On the spherical space form problem

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Dedicated to Professor Wu-Chung Hsiang on his sixtieth birthday.

ABSTRACT. We prove that the finite fundamental groups of closed, oriented three dimensional manifolds are just the finite groups which act freely and linearly on S^3.

1. INTRODUCTION

A well-known problem in three dimensional topology is to list all the finite groups which occur as the fundamental group of some closed 3-manifold. So far, all the known examples come from the finite subgroups $\Gamma \subset SO(4)$ which operate freely on the 3-sphere. The associated 3-manifolds S^3/Γ admit Riemannian metrics of constant positive curvature, and are known as the (orthogonal) spherical space forms. In this paper we prove that these examples exhibit *all* the finite fundamental groups of oriented 3-manifolds.

The classification of orthogonal spherical space forms up to isometry was first proposed by Killing in 1891, and the problem attracted the attention of famous mathematicians of the time, such as Clifford, Hopf, Klein, and Poincaré. According to H. Hopf's 1925 paper [17], the following is a list of all finite fixed-point free subgroups of $SO(4)$:

(1.1) The cyclic group $C(n)$, the generalized quaternion group $Q(4n)$, the binary tetrahedral group $T^*(24)$, the binary octahedral group $O^*(48)$, and the binary icosahedral group $I^*(120)$.

(1.2) The semidirect product $C(2n+1) \rtimes C(2^k)$ of an odd order cyclic group with a cyclic 2-group. More explicitly $C(2n+1) \rtimes C(2^k)$ is given by the presentation $\{A, B : A^{2^k} = B^{2n+1} = 1, ABA^{-1} = B^{-1}\}$ where $k \geq 2, n \geq 1$.

(1.3) The semidirect product $T^*(24) \rtimes C(3^k)$ of the binary tetrahedral group $T^*(24)$ with a cyclic 3-group. More explicitly, $T(24) \rtimes C(3^k)$ is given by the presentation $\{P, Q, X : P^2 = (PQ)^2 = Q^2, X^{3^k} = 1, XPX^{-1} = Q, XQX^{-1} = PQ\}$ where $k \geq 1$.

(1.4) The product of any of the above groups in (1.1)-(1.3) with a cyclic group of coprime order.

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At first glance, the above list may appear to be random. In the forties and fifties, efforts were made to interpret Hopf's list using group cohomology [4] and it was discovered that all these groups have periodic Tate cohomology of period four. In general, a finite group has periodic cohomology if and only if it satisfies the p^2-conditions ("any subgroup of order p^2 is cyclic") for all primes p. From the viewpoint of group theory, this condition means that the odd Sylow subgroup is cyclic and the 2-Sylow subgroup is cyclic or generalized quaternion. If the cohomology has period four then, in addition, the pq-conditions hold ("every subgroup of order pq is cyclic") for p and q distinct odd primes.

The necessity of the $2q$-conditions was established by J. Milnor [23] in 1957, when he showed that the dihedral group cannot operate freely on any $\mathbb{Z}/2$-homology sphere despite the fact that it has periodic cohomology of period 4. In this paper, Milnor also compiled the following list of all finite groups, not in Hopf's list (1.1)-(1.4), but satisfying the restrictions known at the time on fundamental groups of 3-manifolds.

(1.5) The semidirect product $Q(8m,k,l)$ of the odd cyclic group $C(kl)$ with the generalized quaternion group $Q(8n)$. More explicitly, $Q(8n,k,l)$ has the presentation: $\{X,Y,Z : X^2 = Y^{2n} = (XY)^2,\ Z^{kl} = 1,\ XZX^{-1} = Z^r,\ YZY = Z^{-1}\}$. Here n,k,l are all odd integers and relatively prime to each other, $n > k > l \geq 1$, and r satisfies $r \equiv -1 \pmod{k}$, $r \equiv 1 \pmod{l}$. If $l = 1$, we set $Q(8n,k) \equiv Q(n,k,1)$.

(1.6) The group $Q(8n,k,l)$ with the same presentation as (1.5), but with n even.

(1.7) The semidirect product $O(48;3^{k-1},l)$ of the odd order cyclic group $C(3^{k-1}l)$, $3 \nmid l$, with the binary octahedral group $O^*(48)$. More precisely, $O(48;3^{k-1},l)$ has five generators X,P,Q,R,A and the following relations:

$$X^{3^k} = P^4 = A^l = 1,\ P^2 = Q^2 = R^2,\ PQP^{-1} = Q^{-1}$$
$$XPX^{-1} = Q,\ XQX^{-1} = PQ,\ RXR^{-1} = X^{-1},\ RPR^{-1} = QP$$
$$RQR^{-1} = Q^{-1},\ AP = PA,\ AQ = QA,\ RAR^{-1} = A^{-1}.$$

(1.8) The product of any of the above groups in (1.5)-(1.7) with a cyclic group of coprime order.

Thus to establish our main result, it is enough to prove that groups in the above list (1.5)-(1.8) do not act freely on homotopy 3-spheres.

In the late sixties, C. T. C. Wall asked whether Milnor's result could be interpreted using the new theory of nonsimply connected surgery. Ronnie Lee [19] answered this question in 1973 by defining a "semicharacteristic" obstruction for the problem. As well as recovering the previous result of Milnor, the semicharacteristic rules out the family of groups $Q(8n,k,l)$, n even, in (1.6). Later in [35], C. B. Thomas observed that this also eliminates the family of groups $O(48,3^{k-1},l)$ in (1.7) because groups of this type always contain a subgroup isomorphic to $Q(16,3^{k-1},1)$. These results leave undecided only the groups $Q(8n,k,l)$, n odd, in (1.5) and their products with cyclic groups of coprime order in (1.8) from Milnor's original list.

In this this paper, we settle the remaining cases by proving the following:

Theorem A. *For p,q distinct odd primes, the group $Q(8p,q)$ does not operate freely on any homotopy 3-sphere.*

Remark. At the ICM in Zürich (August, 1994) J. H. Rubinstein [27] announced that $Q(8p, q)$ does not act freely on the standard 3-sphere S^3. The methods outlined seem completely different from those in the present work, but we have not seen the details.

Notice that a group $Q(8n, k, l)$ in the family (1.5) always contains a subgroup of the form $Q(8p, q)$. Hence Theorem A eliminates the family (1.7) in Milnor's list and also the corresponding products in (1.8). In other words, we establish the following:

Theorem B. *A necessary and sufficient condition for a finite group to be the fundamental group of a closed, oriented 3-manifold is that it belongs to Hopf's list (1.1)-(1.4).*

Remark: Orthogonal spherical space forms in dimension 3 were classified up to isometry by Seifert and Threfall (in 1930), and the higher dimensional cases by Wolf [39, Part III], completing the work of Vincent. The homeomorphism classification is not yet known, even assuming the Poincaré conjecture, although partial results have been obtained by J. H. Rubinstein and J. T. Pitts [26], [27] using 3-manifold techniques. The possible homotopy types for S^3/G were determined by C. B. Thomas [33], [34]. Another result in this direction was obtained by R. S. Hamilton using methods from differential geometry: if a 3-manifold admits a metric of positive Ricci curvature, then it is an orthogonal space form [16].

We will now outline some of the techniques involved in proving Theorem A. It is useful to start with the analogous spherical space form problem in higher dimensions: namely, the classification of finite group actions (G, Σ^{2n-1}) on homotopy spheres Σ^{2n-1} of dimension $2n-1$, $n \geq 3$. This problem was both a motivation and an important test case for the techniques of algebraic and geometric topology developed in the period 1960–1985. P. A. Smith had already shown in 1944 that the p^2 conditions were necessary for a G-action on any homology sphere Conversely, Swan [29] proved that every group with periodic cohomology acts freely and simplicially on a CW complex homotopy equivalent to a sphere, and asked whether there was always a *finite* simplicial action. Throughout the 1970's remarkable progress was made on the higher dimensional space form problem, culminating in the paper of Madsen, Thomas and Wall [21]. They used the surgery theory of Browder, Novikov, Sullivan and Wall to show that any finite group G satisfying the p^2 and $2p$ conditions (for all primes p) acts freely and smoothly on a homotopy sphere of *some* odd dimension $2n-1>3$. The precise dimensional bounds were not determined (although for G of period d, either $n = d$ or $n = 2d$ is realizable).

The next big step forward was the explicit calculation by Milgram [22] in 1979 of the finiteness obstruction for some of the period 4 groups $G = Q(8p, q)$, following the method of [37]. In particular, Milgram showed that some of these groups are not fundamental groups of 3-manifolds. After this followed a sequence of papers by Milgram (see the survey in [5]), and independently by Madsen [20], aiming at the calculation of the relevant surgery obstruction. Here the problem is to determine which of the groups $Q(8p, q)$ act freely on Σ^{8k+3}, for $k > 0$, since they act linearly on S^{8k+7} for all $k \geq 0$. It turned out that the answer is computable in principle, but depends sporadically on the number theory of the primes p, q.

Despite these spectacular breakthroughs in high dimensions, virtually no further progress was made using these methods on the space form problem in dimension 3. On the other hand, since the mid-eighties a new method for studying the smooth structure of a 4-manifold X has been developed by S. Donaldson and others [7], using the moduli space $\mathcal{M}(P)$ of Yang-Mills connections on an $SU(2)$-bundle P over X. Striking results (such as the existence of non-diffeomorphic, simply-connected, h-cobordant 4-manifolds) follow from studying the geometry and topology of this moduli space.

Our strategy for proving Theorem A comes in two parts. Assuming the existence of a nonlinear space form Σ/G with fundamental group $G = Q(8p, q)$, we will construct a G-equivariant $SU(2)$-bundle P over a certain smooth 4-manifold (G, Z). Then we will apply the theory of equivariant moduli spaces $(\mathcal{M}(P), G)$ developed in [13], [14] to derive a contradiction.

Most of the work in the first part of our argument is to construct a suitable 4-dimensional, framed, cobordism Y with a reference map $c\colon Y \to BG$. The boundary $\partial Y = \partial_0 Y \cup N$, where N is a connected 3-manifold and the composite $N \hookrightarrow Y \xrightarrow{c} BG$ is null-homotopic. The remaining boundary components $\partial_0 Y$ consist of two copies of Σ/G with opposite orientation, spherical space forms $S/Q(4pq)$, and a number of copies of "almost space forms" S'/H, for certain subgroups $H = Q(8p)$, $Q(8q)$, $C(2pq)$ of $Q(8p, q)$. By an almost space form S'/H we mean the quotient of an integral homology sphere S' by a free action of the group H. The induced homomorphism $c_{\#}$ on the fundamental groups $\pi_1(\partial Y) \to \pi_1(Y) \to G$ maps each $\pi_1(S'/H)$ onto $H \subset G$. Furthermore, we require an epimorphism $c_{\#}\colon \pi_1(Y) \to G$, with kernel normally generated by $\pi_1(N)$. For gauge theory, the key additional property is that $b^+(\widetilde{Y}) = 0$, or equivalently, that $H_2(Y; \mathbb{Z}G)$ contains no positive definite subspaces with respect to the intersection pairing.

The construction of the above cobordism Y occupies §§2-8 of this paper. Basically, we start with a framed cobordism $(U, \partial U) \to BG$ with boundary some appropriate collection of linear and nonlinear space forms $\pm\Sigma/G$, and S/H for $H = Q(4pq)$, $Q(8p)$, $Q(8q)$, or $C(2pq)$. By re-attaching the top dimensional cell, we can modify U to a 4-dimensional Poincaré complex V with $\partial V = \partial U$ such that the cup product pairing on $H^2(V, \partial V; \mathbb{Z}G)$ is negative definite. In this step, we use the description of $\mathbb{Z}[Q(8p, q)]$-hermitian forms by means of the "arithmetic square" [38]. Associated to $(V, \partial V)$, there is a surgery problem whose surgery obstruction group $L_4(\mathbb{Z}G)$ has been computed by Madsen [20]. Using this result, we describe in §§7-8 how to eliminate the surgery obstruction. We modify V to construct a new Poincaré complex W, together with a new surgery problem $X \to W$ where some of the boundary components are changed to almost spherical space forms $S'/Q(8p)$, $S''/Q(8q)$, or $S'''/C(2pq)$. The domain of the surgery problem is a compact, smooth, 4-manifold $(X, \partial X)$, such that $\partial X \to \partial W$ is an integral homology equivalence.

Since the surgery obstruction is zero, , the intersection pairing on $H_2(X; \mathbb{Z}G)$ is the orthogonal direct sum of the pairing on W and some free hyperbolic summands. In dimension four we may not be able to complete the smooth surgeries suggested by this algebraic data. Instead, to get rid of the excess hyperbolic summands we use the Disk Embedding Theorem of Freedman [9], [10] to represent these hyperbolic summands by topologically embedded copies of $S^2 \times S^2$ in the interior of X. Then we split open the manifold X along a suitable 3-manifold N. On one side of this splitting, we have the manifold Y with $\partial Y = \partial_0 Y \cup N$ where $\partial_0 Y = \partial X$. In addition, the intersection pairing $H_2(Y; \mathbb{Z}G)$ (modulo its null space) is negative definite as required.

In §§9, 10 of this paper, we consider the equivariant moduli space $(\mathcal{M}(P), G)$ over the manifold Z constructed from the G-covering $\tilde{Y}$ of Y by filling all the spherical boundary components $(Q(4pq), S^3)$ with linear 4-disks $(Q(4pq), D^4)$. Over (G, Z) there is an equivariant $SU(2)$-bundle P with equivariant trivialization along ∂Z such that the relative Chern number $c_2(P, \partial Z) = 1$. By using connections which have L^2-finite energy along the cylindrical ends ∂Z, as in [30], [3, §1], or [24, Ch. 7], we obtain a 5-dimensional moduli space $\mathcal{M}(P)$ with an action of G. Notice that the G-action on Z has singular points with isotropy subgroup $Q(4pq)$ located at the centres $x_i \in D$ of the attached 4-disks. From this we deduce that the induced G-action on $(G, \mathcal{M}(P))$ has a 1-dimensional singular subspace with $Q(4pq)$ as its isotropy subgroup. Geometrically, this subspace of $\mathrm{Fix}(\mathcal{M}(P), Q(4pq))$ represents a 1-parameter family of flows of ASD connections in Z, emitting from particle-like connections at the singular points. By the Uhlenbeck compactness theorem, this flow of instantons has to converge at one of the cylindrical ends of Z (see [8], [31]), which by symmetry reasoning must be the end associated to (G, Σ). Then we show that this process gives rise to a $U(2)$-representation of $Q(8p, q)$, which extends the inclusion of $Q(4pq)$ into $SU(2)$. Since this last statement contradicts the representation theory of $Q(4pq)$, we conclude that it is impossible to have a free $Q(8p, q)$-action on the homotopy sphere Σ.

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2. A Framed Cobordism

We will now start to change the 3-dimensional spherical space form problem into a 4-dimensional problem. We begin by assuming the existence of a free $Q(8p, q)$-action $(Q(8p, q), \Sigma)$ on a homotopy 3-sphere Σ where p and q are two distinct odd primes.

The group $Q(8p, q)$ has the following presentation:

$$Q(8p,q) = \left\langle A, B, X, Y \;\middle|\; \begin{matrix} A^p = B^q = 1, X^2 = Y^2 = (XY)^2, XAX^{-1} = A^{-1} \\ XBX^{-1} = B, YAY^{-1} = A, YBY^{-1} = B^{-1} \end{matrix} \right\rangle. \tag{2.1}$$

In other words, $Q(8p, q)$ is a semidirect product $C(pq) \rtimes Q(8)$ of the cyclic group $C(pq)$ with the quaternion group $Q(8)$. Here the characteristic homomorphism

$\varphi: Q(8) \to Aut(C(pq)) = \mathbb{Z}/p-1 \times \mathbb{Z}/q-1$ is given in the following table.

(2.2)

φ	$\mathbb{Z}/p-1$	$\mathbb{Z}/q-1$
X	-1	1
Y	1	-1
XY	-1	-1

From this description, we see the following three maximal subgroups:

$$Q(8p) = \langle X, Y, A\rangle,\ Q(8q) = \langle X, Y, B\rangle,\ Q(4pq) = \langle XY, A, B\rangle.$$

Moreover, by sending the elements X, Y, XY to appropriate quaternions in $\{i, j, k\}$, we see that $Q(8p), Q(8p), Q(4pq)$ respectively are isomorphic to the following subgroups of the unit quaternions S^3:

$$\begin{aligned} Q(8p) &\cong \langle \pm 1, \pm i, \pm j, \pm k, e^{2\pi i/p}\rangle \\ Q(8q) &\cong \langle \pm 1, \pm i, \pm j, \pm k, e^{2\pi i/q}\rangle \\ Q(4pq) &\cong \langle \pm k, e^{2\pi i/pq}\rangle. \end{aligned}$$

In particular, there exist free linear actions $(Q(8p), S^3), (Q(8q), S^3), (Q(4pq), S^3)$ on the 3-sphere S^3 and hence spherical space forms $S/Q(8p), S/Q(8q), S/Q(4pq)$.

For our application, we also need the maximal cyclic subgroup $C(2pq)$ generated by the elements $A, B, (XY)^2$. By identifying $C(2pq)$ with the cyclic subgroup $\langle \pm e^{2\pi i/pq}\rangle$ in $SU(2)$, we obtain the free linear action $(C(2pq), S^3)$ on S^3 which has the lens space $L(2pq, 1) = S^3/C(2pq)$ as quotient space.

Proposition 2.3. *Assume the existence of a nonlinear space form $\Sigma/Q(8p,q)$. Then there exists a framed, compact, 4-manifold U with the following properties:*

(i) $\pi_1(U) = Q(8p, q)$.

(ii) *The boundary ∂U of U consists of two copies of $\Sigma/Q(8p,q)$ with opposite orientation,* **a** *copies of $S/Q(4pq)$,* **b** *copies of $S/Q(8p)$,* **c** *copies of $S/Q(8p)$, and* **d** *copies of $S/C(2pq)$ where* **a, b, c, d** *are all non-zero and divisible by 48.*

(iii) *The induced homomorphism $\pi_1(\partial U) \to \pi_1(U)$ on the fundamental groups sends $\pi_1(\Sigma/Q(8p,q))$ or $\pi_1(S/H)$ for $H = Q(4pq)$, $Q(8p)$, $Q(8q)$, $C(2pq)$ to the corresponding subgroups $Q(8p,q)$ or $H \subset Q(8p,q)$.*

Proof. As is well-known, the tangent bundle of an oriented 3-manifold is trivial and hence can be provided with a framing. In particular, we can choose a framed manifold structure for each of the linear and nonlinear space forms: $\Sigma/Q(8p,q)$, $S/Q(4pq)$, $S/Q(8p)$, $S/Q(8q)$, $S/C(2pq)$. As a result, we can view the expression for ∂U in terms of these space forms as the following relation in the framed bordism group $\Omega_3^{fr}(BQ(8p,q))$:

$$a[S/Q(4pq)] + b[S/Q(8p)] + c[S/Q(8q)] + d[S/C(2pq)] = 0 \tag{2.4}$$

since the terms $[\Sigma/Q(8p,q)] - [\Sigma/Q(8p,q)]$ cancel out. If we can find a solution of (2.4) by nonzero integers a, b, c, d with $a \equiv b \equiv c \equiv d \equiv 0 \pmod{48}$, then it follows that there exists a framed 4-manifold U' satisfying:

(iv) $\partial U' = \Sigma/Q(8p,q) \cup -\Sigma/Q(8p,q) \cup \mathbf{a}S/Q(4pq) \cup \mathbf{b}S/Q(8p) \cup \mathbf{c}S/Q(8q) \cup \mathbf{d}S/C(2pq)$

(v) the classifying map $c\colon U' \to BQ(8p,q)$ restricted to $\partial U'$ gives the corresponding classifying map on each of the boundary components.

Note that $c_{\#}\colon \pi_1 U' \to Q(8p,q)$ is a surjection. By framed surgery, we can kill the kernel of $c_{\#}$ and obtain a framed 4-manifold U satisfying (2.3) (i)-(iii).

To solve (2.4), we compute $\Omega_3^{fr}(BG)$ using the spectral sequence with E_2 term given by

$$E_{i,j}^2 = H_i(G; \Omega_j^{fr}).$$

The coefficient groups are $\Omega_i^{fr} = \mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/24$ for $i = 0,1,2,3$ respectively.

We first study the image of our relation under the Hurewicz map

$$\Omega_3^{fr}(BQ(8p,q)) \to H_3(Q(8p,q);\mathbb{Z}). \tag{2.5}$$

Since $H_3(Q(8p,q);\mathbb{Z})$ equals $\mathbb{Z}/|Q(8p,q)| = \mathbb{Z}/8pq$, we have a congruence equation in $\mathbb{Z}/8pq$. In fact, by considering $\Sigma/Q(8p,q)$ as the 3-skeleton of the classifying space $BQ(8p,q)$, we can deform the classifying maps for $\Sigma/Q(8p,q)$,$S/Q(8p)$,$S/Q(8q)$, $S/Q(4pq)$, and $S/C(2pq)$ to factor through $\Sigma/Q(8p,q)$:

$$\begin{aligned} f_a\colon S/Q(4pq) &\to \Sigma/Q(8p,q)\\ f_b\colon S/Q(8p) &\to \Sigma/Q(8p,q)\\ f_c\colon S/Q(8q) &\to \Sigma/Q(8p,q)\\ f_d\colon S/C(2pq) &\to \Sigma/Q(8p,q). \end{aligned}$$

Then the contribution of $[S/Q(4pq)]$, $[S/Q(8p)]$, $[S/Q(8q)]$, $[S/C(2pq)]$ to the factor $H_3(Q(8p,q);\mathbb{Z})$ amounts to counting the degrees of the mappings $\deg f_a$, $\deg f_b$, $\deg f_c$, and $\deg f_d$ modulo $8pq$.

From the theory of covering spaces, the maps f_b and f_c factor through the coverings $\Sigma/Q(8p) \to \Sigma/Q(8p,q)$, $\Sigma/Q(8q) \to \Sigma/Q(8p,q)$.

$$\begin{aligned} f_b\colon S/Q(8p) &\xrightarrow{f_b'} \Sigma/Q(8p) \xrightarrow{\pi_p} \Sigma/Q(8p,q)\\ f_c\colon S/Q(8q) &\xrightarrow{f_c'} \Sigma/Q(8q) \xrightarrow{\pi_q} \Sigma/Q(8p,q) \end{aligned}$$

Hence we have

$$\begin{aligned} \deg f_b &= \deg f_b' \cdot \deg \pi_p = p \deg f_b'\\ \deg f_c &= \deg f_c' \cdot \deg \pi_q = q \deg f_c'. \end{aligned}$$

On the other hand, $\deg f_b'$ and $\deg f_c'$ can be taken to be units (mod $8pq$) [29]. Since $(p,q) = 1$, there exist integers r and s such that $1 = rq \deg f_b' + sp \deg f_c'$. From this last equation it follows that given nonzero numbers a', d' there exist non-zero integers b' and c' such that the expression

$$a'[S/Q(4pq)] + b'[S/Q(8p)] + c'[S/Q(8q)] + d'[S/C(2pq)] = 0 \tag{2.6}$$

and so gives no contribution in $H_3(Q(8p,q))$.

The $E^2_{i,3-i}$ terms of the spectral sequence for $i = 1, 2$ are given by:

$$H_2(Q(8p,q); \Omega_1^{fr}) = H_2(Q(8p,q); \mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2,$$
$$H_1(Q(8p,q); \Omega_2^{fr}) = H_1(Q(8p,q); \mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2,$$

and there is a splitting $\Omega_3^{fr}(BG) = \widetilde{\Omega}_3^{fr}(BG) \oplus \Omega_3^{fr}$. Since $\Omega_3^{fr} = \mathbb{Z}/24$ and the first summand is annihilated by 16, we obtain a solution of the bordism equation (2.4) from (2.6) after multiplying the coefficients by 48. This completes the proof. □

The framed bordism U constructed above represents only the first step of the transition from dimension 3 to 4. To apply our equivariant gauge theory, we would like for instance to have $\mathbb{Z}[Q(8p,q)]$-hermitian intersection pairing

$$h\colon H_2(U; \mathbb{Z}[Q(8p,q)]) \times H_2(U; \mathbb{Z}[Q(8p,q)]) \to \mathbb{Z}[Q(8p,q)]$$

negative definite. To approach this condition, we will modify U in a number of steps. This will be carried out in the next four sections.

3. A Poincaré complex

Let $(U, \partial U)$ be a 4-dimensional, framed, cobordism satisfying Diagram 2.4 (i)-(iii). Let $G = Q(8p,q)$ and let

$$b\colon H^2(U, \partial U; \mathbb{Z}G) \times H^2(U, \partial U; \mathbb{Z}G) \to \mathbb{Z}$$

denote the non-singular symmetric bilinear form induced by cup product and evaluation against the fundamental class. Notice that b is a G-invariant form: $b(gx, gy) = b(x, y)$ for all $g \in G$ and all $x, y \in H^2(U, \partial U; \mathbb{Z}G)$.

In this section we show how to modify U by removing a cell e^4 in the interior of U and then re-attaching this cell e^4 by a map $f\colon \partial e^4 \to U - e^4$. The result is a CW complex

$$V = (U - e^4) \cup_f e^4$$

which contains ∂U as a subcomplex, denoted by ∂V.

Variation of the attaching map of the top cell does not change the 3-skeleton, and hence has no effect on the fundamental group and homology in dimensions ≤ 2. By Poincaré duality,

$$H^2(U, \partial U; \mathbb{Z}G) \cong H^2(V, \partial V; \mathbb{Z}G)$$

so we can identify these two groups.

The main result of this section is:

Proposition 3.1. *Let* $b': H^2(U, \partial U; \mathbb{Z}G) \times H^2(U, \partial U; \mathbb{Z}G) \to \mathbb{Z}$ *be a non-singular, G-invariant, symmetric bilinear form, with* $b' \equiv b \pmod{|G|}$. *Then there exists an attaching map f such that the pair* $(V, \partial V)$ *is an oriented, finite, 4-dimensional Poincaré pair with* $\pi_1(V) = G$ *and cup product form* b'.

We will first give a description of $H_2(U; \mathbb{Z}G)$ as a $\mathbb{Z}G$-module. Note that the framed cobordism U is not uniquely determined by (2.4) (i)-(iii). We can, for example, alter the cobordism U by taking the connected sum with copies of $S^2 \times S^2$ away from ∂U. This has the effect of changing $H_2(U; \mathbb{Z}G)$ by taking a sum with a free $\mathbb{Z}G$-module of even rank, and we will refer to this as "stabilization" of the cobordism U.

Let $(\widetilde{U}, \partial\widetilde{U})$ be the universal covering space of $(U, \partial U)$. On $\widetilde{U}$, there is a free action of $Q(8p, q)$ and hence an induced action on its homology $H_2(\widetilde{U})$. By definition, the $\mathbb{Z}G$-module structure on $H_2(\widetilde{U})$ is the same as $H_2(U; \mathbb{Z}G)$.

Note that $\partial\widetilde{U}$ consists of a collection of homotopy 3-spheres. For each of these 3-spheres, we form a cone and extend the G-action to the cone in an obvious manner. In this way, we obtain a 4-dimensional Poincaré complex $\widetilde{U}'$,

$$\widetilde{U}' = \widetilde{U} \cup \text{ (cones over boundary spheres)}$$

where the action of G is no longer free. In fact, for each of the cone points a_λ, we have an isotropy subgroup $G_\lambda \subseteq G$. The cone points, denoted by a_0, a_1, over the components (G, Σ), $(G, -\Sigma)$ are somewhat special because they are G-fixed points.

The above construction of $\widetilde{U}'$ can be compared with the following. Let $\Sigma \times I$ denote the product of Σ with the interval $I = [0,1]$. Then on the two boundary components $\Sigma \times 0$, $\Sigma \times 1$, we can attach two cones to get the suspension $S^1 \wedge \Sigma$ of Σ. The action of G on $\Sigma \times I$ can be extended naturally to $S^1 \wedge \Sigma$ with the upper and lower cone points as fixed points. From equivariant obstruction theory, there exists a degree 1, G-equivariant map

$$\varphi: \widetilde{U}' \to S^1 \wedge \Sigma$$

which sends the free orbits to free orbits, a_0 to the lower cone point and all other a_λ to the upper cone point.

Let $K_*(\varphi)$ denote the kernel of the natural homorphism

$$K_*(\varphi) = Ker\{\varphi_*: H_*(\widetilde{U}') \to H_*(S^1 \wedge \Sigma)\}.$$

Then from the degree 1 property of φ there is an exact sequence

$$0 \to K_*(\varphi) \to H_*(\widetilde{U}') \to H_*(S^1 \wedge \Sigma) \to 0$$

of $\mathbb{Z}G$-modules. From this sequence it is easy to see that $K_*(\varphi) = 0$ for all but the middle homology $K_2(\varphi)$. Since adding points or deleting points does not affect the seond homology, we have

$$K_2(\varphi) = H_2(\widetilde{U}') = H_2(\widetilde{U}).$$

Thus we can shift the calculation of the homology $H_2(U;\mathbb{Z}G)$ to $K_2(\varphi)$ which has the advantage of being the only nonzero homology group of the relative chain complex $C_*(\varphi)$.

The relative chain complex

$$C_*(\varphi) = Ker\{\varphi_*: C_*(\widetilde{U}') \to C_*(S^1 \wedge \Sigma)\}$$

can be calculated by taking equivariant triangulations on $\widetilde{U}'$ and $S^1 \wedge \Sigma$ and cellular maps between them. Since the cone points can be taken to be the vertices and the action are free away form these points, we see that $C_*(\varphi)$ consists of finitely generated free $\mathbb{Z}G$-modules for $* \neq 0$ and

$$C_0(\varphi) = F \oplus \bigoplus_{\lambda \neq 0,1} \mathrm{Ind}_{G_\lambda}^G(\mathbb{Z})$$

for some finitely generated free $\mathbb{Z}G$-module F. Here $\mathrm{Ind}_{G_\lambda}^G(\mathbb{Z}) = \mathbb{Z} \otimes_{\mathbb{Z}G_\lambda} \mathbb{Z}G = \mathbb{Z}[G/G_\lambda]$ stands for the induced representation from the trivial G_λ-representation $\mathbb{Z}$ to G, and the indices in the sum go through all the cone points a_λ except for the G-fixed points a_0, a_1.

Proposition 3.2. *After stabilization, there is an isomorphism:*

$$H_2(U;\mathbb{Z}G) \;\cong\; (\mathbb{Z}G)^r \oplus \bigoplus_{\lambda \neq 0,1} \Omega^2 Ind_{G_\lambda}^G(\mathbb{Z}).$$

Here we use notation $\Omega^2 L$ to denote the first term in an exact sequence:

$$0 \to \Omega^2 L \to F_2 \to F_1 \to L \to 0$$

of finitely generated $\mathbb{Z}G$-modules with F_1, F_2 free over $\mathbb{Z}G$. Since tensoring with $\mathbb{Z}G$ over $\mathbb{Z}G_\lambda$ preserves exactness, we have a stable isomorphism

$$\Omega^2 \mathrm{Ind}_{G_\lambda}^G(\mathbb{Z}) \cong \mathrm{Ind}_{G_\lambda}^G(\Omega^2 \mathbb{Z}).$$

A standard argument in homological algebra proves that the Ω-construction is well-defined up to stabilizing by free $\mathbb{Z}G$-modules.

Corollary 3.3. *After stabilization, the rank of $H_2(U;\mathbb{Q}G)$ is divisible by 16 at each simple factor of $\mathbb{Q}G$.*

Proof of (3.3). After replacing U by a connected sum with copies of $S^2 \times S^2$ if necessary, we may assume that the $r \equiv 0 \pmod{16}$ in the given expression for $H_2(U;\mathbb{Z}G)$. Since the number of boundary components is divisible by 16, the Ω^2-summands also have ranks $\equiv 0 \pmod{16}$. □

Proof of (3.2). We have an exact sequence of $\mathbb{Z}G$-modules

$$0 \to Z_2(\varphi) \to C_2(\varphi) \to C_1(\varphi) \to F \oplus \bigoplus_{\lambda \neq 0,1} \mathrm{Ind}_{G_\lambda}^G \mathbb{Z} \to 0 \tag{3.4}$$

so it follows that

$$\mathcal{Z}_2(\varphi) \oplus (\mathbb{Z}G)^{\ell'} \cong (\mathbb{Z}G)^{\ell} \oplus \bigoplus_{\lambda\neq 0,1} \Omega^2 \mathrm{Ind}_{G_\lambda}^{G}(\mathbb{Z}).$$

On the other hand, $C_*(\varphi)$ with fundamental class $[\widetilde{U}']$ can be viewed as a PD chain complex. Using the same argument as in [20, p. 199], since $K_i(\varphi) = H_i(C_*(\varphi)) = 0$ for $i \geq 3$ we can contract this complex down to a complex $C'_*(\varphi)$ concentrated in dimensions $* \leq 2$ without changing the homology. Then

$$K_2(\varphi) \oplus (\mathbb{Z}G)^{r'} \cong (\mathbb{Z}G)^{r} \oplus \bigoplus_{\lambda\neq 0,1} \Omega^2 \mathrm{Ind}_{G_\lambda}^{G}(\mathbb{Z}).$$

Since $K_2(\varphi) = H_2(U;\mathbb{Z}G)$, this proves (3.2). □

Proof of Proposition 3.1. We must see how the symmetric bilinear form b' leads to a suitable choice for the re-attaching map f. First we note that the conditions

$$H_3(V;\mathbb{Z}G) = H^3(V,\partial V;\mathbb{Z}G)$$
$$H_4(V;\partial V;\mathbb{Z}G) = \mathbb{Z}$$

and the non-singularity of the cup-product form are necessary for $(V,\partial V)$ to be a Poincaré complex.

Re-attaching maps may constructed as follows. First we map ∂e^4 to a wedge of two 3-spheres $\partial e^4 \to S^3 \vee S^3$ by collapsing the boundary of a 3-cell in $S^3 = \partial e^4$ to a point. Then we map $S^3 \vee S^3$ by sending the copy $S^3 \vee *$ by the inclusion map $\gamma\colon \partial e^4 \to U - e^4$ and sending the copy $* \vee S^3$ by a map $\delta\colon S^3 \to U^{(2)}$ from S^3 to the 2-skeleton $U^{(2)} = (U-e^4)^{(2)} \subseteq (U-e^4)$. In other words, f is the composite mapping

$$f\colon \partial e^4 \to S^3 \vee S^3 \xrightarrow{\gamma\vee\delta} U - e^4.$$

The choice $\delta = 0$ just gives the original complex $(U,\partial U)$.

Since $H_3(U^{(2)};\mathbb{Z}G) = 0$, it follows that δ has no effect on homology and, so as far as homology is concerned, f is the same as the original attaching map. As a result, for any such map f the complex $(V,\partial V)$ is a finite Poincaré pair provided that the cup-product form is non-singular.

Variation of the map δ has an effect on the cap product by the fundamental class $[V,\partial V]$ which in turn changes the cup product pairing $b\colon H^2(U,\partial U;\mathbb{Z}G) \times H^2(U,\partial U;\mathbb{Z}G) \to \mathbb{Z}$. From the exact sequence in (3.4) we have

$$H_2(U^{(2)};\mathbb{Z}G) = \mathcal{Z}_2(\varphi).$$

Comparing with the expression for $H_2(U;\mathbb{Z}G) \cong H^2(U,\partial U;\mathbb{Z}G)$ obtained in (3.2), we obtain

$$H_2(U^{(2)};\mathbb{Z}G) = F \oplus H_2(U;\mathbb{Z}G),$$

where F is a free $\mathbb{Z}G$-module given by the image of the boundary operator from the complex $C_*(\varphi)$, $\partial\colon C_3(\varphi) \to C_2(\varphi)$. Note that

$$\pi_2(U^{(2)}) = H_2(U^{(2)};\mathbb{Z}G) = F \oplus H_2(U;\mathbb{Z}G),$$

and by a theorem of Whitehead $\pi_3(U^{(2)})$ is just the space of symmetric pairings on $\mathrm{Hom}_{\mathbb{Z}}(\pi_2(U), \mathbb{Z})$. In particular, we can interpret δ as a symmetric pairing on $F \oplus H^2(U, \partial U; \mathbb{Z}G)$.

For any such pairing, the original cup product form

$$b: H^2(U, \partial U; \mathbb{Z}G) \times H^2(U, \partial U; \mathbb{Z}G) \to \mathbb{Z}$$

is changed by re-attaching the 4-cell to

$$(b + \sum g^*\delta): H^2(V, \partial V; \mathbb{Z}G) \times H^2(V, \partial V; \mathbb{Z}G) \to \mathbb{Z}$$

(see [36, pp. 240-241], [12,§1]). Here $g^*\delta$ is the translate of the symmetric pairing δ by the action of the group element $g \in G$, $g^*\delta(x, y) = \delta(gx, gy)$, and $\sum g^*\delta$ is the sum of these translates as we go through all the group elements in G. Given b' in the statement of Proposition 3.1, we need to find δ so that $b' - b = \sum g^*\delta$.

Let H denote the $\mathbb{Z}G$-module $H^2(U, \partial U; \mathbb{Z}G)$, and $\mathrm{Sym}(H)$ the space of symmetric pairings on H. Then $b' - b$ is an element in $\mathrm{Sym}(H)$ which is invariant under the induced group action. However, the quotient of the group of G-invariant pairings, by those of the form $\sum g^*\delta$ is just the Tate cohomology $\widehat{H}^0(G; \mathrm{Sym}(H))$, which is a torsion group of exponent $8pq = |G|$. But $b': H \times H \to \mathbb{Z}$ on H has the additional property that $b' \equiv b \pmod{|G|}$. Therefore we can write $b' = b + \sum g^*\delta$ for some symmetric pairing δ. We then use the associated map $f = \gamma \vee \delta$, to construct a Poincaré complex $(V, \partial V)$ with b' as its cup product pairing. □

4. Hermitian modules

In this section we will consider the patching construction for $\mathbb{Z}[Q(8p, q)]$-hermitian modules by means of the arithmetic square:

$$\begin{array}{ccc} \mathbb{Z}G & \longrightarrow & \mathbb{Q}G \\ \downarrow & & \downarrow \\ \hat{\mathbb{Z}}G & \longrightarrow & \hat{\mathbb{Q}}G \end{array} \tag{4.1}$$

Here $\hat{\mathbb{Z}}G$ is the product $\prod_\ell \hat{\mathbb{Z}}_\ell G$ of the ℓ-adic group rings and $\hat{\mathbb{Q}}G$ the corresponding weak product of group algebras. Applying the homology functor $H_*(U; -)$ to the above diagram, we have

$$\ldots \to H_*(U; \mathbb{Z}G) \to H_*(U; \hat{\mathbb{Z}}G) \oplus H_*(U; \mathbb{Q}G) \to H_*(U; \hat{\mathbb{Q}}G) \to \ldots \tag{4.2}$$

To simplify our notation, we denote by $H(\mathbb{Z}G), H(\mathbb{Q}G), H(\hat{\mathbb{Z}}G), H(\hat{\mathbb{Q}}G)$ the degree 2 homology of U with the corresponding coefficients in $\mathbb{Z}G$, $\mathbb{Q}G$, $\hat{\mathbb{Z}}G$, or $\hat{\mathbb{Q}}G$. In particular, we can view the module $H(\mathbb{Z}G)$ as patching $H(\hat{\mathbb{Z}}G) = H(\mathbb{Z}G) \otimes \hat{\mathbb{Z}}$ and $H(\mathbb{Q}G) = H(\mathbb{Z}G) \otimes \mathbb{Q}$ together over $H(\hat{\mathbb{Q}}G) = H(\mathbb{Z}G) \otimes \hat{\mathbb{Q}}$, with some isomorphisms

$$H(\hat{\mathbb{Z}}G) \otimes \mathbb{Q} \to H(\hat{\mathbb{Q}}G) \leftarrow H(\mathbb{Q}G) \otimes \hat{\mathbb{Z}}. \tag{4.3}$$

In the same manner, we can describe the $\mathbb{Z}G$-hermitian intersection pairing

$$h: H(U;\mathbb{Z}G) \times H(U;\mathbb{Z}G) \to \mathbb{Z}G$$

as a pull-back. There are intersection pairings over $H(\hat{\mathbb{Z}}G), H(\hat{\mathbb{Q}}G), H(\mathbb{Q}G)$ by the pull-back

$$\begin{aligned} h_{\hat{\mathbf{Z}}}&:H(\hat{\mathbb{Z}}G) \times H(\hat{\mathbb{Z}}G) \to \hat{\mathbb{Z}}G \\ h_{\hat{\mathbf{Q}}}&:H(\hat{\mathbb{Q}}G) \times H(\hat{\mathbb{Q}}G) \to \hat{\mathbb{Q}}G \\ h_{\mathbf{Q}}&:H(\mathbb{Q}G) \times H(\mathbb{Q}G) \to \mathbb{Q}G \end{aligned}$$

and they are patched together by isometries

$$(4.4) \qquad (H(\hat{\mathbb{Z}}G), h_{\hat{\mathbf{Z}}}) \otimes \mathbb{Q} \xrightarrow{\psi} (H(\hat{\mathbb{Q}}G), h_{\hat{\mathbf{Q}}}) \xleftarrow{\phi} (H(\mathbb{Q}G), h_{\mathbf{Q}}) \otimes \hat{\mathbb{Z}}.$$

We want to use this description later in §5, Proposition 5.1 to construct a new interection pairing on the same module $H(\mathbb{Z}G)$. Our strategy is to keep the pairing and isometry

$$(H(\hat{\mathbb{Z}}G), h_{\hat{\mathbf{Z}}}) \otimes \mathbb{Q} \xrightarrow{\psi} (H(\hat{\mathbb{Q}}G), h_{\hat{\mathbf{Q}}})$$

on the left of (4.4) unchanged, vary the pairing $(H(\mathbb{Q}G), h_{\mathbf{Q}})$ to a negative definite one $(H(\mathbb{Q}G), h'_{\mathbf{Q}})$, and then use local classification theory to patch everything together by a new isometry ϕ'

$$(4.5) \qquad (H(\hat{\mathbb{Z}}G), h_{\hat{\mathbf{Z}}}) \otimes \mathbb{Q} \xrightarrow{\psi} (H(\hat{\mathbb{Q}}G), h_{\hat{\mathbf{Q}}}) \xleftarrow{\phi'} (H(\mathbb{Q}G, h'_{\mathbf{Q}}) \otimes \hat{\mathbb{Z}}.$$

The new pairing $(H(\mathbb{Z}G), h')$ on $H(\mathbb{Z}G)$ is obtained by means of the pull back diagram as in [38] or [20].

The first step involves only the rational intersection form.

Proposition 4.6. *Let $(H(\mathbb{Q}G), h_{\mathbb{Q}})$ be a non-singular form with*

(a) *hyperbolic rank ≥ 8,*
(b) *rank $H(\mathbb{Q}G) \equiv 0 \pmod{16}$, and*
(c) *sign $h_{\mathbf{Q}} \equiv 0 \pmod{16}$ at every simple factor of $\mathbb{Q}G$.*

Then there exists a hermitian pairing $(H(\mathbb{Q}G), h'_{\mathbf{Q}})$ such that

(i) *$h'_{\mathbb{Q}}$ is negative definite at all of the real representations of $\mathbb{Q}G$,*
(ii) *$(H(\mathbb{Q}G), h'_{\mathbf{Q}}) \otimes \hat{\mathbb{Z}} \cong (H(\mathbb{Q}G), h_{\mathbf{Q}}) \otimes \hat{\mathbb{Z}}$ over $\hat{\mathbb{Q}}G$,*
(iii) *$\det h'_{\mathbf{Q}} = \det h_{\mathbf{Q}}$, and*
(iv) *$(H(\mathbb{Q}G), h'_{\mathbf{Q}})$ contains $\langle -1 \rangle$ as an orthogonal summand.*

The proof of Proposition 4.6 follows from well-known techniques in quadratic forms (see [28, Ch. 10] for the existence of global forms with prescribed local invariants). First, we recall that $S = \mathbb{Q}[Q(8p, q)]$ is a semi-simple algebra and hence can be decomposed into a product $\prod_{\chi}(\mathbb{Q}G)_{\chi}$ of simple algebras $(\mathbb{Q}G)_{\chi}$ where χ goes through all the irreducibles of G. Since

$$\mathbb{Q}[C(pq)] = \mathbb{Q} \times \mathbb{Q}(\zeta_p) \times \mathbb{Q}(\zeta_q) \times \mathbb{Q}(\zeta_{pq})$$

it follows that $S = \prod_{d|pq} S(d)$ where

$$S(d) = \mathbb{Q}(\zeta_d)^t[X, Y \mid X^2 = Y^2 = (XY)^2]$$

is a twisted group algebra. From the presentation of $G = Q(8p,q)$ given in (2.1) we see that the element $X^2 = (XY)^2 = Y^2$ is central of order two, so the group algebra $S = \mathbb{Q}[Q(8p,q)]$ contains the central idempotent $\frac{1}{2}(1+X^2)$ and splits into a product of two simple algebras $S = S_+ \times S_-$. The first factor $S_+ = \mathbb{Q}[D(2p) \times D(2q)]$ is the group algebra of the product of the two dihedral groups subgroups $D(2p) = \langle A, X\rangle$ and $D(2q) = \langle B, Y\rangle$. From the representation theory of these groups, it follows that

$$\begin{aligned} \mathbb{Q}[D(2p)] &= \mathbb{Q}_+ \times \mathbb{Q}_- \times M_2[\mathbb{Q}(\zeta_p + \zeta_p^{-1})] \\ \mathbb{Q}[D(2q)] &= \mathbb{Q}_+ \times \mathbb{Q}_- \times M_2[\mathbb{Q}(\zeta_q + \zeta_q^{-1})] \end{aligned} . \tag{4.7}$$

Therefore

$$S(1)_+ = \mathbb{Q}_{++} \times \mathbb{Q}_{+-} \times \mathbb{Q}_{-+} \times \mathbb{Q}_{--}$$

while

$$\begin{aligned} S(p)_+ &= M_2[\mathbb{Q}(\zeta_p + \zeta_p^{-1})] \otimes \mathbb{Q}_{++} \times M_2[\mathbb{Q}(\zeta_p + \zeta_p^{-1})] \otimes \mathbb{Q}_{+-} \\ S(q)_+ &= M_2[\mathbb{Q}(\zeta_q + \zeta_q^{-1})] \otimes \mathbb{Q}_{++} \times M_2[\mathbb{Q}(\zeta_q + \zeta_q^{-1})] \otimes \mathbb{Q}_{-+} \end{aligned}$$

and

$$S(pq)_+ = M_4[\mathbb{Q}(\zeta_p + \zeta_p^{-1}, \zeta_q + \zeta_q^{-1})]. \tag{4.8}$$

The subscripts $+, -$, indicate the appropriate sign representations of $Q(8p,q)$ and $(\zeta_p, \zeta_q$ are respectively primitive p^{th} roots and q^{th} roots of unity (see [20, p. 211]). There is a similar decomposition for the second factor S_- into simple algebras which are *non-split* at all the real places:

$$\begin{aligned} S(1)_- &= \mathbb{Q}[i,j,k], & S(p)_- &= \mathbb{Q}(\zeta_p)^t[i,j,k] \\ S(q)_- &= \mathbb{Q}(\zeta_q)^t[i,j,k] & S(pq)_- &= M_2(\mathbb{Q}(\zeta_{pq} + \zeta_{pq}^{-1})^t[i,j,k]) \end{aligned} . \tag{4.9}$$

It is easy to see that all the factors in the above decomposition are preserved under the canonical involution $\alpha\colon \overline{\sum a_g g} \mapsto \sum a_g g^{-1}$ of the group algebra $\mathbb{Q}G$. As an algebra with involution, all the factors in S_+ belong to the type $OK(\mathbb{R})$ while the factors in S_- belong to the type $SpD(\mathbb{H})$. Here we use the classification of [15, p. 549]. A simple algebra (D, α) of dimension n^2 over its centre E has type O (resp. Sp) if E is fixed by α and the fixed set of α on D has dimension $\frac{1}{2}(n^2+n)$ (resp. $\frac{1}{2}(n^2-n)$) over E. We further divide into

(i) type $OK(\mathbb{R})$ if (D,α) has type O, $D = E$ and E has a real imbedding, or
(ii) type $SpD(\mathbb{H})$ if (D,α) has type Sp, $D \neq E$, and D is nonsplit at infinite primes.

We wish to reconstruct the pairing on $(H(\mathbb{Q}G), h_{\mathbf{Q}})$ so that it becomes negative definite. In view of the decomposition above, it is enough to construct a negative definite pairing over each of the simple factors $(H(\mathbb{Q}G)_\chi, h'_\chi)$ with the prescribed local data $(H(\hat{\mathbb{Q}}G)_\chi, h_\chi)$.

For simple factors of type OK we will use the Hasse-Minkowski Theorem. Its proof can be found in many textbooks on quadratic forms (e.g. [28, p. 225]).

Theorem 4.10. *Let E be a global field. For each prime spot ℓ of E let an n-dimensional form ψ_ℓ over E_ℓ be given. Then there exists a form ϕ over E with $\phi_\ell \cong \psi_\ell$ for all ℓ if and only if the following conditions are satisfied:*

(i) *There exists $d \in E^\times$ with $d = \det(\psi_\ell)$ in $E_\ell^\times / E_\ell^{\times 2}$ for all ℓ.*
(ii) *The number of ℓ for which $s(\psi_\ell) = -1$ is finite and even.*

For the remaining simple factors, of type $SpD(\mathbb{H})$, we have the following version of the local to global correspondence:

Theorem 4.11. *Let D be an quaternion skew field with centre E, and let $(D, *)$ be the canonical involution which fixes exactly the elements of E. Given a $(D,*)$-hermitian form $h\colon V \times V \to D$ over the vector space V, the formula $x \longmapsto h(x,x)$ defines a quadratic form known as the trace form $q_h\colon V \to K$ of h*

(i) *Two hermitian forms over $(D,*)$ are isometric if and only if their trace forms are isometric.*
(ii) *If E is a ℓ-adic field, then non-degenerate hermitian forms over D are classified by their dimension.*
(iii) *If E is an algebraic number field then non-degenerate hermitian forms over D are classified by their dimension and their signatures at the real places where D is definite.*

Proof. The proof of (i) is in [28, Thm 10.1.7] and [28, 10.1.8(iii)]. Recall that the canonical involution on $\mathbb{Q}[i,j,k]$ is the one which is type Sp (see [28, p.75]). As is well known, a nondegenerate quadratic form q over an algebraic number field E is completely determined by its rank, $\dim(q)$, determinant $\det(q)$, Hasse symbols $s(q)$, and signatures $\operatorname{sign}(q_\ell)$ at all real places. For $h = \langle \alpha_1, \dots, \alpha_n \rangle$, its trace form q_h is of the form

$$q_h = \oplus\langle \alpha_i, -\alpha_i a, -\alpha_i b, \alpha_i ab \rangle$$

where a, b are elements in E with $D = (a,b)$. From this it is easy to see that

$$\dim(q_h) = 4\dim h, \quad \det(q_h) = 1, \quad \operatorname{sign}(q_h) = 4\operatorname{sign}(h).$$

These invariants are determined by the dimension and signature, and a short computation shows that

$$s_\ell(q_h) = \left(\frac{a,b}{\ell}\right)^n \left(\frac{-1,(-1)^n}{\ell}\right)$$

so the Hasse invariants are also determined . □

Proof of Proposition 4.6. We will begin with the type OK factors $(\mathbb{Q}G)_\chi$ and explain the method by working out the simplest case. Let χ be the trivial representation and $(\mathbb{Q}G)_\chi = \mathbb{Q}_{++} = \mathbb{Q}$. Since the involution is trivial, the hermitian pairing $(H(\mathbb{Q}_{++}), h_{\mathbf{Q}_{++}}) = (H(\mathbb{Q}), b)$ is nothing but a non-singular symmetric bilinear form over the rational vector space $H(\mathbb{Q})$.

We will construct a new bilinear form $(H(\mathbb{Q}), b')$ with the same localizations $(H(\mathbb{Q}_\ell), b_\ell)$, $\ell = 2, 3, \dots, \infty$ as the given form $(H(\mathbb{Q}), b)$ Over the real place, the form $(H(\mathbb{Q}_\infty), b_\infty) = (H(\mathbb{R}), b_{\mathbf{R}})$ is not necessarily negative definite but its rank and signature are multiples of 16 by Corollary 3.3. As a result, we see that

$$s(b_{\mathbf{R}}) = (-1)^{s(s-1)/2} = 1$$

since $s \equiv 0 \pmod 8$ is the number of negative in a diagonal form equivalent to $b_{\mathbf{R}}$. It follows that

$$\det(b_{\mathbf{R}}) = 1 \quad \text{in} \quad \mathbb{R}^{\times}/\mathbb{R}^{\times 2}.$$

If we replace $b_{\mathbf{R}}$ by a negative definite form $b'_{\mathbf{R}}$, then the same equations are satisfied:

$$\det(b'_{\mathbf{R}}) = \det(b_{\mathbf{R}}), \quad s(b'_{\mathbf{R}}) = s(b_{\mathbf{R}}).$$

For the rest of the primes, we let b'_ℓ equal b_ℓ. Then the collection $\{b'_2, \dots, b'_\infty\}$ with $d = \det b$ satisfies the conditions of Theorem 4.10to be the local data for a global form. It follows that we have a bilinear form $(H(\mathbb{Q}), b')$ which is negative definite at the real place and is the same as $(H(\mathbb{Q}_\ell), b_\ell)$ for all other primes.

For the other simple factors $(\mathbb{Q}G)_\chi$ of type OK , the modification of the hermitian pairing $(H(\mathbb{Q}G_\chi), h'_\chi)$ into a negative definite one can be achieved in the same manner, after applying Morita equivalence to translate from forms over $M_2(E)$ to forms over E.

Next we consider the case of simple factor of type $SpD(\mathbb{H})$, and we begin again with the simplest case when $(\mathbb{Q}G)_\chi$ is a division ring. To reconstruct $(H(\mathbb{Q}G_\chi), h)$ we first express h as a diagonal form $\langle a_1, \dots, a_n\rangle$ over the division ring $D = \mathbb{Q}G_\chi$ and define $h' = \langle -1, \dots, -1\rangle$ where h' has the same rank as h. By Theorem 4.11(ii), the forms $h_\ell \cong h'_\ell$ at all finite primes ℓ. On the other hand, h' is negative definite at the real places.

For a general type $SpD(\mathbb{H})$-factors, we have a matrix ring $M_2(D_\chi)$ over a division algebra $(D_\chi, *)$ with an involution defined by the transpose-conjugation operation:

$$(a_{ij}) \longmapsto (a^*_{ji}).$$

By Morita equivalence, the classification of hermitian forms over such simple factors can be reduced to the classification over D_χ. As a result the reconstruction problem of $(H(\mathbb{Q}G_\chi), h_\chi)$ can be treated as the corresponding problem over D_χ, which we have just considered. We complete the proof of parts (i)-(iii) by putting all the modified hermitian forms $(H(\mathbb{Q}G_\chi), h'_\chi)$ together. For part (iv), we use the assumption that form $(H(\mathbb{Q}G), h)$ contains a hyperbolic form of rank ≥ 8, and a special case of the above construction: let $L = (\mathbb{Q}G)^{16}$, take b the hyperbolic form, and b' the diagonal $\langle -1\rangle$ form of rank 16. Then $(L, b) \otimes \hat{\mathbb{Z}} \cong (L, b') \otimes \hat{\mathbb{Z}}$. □

5. Strong Approximation

In Proposition 4.6, we constructed a negative definite hermitian form $(H(\mathbb{Q}G), h'_{\mathbf{Q}})$ such that its completion $(H(\mathbb{Q}G), h'_{\mathbf{Q}}) \otimes \hat{\mathbb{Z}}$ is isometric to the adelic completion $(H(\hat{\mathbb{Q}}G), h_{\hat{\mathbf{Q}}})$ of the original hermitian form. In particular, this implies $\det h'_{\mathbf{Q}} = \det h_{\mathbf{Q}} \in K_1(\mathbb{Q}G)$. Each choice of isometry

$$(H(\mathbb{Q}G, h'_{\mathbf{Q}}) \otimes \hat{\mathbb{Z}} \xrightarrow{\phi'} (H(\hat{\mathbb{Q}}G), h_{\hat{\mathbf{Q}}})$$

gives rise to a form (H', h') on some module over $\mathbb{Z}G$ by pull-back, but there are many possible choices.

Proposition 5.1. *Let $H = H^2(U;\mathbb{Z}G)$ and h denote the $\mathbb{Z}G$-hermitian cup product pairing $h: H \times H \to \mathbb{Z}G$. Then there exists an isometry $\phi': (H(\mathbb{Q}G, h'_{\mathbb{Q}}) \otimes \hat{\mathbb{Z}} \to (H(\hat{\mathbb{Q}}G), h_{\hat{\mathbb{Q}}})$ and a hermitian pairing $h': H \times H \to \mathbb{Z}G$ such that*

(i) *h' is the pull-back $(h_{\hat{\mathbb{Z}}}, \phi', h'_{\mathbb{Q}})$*
(ii) *$h' \equiv h \pmod{|G|}$, and*
(iii) *h' is negative definite at all of the real representations of $\mathbb{Z}G$.*

When the form satisfies the conditions of Proposition 5.1, we can use this data as explained in Section 3 to construct a finite Poincaré pair $(V, \partial V)$ with negative definite intersection form.

Proposition 5.2. *There exists an attaching map f for $V = (U - e^4) \cup_f e^4$ such that the pair $(V, \partial V)$ is an oriented, finite, weakly simple, 4-dimensional Poincaré pair with $\pi_1(V) = G$ and orientation class $[V] \in H_4(V, \partial V; \mathbb{Z}G)$ Moreover the non-singular $\mathbb{Z}G$-hermitian pairing*

$$H^2(V, \partial V; \mathbb{Z}G) \times H^2(V, \partial V; \mathbb{Z}G) \to \mathbb{Z}G$$

induced by cup product and the evaluation against the fundamental cycle $[V]$ is negative definite.

The condition "weakly simple" means that the Whitehead torsion of the Poincaré duality map is zero measured in $Wh'(\mathbb{Z}G) \cong \operatorname{Im}(Wh(\mathbb{Z}G) \to Wh(\hat{\mathbb{Q}}G))$. This is automatically true for manifolds and we will preserve this property in our construction of V from U using (4.6)(iv).

Over each simple factor of $\mathbb{Q}G$ or $\hat{\mathbb{Q}}G$, every module is a direct sum of copies of an irreducible simple module, so we can choose a basis (see [20, §2]), and then compute the determinant of an isometry. Over non-commutative factors, the determinant must be interpreted as the reduced norm. An isometry of based forms with determinant 1 is called a *simple* isometry, and such forms are then called SU-equivalent.

The manifold $(U, \partial U)$ has a basis for its chain complex given by its associated piecewise smooth triangulation. To express the Whitehead torsion of its simple Poincaré duality map in terms of Reidemeister torsions, it is necessary to base the homology groups. Let $\underline{b} = \{e_i\}$ denote a basis of $H(\hat{\mathbb{Z}}G) \otimes \mathbb{Q}$. Using the given isomorphism

$$\Phi: H(\hat{\mathbb{Z}}G) \otimes \mathbb{Q} \xrightarrow{\psi} H(\hat{\mathbb{Q}}G) \xrightarrow{\phi^{-1}} H(\mathbb{Q}G) \otimes \hat{\mathbb{Z}}$$

we have a corresponding basis $\Phi(\underline{b}) = \{\Phi(e_i)\}$ on $H(\mathbb{Q}G) \otimes \hat{\mathbb{Z}})$ under Φ. In particular, Φ is a simple isometry of the given hermitian forms with respect to these bases.

Lemma 5.3. *There exists an isometry*

$$\phi': (H(\mathbb{Q}G), h'_{\mathbb{Q}}) \otimes \hat{\mathbb{Z}} \xrightarrow{\approx} (H(\hat{\mathbb{Q}}G), h_{\hat{\mathbb{Q}}})$$

such that the composite

$$\Phi': H(\hat{\mathbb{Z}}G) \otimes \mathbb{Q} \xrightarrow{\psi} H(\hat{\mathbb{Q}}G) \xrightarrow{(\phi')^{-1}} H(\mathbb{Q}G) \otimes \hat{\mathbb{Z}}$$

is a simple isometry with respect to the bases $\underline{b}$ *and* $\Phi(\underline{b})$.

Proof. It follows from Proposition 4.6 (iv) that $(H(\mathbb{Q}G), h'_{\mathbf{Q}})$ contains the form $\langle -1 \rangle$ on some basis element $e \in H(\mathbb{Q}G)$, in the given basis. This allows us to pre-compose any ϕ' with an isometry of the form $e \mapsto ue$, where $u \in \hat{\mathbb{Q}}G$ and $u\bar{u} = 1$. This realizes all possible values of the reduced norm for an isometry since $\det h_{\mathbf{Q}} = \det h'_{\mathbf{Q}}$. □

Proof of Proposition 5.2. Our new form $(H(\mathbb{Z}G), h')$ is constructed in Proposition 5.1by pull-back using the simple isometry ϕ' in Lemma 5.3. We then apply Proposition 3.1 to construct V from U. It follows that the based chain complex used to compute the adelic Reidemeister torsion of $(V, \partial V)$ is simple chain homotopy equivalent to the one for $(U, \partial U)$. Therefore the image of the Whitehead torsion $\tau(V, \partial V)$ is zero in $Wh(\hat{\mathbb{Q}}G)$ and the Poincaré complex $(V, \partial V)$ is weakly simple. □

To prove Proposition 5.1 we will need the following:

Lemma 5.4. *There exist isomorphisms*

$$\psi_1: H(\hat{\mathbb{Z}}G) \to H(\hat{\mathbb{Z}}G)$$
$$\psi_2: H(\mathbb{Q}G) \to H(\mathbb{Q}G)$$

such that $\Phi = (\psi_2 \otimes id)^{-1} \circ \Phi' \circ (\psi_1 \otimes id)$.

Lemma 5.5. *For every divisor* ℓ *of* $|G|$, *the reduction of* ψ_1 *modulo* ℓ

$$\bar{\psi}_1: H(\hat{\mathbb{Z}}G) \otimes \mathbb{Z}/\ell \to H(\hat{\mathbb{Z}}G) \otimes \mathbb{Z}/\ell$$

is an isometry of the hermitian module $(H(\hat{\mathbb{Z}}G), h_{\hat{\mathbf{Z}}}) \otimes \mathbb{Z}/\ell$.

Proof of Proposition 5.1. Assuming these two assertions (5.4) and (5.5), we can complete the proof of Proposition 5.1. Let (H', h') be the pull-back of our original ℓ-adic form $(H(\mathbb{Z}G), h) \otimes \hat{\mathbb{Z}} = (H(\hat{\mathbb{Z}}G), h_{\hat{\mathbf{Z}}})$ and the new rational form $(H(\mathbb{Q}G), h'_{\mathbf{Q}})$ given by Proposition 4.6, pulled back using the isometry ϕ' of Lemma 5.3. This form will satisfy (5.1)(i) and (5.1)(iii) once we prove that $H' \cong H(\mathbb{Z}G)$ as a $\mathbb{Z}G$-module. The remaining property (5.1)(ii) will follow from Lemma 5.5.

Recall from (4.3) that the module $H(\mathbb{Z}G)$ is obtained by forming the pull-back of the diagram:

$$H(\hat{\mathbb{Z}}G) \to H(\hat{\mathbb{Z}}G) \otimes \mathbb{Q} \xrightarrow{\Phi} H(\mathbb{Q}G) \otimes \hat{\mathbb{Z}} \longleftarrow H(\mathbb{Q}G).$$

Lemma 5.4 gives us a commutative diagram:

$$(5.6) \qquad \begin{array}{ccccccc} H(\hat{\mathbb{Z}}G) & \longrightarrow & H(\hat{\mathbb{Z}}G) \otimes \mathbb{Q} & \xrightarrow{\Phi} & H(\mathbb{Q}G) \otimes \hat{\mathbb{Z}} & \longleftarrow & H(\mathbb{Q}G) \\ \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_1 \otimes id} & & \downarrow{\scriptstyle \psi_2 \otimes id} & & \downarrow{\scriptstyle \psi_2} \\ H(\hat{\mathbb{Z}}G) & \longrightarrow & H(\hat{\mathbb{Z}}G) \otimes \mathbb{Q} & \xrightarrow{\Phi'} & H(\mathbb{Q}G) \otimes \hat{\mathbb{Z}} & \longleftarrow & H(\mathbb{Q}G) \end{array}$$

From this, it follows that there exists an isomorphism

$$\Psi: H(\mathbb{Z}G) \to H(\mathbb{Z}G)'$$

between the pullback $(H(\mathbb{Z}G)$ of the top row in (5.6) and the corresponding pullback $H(\mathbb{Z}G)'$ of the bottom row. Furthermore, this isomorphism Ψ is compatible with ψ_1 after taking the completion

$$\begin{array}{ccc} H(\mathbb{Z}G) & \longrightarrow & H(\hat{\mathbb{Z}}G) \\ \downarrow \Psi & & \downarrow \psi_1 \\ H(\mathbb{Z}G)' & \longrightarrow & H(\hat{\mathbb{Z}}G) \end{array} \tag{5.7}$$

Now the pullback diagram

$$(H(\hat{\mathbb{Z}}G), h_{\hat{\mathbb{Z}}}) \to (H(\hat{\mathbb{Z}}G), h_{\hat{\mathbb{Z}}}) \otimes \mathbb{Q} \xrightarrow{\Phi'} (H(\mathbb{Q}G), h'_{\mathbb{Q}}) \otimes \hat{\mathbb{Z}} \leftarrow (H(\mathbb{Q}G), h'_Q)$$

gives rise to the desired hermitian pairing $(H(\mathbb{Z}G)', h')$ over $H(\mathbb{Z}G)'$. In addition, we have a hermitian pairing $(H(\mathbb{Z}G)', h') \otimes \mathbb{Z}/|G|$ after taking the tensor product with $\mathbb{Z}/|G|$.

From (5.7), we have a commutative diagram:

$$\begin{array}{ccc} (H(\mathbb{Z}G), h) \otimes \mathbb{Z}/|G| & \xrightarrow{\approx} & (H(\hat{\mathbb{Z}}G), h_{\hat{\mathbb{Z}}}) \otimes \mathbb{Z}/|G| \\ \downarrow \Psi \ (\mathrm{mod}\ |G|) & & \downarrow \psi_1 \ (\mathrm{mod}\ |G|) \\ (H(\mathbb{Z}G)', h') \otimes \mathbb{Z}/|G| & \xrightarrow{\approx} & (H(\hat{\mathbb{Z}}G), h_{\hat{\mathbb{Z}}}) \otimes \mathbb{Z}/|G| \end{array}$$

where the two horizontal arrows are isometries. Since ψ_1 (mod $|G|$) is an isometry by Lemma 5.5, it follows that the isomorphism Ψ is an isometry after reduction modulo $|G|$. Or in other words, the pullback hermitian pairing $\Psi^*(h') \equiv h$ (mod $|G|$). This proves (5.1) (ii) and the proof of (5.1) is complete. □

To prove (5.4) and (5.5), we need the following version of the Strong Approximation Theorem for special linear groups due to Eichler and Kneser (see [28, 10.5.1]). Let R be a Dedekind domain with the global field K as quotient field. Let D be a finite-dimensional skew field with centre K and $A = M_n(D)$ and let Γ be an R-order on A. The special linear group $SL(\Gamma)$ is the subgroup of $SL(n, D)$ preserving Γ.

Theorem 5.8. *Let $\mathfrak{P}$ be a finite set of non-archimedean primes , $\mathcal{T} \in SL(n, \hat{D})$ and $\epsilon > 0$. Then there exists $T \in SL(n, D)$ and $\mathcal{S} \in SL(\hat{\Gamma})$ such that $\mathcal{T} = T \circ \mathcal{S}^{-1}$, and $\|S_\mathfrak{p} - Id\|_\mathfrak{p} < \epsilon$ for all $\mathfrak{p} \in \mathfrak{P}$.*

Proof. Consider the element $\mathcal{T} = \{T_\mathfrak{p} : S_\mathfrak{p} \in SL(n, \hat{D}_\mathfrak{p})\}$ in the adelic special linear group $SL(n, \hat{D})$. Then, by definition, for all but finitely many primes $\mathfrak{P}_0 = \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}$ the component $T_\mathfrak{p} \in SL(\hat{\Gamma}_\mathfrak{p})$ for $\mathfrak{p} \neq \mathfrak{p}_1, \dots, \mathfrak{p}_k$. We enlarge $\mathfrak{P}$ if necessary to assume that it contains all primes $\mathfrak{p} \in \mathfrak{P}_0$.

Using [28, 10.5.1] (with $\mathfrak{q}$ one of the infinite primes), and any given $\delta > 0$ we have $T \in SL(n, D)$ such that

$$\|T - T_{\mathfrak{p}_i}\| < \delta$$

for $\mathfrak{p}_i \in \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}$ and $T \in SL(\hat{\Gamma}_\mathfrak{p})$ elsewhere. In particular, by choosing δ small enough we can ensure that $T_{\mathfrak{p}_i}^{-1} \circ T$ is in any given ϵ-neighborhood of the identity.

Since $SL(\hat{\Gamma}_{\mathfrak{p}_i})$ is open in $SL(n, \hat{D}_{\mathfrak{p}}))$, it follows that for $\delta > 0$ sufficiently small $T_{\mathfrak{p}_i}^{-1} \circ T = S_{\mathfrak{p}_i}$ is in $SL(\hat{\Gamma}_{\mathfrak{p}_i})$. Define $S_{\mathfrak{p}} = T_{\mathfrak{p}}^{-1} \cdot T$ for the other primes as well. Then $\mathcal{S} = \{S_p\}$ is an integral adele, $\mathcal{S} \in SL(\hat{\Gamma})$, and $\mathcal{T} = T \circ \mathcal{S}^{-1}$. □

Proof of Lemma 5.4. To apply the above, we recall that $\Phi' \circ \Phi^{-1}$ is a simple automorphism of the vector space $H(\mathbb{Q}G) \otimes \hat{\mathbb{Z}}$. Note that $H(\hat{\mathbb{Q}}G) \otimes \hat{\mathbb{Z}}$ is decomposed as a product $\prod H(\mathbb{Q}G_\chi) \otimes \hat{\mathbb{Z}}$ of simple modules over each of the simple factors $(\mathbb{Q}G)_\chi$ of $\mathbb{Q}G$. In each factor we will take Γ_χ to be the image of $\mathbb{Z}G$ in $(\mathbb{Q}G)_\chi$. Since we can apply the above theorem to each of these factors and multiply them together, we will not distinguish between $H(\mathbb{Q}G) \otimes \hat{\mathbb{Z}}$ and its factors.

For all but a finite set of primes $\mathfrak{P} = \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}$, the automorphism $\Phi' \circ \Phi^{-1}$ preserves the lattice $H(\hat{\mathbb{Z}}G)$. By enlarging this set $\mathfrak{P}$ if necessary, we can assume that it contains all the prime divisors of $|G|$. By (5.8), there exist simple automorphisms ψ_1 of $H(\hat{\mathbb{Z}}G)$ and ψ_2 of $H(\mathbb{Q}G)$ such that

$$\Phi' \circ \Phi^{-1} = (\psi_2 \otimes id) \circ \Phi \circ (\psi_1 \otimes id)^{-1} \circ \Phi^{-1}.$$

As usual "simplicity" of ψ_1, ψ_2 is measured by reduced norms with respect to our fixed bases, after tensoring to $H(\hat{\mathbb{Z}}G) \otimes \mathbb{Q}$ or $H(\mathbb{Q}G) \otimes \hat{\mathbb{Z}}$. This establishes (5.4). □

Proof of Lemma 5.5. Since Φ and Ψ' are isometries between the hermitian forms, by choosing $\epsilon > 0$ small enough we can conclude that $\psi_1 \colon H(\hat{\mathbb{Z}}G) \to H(\hat{\mathbb{Z}}G)$ induces an isometry of the hermitian form $(H(\hat{\mathbb{Z}}G), h_{\hat{\mathbb{Z}}})$ modulo $|G|$. Thus condition (5.5) is also satisfied. □

In Section 8, we will need to vary the construction of $(V, \partial V)$. Recall that the attaching map f for $V = (U - e^4) \cup_f e^4$ is determined by the hermitian form $(H(\mathbb{Z}G), h')$, which is a pull-back of forms over $H(\mathbb{Q}G)$ and $H(\hat{\mathbb{Z}}G)$ identified by the simple isometry ϕ' given in Lemma 5.3.

Proposition 5.9. *Let ϕ' be a simple isometry as in (5.3). For any unitary automorphism $\beta \in SU(H(\hat{Q}G), h_{\hat{\mathbb{Q}}})$, the Poincaré complex $(V_\beta, \partial V_\beta)$ constructed from $\phi'_\beta = \beta \circ \phi'$ is also weakly simple and has negative definite intersection form.*

Proof. The image of the Whitehead torsion $\tau(V, \partial V)$ in $Wh(\hat{Q}G)$ is computed by reduced norms. By construction, these values are the same as those for $\tau(U, \partial U)$. Now we can repeat the proof of Proposition 5.2 using $\Phi'_\beta = \psi \circ (\phi')^{-1} \circ \beta^{-1}$ to construct a hermitian form h'_β, and then re-attach the top cell to get V_β. □

6. Four-Dimensional Surgery

In Sections 2-5 we constructed a collection of weakly simple Poincaré complexes $(V, \partial V)$ with $\pi_1(V) = G$ and negative definite intersection forms. The boundary $\partial V = \partial U$ is the disjoint union of linear and non-linear space forms. These complexes are parametrized by elements $\beta \in SU(H(\hat{Q}G), h_{\hat{\mathbb{Q}}})$, but this dependence will be suppressed for the moment. In this section, we will show that each of these Poincaré complexes admits a degree 1 normal map from a smooth 4-manifold. We then begin to study the surgery obstruction.

Proposition 6.1. *The Spivak normal fiber space $\xi \to V$ is trivial. There exists a trivialization $p: E(\xi) \to \mathbb{R}^\ell$, $\ell = \dim \xi + 1$, and an associated degree 1 normal map $f: (X, \partial X) \to (V, \partial V)$, $b: \nu_X \to \xi$, such that*

(i) *$(X, \partial X)$ is a compact, smooth, oriented 4-manifold with $\pi_1(X) = G$,*
(ii) *$\partial X = \partial V$ and $(f, b) \mid \partial X = id$,*
(iii) *$sign(X) = sign(V)$, and*
(iv) *the surgery obstruction $\lambda(f, b)$ lies in the "weakly simple" surgery obstruction group $L_0'(\mathbb{Z}G)$.*

Proof. Note that $\partial V = \partial U$ is a union of framed manifolds. Hence $\xi \mid \partial V$ has a vector bundle reduction and in fact a framed structure over ∂V. This smooth structure can be extended to give a vector bundle reduction for ξ over V since the first exotic spherical characteristic class is zero on oriented 4-dimensional Poincaré complexes. Then since $w_2(V) = w_3(V) = 0$ and V is homotopy equivalent to a 3-complex, we conclude that ξ is the trivial bundle. We fix a trivialization of ξ to serve as a base-point for normal invariants.

Let $p : E(\xi) \to \mathbb{R}^\ell$, $\ell = \dim \xi + 1 >> 0$, be any fibre homotopy trivialization of ξ extending the given trivialization of $\xi | \partial V$. By making p transverse to $0 \in \mathbb{R}^\ell$, we obtain a compact, smooth 4-manifold X_ξ with $\partial X_\xi = \partial V$ and a degree 1 normal map $f_\xi : X_\xi \to V$ covered by a bundle map $b_\xi : \nu(X_\xi) \to \xi$

$$\begin{array}{ccc} \nu(X_\xi) & \xrightarrow{b_\xi} & \xi \\ \downarrow & & \downarrow \\ X_\xi & \xrightarrow{f_\xi} & V \end{array} \tag{6.2}$$

and $(f_\xi, b_\xi) | \partial X_\xi = id$.

By varying within the normal cobordism class of (f_ξ, b_ξ) if necessary, we may assume that f_ξ induces an isomorphism of fundamental groups, so $\pi_1(X_\xi) = G$. Furthermore, since $(V, \partial V)$ is a weakly simple Poincaré pair, the surgery obstruction $\lambda(f_\xi, b_\xi)$ for (6.2) lies in group $L_0'(\mathbb{Z}G)$ computed in [38]. As a simply connected surgery problem, (6.2) has an obstruction given by the difference $\mathrm{sign}(X_\xi) - \mathrm{sign}(V)$ of two signatures. However we can get rid of this obstruction by the following modification.

Consider a degree 1 map $\varphi : V/\partial V \to S^4$. From [18], it is known that $\pi_4(G/PL) = \mathbb{Z}$ and its generator is represented by a vector bundle η over S^4 with a homotopy trivialization $p : E(\eta) \to \mathbb{R}^\ell$ and $\frac{1}{3}p_1(\eta)[S^4] = -16$. Pulling back this G/PL-structure to V via φ, we can add this to ξ to get a new G/PL-structure $\xi \sharp \varphi^* \eta$. Note that the relative Pontrjagin class $\frac{1}{3}p_1(\xi \sharp \varphi^* \eta)[V/\partial V] = \frac{1}{3}p_1(\eta)[S^4] = -16$ with respect to our base-point trivialization on ξ. Therefore by repeating this construction k-times, $k = \mathrm{sign}(V)/16$, we arrive at a G/PL-structure ξ' over $V/\partial V$ with $\frac{1}{3}p_1(\xi') = -\mathrm{sign}(V)$. Using ξ' instead of ξ, we obtain a corresponding surgery problem:

$$\begin{array}{ccc} \nu(X_{\xi'}) & \xrightarrow{b_{\xi'}} & \xi' \\ \downarrow & & \downarrow \\ X_{\xi'} & \xrightarrow{f_{\xi'}} & V \end{array}$$

Since we have

$$\operatorname{sign}(X_{\xi'}) = \frac{1}{3}p_1(\tau(X_{\xi'}))[X_{\xi'}/\partial X_{\xi'}] = -\frac{1}{3}p_1(\xi')[V/\partial V] = \operatorname{sign}(V)$$

it follows that the simply connected surgery obstruction equals zero. □

There are other surgery obstructions for our problem $(f, b) : X \to V$, independent of the simply-connected signature obstruction. In fact, the relevant surgery obstruction group $L_0'(\mathbb{Z}Q(8p,q))$ has been computed by Madsen in [20] following the methods of Wall [38].

As in [20, p. 208], let

$$L_n'(\mathbb{Z}G) = \begin{cases} L_n^Y(\mathbb{Z}G, \alpha, 1) & \text{for } n \equiv 0 \pmod 2, \text{and} \\ L_n^Y(\mathbb{Z}G, \alpha, 1)/\langle \begin{smallmatrix} 0 & 1 \\ \pm 1 & 0 \end{smallmatrix} \rangle & \text{for } n \equiv 1 \pmod 2, \end{cases}$$

where the decoration $Y = SK_1(\mathbb{Z}G) \oplus \langle \pm g \mid g \in G \rangle$.

Theorem 6.3. *There is a natural splitting:*

$$L_n^Y(\mathbb{Z}G) = \sum{}^{\oplus} \{L_n^Y(\mathbb{Z}G)(d) : d \mid pq\}$$

such that

(i) *for* $d \neq 1, L_n^Y(\mathbb{Z}G)(d) = L_n^X(\mathbb{Z}G)(d)$ *where the decoration* X *stands for* $SK_1(\mathbb{Z}G)$.
(ii) $L_n^X(\mathbb{Z}G)(d) \cong L_n^X(\mathbb{Z}[\mathbb{Z}/d \rtimes Q(8)])(d)$
(iii) *for each* $d \mid pq$*, there is a long exact sequence:*

$$\ldots \to CL_{n+1}^X(S(d)) \to L_n^Y(\mathbb{Z}G)(d) \to L_n^X(T(d)) \oplus \prod_{\ell \nmid d} L_n^X(\hat{R}_\ell(d)) \ldots$$

where

$$R(d) = \mathbb{Z}[\zeta_d]^t Q(8) \qquad S(d) = \mathbb{Q}[\zeta_d]^t Q(8)$$
$$T(d) = \mathbb{R} \otimes S(d) \qquad \hat{R}_\ell(d) = R(d) \otimes \hat{\mathbb{Z}}_\ell$$

and

$$CL_n^X(S(d)) = L_n^X(S(d) \to \hat{S}(d) \oplus T(d))$$

(iv) *The* K*-theory decorations are given by*

$$X(S(d)) = X(T(d)) = X(\hat{R}_\ell(d)) = \{0\}, \ (\ell \text{ odd}).$$

Since the calculations of $L_*^Y(\mathbb{Z}G))(d)$ for different $d \mid pq$ are quite similar, Madsen concentrated on the most difficult case when $d = pq$. For this he proved the following [20, Thm. 4.16]:

Theorem 6.4. *There is an exact sequence*

$$0 \to Coker\psi_1 \to L_0^X(\mathbb{Z}G)(pq) \to Ker\psi_0 \to 0,$$

where $Ker\psi_0$ is the free abelian group detected by the signature invariants corresponding to real places of $F = \mathbb{Q}[\zeta_p + \zeta_p^{-1}, \zeta_q + \zeta_q^{-1}]$. The term $Coker\psi_1$ is determined by the exact sequence

$$(6.5)\quad 0 \to Ker\tilde{\psi}_1^F \to F^{(2)}/F^{\times 2} \to H^0((A/pq)^\times) \to Coker\psi_1^F \to H^0(\Gamma(F)) \to 0,$$

where $A = \mathbb{Z}[\zeta_p + \zeta_p^{-1}, \zeta_q + \zeta_q^{-1}]$, $F = \mathbb{Q}[\zeta_p + \zeta_p^{-1}, \zeta_q + \zeta_q^{-1}]$, $\Gamma(F) = I(F)/F^\times$ is the ideal class group of F, $I(F) = F_A^\times / F_\infty^\times \cdot \hat{A}^\times$ is the ideal class group, and $F^{(2)} \subset F^\times$ consists of elements with even evaluation at all finite primes.

For a geometric surgery problem (f, b), the image of the surgery obstruction $\lambda(f, b)$ in the group $\mathrm{Ker}\,\psi_0$ can be interpreted as the difference $sign_\alpha(V) - sign_\alpha(X)$ between the multi-signatures of X and V.

Proof of Theorem 6.4. The exact sequence of (6.4) comes from the calculation of L-groups [38]: we substitute

$$\prod_{\ell \nmid pq} L_1^X(\hat{A}_\ell) = \prod_{\ell \nmid pq} H^0(A_\ell^\times) \times A/2,$$

and $L_1^X(F_\infty) = H^0(F_\infty^\times)$ together with $CL_1^X(F) = H^0(C(F))$ into the commutative diagram

$$\begin{array}{ccc}
\prod_{\ell \nmid pq} H^0(A_\ell^\times) \times A/2 \times H^0(F_\infty^\times) & \xrightarrow{\psi_1} & H^0(C(F)) \\
\downarrow \approx & & \downarrow \approx \\
\prod_{\ell \nmid pq} L_1^X(\hat{A}_\ell) \times L_1^X(F_\infty) & \xrightarrow{\psi_1} & CL_1^X(F)
\end{array}$$

where $C(F) = \hat{F}^\times / F^\times$ is the idèle class group and the vertical maps are induced by the "change of decoration" Rothenberg sequences in L-theory comparing L^X with L^K. In describing the cokernel of ψ_1, it is convenient to compare with the natural homomorphism

$$H^0(\hat{A}^\times) \times H^0(F_\infty^\times) \times H^0(F^\times) \to H^0(F_A^\times)$$

which has kernel $F^{(2)}/F^{\times 2}$ and cokernel $H^0(\Gamma(F))$. Putting this information together we have the commutative diagram:

$$\begin{array}{ccccccccc}
0 \to & \mathrm{Ker}\,\tilde{\psi}_1^F & \to & H^0(\hat{A}_{d'}^\times) \times H^0(F_\infty^\times) \times H^0(F^\times) & \to & H^0(\hat{F}_A^\times) & \to & \mathrm{Coker}\,\tilde{\psi}_1^F & \to 0 \\
& \downarrow & & \downarrow & & | & & \downarrow & \\
(6.6) \quad 0 \to & F^{(2)}/F^{\times 2} & \to & H^0(\hat{A}^\times) \times H^0(F_\infty^\times) \times H^0(F^\times) & \to & H^0(\hat{F}_A^\times) & \to & H^0(\Gamma(F)) & \to 0 \\
& \downarrow & & \downarrow & & & & & \\
0 \to & H^0(\hat{A}_d^\times) & = & H^0(\hat{A}_d^\times) & & & & &
\end{array}$$

Here $H^0(\hat{A}_d^\times) = \prod_{\ell | pq} H^0(\hat{A}_\ell^\times)$ and $H^0(\hat{A}_{d'}^\times) = \prod_{\ell \nmid pq} H^0(\hat{A}_\ell^\times)$. The snake lemma yields the exact sequence in (6.5). $\square$

We will now apply these calculations to study the surgery obstructions which lie in $\mathrm{Coker}\,\psi_1$. Let $SU_r(\hat{\mathbb{Q}}G)$ denote the group of unitary automorphisms of the hyperbolic form of rank r over $\hat{\mathbb{Q}}G$.

Lemma 6.7. *There is a natural projection $SU_r(\hat{\mathbb{Q}}G) \to Coker\psi_1$, for $r \geq 3$.*

We will denote the image of $\beta \in SU_r(\hat{\mathbb{Q}}G)$ under this projection by $[\beta]$.

Proof. Since

$$CL_n^X(S(d)) = L_n^X(S(d) \dashrightarrow \hat{S}(d) \oplus T(d))$$

is actually a quotient of $L_n^S(\hat{S}(d) \oplus T(d))$ by [38,1.2] and $L_1^S(T(d)) = 0$, we see that $CL_1^S(S(d))$ and hence Coker ψ_1 is a quotient of $L_1^S(\hat{S}(d))$. However, by definition $L_1^S(\hat{\mathbb{Q}}G)$ is a quotient of the stablized special unitary group $SU(\hat{\mathbb{Q}}G)$ and the projection

$$SU_r(\hat{\mathbb{Q}}G)/RU_r(\hat{\mathbb{Q}}G) \to L_1^S(\hat{\mathbb{Q}}G)$$

is an epimorphism for $r \geq 3$. □

Recall from (5.9) that we can vary our Poincaré complex $(V, \partial V)$ to $(V_\beta, \partial V_\beta)$ for any $\beta \in SU(H(\hat{\mathbb{Q}}G), h_{\hat{\mathbb{Q}}})$. Exactly the same process can be used to vary any algebraic quadratic Poincaré complex (as defined in [25]).

Any 4-dimensional quadratic $\mathbb{Z}G$ Poincaré complex $C(\psi)$ can be stabilized by adding a hyperbolic form $\mathbb{H}(q(r)) = \mathbb{H}(\mathbb{Z}G)^r$ of rank r over $\mathbb{Z}G$ (considered as a 4-complex concentrated in the middle dimension). This is just the algebraic analogue of adding copies of $S^2 \times S^2$ to the domain of a geometric surgery problem. Now if $C(\psi(r)) = C(\psi) \oplus \mathbb{H}(q(r))$ is the r-stabilization of $C(\psi)$ and $\beta \in SU_r(\hat{\mathbb{Q}}G)$, we can construct a new quadratic Poincaré complex $C(\psi_\beta(r))$ by pulling back using the same rational and ℓ-adic pieces as $C(\psi) \oplus \mathbb{H}(q(r))$. The identification over $\hat{\mathbb{Q}}G$ is altered by composing with β (just as in Proposition 5.9).

Lemma 6.8. *For any $\beta \in SU_r(\hat{\mathbb{Q}}G)$, $r \geq 3$, and any 4-dimensional quadratic Poincaré complex $C(\psi)$ over $\mathbb{Z}G$, the surgery obstruction $\lambda(C(\psi_\beta(r))) \in L'_0(\mathbb{Z}G)$ is independent of r and given by $\lambda(C(\psi_\beta(r))) = \lambda(C(\psi)) + [\beta]$.*

Proof. Stabilization does not change the surgery obstruction of $C(\psi)$ so

$$\lambda(C(\psi(r))) = \lambda(C(\psi)).$$

Similarly, $\lambda(C(\psi_\beta(r)))$ is independent of r since $r \geq 3$. We can also assume that the patching over $\hat{\mathbb{Q}}G$ used to construct $C(\psi)$, and the action of β, take place in orthogonal direct summands of $C(\psi(r))$. Therefore

$$C(\psi_\beta(r)) = C(\psi(r)) \oplus \mathbb{H}(q_\beta(r)).$$

Since the surgery obstruction is just the algebraic Poincaré cobordism class of $C(\psi_\beta(r))$, and $\lambda(\mathbb{H}(q_\beta(r))) = [\beta]$ by definition, the given formula holds. □

7. Induction Maps

This section contains an algebraic result we will need to handle the multisignature surgery obstruction. Let $R(G)$ denote the real representation ring of G, and recall that there is a natural transformation [38, 2.2] of Mackey functors

$$\sigma_G\colon L'_0(\mathbb{Z}G) \to R(G)$$

given by diagonalizing a hermitian form (H, h) over each irreducible representation α of $\mathbb{R}G$ (see [38, §2.2]) and then taking the formal difference $\mathrm{sign}_\alpha(H, h)$ of the maximal positive and negative definite G-invariant subspaces.

In particular, the homomorphisms $\sigma_H: L'_0(\mathbb{Z}H) \to R(H)$ for $H = Q(8p), Q(8q)$, and $C(2pq)$ are compatible with the induction maps between the surgery obstruction groups, and the corresponding induction homomorphisms:

$$\begin{aligned} i_{1*} &: R(Q(8p)) \to R(Q(8p,q)) \\ i_{2*} &: R(Q(8q)) \to R(Q(8p,q)) \\ i_{3*} &: R(C(2pq)) \to R(Q(8p,q)). \end{aligned}$$

The reduced representation ring $\widetilde{R}(G) = \ker(R(G) \to R(1))$ is generated by elements of the form $(\alpha - \dim\alpha \cdot \mathbf{1})$ for all real G-representations α. This ideal of $R(G)$ is closed under induction and restriction. The transformation σ_G induces

$$\sigma: \tilde{L}'_0(\mathbb{Z}G) \to \widetilde{R}(G),$$

which is again compatible with the Mackey structure, and we have the commutative diagram

$$\begin{array}{ccc} \tilde{L}'_0(Q(8p)) \oplus \tilde{L}'_0(Q(8q)) \oplus \tilde{L}'_0(C(2pq)) & \xrightarrow{I_{1*}\oplus I_{2*}\oplus I_{3*}} & \tilde{L}'_0(\mathbb{Z}G) \\ \downarrow \sigma & & \downarrow \sigma \\ \widetilde{R}(Q(8p)) \oplus \widetilde{R}(Q(8q)) \oplus \widetilde{R}(C(2pq)) & \xrightarrow{i_{1*}\oplus i_{2*}\oplus i_{3*}} & \widetilde{R}(G). \end{array}$$

The main result of this section is:

Proposition 7.1. *The image of $\sigma: \tilde{L}'_0(\mathbb{Z}G) \to \widetilde{R}(G)$ restricted to $Im\,(I_{1*} \oplus I_{2*} \oplus I_{3*})$ contains the subgroup $16 \cdot \widetilde{R}(G)$.*

We will describe the splitting used in [38, §4] and Theorem 6.3, in order to study the induction homomorphisms between these groups.

Lemma 7.2. *The map σ_G has a direct sum decomposition $\sigma = \oplus_{d|pq}\sigma(d)$ where $\sigma(d): L'_0(\mathbb{Z}G)(d) \to R(G)(d)$. A similar splitting exists for the subgroup $C(2pq)$, and the induction map $I_3: L'_0(\mathbb{Z}C(2pq)) \to L'_0(\mathbb{Z}G)$ preserves the components.*

Proof. Note that the group algebra $\mathbb{Q}[C(2pq)]$ decomposes into the product of four different fields $\mathbb{Q}$, $\mathbb{Q}(\zeta_p)$, $\mathbb{Q}(\zeta_q)$, and $\mathbb{Q}(\zeta_{pq})$. This induces a corresponding decomposition on $\mathbb{Q}G = \mathbb{Q}[C(2pq)]^t Q(8)$ and hence on every functor of $\mathbb{Q}G$. In fact, for every covariant functor $A(-)$ from finite subgroups of G to abelian groups, an analogous decomposition exists for $A(G)$. Let $f_p, f_q : C(2pq) \to C(2pq)$ denote the endomorphisms which project onto $C(p)$ and $C(q)$ respectively. They extend to endomorphisms $\hat{f}_p, \hat{f}_q$ of $Q(8p,q)$ by setting $\hat{f}_p \,|\, Q(8) = \hat{f}_q \,|\, Q(8) = id$. Since $\hat{f}_p^2 = \hat{f}_p, \hat{f}_q^2 = \hat{f}_q$, we obtain idempotent endomorphisms $F_p = (\hat{f}_p)_*$ and $F_q = (\hat{f}_q)_*$ of $A(G)$. Hence there is a decomposition

$$A(G) = A(G)(1) \oplus A(G)(p) \oplus A(G)(q) \oplus A(G)(pq)$$

where

$$\begin{aligned}A(G)(1) &= F_pF_q(A(G))\\ A(G)(p) &= F_p(1-F_q)(A(G))\\ A(G)(q) &= F_q(1-F_p)(A(G))\\ A(G)(pq) &= (1-F_p)(1-F_q)(A(G)).\end{aligned}$$

Applying this splitting to the surgery obstruction group $L'_0(\mathbb{Z}G)$, we have

$$L'_0(\mathbb{Z}G) = L'_0(\mathbb{Z}G)(1) \oplus L'_0(\mathbb{Z}G)(p) \oplus L'_0(\mathbb{Z}G)(q) \oplus L'_0(\mathbb{Z}G)(pq).$$

Similarly for $R(G)$, we have

$$R(G) = R(G)(1) \oplus R(G)(p) \oplus R(G)(q) \oplus R(G)(pq).$$

Since the splittings are given by idempotents, we get a corresponding direct sum decomposition for σ.

The idempotent endomorphisms $\hat{f}_p, \hat{f}_q$ also exist on the subgroup $C(2pq)$ and hence give the corresponding decompositions on $L'_0(C(2pq))$ and $R(C(2pq))$. The commutativity of the following diagram $(d \mid pq)$

(7.3)
$$\begin{array}{ccc} L'_0(\mathbb{Z}C(2pq))(d) & \xrightarrow{\sigma} & R(C(2pq))(d) \\ \downarrow{\scriptstyle I_{3*}} & & \downarrow{\scriptstyle i_{3*}} \\ L'_0(\mathbb{Z}G)(d) & \xrightarrow{\sigma} & R(G)(d) \end{array}$$

means that the induction maps from $C(2pq)$ preserve the components. □

There is one more functorial fact which simplifies our problem. Both $Q(8p,q)$ and $C(2pq)$ contain a unique order 2 element $X^2 = Y^2 = (XY)^2$ in the centre. By Schur's lemma the action of this element on the irreducible are either $+1$ or -1. Accordingly the representation rings decompose into two components:

$$\begin{aligned}R(Q(8p,q))(pq) &= R(Q(8p,q))(pq)_+ \oplus R(Q(8p,q))(pq)_-\\ R(C(2pq))(pq) &= R(C(2pq))(pq)_+ \oplus R(C(2pq))(pq)_-\end{aligned}$$

and the homomorphism i_{3*} preserves these decompositions.

Proposition 7.4. *On the (-1)-component the homomorphism*

$$(i_3)_* : R(C(2pq))(pq)_- \to R(Q(8p,q))(pq)_-$$

is surjective, and on the $(+1)$-component the image of the homomorphism

$$(i_3)_* : R(C(2pq))(pq)_+ \to R(Q(8p,q))(pq)_+$$

equals $2 \cdot R(Q(8p,q))(pq)_+$

Proof. Recall that the splittings on $R(C(2pq))$ and $R(Q(8p,q))$ can be achieved by first applying the splittings to the group algebras $\mathbb{Q}[C(2pq)], \mathbb{Q}[Q(8p,q)]$. By (4.8) and (4.9) we see that the rational representations in the top components $S(pq)_+$ and $S(pq)_-$ are induced up from $\mathbb{Q}[C(2pq)]$, and become the twisted group algebras $\mathbb{Q}(\zeta_{pq})^t[X,Y]_\pm$ which have dimension $4(p-1)(q-1)$. In the regular representation of $\mathbb{Q}G$, this factor decomposes as the direct sum of 4 copies (respectively 2 copies) of the simple module for $S(pq)_+ = M_4(F)$ (resp. $S(pq)_- = M_2(D)$). Note that $\mathbb{Q}(\zeta_{pq})\otimes\mathbb{R}$ splits into $(p-1)(q-1)/2$ copies of the complex numbers $\mathbb{C}$, and the centre field $F = \mathbb{Q}(\zeta_p + \zeta_p^{-1}, \zeta_q + \zeta_q^{-1})$ splits into $(p-1)(q-1)/4$ copies of $\mathbb{R}$. By counting dimensions (over $\mathbb{R}$), we see that each one of these irreducible representations $\mathbb{C}$ of $C(2pq)$ induces up to a real 8-dimensional representation. Since the irreducible module $L_\pm$ for a simple factor of $M_4(F)\otimes\mathbb{R}$ (resp. $M_2(D)\otimes\mathbb{R}$) has real dimension 4 (resp. 8), we conclude that the induced real representations is $L_+ \oplus L_+$ in the $(+1)$-component and L_- in the (-1)-component. □

Proof of Proposition 7.1. The endomorphism $\hat{f}_p$ factors through the subgroup $Q(8p)$, and we have the inclusion $\operatorname{Im} F_p \subseteq \operatorname{Im} i_{1*}$. On the other hand, because $\hat{f}_p \circ i_1 = i_1$, we have $(F_p - 1)\operatorname{Im} i_{1*} = 0$. Similar relations hold for F_q and i_{2*}. From the definition of the summands $R(G)(d)$ in terms of these idempotents, it follows that

$$R(G)(1) \oplus R(G)(p) \oplus R(G)(q) = \operatorname{Im}(i_{1*}) + \operatorname{Im}(i_{2*}). \tag{7.5}$$

On the other hand, by Proposition 7.4, we have

$$2 \cdot R(G)(pq) \subseteq \operatorname{Im}(i_{3*})$$

and we can conclude that

$$2 \cdot R(G) \subseteq \operatorname{Im}(i_{1*} \oplus i_{2*} \oplus i_{3*}) \subseteq R(G).$$

Moveover, it follows from the results of [15], [38,2.2.1] on the divisibility of the signature invariants, that

$$\sigma_H : L_0'(\mathbb{Z}H) \to R(H)$$

has image containing the subgroup $8 \cdot R(H)$ for $H = Q(8p)$, $Q(8q)$, or $C(2pq)$. By naturality of σ,

$$16 \cdot R(G) \subseteq \sigma_G(\operatorname{Im}(I_{1*} \oplus I_{2*} \oplus I_{3*})) \subseteq R(G).$$

□

8. Almost Spherical Space Forms

We are now ready to consider the surgery obstructions of the degree 1 normal maps constructed in Proposition 6.1.

Proposition 8.1. *Let $f_\xi : (X, \partial X) \to (V, \partial V)$, $b_\xi : \nu_X \to \xi$ be a degree 1 normal map satisfying the conditions in (6.1). Then there exists an element $\beta \in SU_r(\hat{\mathbb{Q}}G)$, $r \geq 3$, and a degree 1 normal map $f'_{\xi,\beta} : (X', \partial X') \to (V_\beta, \partial V_\beta)$, $b'_{\xi,\beta} : \nu_{X'} \to \xi$ such that*

(1) *$f'_{\xi,\beta} \mid \partial X'$ is an integral homology equivalence, and*
(ii) *$\lambda(f'_{\xi,\beta}, b'_{\xi,\beta}) = 0 \in L_0'(\mathbb{Z}G)$.*

After giving the proof of this result, we will use it to construct the smooth 4-manifold $(Y, \partial Y)$ described in the Introduction.

Proof. We will first consider the multisignature obstruction $\operatorname{sign}_\alpha(V) - \operatorname{sign}_\alpha(X_\xi)$ given by irreducible real representations α, $\alpha \neq 1$, of $G = Q(8p, q)$. Note that this set of surgery obstructions generates the group $\operatorname{Ker}\psi_0$ and so if $\operatorname{sign}_\alpha(V) - \operatorname{sign}_\alpha(X_\xi) = 0$ for all α then $\lambda(f_\xi, b_\xi) \in \operatorname{Coker}\psi_1$.

We begin with the ρ-invariant $\rho_\alpha(N)$ of a 3-manifold N with a unitary representation $\alpha : \pi_1(N) \to U(n)$. Suppose $N = \partial M$ and α extends to a representation of $\pi_1(M)$. Then

$$\rho_\alpha(N) = n \cdot \operatorname{sign}(M) - \operatorname{sign}_\alpha(M).$$

As a consequence of this formula, we have

$$\begin{aligned} \operatorname{sign}_\alpha(V) - \operatorname{sign}_\alpha(X) &= \operatorname{sign}_\alpha(V) - [n \cdot \operatorname{sign}(X) - \rho_\alpha(\partial X)] \\ &= \operatorname{sign}_\alpha(V) - [n \cdot \operatorname{sign}(V) - \rho_\alpha(\partial X)] \end{aligned}$$

or, in other words, the vanishing of the obstruction $\operatorname{sign}_\alpha(V) - \operatorname{sign}_\alpha(X)$ is the same as requiring that the following equation

$$\rho_\alpha(\partial X) = n \cdot \operatorname{sign}(V) - \operatorname{sign}_\alpha(V) \tag{8.2}$$

is satisfied by the domain $(X, \partial X)$ of our degree one normal map. Note that this equation, and the fact that $\operatorname{sign} X = \operatorname{sign} V$, implies that the multisignature difference depends *only* on $\partial X = \partial V$.

In general, equation (8.2) may not be satisfied and so these are nontrivial obstructions for our surgery problem. To get rid of these obstructions, the idea is to replace copies of the spherical space forms $S/Q(8p)$, $S/Q(8q)$, or $S/C(pq)$ on the boundary ∂X by almost spherical space forms $S'/Q(8p)$, $S'/Q(8q)$, $S'/C(2pq)$ and therefore change the ρ-invariants. After this process, our new normal map will no longer restrict to the identity on the boundary, but just to an integral homology equivalence.

One way to construct an almost space form S'/H is to start with an element $\sigma \in L'_0(\mathbb{Z}H)$ and apply the Wall realization theorem to construct a degree 1 normal map

$$(f, b) \colon (M^4, \partial_0 M^4, \partial_1 M^4) \to (S^3/H \times I, S^3/H \times 0, S^3/H \times 1)$$

such that $\lambda(f, b) = \sigma$. More explicitly, this surgery problem is constructed by representing σ by a quadratic form on a free $\mathbb{Z}H$-module and using this algebraic data as a prescription for attaching 2-handles to $S^3/H \times I$. By construction, the lower boundary component $\partial_0 M^4 = S^3/H$ and the restriction of (f, b) is the identity. The upper boundary component $\partial_1 M^4 = S'/H$ is an almost space form. On this boundary component the restriction $f \colon S'/H \to S^3/H$ is just an integral homology equivalence, and a surjection on fundamental groups. The fact that we have lost some control of $\pi_1(S'/H)$ is a typical problem with surgery in dimension 3, but at least S' is an integral homology sphere.

Now suppose that we start with σ_1, σ_2, and σ_3 in $L'_0(Q(8q))$, $L'_0(Q(8q))$, and $L'_0(C(2pq))$ respectively. Then we construct 4-manifolds M_1, M_2, M_3 whose boundary components are the spherical space forms S^3/H_i and the almost space forms

S'/H_i with $H_i = Q(8p)$, $Q(8q)$, or $C(2pq)$ for $i = 1, 2$ or 3. Let $g_i: S'/H_i \to S^3/H_i$, $1 \le i \le 3$, denote the integral homology equivalence obtained by restricting the degree 1 normal map (f_i, b_i) used to construct $(M_i, \partial M_i)$ to the top boundary component.

Next we attach these surgery problems (f_i, b_i) to our degree 1 normal map $X \to V$ along the appropriate boundary components of $\partial X = \partial V$. More precisely, we attach the 4-manifold M_i to a component of ∂X with boundary S^3/H_i and extend the degree 1 map by using the normal maps (f_i, b_i) to collars $S^3/H_i \times I$ on the same component of ∂V. This produces a new degree 1 normal map (which we will consider to be a relative surgery problem):

$$(f'_\xi, b'_\xi): (X', \partial X') \to (V, \partial V), \tag{8.3}$$

where the domain is

$$X' = X \cup M_1 \cup M_2 \cup M_3,$$

and f'_ξ restricted to $\partial X'$ is an integral homology equivalence.

Moreover, if we choose $\sigma, \sigma_2, \sigma_3$ to lie in the reduced surgery obstruction groups $\tilde{L}'_0(Q(8p))$, $\tilde{L}'_0(Q(8q))$, or $\tilde{L}'_0(C(2pq))$, then the simply-connected signature invariants $\mathrm{sign}(M_i) = 0$ for $i = 1, 2$ and 3. It follows that $\mathrm{sign}(X') = \mathrm{sign}(V)$.

We can now compute the effect on the multi-signature obstruction

$$\mathrm{sign}_\alpha(V) - \mathrm{sign}_\alpha(X') = \rho_\alpha(\partial X') - n \cdot \mathrm{sign}(V) + \mathrm{sign}_\alpha(V). \tag{8.4}$$

We have changed the ρ-invariants on the boundary by the formula:

$$\rho_\alpha(\partial X') = \rho_\alpha(\partial X) + \mathrm{sign}_\alpha[I_{1*}(\sigma_1) + I_{2*}(\sigma_2) + (I_{3*}(\sigma_3)]. \tag{8.5}$$

Here I_{1*}, I_{2*}, I_{3*} are the induction homomorphisms between the surgery obstruction groups:

$$\begin{aligned} I_{1*} &: L'_0(Q(8p)) \to L'_0(Q(8p, q)) \\ I_{2*} &: L'_0(Q(8q)) \to L'_0(Q(8p, q)) \\ I_{3*} &: L'_0(C(2pq)) \to L'_0(Q(8p, q)) \end{aligned}$$

already used in Section 7. Substituting (8.5) into (8.4), we have the equation

$$\rho_\alpha(\partial X) + \sum_{1 \le k \le 3} \mathrm{sign}_\alpha[(I_{k*}(\sigma_k)] = n \cdot \mathrm{sign}(V) - \mathrm{sign}_\alpha(V) \tag{8.6}$$

as the requirement for vanishing of the multisignature obstruction for the surgery problem of (f'_ξ, b'_ξ). Therefore our goal is to choose $\sigma_1, \sigma_2, \sigma_3$ in such a manner that the expression (8.6) is satisfied.

The nonsingular hermitian pairing (H, h) for $H = H_2(V; \mathbb{Z}G)$ gives us an element in $R(G)$, whose α-component is $\mathrm{sign}_\alpha(V)$. Therefore we can interpret the expression $n \cdot \mathrm{sign}(V) - \mathrm{sign}_\alpha(V)$, $n = \dim \alpha$, in (8.2) as the α-component of an element

$\sigma(V) \in \widetilde{R}(G)$. Similarly, we have $\sigma(X) \in \widetilde{R}(G)$. In addition, from the construction of $(V, \partial V)$ both $\sigma(V)$ and $\sigma(X)$ are divisible by 16, or in other words

$$\sigma(V) - \sigma(X) \in 16 \cdot \widetilde{R}(G) \subseteq \widetilde{R}(G).$$

Since $\rho_\alpha(\partial X) = n \cdot \mathrm{sign}(X) - \mathrm{sign}_\alpha(X)$, we can rewrite (8.6) as an equation in the reduced representation ring:

$$\sigma_G(I_{1*}(\sigma_1) + I_{2*}(\sigma_2) + I_{3*}(\sigma_3)) = \sigma(V) - \sigma(X) \in 16 \cdot \widetilde{R}(G) \subseteq \widetilde{R}(G) \tag{8.7}$$

where

$$\sigma_G : L_0'(\mathbb{Z}G) \to R(G)$$

is the multisignature natural transformation from Section 7. But the main result of that section, Proposition 7.1, states that equation (8.6) has a solution σ_1, σ_2, σ_3. We may therefore use these elements to construct a degree 1 normal map $(f_\xi', b_\xi'): (X', \partial X') \to (V, \partial V)$ as in (8.3) with $\lambda(f_\xi', b_\xi') \in \mathrm{Coker}\,\psi_1$. Since the multisignature vanishes, this surgery obstruction is independent of the choice of normal map (i.e. depends only on the range $(V, \partial V)$).

To complete the proof of Proposition 8.1, we pick $\beta \in SU_r(\hat{\mathbb{Q}}G)$, for $r \geq 3$, projecting to $\lambda(f_\xi', b_\xi') \in \mathrm{Coker}\,\psi_1$. This is possible by Lemma 6.7(note that the obstructions are now 2-torsion). We then consider the new Poincaré pair $(V_\beta, \partial V_\beta)$ as constructed in Proposition 5.9. By construction, the boundary $\partial V_\beta = \partial V$ and the multisignature of $(V_\beta, \partial V_\beta)$ equals that of $(V, \partial V)$. It follows from Proposition 6.1 that there is a degree 1 normal map $(f_{\xi,\beta}, b_{\xi,\beta})$ onto $(V_\beta, \partial V_\beta)$ which is the identity on the boundary. Since $\partial V_\beta = \partial V$

$$\mathrm{sign}_\alpha(f_{\xi,\beta}, b_{\xi,\beta}) = \mathrm{sign}_\alpha(f_\xi, b_\xi)$$

and we may use the same elements $\sigma_1, \sigma_2, \sigma_3$ to construct a modified normal map $(f_{\xi,\beta}', b_{\xi,\beta}')$, inducing an integral homology equivalence on the boundary, with zero multisignature obstruction

The surgery obstruction $\lambda(f_{\xi,\beta}, b_{\xi,\beta})$ is determined by the induced quadratic structure [25] on the mapping cone complex $C_*(f_{\xi,\beta}')$:

$$0 \to C_*(X_{\xi,\beta}') \to C_*(V_{\xi,\beta}) \to C_*(f_{\xi,\beta}') \to 0.$$

This sequence can be analysed as in §4 by means of the arithmetic square. Again by stabilizing our Poincaré complexes, we can assume that the new identification over $\hat{\mathbb{Q}}G$ given by β takes place on some hyperbolic factors of $(H(\hat{\mathbb{Q}}G), h_{\hat{\mathbb{Q}}})$ orthogonal to those summand used in constructing the map $f_{\xi,\beta}'$. It follows that the quadratic Poincaré complex $C_*(f_{\xi,\beta}')$ can be constructed from the exact sequence of chain complexes

$$0 \to C_*(X') \to C_*(V_\xi) \to C_*(f_\xi') \to 0$$

by re-mixing the complexes $C_*(V_\xi)$ and $C_*(f_\xi')$ simulaneously with β to produce $C_*(V_{\xi,\beta}) \to C_*(f_{\xi,\beta}')$. From Lemma 6.8, it follows that

$$\lambda(f_{\xi,\beta}', b_{\xi,\beta}) = \lambda(f_\xi', b_\xi) + [\beta] = 0$$

and the proof is complete. □

The final result of this section is an application of (8.1):

Proposition 8.8. *Assume the existence of a nonlinear space $\Sigma/Q(8p,q)$. There is a framed, compact, oriented 4-manifold Y with the following properties:*

(i) *The boundary $\partial Y = N \cup \partial_0 Y$, where N is a connected 3-manifold, and $\pi_1(Y)$ modulo the normal closure of $\pi_1(N)$ is isomorphic to $G = Q(8p,q)$.*
(ii) $H_1(Y;\mathbb{R}G) = 0$.
(iii) *The boundary components $\partial_0 Y$ of Y consist of copies of the nonlinear space form Σ/G, spherical space forms $S/Q(4pq)$ and almost space forms S'/H for $H = Q(8p)$, $Q(8q)$, or $C(2pq)$.*
(iv) *The induced homomorphism $\pi_1(\partial_0 Y) \to \pi_1(Y)$ on the fundamental groups sends $\pi_1(\Sigma/G)$, $\pi_1(S/Q(4pq))$, and $\pi_1(S'/H)$ for $H = Q(8p)$, $Q(8q)$, or $C(2pq)$ onto the corresponding subgroups $Q(8p,q)$, $Q(4pq)$ or $H \subseteq Q(8p,q)$.*
(v) *the induced $Q(8p,q)$-hermitian intersection pairing*

$$h\colon H_2(Y;\mathbb{Z}G) \times H_2(Y;\mathbb{Z}G) \to \mathbb{Z}G$$

has radical Im $(H_2(N;\mathbb{Z}G) \rightarrowtail H_2(Y;\mathbb{Z}G))$ and is negative definite on the orthogonal complement of this submodule.

Proof of (8.8). In Proposition 8.1 we have constructed a surgery problem

$$(f,b)\colon (X,\partial X) \to (W,\partial W)$$

with $\lambda(f,b)=0$, where we take $(W,\partial W) = (V_\beta, \partial V_\beta)$, $X = X'_{\beta,\xi}$ and $f = f'_{\beta,\xi}$.

Since the surgery obstruction (f,b) is zero, the intersection form on $H_2(X;\mathbb{Z}G)$ is the orthogonal direct sum of the negative definite form $(H_2(W;\mathbb{Z}G),h)$ and a hyperbolic form $H((\mathbb{Z}G)^r)$ on a free $\mathbb{Z}G$ module of rank $2r$. By [9], [10] we can represent this hyperbolic form geometrically by a topological locally flat embedding of $\sharp r(S^2 \times S^2)$ in the interior of X. Removing a closed tubular neighbourhood of the pairs of topologically embedded 2-spheres in this connected sum produces an open 4-manifold X'' with one end proper homotopy equivalent to $S^3 \times (0,\infty)$. Note that $\pi_1(X'') = \pi_1(X)$ and $H_2(X'';\mathbb{Z}G) = H_2(X;\mathbb{Z}G)$. Now we pick a compact subset $C \subset X''$ such that $\partial X'' = \partial X \subset C$ and the complement $X'' - C$ admits a proper homotopy equivalence $p\colon X'' - C \to S^3 \times (0,\infty)$. Let $N = p^{-1}(S^3 \times t_0)$ be a transverse pre-image for some large value $t = t_0$, and set $Y = X'' - p^{-1}(S^3 \times (t_0,\infty)$. We can assume that map $H_2(Y;\mathbb{Z}G) \to H_2(X'';\mathbb{Z}G)$ induced by the inclusion is surjective. Then by attaching 2-handles to Y along N if necessary, we can assume that $H_1(Y;\mathbb{R}G) = 0$. It also follows that $\pi_1(N)$ normally generates the kernel of the classifying map $c_\#\colon \pi_1(Y) \to Q(8p,q)$ and the inclusion induces an injection $H_2(N;\mathbb{Z}G) \to H_2(Y;\mathbb{Z}G)$. The subspace $\mathcal{N} = \mathrm{Im}\,(H_2(N;\mathbb{Z}G) \to H_2(Y;\mathbb{Z}G))$ is the null space or radical of the intersection form on $H_2(Y;\mathbb{Z}G)$ and the induced form on the quotient $H_2(Y;\mathbb{Z}G)/\mathcal{N}$ is isometric to $(H_2(W;\mathbb{Z}G),h)$, hence is negative definite. □

9. An Equivariant Moduli Space

In the previous sections, assuming the existence of a nonlinear space $\Sigma/Q(8p,q)$, we have constructed a framed, compact, oriented, smooth, 4-manifold Y with boundary $\partial Y = N \cup \partial_0 Y$ satisfying all the requirements of Proposition 8.8. In this section we will describe the equivariant moduli space $(\mathcal{M}(P), G)$ of ASD connections on a $SU(2)$-bundle over a closely related 4-manifold (Z, G). Analysis of the $Q(4pq)$ fixed-point strata in $(\mathcal{M}(P), G)$ will give the proof of our main result, Theorem A.

Let $\widetilde{Y}$ be the $Q(8p, q)$-covering of Y. Then on $\widetilde{Y}$ there is a free action of the group $G = Q(8p, q)$, and on the boundary $\partial_0 \widetilde{Y}$ of $\widetilde{Y}$ we have a union of homotopy spheres $(\Sigma, Q(8p, q))$, standard 3-spheres $(S^3, Q(4pq))$, and integral homology spheres (S', H) for $H = Q(8p)$, $Q(8q)$, or $C(2pq)$ which are invariant under the free action of the given subgroups. In addition, we have a free G-orbit $G \times N$ of boundary components of $\widetilde{Y}$ where each copy of N in $\partial \widetilde{Y}$ is invariant only under the trivial subgroup $\{1\} \subset G$. Among these components, there are the copies $(S^3, Q(4pq))$ of a linear action of $Q(4pq)$ on the standard sphere S^3. For each of these copies, we can extend the action to a corresponding linear action $(D^4, Q(4pq))$ on the disks.

Let (Z, G) denote the 4-manifold with G-action obtained from $(\widetilde{Y}, G)$ by gluing in equivariantly the above 4-disks $(D^4, Q(4pq))$ to each of the boundary components $(S^3, Q(4pq))$. To each other boundary component S' or N we attach a cylindrical end $S' \times [0, \infty)$ or $N \times [0, \infty)$ and extend the G-action in the obvious way (trivially on the $\mathbb{R}^+ = [0, \infty)$ factor). The G-manifold (Z, G) is now non-compact and the G-action on Z is free except for isolated singular points at the centres of the attached 4-disks with isotropy subgroup $Q(4pq)$. The cylindrical ends of Z are permuted in G/H-orbits where $H = Q(8p)$, $Q(8q)$, or $C(2pq)$ for the ends of the form $S' \times [0, \infty)$ and $H = 1$ for the ends $N \times [0, \infty)$.

Clearly this procedure does not affect the homology $H_*(Z)$ of degree $* < 3$. Hence the condition (8.8)(v) on the intersection form of $H_2(Y; \mathbb{Z}G)$ implies that $b^+(Z) = 0$. Here we use the usual notation for the ranks b^+ (resp. b^-) of the maximal positive (resp. negative) definite subspaces of $H_2(Z; \mathbb{R})$ with respect to the intersection form. The rank $b_2(Z) = \operatorname{rank} H_2(Z; \mathbb{R})$ is given by $b_2 = b^+ + b^- + b^0$, where b^0 is the rank of the null space (i.e. the radical) of the intersection form.

Our next task is to describe the equivariant moduli space $(\mathcal{M}(P), G)$ of Yang Mills connections on Z. In [30], [32], C. Taubes constructed a moduli space $\mathcal{M}(P)$ for a general 4-manifold $M = M_0 \coprod \operatorname{End} M$ with cylindrical ends $\operatorname{End} M \cong \partial M_0 \times [0, \infty)$, but without considering possible G-actions. Here M_0 is a compact 4-manifold with boundary. The data consists of a principal $SU(2)$-bundle P on M with a fixed trivialization $(\theta, P \,|\, \partial M_0 \times \mathbb{R}^+)$ of this bundle P over the end. This trivialization provides us an integer $k = c_2(P, \theta)$, known as the relative Chern number [30, Lemma 7.1]

$$c_2(P, \theta) = \frac{1}{8\pi^2} \int tr(F_A \wedge F_A),$$

and a trivial flat connection A_0 over $\partial Z_0 \times \mathbb{R}^+$. Let $\tau \colon Z \to \mathbb{R}$ be a smooth function which is zero on Z_0 and the real parameter on $\partial Z_0 \times \mathbb{R}^+$. Fix this connection A_0 and a constant $\delta > 0$ to be specified later, and consider the space $\mathcal{A}(P, \delta)$ of connections on P which satisfy an exponential decay condition:

$$\mathcal{A}(P, \delta) = \left\{ A_0 + a \;\middle|\; \begin{array}{l} a \in L^2_{2,loc}(AdP \otimes T^*Z) \\ \int_Z e^{\delta\tau}\{|\nabla^2_{A_0} a|^2 + |\nabla_{A_0} a|^2 + |a|^2\} < \infty \end{array} \right\}.$$

The corresponding gauge group $\mathcal{G}(P, \delta)$ acts smoothly on this Banach affine space $\mathcal{A}(P, \delta)$, with quotient space $B(P, \delta) = \mathcal{A}^*(P, \delta)/\mathcal{G}(P, \delta)$ a C^∞-Banach manifold [30,

§7]. For our application, we actually want to use the "thickened" version of this moduli space constructed in [24, Chap. 7] because of the presence of our exceptional ends $G \times N \times [0, \infty)$. Since the trivial flat connection is isolated in the representation variety of a homology 3-sphere [31, Lemma 1.3], the thickened moduli space and the one described above are the same over all of the other cylindrical ends $S' \times [0, \infty)$.

Let Ξ denote the set of conformal structure on M which extends the product conformal structures on the ends. Associated to $\sigma \in M$, the star operation $*$ on $\Omega^2(Ad\,P)$ gives a splitting of $\Omega^2(Ad\,P)$ into its (± 1)-eigenspaces $\Omega^2_\pm(Ad\,P)$. From this decomposition we have the anti-self-duality equation $(F_A)_+ = 0$ whose solutions can be regarded as the zero set of a section $\mathcal{A}(P, \delta) \to L^2_{2,loc}(\Omega^2_+(Ad\,P))$. In addition to the perturbation of conformal structures, we can also perform the "Wilson loop-perturbation" as in [9], [31], [6, §2(b)]. Let $\Gamma(m)$ denote a set of m framed, imbedded circles $\gamma = \{\gamma_i\}$ in M_0,

$$\gamma\colon \coprod_{i=1}^{m} S^1_i \times D^3 \to M_0.$$

For each $\gamma \in \Gamma(m)$, connection A, and $v \in D^3$, we have a map

$$\gamma_A(v)\colon B(P, \delta) \longrightarrow L_m = \prod_i SU(2)/Ad\,SU(2)$$

defined by the collection of the holonomies $\gamma_{i,A}(v)$, $i = 1, \ldots, m$ of the parallel translation via the connection A along the i^{th} loop. We let ω be a 2-form on Z supported in the product neighborhood $\gamma_i(S^1 \times D^3)$ and pulled back from a fixed 2-form on D^3. Then, for each smooth Ad-invariant vector field $p = (p_i)$ on $SU(2)^m$, we have the adjoint-valued 2-form $\pi(A) \in \Omega^2_+(M, Ad\,P)$ given by

$$\pi(A) = \sum_i \omega_+ \otimes p_i[\gamma_{i,A}(v)] \tag{9.1}$$

where ω_+ is the projection of ω on $\Omega^2_+(M)$. In this way, we obtain a set Π of $\mathcal{G}(P, \delta)$-equivariant maps

$$\pi\colon \mathcal{A}(P, \delta) \to L^2_{2,loc}(\Omega^2_+(Ad\,P))$$

and, combining with the conformal perturbation $\sigma \in \Xi$, we have the perturbed ASD equation

$$F_{\sigma\pi}\colon \mathcal{A}(P, \delta) \to L^2_{2,loc}(\Omega^2_+(Ad\,P)), \qquad F_{\sigma\pi}(A) = F(A)_+ + \pi(A).$$

Proposition 9.2. *Let $\mathcal{M}(P)$ denote the thickened moduli space of finite energy ASD connections on P with asymptotic boundary value the trivial flat connection θ along the cylindrical ends. Then there exists a $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$, the following holds:*

(i) *a Baire set of perturbation data $(\sigma, \pi) \in \Xi \times \Pi$ exists for which the moduli space $\mathcal{M}(P, \delta) = F^{-1}_{\sigma,\pi}(0)$ is a smooth manifold of dimension*

$$8k - \frac{3}{2}(\chi(M) + sign(M)) + \frac{1}{2}(h^1(\partial M_0) - h^0(\partial M_0))$$

where k is the relative Chern number and $h^i(\partial M_0) = \operatorname{rank} H^i(\partial M_0; ad\theta)$.

(ii) *the moduli space $\mathcal{M}(P, \delta)$ is orientable and has a preferred orientation when we fix an orientation on the homology $H^1(M; \mathbb{R}) \times H^2(M; \mathbb{R}) \times H^2_+(M; \mathbb{R})$.*

For the proof of (9.2), we refer to [30], [9] and [24, 8.5.1].

Remark 9.3. It is worthwhile to point out in the above perturbation of the ASD equation, $F_{\sigma\pi} = 0$, the perturbation takes place in the compact sub-manifold M_0 and so the equation remains the same over the cylindrical ends $\partial M_0 \times \mathbb{R}$. In particular over these ends, the solutions will be invariant under translation.

In addition, we can make the perturbations arbitrarily small and still have the same effect. Therefore if we know that a one dimensional flow of ASD connection decays to a flat connection on $\operatorname{End} M$, the family of perturbed ASD connections will still have the same decay property. □

In our application, $M = Z$ was constructed from the manifold Y given by Proposition 8.8 so $b_3 = \operatorname{rank} H^0(\partial\widetilde{Y}; \mathbb{R}) - 1$ and $b_1(Z) = 0$. Also $b_2(Z) = b^+ + b^- + b^0$, where $b^+(Z) = 0$ and $b^0(Z) = \operatorname{rank}\ H^1(N; \mathbb{R}G)$. Note that $\operatorname{rank} H^i(N; \operatorname{ad}\theta) = 3 \cdot \operatorname{rank} H^i(N, \mathbb{R})$, since θ is the trivial flat connection, and $\operatorname{rank} H^i(N; \mathbb{R}G) = |G| \cdot \operatorname{rank} H^i(N; \mathbb{R})$, since N is trivially G-covered by $\widetilde{Y} \to Y$. Now we substitute these values into the given dimension formula for $\mathcal{M}_k(P)$ in (9.1) and get $\dim \mathcal{M}_k(P) = 8k - 3$. In fact we will concentrate on the case when $k = 1$ and so we get a 5-dimensional moduli space $\mathcal{M}(P)$.

For a closed 4-manifold M with finite group action (M, G), we previously constructed a moduli space $(\mathcal{M}, G)$ of ASD connections with G-action [13], [14]. The techniques used there can be easily adapted to the non-compact case, to produce a group action $(\mathcal{M}, G)$ on the moduli space $\mathcal{M}(P, \delta)$ with decay condition.

First we fix a real analytic structure on M compatible with the group action (M, G) and product-like along the cylindrical ends $\operatorname{End} M = \partial M_0 \times \mathbb{R}$. With respect to this analytic structure, we have a real analytic equivariant metric which is again product-like on the end. The existence of such analytic structure and analytic metric follows from a general argument as in the closed manifold case.

Next we need an equivariant $SU(2)$-bundle $P \to M$ such that $P\,|\,\partial M_0 \times \mathbb{R}^+$ has an equivariant trivialization $(\theta, P\,|\,\partial M_0 \times \mathbb{R}^+)$ with Chern number $c_2(P, \theta) = 1$. In the present situation, $M = Z$, this is taken care of by the following:

Proposition 9.4. *There exists a $Q(8p, q)$-equivariant $SU(2)$-bundle P over Z with an equivariant trivialization $(\theta, P\,|\,\partial Z_0 \times \mathbb{R}^+)$ such that $c_2(P, \theta) = 1$.*

Proof. From the construction, there exists a degree 1 map $\phi: Y \to W$ sending the 3-manfold N to a point in W and the almost space forms $S'/Q(8p)$, $S'/Q(8q)$, $S'/C(2pq)$ to the spherical space forms $S/Q(8p)$, $S/Q(8q)$, $S/C(2pq)$. Let $\widetilde{W}$ denote the universal covering space of W with the free action (W, G). We can compactify $(\widetilde{W}, G)$ by filling in the spherical boundary components $(S^3, Q(4pq))$, $(S^3, Q(8p))$, $(S^3, Q(8q))$, $(S^3, C(2pq))$ by the corresponding linear actions on D^4. Over (Σ, G) we attach a cone and extend the action in an obvious manner to the cone point. The result is a closed 4-dimensional complex $\widehat{W}$ which has a group action $(\widehat{W}, G)$ with isolated singular points. Moreover, there is a degree one, equivariant map $\phi: Z \to \widehat{W}$ sending the cylindrical ends $\partial Z_0 \times \mathbb{R}^+$ to the various cone points in $\widehat{W}$.

To prove (9.4), it suffices to construct an G-equivariant $SU(2)$-principal bundle $P \to \widehat{W}$ over $\widehat{W}$ with relative Chern number $c_2(P) = 1$. For using ϕ we can form the pull back bundle $\phi^* P$ which is equivariant and with pull back trivialization $\phi^*(P \mid \text{cone points}))$ over the ends and $c_2(P, \theta) = \phi^* c_2(P) = 1$.

The product bundle $P \doteq \widehat{W} \times SU(2)$ over $\widehat{W}$ has a G-action only on the base manifold, and Chern number zero. Note that around a singular point, for example a fixed point x_0 of $Q(4pq)$, the action of the isotropy subgroup on the fiber $x_0 \times SU(2)$ is trivial. Let $(Q(4pq), D^4)$ be an invariant neighborhood of x_0. Then there exists an equivariant bundle structure $(Q(4pq), D^4 \times SU(2))$ on $D^4 \times SU(2)$ with nontrivial structure on the fiber $x_0 \times SU(2)$, namely the diagonal action on the base D^4 and the fiber $SU(2)$. When we restrict the above bundle $(Q(4pq), D^4 \times SU(2))$ to the boundary 3-sphere $(Q(4pq), S^3 \times SU(2))$, we have an equivariant bundle isomorphism

$$\alpha \colon S^3 \times SU(2) \to P \mid S^3 = S^3 \times SU(2).$$

To define this isomorphism, we identify the base 3-sphere S^3 with $SU(2)$ and let $\alpha(x, y) = (x, x^{-1} \cdot y)$. Using α as the clutching isomorphism, we can modify the bundle $(Q(4pq), P)$ to a new equivariant bundle $(Q(4pq), P(\alpha))$,

$$P(\alpha) = \left(P - P \mid D^4\right) \cup_\alpha D^4 \times SU(2).$$

After this modification, the Chern number $c_2(P(\alpha)) = 1$.

Note that the above bundle $P(\alpha)$ is equivariant with respect to $Q(4pq)$. To make it equivariant with respect to the whole group G we have to make a corresponding modification on the translate $J x_0$ of x_0 under the action of $J \in Q(8p, q) \setminus Q(4pq)$, $J \neq id$. Since the above procedure is local, we can carry out the same construction at $J x_0$ and obtain a G-equivariant bundle $P(\alpha, J\alpha)$ over $\widehat{W}$ with Chern number $c_2(P(\alpha, J\alpha)) = 2$.

Instead of using the orbit $\{x_0, Jx_0\}$, we can also carry out the same construction around other singular orbits G/G_0 with isotropy subgroup $G_0 = Q(8p)$ or $Q(8q)$. Each of these operations gives a new equivariant bundle with Chern number adding or subtracting $|G/G_0| = q$ or p. Since $(p, q) = 1$ we can write $1 = kp + lq + 2$ for some integers k, l. Then after making the corresponding equivariant bundle modifications around k orbits fixed under $Q(8q)$ and l orbits fixed under $Q(8p)$, we obtain a G-equivariant bundle P over $\widehat{W}$ with $c_2(P) = 1$. □

With the equivariant trivialization given as above, we have the space $\mathcal{A}(P, \delta)$ of connections with decay condition δ along the boundary. Since we have an equivariant trivialization, we can choose the same δ for all the ends and obtain an action of an extended gauge group by $\mathcal{G}(P, \delta, G)$ on this space

$$1 \to \mathcal{G}(P, \delta) \to \mathcal{G}(P, \delta, G) \to G \to 1.$$

In particular, as we factor out the action of $\mathcal{G}(P, \delta)$, we have an induced action of the finite group G on $\mathcal{M}(P, \delta)$.

Next we consider the set $\Xi(G)$ of equivariant, real-analytic metrics on Z which are product-like over the end. Given such a metric we have a curvature section

$$\begin{aligned} F \colon \mathcal{A}(P, \delta) &\longrightarrow L^2_{2,loc}(\Omega^2_+(Ad\,P)) \\ A &\longmapsto F_A \end{aligned}$$

which is equivariant with respect to the action of $\mathcal{G}(G, P, \delta)$.

As above, we let Z_0 be the compact, G-invariant submanifold obtained from Z by deleting its cylindrical ends. Then by choosing homotopically trivial, framed, loops in Z_0/G, we have a collection of framed loops $\{g\gamma_i \mid i = 1, \dots, n, g \in G\}$ in Z_0 which are disjoint from each other and permuted by the group action. Using this collection of loops $\{g\gamma_i\}$ we can perform the previous Wilson loop perturbation $\pi: \mathcal{A}(P, \delta) \to L^2_{2,loc}(\Omega^2_+(Ad\,P))$ in an equivariant manner.

In [13], we studied a method for perturbing the curvature section into equivariant general position. Let $\Psi: H_1 \to H_2$ be an equivariant G-Fredholm map between G-Hilbert spaces H_1, H_2 with $\Psi(0) = 0$. Then there is a decomposition $H_1 = \operatorname{Ker} T \oplus H_1'$, $H_2 = \operatorname{Im} T \oplus H_2'$ where $T = (d\Psi)_0$. Furthermore, the map Ψ is locally equivalent to $T + \Phi$, where $\Phi: H_1 \to H_2'$, $\Phi(0) = 0$, and $d\Phi\,|_0$ (c.f. [13, Lemma 2.7]). Associated to Φ, we have a map

$$\hat{\Phi}: \operatorname{Ker} T \hookrightarrow H_1 \xrightarrow{\Psi} H_2' \to \operatorname{Coker} T$$

between the finite dimensional G-vector spaces. The key idea [13, Lemma 2.9] is that deforming Φ to equivariant general position amounts to perturbing $\hat{\Phi}$ to Bierstone general position [2].

Suppose A is a connection in $\mathcal{A}(P, \delta)$ where the curvature section $F: \mathcal{A}(P, \delta) \to L^2_{2,loc}(\Omega^2_+(Ad\,P))$ fails to be in equivariant general position. By [8, Lemma 2C.1 and 2C.2], $\operatorname{Ker} dF_A = \operatorname{Ker} D_A$ is generated by the holonomy of a collection of framed loops $\{g\gamma_i\}$ and a finite dimensional subspace of sections over these framed loops which maps onto the cokernel space $\operatorname{Coker} F_A = \operatorname{Coker} D_A$. . It follows that there exists an equivariant map from $\operatorname{Ker} D_A$ to $\operatorname{Coker} D_A$, which we can write in the form $\pi = \sum \omega_+ \otimes p_i[\gamma_i(v)]$. Then it is not difficult to see that there exists a Baire set of perturbations $(\Xi \times \pi)^G$ for which the perturbed curvature section $F_\sigma(A) + \pi(A)$ is in general position with respect to the zero section. This is the equivariant analogue of the first assertion in Proposition 9.2.

Remark 9.5. In [6, §2], Donaldson used a similar method to perturb away much of the stratum $\mathcal{M}_0(P)$ of flat $SU(2)$-connections on M. This step is unnecessary in our case, since connections in $\mathcal{M}(P)$ are product-like near the boundary and $\pi_1(Z)$ is normally generated by $\pi_1(\partial Z)$. For this reason, the holonomy map $p: \pi_1(Z) \to SU(2)$ for a limiting flat $SU(2)$-connection is determined by its restriction $p \mid \pi_1(\partial Z)$ which is trivial. It follows that p is trivial and so $\mathcal{M}_0(P)$ consists of one element, namely the trivial flat connection.

10. The Proof of Theorem A

We will now use the equivariant moduli space $(\mathcal{M}(P), G)$ described in the last section to prove our main result. Recall that on the four manifold Z the action of $Q(8p, q)$ is not free but has isolated singular points $\{x_0\}$ with isotropy group $Q(4pq)$. In [14, §3] we described the Taube's construction of concentrated connections in an equivariant manner. Using D. Austin's work [1] on equivariant instantons on S^4 and a background flat connection on Z, we produce a 1-dimensional $Q(4pq)$-fixed point stratum $\mathcal{N} = \operatorname{Fix}(\mathcal{M}(P), Q(4pq))$ in $\mathcal{M}(P)$ (see also [3, §1]). One one end of $\mathcal{N}$, we have a particle-like connection emitting from x_0. This singular stratum $\mathcal{N}$ has an equivariant normal slice which is isomorphic to the linear action $(Q(4pq), D^4)$ on the 4-disk around x_0.

From this construction, it follows that $\mathcal{N}$ can be interpreted as 1-parameter family of ASD connections A_t, invariant under the $Q(4pq)$-action (not just fixed modulo the gauge group transformations in $\mathcal{A}(P;\delta)/\mathcal{G}(P,\delta)$). As we perturb this submoduli space $\mathcal{N}$, the resulting space still consists of connections which are invariant with respect to the group action $Q(4pq)$ on P and Z.

For each of the $Q(4pq)$-fixed point $x_0 \in \mathrm{Fix}(Z, Q(4pq))$ we have a corresponding 1-parameter family $\mathcal{N}(x_0) = \{A_t\}$ of ASD connections emitting from x_0. Such a family $\mathcal{N}(x_0)$ cannot concentrate itself again to become a particle-like connection at some other point $y_0 \in Z$. This is because y_0 has to be a fixed point of $Q(4pq)$, and we have the following "no-return" argument (compare [14, Lemma 17]):

Proposition 10.1. *The $Q(4pq)$-fixed stratum $\mathcal{N}(x_0)$ cannot converge to a particle-like connection at a $Q(4pq)$-fixed point y_0.*

Proof of (10.1). Let J denote a fixed order 4 element in $Q(8p,q)$ which maps to the nontrivial element in $Q(8p,q)/Q(4pq) = \mathbb{Z}/2$. Note that J^2 is the order two element in $Q(4pq)$. Given a $Q(4pq)$-fixed point $x_0 \in \mathrm{Fix}(Z, Q(4pq))$ the action of J brings x_0 to another $Q(4pq)$-fixed point $z_0 = Jx_0$. If $p\colon Q(4pq) \to SU(2)$ denote the isotropy representation of $Q(4pq)$ on the normal slice of x_0, then the composite

$$p^J\colon Q(4pq) \xrightarrow{c_J} Q(4pq) \xrightarrow{p} SU(2),$$

where c_J denotes conjugation by J, gives us the isotropy representation at z_0. In our construction of Y, we have copies of the spherical space form $S/Q(4pq)$ with the same orientation. It follows that on Z the isotropy representation at any $Q(4pq)$-fixed point $y_0 \in \mathrm{Fix}(Z, Q(4pq))$ is isomorphic to either p or p^J.

In order for the moduli space $\mathcal{N}(x_0)$ emitting from x_0 to converge at some other fixed point x_0, the normal slice representations at x_0 and y_0 have to differ from each other by an orientation reversing isomorphism. From representation theory, it is impossible to have an intertwining operation between p and itself, or between p and p^J, or between p^J and itself, which is orientation reversing. □

We now need the fact that the moduli space $\mathcal{M}(P,\delta)$ of (9.2) has a compactification $\overline{\mathcal{M}(P,\delta)}$ (see [11], [30]). To describe this compactification, we consider a sequence $\{A_n\}$ of ASD connections in $\mathcal{M}(P,\delta)$ which fails to converge. Then there are two possibilities: in the first case, the curvature can concentrate in the neighborhood of a point, leading to a particle-like connection at the point. In the situation of instanton number 1 or $k = 1$ in Proposition 9.2, we can have at most a single instanton bubbling off from the concentration at a point. This can be accounted for by providing a copy of the original manifold M in $\overline{\mathcal{M}(P,\delta)}$ and attaching this copy to a neighborhood $M \times (0,\lambda)$ in $\mathcal{M}(P,\delta)$. In the second case a nontrivial amount of curvature F_{A_n} can concentrate on a region in the end $\mathrm{End}\, M \cong \partial M_0 \times \mathbb{R}^+$. The connection A_n over this region looks like an ASD connection on $\partial M_0 \times \mathbb{R}^+$ asymptotic to flat connections a, b at two ends, or in other words, an element in $\mathcal{M}(\partial M_0 \times \mathbb{R}; a, b) = \mathcal{M}(a,b)$ in the notation of Floer. As the connection $\{A_n\}$ moves towards the boundary $\overline{\mathcal{M}(P,\delta)} \setminus \mathcal{M}(P,\delta)$, this region moves towards the end of M. In terms of $\mathcal{M}(a,b)$, this motion can be thought of as the natural action by R_+ on $\partial M_0 \times \mathbb{R}^+$. Thus to account for these phenomena, we have a stratified space structure in $\overline{\mathcal{M}(P,\delta)} \setminus (\mathcal{M}(P,\delta) \cup M)$ with the strata consisting of products:

$$(10.2) \qquad \mathcal{M}(M,a_0) \times \widehat{\mathcal{M}}(a_0,a_1) \times \mathbb{R} \times \widehat{\mathcal{M}}(a_1,a_2) \times \mathbb{R} \times \ldots \times \widehat{\mathcal{M}}(a_{l-1},\theta) \times \mathbb{R},$$

where $\widehat{\mathcal{M}}(a_i,a_{i+1})$ is the quotient of $\mathcal{M}(a_i,a_{i+1})$ by the translation symmetry.

The above theory of compactification can be easily extended to our equivariant setting. For the stratum of particle-like connections and also those in (10.2) are naturally endowed with group actions.

We return now to the situation where $M = Z$, and consider the 1-dimensional, $Q(4pq)$-fixed, subspace $\mathcal{N}$ in $\mathcal{M}(P,\delta)$. Since this fixed subspace has a nontrivial tangent space, it follows by equivariant general position that it cannot terminate in the interior of $\overline{\mathcal{M}(P,\delta)}$ or in other words it has to move towards the boundary $\overline{\mathcal{M}(P,\delta)} - \mathcal{M}(P,\delta)$. In (10.1), we have eliminated the possibility that the other end of $\mathcal{N}$ becomes a particle-like connection. Thus, as we move along $\mathcal{N}$, we have a flow of ASD connection $\{A_t\}$ with energy (curvature) concentrating on some of the cylindrical ends $(Q(8p), S' \times \mathbb{R}^+)$, $(Q(8q), S' \times \mathbb{R}^+)$, $(C(2pq), S' \times \mathbb{R}^+)$, $(1, N \times \mathbb{R}^+)$, or $(G, \Sigma \times \mathbb{R}^+)$.

Proposition 10.3. *As $t \to \infty$, the family of connections $\{A_t\} = \mathcal{N}$ has the energy $\frac{1}{8\pi^2}\int tr(F_{A_t} \wedge * F_{A_t})$ concentrated on one of the cylindrical end $(G, \Sigma \times \mathbb{R}^+)$ and for the rest of the manifold $Z \setminus \Sigma \times \mathbb{R}^+$ the energy tends to zero.*

Proof. Suppose on the contrary a certain amount of the energy appears in an end whose isotropy subgroup $H \neq G = Q(8p,q)$. For definiteness in the argument, let us assume that this is the end $(1, N \times \mathbb{R}^+)$ with the trivial isotropy subgroup $H = \{1\}$. Since the family $\{A_t\}$ is invariant, under the subgroup $Q(4pq)$, an equal amount of the energy has to appear in the translation $g(N \times \mathbb{R}^+)$ of $N \times \mathbb{R}^+$ for every element $g \in Q(4pq)$. In terms of the equivariant compactification of (10.2), the family $\{A_t\}$, as $t \to \infty$, enters a neighborhood of the boundary which has the form $\mathcal{M}(Z,\alpha) \times (G \times \mathcal{M}(\alpha,\theta))$ where α is some nontrivial flat connection of S'. In fact, if we think of $Q(4pq) \times \mathcal{M}(\alpha,\theta)$ as the product $\prod \mathcal{M}(\alpha,\theta)$ of $4pq$ copies of $\mathcal{M}(\alpha,\theta)$, then $\{A_t\}$ is contained in the diagonal of this product space. Note each $\mathcal{M}(\alpha,\theta)$ has an action of $\mathbb{R}$ induced by the translation symmetry of $N \times \mathbb{R}$, i.e. $\mathcal{M}(\alpha,\theta) = \widehat{\mathcal{M}}(\alpha,\theta) \times \mathbb{R}$. Their product $\prod \mathcal{M}(\alpha,\theta)$ has an action of $\prod \mathbb{R}$, i.e. $\mathcal{M}(\alpha,\theta) = \prod \widehat{\mathcal{M}}(\alpha,\theta) \times \prod \mathbb{R}$ and the t-parameter of the family $\{A_t\}$ coincides with the diagonal action in $\prod \mathbb{R}$. Thus, normal to the family $\mathcal{N} = \{A_t\}$, there is a representation of $Q(4pq)$ on $\mathbb{R}^{4pq-1}$. However, by the construction of $\mathcal{N}$ as a particle-like connection emitting from x_0, the normal slice representation of $Q(4pq)$ is a complex 2-dimensional, faithful, representation. Since the normal slice representation can be expressed as an equivariant index, this is the same all the way along the family. However, by a dimension count, this complex representation cannot contain $\mathbb{R}^{4pq-1}$ as a subrepresentation.

The above argument rules out the possibility that a nontrivial amount of energy of $\{A_t\}$ is contained in the end $(1, N \times \mathbb{R}^+)$. Similar arguments also rule out the possibility that a certain amount of energy of $\{A_t\}$ goes down the cylinders $(Q(8p), S' \times \mathbb{R}^+)$, $(Q(8q), S' \times \mathbb{R}^+)$, or $(C(2pq), S' \times \mathbb{R}^+)$. In each of these cases we examine the normal slice representation of $\mathcal{N} = \{A_t\}$ and show that it is not compatible with the slice representation from the cylindrical ends.

Since the manifold Z has two cylindrical ends $\pm\Sigma \times \mathbb{R}^+$ stabilized by G, the possibility remains that the energy of $\{A_t\}$ is spread out around these two ends. As Σ is simply connected, a flat $SU(2)$-connection on Σ is, up to a gauge transformation, a product connection. It follows that the energy is an integer and the smallest of such integers is 1. From the fact that the integral

$$\frac{1}{8\pi^2}\int tr\,(F_A \wedge F_A) \geq 0$$

for an ASD connection A, it follows that the energy can concentrate only on one of the cylindrical end $(G, \Sigma \times \mathbb{R}^+)$ and on the rest of Z it approaches zero as $t \to \infty$. This proves (10.3). □

Let $\Sigma = \Sigma \times *$ be a slice of the cylindrical end $(G, \Sigma \times \mathbb{R}^+)$, invariant under the group action. Let x be a point on Σ and let $\{gx \mid g \in Q(4pq)\}$ denote the orbit of x under the translation of the subgroup $Q(4pq)$. Let $E = P \times_{SU(2)} \mathbb{C}^2$ be the complex 2-plane bundle associated to the principal $SU(2)$-bundle P and the the standard $SU(2)$-representation on $\mathbb{C}^2$. Then P has a G-bundle structure and so does the corresponding 2-plane bundle E. In particular there exists an isomorphism

$$b(g)\colon E_x \to E_{g\cdot x}$$

which brings the fiber E_x over x to $E_{g\cdot x}$ over $g \cdot x$.

On the other hand, for each t, the $SU(2)$-connection A_t on P gives rise to a connection A'_t on E. If we connect x to $g \cdot x$ by an arc $\gamma \subset \Sigma$, and parallel translate from the fiber E_x to $E_{g\cdot x}$ via A'_t, we obtain another isomorphism

$$T(A_t, \gamma)\colon E_x \to E_{g\cdot x}.$$

By taking the composite, we obtain an automorphism

$$\phi_t(g) = T(A_t, \gamma)^{-1} \circ b(g)\colon E_x \to E_x \tag{10.4}$$

of the same fiber E_x, yielding an element of $SU(2)$.

In general, the above automorphism $\phi_t(g)$ depends on the choice of the path γ. In the present situation, as $t \to \infty$, the connection $A_t \mid \Sigma \times *$ is asymptotically flat and we define $\phi(g)$ by

$$\phi(g) = \lim_{t\to\infty} \phi_t(g) \in SU(2).$$

Then

Lemma 10.5. *The definition of $\phi(g)$ is independent of the choice of path γ joining x to $g \cdot x$. In addition, $\phi(gg') = \phi(g) \cdot \phi(g')$ for any $g, g' \in Q(4pq)$.*

Proof. If γ' is another arc connecting x to $g\cdot x$, then, because Σ is simply connected, $\gamma\cdot(\gamma')^{-1}$ bounds a singular disk Δ in Σ. By the generalized Gauss-Bonnet theorem, the automorphism $T(A_t, \gamma) \cdot T(A_t, \gamma')^{-1}$ obtained by taking parallel translation around the loop $\gamma \cdot (\gamma')^{-1}$ can be computed by integrating the curvature 2-form

F_{A_t} over the disk Δ. As $t \to \infty$, the curvature $F_{A_t} \,|\, \Delta$ tends to zero and so $T(A_t, \gamma) = T(A_t, \gamma')$.

For $g, g' \in Q(4pq)$, let γ be a path joining x to $g\dot{x}$ and γ' a path joining x to $g' \cdot x$. Then $g\gamma' \cdot \gamma$ is a path joining x to $(gg') \cdot x$, so

$$\phi(gg') = T(A, g\gamma' \cdot \gamma)^{-1} b(gg') = T(A, \gamma)^{-1} T(A, g\gamma' \cdot)^{-1} b(g) b(g')$$

But, for any $g \in Q(4pq)$, the connection $\{A_t\}$ is fixed under the action of g. Therefore we have a commutative diagram

$$\begin{array}{ccc} E_x & \xrightarrow{T(A,\gamma')} & E_{g'x} \\ {\scriptstyle b(g)_x}\downarrow & & \downarrow{\scriptstyle b(g)_{g'x}} \\ E_{gx} & \xrightarrow{T(A,g\gamma')} & E_{gg'\cdot x} \end{array}$$

and it follows that

$$\phi(gg\prime) = \phi(g) \cdot \phi(g\prime).$$

□

This procedure for obtaining a representation

$$\phi \colon Q(4pq) \to Aut(E_x) = SU(2)$$

can be extended to the manifold $Z \setminus (\Sigma \times \mathbb{R}^+)$ since the energy of $\{A_t\}$ approaches zero as $t \to \infty$. Note that

$$\pi_1(Z \setminus (\Sigma \times \mathbb{R}^+)) = \pi_1(Z)$$

is not simply connected and so this procedure leads to a representation

$$\psi \colon \pi_1(Z) \rtimes Q(4pq) \to SU(2).$$

On the other hand, as explained in (9.5), $\pi_1(Z)$ is normally generated by the fundamental group $\pi_1(N)$ of the 3-manifold $(1, N)$ on the boundary. From the construction of $\mathcal{M}(P, \delta)$, the holonomy $\psi \,|\, \pi_1(N)$ restricted to $\pi_1(N)$ is trivial. Hence $\psi \,|\, \pi_1(Z) = 1$ and so the representation $\psi \colon \pi_1(Z) \rtimes Q(4pq) \to SU(2)$ in fact factors through $\phi \colon Q(4pq) \to SU(2)$.

Lemma 10.6. *The definition of $\psi \colon \pi_1(Z) \rtimes Q(4pq) \to SU(2)$ is independent of the choice of base-point $x \in Z$, up to equivalence of representations.*

Proof. If $x, x' \in Z$ are any two base-points, we connect them by a path u and obtain the relation

$$b(g)_{x'} \circ T(A, u)_x = T(A, gu)_{gx} \circ b(g)_x.$$

If γ (resp. γ') are paths joining x to gx (resp. x' to gx'), then

$$T(A, \gamma)_x = T(A, gu)^{-1}_{gx} \circ T(A, \gamma')_{x'} \circ T(A, u)_x.$$

Therefore

$$\begin{aligned} \phi_x(g) &= T(A, \gamma)^{-1}_x b(g)_x = T(A, u)^{-1}_x T(A, \gamma')^{-1}_{x'} T(A, gu)_{gx} b(g)_x \\ &= T(A, u)^{-1}_x T(A, \gamma')^{-1}_{x'} b(g)_{x'} T(A, u)_x = T(A, u)^{-1}_x \phi_{x'}(g) T(A, u)_x \end{aligned}$$

and the two different base-points give equivalent representations. □

From this Lemma, we can use any convenient $Q(4pq)$-orbit $\{g \cdot x \,|\, g \in Q(4pq)\}$ in $Z \setminus (\Sigma \times \mathbb{R}^+)$ and we recover the representation ϕ. This leads to the following result:

Proposition 10.7. *Let $\phi: Q(4pq) \to SU(2)$ be defined as above. Then ϕ is an irreducible, faithful 2-dimensional representation of $Q(4pq)$. This representation does not extend to any $U(2)$ representation of G.*

Proof. To define ϕ, we can take our base-point $x \in Z$ to be the $Q(4pq)$-fixed point x_0 from which the particle-like connection starts to emerge in the definition of $\mathcal{N} = \mathcal{N}(x_0)$. From the equivariant Taubes construction, it is not difficult to see that $\phi: Q(4pq) \to SU(2)$ is given by the bundle automorphism of the equivariant bundle $(Q(4pq), E)$ over the fixed point x_0.

Since $J \in G \setminus Q(4pq)$ has the property that $J^2 \neq 1$ in $Q(4pq)$, we see that any extension of ϕ to a $U(2)$-representation of G would also be fixed-point free. But G has no fixed-point free $U(2)$-representations, hence no such extension of ϕ exists. □

Since the end $\Sigma \times \mathbb{R}^+$ is stabilized by the group G, the image $J \cdot \mathcal{N} = \{J^* \cdot A_t\}$ of $\mathcal{N}$ under the action of $J \in G \setminus Q(4pq)$ has similar property with respect to $\Sigma \times \mathbb{R}^+$. More precisely, for $J^* A_t$, the energy is also distributed in the cylinder $\Sigma \times \mathbb{R}^+$ and becomes zero elsewhere as $t \to \infty$. In other words, the 1-parameter family of connections $\{J^* A_t\}$ also enters the neighborhood $\mathcal{M}(\Sigma \times \mathbb{R})$. In addition, using the bundle automorphism $(Q(4pq)^J, P)$ and parallel translation via $J^* A_t$, we obtain a representation $\phi^J: Q(4pq) \to SU(2)$. Since

$$\begin{aligned} T(J^* A_t, \gamma)^{-1} b(g) &= b(J)^{-1} T(A_t, \gamma)^{-1} b(J) b(g) \\ &= b(J)^{-1} T(A_t, \gamma)^{-1} b(JgJ^{-1}) b(J) \\ &= b(J)^{-1} \phi(JgJ^{-1}) b(J), \end{aligned}$$

we see that the representation ϕ^J is equivalent in $SU(2)$ to the composition

$$Q(4pq) \xrightarrow{c_J} Q(4pq) \xrightarrow{\phi} SU(2)$$

of ϕ with the self automorphism c_J given by conjugation by J.

To proceed further, we need the following result of Taubes [30, Prop. 10.1]:

Proposition 10.8. *Let M be a negative definite 4-manifold with a cylindrical end $End M = \partial M_0 \times \mathbb{R}^+$. Let $\mathcal{M}(P, \delta)$ be the moduli space of ASD connections on $SU(2)$-bundle P with decay condition δ as defined before. Then there is an open set $U \subset \mathcal{M}(P, \delta)$ with the property that for some $\lambda_1 > 0, U$ is diffeomorphic to $M \times (0, \lambda_1)$ and isotopic in $B(P, \delta)$ to the image of the Taubes map $T: M \times (0, \lambda_1) \to B(P, \delta)$. If $\{A_n\} \subset \mathcal{M}(P, \delta) \setminus U$ has no convergent subsequence, then for all $a < \infty$*

$$\overline{\lim_{n \to \infty}} \, [\sup\{|F_{A_n}(x)| : x \in \partial M_0 \times [0, a]\}] = 0.$$

In our application, the fundamental group of some of the boundary components of Z may have nontrivial $SU(2)$-representation. However, we have already shown that we can restrict our attention to the end $\Sigma \times \mathbb{R}^+$ which is simply connected. Applying Proposition 10.8 to this situation we see that for any $a < \infty$ and some sequence $A_n = A_{t_n}$ of connections in $\mathcal{N} = \{A_t\}$, we have

$$\overline{\lim_{n \to \infty}} \{\sup\{|F_{A_n}(x)| : x \in \Sigma \times [0, a]\} = 0.$$

Since Σ is simply connected, any flat connection over $\Sigma \times [0, a]$ is a product connection. In particular the family of connections $\{A_n\}$ approaches the product connection θ on $\Sigma \times [0, a]$ for any $a > 0$ and approaches a δ-like connection at infinity.

It follows that over the compact submanifolds $\Sigma \times [0, a]$, the two families of connections A_n and $J^* A_n$ converge to the same flat connection θ with trivial holonomy. We can therefore find gauge transformations $\{\alpha_n\}$ on P such that the distance

$$\| \alpha_n^*(J^* A_n) - A_n \|$$

tends to zero as $t \to \infty$. Since our representations ϕ and ϕ^J are defined over a compact space $\Sigma \times SU(2)$, we may assume without loss of generality that $\lim_{n\to\infty} \alpha_n$ converges to a single gauge transformation α. Without loss of generality, we can assume that $\alpha(x_0) = id$, where x_0 is a $Q(4pq)$-fixed point in Z associated to $\mathcal{N}$.

Let $J' = (b(J) \circ \alpha)$, considered as an extended bundle automorphism of E covering the action of J on Z. Consider the representation $\phi' \colon Q(4pq) \to SU(2)$ defined by the formula

$$\phi'_x(g) = \lim_{n\to\infty} T((J')^* A_n, \gamma)_x^{-1} b(g)_x$$

with respect to a given base-point $x \in Z$.

Since $\alpha(x_0) = id$, we have the simpler expression

$$\phi'_{x_0}(g) = \lim_{t\to\infty} T(J^* A_n, \gamma)^{-1} b(g)$$

at the base-point x_0. In this formula the right-hand side is just the definition of ϕ^J, hence ϕ'_{x_0} is equivalent to the J-conjugate representation $\phi \circ c_J$.

On the other hand, when we use a base-point $x \in \Sigma$ it follows that $\phi'_x(g) = \phi_x(g)$ for all $g \in Q(4pq)$, since $(J')^* A_n)$ converges to A_n.

Now by Lemma 10.6 the representations ϕ'_{x_0} and ϕ'_x are equivalent. Therefore ϕ and its J-conjugate $\phi \circ c_J$ are equivalent as $U(2)$ representations.

Suppose $B \in U(2)$ is any matrix such that $\phi(g) = B\phi(JgJ^{-1})B^{-1}$ for all $g \in Q(4pq)$. Since B^2 commutes with the irreducible representation ϕ, Schur's Lemma implies $B^2 = \pm 1$. We can ensure that $B^2 = -1$ by multiplying B with the central matrix $(i \cdot I)$ if necessary. As a result, we have a $U(2)$-representation $\{\phi, B\}$ of G, contradicting the representation theory of G by Proposition 10.7. $\square$

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