FOURIER COEFFICIENTS OF MODULAR FORMS OF HALF-INTEGRAL WEIGHT

Winfried Kohnen

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3
Fed. Rep. of Germany

MPI/SFB 84-23

Fourier coefficients of modular forms of half-integral weight

Winfried Kohnen, Max-Planck-Institut für Mathematik, Gottfried-Claren-Str. 26,
5300 Bonn 3, Federal Republic of Germany

Introduction

In two important papers [18], [19] Waldspurger showed that under the Shimura correspondence between Hecke eigenforms of weight $k+\frac{1}{2}$ and weight $2k$ the square of the m^{th} Fourier coefficient (m squarefree) of a form of half-integral weight is essentiall proportional to the value at $s=k$ (center of the critical strip) of the L-series of the corresponding form of integral weight twisted with the quadratic character of $\mathbb{Q}(\sqrt{(-1)^k m})$.

The main purpose of the present paper is to give a formula for the product $c(m)\overline{c(n)}$ of two arbitrary Fourier coefficients of a Hecke eigenform g of half-integral weight and of level $4N$ with N odd and squarefree in terms of certain cycle integrals of the corresponding form f of integral weight. This includes as a special case ($m=n$) Waldspurger's result for odd squarefree level and also generalizes [6], where for level 1 the constant of proportionality between the values of the twists and the squares of the Fourier coefficients in Waldspurger's theorem was given explicitly.

As corollaries we obtain results already proved for level 1 in [6], e.g. the non-negativity of the values of the twists at $s=k$ or the fact that the square of the Petersson norm of g divided by one of the periods of f is algebraic (the latter result was also obtained by Shimura [15]). As another corollary we also deduce that $c(m)=O(m^{k/2}\log m)$ and so in particular $c(m)=O(m^{k/2+\varepsilon})$ for every $\varepsilon>0$. The latter estimate has been previously proved by different methods, namely by combining Waldspurger' theorem with estimates for L-series on the critical line à la Phragmén-Lindelöf (cf. the remark in [2], Introd.). Note that the usual Hecke estimate for g gives only

$c(m)=O(m^{k/2+1/4})$, while the Ramanujan-Petersson conjecture for modular forms of half-integral weight would predict $c(m)=O(m^{k/2-1/4+\varepsilon})$.

Our method of proof is roughly speaking the following. We first construct the holomorphic kernel functions for the Shimura and Shintani liftings (Theorems 1 and 2; this is done for arbitrary odd level). Combined with a "multiplicity 1 theorem" as proved in [5] for odd squarefree level one easily deduces a formula for $c(m)\overline{c(n)}$ of the above-mentioned type (Theorem 3). We remark that for level 1 and m=1 this formula was already obtained in the author's thesis [4], and that for level 2 and m a prime a similar identity has been independently proved by Niwa [10].

Some of our methods and results have interesting implications for Heegner points on modular curves, when combined with the recent work of Gross and Zagier on Heegner points and derivatives of L-series of modular forms at the center of the critical strip [3]. This will be discussed in a later joint paper with Gross and Zagier.

Notations

For $z\in\mathbb{C}^*$ and $x\in\mathbb{C}$ we put $z^x=e^{x\log z}$, where $\log z=\log|z| + i\arg z$ and the argument is determined by $-\pi<\arg z\leq\pi$. For z in the upper half-plane H we set $q=e^{2\pi iz}$. For integers x,y with $y\neq 0$ we also write $e_y(x)$ instead of $e^{2\pi ix/y}$.

The symbol $(\frac{c}{d})$ defined for $c,d\in\mathbb{Z}$, $d\neq 0$ is used as in [14],[4]. We extend the definition to all pairs $(c,d)\in\mathbb{Z}^2\setminus\{0,0\}$ by requiring $(\frac{c}{0})$ to be ± 1 or 0 according as c is ± 1 or not.

By N we understand a natural number. We let $\Gamma_0(N) = \{(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix})\in SL_2(\mathbb{Z}) \mid \gamma\equiv 0(N)\}$ and write $\Gamma(1)$ for $SL_2(\mathbb{Z})$.

If $k\in\mathbb{Z}$ we write $M_k(N)$ ($S_k(N)$) for the space of modular forms (cusp forms) of weight k on $\Gamma_0(N)$. For N odd we denote by $S_{k+1/2}(N)$ the space of cusp forms of weight $k+1/2$ on $\Gamma_0(4N)$, which have a Fourier expansion $\sum_{n\geq 1} c(n)q^n$ with $c(n)=0$ unless $(-1)^k n\equiv 0,1(\mathrm{mod}\,4)$ (cf.[5]).

If f and g are cusp forms of weight $\kappa\in\frac{1}{2}\mathbb{Z}$ on some subgroup Γ of finite index in $\Gamma(1)$ we denote by

$$\langle f,g\rangle = \frac{1}{[\Gamma(1):\Gamma]} \int_{\Gamma\backslash H} f(z)\overline{g(z)}y^{\kappa-2}dxdy \qquad (x=\mathrm{Re}z, y=\mathrm{Im}z)$$

their Petersson product.

1. Statement of results

We let $\Gamma(1)$ act on integral binary quadratic forms $[a,b,c](x,y) = ax^2+bxy+cy^2$ by

(1) $$[a,b,c]\circ\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}(x,y) = [a,b,c](\alpha x+\beta y, \gamma x+\delta y).$$

For an integer D with $D\equiv 0,1 \pmod 4$ and a form $Q=[a,b,c]$ whose discriminant b^2-4ac is divisible by D we put

$$\omega_D(Q) = \begin{cases} 0 & \text{if } (a,b,c,D)>1\\ \left(\frac{D}{r}\right) & \text{if } (a,b,c,D)=1, \text{ where Q represents r, } (r,D)=1.\end{cases}$$

If Q represents both r and s, then 4rs may be written as x^2-Dzy^2 for some $x,y,z\in\mathbb{Z}$. Therefore $\left(\frac{D}{r}\right)=\left(\frac{D}{s}\right)$ so that $\omega_D(Q)$ is well-defined. Note that the value $\omega_D(Q)$ depends only on the $\Gamma(1)$-equivalence class of Q.

For integers $k\geq 2$. $N\geq 1$ and integers D,D' with $D,D'\equiv 0,1 \pmod 4$ and $DD'>0$ we set

(2) $$f_{k,N}(z;D,D') = \sum_{\substack{a,b,c\in\mathbb{Z}\\ b^2-4ac=DD'\\ N|a}} \omega_D(a,b,c)(az^2+bz+c)^{-k} \qquad (z\in H).$$

The series converges absolutely uniformly on compact sets, and $f_{k,N}(z;D,D')$ is a cusp form of weight 2k on $\Gamma_0(N)$ (identically zero for $(-1)^kD<0$). For D=1 these functions were introduced by Zagier ([20],App. 2) in connection with the Doi-Naganuma lifting.

For k=1 the series in (2) is not absolutely convergent. However, according to "Hecke's convergence trick" put

$$f_{1,N}(z,s;D,D') = y^s \sum_{\substack{a,b,c\in\mathbb{Z}\\ b^2-4ac=DD'\\ N|a}} \omega_D(a,b,c)(az^2+bz+c)^{-1}|az^2+bz+c|^{-s}$$

$$(z\in H;\ s\in\mathbb{C},\ \mathrm{Re}s>0).$$

We shall prove that for every $z\in H$ this function has a holomorphic continuation to $s=0$ and that

$$f_{1,N}(z;D,D') = f_{1,N}(z,0;D,D')$$

is a modular form of weight 2 on $\Gamma_0(N)$ (it tuns out to be a cusp form for N cubefree, but not in general).

Now let N be odd, k an integer ≥ 1 and let D be a fundamental discriminant (i.e. D is 1 or the discriminant of a quadratic field) with $(-1)^k D>0$. For $z,\tau\in H$ put

$$\Omega_{k,N}(z,\tau;D) = i(N)c_{k,D}^{-1} \sum_{\substack{m\geq 1\\ (-1)^k m\equiv 0,1(4)}} m^{k-1/2} \Big(\sum_{t|N} \mu(t)\left(\tfrac{D}{t}\right)t^{k-1} \qquad (3)$$

$$f_{k,N/t}(tz;D,(-1)^k m)\Big)\ e^{2\pi i m\tau}$$

with

$$c_{k,D} = (-1)^{[k/2]}|D|^{-k+1/2}\pi\binom{2k-2}{k-1}2^{-3k+2}, \quad i(N) = [\Gamma(1):\Gamma_0(N)].$$

Then $\Omega_{k,N}(\ ,\tau;D)$ is in $M_{2k}(N)$ and is in $S_{2k}(N)$ if either $k\geq 2$ or if $k=1$ and N is cubefree.

For $m\in\mathbb{N}$ with $(-1)^k m\equiv 0,1(\mathrm{mod}4)$ let $P_{k,N,m}$ be the m^{th} Poincaré series in $S_{k+1/2}(N)$ characterized by

$$(4)\qquad \langle g,P_{k,N,m}\rangle = i(4N)^{-1}\frac{\Gamma(k-1/2)}{(4\pi m)^{k-1/2}}a_g(m) \qquad (\forall g=\sum_{n\geq 1} a_g(n)q^n\in S_{k+1/2}(N)).$$

Theorem 1. <u>The function</u> $\Omega_{k,N}(z,\tau;D)$ <u>defined by (3) is for each fixed</u> $\tau\in H$ <u>a cusp form in</u> $S_{k+1/2}(N)$ <u>with respect</u> to τ. <u>More precisely one has</u>

$$\Omega_{k,N}(z,\tau;D) = i(N)c_{k,D}^{-1}\frac{(-1)^{[k/2]}3(2\pi)^k}{(k-1)!}\sum_{n\geq 1} n^{k-1}\Big(\sum_{\substack{d|n\\(d,N)=1}} \left(\tfrac{D}{d}\right)(n/d)^k \qquad (5)$$

$$P_{k,N,n^2|D|/d^2}(\tau)\Big)\ e^{2\pi i n z}.$$

Note that Theorem 1 is the precise analogue of the main theorem in [20] for modular forms of half-integral weight.

Let D be a fundamental discriminant with $(-1)^k D>0$. We define the D^{th} Shimura lifting of a function $g=\sum_{n\geq 1} c(n)q^n\in S_{k+1/2}(N)$ by

$$(6) \qquad g|\underline{S}_{k,N,D}(z) = \sum_{n\geq 1} \Big(\sum_{\substack{d|n \\ (d,N)=1}} (\tfrac{D}{d}) d^{k-1} c(n^2|D|/d^2) \Big) q^n.$$

For $f \in S_{2k}(N)$ and D and m as above set

$$(7) \qquad r_{k,N}(f;D,(-1)^k m) = \sum_{\substack{Q \bmod \Gamma_0(N) \\ |Q| = |D|m, Q(1,0)\equiv 0(N)}} \omega_D(Q) \int_{C_Q} f(z) d_{Q,k} z$$

where $Q=[a,b,c]$ runs through a set of $\Gamma_0(N)$-inequivalent integral binary quadratic forms of discriminant $|Q|=|D|m$ and with $N|a$, C_Q is the image in $\Gamma_0(N)\backslash H$ of the semicicle $a|z|^2+b\mathrm{Re}z+c=0$ (oriented from left to right if $a>0$, from right to left if $a<0$ and from $\frac{-c}{b}$ to $i\infty$ if $a=0$) and $d_{Q,k}z=(az^2+bz+c)^{k-1}dz$.

For $f \in S_{2k}(N)$ we now define the D^{th} Shintani lifting of f by

$$(8) \quad f|\underline{S}^*_{k,N,D}(z) = \sum_{\substack{m\geq 1 \\ (-1)^k m\equiv 0,1(4)}} \Big(\sum_{t|N} \mu(t)(\tfrac{D}{t}) t^{k-1}\, r_{k,Nt}(f;D,(-1)^k mt^2) \Big) e^{2\pi i m\tau}.$$

Theorem 2. <u>The mappings</u> $\underline{S}_{k,N,D}$ <u>and</u> $\underline{S}^*_{k,N,D}$ <u>defined by (6) and (8) map</u> $S_{k+1/2}(N)$ <u>to</u> $M_{2k}(N)$ <u>(to</u> $S_{2k}(N)$ <u>if either</u> $k\geq 2$ <u>or if</u> $k=1$ <u>and</u> N <u>is cubefree) and map</u> $S_{2k}(N)$ <u>to</u> $S_{k+1/2}(N)$, <u>respectively. They are adjoint on cusp forms with respect to the Petersson product. More precisely, one has</u>

$$(9) \qquad \langle g, \Omega_{k,N}(-\bar{z},\ ;D)\rangle = g|\underline{S}_{k,N,D}(z) \qquad (\forall\, g\in S_{k+1/2}(N))$$

<u>and</u>

$$(10) \qquad \langle f, \Omega_{k,n}(\ ,-\bar{\tau};D)\rangle = f|\underline{S}^*_{k,N,D}(\tau) \qquad (\forall\, f\in S_{2k}(N)).$$

Lifting maps from modular forms of half-integral weight to integral weight were first introduced by Shimura [14] and later studied by Niwa [9] who gave a non-holomorphic kernel function for them and showed that forms of level 4N are always mapped to level 2N. By Theorem 2 the subspace $S_{k+1/2}(N)$ is mapped all the way down to level N.

Lifting maps from integral to half-integral weight were first introduced by Shintani [16]. In the special case N=1, D=1 Theorems 1 and 2 were proved in [4].

That $\Omega_{k,N}(\ ,\tau;D)$ for k=1 in general is not a cusp form reflects the fact that for weight 3/2 the Shimura liftings as defined in [14] map precisely the orthogonal complement of the "space of theta functions" (cf. [14]) to cusp forms; this was conjectured by Shimura and proved later by Cipra, Kojima, Sturm a.o. Note that the "space of theta functions" is zero either for squarefree level and arbitrary character or for cubefree level and trivial character.

We now come to the main result of the paper. Let N be squarefree. We proved in [5] that in this case one can set up a theory of newforms à la Atkin-Lehner-Li-Miyake for $S_{k+1/2}(N)$; there is a canonically defined subspace $S^{new}_{k+1/1}(N) \subset S_{k+1/2}(N)$, and $S^{new}_{k+1/2}(N)$ and $S^{new}_{2k}(N)$ (subspace of newforms in $S_{2k}(N)$) are isomorphic as modules over the Hecke algebra. The lifting map $\underline{S}_{k,N,D}$ preserves old- and newforms and commutes with all Hecke operators. If $f=\sum_{n\geq 1} a(n)q^n \in S^{new}_{2k}(N)$ is a normalized Hecke eigenform ($a(1)=1$) and $g=\sum_{n\geq 1} c(n)q^n \in S^{new}_{k+1/2}(N)$ a corresponding form of half-integral weight, then the Fourier coefficients of f and g are related by

$$c(n^2|D|) = c(|D|) \sum_{\substack{d|n \\ (d,N)=1}} \mu(d)\left(\frac{D}{d}\right)d^{k-1}a(n/d) \tag{11}$$

for every fundamental discriminant D with $(-1)^kD>0$.

Theorem 3. Let m and n be positive integers with $(-1)^km, (-1)^kn \equiv 0,1 \pmod 4$ and suppose that $(-1)^kn$ is a fundamental discriminant. Then

$$\frac{c(m)\overline{c(n)}}{\langle g,g\rangle} = \frac{(-1)^{[k/2]}2^k}{\langle f,f\rangle} r_{k,N}(f;(-1)^kn,(-1)^km) \tag{12}$$

with the period integral $r_{k,N}$ defined by (7).

Remarks. i) If $(-1)^kn$ is not fundamental, then the formula becomes

$$\frac{c(m)\overline{c(n)}}{\langle g,g\rangle} = \frac{(-1)^{[k/2]}2^k}{\langle f,f\rangle} \sum_{\substack{d|h^2,\, d^2|h^2m \\ (d,N)=1}} \Bigg(\sum_{t|(d,h^2/d)} \mu(t)\left(\frac{D}{t}\right)\varepsilon_{(-1)^km}\left(\frac{d}{t}\right) r_{k,N}(f;D,(-1)^kmh^2/d^2) \Bigg),$$

where $(-1)^kn=Dh^2$ with D fundamental and $\varepsilon_{(-1)^km}$ defined as follows (cf. [21]). Write $m=m_0s^2$ with $(-1)^km_0$ fundamental. Then for any $x\in\mathbb{Z}$

$$\varepsilon_{(-1)^k m}(x) = \begin{cases} \left(\frac{(-1)^k m_0}{x_0}\right) \ell & \text{if } x=x_0\ell^2,\ \ell|s,(\frac{s}{\ell},x_0)=1 \\ 0 & \text{if } (x,s^2) \text{ is not a perfect square.} \end{cases}$$

This can be derived from the Theorem by considering the action of the Hecke operators $T_{2k}(r)$, $r|h$.

ii) Let χ be a quadratic Dirichlet character mod N with conductor $\underline{f}$ and denote by $S_{k+1/2}(N,\chi)$ the space of cusp forms of weight k+1/2 on $\Gamma_0(4N)$ with character $(\frac{4\chi(-1)}{\cdot})\chi$ which have a Fourier expansion $\sum_{n\geq 1} c(n)q^n$ with $c(n)=0$ unless $\chi(-1)(-1)^k n\equiv 0,1 \pmod 4$ (cf. [5]). Then the map $U(\underline{f})$ which replaces the n^{th} Fourier coefficient by the $\underline{f}n^{th}$ one gives an isomorphism between $S^{new}_{k+1/2}(N)$ and $S^{new}_{k+1/2}(N,\chi)$ and converseley, where $S^{new}_{k+1/2}(N,\chi)$ is an appropriateley defined space of newforms in $S_{k+1/2}(N,\chi)$ ([5]). Using this isomorphism and the multiplicative relations for the Fourier coefficients one can obtain a formula analogous to (12) for a Hecke eigenform in $S^{new}_{k+1/2}(N,\chi)$.

For each prime ℓ dividing N we let W_ℓ be the Atkin-Lehner involution on $S^{new}_{2k}(N)$ associated to ℓ and defined by

$$f|W_\ell = f|_{2k}\frac{1}{\sqrt{\ell}}\begin{pmatrix}\ell & \alpha\\ N & \ell\beta\end{pmatrix} \qquad (\alpha,\beta\in\mathbb{Z},\ell^2\beta-N\alpha=\ell),$$

where as usual for $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in SL_2(\mathbb{R})$ we put

$$f|_{2k}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = (\gamma z+\delta)^{-2k}f(\frac{\alpha z+\beta}{\gamma z+\delta}).$$

For every prime ℓ with $\ell|N$ we define $w_\ell\in\{\pm 1\}$ by

$$f|W_\ell = w_\ell f.$$

For D a fundamental discriminant with (D,N)=1 we denote by

$$L(f,D,s) = \sum_{n\geq 1} (\frac{D}{n})a(n)n^{-s} \qquad (\text{Re}s>>0)$$

the L-series of f twisted by the quadratic character $(\frac{D}{\cdot})$. Recall that L(f,D,s) has a holomorphic continuation to $\mathbb{C}$ and that

$$L^*(f,D,s)=(2\pi)^{-s}(ND^2)^{s/2}\Gamma(s)L(f,D,s)$$

satisfies the functional equation

$$L^*(f,D,s) = (-1)^k(\tfrac{D}{-N})L^*(f|W_N,D,s),$$

where

$$f|W_N = f|\prod_{\ell|N} W_\ell = f|_{2k}\frac{1}{\sqrt{N}}\begin{pmatrix}0 & -1\\ N & 0\end{pmatrix}.$$

Note that $L(f,D,k)=0$ for $(-1)^k(\frac{D}{-N})=-1$.

More generally, for an arbitrary non-zero discriminant D we may define

$$L(f,D,s) = \sum_{n\geq 1} \epsilon_D(n)a(n)n^{-s} \qquad (\mathrm{Re}s>>0)$$

with $\epsilon_D(n)$ defined after Theorem 3 (cf.[6]). Then $L(f,D,s)$ equals $\sum_{n\geq 1} (\frac{D}{n})a(n)n^{-s}$ if D is a fundamental discriminant and differs from $L(f,D_0,s)$ by a finite Euler product over the prime divisors of h if $D=D_0h^2$ with D_0 a fundamental discriminant.

Corollary 1. Let D be a discriminant with $(-1)^kD>0$ and suppose that for all prime divisors ℓ of N we have $(\frac{D}{\ell})=w_\ell$. Then

$$(13) \qquad \frac{|c(|D|)|^2}{\langle g,g\rangle} = 2^{\nu(N)}\frac{(k-1)!}{\pi^k}|D|^{k-1/2}\frac{L(f,D,k)}{\langle f,f\rangle},$$

where $\nu(N)$ denotes the number of different prime divisors of N.

For $N=1$ this the main result of [6].

For the proof, assume first that D is a fundamental discriminant. Then by (12)

$$\frac{|c(|D|)|^2}{\langle g,g\rangle} = \frac{(-1)^{[k/2]}2^k}{\langle f,f\rangle} r_{k,N}(f;D,D)$$

with

$$r_{k,N}(f;D,D) = \sum_{\substack{Q \bmod \Gamma_0(N)\\ |Q|=|D|^2,\,Q(1,0)\equiv 0(N)}} \omega_D(Q)\int_{C_Q} f(z)d_{Q,k}z.$$

Now a set of $\Gamma_0(N)$-representatives of quadratic forms Q with discriminant D^2 and with $Q(1,0)\equiv 0(\mathrm{mod}N)$ is given by

$$\{\, Q_\mu\circ W_t \,|\, \mu(\mathrm{mod}D),\, t|N,\, t>0\},$$

where $Q_\mu = [0,D,\mu]$, $W_t = \prod_{\ell|t} W_\ell$ and W_ℓ operates on integral binary quadratic forms according to

$$[a,b,c]\circ W_\ell = [a,b,c]\circ \frac{1}{\sqrt{\ell}}\begin{pmatrix} \ell & \alpha \\ N & \ell\beta \end{pmatrix} \qquad (\alpha,\beta\in\mathbb{Z},\ \ell^2\beta - N\alpha = \ell)$$

and (1).

One easily checks that $\omega_D(Q\circ W_t) = (\frac{D}{t})\omega_D(Q)$ for any form Q. Since $\omega_D(Q_\mu) = (\frac{D}{\mu})$ and $f|W_t = (\frac{D}{t})f$ by assumption we get

$$\begin{aligned} r_{k,N}(f;D,D) &= \sum_{t|N} \sum_{\mu(D)} \omega_D(Q_\mu\circ W_t) \int_{C_{Q_\mu}} (f|W_t)(z)\, d_{Q_\mu,k}(W_t z) \\ &= (\sum_{t|N} 1) \sum_{\mu(D)} (\frac{D}{\mu}) \int_{-\mu/D}^{i\infty} f(z)(Dz+\mu)^{k-1}dz \\ &= 2^{\nu(N)}(Di)^{k-1} i \int_0^\infty \sum_{\mu(D)} (\frac{D}{\mu}) f(\frac{\mu}{|D|}+it)t^{k-1}dt \\ &= 2^{\nu(N)}(Di)^{k-1} i (\frac{D}{-1})^{1/2} |D|^{1/2} \int_0^\infty \sum_{n\geq 1} (\frac{D}{n}) a(n) e^{-2\pi nt} t^{k-1}dt \\ &= 2^{\nu(N)}(-1)^{[k/2]}(2\pi)^{-k}\Gamma(k)|D|^{k-1/2}L(f,D,k), \end{aligned}$$

where in the last line we have used analytic continuation, and (13) follows.

If D is not fundamental, identity (13) can be proved along the same lines (cf. Remark i) after (12)). However, it seems easier to exploit that (13) has already been proved for fundamental discriminants and to apply the same arguments as in [6], p. 188f.

Remark. If for some prime divisor ℓ of N we have $w_\ell = -(\frac{D}{\ell})$, then $r_{k,N}(f;D,$ $=0$ and so the left-hand side of (13) is zero, too. This checks with [5], where it was shown that $c(|D|)=0$ if $w_\ell = -(\frac{D}{\ell})$ for some ℓ.

Corollary 2. <u>Under the same assumptions as in Corollary 1 we have</u> $L(f,D,k)\geq 0$.

If we square both sides of (12) and then apply (13) we get

Corollary 3. *Let* D *and* D' *be two fundamental discriminants with* $(-1)^k D, (-1)^k D' > 0$ *and* $(\frac{D}{\ell}) = (\frac{D'}{\ell}) = w_\ell$ *for all primes* ℓ *dividing* N. *Then*

$$(DD')^{k-1/2} L(f,D,k) L(f,D',k) = \frac{(2\pi)^{2k}}{(k-1)!^2} 2^{-2\nu(N)} |r_{k,N}(f;D,D')|^2$$

with $r_{k,N}(f;D,D')$ *defined by* (7).

As in [6] for level 1 formula (13) also gives information about the algebraic nature of $\langle g,g\rangle$. In fact, by Manin [8] and Shimura [13] one can attach to f two real numbers ω_+ and ω_- such that the values of $\pi^{-s} L(f,D,s)/\omega_\pm$ $(s\in\mathbb{Z}, 0<\mathrm{Re}\,s<2k, (-1)^s(\frac{D}{-1}) = \pm 1)$ are contained in the field generated over $\mathbb{Q}$ by the Fourier coefficients of f and $\sqrt{|D|}$. On the other hand, by [11],[21] $\langle f,f\rangle$ is an algebraic multiple of $\omega_+\omega_-$. Combining this with (13) gives (cf. also [15]):

Corollary 4. *Assume* g *has algebraic Fourier coefficients. Then* $\langle g,g\rangle$ *is an algebraic multiple of one of the periods* ω_+, ω_- *attached to* f.

We mention that by applying Rankin's method to g one can show using (13) (analogous argument as in [6]) that the mean value of $L(f,D,k)$ $(D\in\mathbb{Z}, (-1)^k D>0, (\frac{D}{\ell}) = w_\ell$ for all primes ℓ dividing N) essentially equals $\langle f,f\rangle$.

Finally we would like to apply (12) to estimate the size of the Fourier coefficients of g. For a fundamental discriminant D we denote by

$$L_D(s) = \sum_{n\geq 1} (\frac{D}{n}) n^{-s} \qquad (\mathrm{Re}\,s>0)$$

its L-series. We put

$$C(f) = \max_{z\in H} y^k |f(z)| \quad . \tag{14}$$

Corollary 5. *Let* D *and* D' *be two different relatively prime fundamental discriminants with* $(\frac{D}{\ell}) = (\frac{D'}{\ell}) = w_\ell$ *for all prime divisors* ℓ *of* N *and* $(-1)^k D, (-1)^k D' > 0$. *Then the following estimates hold:*

i) $|c(|D|)|\,|c(|D'|)| \leq 2^{k+\nu(N)+1} C(f) \frac{\langle g,g\rangle}{\langle f,f\rangle} (DD')^{k/2} L_{DD'}(1);$

ii) $L(f,D,k)L(f,D',k) \le \frac{4(2\pi)^{2k}}{(k-1)!^2} C(f)^2 (DD')^{1/2} L_{DD'}(1)^2.$

Here $C(f)$ is defined by (14) and $\nu(N)$ denotes the number of different prime divisors of N.

Taking into account that $L_{DD'}(1)<3\log DD'$ (cf. e.g. [17],Hilfssatz 4) and the fact that there always exists D' with $c(|D'|)$ (resp. $L(f,D',k)$) different from zero we deduce:

Corollary 6. Let D be a fundamental discriminant with $(-1)^k D>0$ and $(\frac{D}{\ell})=w_\ell$ for all primes ℓ dividing N. Then

(15) $c(|D|) = O(|D|^{k/2}\log|D|)$

and

(16) $L(f,D,k) = O(|D|^{1/2}\log^2|D|)$.

Using Deligne's theorem (previously the Ramanujan-Petersson conjecture) for f (actuall a weaker estimate would be sufficient) and the relation (11) one can show that (15) ar (16) hold for arbitrary $D\in\mathbb{Z}$ with $(-1)^k D>0$, $(\frac{D}{\ell})=w_\ell$ for all $\ell|N$. For the connection with the Ramanujan-Petersson conjecture we refer to the remarks in the Introduction.

To prove Corollary 5,i) we write for Q a quadratic form with $|Q|=DD'$

$$|f(z)Q(z,1)^{k-1}dz| = y^k|f(z)|\left(\frac{|Q(z,1)|}{y}\right)^{k-1}\frac{|dz|}{y}$$

and observe that

$$\frac{|Q(z,1)|}{y} = \sqrt{DD'} \qquad \text{(for all } z \text{ on } C_Q\text{)},$$

and that the hyperbolic length of C_Q equals

$$\int_{C_Q} \frac{|dz|}{y} = \log \varepsilon_{DD'}^2,$$

where $\varepsilon_{DD'}$ is the fundamental unit of $\mathbb{Q}(\sqrt{DD'})$ (cf. [12]). Therefore

$$\left| \int_{C_Q} f(z)d_{Q,k}z \right| \leq C(f)(DD')^{(k-1)/2}\log\varepsilon^2_{DD'}.$$

Since the number of $\Gamma_0(N)$-inequivalent primitive binary quadratic forms Q with $|Q|=DD'$ and $Q(1,0)\equiv 0(N)$ is $2^{\nu(N)}h(DD')$, where $h(DD')$ is the number of primitive $\Gamma(1)$-inequivalent forms of discriminant DD', and since $\omega_D(Q)$ vanishes if Q ($|Q|=DD'$) is non-primitiv we obtain

$$|r_{k,N}(f;D,D')| \leq 2^{\nu(N)+1}C(f)(DD')^{(k-1)/2}h(DD')\log\varepsilon_{DD'} .$$

Taking into account (12) and

$$\frac{h(DD')\log\varepsilon_{DD'}}{(DD')^{1/2}} = L_{DD'}(1)$$

we obtain i). Then ii) follows by squaring both sides of i) and applying (13).

2. Proofs of Theorems 1 to 3

2.1. Proof of Theorem 1

We shall prove Theorem 1 by applying the method of [20], i.e. we will expand both sides of (5) in a double Fourier series and then compare Fourier coefficients. We begin with

Proposition 1. For any integer $k\geq 1$ the function $f_{k,N}(z;D,(-1)^k m)$ is a modular form of weight 2k on $\Gamma_0(N)$. It is a cusp form if either $k\geq 2$ or if $k=1$ and N is cubefree.

Proposition 2. The function $f_{k,N}(z;D,(-1)^k m)$ has the expansion

$$f_{k,N}(z;D,(-1)^k m) = \sum_{n\geq 1} c_{k,N}(n;D,(-1)^k m)\, e^{2\pi i n z} \tag{17}$$

with

$$c_{k,N}(n;D,(-1)^k m) = \frac{2(-2\pi)^k}{(k-1)!}(n^2/|D|m)^{(k-1)/2}\Bigg[(-1)^{[(k+1)/2]}\left(\frac{D}{n/\sqrt{m/|D|}}\right)\delta\left(\frac{n}{\sqrt{m/|D|}}\right)|D|^{-1/2} + \pi\sqrt{2}\,(n^2/|D|m)^{1/4}\sum_{a\geq 1, N|a} a^{-1/2}S_{a,D,(-1)^k m}(|D|m,n)\,J_{k-1/2}\left(\frac{\pi n\sqrt{|D|m}}{a}\right)\Bigg];$$

here

$$\delta(x) = \begin{cases} 1 & (x\in\mathbb{Z}) \\ 0 & (x\notin\mathbb{Z}), \end{cases}$$

(18) $$S_{a,D,(-1)^k m}(|D|m,n) = \sum_{b(2a),\,b^2\equiv|D|m(4a)} \omega_D\Big(a,b,\frac{b^2-|D|m}{4a}\Big)\, e_{2a}(nb)$$

is a finite exponential sum and

$$J_{k-1/2}(t) = \Big(\frac{t}{2}\Big)^{k-1/2} \sum_{r\geq 0} (-1)^r \frac{(t/2)^{2r}}{r!\Gamma(k+1/2+r)}$$

is the Bessel function of order k-1/2.

Proof. The calculations are essentially the same as those in [20],App. 2. We shall abbreviate $f_{k,N}(z) = f_{k,N}(z;D,(-1)^k m)$.

First suppose $k\geq 2$. Then the assertions of Proposition 1 are obvious. As for (17), noticing $\omega_D(-a,-b,-c)=(\frac{D}{-1})\omega_D(a,b,c)$ and $(\frac{D}{-1})=(-1)^k$ we write

$$f_{k,N}(z) = f^0_{k,N}(z) + 2 \sum_{a\geq 1,\,N|a} f^a_{k,N}(z)$$

with

$$f^0_{k,N}(z) = 2 \sum_{b>0,\,c\in\mathbb{Z},\,b^2=|D|m} \omega_D(0,b,c)\ (bz+c)^{-k}$$

and

$$f^a_{k,N}(z) = \sum_{b\in\mathbb{Z},\,b^2\equiv|D|m(4a)} \omega_D\Big(a,b,\frac{b^2-|D|m}{4a}\Big)\ \Big(az^2+bz+\frac{b^2-|D|m}{4a}\Big)^{-k}.$$

If $|D|m=b^2$ is a square, then since D is a fundamental discriminant we must have $D|b$, s $m=|D|f^2$ for a positive integer f. Then observing $\omega_D(0,b,c)=(\frac{D}{c})$ we have

$$\begin{aligned} f^0_{k,N}(z) &= 2 \sum_{c\in\mathbb{Z}} \Big(\frac{D}{c}\Big)(|D|fz+c)^{-k} \\ &= 2 \sum_{r(D)} \Big(\frac{D}{r}\Big)|D|^{-k} \sum_{n\in\mathbb{Z}} \Big(fz+\frac{r}{|D|}+n\Big)^{-k}. \end{aligned}$$

Using Lipschitz's formula

$$\sum_{n\in\mathbb{Z}} (\tau+n)^{-s} = \frac{e^{-\pi is/2}(2\pi)^s}{\Gamma(s)} \sum_{n\geq 1} n^{s-1}e^{2\pi in\tau} \qquad (\tau\in H, s\in\mathbb{C}, \mathrm{Re}\,s>1)$$

with $\tau=fz+r/|D|$ and $s=k$ and the value of the Gauss sum

$$\sum_{r(D)} \left(\frac{D}{r}\right) e^{2\pi i n r/|D|} = \left(\frac{D}{n}\right)\left(\frac{D}{-1}\right)^{1/2} |D|^{1/2}$$

we obtain

(19) $$f^0_{k,N}(z) = (-1)^{[k/2]} |D|^{-k+1/2} \frac{2(2\pi)^k}{(k-1)!} \sum_{n\geq 1} n^{k-1} e^{2\pi i n f z} .$$

The sum $f^a_{k,N}(z)$ we write as

$$f^a_{k,N}(z) = \sum_{b(2a),\, b^2\equiv |D| m (4a)} \omega_D\left(a,b,\frac{b^2-|D|m}{4a}\right) \sum_{n\in\mathbb{Z}} \left(a(z+n)^2+b(z+n)+\frac{b^2-|D|m}{4a} \right)^{-k} .$$

The n^{th} Fourier coefficient ($n\geq 1$) of the inner sum is

$$\int_{-\infty+iC}^{\infty+iC} \left(az^2+bz+\frac{b^2-|D|m}{4a}\right)^{-k} e^{-2\pi i n z} dz \qquad (C>0)$$

$$= (-1)^{k+1} i e^{\pi i n b/a} \int_{C-i\infty}^{C+i\infty} e^{2\pi n t}\left(at^2+\frac{|D|m}{4a}\right)^{-k} dt \qquad \left(t=-i\left(z+\frac{b}{2a}\right)\right)$$

(20)

$$= \frac{(-1)^k 2^{k+1/2} \pi^{k+1} n^{k-1/2}}{(|D|m)^{k/2-1/4}\sqrt{a}\,(k-1)!} J_{k-1/2}\left(\frac{\pi n\sqrt{|D|m}}{a}\right) e^{\pi i n b/a} \qquad ([1], 29.3.57).$$

Equation (17) for $k\geq 2$ follows from this and from (19).

Now let us suppose $k=1$. In the same way as above we break up the sum $f_{1,N}(z,s)$ $= f_{1,N}(z,s;D,-m)$ as

$$f_{1,N}(z,s) = f^0_{1,N}(z,s) + 2 \sum_{a\geq 1,\, N|a} f^a_{1,N}(z,s)$$

with

$$f^0_{1,N}(z,s) = 2y^s \sum_{b>0,\, c\in\mathbb{Z},\, b^2=|D|m} \omega_D(0,b,c)\phi_s(bz+c)$$

and

$$f^a_{1,N}(z,s) = y^s \sum_{b\in\mathbb{Z},\, b^2\equiv |D|m(4a)} \omega_D\left(a,b,\frac{b^2-|D|m}{4a}\right)\phi_s\left(az^2+bz+\frac{b^2-|D|m}{4a}\right),$$

where we have used the abbreviation

$$\phi_s(z) = z^{-1}|z|^{-s} .$$

Setting $m=|D|f^2$ with $f\in\mathbb{N}$ we obtain

$$f^{0}_{1,N}(z,s) = 2y^s \sum_{r(D)} \left(\frac{D}{r}\right)\phi_s(|D|) \sum_{n\in\mathbb{Z}} \phi_s\left(fz+\frac{r}{|D|}+n\right)$$

$$= 2y^s \sum_{r(D)} \left(\frac{D}{r}\right)\phi_s(|D|)\left[\phi_s\left(fz+\frac{r}{|D|}\right) + \right.$$

$$\left. + \sum_{n\geq 1}\left\{\phi_s\left(fz+\frac{r}{|D|}+n\right) + \phi_s\left(fz+\frac{r}{|D|}-n\right)\right\}\right].$$

The series in square brackets is absolutely convergent for $s=0$ and at $s=0$ equals

$$\pi\cot\pi\left(fz+\frac{r}{|D|}\right) = -\pi i - 2\pi i \sum_{n\geq 1} e^{2\pi i n(fz+r/|D|)},$$

so

$$\lim_{s\to 0} f^{0}_{1,N}(z,s) = 2\sum_{r(D)}\left(\frac{D}{r}\right)\frac{1}{|D|}\left[-\pi i - 2\pi i \sum_{n\geq 1} e^{2\pi i n(fz+r/|D|)}\right]$$

$$= |D|^{-1/2}\, 4\pi \sum_{n\geq 1}\left(\frac{D}{n}\right)e^{2\pi i n f z}$$

(note that $\sum_{r(D)}\left(\frac{D}{r}\right) = 0$ since $\left(\frac{D}{\cdot}\right)$ is a non-trivial character).

For the sum $f^{a}_{1,N}(z,s)$ with $a\geq 1$ we have

$$f^{a}_{1,N}(z,s) = y^s \sum_{b(2a),\, b^2\equiv |D|m(4a)} \omega_D\left(a,b,\frac{b^2-|D|m}{4a}\right) \sum_{n\in\mathbb{Z}} \phi_s\left(a(z+n)^2+b(z+n) + \frac{b^2-|D|m}{4a}\right).$$

The inner sum has a Fourier expansion

$$(21)\qquad \sum_{n\in\mathbb{Z}} c_s(n;y)e^{2\pi i n z} \qquad\qquad (y=\operatorname{Im} z)$$

with

$$c_s(n;y) = \int_{-\infty+iC}^{\infty+iC} \phi_s\left(az^2+bz+\frac{b^2-|D|m}{4a}\right) e^{-2\pi i n z}dz \qquad\qquad (C>0)$$

$$(22)\qquad = ie^{\pi i n b/a}\int_{C-i\infty}^{C+i\infty} \phi_s\left(at^2+\frac{|D|m}{4a}\right) e^{2\pi n t}dt$$

As is easy to see, the integral converges absolutely, is holomorphic and satisfies a uniform estimate for $\operatorname{Re}s>-\frac{1}{2}$ which suffices to make the series (21) absolutely convergent for all s with $\operatorname{Re}s>-\frac{1}{2}$. Therefore this series has a limit at $s=0$ which is obtained by setting s=0 in each term. Now the poles of the integrand in (22) are on the imaginary axis, to the left of the line of integration; therefore for $n\leq 0$ we can deform the path of integration to the right up to ∞ without crossing any poles and obtain

$$c_0(n;y) = 0 \qquad (n\leq 0).$$

For $n>0$ we can use equation (20) which is valid also for $k=1$ to get

$$c_0(n;y) = \frac{-2^{3/2}\pi^2 n^{1/2}}{(|D|m)^{1/4}\sqrt{a}} J_{1/2}\left(\frac{\pi n\sqrt{|D|m}}{a}\right) e^{\pi i n b/a} \qquad (n>0).$$

Summarizing we have shown that

$$f_{1,N}(z) = \lim_{s\to 0} f_{1,N}(z,s)$$

exists, is holomorphic in z and has a Fourier expansion given by (17).

To see that $f_{1,N}$ is holomorphic at all cusps, note that for $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\Gamma(1)$ we have

$$(23)\qquad (\gamma z+\delta)^{-2} f_{1,N}\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right) = \lim_{s\to 0} \sum_{\substack{Q\\ |Q|=|D|m,\, Q(1,0)\equiv 0(N)}} \omega_D(Q)\phi_s(Q\circ A(z,1))$$

with the sum over all quadratic forms Q with discriminant $|Q|=|D|m$ and $Q(1,0)$ divisible by N. Since $\omega_D(Q)$ depends only on the $\Gamma(1)$-equivalence class of Q the sum in (

is equal to

$$\sum_{\substack{Q\\ |Q|=|D|m,\, (Q\circ A^{-1})(1,0)\equiv 0(N)}} \omega_D(Q)\phi_s(Q(z,1)),$$

and from this by the same type of argument as above one deduces that

$$(\gamma z+\delta)^{-2} f_{1,N}\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)$$

has a Fourier expansion in $e^{2\pi i z/w}$ (with a certain $w\in\mathbb{N}$) with no negative powers and convergent for all $z\in H$.

Since by definition

$$f_{1,N}\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right) = (\gamma z+\delta)^2 f_{1,N}(z) \qquad \left(\forall \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\Gamma_0(N)\right)$$

we have shown that $f_{1,N}$ is a modular form of weight 2 on $\Gamma_0(N)$. Moreover, the consta term in the Fourier expansion of (23) is easily seen to be equal to the constant term in the Fourier expansion of

$$\lim_{s\to 0} \sum_{\substack{Q \\ |Q|=|D|m,\ (Q\cdot A^{-1})(1,0)=0}} \left(\frac{D}{Q(1,0)}\right)\phi_s(Q(z,1)),$$

and this can be evaluated in the same manner as above using the partial fractional decomposition of the cotangent and the identity

$$\pi\cot\pi z = -\pi i - 2\pi i \sum_{n\geq 1} e^{2\pi i n z} \qquad (z\in H).$$

From this one can conclude that $f_{1,N}$ vanishes at all cusps for N cubefree. (This is not true for general N, for example let $p\equiv 3(4)$ be a prime, and let $D=-p$, $N=p^3$, $A=\begin{pmatrix}1 & 0\\ -p & 1\end{pmatrix}$).

Our next task will be to compute the Fourier expansion of the Poincaré series $P_{k,N,m}\in S_{k+1/2}(N)$ $((-1)^k m\equiv 0,1(4))$. Let us first recall some definitions ([14],[5]). Let $\underline{G}_{k+1/2}$ be the group consisting of pairs $(A,\phi(z))$, where $A=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in GL_2^+(\mathbb{R})$ and $\phi(z)$ is a complex valued holomorphic function on H satisfying

$$|\phi(z)| = (\det A)^{-k/2-1/4}|\gamma z+\delta|^{k+1/2},$$

with group law defined by

$$(A,\phi(z))(B,\psi(z)) = (AB,\phi(Bz)\psi(z)).$$

The group algebra of $\underline{G}_{k+1/2}$ over $\mathbb{C}$ acts on functions $g:H\to\mathbb{C}$ by

$$g|\sum_\nu c_\nu(A_\nu,\phi_\nu) = \sum_\nu c_\nu\phi_\nu(z)^{-1}g(A_\nu z).$$

We have an injection $\Gamma_0(4N)\hookrightarrow\underline{G}_{k+1/2}$ given by

$$A\mapsto (A,\left(\tfrac{\gamma}{\delta}\right)\left(\tfrac{-4}{\delta}\right)^{-k-1/2}(\gamma z+\delta)^{k+1/2}).$$

If there is no confusion we shall write A^* for the image of A.

Denote by $\underline{S}_{k+1/2}(4N)$ the space of cusp forms of weight k+1/2 on $\Gamma_0(4N)$ ([14]). Then, as was shown in [5], the orthogonal projection pr from $\underline{S}_{k+1/2}(4N)$ to $S_{k+1/2}(N)$ is given by

$$g|pr = (-1)^{[(k+1)/2]}\frac{1}{3\sqrt{2}}\Big(\sum_{\nu(4)} g|\xi A_\nu^*\Big) + \frac{1}{3}g$$

where

$$\xi = \left(\begin{pmatrix}4 & 1\\ 0 & 4\end{pmatrix}, e^{(2k+1)\pi i/4}\right), \quad A_\nu = \begin{pmatrix}1 & 0\\ 4N\nu & 1\end{pmatrix}.$$

Clearly we have

$$P_{k,N,m} = \underline{P}_{k,N,m}|pr$$

where $\underline{P}_{k,N,m}$ is the m^{th} Poincare series in $\underline{S}_{k+1/2}(4N)$ characterized by

$$\langle g, P_{k,N,m}\rangle = i_{4N}^{-1}\frac{\Gamma(k-1/2)}{(4\pi m)^{k-1/2}}a_g(n) \qquad (\forall\, g = \sum_{n\geq 1} a_g(n)q^n \in S_{k+1/2}(4N)\ ).$$

Proposition 3. <u>Put</u> $\alpha=(\frac{-4}{N})$ <u>and set</u>

$$\eta^{(-\alpha/N)} = \left(\begin{pmatrix}1 & 0\\ -\alpha N & 1\end{pmatrix}, (-\alpha Nz+1)^{k+1/2}\right),\ \eta^{(1/2N)} = \left(\begin{pmatrix}1 & 0\\ 2N & 1\end{pmatrix}, (2Nz+1)^{k+1/2}\right).$$

<u>Let</u> $g\in S_{k+1/2}(4N)$ <u>and write</u>

$$g(z) = \sum_{n\geq 1} a(n)q^n,$$

$$g|\eta^{(-\alpha/N)}(z) = \sum_{n\geq 1} a^{(-\alpha/N)}(n)q^{n/4}$$

$$g|\eta^{(1/2N)}(z) = \sum_{n\geq 1,\,(-1)^k n\equiv 1(4)} a^{(1/2N)}(n)q^{n/4}$$

<u>(expansions of g at the cusps</u> $i\infty$, $-\alpha/N$ <u>and</u> $1/2N$, <u>respectively). Then</u>

$$g|pr(z) = \frac{2}{3}\Bigg[\sum_{n\geq 1,\, n\equiv 0(4)}\left(a(n)+(1-(-1)^k i)2^{2k-1}i^{n/4}a^{(-\alpha/n)}(\tfrac{n}{4})\right)q^n$$

(24)

$$+\sum_{n\geq 1,\,(-1)^k n\equiv 1(4)}\left(a(n)+2^{k-1}(\tfrac{(-1)^k n}{2})a^{(1/2N)}(n)\right)q^n\Bigg].$$

Proof. Since

$$\xi A_2^* = \left(\begin{pmatrix}16 & 0\\ 0 & 16\end{pmatrix}, 1\right)\begin{pmatrix}1+2N & (1+N)/2\\ 8N & 1+2N\end{pmatrix}^* \xi^{-1}$$

we have

$$\begin{aligned} g|\xi A_0^* + g|\xi A_2^* &= g|\xi + g|\xi^{-1}\\ &= e^{-(2k+1)\pi i/4}g(z+\tfrac{1}{4}) + e^{(2k+1)\pi i/4}g(z-\tfrac{1}{4})\\ &= (-1)^{[(k+1)/2]}\sqrt{2}\Bigg(\sum_{n\geq 1,\,(-1)^k n\equiv 0,1(4)} a(n)q^n \end{aligned}$$

$$- \sum_{n\geq 1, (-1)^k n\equiv 0,1(4)} a(n)q^n \Big) ;$$

thus

$$(-1)^{[(k+1)/2]} \frac{1}{3\sqrt{2}} \left(g|\xi A_0^* + g|\xi A_2^* \right) + \frac{1}{3}g = \frac{2}{3} \sum_{n\geq 1, (-1)^k n\equiv 0,1(4)} a(n)q^n.$$

Next from

$$\xi A_{-\alpha}^* = \left(\begin{pmatrix} 4(1-\alpha N) & 1 \\ -16\alpha N & 4 \end{pmatrix}, e^{(2k+1)\pi i/4}(-4\alpha Nz+1)^{k+1/2}\right)$$

$$= \begin{pmatrix} (\frac{1+\alpha N}{2})^2 & \frac{\alpha N-1}{4}+1 \\ N^2+3\alpha N & \alpha N+4 \end{pmatrix}^* \eta^{(-\alpha/N)} \left(\begin{pmatrix} 16 & 1 \\ 0 & 1 \end{pmatrix}, 2^{-2k-1} e^{(2k+1)\pi i/4} \right)$$

we obtain

$$(-1)^{[(k+1)/2]} \frac{1}{3\sqrt{2}} g|\xi A_{-\alpha}^* = \frac{1}{6}(1-(-1)^k i)2^{2k+1} \sum_{n\geq 1} a^{(-\alpha/N)}(n) e^{2\pi i n(16z+1)/4}$$

$$= \frac{2}{3}(1-(-1)^k i)2^{2k-1} \sum_{n\geq 1} i^n a^{(-\alpha/N)}(n) q^{4n}.$$

Finally

$$\xi A_{\alpha}^* = \left(\begin{pmatrix} 4(1+\alpha N) & 1 \\ 16\alpha N & 4 \end{pmatrix}, e^{(2k+1)\pi i/4}(4\alpha Nz+1)^{k+1/2}\right)$$

$$= \begin{pmatrix} \frac{(1+\alpha N)(1+N)}{2} - N & \frac{-\alpha N+1}{4} \\ 2\alpha N(N+1)-4N & -\alpha N+2 \end{pmatrix}^* \eta^{(1/2N)} \left(\begin{pmatrix} 8 & 1 \\ 0 & 2 \end{pmatrix}, 2^{-k-1/2} e^{(2k+1)\pi i/4} \right)$$

and therefore

$$(-1)^{[(k+1)/2]} \frac{1}{3\sqrt{2}} g|\xi A_{\alpha}^* = \frac{1}{6}(1-(-1)^k i)2^{k+1/2} \sum_{n\geq 1, (-1)^k n\equiv 1(4)} a^{(1/2N)}(n) e^{\pi i n/4} q^{n}$$

$$= \frac{2}{3} 2^{k-1} \sum_{n\geq 1, (-1)^k n\equiv 1(4)} \left(\frac{(-1)^k n}{2}\right) a^{(1/2N)}(n) q^n.$$

Using (24) we will now prove

Proposition 4. Let $m\geq 1, (-1)^k m\equiv 0,1(4)$. Then

$$(25)\qquad P_{k,N,m}(z) = \sum_{n\geq 1, (-1)^k n\equiv 0,1(4)} g_{k,N,m}(n)\, e^{2\pi i n z}$$

with

$$g_{k,N,m}(n) = \frac{2}{3}\left[\delta_{m,n} + (-1)^{[(k+1)/2]}\pi\sqrt{2}(n/m)^{k/2-1/4} \sum_{\substack{c\geq 1\\ N|c}} H_c(n,m) J_{k-\frac{1}{2}}\left(\frac{\pi}{c}\sqrt{mn}\right)\right]$$

Here $\delta_{m,n}$ is the Kronecker delta,

$$(26)\qquad H_c(n,m) = (1-(-1)^k i)\left(1+\left(\frac{4}{c}\right)\right)\frac{1}{4c} \sum_{\delta(4c)^*} \left(\frac{4c}{\delta}\right)\left(\frac{-4}{\delta}\right)^{k+1/2} e_{4c}(n\delta+m\delta^{-1})$$

$$(\delta^{-1}\in\mathbb{Z}, \delta\delta^{-1}\equiv 1(4c))$$

is a finite exponential sum (the star in the summation sign shall indicate that δ runs through a primitive residue system mod 4c) and $J_{k-1/2}$ is the Bessel function of order $k-1/2$.

Proof. Write $\underline{P}_m = \underline{P}_{k,N,m}$. With the notation of Proposition 3 let us set

$$\underline{P}_m(z) = \sum_{n\geq 1} g_m(n) q^n,$$

$$\underline{P}_m | \eta^{(-\alpha/N)}(z) = \sum_{n\geq 1} g_m^{(-\alpha/N)}(n) q^{n/4}$$

$$\underline{P}_m | \eta^{(1/2N)}(z) = \sum_{n\geq 1, (-1)^k n\equiv 1(4)} g_m^{(1/2N)}(n) q^{n/4}.$$

Let us first suppose $k\geq 2$. Then P_m is given by

$$\underline{P}_m = \frac{1}{2} \sum_{A\in\Gamma_\infty\backslash\Gamma_0(4N)} j(A,z)^{-2k-1} e(mAz) \qquad (e(z)=e^{2\pi i z})$$

$(\Gamma_\infty = \{\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix} | n\in\mathbb{Z}\}$; $j(A,z) = \left(\frac{\gamma}{\delta}\right)\left(\frac{-4}{\delta}\right)^{-1/2}(\gamma z+\delta)^{1/2}$ for $A = \begin{pmatrix}* & *\\ \gamma & \delta\end{pmatrix}\in\Gamma_0(4N))$.

We shall use the following

Lemma. Let $A = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{R})$. Let $c>0, \gamma>0$ and $k\in\mathbb{Z}, k\geq 2$. Then

$$\sum_{r\in\mathbb{Z}} (c(z+r)+d)^{-k-1/2}\, e\left(\gamma\frac{a(z+r)+b}{c(z+r)+d}\right)$$

(28) $$= \pi\sqrt{2}(-1)^{[(k+1)/2]}(1-(-1)^k i)\,\frac{1}{c}\sum_{n\geq 1} (n/\gamma)^{k/2-1/4}\, J_{k-1/2}\!\left(\frac{4\pi}{c}\sqrt{n\gamma}\right) e_c(\gamma a+nd)\, q^n \qquad (z\in H).$$

Proof. The proof (see also [20]) is a standard application of the Poisson summation formula. It suffices to treat the case $A=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$, i.e. to show that

(29) $$\sum_{r\in\mathbb{Z}} (z+r)^{-k-1/2} e\left(\frac{-\gamma}{z+r}\right) = 2\pi i^{-k} e^{-\pi i/4} \sum_{n\geq 1} (n/\gamma)^{k/2-1/4} J_{k-1/2}(4\pi\sqrt{n\gamma})\, q^n$$

since (28) follows from this after replacing z by $z+\frac{d}{c}$ and γ by γ/c^2 and multiplying the resulting equation with $c^{-k-1/2}e_c(\gamma a)$.

Now observe that for $k\geq 2$ the series on the left-hand side of (29) converges absolutely uniformly on compact subsets of H to a holomorphic function, which has period 1 and vanishes at infinity. It therefore has a Fourier expansion

$$\sum_{n\geq 1} c(n) q^n$$

with

$$c(n) = \int_{iC}^{iC+1} e^{-2\pi inz} \left(\sum_{r\in\mathbb{Z}} (z+r)^{-k-1/2} e\left(\frac{-\gamma}{z+r}\right)\right) dz \qquad (C>0)$$

$$= \int_{C-i\infty}^{C+i\infty} e^{-2\pi inz} z^{-k-1/2} e^{-2\pi i\gamma/z}\, dz\ .$$

After substituting $z=i\sqrt{\gamma/n}\,s$ we obtain

$$c(n) = 2\pi i^{-k} e^{-\pi i/4} (n/\gamma)^{k/2-1/4} \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} s^{-k-1/2} e^{2\pi\sqrt{n\gamma}\,(s-s^{-1})}\, ds$$

$$= 2\pi i^{-k} e^{-\pi i/4} (n/\gamma)^{k/2-1/4} J_{k-1/2}(4\pi\sqrt{n\gamma}\,)$$

(the integral is evaluated in [1],29.3.80; the function $t\mapsto (t/\mu)^{k/2-1/4} J_{k-\frac{1}{2}}(2\sqrt{\mu t}\,)$ is the inverse Laplace transform of $s\mapsto s^{-k-1/2}e^{-\mu/s}$). This proves (28).

We shall now first compute $g_m(n)$. We break up the sum in (27) into the terms with $c=0$ and twice the terms with $c>0$ (note $j(-1,z)=1$). Then we get

$$(30)\quad \underline{P}_m(z) = e(mz) + \sum_{\substack{c>0,\, ad-4Nbc=1\\ (d,4Nc)=1}} \left(\frac{4Nc}{d}\right)\left(\frac{-4}{d}\right)^{k+1/2}(4Ncz+d)^{-k-1/2}e\left(m\frac{az+b}{4Ncz+d}\right),$$

where the sum extends over all pairs $(c,d)\in\mathbb{Z}^2$ with $c>0$, $(d,4Nc)=1$ and to each such pair we have associated a and b in $\mathbb{Z}$ with $ad-4Nbc=1$. Since $\left(\frac{4Nc}{\cdot}\right)$ and $\left(\frac{-4}{\cdot}\right)$ have period 4Nc, we may write

$$\underline{P}_m(z) = e(mz) + \sum_{\substack{c\geq 1,\ d(4Nc)^*\\ ad-4Nbc=1}} \left(\frac{4Nc}{d}\right)\left(\frac{-4}{d}\right)^{k+1/2} \sum_{r\in\mathbb{Z}} (4Nc(z+r)+d)^{-k-1/2}$$
$$e\left(m\frac{a(z+r)+b}{4Nc(z+r)+d}\right).$$

The Fourier expansion of the inner sum is now given by the Lemma, and we obtain

$$(31)\quad g_m(n) = \delta_{m,n} + \pi\sqrt{2}\,(-1)^{[(k+1)/2]}\,(1-(-1)^k i)(n/m)^{k/2-1/4}$$
$$\sum_{c\geq 1} H_{4Nc}(n,m)J_{k-1/2}\left(\frac{\pi}{Nc}\sqrt{mn}\right),$$

where for $u,v\in\mathbb{Z}$ we define

$$\underline{H}_{4Nc}(u,v) = \frac{1}{4Nc}\sum_{\delta(4Nc)^*}\left(\frac{4Nc}{\delta}\right)\left(\frac{-4}{\delta}\right)^{k+1/2}e_{4Nc}(u\delta+v\delta^{-1}) \qquad (\delta^{-1}\in\mathbb{Z},\delta\delta^{-1}\equiv 1(4Nc))$$

Next we compute $g_m^{(-\alpha/N)}(n)$. From (30) and the identity

$$\left(\frac{4Nc}{d}\right)(-\alpha Nz+1)^{1/2}\left(4Nc\frac{z}{-\alpha Nz+1}+d\right)^{1/2} = \left(\frac{4N}{d}\right)\left(\frac{4c-\alpha d}{d}\right)(N(4c-\alpha d)z+d)^{1/2} \qquad (c>0)$$

we obtain

$$\underline{P}_m|\eta^{(-\alpha/N)}(z) = (-\alpha Nz+1)^{-k-1/2}\,P_m\left(\frac{z}{-\alpha Nz+1}\right)$$
$$= (-\alpha Nz+1)^{-k-1/2}e\left(m\frac{z}{-\alpha Nz+1}\right) + \sum_{\substack{c>0,\,d,\,(d,4Nc)=1\\ ad-4Nbc=1}}\left(\frac{4N}{d}\right)\left(\frac{4c-\alpha d}{d}\right)\left(\frac{-4}{d}\right)^{k+1/2}$$
$$(N(4c-\alpha d)z+d)^{-k-1/2}e\left(m\frac{(a-b\alpha N)z+b}{N(4c-\alpha d)z+d}\right).$$

After a substitution $4c-\alpha d\mapsto c$, $a-b\alpha N\mapsto a$ in the sum we see that

$$\underline{P}_m|\eta^{(-\alpha/N)}(z) = (-\alpha Nz+1)^{-k-1/2}e\left(m\frac{z}{-\alpha Nz+1}\right) + \sum_{\substack{c,d\in\mathbb{Z},\,c\equiv-\alpha d(4),\,c>-\alpha d\\ (4Nc,d)=1,\,ad-bNc=1}}\left(\frac{4Nc}{d}\right)\left(\frac{-4}{d}\right)^{k+1/2}$$
$$(Ncz+d)^{-k-1/2}\ e\left(m\frac{az+b}{Ncz+d}\right).$$

We now break up the sum over c into the sum over the terms with c>0 and the sum over the terms with c<0 and then in the latter sum replace a,b,c and d by their negatives. Then we obtain after a short calculation

$$\underline{P}_m|\eta^{(-\alpha/N)}(z) = (-\alpha Nz+1)^{-k-1/2} e(m\frac{z}{-\alpha Nz+1}) + \sum_{\substack{c>0,d,c\neq-\alpha d,c\equiv-\alpha d(4)\\(d,4Nc)=1,ad-bNc=1}} (\frac{4Nc}{d})(\frac{-4}{d})^{k+1/2} (Ncz+d)^{-k-1/2}\, e(m\frac{az+b}{Ncz+d})$$

$$= \sum_{\substack{c>0,d,c\equiv1(2),d\equiv-Nc(4)\\(d,4Nc)=1,ad-bNc=1}} (\frac{4Nc}{d})(\frac{-4}{d})^{k+1/2}(Ncz+d)^{-k-1/2}\, e(m\frac{az+b}{Ncz+d}),$$

and by the periodicity of $(\frac{4Nc}{\ })$ and $(\frac{-4}{\ })$ we get

$$\underline{P}_m|\eta^{(-\alpha/N)}(z) = \sum_{c\geq1,c\equiv1(2)} (\frac{-4}{-Nc})^{k+1/2} \sum_{\substack{d(4Nc)^*,d\equiv-Nc(4)\\ad-bNc=1}} (\frac{4Nc}{d}) \sum_{r\in\mathbb{Z}} (4Nc(\frac{z}{4}+r)+d)^{-k-1/2}\, e(4m\frac{a(z/4+r)\quad+b/4}{4Nc(z/4+r)+d});$$

the Fourier expansion of the inner sum is given by the Lemma, and we see that we pick up a sum

$$(32)\quad \frac{1}{4Nc}(\frac{-4}{-Nc})^{k+1/2} \sum_{\delta(4Nc)^*,\delta\equiv-Nc(4)} (\frac{4Nc}{\delta})\, e_{4Nc}(n\delta+4m\delta^{-1}) \qquad (\delta^{-1}\in\mathbb{Z},\delta\delta^{-1}\equiv1(Nc)).$$

In (32) replace δ by 4δ-Nc. Then after a short calculation we find that (32) is equal to

$$\frac{(-1)^k i}{4}\, i^{-n}(\frac{4}{-Nc})(\frac{-4}{Nc})^{-k-1/2} \sum_{\delta(Nc)^*} (\frac{\delta}{Nc})\, e_{Nc}(n\delta+4^{-1}n\delta^{-1}) \qquad (4^{-1}\in\mathbb{Z},44^{-1}\equiv1(Nc)).$$

We thus conclude

$$(33)\quad g_m^{(-\alpha/N)}(n) = \pi\sqrt{2}\,(-1)^{[(k+1)/2]}(1-(-1)^k i)\, i^{-n}\, \frac{(-1)^k i}{4}\,(n/4m)^{k/2-1/4} \sum_{c\geq1,c\equiv1(2)} \underline{H}'_{Nc}(n,4^{-1}m)\, J_{k-1/2}(\frac{2\pi}{Nc}\sqrt{mn}),$$

where for $u,v\in\mathbb{Z}$ we have put

$$\underline{H}'_{Nc}(u,v) = \left(\frac{4}{-Nc}\right)\left(\frac{-4}{Nc}\right)^{-k-1/2} \sum_{\delta(Nc)^*} \left(\frac{\delta}{Nc}\right) e_{Nc}(u\delta+v\delta^{-1}).$$

Finally we have to compute $g_m^{(1/2N)}(n)$. Arguing analogously as above we obtain

$$\underline{P}_m|\eta^{(1/2N)}(z) = \sum_{\substack{c>0,c\equiv 1(2),d,(d,4Nc)=1\\ ad-2Nbc=1}} \left(\frac{8Nc}{d}\right)\left(\frac{-4}{d}\right)^{k+1/2}(2Ncz+d)^{k+1/2}\, e\left(m\frac{az+b}{2Ncz+d}\right)$$

and from this

$$g_m^{(1/2N)}(n) = \pi\sqrt{2}\,(-1)^{[(k+1)/2]}(1-(-1)^k i)(n/4m)^{k/2-1/4} \tag{34}$$
$$\sum_{c\geq 1,c\equiv 1(2)} H_{8Nc}(n,4m)J_{k-1/2}\left(\frac{\pi}{Nc}\sqrt{mn}\right).$$

By (31), (33) and (34) and Proposition 3 we must still check that

$$H_{Nc}(n,m) = \underline{H}_{4Nc}(n,m) + \begin{cases} 0 & \text{if } c\equiv 0(2) \\ (1-(-1)^k i)\frac{(-1)^k i}{4}\, \underline{H}'_{Nc}(\frac{n}{4},4^{-1}m) & \text{if } c\equiv 1(2), n\equiv 0(4) \\ \frac{1}{\sqrt{2}}\left(\frac{(-1)^k n}{2}\right) \underline{H}_{8Nc}(n,4m) & \text{if } c\equiv 1(2),\ (-1)^k n\equiv 1(4). \end{cases} \tag{35}$$

For $c\equiv 0(2)$ there is nothing to prove. Assume c odd. First observe that if $(4c_1,Nc_2)=1$ we have

$$\underline{H}_{4Nc_1,c_2}(u,v) = \underline{H}_{4c_1}(u,v(N^{-1}c_2^{-1})^2)\, \underline{H}'_{Nc_2}(u,vc_1^{-2}) \tag{36}$$

$(c_1^{-1},N^{-1},c_2^{-1}\in \mathbb{Z}, c_1^{-1}c_1\equiv 1(Nc_2), N^{-1}N\equiv c_2^{-1}c_2\equiv 1(4c_1))$. For the proof one writes $d=4c_1d_2+Nc_2d_1, d_1 \bmod 4c_1, d_2 \bmod Nc_2, (d_1,4c_1)=(d_2,Nc_2)=1$ and uses the identity

$$\left(\frac{4Nc_1c_2}{4c_1d_2+Nc_2d_1}\right)\left(\frac{-4}{4c_1d_2+Nc_2d_1}\right)^{k+1/2} = \left(\frac{4c_1}{d_1}\right)\left(\frac{-4}{d_1}\right)^{k+1/2}\left(\frac{-4}{Nc_2}\right)^{-k-1/2}\left(\frac{d_2}{Nc_2}\right).$$

Now suppose c odd. Then we have by (36)

$$\underline{H}_{4Nc}(n,m) = \underline{H}_4(n,m)\, \underline{H}'_{Nc}(n,m4^{-2}) \tag{37}$$

and

$$\underline{H}_{8Nc}(n,m) = \underline{H}_8(n,m)\, \underline{H}'_{Nc}(n,m4^{-3}), \tag{38}$$

and for $n,m\equiv 0,(-1)^k \pmod 4$ one easily checks that

(39) $\underline{H}_4(n,m) = \frac{1}{4}(-1)^{nm}(1+(-1)^k i)$

and

$$\underline{H}_8(n,m) = \frac{1}{2\sqrt{2}}\left(\frac{m+n}{2}\right)(1+(-1)^k i).$$

Let $n\equiv 0(\mathrm{mod}4)$. Then

$$(1-(-1)^k i)\frac{(-1)^k}{4}\underline{H}'_{Nc}(\tfrac{n}{4},4^{-1}m)$$
$$= (1-(-1)^k i)\frac{(-1)^k i}{4}\underline{H}'_{Nc}(n,4^{-2}m)$$
$$= (1-(-1)^k i)\frac{(-1)^k i}{4}\frac{4}{1+(-1)^k i}\underline{H}_{4Nc}(n,m) \qquad \text{(by (37) and (39))}$$
$$= \underline{H}_{4Nc}(n,m),$$

which proves (35) in this case.

Next let $(-1)^k n\equiv 1(\mathrm{mod}4)$. Then

$$\frac{1}{\sqrt{2}}\left(\frac{(-1)^k n}{2}\right)\underline{H}_{8Nc}(n,4m) = \frac{1}{\sqrt{2}}\left(\frac{(-1)^k n}{2}\right)\frac{1}{2\sqrt{2}}\left(\frac{n+4m}{2}\right)(1+(-1)^k i)\underline{H}'_{Nc}(n,m4^{-2}) \quad \text{(by (3}$$
$$= \frac{1}{4}\left(\frac{(-1)^k n}{2}\right)\left(\frac{n+4m}{2}\right)(1+(-1)^k i)\frac{4(-1)^m}{1+(-1)^k i}\underline{H}_{4Nc}(n,m) \quad \text{(by (3}$$
$$= \underline{H}_{4Nc}(n,m),$$

which proves (35).

This completes the proof of Proposition 3 in the case $k\geq 2$. Finally we have to consider $k=1$, where the series (27) does not converge absolutely. By "Hecke's convergence trick" (in the same manner as in the proofs of Propositions 1 and 2) we ca show, however, that $\underline{P}_{1,N,m}$ is given by

$$\underline{P}_{1,N,m}(z) = \lim_{s\to 0}\frac{1}{2}\sum_{A\in\Gamma_\infty\backslash\Gamma_0(4N)} j(A,z)^{-3}|j(A,z)|^{-4s}\,e(mAz),$$

and that its Fourier expansions at the cusps $i\infty$, $-\alpha/N$ and $1/2N$ are given by formula (31), (33) and (34), respectively. We do not carry out the details. This completes the proof of Proposition 4.

We now proceed to the proof of (5). If we put in identity (17) into the left-hand side of (5) and identity (25) into the right-hand side of (5) we see that for all $n\geq 1$ and $m\geq 1$ with $(-1)^k m\equiv 0,1(\mathrm{mod}4)$ we have to show that

$$\frac{2(-2\pi)^k}{(k-1)!} \sum_{t|N} \mu(t)\left(\frac{D}{t}\right)t^{k-1}(n^2/t^2|D|m)^{(k-1)/2}\, m^{k-1/2} \Big[(-1)^{[(k+1)/2]}\left(\frac{D}{n/t\sqrt{m/|D|}}\right)$$

$$\delta(n/t\sqrt{m/|D|})|D|^{-1/2} + \pi\sqrt{2}\,(n^2/t^2|D|m)^{1/4} \sum_{a\geq 1} \left(\frac{N}{t}a\right)^{-1/2} S_{Na/t,D,(-1)^k m}(|D|m,\frac{n}{t})$$

$$J_{k-1/2}\left(\frac{\frac{n}{t}\sqrt{|D|m}}{Na/t}\right)\Big] =$$

$$(-1)^{[k/2]}\,\frac{3(2\pi)^k}{(k-1)!}\, n^{k-1} \sum_{d|n,(d,N)=1} \left(\frac{D}{d}\right)(n/d)^k\, \frac{2}{3}\Big[\delta_{(n^2/d^2)|D|,m} + (-1)^{[(k+1)/2]}\pi\sqrt{2}$$

$$(md^2/n^2|D|)^{k/2-1/4} \sum_{c\geq 1} H_{Nc}(m,(n^2/d^2)|D|)\, J_{k-1/2}\left(\frac{\pi}{Nc}\sqrt{(n^2/d^2)|D|m}\right)\Big]$$

or equivalently

$$(-1)^{[k/2]}\,(m/|D|)^{k/2}\left(\frac{D}{n/\sqrt{m/|D|}}\right) \sum_{t|N} \mu(t)\delta(n/t\sqrt{m/|D|}) + \pi\sqrt{2}\,(-1)^k\,(m/|D|)^{k/2-1/4}$$

$$n^{1/2} \sum_{a\geq 1} (Na)^{-1/2} \sum_{t|N} \mu(t)\left(\frac{D}{t}\right) S_{Na/t,D,(-1)^k m}(|D|m,\frac{n}{t})\, J_{k-1/2}\left(\frac{\pi n\sqrt{|D|m}}{Na}\right)$$

(40) $=$

$$(-1)^{[k/2]} \sum_{d|n,(d,N)=1} \left(\frac{D}{d}\right)(n/d)^k\, \delta_{n^2|D|/d^2,m} + \pi\sqrt{2}\,(-1)^k(m/|D|)^{k/2-1/4}$$

$$n^{1/2} \sum_{d|n,(d,N)=1} \left(\frac{D}{d}\right)d^{-1/2} \sum_{c\geq 1} H_{Nc}(m,\frac{n^2}{d^2}|D|)\, J_{k-1/2}\left(\frac{\pi n}{Ncd}\sqrt{|D|m}\right).$$

We look at the first terms on both sides of (40). They are zero unless $m=|D|f^2$ for some $f\in\mathbb{N}$ with $f|n$, and in the latter case the first term on the left-hand side equals

$$(-1)^{[k/2]}\left(\frac{D}{n/f}\right)f^k \sum_{t|(n/f,N)} \mu(t) = \begin{cases} (-1)^{[k/2]}\left(\frac{D}{n/f}\right)f^k & \text{if } (n/f,N)=1 \\ 0 & \text{otherwise,} \end{cases}$$

and clearly this is then equal to the first term on the right-hand side of (40).

In the second term on the right-hand side of (40) we substitute $cd=a$ to get

$$\pi\sqrt{2}\,(-1)^k(m/|D|)^{k/2-1/4}\, n^{1/2} \sum_{a\geq 1}\Big(\sum_{d|(a,n),(d,N)=1} \left(\frac{D}{d}\right)d^{-1/2} H_{Na/d}(m,\frac{n^2}{d^2}|D|)$$

$$J_{k-1/2}\left(\frac{\pi n}{Na}\sqrt{|D|m}\right)\Big).$$

We conclude that for the proof of (5) it is sufficient to show that

$$\sum_{t|N} \mu(t)\left(\frac{D}{t}\right)S_{Na/t,D,(-1)^k m}\left(|D|m,\frac{n}{t}\right) = \sum_{\substack{d|(a,n)\\(d,N)=1}} \left(\frac{D}{d}\right)\left(N\frac{a}{d}\right)^{1/2} H_{Na/d}(m,n^2|D|/d^2).$$

Inverting we see that we must prove the following

Proposition 5. <u>Define</u> $S_{a,D,(-1)^k m}(|D|m,n)$ <u>by (18) and</u> $H_c(m,n)$ <u>by (26). Then for all</u> $a\geq 1$, $n\geq 1$ and $m\geq 1$ with $(-1)^k m\equiv 0,1(\mathrm{mod}\,4)$ we have

$$(41)\qquad S_{a,D,(-1)^k m}(|D|m,n) = \sum_{d|(a,n)} \left(\frac{D}{d}\right)(a/d)^{1/2}\, H_{a/d}(m,n^2|D|/d^2).$$

Proof. As functions of n both sides of (41) are periodic with period 2a, so it will be sufficient to show that their Fourier transforms are equal, i.e. we must show that for every $h\in\mathbb{Z}$ we have

$$(42)\qquad \begin{aligned}&\frac{1}{2a}\sum_{n(2a)} e_{2a}(-hn)\, S_{a,D,(-1)^k m}(|D|m,n)\\ &= \frac{1}{4a}\sum_{n(2a)} e_{2a}(-hn) \sum_{d|(a,n)} \left(\frac{D}{d}\right)(4a/d)^{1/2}\, H_{a/d}(m,n^2|D|/d^2).\end{aligned}$$

By definition the left-hand side of (42) equals

$$\frac{1}{2a}\sum_{n(2a)} e_{2a}(-hn) \sum_{b(2a),\,b^2\equiv |D|m(4a)} \omega_D\left(a,b,\frac{b^2-|D|m}{4a}\right) e_{2a}(nb)$$

$$= \frac{1}{2a}\sum_{b(2a),\,b^2\equiv|D|m(4a)} \omega_D\left(a,b,\frac{b^2-|D|m}{4a}\right)\sum_{n(2a)} e_{2a}((b-h)n)$$

$$= \begin{cases} \omega_D\left(a,b,\frac{h^2-|D|m}{4a}\right) & \text{if } h^2\equiv|D|m(4a)\\ 0 & \text{otherwise.}\end{cases}$$

We now must compute the right-hand side of (42). Call it $C_{D,m}(a,h)$. Then

$$C_{D,m}(a,h) = \frac{1}{4a}\sum_{d|a}\left(\frac{D}{d}\right)(4a/d)^{1/2}\left(1+\left(\frac{4}{a/d}\right)\right)\sum_{n(2a/d)} e_{2a/d}(-hn)H_{a/d}(m,|D|n^2).$$

Putting in the definition

$$H_{a/d}(m,|D|n^2) = (1-(-1)^k i)(1+(\frac{4}{a/d}))\frac{1}{4a/d}\sum_{\delta(4a/d)^*}(\frac{4a/d}{\delta})(\frac{-4}{\delta})^{k+1/2}e_{4a/d}(m\delta+|D|n^2\delta^{-1}$$

and making the substitutions $n \mapsto \delta n$, $\delta \mapsto (-1)^k\delta$, $n \mapsto (-1)^k n$ we obtain

$$C_{D,m}(a,h) = \frac{1}{4a}\sum_{d|a}(\frac{D}{d})(4a/d)^{1/2}F_{4a/d}(Q),$$

where $Q(n) = Dn^2-2hn+(-1)^k m$ and for every integer $c\geq 1$ we have put

$$F_{4c}(Q) = \frac{1}{4c}(1+(\frac{4}{c}))\sum_{n(2c),\delta(4c)}(\frac{4c}{\delta})(1-(\frac{-4}{\delta})i)\ e_{4c}(\delta Q(n)).$$

Using the quadratic reciprocity law one checks that

$$F_{4c}(Q) = F_{c_0}(Q)\ F_{c_0'}(Q) \qquad (4c = c_0c_0',\ c_0' \text{ odd});$$

here for c odd we have put

$$F_c(Q) = \frac{1}{c}(\frac{-4}{c})^{-1/2}\sum_{n,\delta(c)}(\frac{\delta}{c})\ e_c(\delta Q(n)).$$

Furthermore, for c odd one has the multiplicative property

$$F_c(Q) = F_{c_1}(Q)\ F_{c_1'}(Q) \qquad (c = c_1c_1',\ (c_1,c_1')=1).$$

From this we see that

$$(43)\qquad C_{D,m}(a,h) = \prod_{p^\nu \| a,\, p^\mu \| 4} C_{D,m}(p^\nu,p^\mu;h)$$

where the product extends over all primes p dividing 4a, p^ν is the exact power of p dividing a ($p^\nu \| a$), μ is 0 or 2 according as p is odd or even ($p^\mu \| 4$) and where we have set

$$C_{D,m}(p^\nu,p^\mu;h) = p^{-\nu-\mu}\sum_{0\leq\lambda\leq\nu}(\frac{D}{p^{\nu-\lambda}})\ p^{(\mu+\lambda)/2}\ F_{p^{\mu+\lambda}}(Q).$$

Put $\Delta = h^2-(-1)^k Dm$ and let us compute $C_{D,m}(p^\nu,p^\mu;h)$ first for p an odd prime, $p\nmid D$. By definition

$$F_{p^\lambda}(Q) = p^{-\lambda}(\frac{-4}{p^\lambda})^{-1/2}\sum_{\delta(p^\lambda)^*}(\frac{\delta}{p^\lambda})\sum_{n(p^\lambda)}e_{p^\lambda}(\delta Q(n)).$$

Since $p\nmid D$ we may write

$$Q(n) = Dn^2-2hn+(-1)^k m \equiv D^{-1}((Dn-h)^2-\Delta) \pmod{p^\lambda}$$

where D^{-1} is an integer inverse to D modulo p^λ, so the sum over n equals

$$e_{p^\lambda}(-\delta D^{-1}\Delta) \sum_{n(p^\lambda)} e_{p^\lambda}(\delta D^{-1}n^2)$$

$$= e_{p^\lambda}(-\delta D^{-1}\Delta) \left(\frac{\delta D}{p^\lambda}\right) \left(\frac{-4}{p^\lambda}\right)^{1/2} p^{\lambda/2},$$

the latter equality being standard. Thus

$$F_{p^\lambda}(Q) = p^{-\lambda/2}\left(\frac{D}{p^\lambda}\right) \sum_{\delta(p^\lambda)^*} e_{p^\lambda}(-\delta D^{-1}\Delta)$$

$$= p^{-\lambda/2}\left(\frac{D}{p^\lambda}\right) \begin{cases} \phi(p^\lambda) & (p^\lambda|\Delta) \\ -p^{\lambda-1} & (p^{\lambda-1}\| \Delta) \\ 0 & (\text{otherwise}). \end{cases}$$

Therefore

$$C_{D,m}(p^\nu,p^\mu;h) = p^{-\nu} \sum_{0\le\lambda\le\nu} \left(\frac{D}{p^{\nu-\lambda}}\right) p^{\lambda/2} F_{p^\lambda}(Q)$$

$$= p^{-\nu}\left(\frac{D}{p^\nu}\right) \begin{cases} \sum_{0\le\lambda\le\nu} \phi(p^\lambda) & (p^\nu|\Delta) \\ \sum_{0\le\lambda\le\rho} \phi(p^\lambda) - p^\rho & (p^\rho\| \Delta, \rho<\nu), \end{cases}$$

i.e. we obtain

$$(44) \qquad C_{D,m}(p^\nu,p^\mu;h) = \begin{cases} \left(\frac{D}{p^\nu}\right) & (p^\nu|\Delta) \\ 0 & (\text{otherwise}) \end{cases} \qquad (p \text{ odd}, p\nmid D).$$

Next assume p odd, $p|D$. In this case

$$\sum_{n(p^\lambda)} e_{p^\lambda}(\delta Q(n))$$

is zero unless $p|h$, as one sees by replacing n with $n+p^{\lambda-1}$. Thus suppose $p|h$ and and put $h_0=h/p$, $D_0=D/p$. Then

$$F_{p^\lambda}(Q) = p^{-\lambda+1}\left(\frac{-4}{p^\lambda}\right)^{-1/2} \sum_{\delta(p^\lambda)^*} \left(\frac{\delta}{p^\lambda}\right) e_{p^\lambda}(\delta(-1)^k m) \sum_{n(p^{\lambda-1})} e_{p^{\lambda-1}}(\delta(D_0 n^2 - 2h_0 n)).$$

In the same way as above the sum over n equals

$$e_{p^{\lambda-1}}(-D_0^{-1}h_0^2\delta)\left(\frac{D_0}{p^{\lambda-1}}\right)\left(\frac{-4}{p^{\lambda-1}}\right)^{1/2} p^{(\lambda-1)/2} \qquad (D_0^{-1}\in \mathbb{Z},\ D_0^{-1}D_0\equiv 1(p^{\lambda-1})),$$

so

$$F_{p^\lambda}(Q) = \left(\frac{D_0}{p^{\lambda-1}}\right)\left(\frac{-4}{p^\lambda}\right)^{-1/2}\left(\frac{-4}{p^{\lambda-1}}\right)^{1/2} p^{(-\lambda+1)/2} \sum_{\delta(p^\lambda)} \left(\frac{\delta}{p}\right) e_{p^\lambda}(\delta((-1)^k m - D_0^{-1}h_0 h))$$

$$= \left(\frac{D_0}{p^\lambda}\right)\left(\frac{-4}{p^\lambda}\right)^{-1/2}\left(\frac{-4}{p^{\lambda-1}}\right)^{1/2} p^{(-\lambda+1)/2} \sum_{\delta(p^\lambda)} \left(\frac{\delta}{p}\right) e_{p^\lambda}((-\delta\Delta/p).$$

Replacing δ with $\delta+p$ we see that $F_{p^\lambda}(Q)=0$ unless $p^\lambda | \Delta$, so let us assume that $p^\lambda | \Delta$.

Then the sum over δ equals

$$p^{\lambda-1} \sum_{\delta(p)} \left(\frac{\delta}{p}\right) e_p\,(-\delta\Delta/p^\lambda)$$

$$= p^{\lambda-1/2}\left(\frac{-4}{p}\right)^{-1/2}\left(\frac{\Delta/p^\lambda}{p}\right).$$

From this we obtain

$$F_{p^\lambda}(Q) = p^{\lambda/2}\left(\frac{D/p}{p^\lambda}\right)\left(\frac{\Delta/p^\lambda}{p}\right) \begin{cases} 1 & (\lambda \text{ even}) \\ \left(\frac{-4}{p}\right) & (\lambda \text{ odd}). \end{cases}$$

If we write $D = p^*.D/p^*$ as a product of two fundamental discriminants with $p^*=\pm p$ we obtain using quadratic reciprocity

$$F_{p^\lambda}(Q) = p^{\lambda/2}\left(\frac{p^*}{\Delta/p^\lambda}\right)\left(\frac{D/p^*}{p^\nu}\right).$$

Thus

$$(45) \qquad C_{D,m}(p^\nu,p^\mu;h) = \begin{cases} \left(\frac{p^*}{\Delta/p^\nu}\right)\left(\frac{D/p^*}{p^\nu}\right) & (p^\nu | \Delta) \\ 0 & (\text{otherwise}) \end{cases} \qquad (p \text{ odd},\ p|D).$$

Now suppose p=2. Then we have to compute

$$C_{D,m}(2^\nu,4;h) = 2^{(-2-\nu)/2}(1+(\frac{4}{2^\nu}))\ F_{2^{\nu+2}}(Q).$$

By definition

$$F_{2^\lambda}(Q) = 2^{-\lambda} \sum_{n(2^{\lambda-1}),\delta(2^\lambda)^*} (\frac{2^\lambda}{\delta})(1-(\frac{-4}{\delta})i)\ e_{2^\lambda}(\delta Q(n)).$$

First assume D≡1(mod4). Then

$$\sum_{n(2^{\lambda-1})} e_{2^\lambda}(\delta Q(n)) = e_{2^\lambda}(-\delta D^{-1}\Delta) \sum_{n(2^{\lambda-1})} e_{2^\lambda}(\delta D^{-1}n^2) \qquad (D^{-1}D\equiv 1(2^\lambda))$$

$$= \frac{1}{2}\, e_{2^\lambda}(-\delta D^{-1}\Delta)\ (\frac{\delta D}{2^\lambda})(\frac{-4}{\delta\Delta})^{-1/2}\ (1+i)2^{\lambda/2}.$$

Therefore

$$F_{2^\lambda}(Q)=2^{-\lambda/2}(\frac{D}{2^\lambda}) \sum_{\delta(2^\lambda)^*} e_{2^\lambda}(-\delta D^{-1}\Delta),$$

and as in the case of an odd prime we obtain

$$(46)\qquad C_{D,m}(2^\nu,4;h) = \begin{cases} (\frac{D}{2^{\nu+2}}) & (2^{\nu+2}|\Delta) \\ 0 & (\text{otherwise}) \end{cases} \qquad (2\nmid D).$$

Next suppose D≡4(mod8) (so D/4≡3(mod4)). First assume $\lambda\geq 4$. Then replacing n by $n+2^{\lambda-3}$ we see that

$$\sum_{n(2^{\lambda-1})} e_{2^\lambda}(\delta Q(n)) = 0$$

unless 4|h, so let us assume the latter condition. Then

$$\sum_{n(2^{\lambda-1})} e_{2^\lambda}(\delta Q(n)) = 2\ e_{2^\lambda}(-D_0^{-1}\delta\Delta/4)\ (\frac{\delta D_0}{2^{\lambda-2}})\ (\frac{-4}{-\delta})^{-1/2}\ (1+i)2^{\lambda/2-1}$$

$$(D_0=\frac{D}{4},\ D_0^{-1}D_0\equiv 1(2^\lambda)),$$

and so

$$F_{2^\lambda}(Q) = 2^{-\lambda/2+1}(-i)\left(\frac{D_0}{2^{\lambda-2}}\right) \sum_{\delta(2^\lambda)^*} \left(\frac{-4}{\delta}\right) e_{2^\lambda}(\delta\Delta/4).$$

Clearly this is zero if $2^\lambda \nmid \Delta$, and for $2^\lambda | \Delta$ we get

$$F_{2^\lambda}(Q) = 2^{\lambda/2-1}\left(\frac{D_0}{2^{\lambda-2}}\right)(-i) \sum_{\delta(4)} \left(\frac{-4}{\delta}\right) e_4\,(\delta\Delta/2^\lambda)$$

$$= 2^{\lambda/2}\left(\frac{-D/4}{2^\lambda}\right)\left(\frac{-4}{\Delta/2^\lambda}\right).$$

Therefore for $\nu \geq 2$

$$(47) \qquad C_{D,m}(2^\nu,4;h) = \begin{cases} \left(\frac{-D/4}{2^{\nu+2}}\right)\left(\frac{-4}{\Delta/2^{\nu+2}}\right) & (2^{\nu+2}|\Delta) \\ 0 & (\text{otherwise}) \end{cases} \qquad (D\equiv 4(8)).$$

One checks that (47) is also true for $\nu=0,1$. Finally, in the same way as above

$$(48) \qquad C_{D,m}(2^\nu,4;h) = \begin{cases} \left(\frac{D/8^*}{2^{\nu+2}}\right)\left(\frac{8^*}{\Delta/2^{\nu+2}}\right) & (2^{\nu+2}|\Delta) \\ 0 & (\text{otherwise}) \end{cases} \qquad (D\equiv 0(8)),$$

where $8^*=\pm 8$ is determined by the condition $D/8^*$ is a fundamental discriminant. Summarizing (43) to (48) we obtain

$$C_{D,m}(a,h) = \begin{cases} \prod_{p^\nu \| 4a} \left(\frac{D/p^*}{p^\nu}\right)\left(\frac{p^*}{(h^2-|D|m)/p^\nu}\right) & \text{if } h^2 \equiv |D|m(4a) \\ 0 & \text{otherwise,} \end{cases}$$

where for every prime p we define $p^* \in \mathbb{Z}$ by the condition that p^* is the exact power of p dividing D and D/p^* is a fundamental discriminant. Thus to prove (42) we are left with showing

Proposition 6. <u>Assume</u> $a,b,c\in\mathbb{Z}$, $a>0$, $b^2-4ac=|D|m$. <u>Then</u>

$$(49) \qquad \omega_D(a,\,b,c) = \prod_{p^\nu \| a} \left(\frac{D/p^*}{p^\nu}\right)\left(\frac{p^*}{ac/p^\nu}\right).$$

Proof. Clearly both sides of (49) are zero if and only if $(a,b,c,D)>1$. Therefore assume $(a,b,c,D)=1$. Then by definition

$$\omega_D(a,b,c) = \left(\frac{D}{r}\right),$$

where r is any integer represented by $[a,b,c]$ with $(r,D)=1$. If $[a,b,c](x,y)=r$, the

(50) $$4ar=(2ax+by)^2-|D|my^2,\ 4cr=(2cx+by)^2-|D|my^2.$$

From (50) we see that cr is a square modulo every odd prime divisor of D; furthermore, since D is a fundamental discriminant we see that for cr odd we must have $cr\equiv 1(\mathrm{mod}\,4)$ if $D\equiv 4(\mathrm{mod}\,8)$, and $cr\equiv 1(\mathrm{mod}\,8)$ if $D\equiv 0(\mathrm{mod}\,8)$. Since by (50) also $4ar$ is a square modulo D, it follows that (49) is equivalent to

$$\left(\frac{D}{\mathrm{sign}\,ar}\right)\prod_{p^\nu\| ar}\left(\frac{D/p^*}{p^\nu}\right)\left(\frac{p^*}{ar/p^\nu}\right) = 1,$$

and that it is sufficient to show that for any $n\in\mathbb{Z}\setminus\{0\}$ such that n is a square modulo D we have

(51) $$\left(\frac{D}{\mathrm{sign}\,n}\right)\prod_{p^\nu\| n}\left(\frac{D/p^*}{p^\nu}\right)\left(\frac{p^*}{n/p^\nu}\right) = 1.$$

We shall prove (51) by induction on the number of prime discriminants dividing D. If $D=1$, there is nothing to prove. Suppose $D=D'q^*$ with q a prime.

i) $q\nmid n$. We must show that

$$\left[\left(\frac{D'}{\mathrm{sign}\,n}\right)\prod_{p^\nu\| n}\left(\frac{D'/p^*}{p^\nu}\right)\left(\frac{p^*}{n/p^\nu}\right)\right]\left[\left(\frac{q^*}{\mathrm{sign}\,n}\right)\prod_{p^\nu\| n}\left(\frac{q^*}{p^\nu}\right)\right] = 1.$$

The expression in the first square-bracket is 1 by induction, while the expression in the second one equals $\left(\frac{q^*}{n}\right)$ which is 1 since n is a square mod q^* by assumption.

ii) $q|n$. Suppose $q^\lambda\| n$. Then we must prove that

$$\left[\left(\frac{D'}{\mathrm{sign}\,n}\right)\prod_{p^\nu\| n}\left(\frac{D/p^*}{p^\nu}\right)\left(\frac{p^*}{n/p^\nu}\right)\right]\left[\left(\frac{q^*}{\mathrm{sign}\,n}\right)\prod_{\substack{p^\nu\| n\\ p\nmid D'q}}\left(\frac{q^*}{p^\nu}\right).\left(\frac{D'}{q^\lambda}\right).\prod_{\substack{p^\nu\| n\\ p|D'}}\left(\frac{q^*}{p^\nu}\right)\cdot\left(\frac{D'}{q^\lambda}\right)\left(\frac{q^*}{n/q^\lambda}\right)\right]$$

$$=1.$$

Again the expression in the first square-bracket is 1 by induction, and the express in the second one equals

$$\left(\frac{q^*}{n/q^\lambda}\right)\left(\frac{q^*}{n/q^\lambda}\right) = 1.$$

Therefore (51) and Proposition 6 is proved. We thus have completed the proof of Theorem 1.

2.2. Proof of Theorem 2

Let $g \in S_{k+1/2}(N)$. Then by (4) and (5)

$$\langle g, \Omega_{k,N,D}(-\bar{z},\ )\rangle = i_N c_{k,D}^{-1} (-1)^{[k/2]} \frac{3(2\pi)^k}{(k-1)!} \sum_{n\geq 1} n^{k-1} \sum_{d|n,(d,N)=1} \left(\frac{D}{d}\right)(n/d)^k \langle g, P_{k,N,n^2|D|/d^2}\rangle e^{2\pi inz}$$

$$= g|\underline{S}_{k,N,D}(z)$$

(note that $i_{4N} = i_N[\Gamma_0(N):\Gamma_0(4N)] = 6i_N$ since N is odd). This proves (9).

For every divisor t of N the map $Q \mapsto Q\circ\begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix}$ gives a bijection between the set of integral binary quadratic forms $[a,b,c]$ with discriminant $|D|m$ and $a\equiv 0(N/t)$ and the set of integral forms $[a,b,c]$ with discriminant $|D|mt^2$ and $a\equiv 0(Nt)$. Moreove if the discriminant of Q is divisible by D we have

$$\omega_D(Q\circ\begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix}) = \left(\frac{D}{t^2}\right)\omega_D(Q).$$

Therefore

$$\sum_{t|N} \mu(t)\left(\frac{D}{t}\right)t^{k-1} f_{k,N/t}(tz;D,(-1)^k m) = \sum_{t|N} \mu(t)\left(\frac{D}{t}\right)t^{k-1} f_{k,Nt}(z;D,(-1)^k mt$$

and to prove (10) it suffices to prove the following

Proposition 7. For any $f\in S_{2k}(N)$ we have

(52) $$\langle f, f_{k,N}(\ ;D,(-1)^k m)\rangle = i_N^{-1}\pi\binom{2k-2}{k-1}2^{-2k+2}(|D|m)^{1/2-k}\, r_{k,N}(f;D,(-1)^k m)$$

with $r_{k,N}(f;D,(-1)^k m)$ defined by (7).

Proof. The proof is given in [4] and [7], cf. also [16]. For the reader's convenience we will briefly repeat the short argument given in [7]. We shall restrict to the case where $|D|m$ is not a perfect square. First suppose $k\geq 2$. We have

$$f_{k,N}(z;D,(-1)^k m) = \sum_A \omega_D(A)\ f_{k,N}(z;A)$$

where the sum is over the finitely many $\Gamma_0(N)$-classes A of forms Q with discriminant $|D|m$ and $Q(1,0)\equiv 0(N)$ and

$$f_{k,N}(z;A) = \sum_{Q\in A} Q(z,1)^{-k}.$$

Clearly $f_{k,N}(\ ;A)\in S_{2k}(N)$, so it is sufficient to compute $\langle f,f_{k,N}(\ ;A)\rangle$. By the usual unfolding argument

$$i_N \langle f,f_{k,N}(\ ;A)\rangle = \int_{\Gamma\backslash H} \sum_{A\in\Gamma_Q\backslash\Gamma} (Q\circ A)(\bar{z},1)^{-k}f(z)y^{2k-2}dxdy$$

$$= \int_{\Gamma_Q\backslash H} f(z)Q(\bar{z},1)^{-k}y^{2k-2}dxdy,$$

where Q is any element of A, $\Gamma=\Gamma_0(N)$ and Γ_Q is the stabilizer of Q in Γ. Let $\theta = \arg\frac{z-\beta}{z-\alpha}$, where α and β are the roots of $Q(z,1)=0$, with $\alpha<\beta$. Then $0<\theta<\pi$ and θ is invariant under Γ_Q. Also

$$d\theta = d\left[\mathrm{Im}\ \log(z-\beta) - \log(z-\beta)\right]$$

$$= \mathrm{Im}\left[\left(\frac{1}{z-\beta} - \frac{1}{z-\beta}\right) dz\right]$$

$$= \sqrt{|D|m}\ \mathrm{Im}\frac{dz}{Q(\bar{z},1)},$$

so

$$dzd\theta = (|D|m)^{1/2}\ Q(\bar{z},1)^{-1}\ dxdy$$

and

$$\frac{y^2}{|Q(z,1)|^2} = \frac{1}{|D|m}\ \sin^2\theta.$$

Therefore

$$f(z)Q(\bar{z},1)^{-k}y^{2k-2}dxdy = (|D|m)^{1/2-k}f(z)Q(z,1)^{k-1}\sin^{2k-2}\theta dzd\theta.$$

Set

$$A_Q = \begin{pmatrix} \frac{1}{2}(t-bu) & -cu \\ au & \frac{1}{2}(t+bu) \end{pmatrix},$$

where $Q=[a,b,c]$ and (t,u) is the smallest positive solution of Pell's equation $t^2-|D|mu^2=4$. Then A_Q generates $\Gamma_Q/\{\pm 1\}$. For each $\theta\in(0,\pi)$ the integral of $f(z)Q(z,1)^{k-1}dz$ from z_0 to $A_Q z_0$, where $z_0\in H$ is any point with $\arg \frac{z_0-\beta}{z_0-\alpha}=\theta$, is $\int_{C_Q} f(z)d_{Q,k}z$, independent of θ. Hence

$$\langle f,f_{k,N}(\ ;A)\rangle = i_N^{-1}\,(|D|m)^{1/2-k}\int_0^{\pi}\sin^{2k-2}\theta d\theta \int_{C_Q} f(z)d_{Q,k}z,$$

and (52) follows.

For k=1 we write

$$f_{1,N}(z,s;D,-m) = \sum_A \omega_D(A)\, f_{1,N}(z,s;A)$$

with

$$f_{1,N}(z,s;A) = y^s \sum_{Q\in A} Q(z,1)^{-1}|Q(z,1)|^{-s} \qquad (z\in H, s\in\mathbb{C}, \mathrm{Re}s>0),$$

and in the same way as above we find

$$\langle f,f_{1,N}(\ ,\bar{s};A)\rangle = i_N^{-1}(|D|m)^{-(1+s)/2}\int_0^{\pi}\sin^{s}\theta d\theta \int_{C_Q} f(z)d_{Q,1}z.$$

Therefore

$$\langle f,f_{1,N}(\ ,0;A)\rangle = i_N^{-1}\,(|D|m)^{-1/2}\pi \int_{C_Q} f(z)d_{Q,1}z,$$

and this proves (52) for k=1.

2.3. Proof of Theorem 3

We use the same arguments already used in [6] and [4]. Let $\underline{S}_{(-1)^k n} = \underline{S}_{k,N,(-1)^k n}$ be the Shimura lifting associated to $(-1)^k n$ (equation (6)) and $\underline{S}^*_{(-1)^k n}$

its adjoint (equation (8)). By (11) we have

(53) $$g|\underline{S}_{(-1)^k n} = c(n)f,$$

and because of the "multiplicity 1 theorem" valid for $S^{new}_{k+1/2}(N)$ ([5])

$$f|\underline{S}^*_{(-1)^k n} = \lambda g$$

for some $\lambda\in\mathbb{C}$. Denoting by $C(f,m,n)$ the m^{th} Fourier coefficient of $f|\underline{S}^*_{(-1)^k n}$ we therefore get

$$\begin{aligned} C(f,m,n) \langle g,g\rangle &= \lambda c(m) \langle g,g\rangle \\ &= c(m) \langle f|\underline{S}^*_{(-1)^k n},g\rangle \\ &= c(m) \langle f,g|\underline{S}_{(-1)^k n}\rangle \\ &= c(m)\overline{c(n)} \langle f,f\rangle , \end{aligned}$$

where in the last line we have used (53). To prove (13) we thus have to show that

$$C(f,m,n) = (-1)^{[k/2]}\, 2^k\, r_{k,N}(f;(-1)^k n,(-1)^k m).$$

Because f is a newform, its scalar product against $f_{k,N/t}(tz;\ (-1)^k n,(-1)^k m)$ is zero for every divisor t of N, $t>1$, so by (3) and (10)

$$C(f,m,n) = c_{k,D}^{-1}\, m^{k-1/2} \langle f,f_{k,N}(\ ;(-1)^k n,(-1)^k m) \rangle ,$$

and the desired equality follows from Proposition 7.

Acknowledgements. Part of this paper was written during a visit to Harvard Universi in 1983/84. The author would like to thank the Department of Mathematics, Harvard, for hospitality and the Max Kade Foundation, New York, for financial support.

References

[1] Abramowitz,M. and Stegun,I.: Handbook of Mathematical Functions, Dover, New York 1965

[2] Goldfeld,D., Hoffstein,J. and Patterson,S.J.: On automorphic functions of half-integral weight with applications to elliptic curves. In: Number Theory related to Fermat's Last Theorem, Proc. of a Conf. sponsored by the Vaughn Foundation, ed. N. Koblitz. Progress in Maths. Vol. 26, Birkhäuser, Boston 1982

[3] Gross,D. and Zagier,D.: Points de Heegner et dérivées des fonctions L, preprint 1983

[4] Kohnen,W.: Beziehungen zwischen Modulformen halbganzen Gewichts und Modulformen ganzen Gewichts. Dissertation. Bonner Math. Schriften 131, Bonn 1981

[5] Kohnen,W.: Newforms of half-integral weight, J. reine angew. Math. 333, 32-72 (1982)

[6] Kohnen,W. and Zagier,D.: Values of L-series of modular forms at the center of the critical strip, Inv. Math. 64, 175-198 (1981)

[7] Kohnen,W. and Zagier,D.: Modular forms with rational periods, to appear in "Modular Forms and Functions", Proc. of a Conf. held at Durham 1983

[8] Manin,Y.: Periods of parabolic forms and p-adic Hecke series, Math. USSR Sbornik 21, 371-393 (1973)

[9] Niwa,S.: Modular forms of half-integral weight and the integral of certain theta-functions, Nagoya Math. J. 56, 147-161 (1974)

[10] Niwa,S.: On certain theta functions and modular forms of half-integral weight, preprint 1983

[11] Rankin,R.A.: The scalar product of modular forms, Proc. Lond. Math. Soc. 2, 198-217 (1972)

[12] Sarnak,P.: Class numbers of indefinite binary quadratic forms, J. of Number Theory 15, 229-247 (1982)

[13] Shimura,G.: Sur les intégrales attachées aux formes automorphes, J. Math. Soc. Japan 11, 291-311 (1959)

[14] Shimura,G.: On modular forms of half-integral weight, Ann. of Math. 97, 440-481 (1973)

[15] Shimura,G.: The critical values of certain zeta functions associated with modular forms of half-integral weight, J. Math. Soc. Japan 33, 649-672 (1981)

[16] Shintani.T.: On construction of holomorphic cusp forms of half-integral weight, Nagoya Math. J. 58, 83-126 (1975)

[17] Siegel,C.L.: Uber die Classenzahl quadratischer Zahlkörper, Acta Arithm. 1, 83-86 (1935)

[18] Waldspurger,J.L.: Correspondances de Shimura et Shintani, J. Math. pures et appl. 59, 1-133 (1980)

[19] Waldspurger,J.L.: Sur les coefficients de Fourier des formes modulaires de poids demi-entier, J. Math. pures et appl. 60, 375-484 (1981)

[20] Zagier,D.: Modular forms associated to real quadartic fields, Inv. Math. 30, 1-46 (1975)

[21] Zagier,D.: Modular forms whose Fourier coefficients involve zeta-functions of quadratic fields. In: Modular Forms of One Variable VI, Springer Lecture Notes 627, 105-169. Berlin-Heidelberg-New York: Springer Verlag 1977