquely determined. The composite of the multiplication $\mathbf{A} \times \mathbf{A} \longrightarrow \mathbf{A}$ and d_ω gives the pairing

$$\mathbf{A} \times \mathbf{A} \longrightarrow \mathbf{A} \longrightarrow \mathbb{F}, \qquad (\alpha, \beta) \mapsto d_\omega(\alpha\beta).$$

As usual, for a closed subgroup B of $\mathbf{A}$ denote by $B^\perp = \{\gamma \in \mathbf{A} : d_\omega(B\gamma) = 0\}$ the subgroup which complements B with respect to d_ω.

There are two two-dimensional properties generalizing the familiar one-dimensional property $\mathrm{k}^\perp = \mathrm{k}$: $A_{01}^\perp = A_{01}$, $A_{02}^\perp = A_{02}$, see [P1,§2 Prop. 1] for repartitions argument which uses a strong approximation property in the second case and reduces to the one-dimensional case in the first case. The adelic argument runs similarly, replacing an analog $f_{m,H} \in K_{m,H}$ of $f_{,m,H}$ of [P1, p. 710] by a sufficiently close element of K, and using an analog of [P1, §1 Prop. 5].

Proposition.

(1) This pairing is symmetric, continuous and non-degenerate.

(2) Every character of the additive group of $\mathbf{A}$ has its image in $\mathbb{F}_p$ and is equal to $\beta \mapsto \mathrm{Tr}_{\mathbb{F}/\mathbb{F}_p} d_\omega(\alpha\beta)$ for an appropriate adele $\alpha \in \mathbf{A}$. The additive group of $\mathbf{A}$ is (non-canonically) self-dual. For every closed subgroup B of $\mathbf{A}$ we have $(B^\perp)^\perp = B$ and B is isomorphic to the group of continuous $\mathbb{F}$-linear maps from $\mathbf{A}/B^\perp$ to $\mathbb{F}$.

(3) We have $A_{12}(D)^\perp = A_{12}(\mathfrak{C} - D)$, $A_{01}^\perp = A_{01}$, $A_{02}^\perp = A_{02}$.

(4) $A_i = A_{ij} \cap A_{ik}$ for every $0 \leqslant i \leqslant 2$, where i, j, k is a permutation of $0, 1, 2$ and we set $A_{ij} = A_{ji}$.

(5) Each of A_*, $A_*(D)$ and any of their sums is closed in $\mathbf{A}$.

(6) $A_0^\perp = A_{01} + A_{02}$, $A_1(D)^\perp = A_{01} + A_{12}(\mathfrak{C} - D)$, $A_2(D)^\perp = A_{01} + A_{12}(\mathfrak{C} - D)$.

(7) Differential maps on $\mathbf{A}$ form a 1-dimensional space $\{d_\omega : \omega \in \Omega^2_{K/\mathbb{F}}\}$ over K.

Proof. Continuity and non-degenerate property follow immediately from the definitions. To construct α in (2), restrict the character on y, find an appropriate α_y and then show that $\alpha = (\alpha_y)$ does the job.

(3) the first property follows from the definitions. For the other two properties see the paragraph preceding the statement of this propostion.

(4) The property for $i = 1, 2$ follows from the definitions. To prove the property for $i = 0$ note that elements $(\alpha_{x,y})$ of A_{01} do not depend on x, and elements of A_{02} do not depend on y, so elements of the intersection $A_{01} \cap A_{02}$ consist of $(a)_{x,y}$ with $a \in K_y$ for all y and $a \in K_x$ for all x. Now take an affine subscheme R of S with the ring of regular function $k[R]$ and use the property that it coincides with the intersection (in an appropriately large ring) of all of its completions of its localizations with respect to its prime ideals of height 1.

(5) follows from the definitions that if $| * | = 2$ then A_*, $A_*(D)$ are closed. For two closed subgroups B, C their sum $B + C$ is the complement of $B^\perp \cap C^\perp$, hence closed.

(6) follows from the previous and $(B \cap C)^\perp = B^\perp + C^\perp$.

(7) using (6), the action of K on differential maps d is $k * d : \alpha \mapsto d(k\alpha)$. The property follows from (2) and (6).

REMARK. Parts (3), (4), (6) of this proposition were stated in [OP3, Prop.2] without proof.

3. Let $H^i(\mathcal{A}_S(D))$ be the cohomology groups of the complex $\mathcal{A}_S(D)$. We have

$$H^0(\mathcal{A}_S(D)) = A_0 \cap A_{12}(D), \quad H^2(\mathcal{A}_S(D)) = A_{012}/(A_{12}(D) + A_{01} + A_{02}),$$
$$H^1(\mathcal{A}_S(D)) = (A_{12}(D) \cap (A_{01} + A_{02}))/(A_1(D) + A_2(D)).$$

Using the previous proposition it is easy to see that there are natural maps from the complex $\mathcal{A}_S(D)$ to the following complexes, each of which is quasi-isomorphic to $\mathcal{A}_S(D)$

$$A_0 \longrightarrow A_{01}/A_1(D) \longrightarrow A_{012}/(A_{12}(D) + A_{02}),$$

$$A_2(D) \longrightarrow A_{12}(D)/A_1(D) \longrightarrow A_{012}/(A_{01} + A_{02}),$$

$$A_2(D) + A_0 \longrightarrow A_{02} \longrightarrow A_{012}/(A_{12}(D) + A_{01}).$$

In particular, there are natural isomorphisms

$$H^1(\mathcal{A}_S(D)) \simeq (A_{02} \cap (A_{12}(D) + A_{01}))/(A_2(D) + A_0) \simeq (A_{01} \cap (A_{12}(D) + A_{02}))/(A_1(D) + A_0).$$

For two topological spaces X, Y over $\mathbb{F}$ denote by $\mathrm{Hom}^c_{\mathbb{F}}(X, Y)$ the $\mathbb{F}$-space of continuous $\mathbb{F}$-linear maps endowed with the corresponding topology.

PROPOSITION. *(1) There are isomorphisms*

$$\mathrm{Hom}^c_{\mathbb{F}}(H^0(\mathcal{A}_S(D)), \mathbb{F}) \simeq H^2(\mathcal{A}_S(\mathfrak{C} - D)),$$
$$\mathrm{Hom}^c_{\mathbb{F}}(H^1(\mathcal{A}_S(D)), \mathbb{F}) \simeq H^1(\mathcal{A}_S(\mathfrak{C} - D)),$$
$$\mathrm{Hom}^c_{\mathbb{F}}(H^2(\mathcal{A}_S(D)), \mathbb{F}) \simeq H^0(\mathcal{A}_S(\mathfrak{C} - D)).$$

(2) Each $\dim_{\mathbb{F}} H^i(\mathcal{A}_S(D))$ *is finite and an invariant of the class of* D *in* $\mathrm{Pic}(S)$*.*

(3) Denote by $\chi_{\mathbf{A}}(D) = \sum (-1)^i \dim_{\mathbb{F}} |H^i(\mathcal{A}_S(D))|$ *the Euler characteristic of* $\mathcal{A}_S(D)$*. Then* $\chi_{\mathbf{A}}(D) = \chi_{\mathbf{A}}(\mathfrak{C} - D)$*.*

Proof. By the previous proposition $H^0(\mathcal{A}_S(D))^\perp = (A_0 \cap A_{12}(D))^\perp = A_{01} + A_{02} + A_{12}(\mathfrak{C} - D)$, so the space of continuous $\mathbb{F}$-linear maps from $H^0(\mathcal{A}_S(D))$ to $\mathbb{F}$ is isomorphic to $A_{012}/H^0(\mathcal{A}_S(D))^\perp = H^2(\mathcal{A}_S(\mathfrak{C} - D))$ and the space of continuous $\mathbb{F}$-linear maps from $H^2(\mathcal{A}_S(D))$to $\mathbb{F}$ is isomorphic to $H^0(\mathcal{A}_S(\mathfrak{C} - D))$.

The space of continuous $\mathbb{F}$-linear maps from $H^1(\mathcal{A}_S(D)) \simeq (A_{01}\cap(A_{12}(D)+A_{02}))/(A_1(D)+A_0)$ to $\mathbb{F}$ is isomorphic to $(A_1(D)^\perp \cap A_0^\perp)/(A_{01}^\perp + (A_{12}(D)^\perp \cap A_{02}^\perp)) = (A_{01} + A_{12}(\mathfrak{C} - D)) \cap (A_{01} + A_{02}))/(A_{01}+A_{02}\cap A_{12}(\mathfrak{C}-D))$ which is equal to $(A_{01}+A_{12}(\mathfrak{C}-D))\cap(A_{01}+A_{02}))/(A_{01}+A_2(\mathfrak{C}-D))$ by the previous proposition. We have a natural map from $(A_{12}(\mathfrak{C}-D)\cap(A_{01}+A_{02}))/(A_1(\mathfrak{C}-D)+A_2(\mathfrak{C}-D))$ to the latter and it is easy to see it is an isomorphism. Thus, the space of continuous $\mathbb{F}$-linear maps from $H^1(\mathcal{A}_S(D))$ to $\mathbb{F}$ is isomorphic to $H^1(\mathcal{A}_S(\mathfrak{C}-D))$.

$H^0(\mathcal{A}_S(0)) = K\cap\mathbf{OA} = \mathbb{F}$. For every divisor D and curve y the quotient $(K\cap A_{12}(D+D_y))/(K+A_{12}(D))$ is finite, since $k(y)\cap\mathbb{A}_{k(y)}(d)$ is finite for every divisor d on y. Hence $H^0(\mathcal{A}_S(D))$ is finite. Due to the first isomorphism so is $H^2(\mathcal{A}_S(D))$.

$H^1(\mathcal{A}_S(D))$ is isomorphic to a closed subspace of a quotient of $A_{12}(D)/A_1(D)$ which is isomorphic to the direct product of compact $\mathbb{A}_{k(y)}/k(y)$ and hence compact, so $H^1(\mathcal{A}_S(D))$ is compact. Since $A_{12}(D)/A_1(D)$ is compact and $H^2(\mathcal{A}_S(D))$ is finite, $A_{012}/(A_{01} + A_{02})$ is compact. It is isomorphic to the space of continuous $\mathbb{F}$-linear maps from A_0 to $\mathbb{F}$, hence A_0 is discrete. Next, $H^1(\mathcal{A}_S(D))$ is isomorphic to $(A_{02} \cap (A_{12}(D) + A_{01}))/(A_2(D) + A_0)$ which is a subquotient of $A_{02}/A_2(D)$, homeomorphic to $A_0/(A_0 \cap A_2(D))$, hence it is discrete. Therefore, $H^1(\mathcal{A}_S(D))$ is finite.

To prove (2), use $A_*(\mathrm{Div}(f)+D)+A_{0*} = A_*(D)-f+A_{0*} = A_*(D)+A_{0*}$, $|A_0\cap A_{12}(\mathrm{Div}(f)+D)| = |A_0 \cap A_{12}(D)|$ for any $f \in K^\times$.

Remarks. 1. The discreteness of K as a topological subspace of $\mathbf{A}_S$ reproduces the well known similar fact in dimension one: the discreteness of a global field k as a topological subspace of the one-dimensional adeles $\mathbb{A}_\mathrm{k}$. Unlike the compactness of the quotient $\mathbb{A}_\mathrm{k}/\mathrm{k}$, the quotient $\mathbf{A}/K$ is not compact but the quotient $\mathbf{A}_S/K^\perp = \mathbf{A}_S/(A_{01} + A_{02})$ is.

2. Finiteness of the cohomologies of the adelic complex has not been given a direct adelic proof in any of the previous papers, instead an isomorphism of H^i with the cohomologies of $\mathcal{O}_S(D)$ was used and then their finiteness, as established in algebraic geometry, was applied.

4. In view of theorems of the cube, it is natural to define for two divisors E, D on S

$$[E, D] := \chi_{\mathbf{A}}(0) - \chi_{\mathbf{A}}(-D) - \chi_{\mathbf{A}}(-E) + \chi_{\mathbf{A}}(-D - E).$$

Proposition.

(1) The pairing $[\ ,\ ]\colon \mathrm{Div}(S) \times \mathrm{Div}(S) \to \mathbb{Z}$ *is a bilinear symmetric form.*

(2) It is invariant with respect to translation by principal divisors.

(3) If D is a divisor whose support does not contain a smooth curve y, then $[D_y, D]$ is equal to $\deg_y(D|_y)$. In particular, if y, z are smooth curves on S with transversal intersection then $[D_y, D_z] = \sum_{x\in y\cap z} |k(x) : \mathbb{F}|$.

(4) The pairing $[\ ,\ ]$ *coincides with the intersection pairing (E, D), in particular,*

$$(E, D) = \chi_{\mathbf{A}}(0) - \chi_{\mathbf{A}}(D) - \chi_{\mathbf{A}}(E) + \chi_{\mathbf{A}}(D + E).$$

Proof. The second property follows from invariance of $\chi_{\mathbf{A}}$ with respect to translation by principal divisors.

To show the third property we use a relation of $\mathcal{A}_S(D)$ and the one-dimensional adelic complex $\mathcal{A}_{k(y)}$ of the curve y on S. Namely, for a smooth curve y and a divisor D whose support does not contain y, we use the following complex $\mathbb{A}_{k(y)}(D|_y) \longrightarrow \mathbb{A}_{k(y)}/k(y)$ as a one-dimensional adelic complex $\mathcal{A}_y(D|_y)$ quasi-isomorphic to the complex $k(y) \oplus \mathbb{A}_{k(y)}(D|_y) \longrightarrow \mathbb{A}_{k(y)}$. We have a natural commutative diagram

$$\begin{array}{ccccc} A_2(D) & \longrightarrow & A_{12}(D)/A_1(D) & \longrightarrow & A_{012}/(A_{01}+A_{02}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{A}_{k(y)}(D|_y) & \longrightarrow & \mathbb{A}_{k(y)}/k(y) & \longrightarrow & 0 \end{array}$$

The two vertical maps are defined using $A_2(D) \longrightarrow A_{12}(D) \longrightarrow \mathbb{A}_{k(y)}(D|_y)$ and $A_{12}(D) \longrightarrow A_{012} \longrightarrow \mathbb{A}_{k(y)}$, the latter is defined using the residue map from $K_{x,z}$ to its first residue field $E_{x,z}$. The kernels of the vertical maps form a complex $\mathcal{A}_S(D-D_y)$

$$A_2(D-D_y) \longrightarrow A_{12}(D-D_y)/A_1(D-D_y) \longrightarrow A_{012}/(A_{01}+A_{02}).$$

Thus, we have an exact sequence of complexes

$$0 \longrightarrow \mathcal{A}_S(D-D_y) \longrightarrow \mathcal{A}_S(D) \longrightarrow \mathcal{A}_y(D|_y) \longrightarrow 0.$$

Hence $[D_y, D] = \chi(\mathcal{A}_y(0)) - \chi(\mathcal{A}_y(-D|_y))$. The latter is equal to virtual dimension $\dim_{\mathbb{F}}(\mathbb{A}_{k(y)}(0) : \mathbb{A}_{k(y)}(-D|_y))$, i.e. to $\deg_y(D|_y)$. In particular, we obtain $[D_y, D_z] = \sum_{x\in y\cap z} |k(x) : \mathbb{F}|$ if y and z are smooth curves intersecting transversally. The third property is proved.

For divisors E_1, E_2, the moving lemma allows to find linearly equivalent divisors E_1', E_2' such that their support does not contain y. Therefore, by the previous material $[D_y, E_1 + E_2] = [D_y, E_1' + E_2'] = [D_y, E_1'] + [D_y, E_2'] = [D_y, E_1] + [D_y, E_2]$. For three divisors D_1, D_2, D_3, the difference $[D_1 + D_2, D_3] - [D_1, D_3] - [D_2, D_3]$ is symmetric in D_i. Since it is 0 when $D_3 = D_y$, $[D_1, D_3] = [D_1 + D_2, D_3] - [D_2, D_3]$ if D_2 is the divisor of a smooth curve, for arbitrary D_1 and D_3. Now represent a divisor D as $D_{y_1} - D_{y_2}$ modulo a principal divisor, where y_i are smooth curves, see e.g. [Ha, Ch.V, p. 359]. We get $[D, E] = [D_{y_1}, E] - [D_{y_2}, E]$ and each of the terms on the right hand side is linear in E, hence so is $[D, E]$.

It is well known (e.g. [Ha, Ch.V, Thm 1.1] that the first two properties and the last sentence of the third property uniquely characterize the intersection pairing. Finally $[E, D] = [-E, -D]$.

Thus, we can compute the intersection number of two divisors entirely in terms of adelic objects associated to the adelic complexes for the divisors.

The formulas

$$(E, D) = \chi_{\mathbf{A}}(0) - \chi_{\mathbf{A}}(D) - \chi_{\mathbf{A}}(E) + \chi_{\mathbf{A}}(D+E), \qquad \chi_{\mathbf{A}}(D) = \chi_{\mathbf{A}}(\mathfrak{C} - D)$$

are the key formulas for surfaces.

As a corollary of the previous two key formulas, we deduce the *adelic Riemann–Roch theorem for S*:

$$(D, \mathfrak{C} - D) = \chi_{\mathbf{A}}(0) - \chi_{\mathbf{A}}(D) - \chi_{\mathbf{A}}(\mathfrak{C} - D) + \chi_{\mathbf{A}}(\mathfrak{C}) = 2(\chi_{\mathbf{A}}(0) - \chi_{\mathbf{A}}(D)).$$

Using the isomorphims $H^i(\mathcal{A}_S(D)) \simeq H^i(S, \mathcal{O}_S(D))$, see [P1, Thm1], [B], [Hu], [Y], we obtain the *Riemann–Roch theorem for 1-cycles on S*:

$$2\big(\chi(S, \mathcal{O}_S(0)) - \chi(S, \mathcal{O}_S(D))\big) = (D, \mathfrak{C} - D).$$

REMARKS. 1. The argument in this proof is immediately extendable to an adelic proof of the Riemann–Roch theorem for smooth projective and even irreducible projective surfaces over an arbitrary field F working with F-linear topology on adeles of the surface over F and using the notion of F-linear compactness, similar to the one-dimensional approaches in [I2, §4] (despite mentioning in the introduction that the proofs use Haar measures, they do not use it for the included proof of the Riemann–Roch theorem), [G].

2. The argument in this proof extends to the case of a quasi-coherent sheaf $\mathcal{F}$ on S and the associated adelic complex $\mathcal{A}_S(\mathcal{F})$, defined in [B1] and [Hu].

3. For another adelic description of the intersection pairing see [P2, §2]; there are also some adelic aspects of intersection theory in [HY].

4. Iwasawa emphasized the importance of the theta-formula as an analytic expression for the self-duality of the adeles, [I1,p.449]. Unlike the adelic proofs of the Riemann–Roch theorems for 1-cocyles on curves and surfaces over fields, which do not actually require more than adelic duality, the Riemann–Roch theorem for number fields and for 0-cycles on surfaces require such an analytic theory (at least we do not currently know proofs which do not use translation invariant measure and integration). See [F4,§3.6] for the theory of analytic adeles $\mathbb{A}$, measure, integration and harmonic analysis on associated objects, and a theta-formula on elliptic surfaces. The latter is closely related to the Riemann–Roch theorem for zero cycles on elliptic surfaces [F4,§56, Rk 3].

5. In the arithmetic case where $S \to \operatorname{Spec} O_{\mathrm{k}}$ is a regular proper scheme of relative dimension one, k a number field, the objects $\mathbf{A}, \mathbf{B}, \mathbf{C}$ were already defined in [F4,§28]. It is useful to find an analogue $\mathcal{A}_S$ of the adelic complex in the arithmetic case so that the Arakelov intersection index $[E, D]$ equals $\chi_{\mathbf{A}}(0) - \chi_{\mathbf{A}}(D) - \chi_{\mathbf{A}}(E) + \chi_{\mathbf{A}}(D+E)$. To achieve that it is natural to pursue the following approach: for a number field, reinterpret the Euler–Minkowski characteristic of a replete (Arakelov) divisor [N, Ch. III, §3] in terms of the Euler characteristic of the one-dimensional adelic complex. In particular, this gives an analog of the formula from section 0 in the number field case: $\deg d = \chi_{\mathcal{A}_{\mathrm{k}}}(d) - \chi_{\mathcal{A}_{\mathrm{k}}}(0)$ for a replete ideal d. Then lift to dimension two with a modified residue map from $\mathcal{A}_S$ to $\mathcal{A}_{\mathrm{k}}$, using in particular [M1] and [M2].

6. For another adelic Riemann–Roch theorem, whose proof uses K-delic structures, for certain finite group bundles on arithmetic surfaces flat over $\mathbb{Z}$, see [CPT].

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