Formal Differential Graded Algebras and Homomorphisms

by

Martin Arkowitz

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
5300 Bonn 3

MPI 86-5

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1. Introduction

As a result of the Sullivan approach to rational homotopy theory, a new and interesting class of spaces has emerged. These are the formal spaces whose very definition requires the theory of minimal models of differential graded algebras. Although not all spaces are formal, the list is large and includes spheres (more generally, suspensions), classifying spaces, Lie groups, Eilenberg-MacLane spaces (more generally, H-spaces), locally symmetric spaces, compact Kähler manifolds, and many homogeneous spaces (including projective spaces) (see [Fe] and [Su 2]). In addition, wedges and products of formal spaces are formal. We take the point of view that formal spaces are a reasonable class of spaces for which to do certain aspects of rational homotopy theory- large enough to include many important examples yet restricted enough to yield concrete results. We are particularly interested in the relationship between the homotopy classes of maps of one formal space into another and the homomorphisms of their rational cohomology algebras (§§ 3,4,and 5) and in the relationship between the group of homotopy classes of homotopy equivalences of a formal space and the automorphisms of its rational cohomology algebra (§ 6). In considering these questions, the formal maps of one formal space to another play a crucial role.

Although the motivation for this paper is topological, it is possible and even advantageous to carry out the investigation in the category of differential graded algebras (DGAs) since rational homotopy theory has shown this category to be closely related to the category of rational topological spaces. We begin in § 2 with several equivalent definitions of formal DGAs. In § 3 we consider homotopy classes of DGA maps of a minimal DGA M into a DGA B. When M and B are formal we define formal homotopy classes of DGA maps from M to B and prove that these are in one-one correspondence with $\mathrm{Hom}(H^*(M), H^*(B))$ (Proposition 3.2). This has several consequences. It shows that any cohomology homomorphism $H^*(M) \longrightarrow H^*(B)$ can be realized as a DGA map $M \longrightarrow B$ when M and B are formal. It also enables us in § 4 to give a new proof of a theorem of Vigué-Poirrier which asserts that any cohomology homomorphism is realizable under suitable connectivity conditions on M and dimension conditions on B. An obstruction theory for formality of maps is then developed in § 5. We give conditions for all the obstructions to vanish, and hence conditions for all maps from one formal space into another to be formal. In § 6 we consider the group $E(M)$ of homotopy classes of homotopy equivalences of a minimal DGA M. When M is formal we apply Proposition 3.2 to express $E(M)$ as a semi-direct product of $\mathrm{Aut}\, H^*(M)$ and $E_*(M)$, the subgroup of $E(M)$ whose elements induce the identity on cohomology. We next use a result of Baues in § 7 to translate results on the function $I:[X,Y] \longrightarrow \mathrm{Hom}(H^*(Y;\mathbb{Q}),H^*(X;\mathbb{Q}))$ obtained in earlier sections into results on the suspension function $\Sigma:[X,Y] \longrightarrow [\Sigma X,\Sigma Y]_{\text{co-H}}$, where the latter is the set of homotopy classes of co-H-maps from ΣX to ΣY. We prove several diverse results about Σ. In § 8 there are some brief remarks on topological implications, generalizations, and duality. The paper concludes with an appendix. Here we sketch an inductive procedure for obtaining information on $\mathcal{E}_\#(\mathcal{M})$ the subgroup of $\mathcal{E}(\mathcal{M})$ consisting of elements which induce the identity on homotopy groups, for any minimal DGA $\mathcal{M}$.

Throughout this paper the term <u>rational space</u> refers to a 1-connected topological space X of the homotopy type of a simplicial complex such that $\pi_n(X)$ is a vector space over the rationals $\mathbb{Q}$ for every $n \geq 1$. A <u>differential graded algebra</u> $\mathcal{A}$ <u>(DGA)</u> is a graded commutative algebra over $\mathbb{Q}$ with a differential $d:\mathcal{A} \to \mathcal{A}$ of degree 1 such that $\mathcal{A}$ is 1-connected $(H^0(\mathcal{A}) = \mathbb{Q},\ H^1(\mathcal{A}) = 0)$. A <u>minimal</u> DGA $\mathcal{M}$ is a DGA which is free as a graded (commutative) algebra and whose differential d is decomposable, i.e., for every positive - dimensional element x, dx is a sum of positive-dimensional elements. Our basic reference for rational homotopy theory and DGAs is [GM].

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2. Formal Differential Graded Algebra

It is well-known that for every DGA A there is an associated minimal DGA denoted M_A and a homomorphism $\rho : M_A \longrightarrow A$ of DGAs such that the induced cohomology homomorphism $\rho^*:H^*(M_A) \longrightarrow H^*(A)$ is an isomorphism [GM,p. 116]. Then M_A is determined up to isomorphism and ρ is determined up to homotopy of DGA maps. For DGAs A and B, a DGA map $\varphi:A \longrightarrow B$ is called a weak equivalence if $\varphi^*:H^*(A) \longrightarrow H^*(B)$ is an isomorphism. Two DGAs A and B are said to have the same homotopy type, written $A \equiv B$, if there is a finite sequence of DGAs $A_0, A_1, \ldots, A_n$ such that $A = A_0$, $B = A_n$ and for each $i = 0,1,\ldots,n-1$ there exists a weak equivalence $A_i \longrightarrow A_{i+1}$ or a weak equivalence $A_{i+1} \longrightarrow A_i$. We write $A \approx B$ if A is isomorphic to B. We note that the cohomology $H^*(A)$ of a DGA A can be regarded as a DGA with trivial differential. Finally, let $Z^*(A)$ denote the DGA of cocycles of A and $\pi:Z^*(A) \longrightarrow H^*(A)$ the natural projection.

Proposition 2.1 The following statements are equivalent for a DGA A and its minimal model M_A with canonical map $\rho:M_A \longrightarrow A$

(i) There exists a DGA map $\lambda:M_A \longrightarrow H^*(A)$ such that $\lambda^* = \rho^*$.

(ii) There exists a DGA map $\Psi:M_A \longrightarrow H^*(M_A)$ such that $\Psi^* = 1$, the identity homomorphism.

(iii) There exists a weak equivalence $\mu: M_A \longrightarrow H^*(A)$.

(iv) $M_A \approx M_{H^*(A)}$

(v) $A \equiv H^*(A)$

(vi) The projection $\pi: Z^*(M_A) \longrightarrow H^*(M_A)$ can be extended to a DGA map $\Psi: M_A \longrightarrow H^*(M_A)$.

Proof. (i) $\Longleftrightarrow$ (ii) is clear. We note that $A \equiv B \Longleftrightarrow M_A \approx M_B$ (see [GM, p. 128]). Thus (iii) $\Rightarrow$ (iv) and (iv) $\Longleftrightarrow$ (v). But (i) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (iii) are both obvious. Now we show (iii) $\Rightarrow$ (i). Define λ to be the composition

$$M_A \xrightarrow{\mu} H^*(A) \xrightarrow{(\mu^*)^{-1}} H^*(M_A) \xrightarrow{\rho^*} H^*(A).$$

This establishes the equivalence of statements (i) through (v). It is also easily shown that (vi) and (ii) are equivalent. This completes the proof.

Definition 2.2 A DGA A which satisfies any of the six equivalent conditions of Proposition 2.1 is called formal.

Proposition 2.1 collects several definitions of formality that have been used by various authors. (iii) and (iv) appear in [LS, p. 111], (ii) is in [DGMS, p. 260], (iii) is in [Su2, p. 315], (iv) is in [FH, p. 578], and (v) is in [HS, p. 236].

Another definition which appears in [Ru, p. 98] turns out not to be equivalent:

Lemma 2.3 If there is a weak equivalence $\varphi: H^*(A) \longrightarrow A$, then A is formal. The converse does not hold.

Proof. Clearly the weak equivalence φ induces a map $M_{H^*(A)} \longrightarrow M_A$ which is a weak equivalence and hence an isomorphism [GM, p. 128]. Thus A is formal. For a counterexample to the converse consider $A = M = \Lambda(x_2, x_3)$, the free commutative algebra on two generators, x_2 of degree 2 and x_3 of degree 3, with $dx_2 = 0$ and $dx_3 = x_2^2$. (M is the minimal model of S^2.) $H^*(M) = \Lambda(y_2)/(y_2^2)$ where y_2 has degree 2 and (y_2^2) is the ideal generated by y_2^2. Clearly A is formal. If $\varphi: H^*(A) \longrightarrow A$ is a weak equivalence, then $\varphi y_2 = qx_2$ for some non-zero rational q. Therefore

$$0 = \varphi(y_2^2) = (\varphi y_2)^2 = q^2 x_2^2.$$

Thus $x_2^2 = 0$ which is a contradiction.

As is well-known, Definition 2.2 leads to a definition of formality of spaces. A 1-connected space X (of the homotopy type of a simplicial complex) is called formal if $A_{PL}(X)$, the PL-DeRham algebra over $\mathbb{Q}$ [GM, chap. VIII] is formal in the sense of Definition 2.2. This notion of formality of spaces has several equivalent formulations in terms of other rational algebraic invariants of X, such as, for example, the Quillen minimal model L_X (see [F],[NM],[LS]). In this paper we concentrate on formality within the DGA category.

3. Formal Differential Graded Algebra Homomorphisms

Let A and B be formal DGAs with DGA maps $\Psi : M_A \longrightarrow H^*(M_A)$ and $\Psi' : M_B \longrightarrow H^*(M_B)$ which induce the identity on cohomology. Let $\alpha : A \to B$ be a DGA map and $\hat{\alpha} : M_A \longrightarrow M_B$ the corresponding DGA map of associated minimal algebras.

Definition 3.1: $\alpha : A \to B$ is called a formal map if the following diagram commutes up to homotopy

$$\begin{tikzcd} M_A \arrow[r, "\hat{\alpha}"] \arrow[d, "\Psi"] & M_B \arrow[d, "\Psi'"] \\ H^*(M_A) \arrow[r, "\hat{\alpha}^*"] & H^*(M_B) \end{tikzcd}$$

(see [GM, Chapter X] for homotopy of DGA maps). This definition depends on the choice of maps Ψ and Ψ' and it would be more precise to call α $[\Psi]$-$[\Psi']$-formal, where $[\Psi]$ and $[\Psi']$ denote the homotopy class of Ψ and Ψ' respectively. We can call $\alpha : A \to B$ formalizable if α is $[\Psi]$-$[\Psi']$-formal for some maps Ψ and Ψ'. We shall deal with formal maps with the understanding that this definition is relative to a fixed choice of Ψ and Ψ'.

If $\alpha, \beta : A \to B$ are homotopic, $\alpha \simeq \beta$, and α is formal, then β is formal. Also notice that α is formal if and only if $\alpha\rho$ is formal, where $\rho : M_A \longrightarrow A$ is the canonical homomorphism. Thus, instead of considering formality of DGA maps $\mathcal{A} \to \mathcal{B}$, we will consider formality of DGA maps $\mathcal{M}_A \to \mathrm{B}$ or rather DGA maps $\mathcal{M} \to \mathcal{B}$, where $\mathcal{M}$ is a minimal DGA. Let $[\mathcal{M},\mathcal{B}]$ denote the homotopy classes of DGA maps $\mathcal{M} \to \mathcal{B}$ and let $\mathrm{Hom}(H^*(\mathcal{M}),H^*(\mathcal{B}))$ denote the algebra homomorphisms $H^*(\mathcal{M}) \longrightarrow H^*(\mathcal{B})$. There is a function

$$I:[\mathcal{M},\mathcal{B}] \longrightarrow \mathrm{Hom}(H^*(\mathcal{M}),H^*(\mathcal{B}))$$

defined by $I[\alpha] = \alpha^*$, where $[\alpha]$ denotes the homotopy class of $\alpha:\mathcal{M} \to \mathcal{B}$. Assume now that $\mathcal{M}$ and $\mathcal{B}$ are formal and let $[\mathcal{M},\mathcal{B}]_f \subseteq [\mathcal{M},\mathcal{B}]$ be the subset of homotopy classes of formal maps. Although we shall not do so, it would be more precise to write $[\mathcal{M},\mathcal{B}]_{[\Psi]-[\Psi']}$ instead of $[\mathcal{M},\mathcal{B}]_f$ to indicate the dependence of $[\mathcal{M},\mathcal{B}]_f$ on the choice of maps Ψ and Ψ'. Set $I' = I|_{[\mathcal{M},\mathcal{B}]_f} : [\mathcal{M},\mathcal{B}]_f \longrightarrow \mathrm{Hom}(H^*(\mathcal{M}),H^*(\mathcal{B}))$.

<u>Proposition 3.2</u> $I': [\mathcal{M},\mathcal{B}]_f \longrightarrow \mathrm{Hom}(H^*(\mathcal{M}),H^*(\mathcal{B}))$ is a bijection.

<u>Proof</u>. I' is one-one: Given formal maps $\alpha,\beta:\mathcal{M} \to \mathcal{B}$ such that $\alpha^* = \beta^*$. Consider the diagram

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "\hat{\alpha},\hat{\beta}"] \arrow[d, "\Psi"] & \mathcal{M}_{\mathcal{B}} \arrow[d, "\Psi'"] \\
H^*(\mathcal{M}) \arrow[r, "\hat{\alpha}^*,\hat{\beta}^*"] & H^*(\mathcal{M}_{\mathcal{B}})
\end{tikzcd}.$$

Then $\Psi'\hat{\alpha} \simeq \Psi'\hat{\beta}$ since $\hat{\alpha}^* = \hat{\beta}^*$. But the function $\Psi'_{\#} :[\mathcal{M},\mathcal{M}_{\mathcal{B}}] \longrightarrow [\mathcal{M},H^*(\mathcal{M}_{\mathcal{B}})]$ induced by Ψ' is a bijection since Ψ' is a weak equivalence [GM,Theorem 10.8]. Thus $\hat{\alpha} \simeq \hat{\beta}$. Since $\rho\hat{\alpha} = \alpha$ and $\rho\hat{\beta} = \beta$, it follows that $\alpha \simeq \beta$.

I' is onto: Given $\varphi : H^*(M) \longrightarrow H^*(B)$ and consider $\lambda = \rho^*\Psi' : M_B \longrightarrow H^*(B)$. Since λ is a weak equivalence, $\lambda_\# : [M, M_B] \longrightarrow [M, H^*(B)]$ is a bijection [GM,Theorem 10.8]. Therefore there exists $\alpha : M \to M_B$ such that $\lambda_\#[\alpha] = [\varphi\Psi]$, and so $\lambda\alpha \simeq \varphi\Psi$. Let $\beta : M \longrightarrow B$ be $\beta = \rho\alpha$. Then $\beta^* = \rho^*\alpha^* = \rho^*\Psi'^*\alpha^* = \lambda^*\alpha^* = \varphi^*\Psi^* = \varphi$. Thus $I[\beta] = \varphi$. It remains only to show that β is formal, i.e., that the following diagram commutes

$$\begin{array}{ccc} M & \xrightarrow{\hat{\beta} = \alpha} & M_B \\ \Big\downarrow \Psi & & \Big\downarrow \Psi' \\ H^*(M) & \xrightarrow{\hat{\beta}^* = \alpha^*} & H^*(M_B) . \end{array}$$

But $\rho^*(\alpha^*\Psi) = \beta^*\Psi = \varphi\Psi \simeq \lambda\alpha = \rho^*(\Psi'\alpha)$, and so $\alpha^*\Psi \simeq \Psi'\alpha$.

We next reformulate Proposition 3.2 in the topological category for later use. Let X and Y be rational spaces which are formal (i.e., $A_{PL}(X)$ and $A_{PL}(Y)$ are formal DGAs). A map $f : X \to Y$ is called <u>formal</u> if $A_{PL}(f) : A_{PL}(Y) \longrightarrow A_{PL}(X)$ is a formal DGA map. This definition clearly depends on a choice of maps $\Psi : A_{PL}(Y) \to M_{A_{PL}(Y)} = M_Y$ and $\Psi' : A_{PL}(X) \to M_{A_{PL}(X)} = M_X$.

Let $[X,Y]_f$ be the collection of homotopy classes of formal maps $X \longrightarrow Y$ and let $[X,Y]$ be the collection of homotopy classes of all maps $X \longrightarrow Y$. Then $I:[X,Y] \longrightarrow \mathrm{Hom}(H^*(Y;\mathbb{Q}),\ H^*(X;\mathbb{Q}))$ assigns to a homotopy class its induced homomorphism of cohomology algebras. The reformulation of Proposition 3.2. is

<u>Proposition 3.3</u>. If X and Y are formal spaces, then $I|_{[X,Y]_f}:[X,Y]_f \longrightarrow \mathrm{Hom}(H^*(Y;\mathbb{Q}),\ H^*(X,\mathbb{Q}))$ is a bijection.

<u>Example 3.4</u> As a simple, illustrative example of Proposition 3.3, let X be the rationalization of the n-sphere $S^n_{\mathbb{Q}}$ and let Y be a formal, rational space of finite type. Then $I|\pi_n(Y)_f:\pi_n(Y)_f \longrightarrow \mathrm{Hom}(H^*(Y;\mathbb{Q}),H^*(S^n_{\mathbb{Q}};\mathbb{Q}))$ is a bijection, where $\pi_n(Y)_f$ consists of homotopy classes of formal maps in $\pi_n(Y)$. By applying vector space duality we obtain $\mathrm{Hom}(H^*(Y;\mathbb{Q}),H^*(S^n_{\mathbb{Q}};\mathbb{Q})) \approx \mathrm{Hom}_{\mathrm{coalg}}(H_*(S^n_{\mathbb{Q}}),H_*(Y))$, where the subscript "coalg" denotes coalgebra homomorphisms. But $\mathrm{Hom}_{\mathrm{coalg}}(H_*(S^n_{\mathbb{Q}}),H_*(Y)) \approx P_n(Y)$, the vector space of primitive elements in $H_n(Y)$. Thus we obtain a bijection $I':\pi_n(Y)_f \longrightarrow P_n(Y)$. It is easily seen that I' is the restriction of the Hurewicz homomorphism $h_n:\pi_n(Y) \longrightarrow H_n(Y)$. In particular, h_n is onto the subspace of primitive elements (see [NM, Proposition 3.4] and [FH]).

We now turn to a few consequences of Proposition 3.2.

<u>Corollary 3.5</u> If $\mathcal{M}$ and $\mathcal{B}$ are formal, then

$$I:[\mathcal{M},\mathcal{B}] \longrightarrow \mathrm{Hom}(H^*(\mathcal{M}),H^*(\mathcal{B}))$$

is onto.

<u>Corollary 3.6</u> If $\mathcal{M}$ is a minimal algebra with $d = 0$ and $\mathcal{B}$ is formal, then

$$I:[\mathcal{M},\mathcal{B}] \longrightarrow \mathrm{Hom}(H^*(\mathcal{M}),H^*(\mathcal{B})) = \mathrm{Hom}(\mathcal{M},H^*(\mathcal{B}))$$

is a bijection.

<u>Proof.</u> One easily shows that $\mathcal{M}$ is formal and $[\mathcal{M},\mathcal{B}]_f = [\mathcal{M},\mathcal{B}]$.

A more general result for spaces has been proved by Scheerer [Sc, Proposition 1].

A simple application of Corollary 3.5 yields a characterization of formal DGAs in terms of the function I.

<u>Corollary 3.7</u> $\mathcal{A}$ is formal $\Leftrightarrow I:[\mathcal{M}_A,\mathcal{B}] \longrightarrow \mathrm{Hom}(H^*(\mathcal{M}_A),H^*(\mathcal{B}))$ is onto for every formal DGA $\mathcal{B}$.

<u>Proof.</u> $\Rightarrow$:Follows from Corollary 3.5.
$\Leftarrow$: Let $\mathcal{B} = H^*(\mathcal{M}_A)$ with $d = 0$. Then $\mathcal{B}$ is formal and so I is onto. Let $1 \in \mathrm{Hom}\,(H^*(\mathcal{M}_A),H^*(\mathcal{M}_A))$ be the identity homomorphism. Then there is a map $\Psi:\mathcal{M}_A \rightarrow H^*(\mathcal{M}_A)$ such that $\Psi^* = 1$. Thus $\mathcal{M}_A$ is formal by Proposition 2.1. Therefore $\mathcal{A}$ is formal.

Next we present some examples to show that it is not possible to weaken the hypothesis or strengthen the conclusion of Corollary 3.5 in an obvious way.

Examples 3.8 a) M and B formal, but $I:[M,B] \longrightarrow \mathrm{Hom}(H^*(M),H^*(B))$ not one-one: Set $M = \Lambda(x_2,x_3)$ with $dx_3 = x_2^2$ and $B = \Lambda(y_3)$, where the subscripts denote the degree of the element. Then $\alpha: M \longrightarrow B$ defined by $\alpha(x_3) = y_3$ is not homotopic to 0 but $I[\alpha] = 0$. (α is the DGA analogue of the Hopf map $S^3 \longrightarrow S^2$.)

b) M is formal, B is not formal, and $I:[M,B] \longrightarrow \mathrm{Hom}(H^*(M),H^*(B))$ is not onto: Let M be the minimal model of $S^3 \vee S^5$ so $M = \Lambda(x_3,x_5,x_7,x_9,\ldots)$ with $dx_3 = 0 = dx_5$, $dx_7 = x_3x_5$, $dx_9 = x_3x_7$, etc. and $B = \Lambda(y_3,y_5,y_7)$ with $dy_3 = 0 = dy_5$ and $dy_7 = y_3y_5$. Then $\varphi: H^*(M) \longrightarrow H^*(B)$ defined by $\varphi\{x_3\} = \{y_3\}$, $\varphi\{x_5\} = \{y_5\}$ is a homomorphism not realizable by a DGA map $M \rightarrow B$.

c) M is not formal, B is formal, and $I:[M,B] \longrightarrow \mathrm{Hom}(H^*(M),H^*(B))$ is not onto: Apply Corollary 3.7.

4. Realizability of Cohomology Homomorphisms

In this section we consider the realizability of cohomology homomorphisms by maps of DGAs. We begin with a simple lemma.

Lemma 4.1 (a) Given a minimal DGA M and an integer $n \geq 1$, there exists a minimal DGA $\widetilde{M}$ and a DGA map $\pi : M \to \widetilde{M}$ such that $\pi^* : H^i(M) \longrightarrow H^i(\widetilde{M})$ is an isomorphism for $i \leq n$ and $H^i(\widetilde{M}) = 0$ for $i > n$.

b) Given a DGA A and an integer $n \geq 1$, there exists a DGA $\hat{A}$ and a DGA map $\chi : \hat{A} \longrightarrow A$ such that $\chi^* : H^i(\hat{A}) \to H^i(A)$ is an isomorphism for $i \geq n$ and $H^i(\hat{A}) = 0$ for $i < n$.

Proof. These constructions can be carried out in the category of DGAs. However, the quickest way to prove the lemma is to perform the constructions in the category of rational spaces where they are well-known, and then take the minimal models of the resulting spaces. For (a) one uses the homotopy decomposition of a space and for (b) the method of killing homology groups (see [Hi, p. 90]).

We next use this lemma and Corollary 3.5 to give a new proof of the following theorem of Vigué-Poirrier [Vi 2].

Proposition 4.2 If M is a minimal DGA with $H^i(M) = 0$ for $1 \leq i \leq \ell$ and B is a DGA with $H^i(B) = 0$ for $i > 3\ell + 1$, then I is onto. That is, every homomorphism $H^*(M) \longrightarrow H^*(B)$ is realizable by a DGA map $M \to B$.

<u>Proof.</u> In this proof we shall use the following result of Halperin-Stasheff [HS,Corollary 5.16]:
If A is a DGA with $H^i(A) = 0$ for $i < m+1$ and $i > 3m+1$, then A is formal.

We apply Lemma 4.1(a) to M with $n = 3\ell + 1$ to obtain $\pi : M \to \widetilde{M}$ with π^* an isomorphism in dimensions $\leq 3\ell + 1$ and $H^i(\widetilde{M}) = 0$ for $i > 3\ell + 1$. Thus $H^i(\widetilde{M}) = 0$ for $i < \ell + 1$ and $i > 3\ell + 1$. By the Halperin-Stasheff result, $\widetilde{M}$ is formal. Next apply Lemma 4.1(b) to B with $n = \ell + 1$ to obtain $\chi : \hat{B} \longrightarrow B$ with χ^* an isomorphism in dimensions $\geq$ $\ell + 1$ and $H^i(\hat{B}) = 0$ for $i < \ell + 1$. Thus $H^i(\hat{B}) = 0$ for $i < \ell + 1$ and $i > 3\ell + 1$. By the Halperin-Stasheff result, $\hat{B}$ is formal. Now consider the commutative diagram

$$\begin{array}{ccc}
[M,B] & \xrightarrow{I} & \mathrm{Hom}(H^*(M),H^*(B)) \\
\uparrow \pi^{\#} & & \approx \uparrow \mathrm{Hom}(\pi^*,1) \\
[\widetilde{M},B] & & \mathrm{Hom}(H^*(\widetilde{M}),H^*(B)) \\
\uparrow \chi_{\#} & & \approx \uparrow \mathrm{Hom}(1,\chi^*) \\
[\widetilde{M},\hat{B}] & \xrightarrow{I} & \mathrm{Hom}(H^*(\widetilde{M}),H^*(\hat{B}))
\end{array}$$

Since $\widetilde{M}$ and $\hat{B}$ are formal, $I : [\widetilde{M},\hat{B}] \longrightarrow \mathrm{Hom}(H^*(\widetilde{M}),H^*(\hat{B}))$ is onto by Corollary 3.5. Therefore $I : [M,B] \longrightarrow \mathrm{Hom}(H^*(M), H^*(B))$ is onto. This concludes the proof.

5. Obstructions to Formality of a Map

Here we develop a simple obstruction theory for formality of a map. This is completely different from the obstruction theory which occurs in [Vi 1]. Let $\mathcal{M}$ be a minimal DGA and let $\mathcal{M}(n)$ denote the minimal sub DGA of $\mathcal{M}$ generated by generators of degree $\leq n$. Then $\mathcal{M} = \cup\mathcal{M}(n)$. Furthermore, $\mathcal{M}(n+1) = \mathcal{M}(n) \otimes \Lambda(V)_{n+1}$, where $\Lambda(V)_{n+1}$ is the free commutative graded algebra generated by the vector space V concentrated in dimension $n+1$. The inclusion $\iota:\mathcal{M}(n) \longrightarrow \mathcal{M}(n+1)$ is called a Hirsch extension [GM, p. 113].

Now recall Definition 3.1: $\alpha:A \longrightarrow B$ is formal if the following diagram commutes up to homotopy

$$\begin{array}{ccc} \mathcal{M}_A & \xrightarrow{\hat{\alpha}} & \mathcal{M}_B \\ \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \Psi'} \\ H^*(\mathcal{M}_A) & \xrightarrow{\hat{\alpha}^*} & H^*(\mathcal{M}_B) \end{array}$$

where Ψ and Ψ' are fixed maps inducing the identity in cohomology. We write $\mathcal{M} = \mathcal{M}_A = \cup\mathcal{M}(n)$ and denote by $\iota:\mathcal{M}(n) \longrightarrow \mathcal{M}(n+1)$ the Hirsch extension and by $\nu_n:\mathcal{M}(n) \longrightarrow \mathcal{M} = \mathcal{M}_A$ the inclusion map.

We assume $\hat{\alpha}^*\Psi\nu_n \simeq \Psi'\hat{\alpha}\nu_n:\mathcal{M}(n) \longrightarrow H^*(\mathcal{M}_B)$, and determine the obstruction to $\hat{\alpha}^*\Psi\nu_{n+1}$ being homotopic to $\Psi'\hat{\alpha}\nu_{n+1}$. Note that $(\hat{\alpha}^*\Psi)^* = \hat{\alpha}^* = (\Psi'\hat{\alpha})^*$, so that the maps $\hat{\alpha}^*\Psi\nu_{n+1}$

and $\Psi'\hat{\alpha}\nu_{n+1}$ induce the same cohomology homomorphism. To simplify notation we set $\beta = \hat{\alpha}^*\Psi\nu_{n+1}$ and $\gamma = \Psi'\hat{\alpha}\nu_{n+1}$. Then $\beta,\gamma : M(n+1) \longrightarrow H^*(M_B)$ with $\beta^* = \gamma^*$ and $\beta\iota \simeq \gamma\iota : M(n) \longrightarrow H^*(M_B)$, and we determine the obstruction to $\beta \simeq \gamma$. For this we use the following result in [GM,pp. 177-78]: There is an exact sequence

$$\mathrm{Hom}(V,H^{n+1}(M_B)) = H^{n+1}(H^*(M_B);V^*) \to [M(n+1),H^*(M_B)] \xrightarrow{\iota^\#} [M(n),H^*(M_B)]$$

and an operation of $\mathrm{Hom}(V,H^{n+1}(M_B))$ on $[M(n+1),H^*(M_B)]$ such that two elements are in the same orbit if and only if they have the same image under $\iota^\#$. Thus there exists $\theta : V \longrightarrow H^{n+1}(M_B)$ such that $[\gamma] = \theta\cdot[\beta] = [\theta\cdot\beta]$. The operation $\theta\cdot\beta$ is defined by $\theta\cdot\beta|M(n) = \beta|M(n)$ and $(\theta\cdot\beta)(x) = \beta(x) + \theta(x)$ for $x \in V$. Now $\beta^* = \gamma^*$, so that if $x \in V \subseteq M(n+1)$ is a cocycle,

$$\gamma^*\{x\} = (\theta\cdot\beta)^*\{x\} = (\theta\cdot\beta)(x) = \beta(x) + \theta(x) = \beta^*\{x\} + \theta(x)$$

where $\{x\}$ denotes the cohomology class of x in $H^{n+1}(M(n+1))$. Thus $\theta(x) = 0$ for every cocycle $x \in V \subseteq M(n+1)$. Let $Z(V)$ be the subspace of V consisting of all x with $dx = 0$. Then θ induces a homomorphism $\bar{\theta} : V/Z(V) \longrightarrow H^{n+1}(M_B)$. We call $\bar{\theta} \in \mathrm{Hom}(V/Z(V),H^{n+1}(M_B))$ <u>the obstruction to β and γ being homotopic</u>. Note that if $\bar{\theta} = 0$, then $\theta = 0$, and so $\beta \simeq \gamma$.

We now interpret this in the category of rational spaces. Here $A = A_{PL}(Y)$ and $B = A_{PL}(X)$ for rational spaces X and Y. We assume that X and Y are of finite type. Then

$H^{n+1}(M_{\mathcal{B}}) \approx H^{n+1}(X;\mathbb{Q})$ and $V \approx \mathrm{Hom}(\pi_{n+1}(Y),\mathbb{Q}) = (\pi_{n+1}(Y))^*$. If $h_{n+1}:\pi_{n+1}(Y) \longrightarrow H_{n+1}(Y)$ is the Hurewicz homomorphism and $h^*_{n+1}:(H_{n+1}(Y))^* \longrightarrow (\pi_{n+1}(Y))^*$ is the dual of h_{n+1}, we can regard h^*_{n+1} as a homomorphism $H^{n+1}(Y;\mathbb{Q}) \longrightarrow V$. Then $Z(V)$ can be identified with image h^*_{n+1} (see [Su 1,p.33]). Thus by duality we obtain

<u>Proposition 5.4.</u> If $f:X \longrightarrow Y$ is a map of rational, formal spaces of finite type, then the successive obstructions to formality of f lie in the vector spaces $\mathrm{Hom}(H_i(X),\ \text{kernel } h_i)$, $i = 2,3\ldots$, where $h_i:\pi_i(Y) \longrightarrow H_i(Y)$ is the Hurewicz homomorphism. In particular, $\mathrm{Hom}(H_i(X),\ \text{kernel } h_i) = 0$ for all i implies that every map $f:X \longrightarrow Y$ is formal.

As a consequence we obtain a result which is a generalization of two theorems in [Vi 1, Theorems 4 and 5].

<u>Corollary 5.5</u> If X and Y are formal, rational spaces of finite type such that $H_i(X) = 0$ for $i \geq 2n+1$ and Y is n-connected, then every map $f:X \longrightarrow Y$ is formal.

<u>Proof.</u> For the n-connected space Y, the Hurewicz homomorphism $h_i:\pi_i(Y) \longrightarrow H_i(Y)$ is known to be a monomorphism for $i \leq 2n$ [AC, p. 546]. Thus all obstructions to formality vanish, and so every map $f:X \longrightarrow Y$ is formal by Proposition 5.4.

The rationalization of the Hopf map $S^3 \longrightarrow S^2$ is not formal, and this shows that the numerical hypotheses of Corollary 5.5 cannot be weakened.

6. The Group of Homotopy Equivalences.

In this section we examine the group $E(M)$ of homotopy classes of homotopy equivalences of a minimal DGA M. An element of $E(M)$ is the homotopy class $[\alpha]$ of a DGA map $\alpha : M \longrightarrow M$ which is a homotopy equivalence. The group operation in $E(M)$ is composition of homotopy classes and the unit of the group is the homotopy class of the identity map. We note that $\alpha : M \longrightarrow M$ is a homotopy equivalence if and only if $\alpha^* : H^*(M) \longrightarrow H^*(M)$ is an automorphism. Let $I : E(M) \longrightarrow \mathrm{Aut}\, H^*(M)$ denote the function defined by $I[\alpha] = \alpha^*$. Then I is a homomorphism of groups. If we now assume that M is formal, we can define a subset $E(M)_f$ of $E(M)$ by $[\alpha] \in E(M)_f \Leftrightarrow \alpha : M \longrightarrow M$ is formal and a homotopy equivalence. As before, formality of maps α is defined with respect to a fixed DGA map $\psi : M \longrightarrow H^*(M)$ with $\psi^* = 1$, i.e., α is $[\psi]-[\psi]$-formal in the sense of Definition 3.1. It is easily seen that $E(M)_f$ is a subgroup of $E(M)$. The first sentence of the following proposition appears in [NM, Corollary 5.5] and elsewhere.

Proposition 6.1. If A is a formal DGA, then $I : E(M_A) \longrightarrow \mathrm{Aut}\, H^*(M_A)$ is an epimorphism. Moreover, there is a homomorphism $J : \mathrm{Aut}\, H^*(M_A) \longrightarrow E(M_A)$ such that $IJ = 1$, and J is an isomorphism of $\mathrm{Aut}\, H^*(M_A)$ onto $E(M_A)_f$.

Proof. Given an automorphism $\phi : H^*(M_A) \longrightarrow H^*(M_A)$. By Proposition 3.2, there exists a DGA formal homomorphism

$\alpha : M_A \longrightarrow M_A$ unique up to homotopy such that $\alpha^* = \phi$. Then α is a homotopy equivalence since ϕ is an automorphism. Define $J(\phi) = [\alpha]$. Clearly J has the desired properties.

Thus if $E_*(M_A)$ denotes the subgroup of $E(M_A)$ consisting of all homotopy classes of homotopy equivalences which induce the identity in cohomology, then there is the following <u>split</u> short exact sequence of groups for a formal DGA A,

$$(6.2) \quad 1 \longrightarrow E_*(M_A) \longrightarrow E(M_A) \underset{J}{\overset{I}{\rightleftarrows}} \operatorname{Aut} H^*(M_A) \longrightarrow 1 .$$

<u>Corollary 6.3</u>. If A is formal, then $E(M_A)$ is isomorphic to the semi-direct product of $E_*(M_A)$ and Aut $H^*(M_A)$,

$$E(M\ ) \approx E_*(M_A) \rtimes \operatorname{Aut} H^*(M_A) .$$

The proof is analogous to the familiar situation in which the subgroup is abelian. One defines an action of $E(M_A)$ on $E_*(M_A)$ by conjugation. By composing this action with J an action of Aut $H^*(M_A)$ on $E_*(M_A)$ is obtained which enables one to define the semi-direct product. The proof that there is an isomorphism proceeds as in the abelian case [Br, pp.87-88].

Thus for formal DGAs A a determination of the group $E(M_A)$ would require knowing (1) the group $E_*(M_A)$, (2) the group Aut $H^*(M_A)$, (3) the semi-direct product of the groups in (1) and (2). Note that (2) and (3) are purely algebraic. In the Appendix we indicate how to obtain information on $E_*(M)$ and its dual for any minimal DGA M.

7. Suspensions and Co-H-maps.

Most of the results in previous sections dealt with the function which assigns to a homotopy class the induced homomorphism of cohomology. By using a theorem of Baues we are able to translate these into results concerning suspensions and co-H-maps. In this section we work in the category of spaces.

Let X and Y denote rational spaces of finite type and let ΣX and ΣY be the suspensions of X and Y respectively. Let $[\Sigma X, \Sigma Y]_{\text{co-H}}$ denote the homotopy classes of co-H-maps $\Sigma X \longrightarrow \Sigma Y$. Then it is shown in [Ba, p.132] that there exists a bijection $P : [\Sigma X, \Sigma Y]_{\text{co-H}} \longrightarrow \mathrm{Hom}(H^*(Y;\mathbb{Q}), H^*(X;\mathbb{Q}))$. By unraveling the definition of P one establishes the following lemma.

Lemma 7.1. The diagram

$$\begin{tikzcd}
{[X, Y]} \arrow[r, "\Sigma"] \arrow[dr, "I"'] & {[\Sigma X, \Sigma Y]_{\text{co-H}}} \arrow[d, "P", "\approx"'] \\
& \mathrm{Hom}(H^*(Y;\mathbb{Q}), H^*(X;\mathbb{Q}))
\end{tikzcd}$$

is commutative, where Σ is the suspension function, P is the Baues bijection, and I is the function which assigns the induced cohomology homomorphism to a homotopy class.

We note in passing that Lemma 7.1 provides an immediate proof of the following result of Lemaire-Sigrist [LS,pp.114-116]:

(7.2) For a map $f : X \longrightarrow Y$, $\Sigma f \simeq 0$ if and only if $f^* = 0 : H^*(Y,\mathbb{Q}) \longrightarrow H^*(X;\mathbb{Q})$.

<u>Proposition 7.3</u>. If X and Y are formal spaces, then the suspension map

$$\Sigma : [X,Y]_f \longrightarrow [\Sigma X, \Sigma Y]_{co\text{-}H}$$

is a bijection. Consequently, any co-H-map $\Sigma X \longrightarrow \Sigma Y$ is homotopic to a suspension.

<u>Proof</u>. The proof is an immediate consequence of Proposition 3.3 and Lemma 7.1.

The following result is also a consequence of Proposition 3.3. and Lemma 7.1.

<u>Corollary 7.4</u>. If X and Y are formal and $f : X \longrightarrow Y$ is any map, then there exists a formal map $g : X \longrightarrow Y$ such that $\Sigma f \simeq \Sigma g$.

Next we note that Proposition 4.2 and Lemma 7.1 yield a rational version of a theorem of Berstein and Hilton [BH, Theorem B].

<u>Corollary 7.5</u>. If X and Y are rational spaces such that $H^i(Y;\mathbb{Q}) = 0$ for $1 \leq i \leq \ell$ and $H^i(X;\mathbb{Q}) = 0$ for $i > 3\ell + 1$, then every co-H-map $\Sigma X \longrightarrow \Sigma Y$ is homotopic to the suspension of some map $X \longrightarrow Y$.

<u>Proof</u>. We consider the commutative diagram of Lemma 7.1 and observe that $I : [X,Y] \longrightarrow \mathrm{Hom}(H^*(Y;\mathbb{Q}), H^*(X;\mathbb{Q}))$ is onto by Proposition 4.2. Thus $\Sigma : [X,Y] \longrightarrow [\Sigma X, \Sigma Y]_{co\text{-}H}$ is onto. This concludes the proof.

We next apply Lemma 7.1 to § 6 . Consider the diagram of Lemma 7.1 in the case $X = Y$:

$$\begin{tikzcd}
{[X,X]} \arrow[r, "\Sigma"] \arrow[dr, "I"'] & {[\Sigma X, \Sigma X]_{co\text{-}H}} \arrow[d, "P", "\approx"'] \\
& \mathrm{Hom}(H^*(X;\mathbb{Q}), H^*(X;\mathbb{Q})) .
\end{tikzcd}$$

From the definition of P it can be shown that (a) if $1 : \Sigma X \longrightarrow \Sigma X$ is the identity map then $P[1]$ is the identity automorphism of $H^*(X;\mathbb{Q})$; (b) if $g,h : \Sigma X \longrightarrow \Sigma X$ are two co-H-maps, then $P([g]\circ[h]) = P[h] \circ P[g]$. Now let $E(X) \subseteq [X,X]$ denote the group of homotopy classes of homotopy equivalences $X \longrightarrow X$ and $E(\Sigma X)_{co\text{-}H} \subseteq [\Sigma X, \Sigma X]_{co\text{-}H}$ denote the group of homotopy classes of homotopy equivalences $\Sigma X \longrightarrow \Sigma X$ which are co-H-maps. The suspension function $\Sigma : E(X) \longrightarrow E(\Sigma X)_{co\text{-}H}$ is then a homomorphism of groups. The function $I : E(X) \longrightarrow \mathrm{Aut}(H^*(X;\mathbb{Q}))$ which assigns the induced cohomology automorphism to a homotopy class is an anti-homomorphism, i.e., $I : E(X) \longrightarrow \mathrm{Aut}\, H^*(X;\mathbb{Q})^{opp}$ is a homomorphism, where $\mathrm{Aut}\, H^*(X;\mathbb{Q})^{opp}$ denotes $\mathrm{Aut}\, H^*(X;\mathbb{Q})$ with the opposite multiplication. The observations (a) and (b) above show that P induces a function $P : E(\Sigma X)_{co\text{-}H} \longrightarrow \mathrm{Aut}\, H^*(X;\mathbb{Q})$ which is an anti-isomorphism of groups. Consequently we have the following results.

Corollary 7.6. The group $E(\Sigma X)_{co\text{-}H}$ is isomorphic to the group $\mathrm{Aut}\, H^*(X; \mathbb{Q})^{opp}$.

Corollary 7.7. The following diagram of groups and homomorphisms is commutative

$$\begin{array}{ccc} E(X) & \xrightarrow{\ \Sigma\ } & E(\Sigma X)_{co\text{-}H} \\ & \searrow^{I} & \approx \downarrow P \\ & & \mathrm{Aut}\, H^*(X; \mathbb{Q})^{opp} \end{array}$$

The analogue of (7.2) is

(7.8) If $f : X \longrightarrow X$, $\Sigma f \simeq 1$ if and only if $f^* = 1 : H^*(X; \mathbb{Q}) \longrightarrow H^*(X; \mathbb{Q})$.

Finally we note that the homomorphism $I : E(X) \longrightarrow \mathrm{Aut}\, H^*(X; \mathbb{Q})^{opp}$ corresponds to the homomorphism $I : E(M_A) \longrightarrow \mathrm{Aut}\, H^*(M_A)$ discussed in § 6 . Thus we obtain the following corollary from Proposition 6.1.

Corollary 7.9. If X is a formal space, then $\Sigma : E(X) \longrightarrow E(\Sigma X)_{co\text{-}H}$ is an epimorphism. Moreover, there exists a homomorphism $\Lambda : E(\Sigma X)_{co\text{-}H} \longrightarrow E(X)$ such that $\Sigma\Lambda = 1$.

Thus we are able to express $E(X)$ as the semi-direct product of the subgroup of $E(X)$ consisting of those homotopy classes which suspend to the identity with $E(\Sigma X)_{co\text{-}H}$ (see (A.1) of the Appendix)).

8. Concluding Remarks.

First of all, we wish to emphasize that, although we have worked within the category of DGAs, our results can be reformulated for the category of rational topological spaces. In several cases such as Proposition 3.3, Proposition 5.4, and § 7, we have done this explicitly. However, all of our results can be restated and yield information about rational topological spaces. For example, the reformulation of Corollary 3.5 is that for formal, rational spaces X and Y, the function $I : [X,Y] \longrightarrow \mathrm{Hom}(H^*(Y;\mathbb{Q}), H^*(X;\mathbb{Q}))$ is onto. This implies that $[X,Y]$ is an infinite set if there is a non-trivial homomorphism of algebras $H^*(Y;\mathbb{Q}) \longrightarrow H^*(X;\mathbb{Q})$.

Secondly, we have assumed for simplicity that all DGAs are 1-connected. It is possible to relax this restriction and deal with connected DGAs which are nilpotent. This leads to a larger class of rational topological spaces, the nilpotent spaces.

Finally, all of our considerations can be dualized. Instead of DGAs, formality, and cohomology homomorphisms, it is possible to work with differential graded Lie algebras, coformality, and homotopy homomorphisms (see [Ta] and [NM] for the relevant definitions). Our results are then valid in the dual situation.

Appendix. The Group of Homotopy Equivalences which Induce the Identity.

In § 6 we determined the group $E(M)$ of homotopy classes of homotopy equivalences of a formal, minimal DGA M in terms of $\mathrm{Aut}\, H^*(M)$ and $E_*(M)$, the subgroup of $E(M)$ consisting of homotopy equivalences which induce the identity homomorphism $H^*(M) \longrightarrow H^*(M)$. In this appendix we sketch a method for obtaining information about $E_*(M)$ and its dual group $E_{\#}(M)$ for any minimal DGA M. We first express Corollary 6.3 in the category of rational topological spaces.

If X is a formal, rational space and $E(X)$ is the group of homotopy classes of homotopy equivalences of X and $E_*(X)$ is the subgroup of those homotopy classes which induce the identity in cohomology with rational coefficients, then there is the semi-direct product decomposition

(A.1) $$E(X) \approx E_*(X) \rtimes \mathrm{Aut}\, H^*(X;\mathbb{Q})^{\mathrm{opp}} .$$

Now for any rational space X, $E(X)$ can also be described as $E(L_X)$ where L_X is the differential graded Lie algebra (DGL) over $\mathbb{Q}$ [Ta, pp.14-17] which is the Quillen minimal model of X [Ta, Chapter III] and $E(L_X)$ is the group of homotopy classes of homotopy equivalences of DGL maps $L_X \longrightarrow L_X$. Then $E_*(X)$ is isomorphic to the subgroup of $E(L_X)$ consisting of those homotopy equivalences which induce the identity on the indecomposables $Q(L_X) = s^{-1}H_*(X;\mathbb{Q})$, where s is the suspension isomorphism [Ta, p.83]. We can present an inductive procedure for expressing $E_*(L_X(n))$ in terms of $E_*(L_X(n-1))$, where $L_X(n)$ is the minimal DGL of the nth section of the rational homology decomposition of X. This would then give information about $\mathcal{E}_*(L_X)$, and hence $\mathcal{E}_*(X)$. Rather than do this, we stay within the category of DGAs and describe the dual procedure.

Thus we consider for a minimal DGA $\mathcal{M}$ the subgroup $\mathcal{E}_\#(\mathcal{M})$ of $\mathcal{E}(\mathcal{M})$ consisting of homotopy classes $[\alpha]$ such that $Q(\alpha) = 1 : Q(\mathcal{M}) \longrightarrow Q(\mathcal{M})$, where $Q(\mathcal{M})$ denotes the graded vector space of indecomposables of $\mathcal{M}$. We write $\mathcal{M} = \cup \mathcal{M}(n)$ where $\mathcal{M}(n)$ is the minimal sub DGA of $\mathcal{M}$ generated by generators of degree $\leq n$. Then $\mathcal{M}(n+1) = \mathcal{M}(n) \otimes \Lambda(V)_{n+1}$, and the inclusion $\iota : \mathcal{M}(n) \longrightarrow \mathcal{M}(n+1)$ is a Hirsch extension (see § 5). We will apply the following lemma to ι.

<u>Lemma A.2</u>. Let $\alpha : \mathcal{A} \longrightarrow \mathcal{B}$ be a DGA map such that $\alpha^* : H^r(\mathcal{A}) \longrightarrow H^r(\mathcal{B})$ is an isomorphism for $r \leq n$ and a monomorphism for $r = n+1$ and let $\mathcal{M}$ be a minimal DGA with no generators in dimensions $> n$ (i.e., $Q^r(\mathcal{M}) = 0$ for $r > n$), then the induced map

$$\alpha_\# : [\mathcal{M},\mathcal{A}] \longrightarrow [\mathcal{M},\mathcal{B}]$$

is a bijection.

<u>Proof</u>. The proof is analogous to the proof of Theorem 10.8 in [G-M] and hence omitted.

For the Hirsch extension $\iota : M(n) \longrightarrow M(n+1)$ we have

$$[M(n+1), M(n+1)] \xrightarrow{\iota^{\#}} [M(n), M(n+1)] \xleftarrow{\iota_{\#}} [M(n), M(n)]$$

with $\iota_{\#}$ an isomorphism by Lemma A.2. Thus there is a function $r : [M(n+1), M(n+1)] \longrightarrow [M(n), M(n)]$ defined by $r = \iota_{\#}^{-1}\iota^{\#}$. Clearly r carries $E_{\#}(M(n+1))$ into $E_{\#}(M(n))$ and gives rise to a homomorphism

$$r : E_{\#}(M(n+1)) \longrightarrow E_{\#}(M(n)) .$$

We next consider the kernel and image of this homomorphism.

Recall from [GM, pp.177-178] that there is an exact sequence

$$\text{(A.3)}\quad \begin{array}{ccccccc} H^{n+1}(A;V^*) & \longrightarrow & [M(n+1),A] & \xrightarrow{\iota^{\#}} & [M(n),A] & \xrightarrow{\mathcal{O}} & H^{n+2}(A;V^*) \\ \| & & & & & & \| \\ \mathrm{Hom}(V,H^{n+1}(A)) & & & & & & \mathrm{Hom}(V,H^{n+2}(A)) \end{array}$$

where $\mathcal{O}$ assigns to a homotopy class the obstruction to extending it over $M(n+1)$. Furthermore, there is an operation of $H^{n+1}(A,V^*)$ on $[M(n+1),A]$ such that two elements in $[M(n+1),A]$ are in the same orbit if and only they have the same $\iota^{\#}$-image.

We consider (A.3) with $A = M(n+1)$. Then if $\alpha \in H^{n+1}(M(n+1);V^*)$ and $[\gamma] \in [M(n+1), M(n+1)]$, the operation is defined by $\alpha\cdot[\gamma] = [\alpha\cdot\gamma]$ where $\alpha\cdot\gamma|M(n) = \gamma|M(n)$ and $(\alpha\cdot\gamma)(x) = \gamma(x) + \tilde{\alpha}(x)$ for $x \in V$, for any lifting

$\tilde{\alpha} : V \longrightarrow Z^{n+1}(M(n+1))$ of α . We now define $s : H^{n+1}(M(n+1);V^*) \longrightarrow [M(n+1), M(n+1)]$ by

$$s(\alpha) = [\alpha \cdot 1] = \alpha \cdot [1]$$

where 1 is the identity map of $M(n+1)$. Let $\lambda : H^{n+1}(M(n+1)) \longrightarrow Q^{n+1}(M(n+1))$ be induced by the projection and $\lambda_* : H^{n+1}(M(n+1);V^*) = \mathrm{Hom}(V, H^{n+1}(M(n+1)) \longrightarrow \mathrm{Hom}(V, Q^{n+1}(M(n+1))) = \mathrm{Hom}(V,V)$ be the homomorphism induced by λ . Set $T^{n+1}(M(n+1);V^*) \subseteq H^{n+1}(M(n+1);V^*)$ equal to kernel λ_*. Then it is easily seen that $\alpha \in H^{n+1}(M(n+1);V^*)$ is in $T^{n+1}(M(n+1);V^*)$ if and only if $s(\alpha) \in [M(n+1), M(n+1)]$ is in $E_\#(M(n+1))$. Thus s induces a function

$$s : T^{n+1}(M(n+1);V^*) \longrightarrow E_\#(M(n+1)) .$$

Now we consider the function $\mathcal{O}$ in (A.3) with $A = M(n)$. Then $\mathcal{O}|E_\#(M(n))$ is a function t from $E_\#(M(n))$ to $H^{n+2}(M(n);V^*)$

$$t = \mathcal{O}|_{E_\#(M(n))} : E_\#(M(n)) \longrightarrow H^{n+2}(M(n);V^*) .$$

We choose a distinguished element in $H^{n+2}(M(n);V^*)$ as follows. The homomorphism $H^{n+2}(M(n), M(n+1)) \longrightarrow H^{n+2}(M(n))$ in the exact cohomology sequence of the inclusion $\iota : M(n) \longrightarrow M(n+1)$ induces a homomorphism $\tau : H^{n+2}(M(n), M(n+1);V^*) \longrightarrow H^{n+2}(M(n);V^*)$. Then $H^{n+2}(M(n), M(n+1);V^*) = \mathrm{Hom}(V, H^{n+2}(M(n), M(n+1))) = \mathrm{Hom}(V,V)$ [GM, p.118] , and we let $b \in H^{n+2}(M(n), M(n+1);V^*)$

correspond to the identity map in $\mathrm{Hom}(V,V)$. Then $\tau(b) = k$ is called the <u>(n+1) st</u> <u>Postnikov</u> <u>invariant</u> and is the chosen distinguished element of the set $H^{n+2}(M(n);V^*)$. The function $t : E_{\#}(M(n)) \longrightarrow H^{n+2}(M(n);V^*)$ carries the distinguished element $[1]$ of $E_{\#}(M(n))$ (the group unit) to the distinguished element k of $H^{n+2}(M(n);V^*)$.

<u>Proposition A.3</u>. For any Hirsch extension $\iota : M(n) \longrightarrow M(n+1)$, the sequence of sets and functions

$$T^{n+1}(M(n+1);V^*) \xrightarrow{s} E_{\#}(M(n+1)) \xrightarrow{r} E_{\#}(M(n)) \xrightarrow{t} H^{n+2}(M(n);V^*)$$

is exact in the following sense:

(a) r and s are homomorphisms of groups and image s $=$ kernel r,

(b) t is a function of sets with distinguished element and image r = kernel $t = t^{-1}(k)$.

<u>Proof</u>. We show that s is a homomorphism (from an additive to a multiplicative group). Note that if $\beta \in T^{n+1}(M(n+1);V^*) \subseteq \mathrm{Hom}(V,H^{n+1}(M(n+1)))$ and $x \in V$, then $\tilde{\beta}(x) \in M(n)$. For $\tilde{\beta}(x) \in Z^{n+1}(M(n+1))$ and $\lambda\beta(x) = 0$ in $Q^{n+1}(M(n+1))$. Thus $\tilde{\beta}(x)$ is an element of degree $n+1$ which is decomposable, i.e., in $(M^+(n+1) \cdot M^+(n+1))^{n+1}$, and so $\tilde{\beta}(x) \in M(n)$. Now

$$s(\alpha+\beta) = [(\alpha+\beta)\cdot 1] \text{ and } s(\alpha)\circ s(\beta) = [(\alpha\circ 1)\circ(\beta\cdot 1)]$$

and for $x \in V$

$$((\alpha+\beta)\cdot 1)(x) = x + \widetilde{\alpha+\beta}(x) = x+\tilde{\alpha}(x)+\tilde{\beta}(x)$$

and

$$(\alpha\cdot 1)\circ(\beta\cdot 1)(x) = (\alpha\cdot 1)(x+\tilde{\beta}(x)) = (\alpha\cdot 1)(x) + (\alpha\cdot 1)(\tilde{\beta}(x))$$
$$= x + \tilde{\alpha}(x) + \tilde{\beta}(x) \ .$$

Thus s is a homomorphism. The proof that image s = kernel r is now any easy consequence of the definitions of s and r and properties of the operation of $H^{n+1}(M(n+1);V^*)$ on $[M(n+1), M(n+1)]$ mentioned earlier.

Finally, the proof that image r = kernel t is a consequence of the following lemma whose proof we omit (Cf. [GM, pp. 113-114]) .

Lemma A.4. Given Hirsch extensions $M(n+1) = M(n) \otimes \Lambda(V)_{n+1}$ and $N(n+1) = N(n) \otimes \Lambda(W)_{n+1}$ and homomorphisms $\alpha : M(n) \longrightarrow N(n)$ and $\rho : V \longrightarrow W$. Then there is a homomorphism $\beta : M(n+1) \longrightarrow N(n+1)$ such that the following diagrams commute

$$\begin{array}{ccc} M(n) & \xrightarrow{\text{incl.}} & M(n+1) \\ \downarrow{\alpha} & & \downarrow{\beta} \\ N(n) & \xrightarrow{\text{incl.}} & N(n+1) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} Q^{n+1}(M(n+1)) & = & V \\ \downarrow{Q^{n+1}(\beta)} & & \downarrow{\rho} \\ Q^{n+1}(N(n+1)) & = & W \end{array}$$

if and only if the following diagram commutes

$$\begin{tikzcd} V \arrow[r, "d"] \arrow[d, "\rho"] & H^{n+2}(\mathcal{M}(n)) \arrow[d, "\alpha^*"] \\ W \arrow[r, "d"] & H^{n+2}(\mathcal{N}(n)) \end{tikzcd} .$$

This concludes the proof of Proposition A.3.

We remark that Proposition A.3 does provide an inductive procedure for obtaining information on $\mathcal{E}_{\#}(\mathcal{M})$. Since $\mathcal{E}_{\#}(\mathcal{M}(1)) = 1$, the induction can always be started. The group $T^{n+1}(\mathcal{M}(n+1);V^*)$ is isomorphic to the group $T^{n+1}(X;\pi_{n+1}(X))$, where $\mathcal{M} = \mathcal{M}_X$, obtained in the category of rational spaces as follows. $T^{n+1}(X;\pi_{n+1}(X))$ is the kernel of the homomorphism which is the composition of the universal coefficient isomorphism $H^{n+1}(X;\pi_{n+1}(X)) \longrightarrow \mathrm{Hom}(H_{n+1}(X),\pi_{n+1}(X))$ and the homomorphism $\mathrm{Hom}(H_{n+1}(X),\pi_{n+1}(X)) \longrightarrow \mathrm{Hom}(\pi_{n+1}(X),\pi_{n+1}(X))$ induced by the Hurewicz map.

Finally we note that Proposition A.3 is a generalization and an adaptation for DGAs of a result which was joint work with C.R. Curjel and whose proof was sketched in [AC 2, Lemma 5.2] .

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Dartmouth College
Hanover, New Hampshire
Max-Planck-Institut
Bonn