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ONE-DIMENSIONAL FIBONACCI TILINGS AND DERIVATIVES OF TWO-COLOUR ROTATIONS OF A CIRCLE

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Abstract. This is a study of the two-colour (composite) rotations $S_\varepsilon(a,b)$, $a,b \in \mathbb{Z}$, of the unit circle C that has a coloration which depends on the continuous parameter $\varepsilon \in C$. We prove that these rotations $S_\varepsilon(a,b)$ are connected with one-dimensional Fibonacci tilings $Til(\varepsilon_m)$ of the unit interval $I \simeq \mathbb{R}/\mathbb{Z}$ and $S_\varepsilon(a,b)$ have interesting global and local structures.

Global structure. The space of the two-colour rotations $S_\varepsilon(a,b)$ is closed under the operation of boundary differentiations or the restrictions of rotations on intervals in the Fibonacci tilings $Til(\varepsilon_m)$. By this invariance, self-similarity properties of the two-colour rotations follow for quadratic irrationals. It means that the derivative rotation is equal to a renormalization-rotation.

Local structure. Among the results here are following. The attractors $Att(S_\varepsilon(a,b)) \subset C$ preserve their measure for the parameter ε in open arcs. It implies that the function $\nu_\varepsilon(\theta)$ of the frequency distribution of points in the $S_\varepsilon(a,b)$-orbits has a plateau. The circle C can be split into open arcs $C^{'} \subset C$ so that the $C^{'}$-restriction of the rotation $S_\varepsilon(a,b)$ is isomorphic to an integral transformation of a simple rotation or a two-colour rotation of C.

Dedicated to Anatoli Nikolaevich Andrianov

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1. Introduction

1. Composite shifts and one-dimensional Fibonacci tilings. Let's identify the interval $I = [0, 1)$ with the unit circle $C \simeq \mathbb{R}/\mathbb{Z}$. Consider *a simple shift* $S : x \mapsto x + \tau \bmod 1$, where $\tau = (1 + \sqrt{5})/2 = 1.618\ldots$ is *the golden section.* We define *derivatives* (induced transformations, the first return maps)

$$S^{(m)} = S\,|I^m$$

as restrictions of the shift S on some intervals $I^m = [\varepsilon_m, 1) \subset I$. Assume that the derivative transformations

$$S^{(m)} \sim S^{\pm 1} \tag{1.1}$$

are similar to the shift S, or *the inverse shift* $S^{-1} : x \mapsto x - \tau \bmod 1$. Then we arrive at a concept of *the branching B-process* (see Section 2). This process generates *the eigenvalues*

$$\varepsilon_m = 1 - \tau^{-m} \tag{1.2}$$

with numbers $m = 0, 1, 2, \ldots$ The property (1.1) exists whenever the parameter ε_m in $I^m = [\varepsilon_m, 1)$ has form (1.2). The power ± 1 in $S^{\pm 1}$ depends on the fact whether the number m is even or odd. Each *differentiation interval*

$$I^m = [\varepsilon_m, 1)$$

is split

$$I^m = G_0^m \oplus E_0^m$$

into intervals G_0^m, E_0^m so that they touch each other. Their *lengths* $g_m = |G_0^m|$, $e_m = |E_0^m|$ are equal to

$$g_m = 1 - \varepsilon_{m+2} = \tau^{-(m+2)}, \;\; e_m = 1 - \varepsilon_{m+1} = \tau^{-(m+1)}$$

for even m, and

$$e_m = 1 - \varepsilon_{m+2} = \tau^{-(m+2)}, \;\; g_m = 1 - \varepsilon_{m+1} = \tau^{-(m+1)}$$

for odd m. In the case of even m , suppose S shifts the intervals G_0^m, E_0^m $F_{m+1} - 1$ and $F_{m+2} - 1$ times, respectively, where F_m are *the Fibonacci numbers*:

$$F_{m+2} = F_{m+1} + F_m$$

and $F_1 = F_2 = 1$ Then we get *the one-dimensional Fibonacci tiling $Til(\varepsilon_m)$* of the unit interval I of *the level m*. For odd m, the number of shifts is equal to $F_{m+2} - 1$ and $F_{m+1} - 1$, respectively.

The following invariancy property

$$S(Til(\varepsilon_m)) = Til^{+}(\varepsilon_m) \tag{1.3}$$

of the Fibonacci tilings is basic. *The dual tiling $Til^{+}(\varepsilon_m)$* of the unit interval I is received from $Til(\varepsilon_m)$ by the rearrangement $I^m = E_0^m \oplus G_0^m$ of its intervals G_0^m and E_0^m. From invariancy (1.3) we get self-similarity property (1.1) for the derivatives $S^{(m)}$.

We can construct the one-dimensional Fibonacci tilings $Til(\varepsilon_m)$ using the following methods [10]:

1) The branching B-process.
2) Substitutions or the division of long intervals in ratio τ (inflation rules).
3) The recurrence formula (see Subsection 5.3)

$$Til^{+}(\varepsilon_{m+2}) = \tau^{-1} \cdot \overline{Til^{+}(\varepsilon_{m+1})} \oplus \tau^{-2} \cdot Til^{+}(\varepsilon_{m}), \tag{1.4}$$

where the line over $\overline{Til^{+}(\varepsilon_{m+1})}$ indicates the inversion of designations for long and short intervals. We denote by $\oplus$ the non-commutative operation that puts the first tiling to the second tiling compressed by τ^{-1} and τ^{-2} times, respectively.

4) The quasi-local coordinates (see Subsection 5.1).
5) The global coordinates (see Subsection 2.3).
6) The differential approach (in this article).

A two-colour shift or a composite shift $S_\varepsilon(a,b): I \longmapsto I$ with integer parameters $a, b \in \mathbb{Z}$ depends on the unit interval *coloring*

$$I = I_\varepsilon^{-} \oplus I_\varepsilon^{+}, \tag{1.5}$$

where $I_\varepsilon^{-} = [0,\varepsilon)$, $I_\varepsilon^{+} = [\varepsilon, 1)$. Then *the composite shift* $S_\varepsilon(a,b)$ is determined by the two-colour vector field (cf. [1])

$$\begin{aligned} x &\longmapsto x + a\tau \bmod 1, \quad \text{if } x \in I_\varepsilon^{-}, \\ x &\longmapsto x + b\tau \bmod 1, \quad \text{if } x \in I_\varepsilon^{+}. \end{aligned} \tag{1.6}$$

Our goal in this paper is to research geometrical and dynamic properties of the composite shifts $S_\varepsilon(a,b)$ and to describe their structure in terms of the one-dimensional Fibonacci tilings $Til(\varepsilon_m)$.

2. Derivatives of composite shifts. Here we use a method of differentiation which means a restriction of the composite shifts $S_\varepsilon(a,b)$ on special intervals I^m, I_m, and J^m connected with the Fibonacci tilings $Til(\varepsilon_m)$. We consider two kinds of derivatives :

1) boundary upper derivatives and boundary lower derivatives,
2) internal derivatives.

The upper derivatives of the order $m = 0, 1, 2, \ldots$

$$S_\varepsilon(a,b)^{(m)} = S_\varepsilon(a,b)\,|I^m \times I^m \tag{1.7}$$

are restrictions of the composite shift $S_\varepsilon(a,b)$ for x and ε both on the intervals $I^m = [\varepsilon_m, 1)$, where ε_m are eigenvalues (1.2) of the B- process.

Note that the parameters a and b in definition (1.6) of the composite shift $S_\varepsilon(a,b)$ are symmetric. Therefore, we can compare the upper derivatives $S_\varepsilon(a,b)^{(m)}$ (1.7) with their dual *lower derivatives*

$$S_\varepsilon(a,b)_{(m)} = S_\varepsilon(a,b)\,|I_m \times I_m,$$

where *the differentiation intervals* $I_m = inv(I^m) = (0, 1-\varepsilon_m]$ are symmetric to the intervals I^m under the involution $inv: x \mapsto 1-x$ of the unit interval $I = [0,1)$.

For the composite shift $S_\varepsilon(a,b)$, we define *internal derivatives* by the following local and global ways:

$$S_\varepsilon(a,b)|I^m(\delta) \times J^m, \qquad S_\varepsilon(a,b)|Orb(I^m(\delta)) \times J^m. \tag{1.8}$$

Here $I^m(\delta) = I^m + \delta \bmod 1$,

$$\delta = |E_0^m|, \; J^m = [1 - |G_0^m| - |L^m|, 1 - |G_0^m|),$$

if the level m is even, and

$$\delta = \varepsilon - (1 - |E_0^m|), \; J^m = [1 - |E_0^m| - |L^m|, 1 - |E_0^m|)$$

in the case of odd m . We denote by $Orb(I^m(\delta))$ *the orbit* of the interval $I^m(\delta)$ under the shift $S_\varepsilon(a,b)$ and by L^m *a lacuna* of the orbit $Orb(I^m(\delta))$ (see Subsection 5.1).

The space $\mathbf{S}$ of two-colour shifts (1.6) is closed under the operation of boundary differentiations (Theorem 1 and Theorem 2). Assume that the composite shift $S_\varepsilon(a,b)$ (1.6) has the integer parameters a, b with the same sign and the Fibonacci numbers F_{m+1}, F_{m+2} satisfy the congruences

$$F_{m+1} \equiv F_{m+2} \equiv F \bmod a,$$

where $1 \leq F \leq |a|$. Suppose $|b| \leq s((|b|F^{-1})_a, m)$ and let $s(k,m)$ be a function from Subsection 4.2. Then the upper derivative of the order m

$$S_\varepsilon(a,b)^{(m)} \sim S_{\varepsilon'}((-1)^m a, (-1)^m sign(b) \cdot b') \tag{1.9}$$

is a composite shift once again. The derivative shift $S_\varepsilon(a,b)^{(m)}$ is similar to the shift on the right-hand side in formula (1.9), where $b' = (|b|F^{-1})_a$ is equal to the least positive solution of the congruence $xF \equiv |b| \, mod \, a$ and $\varepsilon' = (\varepsilon - \varepsilon_m)/(1-\varepsilon_m)$ is a renormalisation of the parameter ε. *The similarity* $\sim$ of the composite shifts in (1.9) means the conjugacy

$$S_\varepsilon(a,b)^{(m)} = h_m \circ S_{\varepsilon'}((-1)^m a, (-1)^m sign(b) \cdot b') \circ h_m^{-1}$$

under the affine isomorphism $h_m(x) = \tau^{-m}(x-1)+1$ of the unit interval I and the differentiation interval I^m.

For the lower derivative $S_\varepsilon(a,b)_{(m)}$ we have the symmetric similarity (cf. (1.9))

$$S_\varepsilon(a,b)_{(m)} \sim S_{\varepsilon'}((-1)^m sign(a) \cdot a', (-1)^m b)).$$

The internal derivatives (1.8) of two-colour shifts have a local non-invariant characteristic. Let ν_ε be a function of *frequency distribution* of $S_\varepsilon(a,b)$-orbits. Here we consider an elementary scenario that explains how plateaus of the function ν_ε can arise.

Let $S_\varepsilon(a,b)$ be composite shift (1.6) with integer coprime parameters $1 \leq b < a$ and let a, b satisfy the congruences

$$F_{m+1} \equiv b \bmod a, \;\; F_{m+2} \equiv 0 \bmod a.$$

Then we have the following *isomorphisms*

$$\begin{array}{ll} S_\varepsilon(a,b)| \quad I^m(\delta) \times J^m & \simeq S^{\pm 1}, \\ S_\varepsilon(a,b)|Orb(I^m(\delta)) \times J^m & \simeq \int S^{\pm 1} \, d_\varepsilon, \end{array} \tag{1.10}$$

with S^{+1} or S^{-1} for even or odd m, respectively. In (1.10), $\int S^{\pm 1} \, d_\varepsilon$ is *an integral transformation* of the simple shift $S^{\pm 1} : \;\; x \longmapsto x \pm \tau \bmod 1$ with respect to *a two-colour distribution* d_ε on the unit interval I.

3. Self-similarity of composite shifts. There is a self-semilarity phenomenon for composite shifts. It follows from two facts:

1) the space $\mathbf{S}$ of composite shifts $S_\varepsilon(a,b)$ (1.6) is closed under the boundary derivatives;

2) the golden section τ is a quadratic irrational number.

Since the upper derivatives $S_\varepsilon(a,b)^{(m)}$ and the lower derivatives $S_\varepsilon(a,b)_{(m)}$ are dual to each other, we formulate some consequences from invariancy (1.9) only for the upper derivatives.

Suppose the integer parameters a, b have the same sign and satisfy the condition $1 \le |b| \le |a|$. Then the composite shift $S_\varepsilon(a,b)$ has *the self-similarity property* (see Corollary 1)

$$S_\varepsilon(a,b)^{(m)} \sim S_{\varepsilon'}(a,b). \tag{1.11}$$

The order m in (1.11) is equal to T or $2T$, if *the period* $T = T(a)$ of the sequence of the Fibonacci numbers $F_1, F_2, \dots, F_n, \dots \bmod a$ is even or odd, respectively.

Let the parameter a be a prime number p satisfying the condition $1 \le b \le p$. Then self-similarity property (1.11) becomes as follows (see Corollary 2):

$$S_\varepsilon(p,b)^{(p-1)} \sim S_{\varepsilon'}(p,b) \tag{1.12}$$

for prime numbers $p \equiv 1$ or $4 \bmod 5$,

$$S_\varepsilon(p,b)^{(2(p+1))} \sim S_{\varepsilon'}(p,b)$$

for prime numbers $p \equiv 2$ or $3 \bmod 5$, and

$$S_\varepsilon(p,b)^{(20)} \sim S_{\varepsilon'}(p,b)$$

for $p = 5$.

Consider the case when a is equal to the Fibonacci number F_m (see Corollary 3). In this case we have

$$S_\varepsilon(F_m,b)^{(2m)} \sim S_{\varepsilon'}(F_m,b) \tag{1.13}$$

for even m, and

$$S_\varepsilon(F_m,b)^{(4m)} \sim S_{\varepsilon'}(F_m,b)$$

for odd m. For $m = 2$, we have a commutative diagram

$$\begin{array}{ccc} S_\varepsilon(1,1)^{(2)} & \sim & S_{\varepsilon'}(1,1) \\ \downarrow\simeq & & \downarrow\simeq \\ S^{(2)} & \sim & S \end{array} \tag{1.14}$$

In particular, from (1.14) it follows the derivative formula $S^{(2)} \sim S$ (1.1) for the simple shift S.

For $m = 3$, we obtain the first non-trivial case of the self-similarity property for composite shifts (see Fig. 1)

$$S_\varepsilon(2,1)^{(6)} \sim S_{\varepsilon'}(2,1). \tag{1.15}$$

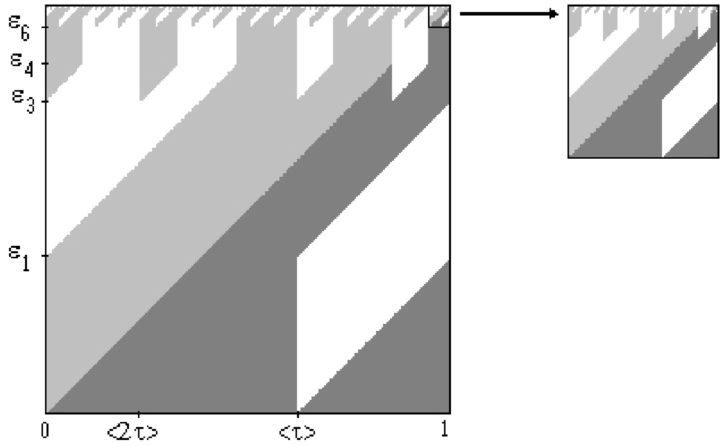


Fig. 1. The attractor development $Att(2,1)$.

In more general case, the composite shift $S_\varepsilon(a,b)$ is *a contracting mapping*

$$I \supset S_\varepsilon(a,b)(I) \supset \cdots \supset S_\varepsilon^k(a,b)(I) \supset \ldots .$$

For any integer parameters a, b , $ab > 0$, the above chain contains only a finite number of strict inclusions. *The attractor* $Att(S_\varepsilon(a,b))$ of the composite shift $S_\varepsilon(a,b)$ is defined as the intersection

$$Att(S_\varepsilon(a,b)) = \bigcap_{k=0}^{\infty} S_\varepsilon^k(a,b)(I),$$

where we denote by $S_\varepsilon^k(a,b)$ the k-multiple composition of the shift $S_\varepsilon(a,b)$.

The unit interval I splits into

$$I = Att(S_\varepsilon(a,b)) + Spir_\varepsilon \tag{1.16}$$

the attractor $Att(S_\varepsilon(a,b))$ and *a spiral* $Spir_\varepsilon$. The attractor $Att(S_\varepsilon(a,b))$ is the minimal set of the shift $S_\varepsilon(a,b)$ and the spiral $Spir_\varepsilon$ is an area any point of which gets in the attractor $Att(S_\varepsilon(a,b))$ through a finite number of iterations. We introduce *the dynamic metric* $dis(\varepsilon)$ as the number of iterations. Then we can say that $dis(\varepsilon) < \infty$, i.e., the maximal distance between the spiral $Spir_\varepsilon$ and the attractor $Att(S_\varepsilon(a,b))$ is finite.

For the composite shift $S_\varepsilon(2,1)$ the following estimations are known:

$$\frac{c_1}{1-\varepsilon} - 2 < dis(\varepsilon) < \frac{c_2}{1-\varepsilon} + 1$$

with the constants $c_1 = \dfrac{\tau^2}{2\sqrt{5}} \approx 0.58...$ and $c_2 = \tau c_1 \approx 0.94...$ In particular, from these estimations it follows that

$$dis(\varepsilon) \longrightarrow \infty \quad \text{as} \quad \varepsilon \uparrow 1.$$

For the shift $S_\varepsilon(2,1)$, Fig. 1 represents *the development* of decomposition (1.16) through the parameter $\varepsilon \in I$

$$I \times I = Att(2,1) + Spir. \tag{1.17}$$

Accoding to the unit interval coloration $I = I_\varepsilon^- \oplus I_\varepsilon^+$ defined in (1.5), the development $Att(2,1)$ of the attractor $Att(S_\varepsilon(2,1))$ consists of *the grey* $Att(2,1)^-$ areas and *the dark* $Att(2,1)^+$ areas, and the development $Spir$ of the spiral $Spir_\varepsilon$ corresponds to *the light* area.

Self-similarity (1.15) of the composite shift $S_\varepsilon(2,1)$ (see Fig. 1) means that the restriction of development (1.17) on the upper right square $I^6 \times I^6$ is similar to complete development (1.17). So the attractor development $Att(2,1)$ is a fractal.

4. Internal derivatives and plateaus. The following equality

$$Orb(I^m(\delta)) = Att(S_\varepsilon(a,b))$$

from Subsection 5.3 is true for all parameters $\varepsilon \in J^m$. It clears up a sense of the second formula in (1.10). Using the above equality, we obtain *an isomorphism*

$$S_\varepsilon(a,b)\,|Att(S_\varepsilon(a,b)) \times J^m \simeq \int S^{\pm 1} d_\varepsilon \tag{1.18}$$

from which it follows the distribution properties of points in the $S_\varepsilon(a,b)$ - orbits (see Theorem 3).

Suppose the composite shift $S_\varepsilon(a,b)$ satisfies condition (1.10) and its parameter ε belongs to the interval J^m. Then for any x, θ in $I = [0,1)$ there exists the limit

$$\nu_\varepsilon(\theta) = \lim_{n\to\infty} \frac{1}{n}\, \sharp\{k:\ S_\varepsilon^k(a,b)(x) \le \theta,\ k = 0,1,\dots,n-1\}$$

which is independent of the choice of the initial point x. The function $\nu_\varepsilon(\theta)$ is equal to *the frequency* when points in the $S_\varepsilon(a,b)$-orbit of x visit the interval $[0,\theta]$. For $\nu_\varepsilon(\theta)$ we prove the formula (see Theorem 3)

$$\nu_\varepsilon(\theta) = \frac{|Att(S_\varepsilon(a,b)) \cap [0,\theta]|}{|Att(S_\varepsilon(a,b))|}, \tag{1.19}$$

where we denote by $|\cdot|$ *a Haar measure* on the torus $C \simeq \mathbb{R}/\mathbb{Z}$.

A diagonal frequency ν_ε is defined as the specialization $\nu_\varepsilon = \nu_\varepsilon(\theta)$ with $\theta = \varepsilon$. For the diagonal frequency ν_ε formula (1.19) has the form

$$\nu_\varepsilon = \frac{|Att(S_\varepsilon(a,b))^-|}{|Att(S_\varepsilon(a,b))|}, \tag{1.20}$$

where $Att(S_\varepsilon(a,b))^-$ is the intersection of the attractor $Att(S_\varepsilon(a,b))$ and the interval $I_\varepsilon^- = [0,\varepsilon)$.

The grey area $Att(2,1)^-$ in Fig. 1 represents the development of the set $Att(S_\varepsilon(2,1))^-$ through the parameter $\varepsilon \in I$. The existence of *plateaus* for the diagonal frequency ν_ε(see Fig. 2) follows from formula (1.20). It means that the frequency ν_ε is independent of the parameter ε, if $\varepsilon \in J^m$ (see Proposition 5).

Let $S_\varepsilon(F_n, F_{n-1})$ be composite shift (1.6) with the parameter $\varepsilon \in J^m$, where $m \equiv n-1 \bmod T(F_n)$, $n \ge 3$, and $T(F_n)$ is the period of the sequence of the Fibonacci numbers $F_1, F_2, \dots, F_m, \dots \bmod F_n$. Then for any initial point x in $I = [0,1)$ the frequency ν_ε, when points in the $S_\varepsilon(F_n, F_{n-1})$-orbit of x visit the interval $I_\varepsilon^- = [0,\varepsilon)$, is equal to (see Proposition 5)

$$\nu_\varepsilon = \frac{1 - F_{n-1}\tau^{-(m+2)}}{1 + F_{n-2}\tau^{-(m+2)}} \tag{1.21}$$

or

$$\nu_\varepsilon = 1 - \frac{\tau^{-(m+2)}}{|Att(S_\varepsilon(F_n, F_{n-1}))|}. \tag{1.22}$$

Formula (1.22) connects the frequency ν_ε with the measure $|Att(S_\varepsilon(F_n, F_{n-1}))|$ of the attractor $Att(S_\varepsilon(F_n, F_{n-1}))$. Thus, if it is known the frequency ν_ε, we can find the measure $|Att(S_\varepsilon(F_n, F_{n-1}))|$ using formula (1.22).

Some corollaries follow from formulas (1.21) and (1.22).

1. Under the above conditions the attractor measure $|Att(S_\varepsilon(F_n, F_{n-1}))|$ is equal to

$$|Att(S_\varepsilon(F_n, F_{n-1}))| = \frac{\tau^{-(m+2)}}{1-\nu_\varepsilon}.$$

So

$$|Att(S_\varepsilon(F_n, F_{n-1}))| \to \frac{1}{F_n}$$

as $\varepsilon \to 1$ and ε lies in the union of the intervals J^m (see (1.8)).

2. Suppose the parameter ε of the composite shift $S_\varepsilon(F_n, F_{n-1})$, $n \geq 3$, belongs to the interval $J^{n-2} = [\tau^{-(n-1)}, 1-\tau^{-n})$. Then the frequency ν_ε is independent of n and is equal to

$$\nu_\varepsilon = \tau^{-1} = 0.618\ldots$$

3. Suppose the parameter ε of the composite shift $S_\varepsilon(2,1)$ belongs to the interval $J^m = [1-\tau^{-m}, 1-\tau^{(m+2)})$ of the level $m \equiv 1 \bmod 3$, $m \geq 1$. Then the frequency ν_ε is equal to (see Fig.2)

$$\nu_\varepsilon = th_\tau\Big(\frac{m}{2}+1\Big). \tag{1.23}$$

Here the function $th_\tau(x) = (\tau^x - \tau^{-x})/(\tau^x + \tau^{-x})$ is a hyperbolic tangent with the basis $\tau = (1+\sqrt{5})/2$.

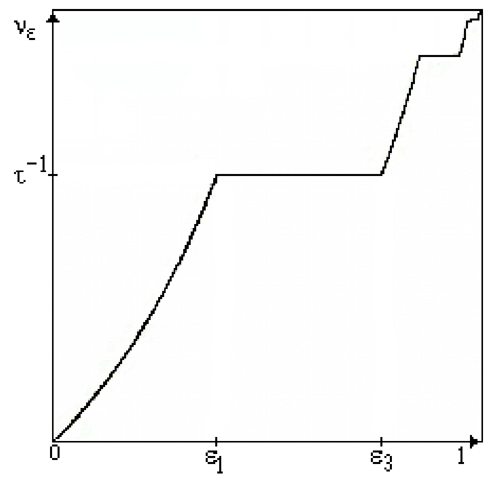


Fig. 2. The diagonal frequency ν_ε for the two-color shift $S_\varepsilon(2,1)$

5. Spectral decompositions and singularities. Let $\mu(\nu)$ be *a measure* of $\varepsilon \in I$ so that the frequency ν_ε is equal to ν, i.e.,

$$\mu(\nu) = |\{\varepsilon \in [0,1): \ \nu_\varepsilon = \nu\}|.$$

The spectrum $sp(a,b)$ of the composite shift $S_\varepsilon(a,b)$ is the set of all $\nu \in I$ satisfying the inequality $\mu(\nu) > 0$, where $\mu(\nu)$ is *the weight* of ν.

For the shift $S_\varepsilon(2,1)$, spectrum $sp(2,1)$ consists of the points (see Fig. 2)

$$\nu_\varepsilon = th_\tau(\frac{m}{2}+1) \tag{1.24}$$

with the weights $\mu^m = \tau^{-(m+1)}$ and the numbers $m \equiv 1 \bmod 3,\ m \geq 1$. A plateau of the frequency function ν_ε corresponds to each spectral point $\nu^m \in sp(2,1)$. The weight $\mu^m = \mu(\nu^m)$ is equal to the length of an interval, where the function ν_ε is constant. Since

$$\nu^m \sim 1 - \tau^{-(m+2)}$$

as $m \to \infty$, the graph of the function $\mu(\nu)$ is asymptotically self-similar with center at the point $(\nu, \mu) = (1, 0)$ and ratio τ^{-3}.

We write the spectrum $sp(2,1)$ of the composite shift $S_\varepsilon(2,1)$ as

$$sp(2,1) = (\,|\,). \tag{1.25}$$

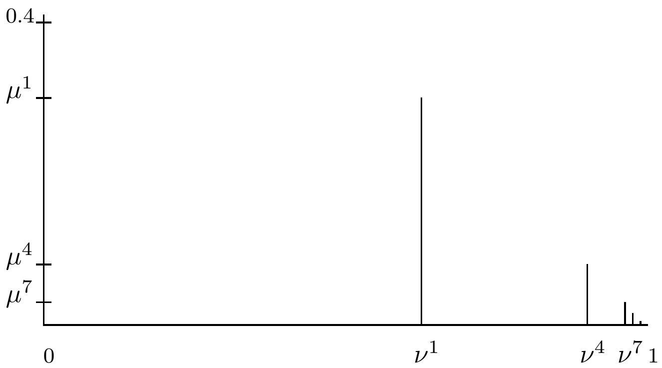


Fig. 3. The spectrum $S_\varepsilon(2,1)$

In (1.25), the symbol $(\,|\,)$ means that the shift $S_\varepsilon(2,1)$ has an elementary spectrum (see Fig. 3) generated by a unique point $\nu^1 = \tau^{-1} = 0.618...$. Moreover, the symbol $(\,|\,)$ means that the spectrum $sp(2,1)$ is asymptoticaly self-similar and the line $|$ is its fundamental domain.

Fig. 5 represents the spectrum of the composite shift $S_\varepsilon(3,1)$. The development $Att(3,1)$ of the attractor $Att(S_\varepsilon(3,1)$ is shown in Fig. 4.

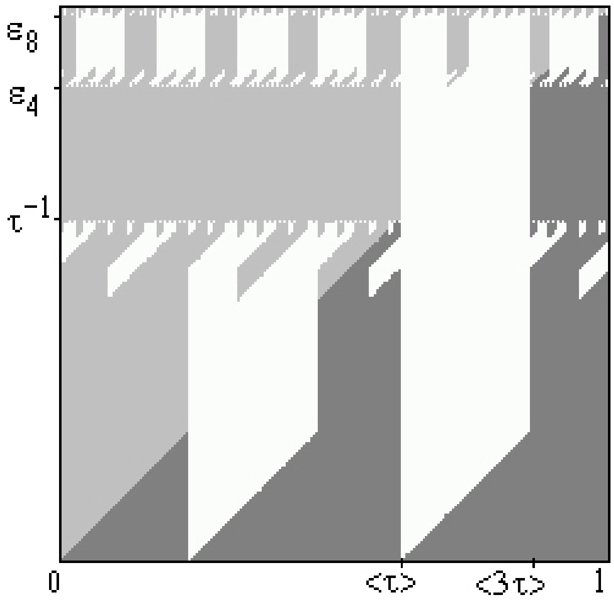


Fig. 4. The attractor development $Att(3,1)$

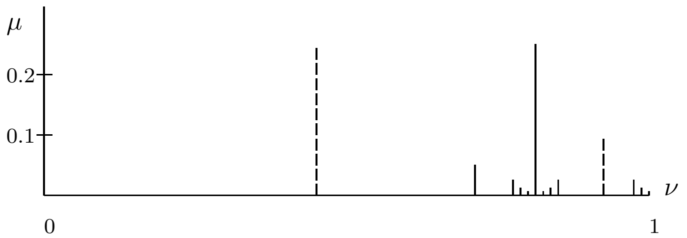


Fig. 5. The spectrum $S_\varepsilon(3,1)$

In the same way as for the spectrum $sp(2,1)$ (1.25),we introduce the following notation for the spectrum $sp(3,1)$:

$$sp(3,1) = ((-2,-1)|(-1,-2)\ |), \tag{1.26}$$

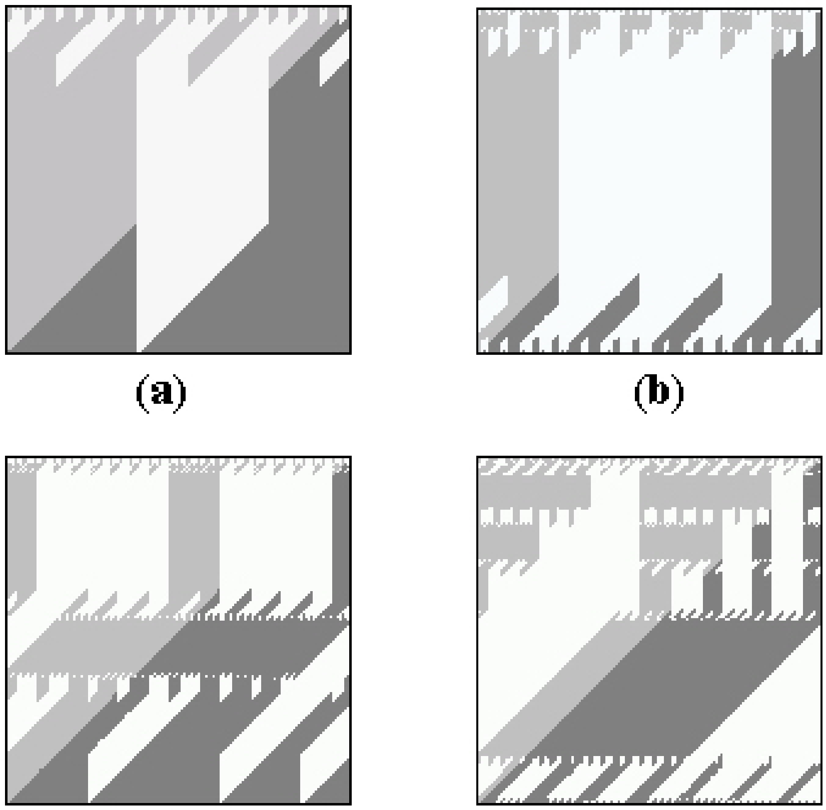
where

$$sp(-2,-1) = (|\ (2,1)). \tag{1.27}$$

The involution $inv: \ \nu \to 1-\nu$ maps the spectrum $sp(2,1)$ to the spectrum $sp(-1,-2)$. Thus, a 180° rotation maps the development $Att(2,1)$ to the development $Att(-1,-2)$ (cf. Fig. 1 and Fig. 4). In spectral decomposition (1.26), the first vertical line $|$ corresponds to a limiting point in the spectrum $sp(3,1)$. The second vertical line $\smile |$ (with the space on the left) corresponds to its discrete point.

The composite shifts $S_\varepsilon(a,b)$ with the integer coprime parameters a, b satisfying the condition $1 \le b < a \le 4$ have the following spectral decompositions:

$$\begin{aligned}
sp(3,1) &= (|\ (2,1)|(-1,-2)\ |),\\
sp(3,2) &= (-1,-2)\ |\ (3,1),\\
sp(3,2) &= ((-1,-2)\ |\ (2,1)|),\\
sp(4,1) &= ((2,1)|(-1,-2)\ |),\\
sp(4,3) &= ((2,3)|(-1,-2)\ |\ (2,1)|(3,2)|).
\end{aligned} \tag{1.28}$$



(a) (b)

(c) (d)

Fig. 6. The attractor developments for two-color shifts
(a) $Att(-2,-1)$, (b) $Att(3,2)$, (c) $Att(4,1)$, (d) $Att(4,3)$

Spectral decompositions (1.28) show that the composite shifts $S_\varepsilon(a,b)$ are decomposed into the sum of elementary transformations. Each transformation is isomorphic to
1) an integral of the simple shift $S^{\pm 1} : x \mapsto x \pm \tau \bmod 1$ or
2) an integral of the two-colour shift $S_\varepsilon(2,1)$ or its symmetric shift $S_\varepsilon(-1,-2)$.
So there exist only two basic shifts S and $S_\varepsilon(2,1)$.

The singularities are the limiting points s of the spectrum $sp(a,b)$ in torus topology $I \sim \mathbb{R}/\mathbb{Z}$. For every such point s we define the degree $deg(s)$. In the spectrum $sp(2,1)$ (see Fig. 3) the singularity $s \equiv 0 \bmod 1$ has degree 1, and in the spectrum $sp(3,1)$ (see Fig. 5) the singularity $s = 0.809\ldots$ has degree 1 and

$s \equiv 0 \bmod 1$ has degree 2. By definition, discrete points of a spectrum have zero degree. *The degree* of the whole spectrum $sp(a, b)$ is equal the maximal degree of its singularities.

The composite shifts $S_\varepsilon(a, b)$ from list (1.28) are distributed according to their degrees $deg(a, b)$ as follows:

$$\begin{aligned}&deg(2,1) = 1,\ deg(-2,-1) = 1,\\ &deg(3,1) = 2,\ deg(3,2) = 2,\ deg(4,1) = 2,\\ &deg(4,3) = 3.\end{aligned}$$

Thus, the degree $deg(a, b)$ of the spectrum $sp(a, b)$ specifies a complexity level of the two-colour shift $S_\varepsilon(a, b)$.

In accordance with Proposition 6, for any $n = 1, 2, 3, \ldots$ there exists a composite shift $S_\varepsilon(a, b)$ with the integer coprime parameters $1 \le b < a \le F_{n+2}$ so that the spectral degree $deg(a, b)$ satisfies the inequality

$$deg(a, b) \ge n.$$

6. Around the Fibonacci tilings. There are several methods for building of the one-dimensional Fibonacci tilings $Til(\varepsilon_m)$ and their limiting tiling

$$Til(\varepsilon_\infty) = \lim_{m\to\infty} \tau^m Til(\varepsilon_m)$$

(see, for example, [2], [6], [10]). In [10] one can find the following methods: the B-process (the branching process), the substitution method (inflation rules), the recurrence Fibonacci method (see (1.4)), the quasi-local method, and the global method.

This article is based on the differential approach (see (1.1), (1.3)). We build the tilings $Til(\varepsilon_m)$ by using the derivatives $S^{(m)} = S\,|I^m$ of the simple shift $S : x \mapsto x + \tau \ mod\ 1$ of the unit interval $I = [0, 1)$. The tilings $Til(\varepsilon_m)$ appear if we consider the following problem:

Let the derivative transformations $S^{(m)} = S\,|I^m$ be similar $S^{(m)} \sim S^{\pm 1}$ to the shift S or its inverse shift S^{-1} (in more general case, we can assume that $S^{(m)}$ is similar to some shift of the unit interval I). What is a differential interval $I^m \subseteq I$?

In the context of number theory derivatives appeared implicitly in the study of regularities of distribution for fractional parts $\langle n\alpha\rangle$, where $n = 1, 2, 3, \ldots$ and α is irrational. An interval $J \subseteq I$ is called a bounded remainder set for the sequence $\langle n\alpha\rangle$, if the sequence of remainders

$$r_{\alpha,J}(N) = \sharp\{n < N : \langle n\alpha\rangle \in J\} - N \mid J \mid$$

is bounded as $N \to \infty$. It is known [3]-[5] that the interval J is a bounded remainder set for $\langle n\alpha\rangle$ if and only if the length $\mid J \mid > 0$ lies in $\alpha\mathbb{Z} + \mathbb{Z}$. Since, by (1.2), the length $\mid I^m \mid$ is equal to $\tau^{-m} = (-1)^m(F_{m+1} - F_m\tau)$ that lies in the set $\tau\mathbb{Z} + \mathbb{Z}$ then the interval I^m is a bounded remainder set for the sequence $\langle n\tau\rangle$. Moreover, any bounded remainder interval J is split into the differentiation intervals I^m.

In this article we discuss a connection between the Fibonacci tilings $Til(\varepsilon_m)$ and the two-colour (composite) shifts $S_\varepsilon(a, b)$ of the interval $I = [0, 1)$ (see (1.6)). The tilings $Til(\varepsilon_m)$ are used to research the derivative shifts $S_\varepsilon(a, b)|*$ (see (1.7), (1.8)).

The composite shifts $S_\varepsilon(a, b)$ by themselves have arisen from the two-dimensional version [8], [9] of the border layer of quasicrystal growth.

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2. One-dimensional Fibonacci tilings

2.1. **Branching process.** We consider a unit circle C and identify it with the interval $I = [0, 1)$, i.e. $I \simeq \mathbb{R}/\mathbb{Z}$. Now we introduce the concept of *branching B-process* on the interval I. It is defined in [10] by the following conditions:

1) Let $P = \{\varepsilon_0, \varepsilon_1\}$ be a set of the marked points $\varepsilon_0 = 0, \quad \varepsilon_1 = 2 - \tau$.
2) A point ε runs through the interval I with a positive constant speed. Its movement begins from the point ε_0.
3) The images ε^α of each marked point ε_α begin movement together with the point ε with the same speed.
4) As soon as one of these points ε^α reaches one of the marked points ε_α, we get a new set P of marked points. The set P contains all the points ε^α and old marked points.
5) The point ε continues its movement after each branching.

From the definition of the B-process it follows that *the set of the marked points* $P = P(\varepsilon)$ depends on ε and the set $P(\varepsilon)$ grows for a discrete set of the parameter's values $\varepsilon \in I$:

$$\varepsilon_0 < \varepsilon_1 < \cdots < \varepsilon_m < \dots, \tag{2.1}$$

where $\varepsilon_0 = 0$. The set $E = \{\varepsilon_m\}$ is *a spectrum* of the B-process and ε_m are *the eigenvalues* of the B-process.

Fig. 2.1. The spectrum of B-process

The spectrum $E = \{\varepsilon_m\}$ (2.1) is infinite. The eigenvalues ε_m satisfy the following equalities and recurrence formulaes:

$$\begin{aligned} &\varepsilon_m = \langle (-1)^m F_m \cdot \tau \rangle = 1 - (\tau - 1)^m = 1 - \tau^{-m}, \\ &\varepsilon_{m+2} = \varepsilon_m - \varepsilon_{m+1} + 1 \end{aligned} \tag{2.2}$$

for $m = 0, 1, 2, \dots$

2.2. **The branching process and tilings.** The set of the marked points $P(\varepsilon)$ tiles the unit interval into small intervals . Let $Til(\varepsilon)$ be an appropriate tiling of I on left-closed, right-open intervals $[*, *) \subset I$. For $\varepsilon = \varepsilon_m$ the tiling

$$Til(\varepsilon_m) = \sum_j G_j^m + \sum_i E_i^m \tag{2.3}$$

is *the eigen Fibonacci tiling*. It consists of intervals of two kinds G_j^m and E_i^m which have lengths g_m and e_m , respectively.. The quotient of these lengths is equal to

$$\begin{aligned} \frac{e_m}{g_m} &= \tau, \quad m = 0, 2, 4, \dots, \\ \frac{g_m}{e_m} &= \tau, \quad m = 1, 3, 5, \dots \end{aligned} \tag{2.4}$$

We see that the quotient of lengths depends on the fact whether the level m of the tiling $Til(\varepsilon_m)$ is even or odd. The spectrum $E = \{\varepsilon_m\}$ and the eigen tilings $Til(\varepsilon_m)$ are connected by the formula

$$\varepsilon_{m+1} = \varepsilon_m + \min\{g^m, e^m\}. \tag{2.5}$$

Here $m \geq 1$ and $\varepsilon_1 = 2 - \tau$. Moreover, we have the convenient formulas for the lengths g_m, e_m of the intervals G_j^m, E_i^m:

$$g_m = 1 - \varepsilon_{m+2} = \tau^{-(m+2)}, \ e_m = 1 - \varepsilon_{m+1} = \tau^{-(m+1)} \tag{2.6}$$

for even levels m, and

$$g_m = 1 - \varepsilon_{m+1} = \tau^{-(m+1)}, \ e_m = 1 - \varepsilon_{m+2} = \tau^{-(m+2)} \tag{2.7}$$

for odd m.

If ε is not an eigenvalue, the tiling $Til(\varepsilon)$ is *not eigen*. If m is even and $\varepsilon_m \leq \varepsilon \leq \varepsilon_{m+1}$, we can construct the tiling

$$Til(\varepsilon) = \sum_j G_j^m + \sum_i E_i^m \tag{2.8}$$

using the eigen tiling $Til(\varepsilon_m)$ (2.3) by the help of *the substitutions*

$$G_j^m = GG_j^m(\varepsilon) \oplus G_j^m(\varepsilon), \quad E_i^m = GE_i^m(\varepsilon) \oplus E_i^m(\varepsilon).$$

Here the left intervals have length $|GG_j^m(\varepsilon)| = |GE_i^m(\varepsilon)| = \varepsilon - \varepsilon_m$. Hence, for $\varepsilon = \varepsilon_m$ the tiling $Til(\varepsilon)$ (2.8) coincides with the eigen tiling $Til(\varepsilon_m)$ (2.3). At $\varepsilon = \varepsilon_{m+1}$ the tiling $Til(\varepsilon)$ passes to the next eigen tiling

$$Til(\varepsilon_{m+1}) = \sum_j G_j^{m+1} + \sum_i E_i^{m+1}, \tag{2.9}$$

where

$$\begin{aligned} \sum_j G_j^{m+1} &= \sum_j G_j^m(\varepsilon_{m+1}) + \sum_i GE_i^m(\varepsilon_{m+1}), \\ \sum_i E_i^{m+1} &= \sum_i E_i^m(\varepsilon_{m+1}). \end{aligned}$$

If the level m is odd, we can obtain tiling $Til(\varepsilon)$ (2.8) by *the substitutions*

$$G_j^m = EG_j^m(\varepsilon) \oplus G_j^m(\varepsilon), \quad E_i^m = EE_i^m(\varepsilon) \oplus E_i^m(\varepsilon)$$

Here $|EG_j^m(\varepsilon)| = |EE_i^m(\varepsilon)| = \varepsilon - \varepsilon_m$. At $\varepsilon = \varepsilon_{m+1}$ eigen tiling $Til(\varepsilon_{m+1})$ (2.9) consists of intervals

$$\begin{aligned} \sum_j G_j^{m+1} &= \sum_j G_j^m(\varepsilon_{m+1}), \\ \sum_i E_i^{m+1} &= \sum_j EG_j^m(\varepsilon_{m+1}) + \sum_i EE_i^m(\varepsilon_{m+1}). \end{aligned}$$

The above construction describes continuous reorganization of the eigen tilings $Til(\varepsilon_m)$ to the next level tilings $Til(\varepsilon_{m+1})$. $Fig.2.2$ presents the reorganisation process for the eigen Fibonacci tilings $Til(\varepsilon_m), m = 0, \dots, 4$.

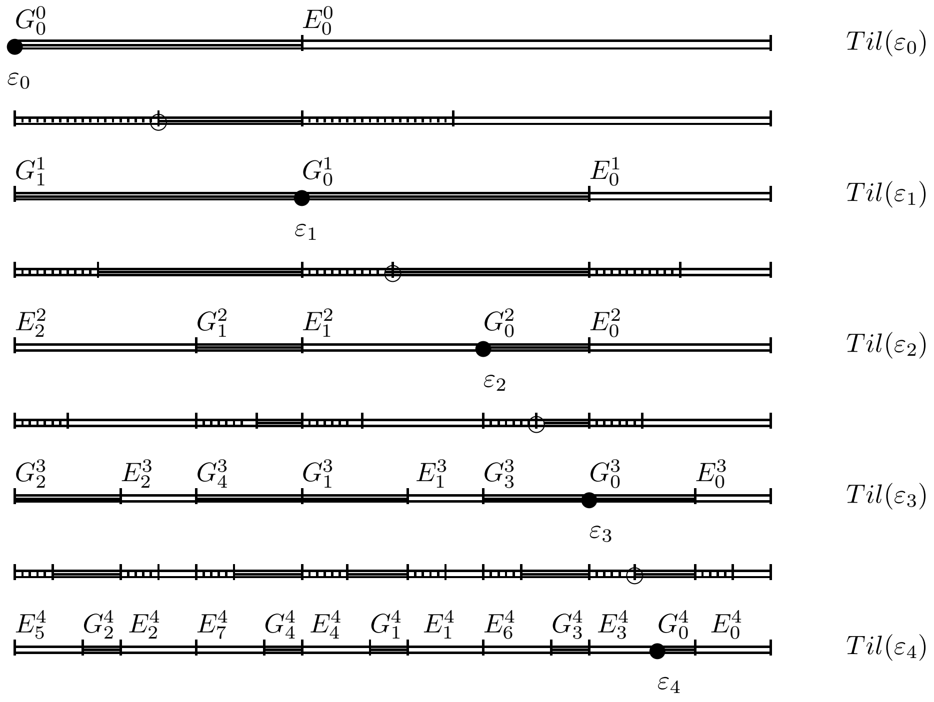

Fig. 2.2. Fibonacci tilings $Til(\varepsilon_m)$ for $m = 0, \dots, 4$

2.3. **Global coordinates.** Let $I[j, i) = [\langle j\tau\rangle, \langle i\tau\rangle)$ be an interval from $I = [0, 1)$ for any integers j, i. We assume that $\langle j\tau\rangle < \langle i\tau\rangle$, where $\langle \cdot \rangle$ is the fractional part. If $i = 0$, we define $I[j, 0) = [\langle j\tau\rangle, 1\rangle)$.

Let m be even. Then eigen Fibonacci tilings $Til(\varepsilon_m)$ (2.3) are

$$Til(\varepsilon_m) = \sum_{0 \le k < F_{m+1}} (G_k^m + E_k^m) + \sum_{F_{m+1} \le k < F_{m+2}} E_k^m, \tag{2.10}$$

split into intervals G_k^m and E_k^m determined by the equalities

$$G_k^m = I[k^+, k^-),\ E_k^m = I[k^-, k). \tag{2.11}$$

Here we use the notation $k^+ = k + F_m$, $k^- = k - F_{m+1}$. The intervals G_k^m, E_k^m from tiling (2.10) have lengths (2.6). The mapping S connects G_k^m, E_k^m and the initial intervals G_0^m, E_0^m by the shift

$$G_{k+1}^m \equiv G_k^m + \tau \bmod 1, \tag{2.12}$$

where $0 \le k < F_{m+1}$ and $G_0^m = I[F_m, -F_{m+1}) = I[\langle F_m\rangle, \langle -F_{m+1}\rangle)$, and

$$E_{k+1}^m \equiv E_k^m + \tau \bmod 1, \tag{2.13}$$

where $0 \le k < F_{m+2}$ and $E_0^m = I[-F_{m+1}, 0) = [\langle -F_{m+1}\tau\rangle, 1)$.

Similarly, for odd levels m eigen tilings $Til(\varepsilon_m)$ (2.3) are represented as the sum

$$Til(\varepsilon_m) = \sum_{0\le k<F_{m+1}} (G_k^m + E_k^m) + \sum_{F_{m+1}\le k<F_{m+2}} G_k^m, \tag{2.14}$$

In this case the intervals G_k^m and E_k^m are equal to

$$G_k^m = I[k^-, k^=), \quad E_k^m = I[k^=, k), \tag{2.15}$$

where $k^- = k - F_m$, $k^= = k - F_{m+2}$. They are connected by the S-shift

$$G_{k+1}^m \equiv G_k^m + \tau \bmod 1, \tag{2.16}$$

where $0 \le k < F_{m+2}$, $G_0^m = I[-F_m, -F_{m+2}) = [\langle -F_m\tau\rangle, \langle -F_{m+2}\tau\rangle)$, and

$$E_{k+1}^m \equiv E_k^m + \tau \bmod 1, \tag{2.17}$$

where $0 \le k < F_{m+1}$ and $E_0^m = I[-F_{m+2}, 0) = [\langle -F_{m+2}\tau\rangle, 1)$.

Formulas (2.11) and (2.15) define *global coordinates* for the intervals G_k^m and E_k^m in the eigen Fibonacci tilings $Til(\varepsilon_m)$.

3. Fibonacci numbers and the golden section

3.1. **Periodicity of Fibonacci numbers** mod a**.** Let us consider a sequence (see [7])

$$F_1, F_2 ..., F_m, ... \bmod a \tag{3.1}$$

of *the Fibonacci numbers* determined by *the recurrence formula*

$$F_{m+2} = F_m + F_{m+1} \tag{3.2}$$

for $m \ge 1$ and the initial conditions $F_1 = 1$, $F_2 = 1$. Let's find in the sequence (3.1) the first repeating pair of the Fibonacci numbers

$$F_{m+T} \equiv F_m \bmod a, \ F_{m+1+T} \equiv F_{m+1} \bmod a. \tag{3.3}$$

Let $T = T(a)$ be the least number for which congruences (3.3) holds for some index m. By $m_1 = m_1(a)$ we denote the least of such m. Since the number of all integer pairs mod$a$ is equal to a^2, therefore the function $T(a)$ is well defined and we have the trivial inequality

$$T(a) \le a^2.$$

Subtracting in (3.3) the first congruence from the second congruence and using the recurrence formula for Fibonacci numbers, we get one more pair of the congruences

$$F_{m-1+T} \equiv F_{m-1} \bmod a, \ F_{m+T} \equiv F_m \bmod a.$$

It means that in congruences (3.3) the index m can be moved on one step to the left, if $m \ge 2$. Similarly, by addition of congruences (3.3), we see that the index m also can be moved to the right. Therefore T is *the period* of sequence (3.1) and

$$m_1 = 1 \quad \text{for any} \quad a \ge 1.$$

Hence, in sequence (3.1) the first repeating pair is

$$F_{T+1} \equiv 1 \bmod a, \ F_{T+2} \equiv 1 \bmod a. \tag{3.4}$$

From (3.4) we obtain the congruence $F_T \equiv 0 \bmod a$. Suppose the congruence

$$F_t \equiv 0 \bmod a$$

holds for some t. If we assume $a > 1$, then using the recurrence formula for Fibonacci numbers we have

$$F_{t+1} \equiv f \bmod a,\ F_{t+2} \equiv f \bmod a, \tag{3.5}$$

and $f \equiv F_{t-1} \bmod a$. Let $t(a)$ be a least t so that a divides the Fibonacci number F_t. The number t is called *a period of self-similarity* of sequence (3.1) and f is *a coefficient of self-similarity* of the sequence. Note that the functions $t(a)$ and $T(a)$ have the following multiplicative property

$$t(a_1 \cdot a_2) = \text{l.c.m.}(t(a_1), t(a_2)), \quad T(a_1 \cdot a_2) = \text{l.c.m.}(T(a_1), T(a_2)), \tag{3.6}$$

if a_1 and a_2 are coprime numbers.

We can write congruences (3.5) as

$$F_{t+1} \equiv fF_1 \bmod a,\ F_{t+2} \equiv fF_2 \bmod a.$$

Using these congruences and the recurrence formula, we obtain the self-similarity property of sequence (3.1):

$$F_{m+t} \equiv fF_m \bmod a$$

for all $m = 1, 2, 3...$ Substituting $t+1$, $t+2$, and $2t+1$, $2t+2$, etc. for m in the last congruence and using (3.5), we get the congruences

$$F_{it+1} \equiv f^i \bmod a,\ F_{it+2} \equiv f^i \bmod a, \tag{3.7}$$

where $i = 1, 2, 3...$ and $f = f(a)$ is the coefficient of self-similarity of sequence (3.1).

Suppose congruence (3.5) holds with some f. We assert that a and f are coprime numbers. If the greatest common divisor of the numbers a and f is equal to d, then from congruence (3.5) follows that $F_{t+2} \equiv 0 \bmod d$, $F_{t+1} \equiv 0 \bmod d, \dots,$ $F_1 \equiv 0 \bmod d$, i.e. $d = 1$.

Since the numbers a and f are coprime, there exists an exponent $\delta > 0$ such that

$$f^\delta \equiv 1 \bmod a.$$

Let $\delta = \delta(a)$ be *the order f mod a*. It is easy to see, that the order δ can accept only the values 1, 2 or 4. Really, from the formula $F_{t-1}^2 = F_{t-2}F_t + (-1)^t$ and the condition $a|F_t$ we get the congruence

$$F_{t-1}^2 \equiv (-1)^t \bmod a.$$

The coefficient of self-similarity satisfies the congruence $f \equiv F_{t-1} \bmod a$. By the above two congruence, we get the following statements:

$$\begin{array}{ll} \delta = 1 \text{ or } 2, & \text{if } t \text{ is even },\\ \delta = 4, & \text{if } t \text{ is odd and } a \geq 3. \end{array} \tag{3.8}$$

Besides, the order $\delta = 1$ for $a = 1$ or 2.

Let us prove that the periods $T = T(a)$, $t = t(a)$ of sequence (3.1) and the order $\delta = \delta(a)$ are connected by the formula

$$T(a) = t(a)\delta(a) \tag{3.9}$$

for all $a = 1, 2, 3...$ First, we notice that $t(a) \leq T(a)$. Let's divide $T(a) = kt(a) + r$, where r is a remainder $0 \leq r < t(a)$. Then by congruences (3.7) and (3.6) we can write

$$F_{kt+1} \equiv F_{kt+2} \bmod a, \; F_{T+1} \equiv F_{T+2} \bmod a.$$

From the first congruence it follows that the module a is coprime with the Fibonacci numbers F_{kt+1} and F_{kt+2}. Therefore

$$F_{r+1} \equiv F_{r+2} \equiv f \bmod a, \quad \text{where} \quad f \equiv F_{T+1} F_{kt+1}^{-1} \bmod a.$$

Hence, $F_r \equiv 0 \bmod a$. As $0 \leq r < t(a)$, from the definition of the period $t(a)$ we obtain $r = 0$.

It is known that the period $T(a)$ for a prime module $a = p$ has properties:

$$T(p) \text{ divides } p-1, \text{ if } p \equiv 1 \text{ or } 4 \bmod 5,$$

$$T(p) \text{ divides } 2(p+1), \text{ if } p \equiv 2 \text{ or } 3 \bmod 5.$$

3.2. **Fibonacci numbers and Euler function.** Consider the real quadratic field $\mathbb{Q}(\sqrt{5})$. Its ring of integers $\mathbb{Z}[\tau]$ has the basis 1, $\tau = (1+\sqrt{5})/2$ over the ring $\mathbb{Z}$. It is an Euclidian ring and τ is a unit of the given ring, because $\tau^{-1} = \tau - 1$. Under the Euler theorem in the ring $\mathbb{Z}[\tau]$ for each $a = 1, 2, 3...$ it holds the congruence

$$\tau^{\Phi(a)} \equiv 1 \bmod a\mathbb{Z}[\tau], \tag{3.10}$$

where Φ is *the Euler function* of the ring $\mathbb{Z}[\tau]$. The restriction Φ onto the set of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ represents *a multiplicative function* with the following values on degrees of prime numbers:

$$\Phi(p^\alpha) = p^{2\alpha-2}\Phi(p) \tag{3.11}$$

for prime numbers $p \in \mathbb{N}$ and $\alpha \geq 1$, and

$$\begin{array}{lll} \Phi(p) = (p-1)^2, & \text{if } p \equiv 1 \text{ or } 4 & \pmod 5, \\ \Phi(p) = p^2 - 1, & \text{if } p \equiv 2 \text{ or } 3 & \pmod 5, \\ \Phi(p) = p(p-1), & \text{if } p = 5. & \end{array}$$

Let $\Delta = \Delta(a)$ be *an order* of τ in the ring $\mathbb{Z}[\tau]$, i.e. the least $\Delta > 0$, $\Delta \in \mathbb{Z}$ with the condition

$$\tau^\Delta \equiv 1 \bmod a\mathbb{Z}[\tau]. \tag{3.12}$$

Let's take the well-known formula

$$\tau^\Delta = F_{\Delta-1} + F_\Delta \tau$$

connecting the golden section τ with the Fibonacci numbers. Comparing this formula with condition (3.12), we get the congruences

$$F_{\Delta-1} \equiv 1 \bmod a, \quad F_\Delta \equiv 0 \bmod a$$

in the ring $\mathbb{Z}$. Conversely, from the last congruences it follows congruence (3.12). Besides, combining the last congruences with the recurrence formula for Fibonacci numbers(3.2) we have two congruences

$$F_{\Delta+1} \equiv 1 \bmod a, \; F_{\Delta+2} \equiv 1 \bmod a.$$

Taking into account minimality of the order $\Delta(a)$ (3.12) we obtain that the order $\Delta(a)$ is equal to the period $T(a)$ of sequence (3.1).

Proposition 1. *1. Suppose a is any natural number, $T(a)$ is the period of the sequence of Fibonacci numbers $F_1, F_2..., F_m, ...$ mod a, and $t(a)$ is its period of self-similarity. Then the following equality is true*

$$T(a) = \Delta(a),$$

where $\Delta(a)$ is the order of τ mod $a\mathbb{Z}[\tau]$. The periods $T(a)$ and $t(a)$ divide $\Phi(a)$, where Φ is the Euler function of the ring $\mathbb{Z}[\tau]$. The multiplicative function Φ is determined by equalities (3.11).

2. For $m \geq 4$ it follows that

$$\begin{aligned} T(F_m) &= 2m, \quad \textit{if} \quad m \textit{ even}, \\ T(F_m) &= 4m, \quad \textit{if} \quad m \textit{ odd}. \end{aligned} \tag{3.13}$$

The equalities $T(F_1) = T(F_2) = 1$ and $T(F_3) = 3$ hold for indexes $m < 4$. In particular, from (3.13) it follows that $2m$ or $4m$ divides $\Phi(F_m)$, if $m \geq 4$ and m is even or odd respectively.

Proof. 1. The equality $T(a) = \Delta(a)$ is already proved. The fact that $T(a)$ and $t(a)$ divide $\Phi(a)$ follows from (3.10), (3.13), and (3.9).

2. Let $a = F_m$. In this case the period of self-similarity t is equal to

$$t(F_m) = m \tag{3.14}$$

for indexes $m \geq 3$, and $t(F_1) = t(F_2) = 1$. The coefficient of self-similarity f (3.5) satisfies the congruence

$$f \equiv F_{m-1} \bmod F_m.$$

Taking into account that the congruence $F_{m-1} \equiv 1 \bmod F_m$ holds only for indexes $m = 1, 2, 3$, formulas (3.8) take the form for the module $a = F_m$

$$\begin{aligned} \delta(F_m) &= 2 \quad \text{for even } m \geq 4, \\ \delta(F_m) &= 4 \quad \text{for odd } \ m \geq 4, \end{aligned} \tag{3.15}$$

and $\delta(F_m) = 1$ for the indexes $m = 1, 2, 3$. Finally, combining (3.14) and (3.15) we transform formula (3.9) to formulas (3.13). □

In Table 3.1 for sequence of Fibonacci numbers mod a (3.1) we give the values of the periods $T = T(a)$, the periods of self-similarity $t = t(a)$, the coefficients of self-similarity $f = f(a) \equiv F_{t-1} \bmod a$, $0 < f \leq a$, the orders $\delta = \delta(a)$ of the coefficients f mod a, and the Euler function Φ of the ring $\mathbb{Z}[\tau]$, where $1 \leq a \leq 21$.

a	t	f	δ	T	Φ
1	1	1	1	1	1
2	3	1	1	3	3
3	4	2	2	8	8
4	6	1	1	6	12
5	5	3	4	20	20
6	12	5	2	24	24
7	8	6	2	16	48
8	6	5	2	12	48
9	12	8	2	24	72
10	15	7	4	60	60

a	t	f	δ	T	Φ
11	10	1	1	10	100
12	12	5	2	24	96
13	7	8	4	28	168
14	24	13	2	48	144
15	20	11	2	40	160
16	12	9	2	24	192
17	9	4	4	36	288
18	12	11	2	24	216
19	18	1	1	18	324
20	30	9	2	60	240
21	8	13	2	16	384

Table 3.1.

3.3. **Distribution of two points on a circle.** We identify the unit interval $I = [0,1)$ with the unit circle $\mathbb{R}/\mathbb{Z}$ and consider *the simple shift*

$$S(x) = x + \tau \bmod 1,$$

where $\tau = (1+\sqrt{5})/2$ is the golden section. The unit interval is tiled

$$Til(\varepsilon_0) = G_0^0 \oplus E_0^0$$

into two intervals $G_0^0 = [0, \tau^{-2})$ and $E_0^0 = [\tau^{-2}, 1)$. By the sign $\oplus$ we denote a non-commutative operation of applying intervals. For any point $x \in I$ we consider its orbit $\{x_i\}_{i=0}^{\infty}$, where $x_i = S^i(x)$. Define the function $\varepsilon(x_i)$ (*the indicator* of G_0^0) by the condition

$$\varepsilon(x_i) = \begin{cases} 1, & \text{if } x \in G_0^0, \\ 0, & \text{if } x \in E_0^0. \end{cases}$$

Let $s_k = \varepsilon(x_1) + \cdots + \varepsilon(x_k)$ for $k = 1, 2, 3...$ and

$$s_k = \max s_k(x), \ x \in I. \tag{3.16}$$

As the fractional part $\langle\tau\rangle$ is greater than the length τ^{-2} of the interval G_0^0, in the sequence $\varepsilon(x_1), \ldots, \varepsilon(x_k)$ two units can not be in succession. From here it follows the trivial upper estimation for the function s_k

$$s_k \le \left[\frac{k+1}{2}\right],$$

where the square brackets denote the integer part of a number.

To prove the exact formula for the function s_k (see (3.16)) we transform the tiling $Til(\varepsilon_0)$ to the second level tiling

$$Til(\varepsilon_2) = E_2^2 \oplus G_1^2 \oplus E_1^2 \oplus G_0^2 \oplus E_0^2,$$

where $G_0^0 = E_0^2 \oplus G_1^2$, and interval $E_0^0 = E_1^2 \oplus G^2 \oplus E_0^2$ is tiled symmetrically. All possible pairs of points $(x, y) \in I \times I$ we split

$$I \times I = M_1 + \cdots + M_5$$

into subsets M_i, $i = 1, \ldots, 5$, consisting of pairs (x, y) for which the fractional part $\langle y - x\rangle$ belongs to i-th interval of the tiling $Til(\varepsilon_2)$. The intervals are numbered from left to right. The pair $(\varepsilon(x)\ \ \varepsilon(y))$ is called *the state* of the pair (x, y). The following graphs specify *an admissible S-transitions*

$$S : (\varepsilon(x), \varepsilon(y)) \mapsto (\varepsilon(S(x)), \varepsilon(S(y)))$$

of a pair (x, y) from one state to another.

$$\begin{array}{ccc} & (1\ 1) & \\ & \uparrow\downarrow & \\ & (0\ 0) & \\ & \circlearrowleft & \\ \swarrow & & \nwarrow \\ (1\ 0) & \longrightarrow & (0\ 1) \\ & (M_1) & \end{array} \qquad \begin{array}{ccc} & (1\ 1) & \\ & \downarrow\uparrow & \\ & (0\ 0) & \\ & \circlearrowleft & \\ \nearrow & & \nwarrow \\ (1\ 0) & \longleftarrow & (0\ 1) \\ & (M_4) & \end{array}$$

$$\begin{array}{ccc} (0\ 0)_0 & & (0\ 0)_1 \\ \downarrow & \nwarrow\!\!\!\!\nearrow & \downarrow \\ (1\ 0)_0 & \rightleftarrows & (0\ 1)_1 \\ & (M_3) & \end{array}$$

$$\begin{array}{ccc} & (1\ 1) & \\ & \uparrow\downarrow & \\ & (0\ 0) & \\ & \circlearrowleft & \\ \swarrow & & \nwarrow \\ (1\ 0) & \rightleftarrows & (0\ 1) \\ & (M_2) & \end{array} \qquad \begin{array}{ccc} & (1\ 1) & \\ & \downarrow\uparrow & \\ & (0\ 0) & \\ & \circlearrowleft & \\ \nearrow & & \nwarrow \\ (1\ 0) & \rightleftarrows & (0\ 1) \\ & (M_5) & \end{array}$$

Fig. 3.1. The graphs (M_i), $i = 1, \dots, 5$ of the admissible S-transitions from one state to another

In the graph (M_3) the state $(0\,0)$ splits into the states $(0\,0)_0$ and $(0\,0)_1$. A pair (x, y) is found in the first state, if $x \in E_1^2$, and it is found in the second state, if $y \in E_1^2$.

For any $y \in I$ we define the difference

$$\Delta_k(0, y) = s_k(0) - s_k(y),$$

where $k = 1, 2, 3....$ We assume $(0, y) \in M_1$. Then the pair $(0, y)$ has the initial state $(1\,1)$. We act on $(0, y)$ by the shift S:

$$(0, y) \to (S(0), S(y)) \to \cdots \to (S^k(0), S^k(y)).$$

By the graph (M_1), under the shift S the pair $(0, y)$ changes its state according to the following chain

$$(1\,1)_= \to \cdots_= \to (1\,0)_\nearrow \to (0\,1)_= \to \cdots_=$$

Here the signs $=$ and $\nearrow$ show that on the given step the difference is equal to 0 or 1 respectively, i.e. $\Delta_k(0, y) = 1$ for k with the state $(1\,0)$ and $\triangle_k(0, y) = 0$ in other cases. Thus, the difference is equal to

$$\Delta_k(0, y) = 0 \ \text{ or } \ 1 \tag{3.17}$$

for all $k = 1, 2, 3...$, if pair $(0, y) \in M_1$. In the case, when the pair $(0, y)$ belongs to other subsets M_i, $i = 2..., 5$, equality (3.17) can be proved similarly by the help of S-transitions graph (M_i).

From the definition of the function s_k (see (3.16) and (3.17)) follows the equality

$$s_k = s_k(0) \tag{3.18}$$

for $k = 1, 2, 3, \ldots$. Given equality specifies that the extremal function s_k admits the global realization at $x = 0$.

Let's find the exact formula for the sum $s_k(0)$. Let $R(k)$ be a number of points of the orbit $\{0_i\}_{i=0}^{\infty}$ with numbers $1 \le i \le k$ that visit the interval E_0^0. The following formula for $R(k)$ is proved in Lemma 6.13 [10]

$$R(k) = [k\tau^{-1}]. \tag{3.19}$$

The unit interval I is split into the intervals G_0^0 and E_0^0. We can write the equality $s_k(0) = k - R(k)$. Using (3.19) we get $s_k(0) = k - [k\tau^{-1}]$. Notice that $[-k\tau^{-1}] = -[k\tau^{-1}] - 1$ and $1 - \tau^{-1} = \tau^{-2}$. Thus we have the required formula

$$s_k(0) = [k\tau^{-2}] + 1 \tag{3.20}$$

for $k = 1, 2, 3...$

Proposition 2. *For extremal function s_k (3.16) the following exact formula takes place*

$$s_k = [k\tau^{-2}] + 1 \tag{3.21}$$

for $k = 1, 2, 3...$

The proof follows from equality (3.18) and formula (3.20). □

4. Boundary derivatives and self-similarity

4.1. **Derivatives of the simple shift.** We write the Fibonacci tiling $Til(\varepsilon_m)$ (see (2.10), (2.11)) as the sum

$$Til(\varepsilon_m) = X \oplus G_0^m \oplus E_0^m. \tag{4.1}$$

Suppose the tiling $Til(\varepsilon_m)$ has an even level m. Then the simple shift $S : x \mapsto x + \tau \bmod 1$ of the unit interval $I = [0, 1)$ translates $Til(\varepsilon_m)$ in the tiling

$$S(Til(\varepsilon_m)) = X \oplus E_{F_{m+2}}^m \oplus G_{F_{m+1}}^m$$

Therefore the action of the shift S on the tiling $Til(\varepsilon_m)$ is reduced to a rearrangement of two right intervals

$$G_0^m \oplus E_0^m = E_{F_{m+2}}^m \oplus G_{F_{m+1}}^m. \tag{4.2}$$

Denote by I^m the sum $G_0^m \oplus E_0^m$. Let $e_m = |E_0^m|$ be the length of the interval E_0^m. By formulas (2.6), (2.7), $e_m = \tau^{-(m+1)}$ and

$$|I^m| = \tau^{-m} \tag{4.3}$$

for any level m. Therefore we have

$$\frac{e_m}{|I^m|} = \tau^{-1} = \tau - 1 \equiv \tau \bmod 1.$$

Hence for even m the derivative $S|I^m$, i.e. restriction of the shift S on the intervals I^m, is similar to the shift S:

$$S|I^m \sim S. \tag{4.4}$$

Similarly, if the level m is odd, using formulas (2.14) and (2.15) we obtain

$$S(Til(\varepsilon_m)) = X \oplus E^m_{F_{m+1}} \oplus G^m_{F_{m+2}}.$$

The action of the shift S, by 4.1, is reduced to the rearrangement of intervals

$$G^m_0 \oplus E^m_0 = E^m_{F_{m+1}} \oplus G^m_{F_{m+2}}. \tag{4.5}$$

By the formulaes (2.7) and (4.3), for odd m we have

$$\frac{e_m}{|I^m|} = \tau^{-2} = 2 - \tau \equiv -\tau \bmod 1.$$

Thus the derivative

$$S|I^m \sim S^{-1} \tag{4.6}$$

is similar to the inverse shift $S^{-1} : x \mapsto x - \tau \bmod 1$ of the unit interval I.

4.2. **Upper derivatives of two-colour shifts.** For any a and b from the set $\mathbb{Z} = \{0, \pm1, \pm2...\}$, $ab > 0$, we define *the two-colour* or *the composite shift* $S_\varepsilon(a, b)$ of the unit interval I by the conditions

$$\begin{aligned} S_\varepsilon(a,b)(x) &= x + a\tau \bmod 1, \quad \text{if} \quad x \in I^-_\varepsilon, \\ S_\varepsilon(a,b)(x) &= x + b\tau \bmod 1, \quad \text{if} \quad x \in I^+_\varepsilon. \end{aligned} \tag{4.7}$$

Here ε is a continuous parameter of the composite shift $S_\varepsilon(a, b)$, that ranges over the unit interval I. Define *the colouring* of the interval I by the tiling

$$I = I^-_\varepsilon \oplus I^+_\varepsilon$$

into the intervals $I^-_\varepsilon = [0, \varepsilon)$ and $I^+_\varepsilon = [\varepsilon, 1)$. The aim of this section is to consider *the upper derivatives*

$$S_\varepsilon(a,b)^{(m)} = S_\varepsilon(a,b)|I^m \times I^m \tag{4.8}$$

of composite shift (4.8) for any $m = 0, 1, 2....$ The upper derivatives are a restriction of the shift $S_\varepsilon(a, b)$ for x and ε both on the interval $I^m = [\varepsilon_m, 1)$. Here $\varepsilon_m = 1-\tau^{-m}$ are eigenvalues (2.2) of the B-process.

First, we consider the composite shift $S_\varepsilon(a, b)$ with the parameters a, b from the set $\mathbb{N} = \{1, 2, 3, \ldots\}$. Write b as

$$b = qa + b_1,$$

where $1 \le b_1 \le a$. For any point $x \in I^m$ denote by $Orb(x)$ its S-*orbit.* It consists of the elements $x_i = S^i(x) \equiv x + i\tau \bmod 1$. Here $i = 0, 1, 2...$ is *the* S-*number* of the points x_i from the orbit $Orb(x)$. Similarly, consider *the* $S_\varepsilon(a, b)$-*orbit* $Orb_\varepsilon(x)$ which consists of the elements $x_{j'} = S^j_\varepsilon(a, b)(x)$, $j = 0, 1, 2....$ Here j is *the* $S_\varepsilon(a, b)$-*number* of the element $x_{j'}$ with the S-number that is equal to j'.

Suppose the point x belongs to the interval $I^m_+ = I^m \cap I^+_\varepsilon$ from I^m. Then by definition (4.7) the orbit $Orb_\varepsilon(x)\backslash\{x\}$ consists of elements with the S-numbers

$$b, b + a, \ldots, b + ka, \ldots \tag{4.9}$$

These numbers represent the whole progression $i \equiv b_1 \bmod a$ beginning from $i = b_1$. We consider all the elements of the orbit $Orb_\varepsilon(x)\backslash\{x\}$ up to the first returning of the point x in the interval I^m. Further we impose additional restrictions on the interval I^m which ensure the returning of x in I^m. If the point x belongs to the interval $I^m_- = I^m \cap I^-_\varepsilon$ from I^m, the elements of its orbit $Orb_\varepsilon(x)\backslash\{x\}$ have the S-numbers

$$a, 2a, \ldots, ka, \ldots \tag{4.10}$$

forming the progression $i \equiv 0 \; mod \, a$ beginning from $i = a$.

According to Subsection 2.3 and equalities (4.2), (4.3), for any point x from the interval $I^m = G_0^m \oplus E_0^m$ the minimal S-number $i > 0$ of elements x_i from $Orb(x) \cap I^m$ is equal to

$$\begin{aligned} i &= F_{m+1} \quad \text{if} \quad x \in G_0^m, \\ i &= F_{m+2} \quad \text{if} \quad x \in E_0^m, \end{aligned} \tag{4.11}$$

for even levels m. Similarly,

$$\begin{aligned} i &= F_{m+2} \quad \text{if} \quad x \in G_0^m, \\ i &= F_{m+1} \quad \text{if} \quad x \in E_0^m, \end{aligned}$$

for odd m.

Now we need to connect the Fibonacci numbers F_{m+1} and F_{m+2} with the shift's parameter a. First, we assume

$$F_{m+1} \equiv F_{m+2} \equiv 1 \bmod a.$$

Let x_i be elements consistently returned by the shift S in the interval I^m. Then their S-numbers are equal to

$$j_1, j_2, \dots, j_k, \dots$$

Here $j_k = i_1 + i_2 + \dots + i_k$ is the number of the element $x_{j_k} \in I^m$ corresponding to the k-th returning of the point x in the interval I^m.

Let's calculate the least numbers k for which j_k gets into the progression $b + a\mathbb{N}$ (see (4.9)) or the progression $a\mathbb{N}$ (see (4.10)). From (4.11) it follows that any number $i_l \equiv 1 \bmod a$. Hence

$$j_k \equiv k \bmod a.$$

We estimate the values of j_k with the help of the function s_k (see (3.16)). If the level m is even, then using (4.4) and (4.11) we obtain the inequality

$$j_k = i_1 + i_2 + \dots + i_k \geq s_k F_{m+1} + (k - s_k) F_{m+2}.$$

Using the notation $s(k,m) = kF_{m+2} - s_k F_m$, we can write this inequality as

$$j_k \geq s(k,m), \tag{4.12}$$

where $k = 1, 2, 3...$ For odd m we use formula (4.6) and the function s_k and replace the shift S by its inverse shift S^{-1}. Repeating the above proof we return to inequality (4.12). For the inverse function s_k we again use exact formula (3.21).

Let's choose $k = b_1$. The number j_k belongs to the progression $b + a\mathbb{N}$ if and only if the inequality $j_k \geq b$ is true. Assume that $b \leq s(b_1, m)$. Using (4.12) we see that this inequality takes place. So, for any point $x \in I_+^m$ the first element $x_{j'}$ from the $S_\varepsilon(a,b)$-orbit $Orb_\varepsilon(x)$, returning to the interval I^m, has the S-number

$$j' = j_{b_1} \quad \text{if} \quad b \leq s(b_1, m). \tag{4.13}$$

Further we choose $k = a$. Then the number j_k belongs to the progression $a\mathbb{N}$ as $j_k \equiv 0 \bmod a$. By (4.11), $j_k = i_1 + i_2 + \dots + i_k \geq a$. Suppose the initial point x belongs to the interval I_-^m and $x_{j'}$ is the first element from the orbit $Orb_\varepsilon(x)$ returning to the interval I^m. Then we have

$$j' = j_a. \tag{4.14}$$

Let's consider more general restrictions for the Fibonacci numbers

$$F_{m+1} \equiv F_{m+2} \equiv F \bmod a,$$

where $1 \leq F \leq a$. In the Subsection 3.1 we proved that the coprimity of the numbers F and a follows from these congruences. If $F = a$, then $a = 1$. We denote by $(bF^{-1})_a$ a least positive solution $x \geq 1$ of the congruence $xF \equiv b \bmod a$. So, if $F = 1$, then $(bF^{-1})_a$ is equal to to the remainder $1 \leq b_1 \leq a$ as before. For $k = (bF^{-1})_a$ we have the congruence

$$j_k \equiv b_1 \bmod a.$$

Hence the S-number j_k belongs to the progression $b + a\mathbb{N}$, if $b \leq s((bF^{-1})_a, m)$. Let $x_{j'}$ be the first element from the $S_\varepsilon(a,b)$-orbit $Orb_\varepsilon(x)$ returning to the interval I^m. Here the initial point x belongs to I^m_+. Then $x_{j'}$ has the number

$$j' = j_{(bF^{-1})_a} \quad \text{if} \quad b \leq s((bF^{-1})_a, m). \tag{4.15}$$

Under the condition $F = 1$ formula (4.15) transforms to formula (4.13). If we choose an initial point x from the interval I^m_-, the S-number j' of the first point $x_{j'}$ returning to the interval I^m again satisfies equality (4.14).

Finally we consider the composite shift $S_\varepsilon(a,b)$ with integer negative parameters a, b. For the negative parameters a, b it is possible to interpret the shift $S_\varepsilon(a,b)$ as composite shift (4.7) with the positive parameters $-a$, $-b$ replacing τ by $-\tau$. Thus it is necessary to replace the simple shift S from Subsection 4.1 by its inverse shift S^{-1} and to change the order of rearrangement (4.2) and (4.5). After the specified changes formula (4.15) takes the form

$$j' = j_{(|b|F^{-1})_a} \quad \text{if} \quad |b| \leq s((|b|F^{-1})_a, m). \tag{4.16}$$

Here the coefficient F satisfies the inequalities $1 \leq F \leq |a|$. Formula (4.14) transforms to the form

$$j' = j_{|a|}.$$

Theorem 1. *Let $S_\varepsilon(a,b)$ be a composite shift (see (4.7)) of the unit interval $I = [0,1)$ with the integer parameters a, b so that $ab > 0$. Suppose for the index $m = 0,1,2...$ the Fibonacci numbers F_{m+1}, F_{m+2} satisfy the congruences*

$$F_{m+1} \equiv F_{m+2} \equiv F \bmod a, \tag{4.17}$$

where $1 \leq F \leq |a|$. Suppose $|b| \leq s((bF^{-1})_a, m)$ for the function $s(k,m)$ from (4.12). Then for the upper derivatives of the composite shift $S_\varepsilon(a,b)^{(m)}$ (see (4.8)) the following statements hold:

$$S_\varepsilon(a,b)^{(m)} \sim S_{\varepsilon'}(a, sign(b) \cdot b'), \tag{4.18}$$

if m is even. Here

$$b' = (|b|F^{-1})_a, \qquad \varepsilon' = (\varepsilon - \varepsilon_m)/(1 - \varepsilon_m) \tag{4.19}$$

are renormalisations of the parameters b, ε, $sign(b) = \pm 1$ is the sign of b, and $\varepsilon_m = 1 - \tau^{-m}$ is the eigenvalues of the B-process (see (2.2));

$$S_\varepsilon(a,b)^{(m)} \sim S_{\varepsilon'}(-a, -sign(b) \cdot b'), \tag{4.20}$$

if m is odd, ε' and b' are the same renormalisations of parameters as in (4.19). The similarity of the composite shifts in (4.18) and (4.20) means their conjugacy

$$S_\varepsilon(a,b)^{(m)} = h_m \circ S_{\varepsilon'}((-1)^m a, (-1)^m sign(b) \cdot b') \circ h_m^{-1}$$

under the affine isomorphism $h_m(x) = \tau^{-m}(x-1)+1$ of the unit interval I and the differentiation interval $I^m = [\varepsilon_m, 1)$.

Proof. Using tilings (4.2) and (4.5) we see that each returning of an initial point $x \in I^m$ to the interval I^m corresponds to rearrangement of the intervals G_0^m and E_0^m. The rearrangement is equivalent to the simple shift S or S^{-1} of the unit interval, if the level m is even or odd, respectively.
The S-numbers of the first returnings of the initial point x to the interval I^m are calculated by formulas (4.13) - (4.16). Thus, using the definition of the composite shift $S_\varepsilon(a,b)$ (see (4.7)), we obtain similarity formulas (4.18) and (4.20), and the renormalisation b' of parameter b. Renormalisation ε' (4.19) of the parameter ε and the affine isomorphism h_m follow from the choice of the differentiation interval $I^m = [\varepsilon_m, 1)$ and from the equality $\varepsilon_m = 1 - \tau^{-m}$.

□

4.3. **Self-similarity of the composite shift.** For the composite shift $S_\varepsilon(a,b)$ we consider the upper derivatives $S_\varepsilon(a,b)^{(m)}$ of the order $m = kt$, where $k = 0, 1, 2...$ and $t = t(a)$ is equal to the self-similarity period of the sequence of the Fibonacci numbers mod a (see (3.1)). By theorem 1, for $m = kt$ we have

$$S_\varepsilon(a,b)^{(kt)} \sim S_{\varepsilon'}((-1)^{kt}a, (-1)^{kt} sign(b) \cdot b^{(k)})$$

with the renormalization $b^{(k)} = (|b|f^{-1})_a$ and the self-similarity coefficient $f = f(a)$ (see (3.5)) of the sequence of the Fibonacci numbers mod a. In particular, for $k = \delta$, where $\delta = \delta(a)$ is the order of f, we obtain $b^{(\delta)} = (|b|)_a$. If we impose the restriction $1 \leq |b| \leq |a|$ for the parameter b, we can write $sign(b) \cdot b^{(\delta)} = b$. Hence,

$$S_\varepsilon(a,b)^{(T)} \sim S_{\varepsilon'}((-1)^T a, (-1)^T b).$$

Here $T = \delta t$ is the period of the sequence of the Fibonacci numbers mod a. In the case, when the period T is even, it is follows the property of *self-similarity* of the composite shift

$$S_\varepsilon(a,b)^{(T)} \sim S_{\varepsilon'}(a,b),$$

i.e. the restriction of $S_\varepsilon(a,b)$ onto the square $I^T \times I^T$ (4.8) is similar to the shift $S_\varepsilon(a,b)$. If the period T is odd, we choose $k = 2\delta$ and again come to the self-similarity

$$S_\varepsilon(a,b)^{(2T)} \sim S_{\varepsilon'}(a,b).$$

So, we have proved

Corollary 1. *The composite shift $S_\varepsilon(a,b)$ (see (4.7)) with the integer parameters a, b so that $ab > 0$ and $1 \leq |b| \leq |a|$ has the following self-similarity property*

$$S_\varepsilon(a,b)^{(m)} \sim S_{\varepsilon'}(a,b). \tag{4.21}$$

The renormalization ε' of the parameter ε is calculated by formula (4.19). The derivative order m in (4.21) is equal to T or $2T$, if the period $T = T(a)$ of the sequences of the Fibonacci numbers mod a *is even or odd, respectively.* □

If the parameter a of the composite shift $S_\varepsilon(a,b)$ is equal to a prime number $p \not\equiv 0 \ mod\ 5$, we have a simple connection between p and the period $T = T(p)$ of the sequence of the Fibonacci numbers mod p (cf. (3.1)). For $p = 5$ the value of the period $T(5)$ is specified in Table 3.1.

Corollary 2. *The upper derivatives $S_\varepsilon(p,b)^{(p)}$ of the prime order p for the composite shift $S_\varepsilon(p,b)$ (see (4.7)) with the parameters $1 \leq b \leq p$ have the following self-similarity properties :*

$$S_\varepsilon(p,b)^{(p-1)} \sim S_{\varepsilon'}(p,b)$$

for the prime numbers $p \equiv 1$ *or* $4 \bmod 5$*,*

$$S_\varepsilon(p,b)^{(2(p+1))} \sim S_{\varepsilon'}(p,b)$$

for the prime numbers $p \equiv 2$ *or* $3 \bmod 5$*, and*

$$S_\varepsilon(p,b)^{(20)} \sim S_{\varepsilon'}(p,b)$$

for $p = 5$*.* □

We know the exact formula for the period $T = T(a)$ of sequence (3.1) only for $a = F_m$. By formulas (3.13), for $m \geq 4$ the period $T(F_m)$ is equal to $2m$ if m is even, and it is equal to $4m$ if m is odd. In other cases the period $T(F_m)$ is equal to 1 for $m = 1,\ 2$ and, by Table 3.1, $T(F_m)$ is equal to 3 for $m = 3$. Thus, from formula (4.21) it follows

Corollary 3. *If* $m \geq 4$ *and* $1 \leq b \leq F_m$*, the following self-similarity properties of the composite shifts* $S_\varepsilon(F_m,b)$ *take place:*

$$S_\varepsilon(F_m,b)^{(2m)} \sim S_{\varepsilon'}(F_m,b)$$

if m *is even, and*

$$S_\varepsilon(F_m,b)^{(4m)} \sim S_{\varepsilon'}(F_m,b)$$

if m *is odd. For* $m = 2$ *the self-similarity formula*

$$S_\varepsilon(1,1)^{(2)} \sim S_{\varepsilon'}(1,1)$$

is equivalent to the formula $S|I^2 \sim S$ *(see (4.4)) for the simple shift* $S : x \mapsto x + \tau \bmod 1$*. For* $m = 3$ *we have the formula*

$$S_\varepsilon(2,1)^{(6)} \sim S_{\varepsilon'}(2,1).$$

□

Comparing Corollary 2 and Corollary 3, we see that the composite shifts $S_\varepsilon(a,b)$ with the parameter $a = F_m$ have very short periods of the self-similarity. We notice that $\delta(F_m) = 2$ (3.15), $f = F_{m-1}$, $F_{m-1}^2 \equiv 1 \bmod F_m$ for even $m \geq 4$ and $\delta(F_m) = 4$, $F_{m-1}^2 \equiv -1 \bmod F_m$ for odd m. Therefore, by theorem 1 we get the following formulas for the upper derivatives $S_\varepsilon(a,b)^{(m)}$ of the composite shift $S_\varepsilon(a,b)$ inside the self-similarity period:

$$\begin{aligned} S_\varepsilon(F_m,F_{m-1})^{(2m)} &\sim S_{\varepsilon'}(F_m,F_{m-2}), \\ S_\varepsilon(F_m,F_{m-2})^{(2m)} &\sim S_{\varepsilon'}(F_m,F_{m-1}) \end{aligned} \tag{4.22}$$

for even $m \geq 4$, and

$$\begin{aligned} S_\varepsilon(F_m,F_{m-1})^{(m)} &\sim S_{\varepsilon'}(F_m,1), \\ S_\varepsilon(F_m,1)^{(m)} &\sim S_{\varepsilon'}(F_m,F_{m-1}) \end{aligned}$$

for odd $m \geq 4$.

Generally, the composite shift $S_\varepsilon(a,b)$ is *a contracting mapping*

$$I \supset S_\varepsilon(a,b)(I) \supset \ldots \supset S_\varepsilon^k(a,b)(I) \supset \ldots$$

For any integer parameters $a,\ b,\ ab > 0$, this chain contains only a finite number of strict inclusions. *The attractor* of the composite shift $S_\varepsilon(a,b)$ is the set

$$Att(S_\varepsilon(a,b)) = \bigcap_{k=0}^{\infty} S_\varepsilon^k(a,b)(I),$$

where we denote by $S_\varepsilon^k(a,b)$ the k-multiple composition of the shift $S_\varepsilon(a,b)$. In Theorem 1 let us choose the parameters $a,b \in \mathbb{Z}$, $ab > 0$, so that a divides b. Assume that in formulaes (4.18) and (4.20) the order of upper derivatives m is divided into the self-similarity period $t = t(a)$ of sequence (3.1) of the Fibonacci numbers . Then we have the similarity

$$S_\varepsilon(a,b)^{(m)} \sim S^a \tag{4.23}$$

for even m, and

$$S_\varepsilon(a,b)^{(m)} \sim S^{-a}$$

for odd m. Here S is the simple shift of the unit interval $I = [0,1)$ from Subsection 3.3. By the above definition of attractor, the equality $Att(S^a) = I$ takes place for all $a \neq 0$. Since τ is irrational, the shift $S^a : x \mapsto x + a\tau \bmod 1$ has a minimality property, i.e. the orbit of any point $x \in I$ is dense in I. In other words, the shift S^a has not closed invariant proper subsets in I. Recall that, by definition (4.8), *the upper derivative* of the composite shift $S_\varepsilon(a,b)^{(m)}$ is a restriction of the shift $S_\varepsilon(a,b)$ and its parameter ε onto the interval $I^m = [1-\tau^{-m}, 1)$. Then from (4.23) it follows

Corollary 4. *Suppose the composite shift $S_\varepsilon(a,b)$ (see (4.7)) has the parameters a, b of the same sign, a divides b, and m is divided by the self-similarity order $t = t(a)$ of sequence (3.1) of the Fibonacci numbers* mod a . *Then for any ε from the interval $I^m = [1-\tau^{-m}, 1)$ we have*

$$I^m \subset Att(S_\varepsilon(a,b)). \tag{4.24}$$

For each point $x \in I$ and its orbit $Orb_\varepsilon(x)$ consisting of the elements $S_\varepsilon^j(a,b)(x)$, $j = 0,1,2,\ldots$, the equality

$$(Orb_\varepsilon(x) \cap I^m)^c = I^m \tag{4.25}$$

is true. In equality (4.25) we denote by the index c a closure operation .

□

4.4. **Lower derivatives of the composite shift.** It is easy to see that the composite shift $S_\varepsilon(a,b)$ (4.7) has *the symmetry*

$$S_\varepsilon(a,b)(x) = S_{inv(\varepsilon)}(-a,-b)(inv(x)) \tag{4.26}$$

that corresponds to the central symmetry

$$(x_1, x_2) \longmapsto (inv(x_1), inv(x_2))$$

of the unit square $I^c \times I^c$, where $inv(x) = 1 - x$ is an involution of the unit interval $I^c = [0,1]$. The definition of upper derivatives $S_\varepsilon(a,b)^{(m)}$ (see (4.8)) is based on the graduation of I by the differentiation intervals $I^m = [\,\varepsilon_m, 1)$ for $\varepsilon_m = 1 - \tau^{-m}$. In the graduation let us replace the intervals I^m by their images

$$I_m = inv(I^m) = (\,0, 1-\varepsilon_m] = (\,0, \tau^{-m}]$$

under the involution inv. Then equality (4.8) transforms to the definition of *lower derivatives* of the composite shift

$$S_\varepsilon(a,b)_{(m)} = S_\varepsilon(a,b)|I_m \times I_m. \tag{4.27}$$

It means a restriction of the parameters x and ε onto the interval I_m. Symmetry (4.26) connects lower derivatives (4.27) and upper derivatives (4.8) by the equality

$$S_\varepsilon(a,b)_{(m)}(x) = S_{inv(\varepsilon)}(-a,-b)^{(m)}(inv(x)). \tag{4.28}$$

Using Theorem 1 we can prove the following result for the lower derivative $S_\varepsilon(a,b)_{(m)}$.

Theorem 2. *Suppose the composite shift $S_\varepsilon(a,b)$ (see (4.7)) of the unit interval I has the integer parameters a, b of the same sign. Let the congruences*

$$F_{m+1} \equiv F_{m+2} \equiv F \bmod b \tag{4.29}$$

be true, where F_{m+1}, F_{m+2} are the Fibonacci numbers, and $1 \le F \le |b|$. Suppose $|a| \le s(a',m)$ for $a' = (|a|F^{-1})_b$ and the function $s(a',m)$ (see (4.19),(4.12)). Under these conditions the following statements hold for the lower derivatives $S_\varepsilon(a,b)_{(m)}$ (see (4.27)).

For even m,

$$S_\varepsilon(a,b)_{(m)} \sim S_{\varepsilon,}(sign(a)\cdot a', b) \tag{4.30}$$

with the renormalization

$$\varepsilon, = \varepsilon/(1-\varepsilon_m); \tag{4.31}$$

for odd m,

$$S_\varepsilon(a,b)_{(m)} \sim S_{\varepsilon,}(-sign(a)\cdot a', -b) \tag{4.32}$$

with the same renormalization $\varepsilon,$ of the parameter ε.

Proof. Let's calculate the lower derivative $S_\varepsilon(a,b)_{(m)}$ with the help of equality (4.28) and Theorem 1. In the case of an even level m, by formula (4.18), we obtain the similarity for the upper derivative in (4.28)

$$S_{inv(\varepsilon)}(-a,-b)^{(m)}(inv(x)) \sim S_{inv(\varepsilon)'}(-b, sign(-a)\cdot a')(inv(x)) \tag{4.33}$$

with the renormalization (see (4.19))

$$inv(\varepsilon)' = \frac{inv(\varepsilon)-\varepsilon_m}{1-\varepsilon_m} = 1 - \frac{\varepsilon}{1-\varepsilon_m}.$$

Thus, we get the conjugate formula

$$inv(\varepsilon)' = inv(\varepsilon,) \tag{4.34}$$

of upper renormalizations (4.19) and lower renormalizations (4.31) of the parameter $\varepsilon \in I$. Using this formula and symmetry (4.26) of the composite shift , we obtain the equality

$$S_{inv(\varepsilon)'}(-b, sign(-a)\cdot a')(inv(x)) =$$

$$= S_{inv(\varepsilon,)}(-b, sign(-a)\cdot a')(inv(x)) = S_{\varepsilon,}(sign(a)\cdot a', b).$$

On the other hand, by formula (4.28), the upper derivative in (4.33) is equal to $S_\varepsilon(a,b)_{(m)}(x)$. Therefore, for even m, similarity (4.30) follows from (4.33) and the last equality.

In the case of an odd level m the upper derivative in equality (4.28) can be calculated by formula (4.20). This completes the proof.

□

For the same reason, with the help of symmetries (4.26) and (4.28), the self-similarity property in Corollaries 1 - 3 is transferred to the lower derivatives of the composite shift $S_\varepsilon(a,b)_{(m)}$ (see (4.27)).

5. Internal derivatives and plateaus

5.1. **Lacunas.** Consider the composite shift $S_\varepsilon(a,b)$ (see (4.7)) with the integer parameters $1 \le b < a$, $(a,b)=1$. Suppose the followings congruences are fulfilled

$$F_{m+1} \equiv b \bmod a,\ F_{m+2} \equiv 0 \bmod a. \tag{5.1}$$

From the Fibonacci tiling $Til(\varepsilon_m)$ of an even level m we allocate a subset $Orb(I^m) = Orb(G_0^m) + Orb(E_0^m)$. This set is a sum of (5.2) and (5.3), where

$$G_0^m \to G_b^m \to G_{b+a}^m \to \cdots \to G_{b+na}^m \to \cdots \to G_{F_{m+1}-a}^m \tag{5.2}$$

is the orbit $Orb(G_0^m)$ of intervals G_j^m with indexes $j=0$ and $j=b+na$, $0 \le n < \dfrac{F_{m+1}-b}{a}$, and

$$E_0^m \to E_a^m \to \cdots \to E_{na}^m \to \cdots \to E_{F_{m+2}-a}^m \tag{5.3}$$

is the orbit $Orb(E_0^m)$ of intervals E_i^m with indexes $i=na$, $0 \le n < \dfrac{F_{m+2}}{a}$.

For the intervals from orbits (5.2) and (5.3) we assume that the inclusions

$$G_0^m \subset I_\varepsilon^+,\ G_j^m \subset I_\varepsilon^- \text{ for } j>0, \tag{5.4}$$

$$E_i^m \subset I_\varepsilon^- \text{ for all } i$$

are true, where $I_\varepsilon^- = [0,\varepsilon)$ and $I_\varepsilon^+ = [\varepsilon,1)$. Then from the definition of the composite shift $S_\varepsilon(a,b)$ (see (4.7)) it follows that the set $Orb(I^m)$ is the disjoint sum of the $S_\varepsilon(a,b)$-orbits of the intervals G_0^m and E_0^m. By equality (4.2) we get

$$S_\varepsilon(a,b)(Orb(I^m)) = Orb(I^m), \tag{5.5}$$

i.e. the set $Orb(I^m)$ is invariant under the transformation $S_\varepsilon(a,b)$.

A lacuna L^m is the greatest possible length interval $L^m \subset I$ that touches on the left the interval G_0^m and disjoints with the set $Orb(I^m)$. Our aim is to find the lacuna L^m of the orbit $Orb(I^m)$.

We apply an Algorithm from [10] for building the Fibonacci tilings $Til(\varepsilon_m)$. This algorithm uses quasi-local coordinates for the generating intervals .

△ From the right to the left we build the sequence

$$\ldots G_{k^*}^m, E_{k^*}^m, G_k^m, E_k^m, \ldots, G_0^m, E_0^m \tag{5.6}$$

which consists of the intervals G_l^m, E_l^m with the indexes $k = 0,1,\ldots,F_{m+2}-1$, by the following rule:
If m is even, the transition $k \mapsto k^\star$ in (5.6) is

$$k \mapsto k^\star = k + F_m \bmod F_{m+2}$$

and from sequence (5.6) all intervals G_k^m with indexes $k \ge F_{m+1}$ are deleted.
If m is odd, the transition $k \mapsto k^\star$ is

$$k \mapsto k^\star = k + F_{m+1} \bmod F_{m+2}$$

and from sequence (5.6) the intervals E_k^m with indexes $k \ge F_{m+1}$ are deleted. ▽

Using Algorithm (5.6), we can write the lacuna L^m for an even level m as the sum

$$L^m = E_{-(a-1)b}^m \oplus G_{-(a-2)b}^m \oplus E_{-(a-2)b}^m \oplus \cdots \oplus G_{-2b}^m \oplus E_{-2b}^m \oplus G_{-b}^m \oplus E_{-b}^m \tag{5.7}$$

of the intervals that touch each other. These intervals E_k^m have indexes

$$k \equiv sF_m \equiv -bs \bmod F_{m+2}, \tag{5.8}$$

where $1 \le s \le a-1$. Under condition (5.1) the parameters a, b are coprime and a divides F_{m+2}. Hence, $k \equiv -bs \not\equiv 0 \bmod a$. According to (5.6) the short intervals G_k^m from decomposition (5.7) have indexes

$$k \equiv tF_m \equiv -bt \bmod F_{m+2}, \quad k < F_{m+1}, \tag{5.9}$$

where $1 \le t \le a-2$. In this case $k \equiv -bt \not\equiv b \bmod a$, as for the same reasons the congruence $-bt \equiv b \bmod a$ is equivalent to $t \equiv a-1 \bmod a$.

From Algorithm (5.6) it follows that the interval L^m (see (5.7)) touches on the left G_0^m. We also see that L^m disjoints with the set $Orb(I^m)$ under conditions (5.4). The interval L^m has the maximal possible length with preservation of the specified properties. The lacuna L^m contains $a-1$ intervals E_k^m and

$$t(a,m) = \sharp\Big\{1 \le t \le a-2 : \Big\{\frac{tF_m}{F_{m+2}}\Big\} < \frac{F_{m+1}}{F_{m+2}}\Big\} \tag{5.10}$$

intervals G_k^m. It has the length

$$|L^m| = (a-1)|E_0^m| + t(a,m)|G_0^m|. \tag{5.11}$$

In the case of an odd level m we consider the set $Orb(I^m) = Orb(E_0^m) + Orb(G_0^m)$ consisting of the sets

$$\begin{array}{l} E_0^m \to E_b^m \to E_{b+a}^m \to \cdots \to E_{b+na}^m \to \cdots \to G_{F_{m+1}-a}^m, \\ G_0^m \to G_a^m \to \cdots \to G_{na}^m \to \cdots \to G_{F_{m+2}-a}^m \end{array} \tag{5.12}$$

that are the orbits $Orb(E_0^m)$ and $Orb(G_0^m)$. The intervals E_j^m and G_i^m have indexes (5.2) and (5.3), respectively.

Assume that the intervals from orbits (5.12) satisfy the conditions

$$\begin{array}{l} E_0^m \subset I_\varepsilon^+, E_j^m \subset I_\varepsilon^- \text{ for } j > 0, \\ G_i^m \subset I_\varepsilon^- \text{ for all } i. \end{array} \tag{5.13}$$

In these conditions the set $Orb(I^m)$ is equal to the sum of the $S_\varepsilon(a,b)$-orbits of the intervals E_0^m, G_0^m. $Orb(I^m)$ is an invariant set (see (5.5)) under the composite shift $S_\varepsilon(a,b)$.

If m is odd, the lacuna L^m is situated on the right-side of the interval E_0^m. By Algorithm (5.6) we write L^m as the sum

$$L^m = G_{-b}^m \oplus E_{-b}^m \oplus G_{-2b}^m \oplus E_{-2b}^m \oplus \cdots \oplus G_{-(a-2)b}^m \oplus E_{-(a-2)b}^m \oplus G_{-(a-1)b}^m \tag{5.14}$$

of the intervals G_0^m and E_0^m with indexes (5.8) and (5.9), respectively. Therefore, the length of L^m is equal to

$$|L^m| = (a-1)|G_0^m| + t(a,m)|E_0^m|, \tag{5.15}$$

where $t(a,m)$ is the function in (5.10).

Now let us find when inclusions (5.4) and (5.13) hold. We require an embedding of the Fibonacci tiling $Til(\varepsilon_m)$ to the unit interval $I = [0,1)$. This *embedding* depends on the parameters m and ε and is defined by the equation

$$Til(\varepsilon_m, \delta) = t_\delta(Til(\varepsilon_m)). \tag{5.16}$$

Here

$$I \xrightarrow{t_\delta} I : \ x \longmapsto x + \delta \bmod 1$$

is the simple shift of the interval I. If m is even, we choose

$$\delta = |E_0^m|, \ \varepsilon \in J^m = [1 - |G_0^m| - |L^m|, 1 - |G_0^m|). \tag{5.17}$$

In the case of odd m we assume the following restrictions on the parameters δ and ε

$$\delta = \varepsilon - (1 - |E_0^m|),\ \varepsilon \in J^m = [1 - |E_0^m| - |L^m|, 1 - |E_0^m|). \tag{5.18}$$

In both cases, according to equalities (2.6), (2.7), (5.11), and (5.15), the length of the lacuna $L^m(\delta)$ is equal to

$$|L^m(\delta)| = (a-1)\tau^{-(m+1)} + t(a,m)\tau^{-(m+2)}. \tag{5.19}$$

Embending t_δ (5.16) generates the set

$$Orb(I^m(\delta)) = Orb(G_0^m(\delta)) + Orb(E_0^m(\delta))$$

for the shifted interval $I^m(\delta) = t_\delta(I^m) = G_0^m(\delta) + E_0^m(\delta)$. It is easy to see that for $\varepsilon \in J^m$ the intervals from the orbits $Orb(G_0^m(\delta))$ and $Orb(E_0^m(\delta))$ satisfy conditions (5.4), (5.13). Hence, the specified set is an orbit of the interval $I^m(\delta)$ under the shift $S_\varepsilon(a,b)$.

Now we need to introduce the concept of *integrated transformation*

$$T_\varepsilon^\pm = \int S^\pm d_\varepsilon$$

acting on the space $\widehat{I}_\varepsilon = \{(x,i) : x \in I,\ 0 \le i < d_\varepsilon(x)\}$ by the rule

$$T_\varepsilon^\pm = \begin{cases} (x, i+1) & \text{if } i+1 < d_\varepsilon, \\ (S^\pm(x), 0) & \text{if } i+1 = d_\varepsilon. \end{cases}$$

Here $S^{\pm 1}$ is the simple shift $x \mapsto x \pm \tau \bmod 1$ of the unit interval I. *The distribution function* $d_\varepsilon : I \to \mathbb{N} = \{1, 2, \ldots\}$ is equal to

$$d_\varepsilon(x) = \begin{cases} orb(G_0^m(\delta)) & \text{if } x \in [0, \langle -\tau \rangle), \\ orb(E_0^m(\delta)) & \text{if } x \in [\langle -\tau \rangle, 1) \end{cases}$$

for even m, and

$$d_\varepsilon(x) = \begin{cases} orb(G_0^m(\delta)) & \text{if } x \in [0, \langle \tau \rangle), \\ orb(E_0^m(\delta)) & \text{if } x \in [\langle \tau \rangle, 1) \end{cases}$$

for odd m.

According to (5.2), (5.3), (5.12) the number of various intervals in the orbits $Orb(G_0^m(\delta))$ and $Orb(E_0^m(\delta))$ is equal to

$$orb(G_0^m(\delta)) = 1 + \frac{F_{m+1} - b}{a}, \qquad orb(E_0^m(\delta)) = \frac{F_{m+2}}{a}$$

for an even level m, and

$$orb(G_0^m(\delta)) = \frac{F_{m+2}}{a}, \qquad orb(E_0^m(\delta)) = 1 + \frac{F_{m+1} - b}{a}$$

for odd m, respectively.

Proposition 3. *Let $S_\varepsilon(a,b)$ be composite shift (4.7) with the integer coprime parameters $1 \le b < a$ satisfying the congruences $F_{m+1} \equiv b \bmod a$, $F_{m+2} \equiv 0 \bmod a$. Then we have the following isomorphisms*

$$S_\varepsilon(a,b)|I^m(\delta) \times J^m \simeq S^{\pm 1}, \tag{5.20}$$

$$S_\varepsilon(a,b)|Orb(I^m(\delta)) \times J^m \simeq \int S^{\pm 1} d_\varepsilon \tag{5.21}$$

for the restrictions of the composite shift $S_\varepsilon(a,b)$ for the parameter x onto the interval $I^m(\delta) = G_0^m(\delta) + E_0^m(\delta)$ and its orbit $Orb(I^m(\delta))$, and for the parameter ε onto the interval J^m (5.17), (5.18). In the right-hand side (5.20) and (5.21) there is the simple shift $S : x \mapsto x + \tau \bmod 1$ or its inverse shift $S^{-1} : x \mapsto x - \tau \bmod 1$ for m even or odd, respectively.

Proof. As above, for ε from the interval J^m (5.17), (5.18) the $S_\varepsilon(a,b)$-orbits $Orb(I^m(\delta))$ preserve the structure of orbits $Orb(I^m)$ (5.2), (5.3) and (5.12). Thus, isomorphisms (5.20) and (5.21) follow from the similarity property (see (4.4) and (4.6)) for the derivatives $S|I^m \sim S^{\pm 1}$.

□

5.2. **Spirals.** By Proposition 3, the set $Orb(I^m(\delta))$ is invariant under the action of the composite shift $S_\varepsilon(a,b)$ with the parameter $\varepsilon \in J^m$. Now we need to prove that $Orb(I^m(\delta))$ is an attraction area for the unit interval I and the dynamic distance from any point $x \in I$ to the set $Orb(I^m(\delta))$ is finite. It is enough to prove both properties for the points x from *the spiral set*

$$Spir_\varepsilon = I^m \backslash Orb(I^m(\delta)). \tag{5.22}$$

Explanation of this name for the supplementation $Spir_\varepsilon$ of the set $Orb(I^m(\delta))$ follows from the proof of the following result.

Proposition 4. *Suppose composite shift $S_\varepsilon(a,b)$ (4.7) satisfies the conditions in Proposition 3. We conjecture that the parameter ε belongs to the interval J^m (see (5.17), (5.18)). Then for any point x from the unit interval I there exists a number $k = 0, 1, 2, \ldots$ for which $S_\varepsilon^k(a,b)(x)$ visits the set $Orb(I^m(\delta))$ which is the $S_\varepsilon(a,b)$-orbit of the interval $I^m(\delta)$ from Subsection 5.1. The number k depends on the parameters ε, a, b.*

Proof. Let us recall the conditions from proposition 3 for the parameters a and b:

$$\begin{aligned} &1 \le b < a, \quad (a,b) = 1, \\ &b \le F_{m+1}, \quad a \le F_{m+2}. \end{aligned} \tag{5.23}$$

As $F_{m+1} \equiv b \bmod a$, and $F_{m+1} > 0$, we obtain that $F_{m+1} = b + ka$ for some integer $k \ge 0$. Thus, the third condition in (5.23) holds. The fourth condition in (5.23) holds, because the Fibonacci number F_{m+2} is divided themselves on a.

Lacunas L^m (5.7) and (5.14) are symmetric for even and odd levels m. Therefore, without loss of generality we can consider only the case of even m.

Using decomposition (5.7) and embedding t_δ (5.16), we write the lacuna $L^m(\delta)$ as the sum

$$L^m(\delta) = E^m_{k_{a-1}}(\delta) \oplus G^m_{k_{a-2}}(\delta) \oplus E^m_{k_{a-2}}(\delta) \oplus \cdots \oplus G^m_{k_1}(\delta) \oplus E^m_{k_1}(\delta). \tag{5.24}$$

Here the intervals $E_k^m(\delta)$ have the indexes (see (5.8))

$$k_s = sF_m = -bs \bmod F_{m+2}, \quad 1 \le s \le a-1.$$

By (5.9), we impose the additional restrictions

$$k_s < F_{m+1}, \quad 1 \le s \le a-2,$$

for the indexes of the intervals $E_k^m(\delta)$.

Now we show that any ε from interval J^m (5.17) belongs to one of the intervals $E^m_{k_s}(\delta)$ or $G^m_{k_s}(\delta)$ from decomposition (5.24). We use the induction on $s =$

$1,2,\ldots,a-1$.Assume that the parameter ε runs the intervals $E^m_{k_1}(\delta), G^m_{k_1}(\delta),\ldots,E^m_{k_{a-1}}(\delta)$ in the above order.

Let's accept the following agreement. The shift $S_\varepsilon(a,b)$ translates any point x in the interval $I^-_\varepsilon=[0,\varepsilon)$ to the point $x'\equiv x+a\tau \bmod 1$, where $a=1,2,\ldots$. Since $\tau=(1+\sqrt{5})/2$ is irrational, then the point x visits the interval $I^+_\varepsilon=[\varepsilon,1)=I\backslash I^-_\varepsilon$ through a finite number of iterations. Without loss of generality, we can assume that the point x in Proposition 4 belongs to the set

$$Spir_\varepsilon\cap I^+_\varepsilon=L^m(\delta)\cap I^+_\varepsilon,$$

where $Spir_\varepsilon$ is the spiral set (see (5.22)).

Case E. Suppose $s=1$, $\varepsilon\in E^m_{k_1}(\delta)$, and the initial point x belongs to $E^m_{k_1}(\delta)\cap I^+_\varepsilon$. According to (5.8) the index k_1 is equal to F_m . By (5.23), we obtain $k_1+b\le F_{m+2}$. The following two cases are possible. Case 1) $k_1+b<F_{m+2}$. Case 2) $k_1+b=F_{m+2}$.

In the first case the point x is translated by the shift $S_\varepsilon(a,b)$ to a point x' from the interval $E^m_{k_1'}(\delta)$ with the index $k_1'=k_1+b$. As $k_1'\equiv 0 \bmod a$, then using (5.3) we see that the point x' visits the orbit $Orb(I^m(\delta))$.

In the second case, from the tiling $G^m_0\oplus E^m_0=E^m_{F_{m+2}}\oplus G^m_{F_{m+1}}$ (see (4.2)) it follows that x is translated to a point x' in the interval $I^m(\delta)=G^m_0\oplus E^m_0$ so that x' again visits the orbit $Orb(I^m(\delta))$.

Therefore, in both cases the composite shift $S_\varepsilon(a,b)$ moves the point x through the intervals which form the chains

$$\begin{aligned}&E^m_{k_1}(\delta)\Longrightarrow E^m_{k_1'}(\delta)\subset Orb(I^m(\delta)),\\ &E^m_{k_1}(\delta)\Longrightarrow I^m(\delta)\subset Orb(I^m(\delta)).\end{aligned}\tag{5.25}$$

Here we denote by the double arrows the shift from the interval I^+_ε.

Case GE. Assume that $s=1$, $\varepsilon\in G^m_{k_1}(\delta)$, and the initial point x belongs to the set $G^m_{k_1}(\delta)\cap I^+_\varepsilon$. If the point x belongs to the previous set $E^m_{k_1}(\delta)\cap I^+_\varepsilon$ that is equal to $E^m_{k_1}(\delta)$, its orbit (5.25) does not change. Let us consider two cases 1) $k_1+b<F_{m+1}$, and 2) $k_1+b\ge F_{m+1}$.

In the first case the point x is translated to a point x' in the interval $G^m_{k_1'}(\delta)$ with the index $k_1'=k_1+b<F_{m+1}$, $k_1'\equiv 0 \bmod a$. By the condition in Proposition 4 $F_{m+1}\equiv b \bmod a$, hence there exists the least integer $t_{\min}\ge 0$ satisfying the inequalities $b+at_{\min}<F_{m+1}$ and $k_1+b+at_{\min}\ge F_{m+1}$. By (4.2), the point x' moves to a point x'' in the interval $E^m_{k_1''}(\delta)$ with the index

$$k_1''=k_1'+at_{\min}-F_{m+1}=k_1+(b+at_{\min}-F_{m+1})\le k_1.$$

From $k_1^{''}\equiv k_1 \bmod a$ we conclude that the point x'' is translated to a point x''' in the previous interval $E^m_{k_1}(\delta)$.

In the second case, by (4.2), we see that the point x' belongs to the interval $E^m_{k_1'}(\delta)$ with the index $k_1'=k_1+(bF_{m+1})\le k_1$. Hence, x' is translated to a point x'' again in the interval $E^m_{k_1}(\delta)$. In the case GE the composite shift $S_\varepsilon(a,b)$ moves the point x through the intervals which form the chains

$$\begin{aligned}&G^m_{k_1}(\delta)\Longrightarrow G^m_{k_1'}(\delta)\longrightarrow E^m_{k_1''}(\delta)\longrightarrow E^m_{k_1}(\delta),\\ &G^m_{k_1}(\delta)\Longrightarrow E^m_{k_1'}(\delta)\longrightarrow E^m_{k_1}(\delta).\end{aligned}\tag{5.26}$$

Here we denote by the double arrows and the simple arrows shifts from the intervals I^+_ε and I^-_ε, respectively.

Let us remark that proof of the induction step $s \mapsto s+1$ for $s = 1, 2, \dots, a-2$ depends on the fact

$$E^m_{k_{s+1}}(\delta) \oplus G^m_{k_s}(\delta), \quad G^m_{k_{s+1}}(\delta) \oplus E^m_{k_{s+1}}(\delta) \oplus G^m_{k_s}(\delta) \tag{5.27}$$

$$E^m_{k_{s+1}}(\delta) \oplus E^m_{k_s}(\delta), \quad G^m_{k_{s+1}}(\delta) \oplus E^m_{k_{s+1}}(\delta) \oplus E^m_{k_s}(\delta) \tag{5.28}$$

whether the lacuna $L^m(\delta)$ contains the interval $G^m_{k_s}(\delta)$ (see (5.27)) or not(see (5.28)).

In the case EG the parameter ε belongs to the interval $E^m_{k_{s+1}}(\delta)$. The initial point belongs to the intersection $E^m_{k_{s+1}}(\delta) \cap I^+_\varepsilon$. As the lacuna $L^m(\delta)$ contains the interval $G^m_{k_s}(\delta)$ (5.27), we have $k_s < F_{m+1}$ and $k_{s+1} = k_s + F_m < F_{m+2}$. Let us consider two cases 1) $k_{s+1} + b < F_{m+2}$, and 2) $k_{s+1} + b \geq F_{m+2}$.

In the first case the point x is translated to a point x' in the interval $E^m_{k'_{s+1}}(\delta)$ with the index $k'_{s+1} = k_{s+1} + b = k_s + F_m + b$. Therefore, there exists $t_{\min} \geq 0$ so that $b + at_{\min} \leq F_{m+1}$ $k_s + F_m + b + at_{\min} \geq F_{m+2}$. Hence, by (4.2), we obtain that the point x' is translated by the shift $S_\varepsilon(a, b)$ to a point x'' which belongs to the sum $G^m_{k''_{s+1}}(\delta) \oplus E^m_{k''_{s+1}}(\delta)$. Here the intervals $G^m_{k''_{s+1}}(\delta)$ and $E^m_{k''_{s+1}}(\delta)$ touch each other and have the index $k''_{s+1} = k_s + F_m + b + at_{\min} - F_{m+2}$. By the choice of $t_{\min}$, we have $k''_{s+1} = k_s + (b + at_{\min} - F_{m+1}) \leq k_s$. Since $k''_{s+1} \equiv k_s \bmod a$, the point x'' is translated to x''' in the interval sum $G^m_{k_s}(\delta) \oplus E^m_{k_s}(\delta)$.

In the second case the point x at once gets in the set $G^m_{k'_{s+1}}(\delta) \oplus E^m_{k'_{s+1}}(\delta)$, where the index $k'_{s+1} = k_{s+1} + b - F_{m+2} = k_s + (b - F_{m+1})$ is less than or equal to k_s.

In any case the initial point x moves through the following chains

$$\begin{aligned} &E^m_{k_{s+1}}(\delta) \Longrightarrow E^m_{k'_{s+1}}(\delta) \longrightarrow G^m_{k''_{s+1}}(\delta) \oplus E^m_{k''_{s+1}}(\delta) \longrightarrow G^m_{k_s}(\delta) \oplus E^m_{k_s}(\delta), \\ &E^m_{k_{s+1}}(\delta) \Longrightarrow G^m_{k'_{s+1}}(\delta) \oplus E^m_{k'_{s+1}}(\delta) \longrightarrow G^m_{k_s}(\delta) \oplus E^m_{k_s}(\delta). \end{aligned} \tag{5.29}$$

By the scheme (5.27), in the case GEG the parameter ε lies in $G^m_{k_{s+1}}(\delta)$, x lies in the set $G^m_{k_{s+1}}(\delta) \cap I^+_\varepsilon$. Since ε lies in the interval J^m (see (5.17)), then $s \leq a-3$. As in the above case the index k_{s+1} is equal to $k_s + F_m$. The existence of the interval $G^m_{k_{s+1}}(\delta)$ in the lacuna $L^m(\delta)$ implies the inequality $k_{s+1} < F_{m+1}$. Consider two cases: 1) $k_{s+1} + b < F_{m+1}$, 2) $k_{s+1} + b \geq F_{m+1}$.

In the first case the point x is translated to a point x' in the interval $G^m_{k'_{s+1}}(\delta)$ with the index $k'_{s+1} = k_{s+1} + b$. Further, we choose $t_{\min} \geq 0$ under the conditions $b + at_{\min} \leq F_{m+1}$, $k_{s+1} + b + at_{\min} \geq F_{m+1}$. Then the point x' moves to a point x'' in the interval $E^m_{k''_{s+1}}(\delta)$ with the index $k''_{s+1} = k''^{s+1} + at_{\min} - F_{m+1}$. This index satisfies the inequality $k''^{s+1} = k_{s+1} + (b + at_{\min} - F_{m+1}) \leq k_{s+1}$ and the congruence $k''_{s+1} \equiv k_{s+1} \bmod a$.

In the second case the point x' visits the interval $E^m_{k'_{s+1}}(\delta)$. By (5.24), the index $k''^{s+1} = k_{s+1} + b - F_{m+1}$ satisfies the inequality $k''^{s+1} \leq k_{s+1}$.

The moving ways of the initial point x look as following:

$$\begin{aligned} &G^m_{k_{s+1}}(\delta) \Longrightarrow G^m_{k'_{s+1}}(\delta) \longrightarrow E^m_{k''_{s+1}}(\delta) \longrightarrow E^m_{k_{s+1}}(\delta), \\ &G^m_{k_{s+1}}(\delta) \Longrightarrow E^m_{k'_{s+1}}(\delta) \longrightarrow E^m_{k_{s+1}}(\delta). \end{aligned} \tag{5.30}$$

To complete the proof of Proposition 4 we need to consider the induction step in two cases (5.28).

In the case EE we have ε in the interval $E^m_{k_{s+1}}(\delta)$. An initial point x belongs to the set $E^m_{k_{s+1}}(\delta)\cap I^+_\varepsilon$. Since the lacuna $L^m(\delta)$ does not contain the interval $G^m_{k_s}(\delta)$, we obtain $k_s \geq F_{m+1}$. Hence, the next index is equal to $k_{s+1} = k_s + F_m - F_{m+2} = k_s F_{m+1}$. The index k_{s+1} is less than or equal to F_{m+2}. Therefore, it follows $k_{s+1} = k_s - F_{m+1} < F_m$. By (5.24), we obtain that the initial point x moves to a point x' in the interval $E^m_{k'_{s+1}}(\delta)$ with the index $k'_{s+1} = k_{s+1} + b = k_s + (b - F_{m+1})$. This index is less than or equal to k_s, and $k'_{s+1} \equiv k_s \bmod a$. Thus, we conclude that the moving of the initial point x is possible only along an unique chain

$$E^m_{k_{s+1}}(\delta) \Longrightarrow E^m_{k'_{s+1}}(\delta) \longrightarrow E^m_{k_s}(\delta). \tag{5.31}$$

In the last case GEE from (5.28) it follows $\varepsilon \in G^m_{k_{s+1}}(\delta)$ and the initial point x belongs to the intersection $G^m_{k_{s+1}}(\delta)\cap I^+_\varepsilon$. For the same reasons, as in the case EE, the index k_{s+1} is less than F_m. Hence, the interval $G^m_{k_{s+1}}(\delta)$ belongs to the lacuna $L^m(\delta)$. Consider two cases: 1) $k_{s+1} + b < F_{m+1}$, and 2) $k_{s+1} + b \geq F_{m+1}$.

In the first case the point x is translated by the shift $S_\varepsilon(a,b)$ to a point x' in the interval $G^m_{k'_{s+1}}(\delta)$ with the index $k'_{s+1} = k'_{s+1} + b$. Further we choose $t_{\min} \geq 0$ under the conditions $b + at_{\min} \leq F_{m+1}$ and $k_{s+1} + b + at_{\min} \geq F_{m+1}$. Then the next point x'' lies in the interval $E^m_{k''_{s+1}}(\delta)$ with the index

$$k''_{s+1} = k'_{s+1} + at_{\min} - F_{m+1} = k_{s+1} + (b + at_{\min} - F_{m+1}) \leq k_{s+1}.$$

This index is congruent to the index $k_{s+1} \bmod a$.

In the second case we have $k''_{s+1} = k_{s+1} + (b - F_{m+1}) \leq k_{s+1}$. Hence, the moving of the initial point x is possible only along two chains

$$\begin{aligned} &G^m_{k_{s+1}}(\delta) \Longrightarrow G^m_{k'_{s+1}}(\delta) \longrightarrow E^m_{k''_{s+1}}(\delta) \longrightarrow E^m_{k_{s+1}}(\delta),\\ &G^m_{k_{s+1}}(\delta) \Longrightarrow E^m_{k'_{s+1}}(\delta) \longrightarrow E^m_{k_{s+1}}(\delta). \end{aligned} \tag{5.32}$$

By (5.25) - (5.32), we obtain the following result. For any point x in the set $L^m(\delta)\cap I^+$, the beginning of the $S_\varepsilon(a,b)$-orbit $x \to x' \to x'' \to \cdots$ does not depend on the choice of the parameter ε in the interval J^m (see (5.17)). Using this property of orbits, we complete the proof of Proposition 4.

□

5.3. **Frequency distribution and plateaus.** Suppose that ε lies in the interval J^m. By Proposition 4, we have the equality

$$Orb(I^m(\delta)) = Att(S_\varepsilon(a,b)). \tag{5.33}$$

Here $Att(S_\varepsilon(a,b))$ is the attractor of the composite shift $S_\varepsilon(a,b)$ from Section 4.3. By Proposition 4, the attractor has the property

$$Att(S_\varepsilon(a,b)) = \bigcap_{0\leq k\leq k(\varepsilon,a,b)} S^k_\varepsilon(a,b)(I).$$

Applying these equalities and Proposition 3, we obtain the following theorem.

Theorem 3. *Suppose the composite shift $S_\varepsilon(a,b)$ (see (4.7)) satisfies the conditions in Proposition 3. Suppose the parameter ε lies in the interval J^m (see (5.17), (5.18)). Then for any x, θ in the unit interval $I = [0,1)$ there exists the limit*

$$\nu_\varepsilon(\theta) = \lim_{n\to\infty} \frac{1}{n} \sharp\{k : \ S^k_\varepsilon(a,b)(x) \leq \theta,\ k = 0,1,\ldots,n-1\}$$

that is equal to the frequency of which points in the $S_\varepsilon(a,b)$-orbit of x visit the interval $[0,\theta]$. The limit $\nu_\varepsilon(\theta)$ does not depend on the choice of the initial point $x \in I$ and $\nu_\varepsilon(\theta)$ can be calculated by the formula

$$\nu_\varepsilon(\theta) = \frac{|Att(S_\varepsilon(a,b)) \cap [0,\theta]|}{|Att(S_\varepsilon(a,b))|}, \tag{5.34}$$

where $|\cdot|$ is the Haar measure on the torus $I \simeq \mathbb{R}/\mathbb{Z}$.

□

Further we shall be interested in *the diagonal frequency* $\nu_\varepsilon = \nu_\varepsilon(\theta)$. We write (5.34) as the following formula

$$\nu_\varepsilon = \frac{|Att(S_\varepsilon(a,b))^-|}{|Att(S_\varepsilon(a,b))|}, \tag{5.35}$$

where $Att(S_\varepsilon(a,b))^-$ is equal to the intersection of the attractor $Att(S_\varepsilon(a,b))$ with the interval $I_\varepsilon^- = [0,\varepsilon)$ in Section 5.1.

Consider an infinite series of the composite shifts $S_\varepsilon(F_n, F_{n-1})$ for $n \geq 3$. The sequence of the Fibonacci numbers $F_1, F_2, \ldots, F_m, \ldots \bmod F_n$ has *the period* (see (3.13))

$$T(F_n) = \begin{cases} 3, & \text{if } n = 3, \\ 2k, & \text{if } n \geq 4,\ n \text{ even}, \\ 4k, & \text{if } n \geq 4,\ n \text{ odd}. \end{cases} \tag{5.36}$$

Therefore, the congruences

$$F_{m+1} \equiv F_{n-1} \bmod F_n\ ,\ F_{m+2} \equiv 0 \bmod F_n$$

hold if and only if the level m of the Fibonacci tiling $Til(\varepsilon_m)$ belongs to the progression

$$m \equiv n-2 \bmod T(F_n). \tag{5.37}$$

The orbits $Orb(I^m(\delta))$ (see (5.2), (5.3), (5.12)), the lacunas $L^m(\delta)$ (see (5.7), (5.14)), and the internal derivatives of the composite shifts $S_\varepsilon(F_n, F_{n-1})$ (see (5.20), (5.21)) are symmetric for even and odd levels m. Therefore, without loss of generality, we can assume that m is even.

The length of the lacuna $L^m(\delta)$ is calculated by formula (5.10)

$$|L^m(\delta)| = (F_n - 1)\tau^{-(m+1)} + t(F_n, m)\tau^{-(m+2)}.$$

Here, by (5.10), the function $t(F_n, m)$ is equal to

$$t(F_n, m) = \sharp\Big\{1 \leq t \leq F_n - 2 :\ \left\{\frac{tF_m}{F_{m+2}}\right\} < \frac{F_{m+1}}{F_{m+2}}\Big\}.$$

Let us prove that the value of the function $t(F_n, m)$ does not depend on the parameter m, if m satisfy the condition in (5.37). If $m = n-2$, we arrive at the formula

$$t(F_n, n-2) = \sharp\Big\{1 \leq t \leq F_n - 2 :\ \left\{\frac{tF_{n-2}}{F_n}\right\} < \frac{F_{n-1}}{F_n}\Big\}.$$

Let us calculate $t(F_n, n-2)$ with the help of the Algorithm in Section 5.1 that builds the Fibonacci tilings $Til(\varepsilon_m)$. First, expand the interval $0 \leq t \leq F_n - 1$. Then the right part in the equality for $t(F_n, n-2)$ becomes equal to F_{n-1}. It is easy to see that the inequality for the fractional parts holds, if $t = 0$, and it does not hold, if $t = F_n - 1$. Thus, we obtain the equality

$$t(F_n, n-2) = F_{n-1} - 1$$

for $n \geq 3$.

The Fibonacci tiling $Til(\varepsilon_m)$ consists of F_{m+1} short intervals and F_{m+2} long intervals. In the paper [10] the author proved that the Fibonacci tilings $Til(\varepsilon_m)$ satisfy *the recurrence formula*

$$Til^+(\varepsilon_{m+2}) = \tau^{-1} \cdot \overline{Til^+(\varepsilon_{m+1})} \oplus \tau^{-2} \cdot Til^+(\varepsilon_m)$$

of the sort as for the Fibonacci numbers F_m, i.e. $F_{m+2} = F_{m+1} + F_m$. The line over $\overline{Til^+(\varepsilon_{m+1})}$ indicates the inversion of designations for long and short intervals. We denote by $\oplus$ the non-commutative operation that puts the first tiling to the second tiling compressed by τ^{-1} and τ^{-2} times, respectively.

By the construction of lacunas L^m, the Algorithm in Section 5.1, and the recurrence formula, we get that the function $t(F_n, m)$ does not depend on the parameter m in (5.37). Using the specified invariancy and the equality for the function $t(F_n, n-2)$, we obtain the formula

$$t(F_n, m) = F_{n-1} - 1. \tag{5.38}$$

This formula takes place for all $n \geq 3$ and $m \equiv n-2 \bmod T(F_n)$. With the help of formula (5.38), we find the lacuna length

$$|L^m(\delta)| = \tau^{-m}(\tau^{n-2} - 1). \tag{5.39}$$

By the definition of the orbit $Orb(I^m)$ (see (5.2), (5.3)), we can write its measure as

$$|Orb(I^m(\delta))| = |G_0^m(\delta)| \cdot orb(G_0^m(\delta)) + |E_0^m(\delta)| \cdot orb(E_0^m(\delta)),$$

where the factors

$$orb(G_0^m(\delta)) = 1 + \frac{F_{m+1} - F_{n-1}}{F_n}, \quad orb(E_0^m(\delta)) = \frac{F_{m+2}}{F_n}$$

are equal to the numbers of intervals in the orbits $Orb(G_0^m(\delta))$ and $Orb(E_0^m(\delta))$, respectively. The lengths $|G_0^m(\delta)| = |G_0^m|$and $|E_0^m(\delta)| = |E_0^m|$ can be calculated by formulas (2.6). By equality (5.33), we find the attractor measure

$$|Att(S_\varepsilon(F_n, F_{n-1}))| = \frac{1}{F_n}(1 + F_{n-2}\tau^{-(m+2)}) \tag{5.40}$$

for the composite shift $S_\varepsilon(F_n, F_{n-1})$, if its parameter ε lies in the interval (see (5.17))

$$J^m = [1 - \tau^{-(m+2)} - \tau^{-m}(\tau^{n-2} - 1), 1 - \tau^{-(m+2)}). \tag{5.41}$$

Using the equal

$$Att(S_\varepsilon(F_n, F_{n-1}))^- = Att(S_\varepsilon(F_n, F_{n-1}))\backslash\{G_0^m(\delta)\}$$

and (5.40), we have

$$\begin{aligned} |Att(S_\varepsilon(F_n, F_{n-1}))^-| &= \frac{1}{F_n}(1 + F_{n-2}\tau^{-(m+2)}) - \tau^{-(m+2)} = \\ &= \frac{1}{F_n}(1 - F_{n-1}\tau^{-(m+2)}). \end{aligned} \tag{5.42}$$

Proposition 5. *Let $S_\varepsilon(F_n, F_{n-1})$ be composite shift (4.7). Suppose the parameter ε lies in the interval J^m (see (5.41)), where $m \equiv n-2 \bmod T(F_n)$, $n \geq 3$, and $T(F_n)$ is the period determined by formula (5.36). Then for all initial points x in the unit interval $I = [0,1)$ the frequency ν_ε of which points in the $S_\varepsilon(F_n, F_{n-1})$-orbit of x visit the interval $I_\varepsilon^- = [0,\varepsilon)$ is calculated by the formula*

$$\nu_\varepsilon = \frac{1 - F_{n-1}\tau^{-(m+2)}}{1 + F_{n-2}\tau^{-(m+2)}} \tag{5.43}$$

or by the formula

$$\nu_\varepsilon = 1 - \frac{\tau^{-(m+2)}}{|Att(S_\varepsilon(F_n, F_{n-1}))|} \tag{5.44}$$

which connects the frequency ν_ε with the attractor measure $|Att(S_\varepsilon(F_n, F_{n-1}))|$ (see (5.40)).

Proof. Substituting equality (5.40) and the second equality from (5.42) in formula (5.35) for the diagonal frequency ν_ε, we prove formula (5.43). Using (5.40) and the first equality from (5.42), we get formula (5.44).

□

Using formulae (5.43) and (5.44), we get

Corollary 5. *Under the conditions in Propositon 5, the attractor measure $|Att(S_\varepsilon(F_n, F_{n-1}))|$ is equal to*

$$|Att(S_\varepsilon(F_n, F_{n-1}))| = \frac{\tau^{-(m+2)}}{1-\nu_\varepsilon}.$$

So (cf. (5.40))

$$|Att(S_\varepsilon(F_n, F_{n-1}))| \to \frac{1}{F_n}$$

as $\varepsilon \to 1$ and ε lies in the union of the intervals J^m.

Consider two special cases of formula (5.43): 1) $m - n - 2$ and 2) $n = 3$.

Corollary 6. *Suppose the parameter ε of the composite shift $S_\varepsilon(F_n, F_{n-1})$, $n \geq 3$, lies in the interval $J^{n-2} = [\tau^{-(n-1)}, 1 - \tau^{-n})$. Then the frequency ν_ε does not depend on the number n of the shift series and ν_ε is equal to*

$$\nu_\varepsilon = \tau^{-1} = 0.618\ldots \tag{5.45}$$

Corollary 7. *Suppose the parameter ε of the composite shift $S_\varepsilon(2,1)$ lies in the interval $J^m = [1 - \tau^{-m}, 1 - \tau^{(m+2)})$ of the level $m \equiv 1 \bmod 3$, $m \geq 1$. Then the frequency ν_ε is calculated by the following formula (see Fig.2)*

$$\nu_\varepsilon = th_\tau\Big(\frac{m}{2} + 1\Big). \tag{5.46}$$

Here the function $th_\tau(x) = (\tau^x - \tau^{-x})/(\tau^x + \tau^{-x})$ is the hyperbolic tangent with the basis equal to the golden section $\tau = (1+\sqrt{5})/2$.

6. Spectral decomposition and singularities

6.1. **The composite shift** $S_\varepsilon(3,1)$**.** We apply the differentiation method from Subsections 4.4 to researching the structure of the composite shift $S_\varepsilon(3,1)$. By Corollary 2, we have the self-similarity property

$$S_\varepsilon(3,1)^{(8)} = S_\varepsilon(3,1)|I^8 \times I^8 \sim S_{\varepsilon'}(3,1), \tag{6.1}$$

where $I^8 = [\varepsilon_8, 1)$ and $\varepsilon_8 = 1 - \tau^{-8}$. Therefore, without loss of generality, we can consider a behaviour of the shift $S_\varepsilon(3,1)$ only for ε from the fundamental area $F = [0, \varepsilon_8)$ of the unit interval $I = [0,1)$. Various types of behaviour of the shift $S_\varepsilon(3,1)$ correspond to the intervals $F_1, \ldots, F_4$ from *the tiling*

$$F = F_1 \oplus F_2 \oplus F_3 \oplus F_4 =$$
$$= [0, \tau^{-1}) \oplus [\tau^{-1}, \varepsilon_4) \oplus [\varepsilon_4, \varepsilon_4 + \tau^{-7}) \oplus [\varepsilon_4 + \tau^{-7}, \varepsilon_8).$$

Since the congruences $F_7 \equiv 1 \bmod 3$, $F_8 \equiv 0 \bmod 3$ are fulfilled, we apply Proposition 3 for even $m = 6$ to the shift $S_\varepsilon(3,1)$. Accordingly (5.17) the interval J^6 coincides with F_4. Hence, by formula (5.21), we obtain *the isomorphism*

$$S_\varepsilon(3,1)|Orb(I^6(\delta)) \times F_4 \simeq \int S\, d_\varepsilon. \tag{6.2}$$

In the integral there is *the distribution function*

$$d_\varepsilon = \begin{cases} 5, & \text{if } x \in [0, \langle -\tau \rangle), \\ 7, & \text{if } x \in [\langle -\tau \rangle, 1). \end{cases}$$

For any $\varepsilon \in F_4$ and any initial point $x \in I$ we denote by ν_ε the frequency that points from the $S_\varepsilon(3,1)$-orbit of x visit the interval $I_\varepsilon^- = [0, \varepsilon)$. By Theorem 3, the frequency ν_ε is calculated by the formula

$$\nu_\varepsilon = \frac{|Att(S_\varepsilon(3,1))^-|}{|Att(S_\varepsilon(3,1))|},$$

Using equality (5.33) and formulas (2.6), we see that the above measures are equal to

$$\begin{aligned} |\ Att(S_\varepsilon(3,1))\ | &= |Orb(I^6(\delta))| = 5\tau^{-8} + 7\tau^{-7}, \\ |Att(S_\varepsilon(3,1))^-| &= 4\tau^{-8} + 7\tau^{-7}. \end{aligned}$$

Hence, for these ε and x the frequency ν_ε is constant. It is equal to

$$\nu_\varepsilon = 0.9397\ldots \tag{6.3}$$

By Theorem 1 for the level $m = 4$ and Theorem 2 for the level $m = 3$, we calculate the mixed derivative

$$S_\varepsilon(3,1)|\, F_3 \times F_3 = (S_\varepsilon(3,1)^{(4)})_{(4)} \sim S_{\varepsilon"}(-1,-2) \tag{6.4}$$

of the composite shift $S_\varepsilon(3,1)$.

Let us show that the behaviour of the shift $S_\varepsilon(3,1)$, when the parameter ε belongs to the intervals F_1 and F_2, differs from the above behaviour of the shift on the intervals F_3 and F_4, .

If $\varepsilon \in F_2$, we use the embedded Fibonacci tiling $Til(\varepsilon_2, \delta)$ of the level $m = 2$ (see (5.16)), where the shift δ is equal to $|E_0^2| = \tau^{-3}$. We find the internal derivative

$$S_\varepsilon(3,1)|Att(S_\varepsilon(3,1)) \times F_2 \ \sim\ S\Big(\frac{\tau}{2}\Big). \tag{6.5}$$

Here

$$\begin{aligned} &Att(S_\varepsilon(3,1)) = Til(\varepsilon_2,\delta)\backslash\{E_1^2(\delta)\} = I\backslash F_2,\\ &Spir_\varepsilon = \{E_1^2(\delta)\} = F_2, \end{aligned}$$

and $S\left(\dfrac{\tau}{2}\right)$ is the shift of the unit interval $I \to I : x \mapsto x + \frac{\tau}{2} \bmod 1$. Since

$$\begin{aligned} &|Att(S_\varepsilon(3,1))| = 2\tau^{-3} + 2\tau^{-4},\\ &|Att(S_\varepsilon(3,1))^-| = 2\tau^{-3} + \tau^{-4}, \end{aligned}$$

the frequency ν_ε does not depend on the parameter ε from the interval F_2, and ν_ε is equal to

$$\nu_\varepsilon = 0.809\ldots \tag{6.6}$$

In the case $\varepsilon \in F_1$, we apply the tiling $Til(\varepsilon_1,\delta)$ of the level $m = 1$ with $\delta = -|GE_0^1| = -\tau^{-2}$. Then we obtain

$$\begin{aligned} &S_\varepsilon(3,1)|F_1 \times F_1 \sim S_{\varepsilon'}(-2,-1),\\ &S_\varepsilon(3,1)|\, I \times F_1 \;\simeq\; \textstyle\int S_{\varepsilon'}(-2,-1)\, d_{\varepsilon'}. \end{aligned} \tag{6.7}$$

Here $x \in I$, $\epsilon \in F_1$, $\varepsilon' = \tau \cdot \varepsilon$ is *a renormalization* of the parameter ε, and $d_{\varepsilon'} :\ I \times I \to \mathbb{N}$ is *the distribution function*

$$d_{\varepsilon'}(x) = \begin{cases} 1 & \text{if } (x,\varepsilon') \in I \times I\backslash D,\\ 2 & \text{if } (x,\varepsilon') \in D, \end{cases}$$

where D is a non-crossing sum of two rectangulars

$$[0,\tau^{-1}) \times [0,\tau^{-3}),\ [0,\tau^{-3}) \times [\tau^{-3},1)$$

and the triangle

$$\{(x,\varepsilon') :\ x \in [\varepsilon',\tau^{-1}),\ \varepsilon' \in [\tau^{-3},\tau^{-1})\}.$$

6.2. **Spectrum.** For any composite shift $S_\varepsilon(a,b)$ (see (4.7)) we define the function $\mu(\nu)$ for $\nu \in [0,1)$ by the equality

$$\mu(\nu) = |\{\varepsilon \in [0,1) :\ \nu_\varepsilon = \nu\}|.$$

The spectrum $sp(a,b)$ is the set of all $\nu \in [0,1)$ for which *the weights* $\mu(\nu) > 0$.

It is possible to show that for the composite shift $S_\varepsilon(2,1)$ formula (5.46) includes all spectrum $sp(2,1)$. This spectrum consists of the points

$$\nu^m = th_\tau\Big(\frac{m}{2} + 1\Big)$$

with the numbers $m \equiv 1 \bmod 3$, $m \geq 1$, and the weights $\mu^m = \tau^{-(m+1)}$. Thus, each spectral point $\nu^m \in sp(2,1)$ corresponds to *a plateau* of the frequency function ν_ε. The weight $\mu^m = \mu(\nu^m)$ is equal to the length of an interval for which the function ν_ε is constant. Taking into account that

$$\nu^m \sim 1 - \tau^{-(m+2)} \quad \text{as} \quad m \to \infty,$$

we obtain *the asymptotic self-similarity property* for the graph of the function $\mu(\nu)$ (see Fig. 3). It means that the function $\mu(\nu)$ is asymptotic invariant under the homothetic transformation with center at the point $(\nu,\mu) = (1,0)$ and ratio τ^{-3} . We write this self-similarity property of the shift $S_\varepsilon(2,1)$ by *the symbol*

$$sp(2,1) = (\,|). \tag{6.8}$$

Similarly to (6.8), from (6.3), (6.4), (6.6), and (6.7) it follows that the composite shift $S_\varepsilon(3,1)$ has the spectral decomposition

$$sp(3,1) = (|\ (2,1)|(-1,-2)\ |).$$

Here the first vertical line corresponds to a limiting point of the spectrum $sp(3,1)$, the second line corresponds to its discrete point (see Fig. 5).

Here is the list of spectral decompositions for composite shifts $S_\varepsilon(a,b)$ with the integer coprime parameters a, b, satisfying the conditions $1 \le b < a \le 4$:

$$\begin{aligned}
&sp(3,1) = (|\ (2,1)|(-1,-2)\ |),\\
&sp(3,2) = (-1,-2)\ |\ (3,1),\\
&sp(3,2) = ((-1,-2)\ |\ (2,1)|),\\
&sp(4,1) = ((2,1)|(-1,-2)\ |),\\
&sp(4,3) = ((2,3)|(-1,-2)\ |\ (2,1)|(3,2)|).
\end{aligned} \tag{6.9}$$

The above list does not contain the composite shift $S_\varepsilon(4,2)$ with not coprime parameters $a = 4$, $b = 2$. The shift $S_\varepsilon(4,2)$ can be interpreted as a composite shift with the coprime parameters $a = 2$, $b = 1$. For this aim we need to replace τ with 2τ in definition (4.7). The shift $S_\varepsilon(4,2)$ belongs to the α-family of composite shifts

$$\begin{aligned}
S_\varepsilon(a,b)(x) &= x + a\alpha \bmod 1 \quad \text{if} \quad x \in I_\varepsilon^-,\\
S_\varepsilon(a,b)(x) &= x + b\alpha \bmod 1 \quad \text{if} \quad x \in I_\varepsilon^+,
\end{aligned} \tag{6.10}$$

where $\alpha = 2\tau$. To research these composite shifts we require a tiling Til of the unit interval $I = [0,1)$ that corresponds to the irrationality $\alpha = 2\tau$ (cf. Section 2).

6.3. **Singularities.** Let $sp(a,b)^c$ be *a closure* of the spectrum $sp(a,b)$ by a topology of the tours $I \sim \mathbb{R}/\mathbb{Z}$. For example, the set $sp(2,1)^c$ (see (6.8)) consists of the discrete points ν^m and one limiting point $s \equiv 0 \bmod 1$. The last point is *a singularity* of the degree $deg(s) = 1$. Discrete points of a spectrum we consider as singularities of zero degree.

The closure $sp(3,1)^c$ (see (6.9)) is more difficultly arranged. It includes the discrete points of the spectrum $sp(3,1)$ ($\nu = 0.9387\ldots$ (6.3), and others), the singularities of the first degree ($\nu = 0.809\ldots$ (6.6) etc.), and the singularity $s \equiv 0 \bmod 1$ of the second degree. By self-similarity property (6.1), the singularity of the second degree is a limiting point of the singularities of the degree 1.

Let's define *the degree* of a singularity by the induction on deg. Then we see that the singularity $s \equiv 0 \bmod 1$ in the spectrum closure $sp(4,1)^c$ (see (6.9)) has the degree 3.

The degree $deg(a,b)$ of the spectrum $sp(a,b)$ is equal to the maximal degree of singularities of the composite shift $S_\varepsilon(a,b)$. For the composite shift $S_\varepsilon(a,b)$, the degree $deg(a,b)$ characterizes complexity of the spectral decomposition (see (6.9) or the length of monotone chains

$$S_\varepsilon(a_1,b_1) \hookrightarrow S_\varepsilon(a_2,b_2) \hookrightarrow \cdots \hookrightarrow S_\varepsilon(a_g,b_g) \hookrightarrow S_\varepsilon(a,b).$$

Proposition 6. *Let $S_\varepsilon(a,b)$ be a composite shift (see (4.7))with the integer coprime parameters $1 \le b < a \le F_{n+2}$. Then for any $n = 1,2,3,\ldots$ there exists the shift $S_\varepsilon(a,b)$ with the spectral degree*

$$deg(a,b) \ge n.$$

Remark 1. *In Proposition 6 the lower estimate $deg(a,b) \ge n$ is not exact, because the composite shift $S_\varepsilon(5,1)$ has the degree $deg(5,1) = 4$.*

Proof. Let us prove that the composite shift $S_\varepsilon(a, b)$ with the parameters $a = F_{n+2}$, $b = F_{n+1}$ has the spectral degree $deg(a, b) \geq n$. By Theorem 2 we calculate the lower derivative

$$S_\varepsilon(F_{n+2}, F_{n+1})_{(2T(F_{n+1}))} \sim S_{\varepsilon'}(F_n, F_{n+1}) \tag{6.11}$$

with the order equal to the double period $T(F_{n+1})$ (see (5.36)) of the sequence of the Fibonacci numbers mod F_{n+1}. Then, applying Theorem 1, we get the upper derivative

$$S_{\varepsilon'}(F_n, F_{n+1})^{(2T(F_n))} \sim S_{\varepsilon''}(F_n, F_{n-1}).$$

In the same way we come to one of the shifts $S_\varepsilon(F_2, F_3)$ or $S_\varepsilon(F_3, F_2)$, if n is even or odd, respectively. By Corollary 1, the initial shift $S_\varepsilon(F_{n+2}, F_{n+1})$ has the self-similarity property

$$S_\varepsilon(F_{n+2}, F_{n+1})^{(2T(F_{n+2}))} \sim S_{\varepsilon'}(F_{n+2}, F_{n+1}).$$

Using this property and (6.11), we come to the following conclusion: if the derivative shift $S_{\varepsilon'}(F_n, F_{n+1})$ has a degree d_{n+1}, the initial shift $S_\varepsilon(F_{n+2}, F_{n+1})$ has the degree

$$d_{n+2} \geq d_{n+1} + 1.$$

The composite shift $S_\varepsilon(F_2, F_3) = S_\varepsilon(1, 2)$ and its inverse shift $S_\varepsilon(F_3, F_2) = S_\varepsilon(2, 1)$ have the degree 1. Therefore, by the induction on n, we complete the proof of the proposition.

□

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