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SPLITTING OF OPERATIONS, MANIN PRODUCTS AND ROTA-BAXTER OPERATORS

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Abstract. This paper provides a general operadic definition for the notion of splitting the operations of algebraic structures. This construction is proved to be equivalent to some Manin products of operads and it is shown to be closely related to Rota-Baxter operators. Hence, it gives a new effective way to compute Manin black products. The present construction is shown to have symmetry properties. Finally, this allows us to describe the algebraic structure of square matrices with coefficients in algebras of certain types. Many examples illustrate this text, including the case of Jordan algebras.

Contents

1. Introduction

Since the late 1990s, several algebraic structures with multiple binary operations have emerged: first the dendriform dialgebra of Loday [34], then the dendriform trialgebra of Loday and Ronco [38], discovered from studying algebraic K-theory, operads and algebraic topology. These were followed by quite a few other related structures, such as the quadri-algebra [3], the ennea-algebra, the NS-algebra, the dendriform-Nijenhuis and octo-algebra [29, 30, 31]. All these algebraic structures have a common property of "splitting the associativity", i.e., expressing the multiplication of an associative algebra as the sum of a string of binary operations. For example, a dendriform dialgebra has a string of two operations and satisfies three axioms, and it can be seen as an associative algebra which multiplication can be decomposed into two operations "in a coherent way". The constructions found later have increasing complexity in their definitions. For example the quadri-algebra [3] has a string of four operations satisfying nine axioms and the octo-algebra [30] has a string of eight operations satisfying 27 axioms. As shown in [14], these constructions can be put into the framework of operad (black square) products for nonsymmetric operads [14, 35, 47]. By doing so, they proved that these newer algebraic structures can be obtained from the known ones by the Manin black square product.

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It has been observed that a crucial role in the splitting of associativity is also played by the Rota-Baxter operator which originated from the probability study of G. Baxter [8], promoted by the combinatorial study of G.-C. Rota [42] and found many applications in the last decade in mathematics and physics [1, 4, 5, 15, 20, 21, 44], especially in the Connes-Kreimer approach of renormalization in quantum field theory [10, 17, 18, 24, 40]. The first such instance is the fact that a Rota-Baxter operator of weight zero on an associative algebra gives a dendriform dialgebra [1, 2]. Further instances were discovered later [3, 13, 29, 30, 31]. It was then shown that, in general, a Rota-Baxter operator on a class of binary quadratic nonsymmetric operads gives the black square product of dendriform algebra with these operads [14].

More recently, analogues of the dendriform dialgebra, quadri-algebra and octo-algebra for the Lie algebra, Jordan algebra, alternative algebra and Poisson algebra have been obtained [2, 6, 26, 32, 41]. They can be regarded as "splitting" of the operations in these latter algebras. On the other hand, it has been observed [47] that taking the Manin black product with the operad *PreLie* of preLie algebras also plays a role of splitting the operations of an operad. For example, the Manin black product of *PreLie* with the operad of associative algebras (resp. commutative algebras) gives the operad of dendriform dialgebras (resp. Zinbiel algebras).

Our goal in this paper is to set up a general framework to make precise the notion of "splitting" any binary algebraic operad, and to generalize the aforementioned relationship of "splitting" an operad with the Rota-Baxter operator and Manin black product. We achieve this through defining and studying the **successors** of binary algebraic operads defined by generating operations and relations. Thus we can go far beyond the scope of binary quadratic nonsymmetric operads [14] and can apply the construction for example to the operads of Lie algebras, Poisson algebras and Jordan algebras. This gives a quite general way to relate known operads and to produce new operads from the known ones.

We then explain the relationship between the three constructions applied to a binary operad $\mathcal{P}$: taking its bisuccessor (resp. trisuccessor) is equivalent to taking its Manin black product $\bullet$ with the operad *PreLie* (resp. *PostLie*), when the operad is quadratic. Both constructions can be obtained from a Rota-Baxter operator of weight zero (resp. non-zero). This is summed up in the following morphisms of operads.

$$\mathit{PreLie} \bullet \mathcal{P} \cong \mathrm{Su}(\mathcal{P}) \to \mathrm{RB}_0(\mathcal{P})$$

and

$$\mathit{PostLie} \bullet \mathcal{P} \cong \mathrm{TSu}(\mathcal{P}) \to \mathrm{RB}_1(\mathcal{P})\ .$$

Notice that this provides an effective way to compute Manin products for operads.

The space of squared matrices with coefficients in a commutative algebra carries a canonical associative algebra structure. We generalize such a result using the notion of successors: we describe canonical algebraic structures carried by squared matrices with coefficients in algebras over an operad. Finally, the present notion of successors is defined in such a way that it shares nice symmetry properties.

The following is a layout of this paper. In Section 2, the concepts of *successors* are introduced, together with examples and basic properties. The relationship of the successors with the Manin black product is studied in Section 3, establishing the connection indicated by the left-hand side in the above diagram. We apply these results to the study of algebraic properties of square matrices in Section 4. The relationship of the successors with the Rota-Baxter operator is studied in Section 5, establishing the connection indicated by the right-hand side in the above diagram. In

Section 6, we prove symmetric properties of iterated successors. Further examples are provided in the Appendix.

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2. The successors of a binary operad

In this section, we first introduce the concepts of the bisuccessor and trisuccessor of a labeled planar binary tree. These concepts are then applied to define similar concepts for a nonsymmetric operad and a (symmetric) operad. A list of examples are provided, followed by a study of the relationship among an operad and its successors.

2.1. **The successors of a tree.**

2.1.1. *Labeled trees.*

Definition 2.1. (a) Let $\mathfrak{T}$ denote the set of planar binary reduced rooted trees together with the trivial tree $|$. If $t \in \mathfrak{T}$ has n leaves, we call t an n-**tree**.

(b) Let $\mathcal{X}$ be a set. For an n-tree t in $\mathfrak{T}$, let $t(\mathcal{X})$ denote the set of decorations of t on its vertices by elements in $\mathcal{X}$ and of distinct decorations of t on its leaves by elements in $\mathbb{N}$. Let $\mathfrak{T}(\mathcal{X})$ be the set of planar binary rooted trees whose vertices are decorated by elements in $\mathcal{X}$ and leaves are distinctly decorated by elements in $\mathbb{N}$. Thus

$$\mathfrak{T}(\mathcal{X}) = \coprod_{t\in\mathfrak{T}} t(\mathcal{X}). \tag{1}$$

If $\tau \in t(\mathcal{X})$ for a n-tree t, we call τ a labeled n-**tree**.

(c) For $\tau \in \mathfrak{T}(\mathcal{X})$, we let $\mathrm{V}in(\tau)$ (resp. $\mathrm{L}in(\tau)$) denote the set of labels of the vertices (resp. leaves) of τ.

(d) Let $\tau_\ell, \tau_r \in \mathfrak{T}(\mathcal{X})$ with disjoint sets of leaf labels. Let $\omega \in \mathcal{X}$. The **grafting of τ_ℓ and τ_r along ω** is denoted by $\tau_\ell \vee_\omega \tau_r$. It gives rise to an element in $\mathfrak{T}(\mathcal{X})$.

(e) For $\tau \in \mathfrak{T}(\mathcal{X})$ with $|\mathrm{L}in(\tau)| > 1$, we let $\tau = \tau_\ell \vee_\omega \tau_r$ denote the unique decomposition of τ as a grafting of τ_ℓ and τ_r in $\mathfrak{T}(\mathcal{X})$ along $\omega \in \mathcal{X}$.

Let V be a vector space, regarded as an arity graded vector space concentrated in arity 2: $V = V_2$. Recall that the free nonsymmetric operad $\mathcal{T}_{ns}(V)$ on V is given by the vector space

$$\mathcal{T}_{ns}(V) := \bigoplus_{t\in\mathfrak{T}} t[V] ,$$

where $t[V]$ is the treewise tensor module associated to t. This module is explicitly given by

$$t[V] := \bigotimes_{v\in \mathrm{V}in(t)} V_{|\mathrm{In}(v)|} ,$$

where $|In(v)|$ denotes the number of inputs of the vertex v, see Section 5.8.5 of [39]. A basis $\mathcal{V}$ of V induces a basis $t(\mathcal{V})$ of $t[V]$ and a basis $\mathfrak{T}(\mathcal{V})$ of $\mathcal{T}_{ns}(V)$. In particular, any element of $t[V]$ can be represented as a sum of elements in $t(\mathcal{V})$.

2.1.2. *Bisuccessor.*

Definition 2.2. Let V be a vector space with a basis $\mathcal{V}$.

(a) Define a vector space

$$\widetilde{V} = V \otimes (\mathbf{k} \prec \oplus\, \mathbf{k} \succ)\,, \tag{2}$$

where we denote $(\omega \otimes \prec)$ (resp. $(\omega \otimes \succ)$) by $\begin{pmatrix}\omega\\ \prec\end{pmatrix}$ (resp. $\begin{pmatrix}\omega\\ \succ\end{pmatrix}$), for $\omega \in V$. Then $\mathcal{V} \times \{\prec, \succ\}$ is a basis of $\widetilde{V}$.

(b) For a labeled n-tree τ in $\mathfrak{T}(\mathcal{V})$, define $\widetilde{\tau}$ in $\mathcal{T}_{ns}(\,\widetilde{V}\,)$, where $\widetilde{V}$ is seen as an arity graded module concentrated in arity 2, as follows:

- $\widetilde{|} = |$
- when $n \geq 2$, $\widetilde{\tau}$ is obtained by replacing each decoration $\omega \in \mathrm{V}in(\tau)$ by

$$\begin{pmatrix}\omega\\ *\end{pmatrix} := \begin{pmatrix}\omega\\ \prec\end{pmatrix} + \begin{pmatrix}\omega\\ \succ\end{pmatrix}.$$

We extend this definition to $\mathcal{T}_{ns}(V)$ by linearity.

Definition 2.3. Let V be a vector space with a basis $\mathcal{V}$. Let τ be a labeled n-tree in $\mathfrak{T}(\mathcal{V})$. The **bisuccessor** $\mathrm{Su}_x(\tau)$ of τ with respect to a leaf $x \in \mathrm{L}in(\tau)$ is an element of $\mathcal{T}_{ns}(\widetilde{V})$ defined by induction on $n := |\mathrm{L}in(\tau)|$ as follows:

- $\mathrm{Su}_x(\,|\,) = |$;
- assume that $\mathrm{Su}_x(\tau)$ have been defined for τ with $|\mathrm{L}in(\tau)| \leq k$ for a $k \geq 1$. Then, for a labeled $(k+1)$-tree $\tau \in \mathfrak{T}(\mathcal{V})$ with its decomposition $\tau_\ell \vee_\omega \tau_r$, we define

$$\mathrm{Su}_x(\tau) = \mathrm{Su}_x(\tau_\ell \vee_\omega \tau_r) = \begin{cases} \mathrm{Su}_x(\tau_\ell) \vee_{\begin{pmatrix}\omega\\ \prec\end{pmatrix}} \widetilde{\tau}_r, & x \in \mathrm{L}in(\tau_\ell), \\ \widetilde{\tau}_\ell \vee_{\begin{pmatrix}\omega\\ \succ\end{pmatrix}} \mathrm{Su}_x(\tau_r), & x \in \mathrm{L}in(\tau_r). \end{cases} \tag{3}$$

For $m \geq 1$, the m-th iteration of Su is denoted by Su^m.

We have the following description of the bisuccessor.

Proposition 2.4. *Let V be a vector space with basis $\mathcal{V}$, τ be in $\mathfrak{T}(\mathcal{V})$ and x be in* $\mathrm{L}in(\tau)$*. The bisuccessor* $\mathrm{Su}_x(\tau)$ *of τ is obtained by relabeling each vertex of τ according to the following rules:*

(a) *we replace the label ω of each vertex on the path from the root the the leave x of τ by*

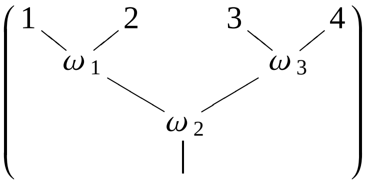
(i) $\begin{pmatrix}\omega\\ \prec\end{pmatrix}$ *if the path turns left at this vertex,*

(ii) $\begin{pmatrix}\omega\\ \succ\end{pmatrix}$ *if the path turns right at this vertex,*

(b) *we replace the label ω of each vertex not on the path from the root the the leave x of τ by* $\begin{pmatrix}\omega\\ \star\end{pmatrix} := \begin{pmatrix}\omega\\ \prec\end{pmatrix} + \begin{pmatrix}\omega\\ \succ\end{pmatrix}$.

Proof. By induction on $|\mathrm{L}in(\tau)| \geq 1$. □

Example 2.5. Su_2



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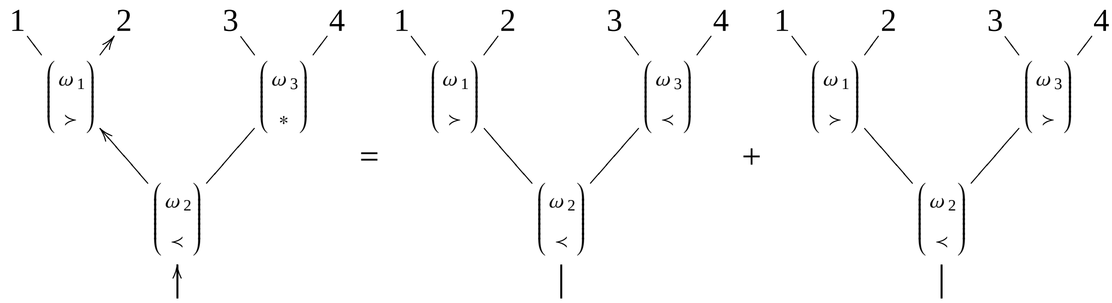


Lemma 2.6. *Let V be a vector space with basis $\mathcal{V}$, τ be a labeled n-tree in $\mathcal{T}(\mathcal{V})$ and x be in* $\mathrm{L}in(\tau)$. *Then the following relation holds*

$$\mathrm{Su}_{\sigma^{-1}(x)}(\tau^{\sigma}) = \mathrm{Su}_x(\tau)^{\sigma}\ , \forall \sigma \in \mathbb{S}_n\ .$$

Proof. By inspection of the action of the symmetric group on a tree. □

2.1.3. *Trisuccessors.*

Definition 2.7. Let V be a vector space with a basis $\mathcal{V}$.

(a) Define a vector space

$$\widehat{V} = V \otimes (\,\mathbf{k} \prec \oplus\, \mathbf{k} \succ \oplus\, \mathbf{k}\,\cdot\,)\ , \tag{4}$$

where we denote $(\omega \otimes \prec)$ (resp. $(\omega \otimes \succ)$, resp. $(\omega \otimes \cdot)$) by $\begin{pmatrix}\omega\\ \prec\end{pmatrix}$ $\Big($resp. $\begin{pmatrix}\omega\\ \succ\end{pmatrix}$, resp. $\begin{pmatrix}\omega\\ \cdot\end{pmatrix}\Big)$, for $\omega \in V$. Then $\mathcal{V} \times \{\prec, \succ, \cdot\}$ is a basis of $\widehat{V}$.

(b) For a labeled n-tree τ in $\mathcal{T}(\mathcal{V})$, define $\widehat{\tau}$ in $\mathcal{T}_{ns}(\widehat{V})$, where $\widehat{V}$ is regarded as an arity graded module concentrated in arity 2, as follows:

- $\widehat{|} = |$
- when $n \geq 2$, $\widehat{\tau}$ is obtained by replacing the label $\omega \in \mathrm{V}in(\tau)$ of each vertex of τ by

$$\begin{pmatrix}\omega\\ \star\end{pmatrix} := \begin{pmatrix}\omega\\ \prec\end{pmatrix} + \begin{pmatrix}\omega\\ \succ\end{pmatrix} + \begin{pmatrix}\omega\\ \cdot\end{pmatrix}.$$

We extend this definition to $\mathcal{T}_{ns}(\widehat{V})$ by linearity.

Definition 2.8. Let V be a vector space with a basis $\mathcal{V}$. Let τ be a labeled n-tree in $\mathcal{T}(\mathcal{V})$ and let J be a nonempty subset of $\mathrm{L}in(\tau)$. The **trisuccessor** $\mathrm{TSu}_J(\tau)$ of τ with respect to J is an element of $\mathcal{T}_{ns}(\widehat{V})$ defined by induction on $n := |\mathrm{L}in(\tau)|$ as follows:

- $\mathrm{TSu}_J(\,|\,) = |$;
- assume that $\mathrm{TSu}_J(\tau)$ have been defined for τ with $|\mathrm{L}in(\tau)| \leq k$ for a $k \geq 1$. Then, for a labeled $(k+1)$-tree $\tau \in \mathcal{T}(\mathcal{V})$ with its decomposition $\tau_\ell \vee_\omega \tau_r$, we define

$$\mathrm{TSu}_J(\tau) = \mathrm{TSu}_J(\tau_\ell \vee_\omega \tau_r) = \begin{cases} \mathrm{TSu}_J(\tau_\ell) \vee_{\begin{pmatrix}\omega\\ \prec\end{pmatrix}} \widehat{\tau_r}, & J \subseteq \mathrm{L}in(\tau_\ell), \\ \widehat{\tau_\ell} \vee_{\begin{pmatrix}\omega\\ \succ\end{pmatrix}} \mathrm{TSu}_J(\tau_r), & J \subseteq \mathrm{L}in(\tau_r), \\ \mathrm{TSu}_{J\cap \mathrm{L}in(\tau_\ell)}(\tau_\ell) \vee_{\begin{pmatrix}\omega\\ \cdot\end{pmatrix}} \mathrm{TSu}_{J\cap \mathrm{L}in(\tau_r)}(\tau_r), & \text{otherwise.} \end{cases} \tag{5}$$

For $m \geq 1$, the m-th iteration of TSu is denoted by TSu^m.

We have the following description of the trisuccessor.

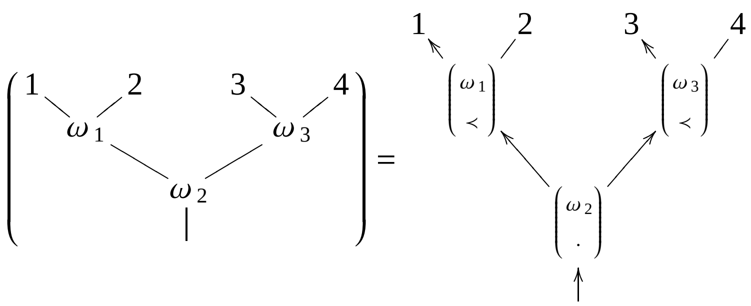
Proposition 2.9. *Let V be a vector space, τ be in $\mathcal{T}(\mathcal{V})$ and J be a nonempty subset of* $\mathrm{L}in(\tau)$. *The trisuccessor* $\mathrm{TSu}_J(\tau)$ *is obtained by relabeling each vertex of τ according to the following rules:*

(a) *we replace the label ω of each vertex on at least one of the paths from the root to the leafs x in J by*
   (i) $\begin{pmatrix}\omega\\ \prec\end{pmatrix}$ *if all such paths turn left at this vertex;*
   (ii) $\begin{pmatrix}\omega\\ \succ\end{pmatrix}$ *if all such paths turn right at this vertex;*
   (iii) $\begin{pmatrix}\omega\\ \cdot\end{pmatrix}$ *if some of such paths turn left and some of such paths turn right at this vertex;*

(b) *we replace the label ω of each other vertex by* $\begin{pmatrix}\omega\\ \star\end{pmatrix} := \begin{pmatrix}\omega\\ \prec\end{pmatrix} + \begin{pmatrix}\omega\\ \succ\end{pmatrix} + \begin{pmatrix}\omega\\ \cdot\end{pmatrix}$.

Proof. The proof follows from the same argument as the proof of Proposition 2.4. □

Example 2.10. $\mathrm{TSu}_{\{1,3\}}$



Lemma 2.11. *Let V be a vector space with basis $\mathcal{V}$, τ be a labeled n-tree in $\mathcal{T}(\mathcal{V})$ and J be a nonempty subset of* $\mathrm{L}in(\tau)$. *Then the following relation holds*

$$\mathrm{TSu}_{\sigma^{-1}(J)}(\tau^{\,\sigma}) = \mathrm{TSu}_J(\tau)^{\,\sigma}\ , \forall \sigma \in \mathbb{S}_n\ .$$

2.2. **The successors of a binary nonsymmetric operad.** Note that the definition of the successors extends linearly from $\mathfrak{T}(\mathcal{V})$ to $\mathcal{T}_{ns}(V)$ and to $\mathcal{T}_{ns}(\widehat{V})$, when $\mathcal{V}$ is a linear basis of V.

Definition 2.12. Let V be a vector space and $\mathcal{V}$ be a basis of V.

(a) An element

$$r := \sum_{i=1}^{r} c_i\tau_i, \quad c_i \in \mathbf{k}, \tau_i \in \mathfrak{T}(\mathcal{V}),$$

in $\mathcal{T}_{ns}(V)$ is called **homogeneous** of arity n if $|\mathrm{L}in(\tau_i)| = n$ for $1 \le i \le r$.

(b) A collection of elements

$$r_s := \sum_{i} c_{s,i}\tau_{s,i}, \quad c_{s,i} \in \mathbf{k}, \tau_{s,i} \in \mathfrak{T}(\mathcal{V}), 1 \le s \le k, k \ge 1,$$

in $\mathcal{T}_{ns}(V)$ is called **locally homogenous** if each element r_s, $1 \le s \le k$, in the system is homogeneous of a certain arity n_s.

Definition 2.13. Let $\mathcal{P} = \mathcal{T}_{ns}(V)/(R)$ be a binary nonsymmetric operad with a basis $\mathcal{V}$ of $V = V_2$. In this case, the space of relations R is the vector space spanned by locally homogeneous elements of the form

$$r_s = \sum_{i} c_{s,i}\tau_{s,i} \ \in \mathcal{T}_{ns}(V)\ , \ \ c_{s,i} \in \mathbf{k},\ \ \tau_{s,i} \in \mathfrak{T}(\mathcal{V}),\ \ 1 \le s \le k, k \ge 1. \tag{6}$$

(a) The **bisuccessor** of $\mathcal{P}$ is defined to be the binary nonsymmetric operad

$$\mathrm{Su}(\mathcal{P}) := \mathcal{T}_{ns}(\widetilde{V})/(\mathrm{Su}(R))\ ,$$

where the space of relations is the vector space spanned by

$$\mathrm{Su}(R) := \left\{ \mathrm{Su}_x(r_s) = \sum_{i} c_{s,i}\mathrm{Su}_x(\tau_{s,i}) \ \middle|\ x \in \mathrm{L}in(\tau_{s,i}),\ 1 \le s \le k \right\} \tag{7}$$

Note that, by our assumption, for a fixed s, $\mathrm{L}in(\tau_{s,i})$ are the same for all i. The **N-th bisuccessor** ($N \ge 2$) of $\mathcal{P}$, which is denoted by $\mathrm{Su}^N(\mathcal{P})$, is defined as the bisuccessor of the **$(N-1)$-th bisuccessor** of the operad, where the **first bisuccessor** of the operad is just the bisuccessor of the operad.

(b) The **trisuccessor** of $\mathcal{P}$ is defined to be the binary nonsymmetric operad

$$\mathrm{TSu}(\mathcal{P}) := \mathcal{T}_{ns}(\widehat{V})/(\mathrm{TSu}(R))\ ,$$

where the space of relations is the vector space spanned by

$$\mathrm{TSu}(R) := \left\{ \mathrm{TSu}_J(r_s) = \sum_{i} c_{s,i}\mathrm{TSu}_J(\tau_{s,i}) \ \middle|\ \emptyset \ne J \subseteq \mathrm{L}in(\tau_{s,i}),\ 1 \le s \le k \right\} \tag{8}$$

The **N-th trisuccessor** ($N \ge 2$) of $\mathcal{P}$, which is denoted by $\mathrm{TSu}^N(\mathcal{P})$, is defined as the trisuccessor of the **$(N-1)$-th trisuccessor** of the operad, where the **first trisuccessor** of the operad is just the trisuccessor of the operad.

Proposition 2.14. *The definition of the successors of a binary non-symmetric operad does not depend on the basis of the vector space of generating operations.*

Proof. Let $\mathcal{P} := \mathcal{T}_{ns}(\widehat{V})/(R)$ be a binary non-symmetric operad. This proposition is straightforward from the linearity of the successors and from the treewise tensor module structure on $\mathcal{T}_{ns}(V)$ and on $\mathcal{T}_{ns}(\widehat{V})$. $\square$

We give some examples of successors.

Example 2.15. The **dendriform dialgebra** of Loday [34] is defined by two bilinear operations $\{\prec, \succ\}$ and relations:

$$(x \prec y) \prec z = x \prec (y \star z),\ (x \succ y) \prec z = x \succ (y \prec z),\ (x \star y) \succ z = x \succ (y \succ z),$$

where $\star := \prec + \succ$. It is easy to check that the corresponding operad *Dend* is the bisuccessor of *Ass*. Similarly, the operad *Quad* of quadri-algebras of Aguiar and Loday [3], is the bisuccessor of *Dend*. Furthermore, the operad *Octo* of octo-algebras of Leroux [30] is the bisuccessor of *Quad*. For $N \geq 2$, the N-th power of *Dend* defined in [14] is the N-th bisuccessor of *Dend*.

Example 2.16. Similarly, the trisuccessor of *Ass* is the operad *TriDend* of dendriform trialgebras defined by Loday and Ronco [38]. The operad *Ennea* of Ennea-algebras of Leroux [31] is the trisuccessor of *TriDend*. For $N \geq 2$, the N-th power of *TriDend* defined in [14] is the N-th trisuccessor of *TriDend*.

2.3. **The successors of a binary operad.** When $V = V(2)$ is an $\mathbb{S}$-module concentrated in arity 2, the free operad $\mathcal{T}(V)$ is generated by the binary trees "in space" with vertices labeled by elements in V. So we have to refine our arguments.

More precisely, the free operad $\mathcal{T}(V)$ on an $\mathbb{S}$-module $V = V(2)$ is given by the $\mathbb{S}$-module

$$\mathcal{T}(V) := \bigoplus_{\mathrm{t}\in\mathbb{T}} \mathrm{t}[V]\ ,$$

where $\mathbb{T}$ denotes the set of isomorphism classes of reduced binary trees, see Appendix C of [39], and where $\mathrm{t}[V]$ is the treewise tensor $\mathbb{S}$-module associated to t. This $\mathbb{S}$-module is explicitly given by

$$\mathrm{t}[V] := \bigotimes_{v\in \mathrm{V}in(\mathrm{t})} V(\mathrm{I}n(v))\ ,$$

see Section 5.5.1 of [39]. Notice that $\mathrm{I}n(v)$ is a set. For any finite set $\mathcal{X}$ of cardinal n, the definition of $V(\mathcal{X})$ is given by the following coinvariant space

$$V(\mathcal{X}) := \left(\bigoplus_{f:\underline{n}\to\mathcal{X}} V(n) \right)_{\mathbb{S}_n}\ ,$$

where the sum is over all the bijections from $\underline{n} := \{1, \ldots, n\}$ to $\mathcal{X}$ and where the symmetric group acts diagonally.

To represent a tree t in $\mathbb{T}$ with a planar tree in $\mathfrak{T}$ consists in choosing a total order on the set of inputs of each vertex of t. We define an equivalence relation $\sim$ on $\mathfrak{T}$ as follows: two planar binary trees in $\mathfrak{T}$ are equivalent if they represent the same tree in $\mathbb{T}$. It induces a bijection $\mathbb{T} \cong \mathfrak{T}/\sim$. Moreover, by Section 2.8 of [25], we have $t[V] \cong \mathrm{t}[V]$, for any planar binary tree t in $\mathfrak{T}$ which represents the binary tree t in $\mathbb{T}$. Therefore, we have

$$\mathcal{T}(V) \cong \bigoplus_{t\in\mathfrak{R}} t[V]\ ,$$

where $\mathfrak{R}$ is a set of representatives of $\mathfrak{T}/\sim$.

Example 2.17. For instance, one set of representatives of $\mathfrak{T}/\sim$ is the set of tree monomials defined in Section 2.8 of [25]. See also Section 3.1 of [12]. Another example is a generalization of the trees I, II and III given in Section 7.6.3 of [39].

Lemma 2.18. *Let $\mathfrak{R}$ be a set of representatives of $\mathfrak{T}/\sim$ and $V = V(2)$ be an $\mathbb{S}$-module concentrated in arity 2, with linear basis $\mathcal{V}$. Then $\mathfrak{R}(\mathcal{V}) := \{\tau \in t(\mathcal{V}) \,|\, t \in \mathfrak{R}\}$ is a linear basis of the free operad $\mathcal{T}(V)$.*

Proof. According to Section 2.1, when t is a planar binary tree, $t(\mathcal{V})$ is a basis of $t[V]$. □

Definition 2.19. Let $\mathcal{P} = \mathcal{T}(V)/(R)$ be a binary operad on the $\mathbb{S}$-module $V = V(2)$, concentrated in arity 2 with a $\mathbf{k}[\mathbb{S}_2]$-basis $\mathcal{V}$, such that R is spanned, as an $\mathbb{S}$-module, by locally homogeneous elements of the form

$$R := \left\{ r_s := \sum_i c_{s,i}\tau_{s,i} \;\middle|\; c_{s,i} \in \mathbf{k}, \tau_{s,i} \in \{t(\mathcal{V}), t \in \mathfrak{R}\},\ 1 \le s \le k, k \ge 1 \right\}, \tag{9}$$

where $\mathfrak{R}$ is a set of representatives of $\mathfrak{T}/\sim$.

(a) The **bisuccessor** of $\mathcal{P}$ is defined to be the binary operad $\mathrm{Su}(\mathcal{P}) = \mathcal{T}(\widetilde{V})/(\mathrm{Su}(R))$ where the $\mathbb{S}_2$-action on $\widetilde{V}$ is given by

$$\begin{pmatrix} \omega \\ \prec \end{pmatrix}^{(12)} := \begin{pmatrix} \omega^{(12)} \\ \succ \end{pmatrix}, \quad \begin{pmatrix} \omega \\ \succ \end{pmatrix}^{(12)} := \begin{pmatrix} \omega^{(12)} \\ \prec \end{pmatrix}, \ \omega \in V, \tag{10}$$

and the space of relations is generated, as an $\mathbb{S}$-module, by

$$\mathrm{Su}(R) := \left\{ \mathrm{Su}_x(r_s) := \sum_i c_{s,i}\mathrm{Su}_x(t_{s,i}) \;\middle|\; x \in \mathrm{L}in(t_{s,i}),\ 1 \le s \le k \right\} \tag{11}$$

Note that, by our assumption, for a fixed s, $\mathrm{L}in(t_{s,i})$ are the same for all i. The N**-th bisuccessor** ($N \ge 2$) of $\mathcal{P}$, which is denoted by $\mathrm{Su}^N(\mathcal{P})$, is defined as the successor of the $(N-1)$**-th bisuccessor** of the operad, where the **first bisuccessor** of the operad is just the bisuccessor of the operad.

(b) The **trisuccessor** of $\mathcal{P}$ is defined to be the binary operad $\mathrm{TSu}(\mathcal{P}) = \mathcal{T}(\widehat{V})/(\mathrm{TSu}(R))$ where the $\mathbb{S}_2$-action on $\widehat{V}$ is given by

$$\begin{pmatrix} \omega \\ \prec \end{pmatrix}^{(12)} := \begin{pmatrix} \omega^{(12)} \\ \succ \end{pmatrix}, \quad \begin{pmatrix} \omega \\ \succ \end{pmatrix}^{(12)} := \begin{pmatrix} \omega^{(12)} \\ \prec \end{pmatrix}, \quad \begin{pmatrix} \omega \\ \cdot \end{pmatrix}^{(12)} := \begin{pmatrix} \omega^{(12)} \\ \cdot \end{pmatrix}, \ \omega \in V, \tag{12}$$

and the space of relations is generated, as an $\mathbb{S}$-module, by

$$\mathrm{TSu}(R) := \left\{ \mathrm{TSu}_J(r_s) := \sum_i c_{s,i}\mathrm{TSu}_J(t_{s,i}) \;\middle|\; \emptyset \ne J \subseteq \mathrm{L}in(t_{s,i}),\ 1 \le s \le k \right\} \tag{13}$$

The N**-th trisuccessor** ($N \ge 2$) of $\mathcal{P}$, which is denoted by $\mathrm{TSu}^N(\mathcal{P})$, is defined as the trisuccessor of the $(N-1)$**-th trisuccessor** of the operad, where the **first trisuccessor** of the operad is just the trisuccessor of the operad.

Proposition 2.20. *The successors of a binary operad $\mathcal{P} = \mathcal{T}(V)/(R)$ depends neither on the $\mathbf{k}[\mathbb{S}_2]$-basis $\mathcal{V}$ of V nor on the set of representatives $\mathfrak{R}$ of $\mathfrak{T}/\sim$.*

Proof. Notice that if $\mathcal{V}$ is a $\mathbf{k}[\mathbb{S}_2]$-basis of V then the set $\mathcal{V} \otimes \mathbb{S}_2$ is a linear basis of V.

The independence with respect to the choice of a $\mathbf{k}[\mathbb{S}_2]$-basis of V is a consequence of the linearity of the successors and of the treewise tensor module structure.

Next let $\mathcal{V}$ be a $\mathbf{k}[\mathbb{S}_2]$-basis of V. Let $\mathfrak{R}$ and $\mathfrak{R}'$ be two sets of representatives of $\mathfrak{T}/\sim$. Let τ in $t(\mathcal{V} \otimes \mathbb{S}_2)$ and τ' in $t'(\mathcal{V} \otimes \mathbb{S}_2)$, where $t \in \mathfrak{R}$ and $t' \in \mathfrak{R}'$, be two labeled planar binary trees which arise from the same element in $\mathcal{T}(V)$, through the bijections given previously in this section.

Then, for all $i \in \mathrm{L}in(\tau) = \mathrm{L}in(\tau')$ (resp. for any nonempty subset $J \subseteq \mathrm{L}in(\tau) = \mathrm{L}in(\tau')$), we have $\mathrm{Su}_i(\tau) = \mathrm{Su}_i(\tau')$ (resp. $\mathrm{TSu}_J(\tau) = \mathrm{TSu}_J(\tau')$). Finally, we conclude the proof using Lemma 2.18 and the linearity of the successors. □

2.4. **Relations with the non-symmetric framework.** We denote by Op (resp. by $\mathsf{Ns\,Op}$) the category of operads (resp. of non-symmetric operads). There is a forgetful functor

$$\begin{array}{ccc} \mathsf{Op} & \to & \mathsf{Ns\,Op} \\ \mathcal{P} & \mapsto & \overline{\mathcal{P}} \end{array} ,$$

where $\overline{\mathcal{P}}_n := \mathcal{P}(n)$. In other words, we forget the $\mathbb{S}_n$-module structure.
This functor admits a left adjoint

$$\begin{array}{ccc} \mathsf{Ns\,Op} & \to & \mathsf{Op} \\ \mathcal{P} & \mapsto & Reg(\mathcal{P}) \end{array} ,$$

where $Reg(\mathcal{P})(n) := \mathcal{P}_n \otimes \mathbf{k}[\mathbb{S}_n]$. Such operads are called *regular operads*, see [39, Section 5.8.12] for more details. Notice that a presentation of the regular operad associated to a binary non-symmetric operad $\mathcal{P} = \mathcal{T}_{ns}(V)/(R)$, where $\mathcal{T}_{ns}(V)$ is the free non-symmetric operad on $V = V(2)$ and $R = \{R_n\}_{n\in\mathbb{N}}$, is given by

$$Reg(\mathcal{P}) = \mathcal{T}(V \otimes \mathbf{k}[\mathbb{S}_2])/(R_n \otimes \mathbf{k}[\mathbb{S}_n], n \in \mathbb{N}) .$$

Proposition 2.21. *Let $\mathcal{P} = \mathcal{T}_{ns}(V)/(R)$ be a binary non-symmetric operad. We have*

$$\mathrm{Su}(Reg(\mathcal{P})) \cong Reg(\mathrm{Su}(\mathcal{P})) .$$

Proof. As $\mathbb{S}_2$-modules, the space of generating operations of $Reg(\mathcal{P})$ is spanned by V, so the space of generating operations of $\mathrm{Su}(Reg(\mathcal{P}))$ is spanned by $\widetilde{V}$. As $\mathbb{S}$-modules, the space of relations of $Reg(\mathcal{P})$ is spanned by R, so the space of relations of $\mathrm{Su}(Reg(\mathcal{P}))$ is spanned by $\mathrm{Su}(R)$. □

2.5. **Examples of successors.** We give some examples of successors of binary operads.
Let $V = V(2)$ be an $\mathbb{S}_2$-module of generating operations. Then we have

$$\mathcal{T}(V)(3) = (V \otimes_{\mathbb{S}_2} (V \otimes \mathbf{k} \oplus \mathbf{k} \otimes V)) \otimes_{\mathbb{S}_2} \mathbf{k}[\mathbb{S}_3].$$

$\mathcal{T}(V)(3)$ can be identify with 3 copies of $V \otimes V$. We denote them by $V \circ_{\mathrm{I}} V$, $V \circ_{\mathrm{II}} V$ and $V \circ_{\mathrm{III}} V$, following the convention in [47]. Then, as a vector space, $\mathcal{T}(V)(3)$ is generated by elements of the form

$$\omega \circ_{\mathrm{I}} \nu (\leftrightarrow (x\,\omega\,y)\,\nu\,z),\ \omega \circ_{\mathrm{II}} \nu (\leftrightarrow (y\,\nu\,z)\,\omega\,x),\ \omega \circ_{\mathrm{III}} \nu (\leftrightarrow (z\,\nu\,x)\,\omega\,y), \forall\, \omega, \nu \in \mathcal{V}. \tag{14}$$

For an operad where the space of generators V is equal to $\mathbf{k}[\mathbb{S}_2] = \mu.\mathbf{k} \oplus \mu'.\mathbf{k}$ with $\mu.(12) = \mu'$, we will adopt the convention in [47, p. 129] and denote the 12 elements of $\mathcal{T}(V)(3)$ by v_i, for $1 \le i \le 12$, in the following table.

v_1	$\mu \circ_{\mathrm{I}} \mu \leftrightarrow (xy)z$	v_5	$\mu \circ_{\mathrm{III}} \mu \leftrightarrow (zx)y$	v_9	$\mu \circ_{\mathrm{II}} \mu \leftrightarrow (yz)x$
v_2	$\mu' \circ_{\mathrm{II}} \mu \leftrightarrow x(yz)$	v_6	$\mu' \circ_{\mathrm{I}} \mu \leftrightarrow z(xy)$	v_{10}	$\mu' \circ_{\mathrm{III}} \mu \leftrightarrow y(zx)$
v_3	$\mu' \circ_{\mathrm{II}} \mu' \leftrightarrow x(zy)$	v_7	$\mu' \circ_{\mathrm{I}} \mu' \leftrightarrow z(yx)$	v_{11}	$\mu' \circ_{\mathrm{III}} \mu' \leftrightarrow y(xz)$
v_4	$\mu \circ_{\mathrm{III}} \mu' \leftrightarrow (xz)y$	v_8	$\mu \circ_{\mathrm{II}} \mu' \leftrightarrow (zy)x$	v_{12}	$\mu \circ_{\mathrm{I}} \mu' \leftrightarrow (yx)z$

2.5.1. *Examples of successors.* Recall that a **(left) Zinbiel algebra** [34] is defined by a bilinear operation $\cdot$ and a relation

$$(x \cdot y + y \cdot x) \cdot z = x \cdot (y \cdot z).$$

Proposition 2.22. *The operad Zinb is the bisuccessor of the opeard Com.*

Proof. Let ω be the generating operation of the operad *Com*. Set $\prec := \begin{pmatrix} \omega \\ \prec \end{pmatrix}$ and $\succ := \begin{pmatrix} \omega \\ \succ \end{pmatrix}$. Since $\begin{pmatrix} \omega \\ \prec \end{pmatrix}^{(12)} = \begin{pmatrix} \omega^{(12)} \\ \succ \end{pmatrix} = \begin{pmatrix} \omega \\ \succ \end{pmatrix}$, we have $\prec^{(12)} = \succ$. The space of relations of *Com* is generated as an $\mathbb{S}_3$-module by

$$v_1 - v_9 = \omega \circ_{\mathrm{I}} \omega - \omega \circ_{\mathrm{II}} \omega \ .$$

Then we have

$$\begin{aligned}
\mathrm{Su}_x(v_1 - v_9) &= z \succ (y \succ x) - (y \succ z + z \succ y) \succ x;\\
\mathrm{Su}_y(v_1 - v_9) &= z \succ (x \succ y) - x \succ (z \succ y);\\
\mathrm{Su}_z(v_1 - v_9) &= (x \succ y + y \succ x) \succ z - x \succ (y \succ z).
\end{aligned}$$

Replacing the operation $\succ$ by $\cdot$, we get $\mathrm{Su}(Com) = Zinb$. □

Also recall that a **right pre-Lie algebra** is defined by one bilinear operation $\cdot$ and one relation:

$$(x \cdot y) \cdot z - x \cdot (y \cdot z) = (x \cdot z) \cdot y - x \cdot (z \cdot y)\ .$$

The associated operad is denoted by *PreLie*.

Proposition 2.23. *The operad PreLie is the bisuccessor of the operad Lie.*

Proof. Let μ be the generating operation of the operad *Lie*. Set $\prec := \begin{pmatrix} \mu \\ \prec \end{pmatrix}$ and $\succ := \begin{pmatrix} \mu \\ \succ \end{pmatrix}$. Since $\begin{pmatrix} \mu \\ \prec \end{pmatrix}^{(12)} = \begin{pmatrix} \mu^{(12)} \\ \succ \end{pmatrix} = -\begin{pmatrix} \mu \\ \succ \end{pmatrix}$, we have $\prec^{(12)} = -\succ$. The space of relations of *Lie* is generated as an $\mathbb{S}_3$-module by

$$v_1 + v_5 + v_9 = \mu \circ_{\mathrm{I}} \mu + \mu \circ_{\mathrm{II}} \mu + \mu \circ_{\mathrm{III}} \mu\ .$$

Then we have

$$\begin{aligned}
\mathrm{Su}_x(v_1 + v_5 + v_9) &= (x \prec y) \prec z - (x \prec z) \prec y - x \prec (y \prec z - z \prec y);\\
\mathrm{Su}_y(v_1 + v_5 + v_9) &= -(y \prec x) \prec z - y \prec (-x \prec z + z \prec x) + (y \prec z) \prec x;\\
\mathrm{Su}_z(v_1 + v_5 + v_9) &= -z \prec (-y \prec x + x \prec y) + (z \prec x) \prec y - (z \prec y) \prec x.
\end{aligned}$$

Replacing the operation $\prec$ by $\cdot$, we get $\mathrm{Su}(Lie) = PreLie$. □

A **Poisson algebra** is defined to be a **k**-vector space with two bilinear operations $\{,\}$ and $\circ$ such that $\{,\}$ is a Lie bracket and $\circ$ is a product of commutative associative algebra, and they are compatible in the sense that

$$\{x, y \circ z\} = \{x, y\} \circ z + y \circ \{x, z\}.$$

A **(left) pre-Poisson algebra** of Aguiar [2] is defined as two bilinear operations $*$ and $\cdot$ such that $*$ is a product of (left) Zinbiel algebra and $\cdot$ is a product of (left) pre-Lie algebra and they are compatible in the sense that

$$\begin{aligned}
(x \cdot y - y \cdot x) * z &= x \cdot (y * z) - y * (x \cdot z),\\
(x * y + y * x) \cdot z &= x * (y \cdot z) + y * (x \cdot z).
\end{aligned}$$

By a similar argument as in Proposition 2.22, we obtain

Proposition 2.24. *The bisuccessor of the operad Poisson is the operad PrePoisson.*

Proof. Straightforward by computation. □

2.5.2. *Examples of trisuccessors.* We similarly have the following examples of trisuccessors of operads.

Example 2.25. A **commutative tridendriform algebra** [36, 38] is a vector space A equipped with a product $\prec$ and a commutative associative product $\cdot$ satisfying the following equations:

$$(x \prec y) \prec z = x \prec (y \prec z + z \prec y + y \cdot z),$$
$$(x \cdot y) \prec z = x \cdot (y \prec z).$$

Proposition 2.26. *The operad ComTriDend is the trisuccessor of the operad Comm.*

A **PostLie algebra** [46] is a vector space A with a product $\circ$ and a skew-symmetric operation $[,]$ satisfying the relations:

$$\begin{aligned} &[[x,y],z] + [[z,x],y] + [[y,z],x] = 0,\\ &(x \circ y) \circ z - x \circ (y \circ z) - (x \circ z) \circ y + x \circ (z \circ y) - x \circ [y,z] = 0,\\ &[x,y] \circ z - [x \circ z, y] - [x, y \circ z] = 0. \end{aligned} \tag{15}$$

If $(A, \circ, [,])$ is a PostLie algebra, then $(A, [,])$ and $(A, \{,\})$ are Lie algebras, where the operation $\{,\}$ is defined by

$$\{x,y\} := x \circ y - y \circ x + [x,y], \quad \forall x,y \in A.$$

Moreover, it is easy to see that if the operation $[,]$ happens to be trivial, then $(A, \circ)$ becomes a preLie algebra.

Proposition 2.27. *The operad PostLie is the trisuccessor of the operad Lie.*

Proof. Let μ be the generating operation of the operad *Lie*. Set $\prec := \begin{pmatrix} \mu \\ \prec \end{pmatrix}$, $\succ := \begin{pmatrix} \mu \\ \succ \end{pmatrix}$ and $\cdot := \begin{pmatrix} \mu \\ \cdot \end{pmatrix}$. Since $\begin{pmatrix} \mu \\ \prec \end{pmatrix}^{(12)} = \begin{pmatrix} \mu^{(12)} \\ \succ \end{pmatrix} = -\begin{pmatrix} \mu \\ \succ \end{pmatrix}$ and $\begin{pmatrix} \mu \\ \cdot \end{pmatrix}^{(12)} = \begin{pmatrix} \mu^{(12)} \\ \cdot \end{pmatrix} = -\begin{pmatrix} \mu \\ \cdot \end{pmatrix}$, we have $\prec^{(12)} = -\succ$ and $\cdot^{(12)} = -\cdot$. The space of relations of *Lie* is generated as an $\mathbb{S}_3$-module by

$$v_1 + v_5 + v_9 = \mu \circ_{\mathrm{I}} \mu + \mu \circ_{\mathrm{II}} \mu + \mu \circ_{\mathrm{III}} \mu\ .$$

Then we have

$$\begin{aligned}
\mathrm{TSu}_{\{x\}}(v_1 + v_5 + v_9) &= (x \prec y) \prec z - (x \prec z) \prec y - x \prec (y \prec z - z \prec y + y \cdot z);\\
\mathrm{TSu}_{\{y\}}(v_1 + v_5 + v_9) &= -(y \prec x) \prec z - y \prec (-x \prec z + z \prec x + z \cdot x) + (y \prec z) \prec x;\\
\mathrm{TSu}_{\{z\}}(v_1 + v_5 + v_9) &= -z \prec (-y \prec x + x \prec y + x \cdot y) + (z \prec x) \prec y - (z \prec y) \prec x;\\
\mathrm{TSu}_{\{x,y\}}(v_1 + v_5 + v_9) &= (x \cdot y) \prec z - (x \prec z) \cdot y - x \cdot (y \prec z);\\
\mathrm{TSu}_{\{y,z\}}(v_1 + v_5 + v_9) &= -(y \prec x) \cdot z - y \cdot (z \prec x) - (y \cdot z) \prec x;\\
\mathrm{TSu}_{\{x,z\}}(v_1 + v_5 + v_9) &= -z \cdot (x \prec y) + (z \cdot x) \prec y - (z \prec y) \cdot x;\\
\mathrm{TSu}_{\{x,y,z\}}(v_1 + v_5 + v_9) &= (x \cdot y) \cdot z + (z \cdot x) \cdot y + (y \cdot z) \cdot x.
\end{aligned}$$

Replacing the operations $\prec$ by $\circ$ and $\cdot$ by $[,]$, we get $\mathrm{TSu}(\mathit{Lie}) = \mathit{PostLie}$. □

2.6. **Properties.** We study the relationship among a binary operad and its successors.

2.6.1. *Operads and their successors.*

Lemma 2.28. *Let V be an $\mathbb{S}$-module concentrated in arity 2 with linear basis $\mathcal{V}$. For a labeled planar binary n-tree $\tau \in \mathfrak{T}(\mathcal{V})$, the following equations hold in $\mathcal{T}(V)$:*

$$\sum_{x\in \mathrm{L}in(\tau)} \mathrm{Su}_x(\tau) = \widetilde{\tau}, \tag{16}$$

$$\sum_{J\subseteq \mathrm{L}in(\tau)} \mathrm{TSu}_J(\tau) = \hat{\tau}. \tag{17}$$

Proof. We prove Eq. (16) by induction on $|\mathrm{L}in(\tau)|$. When $|\mathrm{L}in(\tau)| = 1$, we have

$$\sum_{x\in \mathrm{L}in(\tau)} \mathrm{Su}_x(\tau) = \tau = \widetilde{\tau}.$$

Now assume that Eq. (16) holds for all $\tau \in \mathfrak{T}(\mathcal{V})$ with $\mathrm{L}in(\tau) \leq k$ for a $k \geq 1$ and consider a $(k+1)$-tree τ in $\mathfrak{T}(\mathcal{V})$. Since $\tau = \tau_\ell \vee_\omega \tau_r$ for some $\ell, r \leq k$ and $\omega \in V$. Then by the definition of the bisuccessor of a planar binary tree and the induction hypothesis, we have

$$\begin{aligned}
\sum_{x\in \mathrm{L}in(\tau)} \mathrm{Su}_x(\tau) &= \sum_{x\in \mathrm{L}in(\tau_\ell)} \mathrm{Su}_x(\tau_\ell) \vee_{\left(\begin{smallmatrix}\omega\\ \prec\end{smallmatrix}\right)} \widetilde{\tau}_r + \widetilde{\tau}_\ell \vee_{\left(\begin{smallmatrix}\omega\\ \succ\end{smallmatrix}\right)} \sum_{x\in \mathrm{L}in(\tau_r)} \mathrm{Su}_x(\tau_r)\\
&= \widetilde{\tau}_\ell \vee_{\left(\begin{smallmatrix}\omega\\ \prec\end{smallmatrix}\right)} \widetilde{\tau}_r + \widetilde{\tau}_\ell \vee_{\left(\begin{smallmatrix}\omega\\ \succ\end{smallmatrix}\right)} \widetilde{\tau}_r = \widetilde{\tau}_\ell \vee_{\left(\begin{smallmatrix}\omega\\ *\end{smallmatrix}\right)} \widetilde{\tau}_r = \widetilde{\tau}.
\end{aligned}$$

This completes the induction. The proof of Eq. (17) is similar. □

Proposition 2.29. *Let $\mathcal{P} = \mathcal{T}(V)/(R)$ be a binary operad.*

(a) *There is a morphism of operads from $\mathcal{P}$ to* $\mathrm{Su}(\mathcal{P})$ *which extends the linear map from V to $\widetilde{V}$ defined by*

$$\omega \mapsto \begin{pmatrix}\omega\\ \star\end{pmatrix}, \quad \omega \in V. \tag{18}$$

(b) *There is a morphism of operads from $\mathcal{P}$ to* $\mathrm{TSu}(\mathcal{P})$ *which extends the linear map from V to $\widehat{V}$ defined by*

$$\omega \mapsto \begin{pmatrix}\omega\\ \star\end{pmatrix}, \quad \omega \in V. \tag{19}$$

(c) *There is a morphism of operads from $\mathcal{P}$ to* $\mathrm{TSu}(\mathcal{P})$ *which extends the linear map from V to $\widehat{V}$ defined by*

$$\omega \mapsto \begin{pmatrix}\omega\\ \cdot\end{pmatrix}, \quad \omega \in V. \tag{20}$$

Proof. We assume that R is given by (9).
(a) It is easy to see that the linear map defined in Eq. (19) is $\mathbb{S}_2$-equivariant so it induces a morphism of operads from $\mathcal{T}(V)$ to $\mathrm{Su}(\mathcal{P})$. Moreover, by Lemma 2.28, Eq. (16) holds. Hence we have

$$\sum_i c_{s,i}\widetilde{\tau}_{s,i} = \sum_i \sum_{x\in \mathrm{L}in(\tau_{s,i})} c_{s,i}\mathrm{Su}_x(\tau_{s,i}), 1 \leq s \leq k.$$

Since $L_s := \mathrm{L}in(\tau_{s,i})$ does not depend on i, we have

$$\sum_i c_{s,i}\widetilde{\tau}_{s,i} = \sum_{x\in L_s} \mathrm{Su}_x\left(\sum_i c_{s,i}\tau_{s,i}\right) = 0, \quad 1 \le s \le k.$$

and this ends the proof.
(b) Similar to the proof of Item (a).
(c) It is easy to see that the linear map defined in Eq. (20) is $\mathbb{S}_2$-equivariant so it induces a morphism of operads from $\mathcal{T}(V)$ to $\mathrm{TSu}(\mathcal{P})$. Moreover, by the definition of a trisuccessor the following equations hold:

$$\sum_i c_{s,i}\mathrm{TSu}_{\mathrm{L}in(\tau_{s,i})}(\tau_{s,i}) = 0, \quad 1 \le s \le k.$$

Note that the labeled tree $\mathrm{TSu}_{\mathrm{L}in(\tau_{s,i})}(\tau_{s,i})$ is obtained by replacing the label of each vertex of $\tau_{s,i}$, say ω, by $\begin{pmatrix}\omega\\ \cdot\end{pmatrix}$. Hence the conclusion holds. □

If we take $\mathcal{P}$ to be the operad of associative algebras then we obtain the following results of Loday [34] and Loday and Ronco [38]:

Corollary 2.30. (a) *Let $(A, \prec, \succ)$ be a dendriform dialgebra. Then the operation $* := \prec + \succ$ makes A into an associative algebra.*
(b) *Let $(A, \prec, \succ, \cdot)$ be a dendriform trialgebra. Then the operation $\star := \prec + \succ + \cdot$ makes A into an associative algebra.*
(c) *Let $(A, \prec, \succ, \cdot)$ be a dendriform trialgebra. Then $(A, \cdot)$ carries an associative algebra structure.*

2.6.2. *Relationship between bisuccessors and trisuccessors of a binary operad.*

Lemma 2.31. *Let τ be a labeled n-tree in $\mathcal{T}(\mathcal{V})$. If the operations $\left\{\begin{pmatrix}\omega\\ \cdot\end{pmatrix} \mid \omega \in V\right\}$ are trivial then, for any $x \in \mathrm{L}in(\tau)$, we have*

$$\mathrm{TSu}_{\{x\}}(\tau) = \mathrm{Su}_x(\tau) \text{ in } \mathcal{T}(\widehat{V}) . \tag{21}$$

Proof. There is only one path from the root the the leafs in $\{x\}$ of τ so, by Proposition 2.4 and by Proposition 2.9, if the operations $\left\{\begin{pmatrix}\omega\\ \cdot\end{pmatrix} \mid \omega \in V\right\}$ are trivial then the bisuccessor and the trisuccessor with respect to x coincide. □

The following results relate the bisuccessor and the trisuccessor of a binary algebraic operad.

Proposition 2.32. *Let $\mathcal{P} = \mathcal{T}(V)/(R)$ be a binary algebraic operad.*
(a) *If the operations $\left\{\begin{pmatrix}\omega\\ \cdot\end{pmatrix} \mid \omega \in V\right\}$ are trivial, then there is a morphism of operads from $\mathrm{Su}(\mathcal{P})$ to $\mathrm{TSu}(\mathcal{P})$ which extends the inclusion of $\widetilde{V}$ in $\widehat{V}$.*
(b) *There is a morphism of operads from $\mathrm{TSu}(\mathcal{P})$ to $\mathrm{Su}(\mathcal{P})$ which extends the linear map defined by*

$$\begin{pmatrix}\omega\\ \prec\end{pmatrix} \to \begin{pmatrix}\omega\\ \prec\end{pmatrix}, \quad \begin{pmatrix}\omega\\ \succ\end{pmatrix} \to \begin{pmatrix}\omega\\ \succ\end{pmatrix}, \quad \begin{pmatrix}\omega\\ \cdot\end{pmatrix} \to 0, \quad \omega \in V . \tag{22}$$

Proof. We assume that R is given by (9).
(a) The inclusion $\widetilde{V} \hookrightarrow \widehat{V}$ is $\mathbb{S}_2$-equivariant so it induces a morphism of operads from $\mathcal{T}(V)$ to $\mathrm{TSu}(\mathcal{P})$, which kernel is the ideal generated by $\mathrm{Su}(R)$ by Lemma 2.31.
(b) The linear map defined by Eq. (22) is $\mathbb{S}_2$-equivariant hence it induces a morphism of operads $\varphi : \mathrm{TSu}(\mathcal{P}) \to \mathrm{Su}(\mathcal{P})$, and $\varphi\left(\begin{pmatrix} \omega \\ \star \end{pmatrix}\right) = \begin{pmatrix} \omega \\ * \end{pmatrix}$. Then, we have

$$\varphi(\mathrm{TSu}_{\{x\}}(\tau_{s,i})) = \mathrm{Su}_x(\tau_{s,i}) \ , \ \forall x \in \mathrm{L}in(\tau_{s,i})$$

and

$$\varphi(\mathrm{TSu}_{\{J\}}(\tau_{s,i})) = 0 \ , \ \forall J \subseteq \mathrm{L}in(\tau_{s,i}), |J| > 1 \ .$$

□

If we take $\mathcal{P}$ to be the operad of associative algebra, then we obtain the following results of Loday and Ronco [38]:

Corollary 2.33. (a) *Let $(A, \prec, \succ, \cdot)$ be a dendriform trialgebra. If the operation $\cdot$ is trivial, then $(A, \prec, \succ)$ becomes a dendriform dialgebra.*
(b) *Let $(A, \prec, \succ)$ be a dendriform dialgebra. Then $(A, \prec, \succ, 0)$ carries a dendriform trialgebra structure, where 0 denotes the trivial product.*

3. Successors and Manin black product

We now identify the bisuccessor (resp. trisuccessor) of a binary, quadratic operad $\mathcal{P}$ with the Manin black product of *PreLie* (resp. *PostLie*) with $\mathcal{P}$.

Definition 3.1. *(*[19, 47]*)* Let $\mathcal{P} = \mathcal{T}(V)/(R)$ and $Q = \mathcal{T}(W)/(S)$ be two binary quadratic operads with finite-dimensional generating spaces. Define their **Manin black product** by the formula

$$\mathcal{P} \bullet Q := \mathcal{T}(V \otimes W \otimes \mathbf{k}.\mathrm{sgn}_{\mathbb{S}_2})/(\Psi(R \otimes S)) \ ,$$

where Ψ is defined in Section 4.3 of [47].

According to Proposition 25 of [47], notice that the Manin black product is symmetric and associative. Moreover, it is a bifunctor.

3.1. Bisuccessors as the Manin black product by *PreLie*.

Theorem 3.2. *Let $\mathcal{P}$ be a binary quadratic operad. We have the isomorphism of operads*

$$\mathrm{Su}(\mathcal{P}) \cong PreLie \bullet \mathcal{P}.$$

Proof. Denote the generating operation of *PreLie* by μ and continue with the notations v_i, for $1 \leq i \leq 12$ of the table given in Section 2.5 with $\omega = \nu = \mu$. The space of relations of *PreLie* is generated as a vector space by $v_i - v_{i+1} + v_{i+2} - v_{i+3}$, $i = 1, 5, 9$.
We define an isomorphism of $\mathbb{S}_2$-modules by

$$\begin{array}{rrcl} \eta : & PreLie(2) \otimes \mathcal{P}(2) \otimes \mathbf{k}.\mathrm{sgn}_{\mathbb{S}_2} & \to & \mathrm{Su}(\mathcal{P})(2) \\ & \mu \otimes \omega \otimes 1 & \mapsto & \begin{pmatrix} \omega \\ \prec \end{pmatrix} \end{array} \ , \tag{23}$$

which induces an isomorphism of $\mathbb{S}_3$-modules:

$$\bar{\eta} : 3(PreLie(2) \otimes \mathcal{P}(2) \otimes \mathbf{k}.\mathrm{sgn}_{\mathbb{S}_2})^{\otimes 2} \to 3\mathrm{Su}(\mathcal{P})^{\otimes 2}.$$

Then we just need to prove that, for every relation γ of R, we have

$$\bar{\eta}(\Psi((v_1 - v_2 + v_3 - v_4) \otimes \gamma)) = \mathrm{Su}_x(\gamma), \tag{24}$$

$$\bar{\eta}(\Psi((v_5 - v_6 + v_7 - v_8) \otimes \gamma)) = \mathrm{Su}_z(\gamma), \tag{25}$$

$$\bar{\eta}(\Psi((v_9 - v_{10} + v_{11} - v_{12}) \otimes \gamma)) = \mathrm{Su}_y(\gamma). \tag{26}$$

If Eq. (24) holds, by lemma 2.6, we have

$$\bar{\eta}(\Psi((v_5 - v_6 + v_7 - v_8) \otimes \gamma)) = \bar{\eta}(\Psi((v_1 - v_2 + v_3 - v_4) \otimes \gamma^{\sigma_1^{-1}})^{\sigma_1}) = \mathrm{Su}_z(\gamma)$$

and

$$\bar{\eta}(\Psi((v_9 - v_{10} + v_{11} - v_{12}) \otimes \gamma)) = \bar{\eta}(\Psi((v_1 - v_2 + v_3 - v_4) \otimes \gamma^{\sigma_2^{-1}})^{\sigma_2}) = \mathrm{Su}_y(\gamma),$$

for every relation γ of R, where $\sigma_1 = (132), \sigma_2 = (123)$. Thus we only need to prove Eq. (24) for every $\gamma \in \mathcal{T}(V)(3)$.

By Section 2.5, we only need to prove Eq. (24) for every $\gamma \in \mathcal{T}(V)(3)$ in Eq. (14). To do this, we notice that, for all ω and ν in V, we have

$$\mathrm{Su}_x(\omega \circ_{\mathrm{I}} \nu) = \begin{pmatrix} \omega \\ \prec \end{pmatrix} \circ_{\mathrm{I}} \begin{pmatrix} \nu \\ \prec \end{pmatrix}, \tag{27}$$

$$\mathrm{Su}_x(\omega \circ_{\mathrm{II}} \nu) = \begin{pmatrix} \omega \\ \succ \end{pmatrix} \circ_{\mathrm{II}} \begin{pmatrix} \nu \\ \star \end{pmatrix}, \tag{28}$$

$$\mathrm{Su}_x(\omega \circ_{\mathrm{III}} \nu) = \begin{pmatrix} \omega \\ \prec \end{pmatrix} \circ_{\mathrm{III}} \begin{pmatrix} \nu \\ \succ \end{pmatrix}. \tag{29}$$

Then we obtain

$$\begin{aligned} &\bar{\eta}(\Psi((v_1 - v_2 + v_3 - v_4) \otimes (\omega \circ_{\mathrm{I}} \nu))) \\ &= \bar{\eta}(\Psi((\mu \circ_{\mathrm{I}} \mu) \otimes (\omega \circ_{\mathrm{I}} \nu))) = \bar{\eta}((\mu \otimes \omega \otimes 1) \circ_{\mathrm{I}} (\mu \otimes \nu \otimes 1)) = \begin{pmatrix} \omega \\ \prec \end{pmatrix} \circ_{\mathrm{I}} \begin{pmatrix} \nu \\ \prec \end{pmatrix} \\ &= \mathrm{Su}_x(\omega \circ_{\mathrm{I}} \nu). \end{aligned}$$

In the same way, we prove that equation (24) holds for the monomials $\omega \circ_{\mathrm{II}} \nu$ and $\omega \circ_{\mathrm{III}} \nu$. So, we conclude with

$$\begin{aligned} &\bar{\eta}(\Psi((v_1 - v_2 + v_3 - v_4) \otimes \gamma)) \\ &= \bar{\eta}(\Psi((v_1 - v_2 + v_3 - v_4) \otimes \mu \circ_{\mathrm{I}} \mu - \mu' \circ_{\mathrm{II}} \mu + \mu' \circ_{\mathrm{II}} \mu' - \mu \circ_{\mathrm{III}} \mu')) \\ &= \mathrm{Su}_x(\gamma). \end{aligned}$$

□

Repeated application of the theorem gives $\mathrm{Su}^2(\mathcal{P}) \cong PreLie \bullet PreLie \bullet \mathcal{P}$ and, more generally, $\mathrm{Su}^n(\mathcal{P}) \cong PreLie^{\bullet n} \bullet \mathcal{P}$. Thus we have an action of $\mathbb{S}_2$ on $\mathrm{Su}^2(\mathcal{P})$ by exchanging the two *PreLie* factors and, more generally, an action of $\mathbb{S}_n$ on $\mathrm{Su}^n(\mathcal{P})$ by exchanging the n *PreLie* factors. See Section 6 for symmetries on more general operads.

In the nonsymmetric framework, the analogue of Theorem 3.2 is the following result.

Theorem 3.3. *Let $\mathcal{P}$ be a binary quadratic nonsymmetric operad. There is an isomorphism of nonsymmetric operads*

$$\mathrm{Su}(\mathcal{P}) \cong Dend \blacksquare \mathcal{P},$$

where $\blacksquare$ denotes the black square product in [14, 47].

Proof. The proof is similar to the proof of Theorem 3.2. □

Examples of bisuccessors. Note that Theorem 3.2 gives a convenient way to compute the black Manin product of a binary operad with the operad *PreLie*, as we can see from the following corollary. Further examples are given the Appendix A.

Corollary 3.4. (a) ([47]) *We have PreLie • Com = Zinb and PreLie • Ass = Dend.*
(b) ([45]) *We have PreLie • Poisson = prePoisson.*

Proof. Item (a) follows from Proposition 2.22 and Theorem 3.2 while Item (b) follows from Proposition 2.24 and Theorem 3.2. □

Remark 3.5. Notice that the Manin black product does not commute with the functor of regularization, defined in Section 2.4, whereas the bisuccessor does, according to Proposition 2.21.

3.2. **Trisuccessor and Manin black product by** *PostLie***.**

Theorem 3.6. *Let $\mathcal{P}$ be a binary quadratic operad. We have the isomorphism of operads*

$$\mathrm{TSu}(\mathcal{P}) \cong \mathit{PostLie} \bullet \mathcal{P}.$$

Proof. The sketch of this proof is similar to the one of the proof of Theorem 3.2.
Denote the generating operations $[\,,]$ and $\circ$ of *PostLie* by β and ϵ respectively. Then $\beta' = -\beta$. The space of relations of *PostLie* is generated as a vector space by

$$\beta \circ_{\mathrm{I}} \beta + \beta \circ_{\mathrm{II}} \beta + \beta \circ_{\mathrm{III}} \beta\,, \tag{30}$$

$$\epsilon \circ_{\mathrm{I}} \epsilon - \epsilon' \circ_{\mathrm{II}} \epsilon + \epsilon' \circ_{\mathrm{II}} \epsilon' - \epsilon' \circ_{\mathrm{II}} \beta - \epsilon \circ_{\mathrm{III}} \epsilon'\,, \tag{31}$$

$$\epsilon \circ_{\mathrm{I}} \beta - \beta \circ_{\mathrm{III}} \epsilon' + \beta \circ_{\mathrm{II}} \epsilon\,, \tag{32}$$

$$\epsilon \circ_{\mathrm{I}} \epsilon' - \epsilon' \circ_{\mathrm{III}} \epsilon' - \epsilon \circ_{\mathrm{II}} \epsilon + \epsilon' \circ_{\mathrm{III}} \epsilon + \epsilon' \circ_{\mathrm{III}} \beta\,, \tag{33}$$

$$\epsilon \circ_{\mathrm{II}} \epsilon' - \epsilon' \circ_{\mathrm{I}} \epsilon' - \epsilon \circ_{\mathrm{III}} \epsilon + \epsilon' \circ_{\mathrm{I}} \epsilon - \epsilon' \circ_{\mathrm{I}} \beta\,, \tag{34}$$

$$-\,\epsilon \circ_{\mathrm{II}} \beta - \beta \circ_{\mathrm{III}} \epsilon + \beta \circ_{\mathrm{I}} \epsilon'\,, \tag{35}$$

and

$$-\,\epsilon \circ_{\mathrm{III}} \beta - \beta \circ_{\mathrm{I}} \epsilon + \beta \circ_{\mathrm{II}} \epsilon'\,. \tag{36}$$

We define an isomorphism of $\mathbb{S}_2$-modules by

$$\begin{array}{rcl} \eta: \quad \mathit{PostLie}(2) \otimes \mathcal{P}(2) \otimes \mathbf{k}.\mathrm{sgn}_{\mathbb{S}_2} & \to & \mathrm{TSu}(\mathcal{P})(2) \\ \beta \otimes \omega \otimes 1 & \mapsto & \begin{pmatrix} \omega \\ \cdot \end{pmatrix} \\ \epsilon \otimes \omega \otimes 1 & \mapsto & \begin{pmatrix} \omega \\ \prec \end{pmatrix} \end{array}, \tag{37}$$

which induces an isomorphism of $\mathbb{S}_3$-modules:

$$\overline{\eta} : 3(\mathit{PostLie}(2) \otimes \mathcal{P}(2) \otimes \mathbf{k}.\mathrm{sgn}_{\mathbb{S}_2})^{\otimes 2} \to 3\mathrm{TSu}(\mathcal{P})^{\otimes 2}.$$

Then we just need to prove that, for every relation γ of $\mathcal{P}$, we have

$$\overline{\eta}(\Psi((\beta \circ_{\mathrm{I}} \beta + \beta \circ_{\mathrm{II}} \beta + \beta \circ_{\mathrm{III}} \beta) \otimes \gamma)) = \mathrm{TSu}_{\{x,y,z\}}(\gamma), \tag{38}$$

$$\overline{\eta}(\Psi((\epsilon \circ_{\mathrm{I}} \epsilon - \epsilon' \circ_{\mathrm{II}} \epsilon + \epsilon' \circ_{\mathrm{II}} \epsilon' - \epsilon' \circ_{\mathrm{II}} \beta - \epsilon \circ_{\mathrm{III}} \epsilon') \otimes \gamma)) = \mathrm{TSu}_{\{x\}}(\gamma), \tag{39}$$

$$\bar{\eta}(\Psi((\epsilon \circ_{\mathrm{I}} \epsilon' - \epsilon' \circ_{\mathrm{III}} \epsilon' - \epsilon \circ_{\mathrm{II}} \epsilon + \epsilon' \circ_{\mathrm{III}} \epsilon + \epsilon' \circ_{\mathrm{III}} \beta) \otimes \gamma)) = \mathrm{TSu}_{\{y\}}(\gamma), \tag{40}$$

$$\bar{\eta}(\Psi((\epsilon \circ_{\mathrm{II}} \epsilon' - \epsilon' \circ_{\mathrm{I}} \epsilon' - \epsilon \circ_{\mathrm{III}} \epsilon + \epsilon' \circ_{\mathrm{I}} \epsilon - \epsilon' \circ_{\mathrm{I}} \beta) \otimes \gamma)) = \mathrm{TSu}_{\{z\}}(\gamma), \tag{41}$$

$$\bar{\eta}(\Psi((\epsilon \circ_{\mathrm{I}} \beta - \beta \circ_{\mathrm{III}} \epsilon' + \beta \circ_{\mathrm{II}} \epsilon) \otimes \gamma)) = \mathrm{TSu}_{\{x,y\}}(\gamma). \tag{42}$$

$$\bar{\eta}(\Psi((-\epsilon \circ_{\mathrm{II}} \beta - \beta \circ_{\mathrm{III}} \epsilon + \beta \circ_{\mathrm{I}} \epsilon') \otimes \gamma)) = \mathrm{TSu}_{\{y,z\}}(\gamma). \tag{43}$$

$$\bar{\eta}(\Psi((-\epsilon \circ_{\mathrm{III}} \beta - \beta \circ_{\mathrm{I}} \epsilon + \beta \circ_{\mathrm{II}} \epsilon') \otimes \gamma)) = \mathrm{TSu}_{\{x,z\}}(\gamma). \tag{44}$$

By Lemma 2.11, the same argument as in the preLie case implies that we just need to prove Eq. (38), Eq. (39) and Eq. (42).

By Section 2.5, we only need to prove Eq. (24) for every $\gamma \in \mathcal{T}(V)(3)$ in Eq. (14). To do this, we notice that, for all ω and ν in V, we have

$$\mathrm{TSu}_{\{x\}}(\omega \circ_{\mathrm{I}} \nu) = \begin{pmatrix} \omega \\ \prec \end{pmatrix} \circ_{\mathrm{I}} \begin{pmatrix} \nu \\ \prec \end{pmatrix},\ \mathrm{TSu}_{\{x,y\}}(\omega \circ_{\mathrm{I}} \nu) = \begin{pmatrix} \omega \\ \prec \end{pmatrix} \circ_{\mathrm{I}} \begin{pmatrix} \nu \\ \cdot \end{pmatrix},\ \mathrm{TSu}_{\{x,y,z\}}(\omega \circ_{\mathrm{I}} \nu) = \begin{pmatrix} \omega \\ \cdot \end{pmatrix} \circ_{\mathrm{I}} \begin{pmatrix} \nu \\ \cdot \end{pmatrix}, \tag{45}$$

$$\mathrm{TSu}_{\{x\}}(\omega \circ_{\mathrm{II}} \nu) = \begin{pmatrix} \omega \\ \succ \end{pmatrix} \circ_{\mathrm{II}} \begin{pmatrix} \nu \\ \star \end{pmatrix},\ \mathrm{TSu}_{\{x,y\}}(\omega \circ_{\mathrm{II}} \nu) = \begin{pmatrix} \omega \\ \cdot \end{pmatrix} \circ_{\mathrm{II}} \begin{pmatrix} \nu \\ \prec \end{pmatrix},\ \mathrm{TSu}_{\{x,y,z\}}(\omega \circ_{\mathrm{II}} \nu) = \begin{pmatrix} \omega \\ \cdot \end{pmatrix} \circ_{\mathrm{II}} \begin{pmatrix} \nu \\ \cdot \end{pmatrix}, \tag{46}$$

$$\mathrm{TSu}_{\{x\}}(\omega \circ_{\mathrm{III}} \nu) = \begin{pmatrix} \omega \\ \prec \end{pmatrix} \circ_{\mathrm{III}} \begin{pmatrix} \nu \\ \succ \end{pmatrix},\ \mathrm{TSu}_{\{x,y\}}(\omega \circ_{\mathrm{III}} \nu) = \begin{pmatrix} \omega \\ \cdot \end{pmatrix} \circ_{\mathrm{III}} \begin{pmatrix} \nu \\ \succ \end{pmatrix},\ \mathrm{TSu}_{\{x,y,z\}}(\omega \circ_{\mathrm{III}} \nu) = \begin{pmatrix} \omega \\ \cdot \end{pmatrix} \circ_{\mathrm{III}} \begin{pmatrix} \nu \\ \cdot \end{pmatrix}. \tag{47}$$

Then, we have

- $\bar{\eta}(\Psi((\beta \circ_{\mathrm{I}} \beta + \beta \circ_{\mathrm{II}} \beta + \beta \circ_{\mathrm{III}} \beta) \otimes (\omega \circ_{\mathrm{I}} \nu))) = \mathrm{TSu}_{\{x,y,z\}}(\omega \circ_{\mathrm{I}} \nu)$,
- $\bar{\eta}(\Psi((\epsilon \circ_{\mathrm{I}} \epsilon - \epsilon' \circ_{\mathrm{II}} \epsilon + \epsilon' \circ_{\mathrm{II}} \epsilon' - \epsilon' \circ_{\mathrm{II}} \beta - \epsilon \circ_{\mathrm{III}} \epsilon') \otimes (\omega \circ_{\mathrm{I}} \nu))) = \mathrm{TSu}_{\{x\}}(\omega \circ_{\mathrm{I}} \nu)$,
- $\bar{\eta}(\Psi((\epsilon \circ_{\mathrm{I}} \beta - \beta \circ_{\mathrm{III}} \epsilon' + \beta \circ_{\mathrm{II}} \epsilon) \otimes (\omega \circ_{\mathrm{I}} \nu))) = \mathrm{TSu}_{\{x,y\}}(\omega \circ_{\mathrm{I}} \nu)$.

In the same way, we prove that the equations (38), (39) and (42) hold for the monomials $\omega \circ_{\mathrm{II}} \nu$ and $\omega \circ_{\mathrm{III}} \nu$, which ends the proof. □

Remark 3.7. Notice that we could derive Theorem 3.2 from Theorem 3.6 using the following diagram:

$$\begin{tikzcd} \mathrm{TSu}(\mathcal{P}) \arrow[r, "\cong"] \arrow[d] & \mathit{PostLie} \bullet \mathcal{P} \arrow[d] \\ \mathrm{Su}(\mathcal{P}) & \mathit{PreLie} \bullet \mathcal{P} \end{tikzcd}$$

The above proof implies that the top isomorphism preserves the kernels of the two vertical maps. Then, one just needs to check that the vertical maps are surjective.

The analogue of Theorem 3.6 in the nonsymmetric framework is the following proposition.

Theorem 3.8. *Let $\mathcal{P}$ be a binary quadratic nonsymmetric operad. There is an isomorphism of nonsymmetric operads*

$$\mathrm{TSu}(\mathcal{P}) \cong \mathit{TriDend} \blacksquare \mathcal{P}\ .$$

Proof. The proof is similar to the proof of Theorem 3.6. □

Examples of trisuccessors. As in the case of bisuccessors, Theorem 3.6 makes it easy to compute the black Manin product of *PostLie* with any binary operad $\mathcal{P}$. Others examples are given in the Appendix A.

Corollary 3.9. *We have PostLie $\bullet$ Ass = TriDend.*

Proposition 3.10. *The trisuccessor of the operad PreLie is the operad encoding the following algebraic structure:*

$$\begin{aligned}
(x \prec y) \prec z - x \prec (y \star z) &= (x \prec z) \prec y - x \prec (z \star y), \\
(x \succ y) \prec z - x \succ (y \prec z) &= (x \star z) \succ y - x \succ (z \succ y), \\
(x \cdot y) \prec z - x \cdot (y \prec z) &= (x \prec z) \cdot y - x \cdot (z \succ y), \\
(x \succ y) \cdot z - x \succ (y \cdot z) &= (x \succ z) \cdot y - x \succ (z \cdot y), \\
(x \cdot y) \cdot z - x \cdot (y \cdot z) &= (x \cdot z) \cdot y - x \cdot (z \cdot y),
\end{aligned}$$

where $x \star y = x \prec y + x \succ y + x \cdot y$. It is also the bisuccessor of the operad PostLie.

4. Algebraic structure on square matrices

One knows that the vector space of square n-matrices, for $n \geq 1$, with coefficients in a commutative algebra carries an associative algebra structure. Naturally, one can wonder what happens when the space of coefficients is endowed with another algebraic structure. In this section, we answer this question.

Proposition 4.1. *Let $\mathcal{P}$ be an operad and let A be a $\mathcal{P}$-algebra. Then, the vector space $\mathcal{M}_n(A)$, for $n \geq 1$, of $(n \times n)$-matrices with coefficients in A, carries a canonical $\overline{\mathcal{P}}$-algebra structure given by the family of maps $\alpha_m : \overline{\mathcal{P}}_m \to \mathrm{Hom}(\mathcal{M}_n(A)^{\otimes m}, \mathcal{M}_n(A))$ defined by*

$$\alpha_m(\mu)(M^1 \otimes \cdots \otimes M^m)_{i,j} := \sum_{k_1,\ldots,k_{m-1}}^{m} \alpha_A(\mu)(M^1_{i,k_1}, \ldots, M^m_{k_{m-1},j}) \ , \forall 1 \leq i, j \leq n, \forall m \geq 0 \ ,$$

where $\alpha_A : \mathcal{P} \to \mathrm{End}_A$ is the structure of $\mathcal{P}$-algebra on A.

Proof. We denote $\overline{\alpha}_m(\mu)$ by $\overline{\mu}$. Let $\mu \otimes \nu_1 \otimes \cdots \otimes \nu_d$ be in $\overline{\mathcal{P}}(d) \otimes \overline{\mathcal{P}}(c_1) \otimes \cdots \otimes \overline{\mathcal{P}}(c_d)$, with $c_1 + \cdots + c_d = m$, and let $M^1, \ldots, M^m$ be in $\mathcal{M}_n(A)$. We have

$$\begin{aligned}
&\overline{\mu}(\overline{\nu_1}(M^1, \ldots, M^{c_1}), \ldots, \overline{\nu_d}(M^{\cdot}, \ldots, M^m))_{i,j} \\
&= \sum_{k_1,\ldots,k_{d-1}=1}^{n} \sum_{l^1_1,\ldots,l^1_{c_1-1}=1}^{n} \cdots \sum_{l^d_1,\ldots,l^d_{c_d-1}=1}^{n} \alpha_A(\mu)(\alpha_A(\nu_1)(M^1_{i,l^1_1}, \ldots, M^{c_1}_{l^1_{c_1-1},k_1}), \ldots, \alpha_A(\nu_d)(M^{\cdot}_{k_{d-1},l^d_1}, \ldots, M^m_{l^d_{c_d-1}})) \\
&= \sum_{k_1,\ldots,k_{d-1}=1}^{n} \sum_{l^1_1,\ldots,l^1_{c_1-1}=1}^{n} \cdots \sum_{l^d_1,\ldots,l^d_{c_d-1}=1}^{n} \gamma_{\mathcal{P}}(\mu; \nu_1, \ldots, \nu_d)(M^1_{i,l^1_1}, \ldots, M^{c_1}_{l^1_{c_1-1},k_1}, \ldots, M^{\cdot}_{k_{d-1},l^d_1}, \ldots, M^m_{l^d_{c_d-1}}) \\
&= \gamma_{\overline{\mathcal{P}}}(\mu; \nu_1, \ldots, \nu_d)(M^1, \ldots, M^d)_{i,j} \ , \forall 1 \leq i, j \leq n,
\end{aligned}$$

where $\gamma_{\mathcal{P}} = \gamma_{\overline{\mathcal{P}}}$ denotes the composition maps. So, these maps endow $\mathcal{M}_n(A)$ with a $\overline{\mathcal{P}}$-algebra structure. □

Now, we have to describe the operad $\overline{\mathcal{P}}$. For instance, since $\overline{Com} = As$, we recover the classical associative structure of the space of matrices with coefficients in a commutative algebra. Moreover, in [43], in [7], and in [11], the authors prove respectively that the non-symmetric operads $\overline{Lie}$ and $\overline{PreLie}$ are free. Thus, on the space of matrices with coefficients in a Lie algebra (resp. preLie algebra), there is, in general, no relations among the operations defined in Proposition 4.1.

It is a non-trivial problem to describe the non-symmetric operad $\overline{\mathcal{P}}$ associated to a symmetric operad $\mathcal{P}$. However, when $\mathcal{P}$ turns out to be the bisuccessor of some operads, we have the following result.

Theorem 4.2. *Let $\mathcal{P}$ be a non-symmetric binary operad and O be a symmetric binary operad. And let A be an algebra over $\mathrm{Su}^k(O)$, for $k \geq 0$. Any morphism from $Reg(\mathcal{P})$ to O induces a morphism of non-symmetric operads*

$$\mathrm{Su}^k(\mathcal{P}) \to \overline{\mathrm{Su}^k(O)}\ ,$$

which endows $\mathcal{M}_n(A)$, for $n \geq 1$, with a $\mathrm{Su}^k(\mathcal{P})$-algebra structure.

Proof. Let A be an algebra over $\mathrm{Su}^k(O)$. By Proposition 4.1, $\mathcal{M}_n(A)$ carries a structure of an algebra over $\overline{\mathrm{Su}^k(O)}$. By functoriality of the bisuccessor, a morphism from $Reg(\mathcal{P})$ to O gives rise to a morphism from $\mathrm{Su}^k(Reg(\mathcal{P}))$ to $\mathrm{Su}^k(O)$. Then, the following composite induces a $\mathrm{Su}^k(\mathcal{P})$-algebra structure on $\mathcal{M}_n(A)$:

$$\mathrm{Su}^k(\mathcal{P}) \to \overline{Reg(\mathrm{Su}^k(\mathcal{P}))} \cong \overline{\mathrm{Su}^k(Reg(\mathcal{P}))} \to \overline{\mathrm{Su}^k(O)}\ ,$$

where the left hand-side map is given by the unit of the adjunction between the forgetful and the regularization functors and where the isomorphism is a consequence of Proposition 2.21. ◻

Corollary 4.3. *Let A be an algebra over $\mathrm{Su}^k(Com)$, $k \geq 0$. Then $\mathcal{M}_n(A)$, $n \geq 1$, carries a functorial structure of algebra over $Dend^{\blacksquare k}$.*

More precisely, this structure is given by the following generating operations

$$*_{(i_1,\dots,i_k)} : \mathcal{M}_n(A) \otimes \mathcal{M}_n(A) \to \mathcal{M}_n(A)\ ,$$

with $(i_1, \dots, i_k) \in \{0,1\}^k$, defined by

$$(M *_{(i_1,\dots,i_k)} N)_{i,j} := \sum_{l=1}^{n} M_{i,l} \star_{(i_1,\dots,i_k)} N_{l,j}\ ,$$

where $\{\star_{(i_1,\dots,i_k)}\}_{(i_1,\dots,i_k)\in\{0,1\}^k}$ denote the set of generating operations of $\mathrm{Su}^k(Com)$.

In particular, these operations satisfy

$${}^t(M *_{(i_1,\dots,i_k)} N) = {}^tN *_{(1-i_1,\dots,1-i_k)} {}^tM\ , \quad \forall (i_1, \dots, i_k) \in \{0,1\}^k, \forall M, N \in \mathcal{M}_n(A)\ .$$

Proof. Applying Theorem 4.2, since $\overline{Com} = As$, $\mathcal{M}_n(A)$ carries a structure of algebra over $\mathrm{Su}^k(As)$, which is isomorphic to $Dend^{\blacksquare k} \blacksquare\, As = Dend^{\blacksquare k}$, by Theorem 3.3.

We denote by $\star$ and $*$ the generating operation of the operad Com and As respectively. Then, the space of generating operations of $\mathrm{Su}^k(Com)$ and of $\mathrm{Su}^k(As)$ are respectively spanned by

$$\star_{(i_1,\dots,i_k)} := \star \otimes \mu_1 \otimes \cdots \otimes \mu_k$$

and by

$$*_{(i_1,\dots,i_k)} := * \otimes \mu_1 \otimes \cdots \otimes \mu_k\ ,$$

with $i_l = 0$ if $\mu_j =\prec$ and $i_l = 1$ if $\mu_j =\succ$. When we explicit the composite of the maps given in Proposition 4.1 and in the proof of Theorem 4.2 on the space of generating operations, we have

$$\begin{array}{rcl} \mathrm{Su}^k(As)_2 & \to & \mathrm{Hom}(\mathcal{M}_n(A)^{\otimes 2}, \mathcal{M}_n(A)) \\ *_{(i_1,\dots,i_k)} & \mapsto & *_{(i_1,\dots,i_k)} : M \otimes N \mapsto \left(\sum_{l=1}^{n} M_{i,l} \star_{(i_1,\dots,i_k)} N_{l,j}\right)_{1\le i,j\le n} \end{array} .$$

The last result is a consequence of the $\mathbb{S}_2$-action on the space of generating operations of the operad $\mathrm{Su}^k(Com)$, that is

$$\star_{(i_1,\dots,i_k)}^{(12)} = \star_{(1-i_1,\dots,1-i_k)} .$$

□

Notice that for $k = 2$, according to Proposition 2.22, the space of matrices with coefficients in an Zinbiel algebra $(A, \centerdot)$ carries a natural structure of dendriform algebra given by the following operations

$$M \lhd N = \left(\sum_{l=1}^{n} M_{i,l} \centerdot N_{l,j}\right)_{1\le i,j\le n} ,$$

and

$$M \rhd N = \left(\sum_{l=1}^{n} N_{l,j} \centerdot M_{i,l}\right)_{1\le i,j\le n} .$$

And, these operations satisfy

$${}^t(M \lhd N) = {}^tN \rhd {}^tM .$$

It would be interesting to add the transpose to the generating operations of $Dend^{\blacksquare k}$ and to study this operad.

5. Successors and Rota-Baxter operators on operads

In this section, we establish the relationship between the bisuccessor, respectively the trisuccessor, of an operad and the operads of Rota-Baxter algebras of weight zero, respectively of non-zero weight. We work with operads, but all the results hold for nonsymmetric operads as well.

5.1. Bisuccessors and Rota-Baxter operators of weight zero.

Definition 5.1. Let $V = V(2)$ be an $\mathbb{S}$-module concentrated in arity 2.

(a) Let V_P be the $\mathbb{S}$-module concentrated in arity 1 and arity 2, defined by $V_P(1) = \mathrm{span}_{\mathbf{k}}(P)$ and $V_P(2) = V$, where P is a symbol. Then $\mathcal{T}(V_P)$ is the free operad generated by binary operations V and a unary operation $P \neq \mathrm{id}$.

(b) Define $\widetilde{V}$ by Eq. (2), regarded as an $\mathbb{S}$-module concentrated in arity 2. Define a morphism of $\mathbb{S}$-modules from $\widetilde{V}$ to $\mathcal{T}(V_P)$ by the following correspondence:

$$\xi : \quad \begin{pmatrix} \omega \\ \prec \end{pmatrix} \mapsto \omega \circ (\mathrm{id} \otimes P), \quad \begin{pmatrix} \omega \\ \succ \end{pmatrix} \mapsto \omega \circ (P \otimes \mathrm{id}),$$

where $\circ$ is the operadic composition. By universality of the free operad, ξ induces a homomorphism of operads that we still denote by ξ:

$$\xi : \mathcal{T}(\widetilde{V}) \to \mathcal{T}(V_P).$$

(c) Let $\mathcal{P} = \mathcal{T}(V)/(R_{\mathcal{P}})$ be a binary operad defined by generating operations V and relations $R_{\mathcal{P}}$. Then we define the **operad of Rota-Baxter $\mathcal{P}$-algebra of weight zero** by

$$\mathrm{RB}_0(\mathcal{P}) := \mathcal{T}(V_P)/\left(R_{\mathcal{P}}, RB_{\mathcal{P}}\right),$$

where

$$RB_{\mathcal{P}} := \{\omega \circ (P \otimes P) - P \circ \omega \circ (P \otimes \mathrm{id}) - P \circ \omega \circ (\mathrm{id} \otimes P) \mid \omega \in V\},$$

called the set of Rota-Baxter relations. We denote by $p_1 : \mathcal{T}(V_P) \to \mathrm{RB}_0(\mathcal{P})$ the operadic projection.

Interpreting Theorem 4.2 of [45] at the level of operads, for any binary quadratic operad

$$\mathcal{P} = \mathcal{T}(V)/(R),$$

there is a morphism of operads

$$PreLie \bullet \mathcal{P} \to \mathrm{RB}_0(\mathcal{P}),$$

defined by the following map

$$\begin{array}{rcl} PreLie(2) \otimes \mathcal{P}(2) & \to & \mathrm{RB}_0(\mathcal{P}) \\ \mu \otimes \omega & \mapsto & \omega \circ (\mathrm{id} \otimes P) \\ \mu' \otimes \omega & \mapsto & \omega \circ (P \otimes \mathrm{id}) \end{array},$$

where μ denotes the generating operation of the operad *PreLie*. By Theorem 3.2, this induces the following morphism of operads

$$\begin{array}{rcl} \mathrm{Su}(\mathcal{P}) & \to & \mathrm{RB}_0(\mathcal{P}) \\ \begin{pmatrix} \omega \\ \prec \end{pmatrix} & \mapsto & \omega \circ (\mathrm{id} \otimes P) \\ \begin{pmatrix} \omega \\ \succ \end{pmatrix} & \mapsto & \omega \circ (P \otimes \mathrm{id}) \end{array}.$$

If we take $\mathcal{P}$ to be the operad of associative algebras or the operad of Poisson algebras then we obtain the following results of Aguiar [2]:

Corollary 5.2. (a) *Let $(A, \circ)$ be an associative algebra and let $P : A \to A$ be a Rota-Baxter operator of weight zero. Define two bilinear products on A by*

$$x \prec y := x \circ P(y), \quad x \succ y := P(x) \circ y, \quad x, y \in A.$$

Then $(A, \prec, \succ)$ becomes a dendriform dialgebra.

(b) *Let $(A, \circ, \{\ ,\ \})$ be a Poisson algebra and let $P : A \to A$ be a Rota-Baxter operator of weight zero. Define two bilinear products on A by*

$$x \cdot y := P(x) \circ y, \quad x * y := x \circ P(y), \quad x, y \in A.$$

Then $(A, \cdot, *)$ becomes a pre-Poisson algebra.

5.2. **Trisuccessors and Rota-Baxter operators of non-zero weight.** In this section, we establish a relationship between the trisuccessor of an operad and Rota-Baxter operators of a non-zero weight on this operad. For simplicity, we assume that the weight of the Rota-Baxter operator is one.

Definition 5.3. Let $V = V(2)$ be an $\mathbb{S}$-module concentrated in arity 2.

(a) Define $\widehat{V}$ by Eq. (4), seen as an $\mathbb{S}$-module concentrated in arity 2. Define a morphism of $\mathbb{S}$-modules from $\widehat{V}$ to $\mathcal{T}(V_P)$ by the following correspondence:

$$\eta : \quad \begin{pmatrix} \omega \\ \prec \end{pmatrix} \mapsto \omega \circ (\mathrm{id} \otimes P), \quad \begin{pmatrix} \omega \\ \succ \end{pmatrix} \mapsto \omega \circ (P \otimes \mathrm{id}), \quad \begin{pmatrix} \omega \\ \cdot \end{pmatrix} \mapsto \omega,$$

where $\circ$ is the operadic composition. By universality of the free operad, η induces a homomorphism of operads:

$$\eta : \mathcal{T}(\widehat{V}) \to \mathcal{T}(V_P).$$

(b) Let $\mathcal{P} = \mathcal{T}(V)/(R_{\mathcal{P}})$ be a binary operad defined by generating operations V and relations $R_{\mathcal{P}}$. Then we define the **operad of Rota-Baxter $\mathcal{P}$-algebra of weight one** by

$$\mathrm{RB}_1(\mathcal{P}) := \mathcal{T}(V_P)/\left(R_{\mathcal{P}}, RB_{\mathcal{P}}\right),$$

where

$$RB_{\mathcal{P}} := \{\, \omega \circ (P \otimes P) - P \circ \omega \circ (P \otimes \mathrm{id}) - P \circ \omega \circ (\mathrm{id} \otimes P) - P \circ \omega \mid \omega \in V\},$$

called the set of Rota-Baxter relations of weight one. We denote by $p_1 : \mathcal{T}(V_P) \to \mathrm{RB}_1(\mathcal{P})$ the operadic projection.

Theorem 5.4. *Let $\mathcal{P}$ be a binary quadratic operad.*

(a) *There is a morphism of operads*

$$PostLie \bullet \mathcal{P} \cong \mathrm{TSu}(\mathcal{P}) \to \mathrm{RB}_1(\mathcal{P}),$$

which extends the map η given in Definition 5.3.

(b) *Let A be a $\mathcal{P}$-algebra. Let $P : A \to A$ be a Rota-Baxter operator of weight one. Then the following operations make A into a $(PostLie \bullet \mathcal{P})$-algebra:*

$$x \prec_j y := x \circ_j P(y), \quad x \succ_j y := P(x) \circ_j y, \quad x \cdot_j y := x \circ_j y, \quad \forall \circ_j \in \mathcal{P}(2), \quad x, y \in A.$$

Proof. (a) First, we prove by induction on $|\mathrm{L}in(\tau)| \geq 1$ the following technical results hold for any $\tau \in \mathfrak{T}(V)$ with $\mathrm{L}in(\tau) = n$:

(i) We have

$$P \circ \eta(\overline{\tau}) \equiv \tau \circ P^{\otimes n} \mod (R_{\mathcal{P}}, RB_{\mathcal{P}}). \tag{48}$$

(ii) For $\emptyset \neq J \subseteq \mathrm{L}in(\tau)$ with $|\mathrm{L}in(\tau)| = n$, let $P^{\otimes n, J}$ denote the n-th tensor power of P but with the component from J replaced by the identity map. So, for example, denoting the two inputs of $P^{\otimes 2}$ by x_1 and x_2, then $P^{\otimes 2, \{x_1\}} = P \otimes \mathrm{id}$ and $P^{\otimes 2, \{x_1, x_2\}} = \mathrm{id} \otimes \mathrm{id}$. Then we have

$$\eta(\mathrm{TSu}_J(\tau)) \equiv \tau \circ (P^{\otimes n, J}) \mod (R_{\mathcal{P}}, RB_{\mathcal{P}}). \tag{49}$$

Let $R_{\mathrm{TSu}(\mathcal{P})}$ be the relation space of $\mathrm{TSu}(\mathcal{P})$. By definition, the relations of $\mathrm{TSu}(\mathcal{P})$ are generated by $\mathrm{TSu}_J(r)$ for locally homogeneous $r = \sum_i c_i\tau_i \in R_{\mathcal{P}}$, where $\emptyset \neq J \subseteq \mathrm{L}in(\tau_i)$, the latter independent of the choice of i. By the aforementioned results (48) and (49), we have

$$\eta\left(\sum_i c_i \mathrm{TSu}_J(\tau_i)\right) = \sum_i c_i \eta(\mathrm{TSu}_J(\tau_i)) = \sum_i c_i\tau_i \circ P^{\otimes n,J} = \left(\sum_i c_i\tau_i\right)\circ P^{\otimes n,J} = 0 \mod (R_{\mathcal{P}}, RB_{\mathcal{P}}).$$

Hence $\eta(R_{\mathrm{TSu}(\mathcal{P})}) \subseteq (R_{\mathcal{P}}, RB_{\mathcal{P}})$ and η induces a morphism of operads

$$\bar{\eta} : \mathrm{TSu}(\mathcal{P}) \to \mathrm{RB}_1(\mathcal{P}) .$$

(b) It is the interpretation at the level of algebras of the morphism

$$PostLie \bullet \mathcal{P} \to \mathrm{RB}_1(\mathcal{P}) .$$

□

If we take $\mathcal{P}$ to be the operad *Ass*, resp. the operad *Dend*, then we derive the results [13, 14] that a Rota-Baxter operator on an associative algebra, resp. on a dendriform algebra, gives a dendriform trialgebra by Corollary 3.9, resp. an algebra over the operad *PostLie* • *Dend*.

6. A symmetric property of successors

There are symmetries in the iterations of successors. The first instances of such phenomena were discovered in quadri-algebras [3] and then in ennea algebras [31]. These instances were shown to also follow from symmetries of Manin black square powers of binary quadratic non-symmetric operads [14]. Similar symmetries were recently found in operads, such as those from L-dendriform algebras [6] and L-quadri-algebras [32]. This time the symmetries can also be derived from symmetries of Manin products of binary quadratic operads, as we can see in Section 3. We now show that a symmetry hold for the iterated successors of any binary operad without the quadratic condition.

6.1. A symmetric property of successors.

Definition 6.1. Let V be a vector space and $n \geq 1$.

(a) We define the vector space $V^{\sim n}$ by

$$V^{\sim n} := V \otimes (\mathbf{k} \prec \oplus\, \mathbf{k} \succ)^{\otimes n} .$$

The vector space $V^{\sim n}$ is generated by elements of the form $\omega \otimes \mu_1 \otimes \ldots \otimes \mu_n$, with $\omega \in V$ and $\mu_i \in \{\prec, \succ\}$. It is obtained by iteration of $\tilde{}$ defined by (2).

(b) Let σ be in $\mathbb{S}_n$. We define the map $\phi_\sigma : \mathcal{T}(V^{\sim n}) \to \mathcal{T}(V^{\sim n})$ to be the unique morphism of operads which extends the following morphism of $\mathbb{S}$-modules

$$\begin{array}{ccc} V^{\sim n} & \to & \mathcal{T}(V^{\sim n}) \\ \omega \otimes \mu_1 \otimes \ldots \otimes \mu_n & \mapsto & \omega \otimes \mu_{\sigma(1)} \otimes \ldots \otimes \mu_{\sigma(n)} \end{array} . \tag{50}$$

Theorem 6.2. *Let $\mathcal{P} = \mathcal{T}(V)/(R)$ be a binary operad. For any σ in $\mathbb{S}_n$, there exists an automorphism Φ_σ of the operad* $\mathrm{Su}^n(\mathcal{P})$*. This induces a morphism of groups*

$$\mathbb{S}_n \to \mathrm{Aut}(\mathrm{Su}^n(\mathcal{P})) .$$

Proof. Using the interpretation of the bisuccessor given in Proposition 2.4, when we compute the bisuccessor of a labeled tree τ in $\mathcal{T}(V)$ we do not change the underlying tree but only the labels of the vertices. So, by symmetry and by associativity of the tensor product, we have

$$\mathrm{Su}_{i_{\sigma(1)}}\dots\mathrm{Su}_{i_{\sigma(l)}}(\tau)=\phi_\sigma(\mathrm{Su}_{i_1}\dots\mathrm{Su}_{i_l}(\tau))\ ,$$

where $\sigma\in\mathbb{S}_n$ and where $i_1,\dots,i_l\in \mathrm{L}in(\tau)$ are not necessarily distinct.
Assume that R is given by (9). Then, we obtain that

$$\phi_\sigma(\mathrm{Su}^n(R))=\left\{\sum_j c_{s,i}\phi_\sigma(\mathrm{Su}_{i_1}\dots\mathrm{Su}_{i_k}(\tau_{s,j})),\ i_1,\dots,i_k\in \mathrm{L}in(\tau_{s,j})\ \middle|\ 1\le s\le k,\right\}=\mathrm{Su}^n(R).$$

Thus the composite $V^{\sim n}\xrightarrow{\phi_\sigma}\mathcal{T}(V^{\sim n})\twoheadrightarrow \mathrm{Su}^n(\mathcal{P})$ induces a morphism $\Phi_\sigma:\mathrm{Su}^n(\mathcal{P})\to\mathrm{Su}^n(\mathcal{P})$. And, by definition, we have

$$\phi_\sigma\phi_{\sigma'}=\phi_{\sigma\sigma'},\forall\sigma,\sigma'\in\mathbb{S}_n\ .$$

and we deduce from this the rest of the theorem. □

When $\mathcal{P}$ is taken to be *Ass*, the involution $\Phi_{(12)}:\mathrm{Su}(\mathcal{P})\to\mathrm{Su}(\mathcal{P})$ of Theorem 6.2 gives the following result of Aguiar and Loday [3]:

Corollary 6.3. *Let* $(A,\nwarrow,\swarrow,\nearrow,\searrow)$*, be a quadri-algebra. Then its transpose* $(A,\nwarrow^t,\swarrow^t,\nearrow^t,\searrow^t)$ *is also a quadri-algebra, where*

$$\nwarrow^t:=\nwarrow,\quad \swarrow^t:=\nearrow,\quad \nearrow^t:=\swarrow,\quad \searrow^t:=\searrow\ .$$

Proof. This is clear since, in terms of bisuccessors, we have $Quadri=\mathrm{Su}^2(Ass)$ by Example 2.15 and

$$\nwarrow=\begin{pmatrix}\omega\\ \prec\\ \prec\end{pmatrix},\quad \swarrow=\begin{pmatrix}\omega\\ \prec\\ \succ\end{pmatrix},\quad \nearrow=\begin{pmatrix}\omega\\ \succ\\ \prec\end{pmatrix},\quad \searrow=\begin{pmatrix}\omega\\ \succ\\ \succ\end{pmatrix},$$

where ω denotes the binary operation of associative algebras. □

Next, we provide an example of symmetric property when the double successor functor is applyied to a non-quadratic operad, namely, the operad of Jordan algebra.

Definition 6.4. We now assume that the characteristic of $\mathbf{k}$ is neither two nor three.

(a) A **Jordan algebra** [27] is defined by one bilinear operation $\circ$ and relation:

$$((x\circ y)\circ u)\circ z+((y\circ z)\circ u)\circ x+((z\circ x)\circ u)\circ y=(x\circ y)\circ(u\circ z)+(y\circ z)\circ(u\circ x)+(z\circ x)\circ(u\circ y).$$

(b) A **pre-Jordan algebra** [26] is defined by one bilinear operation $\cdot$ and relations

$$\begin{aligned}(x\odot y)\cdot(z\cdot u)+(y\odot z)\cdot(x\cdot u)+(z\odot x)\cdot(y\cdot u)&=z\cdot((x\odot y)\cdot u)+x\cdot((y\odot z)\cdot u)+y\cdot((z\odot x)\cdot u),\\ x\cdot(y\cdot(z\cdot u))+z\cdot(y\cdot(x\cdot u))+((x\odot z)\odot y)\cdot u&=z\cdot((x\odot y)\cdot u)+x\cdot((y\odot z)\cdot u)+y\cdot((z\odot x)\cdot u),\end{aligned}$$

where $x\odot y:=x\cdot y+y\cdot x$.

It is easy to obtain the following conclusion:

Proposition 6.5. *The bisuccessor of the operad Jordan is the operad PreJordan.*

Moreover, we have the following results.

Proposition 6.6. *The operad* $\mathrm{Su}^2(Jordan) = \mathrm{Su}(PreJordan)$ *is generated by two bilinear operations* $\prec$ *and* $\succ$ *that satisfy following relations:*

$$\begin{aligned}
& (x \prec y + y \succ x) \prec (z \cdot u) + (y \circ z) \succ (x \prec u) + (z \succ x + x \prec z) \prec (y \cdot u) \\
= \ & z \succ ((x \prec y + y \succ x) \prec u) + x \prec ((y \circ z) \cdot u) + y \succ ((z \succ x + x \prec z) \prec u); \\
& (x \circ y) \succ (z \succ u) + (y \circ z) \succ (x \succ u) + (z \circ x) \succ (y \succ u) \\
= \ & z \succ ((x \circ y) \succ u) + x \succ ((y \circ z) \succ u) + y \succ ((z \circ x) \succ u); \\
& x \prec (y \cdot (z \cdot u)) + z \succ (y \succ (x \prec u)) + ((x \prec z + z \succ x) \prec y + y \succ (x \prec z + z \succ x)) \prec u \\
= \ & z \succ ((x \prec y + y \succ x) \prec u) + x \prec ((y \circ z) \cdot u) + y \succ ((z \succ x + x \prec z) \prec u); \\
& x \succ (y \prec (z \cdot u)) + z \succ (y \prec (x \cdot u)) + ((x \circ z) \succ y + y \prec (x \circ z)) \prec u \\
= \ & z \succ ((x \succ y + y \prec x) \prec u) + x \succ ((y \prec z + z \succ y) \prec u) + y \prec ((z \circ x) \cdot u); \\
& x \succ (y \succ (z \prec u)) + z \prec (y \cdot (x \cdot u)) + ((x \succ z + z \prec x) \prec y + y \succ (x \succ z + z \prec x)) \prec u \\
= \ & z \prec ((x \circ y) \cdot u) + x \succ ((y \succ z + z \prec y) \prec u) + y \succ ((z \prec x + x \succ z) \prec u); \\
& x \succ (y \succ (z \succ u)) + z \succ (y \succ (x \succ u)) + ((x \circ z) \circ y) \succ u \\
= \ & z \succ ((x \circ y) \succ u) + x \succ ((y \circ z) \succ u) + y \succ ((z \circ x) \succ u),
\end{aligned}$$

where $x \cdot y := x \prec y + x \succ y, x \circ y := x \cdot y + y \cdot x$. *The operation* $\cdot$ *satisfies the relations defining a preJordan algebra and the operation* $\circ$ *satisfies the relations defining a Jordan algebra.*

Proposition 6.7. *The map* ϕ *that sends* $\prec$ *to* $\prec^{(12)}$, $\prec^{(12)}$ *to* $\prec$ *and leaves the other operations of* $\mathrm{Su}^2(Jordan)$ *invariant induces an involution of the operad* $\mathrm{Su}^2(Jordan)$.

Proof. It is a corollary of Theorem 6.2 with the following identifications:

$$\succ^{(12)} = \begin{pmatrix} \omega \\ \prec \\ \prec \end{pmatrix}, \quad \prec^{(12)} = \begin{pmatrix} \omega \\ \prec \\ \succ \end{pmatrix}, \quad \prec = \begin{pmatrix} \omega \\ \succ \\ \prec \end{pmatrix}, \quad \succ = \begin{pmatrix} \omega \\ \succ \\ \succ \end{pmatrix},$$

where ω denotes the generating operation of *Jordan*. □

6.2. **A symmetric property of trisuccessors.**

Definition 6.8. Let V be a vector space and $n \geq 1$.

(a) We define the vector space $V^{\wedge n}$ by

$$V^{\wedge n} := V \otimes (\mathbf{k} \prec \oplus \mathbf{k} \succ \oplus \mathbf{k} \cdot)^{\otimes n} .$$

The vector space $V^{\wedge n}$ is generated by elements of the form $\omega \otimes \mu_1 \otimes \ldots \otimes \mu_n$, with $\omega \in V$ and $\mu_i \in \{\prec, \succ, \cdot\}$. It is obtained by iteration of $\frown$ defined in (4).

(b) Let σ be in $\mathbb{S}_n$. We define the map $\psi_\sigma : \mathcal{T}(V^{\wedge n}) \to \mathcal{T}(V^{\wedge n})$ to be the unique morphism of operads which extends which extends the following morphism of $\mathbb{S}$-modules

$$\begin{aligned} V^{\wedge n} & \to \mathcal{T}(V^{\wedge n}) \\ \omega \otimes \mu_1 \otimes \ldots \otimes \mu_n & \mapsto \omega \otimes \mu_{\sigma(1)} \otimes \ldots \otimes \mu_{\sigma(n)} \quad . \end{aligned} \tag{51}$$

Theorem 6.9. *Let* $\mathcal{P} = \mathcal{T}(V)/(R)$ *be a binary operad. For any* σ *in* $\mathbb{S}_n$, *there exists an automorphism* Ψ_σ *of the operad* $\mathrm{TSu}^n(\mathcal{P})$. *This induces a morphism of groups*

$$\mathbb{S}_n \to \mathrm{Aut}(\mathrm{TSu}^n(\mathcal{P})) .$$

Proof. This proof follows the same arguments as the proof of Theorem 6.2. $\square$

When $\mathcal{P}$ is taken to be *Ass*, the involution $\Psi_{(12)} : \mathrm{TSu}(\mathcal{P}) \to \mathrm{TSu}(\mathcal{P})$ of Theorem 6.9 gives the following result of Leroux [31]:

Corollary 6.10. *Let* $(A, \nwarrow, \swarrow, \prec, \nearrow, \searrow, \succ, \uparrow, \downarrow, \circ)$ *be an ennea-algebra. Then its transpose* $(A, \nwarrow^t, \swarrow^t, \prec^t, \nearrow^t, \searrow^t, \succ^t, \uparrow^t, \downarrow^t, \circ^t)$ *is also an ennea-algebra, where*

$$\nwarrow^t:=\nwarrow,\ \swarrow^t:=\nearrow,\ \prec^t:=\uparrow,\ \nearrow^t:=\swarrow,\ \searrow^t:=\searrow,\ \succ^t:=\downarrow,\ \uparrow^t:=\prec,\ \downarrow^t:=\succ,\ \circ^t := \circ.$$

Proof. In fact, in this case $Ennea = \mathrm{TSu}^2(Ass)$ and in our terminology, the products of A are reformulated as follows:

$$\nwarrow = \begin{pmatrix}\omega\\ \prec\\ \prec\end{pmatrix},\quad \swarrow = \begin{pmatrix}\omega\\ \prec\\ \succ\end{pmatrix},\quad \prec = \begin{pmatrix}\omega\\ \prec\\ \cdot\end{pmatrix},\quad \nearrow = \begin{pmatrix}\omega\\ \succ\\ \prec\end{pmatrix},\quad \searrow = \begin{pmatrix}\omega\\ \succ\\ \succ\end{pmatrix},\quad \succ = \begin{pmatrix}\omega\\ \succ\\ \cdot\end{pmatrix},\quad \uparrow = \begin{pmatrix}\omega\\ \cdot\\ \prec\end{pmatrix},\quad \downarrow = \begin{pmatrix}\omega\\ \cdot\\ \succ\end{pmatrix}, \circ = \begin{pmatrix}\omega\\ \cdot\\ \cdot\end{pmatrix},$$

where ω denotes the generating operation of *Ass*. $\square$

Appendix A. Examples of successors

A.1. **L-quadri and L-dendriform operads.** An **L-dendriform algebra** [6] is defined to be a **k**-vector space A with two bilinear operations $\prec, \succ : A \otimes A \to A$ that satisfy relations

$$(x \prec y) \prec z + y \succ (x \prec z) = x \prec (y \cdot z) + (y \succ x) \prec z,$$

$$(x \cdot y) \succ z + y \succ (x \succ z) = x \succ (y \succ z) + (y \cdot x) \succ z,$$

where $\cdot = \prec + \succ$.

Proposition A.1. *The operad LDend is the bisuccessor of PreLie, equivalently*

$$PreLie \bullet PreLie = LDend\ .$$

Proof. Let μ be the generating operation of *PreLie*. Set $\prec := \begin{pmatrix}\mu\\ \prec\end{pmatrix}$ and $\succ := \begin{pmatrix}\mu\\ \succ\end{pmatrix}$. The space of relations of *PreLie* is generated as an $\mathbb{S}_3$-module by

$$v_1 - v_2 - v_{12} + v_{11}.$$

Note here we use the left Pre-Lie algebra. The space of relations of *LDend* is generated, as an $\mathbb{S}_3$-module, by

$$\begin{aligned} r_1 : &= (x \prec y) \prec z + y \succ (x \prec z) - x \prec (y \cdot z) - (y \succ x) \prec z, \\ r_2 : &= (x \cdot y) \succ z + y \succ (x \succ z) - x \succ (y \succ z) - (y \cdot x) \succ z. \end{aligned}$$

Then we have

$$\begin{aligned} \mathrm{Su}_x(v_1 - v_2 - v_{12} + v_{11}) &= (x \prec y) \prec z - x \prec (y \prec z + y \succ z) - (y \succ x) \prec z + y \succ (x \prec z); \\ \mathrm{Su}_y(v_1 - v_2 - v_{12} + v_{11}) &= (x \succ y) \prec z - x \succ (y \prec z) - (y \prec x) \prec z + y \prec (x \succ z + x \prec z); \\ \mathrm{Su}_z(v_1 - v_2 - v_{12} + v_{11}) &= (x \succ y + x \prec y) \succ z - x \succ (y \succ z) - (y \prec x + y \succ x) \succ z + y \succ (x \succ z). \end{aligned}$$

Rewriting the relations with the operations $\prec^{(12)}, \succ^{(12)}$ and then, replacing these operations by $<$ and $>$ respectively, we get $\mathrm{Su}(PreLie) = LDend$. $\square$

An **L-quadri-algebra** [32] is a vector space endowed with four binary operations $\swarrow$, $\nwarrow$, $\nearrow$ and $\searrow$ that satisfy the following relations

$$
\begin{aligned}
& x \searrow (y \nwarrow z) - (x \searrow y) \nwarrow z - y \nwarrow (x \nearrow z + x \nwarrow z + x \swarrow z + x \searrow z) + (y \nwarrow x) \nwarrow z = 0\ ;\\
& x \searrow (y \nearrow z) - (x \searrow y + x \swarrow y) \nearrow z - y \nearrow (x \searrow z + x \nearrow z) + (y \nearrow x + y \nwarrow x) \nearrow z = 0\ ;\\
& x \searrow (y \swarrow z) - (x \searrow y + x \nearrow y) \swarrow z - y \swarrow (x \searrow z + x \swarrow z) + (y \swarrow x + y \nwarrow x) \swarrow z = 0\ ;\\
& x \nearrow (y \swarrow z + y \nwarrow z) - (x \nearrow y) \nwarrow z - y \swarrow (x \nearrow z + x \nwarrow z) + (y \swarrow x) \nwarrow z = 0\ ;\\
& x \searrow (y \searrow z) - (x \nearrow y + x \nwarrow y + x \swarrow y + x \searrow y) \searrow z\\
& -y \searrow (x \searrow z) + (y \nearrow x + y \nwarrow x + y \swarrow x + y \searrow x) \searrow z = 0\ .
\end{aligned}
$$

Let *LQuad* denote the operad of L-quadri-algebras.

Proposition A.2. *The bisuccessor of LDend is LQuad, equivalently*

$$PreLie^{\bullet 3} \cong LQuad\ .$$

Proof. By Theorem 3.2, the operad $PreLie^{\bullet n}$, for $n \geq 2$, is given by the $(n-1)$-th bisuccessor of *PreLie*. By Proposition A.1, we obtain $PreLie^{\bullet 2} \cong LDend$. So we just need to prove that $\mathrm{Su}(LDend) \cong LQuad$.

To prove this previous statement, we continue to use the notations in Section 2.5. Let us denote the two generating operations $\prec$ and $\succ$ of *LDend* by μ and ν respectively. Then the space of relations of *LDend* is generated as an $\mathbb{S}_3$-module by

$$r_1 := \mu \circ_{\mathrm{I}} \mu + \nu' \circ_{\mathrm{III}} \mu' - \mu' \circ_{\mathrm{II}} \mu - \mu' \circ_{\mathrm{II}} \nu - \mu \circ_{\mathrm{I}} \nu'$$

and by

$$r_2 := \nu \circ_{\mathrm{I}} \nu + \nu \circ_{\mathrm{I}} \mu + \nu' \circ_{\mathrm{III}} \nu' - \nu' \circ_{\mathrm{II}} \nu - \nu \circ_{\mathrm{I}} \mu' - \nu \circ_{\mathrm{I}} \nu'\ .$$

Under the notations $\nwarrow := \begin{pmatrix} \mu \\ \prec \end{pmatrix}$, $\nearrow := \begin{pmatrix} \mu \\ \succ \end{pmatrix}$, $\swarrow := \begin{pmatrix} \nu \\ \prec \end{pmatrix}$ and $\searrow := \begin{pmatrix} \nu \\ \succ \end{pmatrix}$, we have

$\mathrm{Su}_i(r_j)$	r_1	r_2
Su_1	$\nwarrow \circ_{\mathrm{I}} (\nwarrow - \searrow^{(12)}) + \searrow^{(12)} \circ_{\mathrm{III}} \nwarrow^{(12)} - \nwarrow^{(12)} \circ_{\mathrm{II}} *$	$\swarrow \circ_{\mathrm{I}} (< - >^{(12)}) + \searrow^{(12)} \circ_{\mathrm{III}} \swarrow^{(12)} - \swarrow^{(12)} \circ_{\mathrm{II}} \vee$
Su_2	$\nwarrow \circ_{\mathrm{I}} (\nearrow - \swarrow^{(12)}) + \swarrow^{(12)} \circ_{\mathrm{III}} \wedge^{(12)} - \nearrow^{(12)} \circ_{\mathrm{II}} <$	$\mathrm{Su}_1(r_2)^{(12)}$
Su_3	$\nearrow \circ_{\mathrm{I}} (\wedge - \vee^{(12)}) + \searrow^{(12)} \circ_{\mathrm{III}} \nearrow^{(12)} - \nearrow^{(12)} \circ_{\mathrm{II}} >$	$\searrow \circ_{\mathrm{I}} (* - *^{(12)}) + \searrow^{(12)} \circ_{\mathrm{III}} \searrow^{(12)} - \searrow^{(12)} \circ_{\mathrm{II}} \searrow$

where $< := \swarrow + \nwarrow$, $> := \searrow + \nearrow$, $\vee := \searrow + \swarrow$, $\wedge := \nearrow + \nwarrow$ and $* := \swarrow + \nwarrow + \nearrow + \searrow$. Finally we get

$$\mathrm{Su}(LDend) \cong LQuad\ .$$

□

A.2. **Alternative and pre-alternative operads.** We next assume that the characteristic of **k** is not two. An **alternative algebra** [28] is defined to be a **k**-vector space with one bilinear operation $\circ$ that satisfies the following relations

$$
\begin{aligned}
(x \circ y) \circ z + (y \circ x) \circ z &= x \circ (y \circ z) + y \circ (x \circ z),\\
(x \circ y) \circ z + (x \circ z) \circ y &= x \circ (y \circ z) + x \circ (z \circ y).
\end{aligned}
$$

A **pre-alternative algebra** [41] is defined to be a **k**-vector space with two bilinear operations $\prec$ and $\succ$ and that satisfy the following relations

$$\begin{aligned}
(x \circ y + y \circ x) \succ z &= x \succ (y \succ z) + y \succ (x \succ z),\\
(x \succ z) \prec y + (z \prec x) \prec y &= x \succ (z \prec y) + z \prec (x \circ y),\\
(y \circ x) \succ z + (y \succ z) \prec x &= y \succ (x \succ z) + y \succ (z \prec x),\\
(z \prec x) \prec y + (z \prec y) \prec x &= z \prec (x \circ y + y \circ x),
\end{aligned}$$

where $\circ = \prec + \succ$.

Proposition A.3. *The bisuccessor of the operad Alter is the operad PreAlter, equivalently*

$$PreLie \bullet Alter = PreAlter\ .$$

And the trisuccessor of the operad Alter is the operad encoding the following algebraic structure:

$$\begin{aligned}
(x \star y + y \star x) \succ z &= x \succ (y \succ z) + y \succ (x \succ z),\\
(x \succ z) \prec y + (z \prec x) \prec y &= x \succ (z \prec y) + z \prec (x \star y),\\
(y \star x) \succ z + (y \succ z) \prec x &= y \succ (x \succ z) + y \succ (z \prec x),\\
(z \prec x) \prec y + (z \prec y) \prec x &= z \prec (x \star y + y \star x),\\
(x \cdot y) \prec z + (y \cdot x) \prec z &= x \cdot (y \prec z) + y \cdot (x \prec z),\\
(x \prec y) \cdot z + (y \succ x) \cdot z &= x \cdot (y \succ z) + y \succ (x \cdot z),\\
(x \cdot y) \prec z + (x \prec z) \cdot y &= x \cdot (y \prec z) + x \cdot (z \succ y),\\
(x \succ y) \cdot z + (x \succ z) \cdot y &= x \succ (y \cdot z) + x \succ (z \cdot y),\\
(x \cdot y) \cdot z + (y \cdot x) \cdot z &= x \cdot (y \cdot z) + y \cdot (x \cdot z),\\
(x \cdot y) \cdot z + (x \cdot z) \cdot y &= x \cdot (y \cdot z) + x \cdot (z \cdot y),
\end{aligned}$$

where $x \star y = x \prec y + x \succ y + x \cdot y$.

A.3. **Leibniz and pre-Leibniz operads.** A **Leibniz algebra** [33] is defined to be a **k**-vector space with one bilinear product [,] satisfying the Leibniz identity

$$[[x, y], z] = [[x, z], y] + [x, [y, z]]\ .$$

Proposition A.4. *The bisuccessor of the operad Leibniz is the operad encoding the following algebraic structure:*

$$\begin{aligned}
(x \prec y) \prec z &= (x \prec z) \prec y + x \prec (y \succ z + y \prec z),\\
(x \succ y) \prec z &= (x \succ z + x \prec z) \succ y + x \succ (y \prec z),\\
(x \succ y + x \prec y) \succ z &= (x \succ z) \prec y + x \succ (y \succ z).
\end{aligned}$$

And the trisuccessor of the operad Leibniz is the operad encoding the following algebraic structure:

$$\begin{aligned}
(x \prec y) \prec z &= (x \prec z) \prec y + x \prec (y \star z),\\
(x \succ y) \prec z &= (x \star z) \succ y + x \succ (y \prec z),\\
(x \star y) \succ z &= (x \succ z) \prec y + x \succ (y \succ z),\\
(x \cdot y) \prec z &= (x \prec z) \cdot y + x \cdot (y \prec z),\\
(x \prec y) \cdot z &= (x \cdot z) \prec y + x \cdot (y \succ z),
\end{aligned}$$

$$\begin{aligned}(x \succ y)\cdot z &= (x \succ z)\cdot y + x \succ (y\cdot z),\\ (x\cdot y)\cdot z &= (x\cdot z)\cdot y + x\cdot (y\cdot z),\end{aligned}$$

where $x \star y = x \prec y + x \succ y + x\cdot y$.

A.4. **The operad** *Poisson***.** A **(left) post-Poisson algebra** is a **k**-vector space A equipped with four bilinear operations $([,], \diamond, \cdot, \succ)$ such that $(A, [,], \diamond)$ is a (left) post-Lie algebra, $(A, \cdot, \succ)$ is a commutative tridendriform algebra, and they are compatible in the sense that (for any $x, y, z \in A$)

$$[x, y\cdot z] = [x,y]\cdot z + y\cdot [x,z],$$

$$[x, z\succ y] = z \succ [x,y] - y\cdot (z\diamond x),$$

$$x\diamond (y\cdot z) = (x\diamond y)\cdot z + y\cdot (x\diamond z),$$

$$(y\succ z + z\succ y + y\cdot z)\diamond x = z\succ (y\diamond x) + y\succ (z\diamond x),$$

$$x\diamond (z\succ y) = z\succ (x\diamond y) + (x\diamond z - z\diamond x + [x,z])\succ y.$$

Let *PostPoisson* denote the operad encoding the post-Poisson algebras.

Remark A.5. Let $(A, [,], \diamond, \cdot, \succ)$ be a post-Poisson algebra. If the operations $[,]$ and $\cdot$ are trivial, then it is a pre-Poisson algebra.

Proposition A.6. *The trisuccessor of the operad Poisson is the operad PostPoisson, equivalently*

$$PostLie \bullet Poisson = PostPoisson\ .$$

A.5. **The operad** *Jordan***.** Assume that the characteristic of **k** is neither two nor three.

Proposition A.7. *The trisuccessor of the operad Jordan is the operad encoding the following algebraic structure:*

$$\begin{aligned}
&((x\prec y)\prec u)\prec z + x\prec((y\star z)\star u) + ((x\prec z)\prec u)\prec y\\
=\ &(x\prec y)\prec(u\star z) + (x\prec u)\prec(y\star z) + (x\prec z)\prec(u\star y),\\
&(u\prec(x\star y))\prec z + (u\prec(y\star z))\prec x + (u\prec(z\star x))\prec y\\
=\ &(u\prec z)\prec(x\star y) + (u\prec z)\prec(y\star z) + (u\prec y)\prec(z\star x),\\
&((x\cdot y)\prec u)\prec z + ((y\prec z)\prec u)\cdot x + ((x\prec z)\prec u)\cdot y\\
=\ &(x\cdot y)\prec(u\star z) + (y\prec z)\cdot(x\prec u) + (x\prec z)\cdot(y\prec u),\\
&((x\prec y)\cdot u)\prec z + (u\prec(y\star z))\cdot x + ((x\prec z)\cdot u)\prec y\\
=\ &(x\prec y)\cdot(u\prec z) + (u\cdot x)\prec(y\star z) + (x\prec z)\cdot(u\prec y),\\
&((x\cdot y)\prec u)\cdot z + ((y\cdot z)\prec u)\cdot x + ((z\cdot x)\prec u)\cdot y\\
=\ &(x\cdot y)\cdot(z\prec u) + (y\cdot z)\cdot(x\prec u) + (z\cdot x)\cdot(y\prec u),\\
&((x\cdot y)\cdot u)\prec z + ((y\prec z)\cdot u)\cdot x + ((x\prec z)\cdot u)\cdot y\\
=\ &(x\cdot y)\cdot(u\prec z) + (y\prec z)\cdot(u\cdot x) + (x\prec z)\cdot(u\cdot y),\\
&((x\cdot y)\cdot u)\cdot z + ((y\cdot z)\cdot u)\cdot x + ((z\cdot x)\cdot u)\cdot y\\
=\ &(x\cdot y)\cdot(u\cdot z) + (y\cdot z)\cdot(u\cdot x) + (z\cdot x)\cdot(u\cdot y),
\end{aligned}$$

where $x\star y = x\prec y + y\prec x + x\cdot y$.

Concluding remark. Despite the generality of the approach in this paper, we believe that this study provides a new starting point, rather than the end, on the splitting of operads. There are many other ways to split the associativity than the ones provided by the dendriform dialgebra and trialgebra alluded above, for instance given by the so-called CABQR operads [16]. What are their generalizations to the general operads? The restriction of Manin black product to binary quadratic operads calls for its generalization to non-quadratic operads, so that the successors of the operads of Jordan and pre-Jordan algebras, for example, can be viewed in terms of the Manin black product. On the other hand, the well-known Koszul duality of Manin black product suggests a possible duality for the successors and Rota-Baxter operators, maybe to differential type operators [5, 22, 23]. For some recent progress see [37, 44].

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