HOLOMORPHIC CONFORMAL STRUCTURES AND UNIFORMIZATION OF COMPLEX SURFACES

by

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0. Introduction

The smooth compact complex surfaces with a holomorphic conformal structure are completely classified in S. Kobayshi-Ochiai [Kob-O1]. The importance of such surfaces is clear if one pays attension to the fact that the compact free quotient of the bidisc are included in this category. However it seems to be in general a difficult problem to construct directly examples of such surfaces in a geometric way. One idea is to construct singular holomorphic structures and to ask then whether there would be a desingularization procedure for them. This paper attempts to present a method of the construction of holomorphic conformal structures along this idea and to introduce some non-empty spaces of singular conformal structures on $P_2(\mathbb{C})$ which are desingularizable. In [H1,2], Hirzebruch showed that some Hilbert modular surfaces are obtained by taking the double covering of $P_2(\mathbb{C})$ ramifying exactly along some rigid and symmetric curve configurations. Therefore it will be an interesting problem to construct directly the holomorphic conformal structure on $P_2(\mathbb{C})$ induced from such Hilbert modular surfaces (see section six). The plan of the present paper is as follows. In the first section we review the theory of characterization of surfaces uniformazed by symmetric domains. Our viewpoint is Einstein-Kähler metrics and holomorphic G-structures. In the second section we introduce the notion of generalized holomorphic conformal structure on a smooth compact complex surface. The third section contains the study of complex V-surfaces with at worst isolated quotient singularities with a complete Einstein-Kähler orbifold-metric of negative Ricci curvature and a logarithmic orbifold-holomorphic conformal structure. This will be an "equivariant and logarithmic" version of Theorem (4.1) of [Kob-O1] in the case of dimension two. The fourth section contains the desingularization procedure of generalized holomorphic conformal structure, which is the main idea of this paper. In the fifth section we discuss on generalized holomorphic conformal structure on $P_2(\mathbb{C})$. Examples are presented in the final section, which contain some Hilbert modular surfaces and compact surfaces with rational double points satisfying $\bar{c}_1^2 = 2\bar{c}_2$ in the modified sense (see [Ko] and [M]) but not covered by the bidisc. Here for compact surface of general type X and its canonical model X', $\bar{c}_1$ and $\bar{c}_2$ are defined as

$$\bar{c}_1(X) = c_1(X)$$
$$\bar{c}_2(X) = c_2(X) - \sum_{p\ \mathrm{Sing}\in X'} \left(e(Ep) - \frac{1}{|G(p)|} \right)$$

where Ep is the exceptional divisor of the minimal resolution of $p \in \mathrm{Sing}\ X'$, $e(Ep)$ is the Euler number of Ep and $|G(p)|$ is the order of the local fundamental group $G(p)$ of $p \in \mathrm{Sing}\ X'$. We include also a Hilbert modular surface which can be obtained simply by modifying $P_2(\mathbb{C})$ in the birational category, showing that the formualtion in section 5 is applicable even to such a case.

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1. Complex surfaces uniformized by symmetric domains

Compact complex surfaces uniformized by the symmetric domain Ω satisfy

(1) $$c_1^2 = 3c_2$$

or

(2) $$c_1^2 = 2c_2$$

according as Ω is the ball $\mathbb{B}^2$ or the bidisc $D \times D$.

Now let X be a compact complex surface of general type and X' its canonical model. Then there exists a unique (up to constant multiplication) Einstein-Kähler orbifold metric on X' [Ko]. By using the curvature tensor of this canonical metric, one can compute the generalized Miyaoka-Yau characteristic (see [Ko] and [M]), namely

$$c_2 - \frac{1}{3} c_1^2 - \sum_{p \in \mathrm{Sing}\, X'} \left(e(E_p) - \frac{1}{|G(p)|} \right) = \frac{1}{8\pi^2} \int_{X'} |W_-|^2$$

which is clearly non-negative, where E_p is the exceptional set of the minimal desingularization and $e(E_p)$ is the Euler number of E_p and $G(p)$ is the binary polyhedral group corresponding to the singularity $p \in X'$ and W_- is the anti-self-dual part of the Weyl tensor of the canonical Einstein-Kähler orbifold-metric. Therefore, if X as above satisfies the equality (1) then X' must coincide with X, and $W_- \equiv 0$. This means that X is uniformized by $\mathbb{B}^2$. Combining this with the results in [Kob-O2], we obtain the following three equivalent statements for any compact complex surface X of general type:

(i) $c_1^2(X) = 3c_2(X)$,

(ii) X admits a holomorphic projective connection,

(iii) X is uniformized by $\mathbb{B}^2$.

Now let us consider the similar question for a surface X of general type satisfying one of the following conditions:

(i)' $c_1^2(X) = 2c_2(X)$,

(ii)' X admits a holomorphic conformal structure,

(iii)' X is uniformized by $D \times D$.

There are implications (iii)' $\Rightarrow$ (i)' , (iii)' $\Rightarrow$ (ii)' and (ii)' $\Rightarrow$ (iii)'. We should note that S. Kobayashi and Ochiai [Kob-O1] showed that the existence of a holomorphic conformal structure implies the ampleness of the canonical bundle K_X of X. The equivalence (ii)' $\Leftrightarrow$ (iii)' for any compact complex surface of general type opens a way to construct directly examples of surfaces uniformized by $D \times D$. The difference from the care of $\mathbb{B}^2$ is that the equality (i)' seems to be insufficient to conclude the validity of (ii)' and (iii)'. B. Hunt pointed out that there is no hope to characterize the compact quotients of bounded domains of rank ≥ 2 and complex dimension ≥ 3 only in terms of c_1 and c_2. In fact, $c_1 c_2 = 24$ and $c_1^3 = 54$ for both $P_1(\mathbb{C}) \times P_2(\mathbb{C})$ and $Q_3(\mathbb{C})$.

The following discussion is completely independent of the former but the same point of view will appear in the final section of this paper (see Example 5). Now let X be a compact surface of general type and X' its canonical model. Assume that

$$c_2 - \frac{1}{3} c_1^2 - \sum_{p \in \mathrm{Sing}\, X'} \left(e(E_p) - \frac{1}{|G(p)|} \right) = 0.$$

This means $W_- \equiv 0$ for the canonical Einstein-Kähler orbifold-metric on X'. So, X' is uniformized by $\mathbb{B}^2$. This would probably be understood as follows: From the theory of negligeable singularities ([B]) there is a one-parameter family of compact complex surfaces $\{X_t\}_{t \in D}$ such that $X_0 = X$ and the canonical bundle K_{X_t} of X_t is ample for $t \neq 0$. By the theorem of Aubin [A] and Yau [Y1], there is a unique Einstein-Kähler metric g_t on X_t representing $c_1(K_{X_t})$. As t tends to zero, the anti-self-dual Weyl curvature tensor W_- will localize to (-2)-curves on X and off these (-2)-curves g_t will converge to the canonical Einstein-Kähler orbifold metric on X which is half-conformally flat $W_- \equiv 0$. We will find an example in which this kind of argument is valid in the final section. In the case of the deformation of K3 surfaces with rational double points, the above argument of the convergence of Einstein-Kähler metrics is reasonably justified [Ko-T].

2. Holomorphic conformal structures in dimension 2 .

Definition. A generalized holomorphic conformal structure defined by a holomorphic line bundle L over a smooth compact complex surface X is a primitive holomorphic section τ of $L \otimes S^2T^*(X)$. Here the primitiveness means that, at the germ level, τ is not divisible by any non-units in the structure sheaf of X.

We give a local expression of a generalized holomorphic conformal structure in the following way. Let $\{\Omega_\alpha\}$ be an open Stein covering of X by coordinate neighborhood and assume $H^2(\Omega_\alpha;\mathbb{Z}) = \{0\}$, $\forall_\alpha$. For a generalized holomorphic conformal structure we can find a system of holomorphic sections $\tau_\alpha \in \Gamma(\Omega_\alpha, S^2T^*(X))$ such that if $\tau_\alpha = u\tau'_\alpha$ for $u \in \Gamma(\Omega_\alpha, \mathcal{O})$ and $\tau'_\alpha \in \Gamma(\Omega_\alpha, S^2T^*(X))$ then $u \in \Gamma(\Omega_\alpha, \mathcal{O}^*)$, and

$$\tau_\alpha \doteq h_{\alpha\beta}\,\tau_\beta$$

where $h_{\alpha\beta} \in \Gamma(\Omega_\alpha \cap \Omega_\beta, \mathcal{O}^*)$ and the cocycle $\{h_{\alpha\beta}\}$ defines the holomorphic line bundle L.

<u>Definition.</u> The <u>discriminant divisor</u> D for a generalized holomorphic conformal struture $\{\tau_\alpha\}$ is the divisor on X defined by $\{\det \tau_\alpha\} = 0$, where τ_α is locally given by $\tau_\alpha = \sum_{ij=1}^{2} g_{\alpha ij} dz_\alpha^i \, dz_\alpha^j$ and $\det \tau_\alpha = \det (g_{\alpha ij})$. The divisor D does not depend on the choice of τ_α's. By taking the determinant of

$$ {}^t\!\left(\frac{\partial (z_\alpha)}{\partial (z_\beta)}\right) \left(g_{\alpha ij}\right)\left(\frac{\partial (z_\alpha)}{\partial (z_\beta)}\right) = h_{\alpha\beta}(g_{\beta ij}) \ , $$

we get

$$ [D] = 2(L + K). $$

Thus we can consider a double covering $\tilde{X}$ of X branched exactly over D (we have 2^q such coverings where q denotes the irregularity of X. If one wants $\tilde{X}$ to depend functorially on X, then one may take the double covering associated with $L + K$).

<u>Definition.</u> If the discrimiant divisor D is empty, i.e., $\det \tau_\alpha$ is non-vanishing everywhere, the generalized holomorphic conformal structure $\tau = \{\tau_\alpha\}$ is called <u>non-degenerate</u> or simply a <u>holomorphic conformal structure</u> [Kob-O1].

On the other hand, there are cases where some components of D appear with multiplicity 2. For example, let X be a Hilbert modular surface with cusps and (y,E) its minimal resolution where E is the exceptional divisor. Then the holomorphic conformal structure on $Y - E$ extends as a generalized one on Y with the discriminant $2E$, i.e., every component of E appears with multiplicity 2. See Example 6.4 in Section 6.

3. Einstein-Kähler surfaces with a generalized holomorphic conformal structure

In [Kob-O1], S. Kobayashi and Ochiai proved the following

Theorem ([Kob-O1]) Let X be a compact Einstein-Kähler smooth surface admitting a holomorphic conformal structure. Then X is either $P_1(\mathbb{C}) \times P_1(\mathbb{C})$, or flat, or covered by $D \times D$ according as the Ricci curvature is positive, 0 or negative.

We will use the logarithmic-orbifold version of this theorem. Since the arguments given by S. Kobayshi-Ochiai are typical in concluding that the given metric is locally symmetric, we give here an outline of their proof of the above theorem. We assume $\mathrm{Ric} \neq 0$. Since the given holomorphic conformal structure is non-degenerate, we have a holomorphic section σ of

$$K^{-2} \otimes S^2T^*(X) \subset T_4^4(T(X))$$

by symmetrizing $g^2 = g \otimes g$. Since the metric is Einstein-Kähler we are able to use Bochner's vanishing theorem in an effective way to conclude that σ is covariantly constant. The existence of this covariantly constant object causes reduction on the holonomy group and it follows that X must be locally symmetric.

To formulare a logarithmic-orbifold version of the above theorem, we need to introduce the following:

Definition. Let X be a smooth surface and D a reduced divisor with normal crossings. A generalized holomorphic conformal structure τ behaves logarithmically near D if τ is given locally by

$$y^2P(dx)^2 + 2xyQ(dx)(dy) + x^2R(dy)^2$$

with $Q^2 - PR \neq 0$ in coordinates (x,y) such that D is given by $xy = 0$.

A logarithmic-orbifold version may be formulated as follows, for example.

Theorem 1 Let $\overline{X}$ be a compact complex surface with at worst quotient singularities and E a divisor lying in the regular part of $\overline{X}$. Assume that E consists of the exceptional sets for Hilbert modular cusps and that $\overline{X} - E$ admits a complete Einstein-Kähler orbifold-metric with negative Ricci curvature with quasi bounded geometry and equivalent to the bidisc metric near cusps. If $\overline{X} - E$ admits an orbifold-holomorphic conformal structure which behaves logarithmically near E, then $\overline{X} - E$ is uniformized by the bidisc.

If Z is a (possibly non-compact) surface with at worst isolated quotient singularities, then an orbifold-holomorphic conformal structure on Z means such a conformal structure on Reg (Z) that any local uniformization at any singular point lifts and extends it to the point over the singularity. For the definition of the quasi-bounded geometry of complete Kähler metric, see for example [Ko].

Proof of Theorem 1 We prove the theorem in the special case that $\overline{X}$ is non-singular. The given holomorphic sonformal structure on $\overline{X} - E$ extends to a generalized one on $\overline{X}$ with the discriminant $D = 2E$, i.e., the discriminant is E with each irreducible component multiplicitiy 2. Just as in S. Kobayashi-Ochiai, we obtain a holomorphic section s of

$$(K_{\overline{X}} \otimes [E])^{-2} \otimes S^1 T^*_{\overline{X}}(\log E) \subset \overset{4}{\otimes} T_{\overline{X}}(-\log E) \otimes \overset{4}{\otimes} T^*_{\overline{X}}(\log E),$$

since the given holomorphic conformal structure on $\overline{X} - E$ behaves logarithmically near E. We recall that a Hilbert modular cusp is defined in the following way. Let K be a totally real quadratic field over $\mathbb{Q}$ and

M a free abelian subgroup of rank two and V a totally positive multiplicative group of rank one such that $VM = M$. Let

$$G(M,V) = \left\{ \begin{pmatrix} \varepsilon & \mu \\ 0 & 1 \end{pmatrix} \middle| \; \varepsilon \in V, \; \mu \in M \right\} .$$

Then $G(M,V)$ acts on $H^2 = \{(z_1,z_2) \in \mathbb{C}^2 \mid \text{Im } z_1 > 0, \text{Im } z_2 > 0\}$ by $(z_1,z_2) \longmapsto (\varepsilon z_1 + \mu, \varepsilon^1 z_2 + \mu^1)$, where ε^1 and μ^1 are conjugate of ε and μ over $\mathbb{Q}$, respectively. This action is free and properly discontinuous. The Hilbert modular cusp is obtained by adjoining a point ∞ to the complex manifold $H^2/G(M,V)$ with its neighborhood system $y_1 y_2 > d$, where $z_i = x_i + \sqrt{-1} y_i$ $(i = 1,2)$. Since our complete Einstein-Kähler metric is equivalent to the bidisc metric near Hilbert modular cusps and is of quasi-bounded geometry, we see that $\dfrac{\partial |s|^2}{\partial n}$ is uniformly bounded for large $d > 0$, where $\dfrac{\partial}{\partial n}$ is the unit normal vector to the real hypersurface $y_1 y_2 = d$. We can thus use Stokes' theorem to conclude that

$$\int_{\overline{X}-E} \Delta |s|^2 * 1 = 0.$$

On the other hand, Bochner's formula (see for example [Kob] tells us that

$$\Delta |s|^2 = |\nabla s|^2 .$$

Therefore we obtain

$$\int_{\overline{X}-E} |\nabla s|^2 * 1 = 0,$$

which means $\nabla s \equiv 0$, i.e., s is covariantly constant. As we mentioned earlier, this causes the reduction on the homology group of our complete Einstein-Kähler metric. Therefore $\overline{X} - E$ is complete locally symmetric Kähler manifold with negative Ricci curvature with the relation $\overline{c}_1^2 = 2\overline{c}_2$ between logarithmic Chern numbers. It must be uniformized by the bidisc. Finally, it is not difficult to modify the above arguments to the general case.

Q.E.D.

The existence of an Einstein-Kähler metric with the above properties is proved in [Ko]. So, we obtain:

<u>Theorem 1'</u> Let $\overline{X}$ be a compact complex surface with at worst quotient singularities and E a divisor lying in the regular part of $\overline{X}$ consisting of the exceptional sets for the minimal resolution for Hilbert modular cusps. Assume the adjoint bundle $K_{\overline{X}} + E$ is <u>orbifold-ample</u> outside of E (for the definition, see Theorem 1 of [Ko]). If $\overline{X} - E$ admits an <u>orbifold-holomorphic conformal structure</u> which behaves <u>logarithmically</u> near E, then $\overline{X} - E$ is uniformized by the bidisc.

4. Desingularization of generalized holomorphic conformal structures

Let X be a smooth compact complex surface. Let $\tau = \{\tau_\alpha\}$ be a generalized holomorphic conformal structure on X with the defining line bundle L, i.e., L is defined by the 1-cocycle $h_{\alpha\beta} = \frac{\tau_\alpha}{\tau_\beta}$, and with the discriminant divisor D. We now consider the following

(I) the discriminant divisor D is reduced,

(II) τ_α is of rank 1, i.e., rank $(g_{\alpha ij}) = 1$, along the regular part Reg (D) of D,

(III) for every $p \in \mathrm{Reg}\,(D)$, the one dimensional null space N_p (multiplicity 2) of τ at p coincides with $T_p(D)$.

Definition. A generalized holomorphic conformal structure is called tangential if it satisfies the condition (III) along Reg(D).

Remark 1. Under the following change of variables $(x,y) \longmapsto (x,z)$ where $z = y^2$, the holomorphic conformal structure $(dx)^2 + (dy)^2$ is reduced to the generalized one $4z(dx)^2 + (dz)^2$ which satisfies (I) - (III) locally.

Remark 2. If a given generalized holomorphic conformal structure behaves logarithmically along D, the condition

(I) is not satisfied because every component of D is of multiplicity 2. But (II) and (III) are satisfied along $\mathrm{Reg}\,(D)$.

Since $[D]$ is divisible by 2 in $H^2(X;\mathbb{Z})$, we may consider a double covering $\tilde{X}$ of branching exactly over D. Let $X(\sqrt{D})$ be the double covering of X along D associated to $L + K_X$ (recall $[D] = 2(L + K_X)$. It is well-known that $\mathrm{Sing}\,(\tilde{X}) = \mathrm{Sing}\,(D)$ and any quotient singularity of $\tilde{X}$ is a simple singular point (or a rational double point). We set $D^* := \mathrm{Reg}(D) \cup \{$simple singular points of $D\}$. The following Theorem 2 gives a desingularization procedure of generalized holomorphic conformal strutures by means of the <u>double covering trick</u> (cf. Remark 1).

<u>Theorem 2</u> Assume that the conditions (I)-(III) are fulfilled. Then the induced holomorphic conformal structure on $\tilde{X} - D$ uniquely extends to a <u>non-degenerate</u> holomorphic conformal structure on $\mathrm{Reg}\,(\tilde{X}) = (\tilde{X} - D) \cup \mathrm{Reg}(D)$. It extends automatically to an orbifold-holomorphic conformal struture on $(\tilde{X} - D) \cup D^*$.

For the proof, we need the following:

<u>Lemma</u>. Let B be a neighborhood of 0 in $\mathbb{C}^2 : (x,y)$ and $T(B^0) = L_1^0 \oplus L_2^0$ a splitting of the tangent bundle of $B^0 = B - \{0\}$. Then this splitting extends uniquely to a splitting $T(B) = L_1 \oplus L_2$.

Proof. We set $dx = \xi_1^0 + \xi_2^0$, $dy = \eta_1^0 + \eta_2^0$ to define $\xi_i, \eta_i \in \Gamma(B^0, (L_i^0)^*)$, $i = 1,2$, where $T^*(B)^0 = (L_1^0)^* \oplus (L_2^0)^*$ denotes the dual splitting. Then 1-forms ξ_1^0, η_i^0, $i = 1,2$, extends to holomorphic 1-forms ξ_i, η_i over B by Hartogs-Osgood's theorem. Obviously we have $\xi_1 \wedge \eta_1 =$ $= \xi_2 \wedge \eta_2 = 0$ and

$$dx \quad dy = \xi_1 \wedge \eta_2 + \xi_2 \wedge \eta_1 .$$

Thus at least one of the terms in the right hand side, say $\xi_1 \wedge \eta_2$ must be different from zero at $0 \in \mathbb{C}^2$. Now the equation $\eta_2 = 0$, $\xi_1 = 0$ define the extensions of L_1, L_2, respectively.

Q.E.D.

Proof of Theorem 2. We first note that the second assertion of the theorem is a consequence of the first, because the holomorphic conformal structures on a smooth simply connected surface Z are in one to one correspondence with the local splitting of the tangent bundle $T(Z)$ into sum of two line bundles [Kob-O1]. So, we have only to prove the first assertion of the theorem. Let $(ds)^2 = P(dx)^2 + 2Qdxdy + R(dy)^2$, $\Delta = Q^2 - RP$, locally. Now suppose that $p \in \mathrm{Reg}\ (D)$ and the local coordinates (x,y) in X are so chosen that $p = (0,0)$. By a suitable linear change of x,y we can moreover assume that $Q \neq 0$ $\Delta_x \cdot \Delta_y \neq 0$ at p, where we have used the abreviation $\Delta_x = \frac{\partial \Delta}{\partial x}$, $\Delta_y = \frac{\partial \Delta}{\partial y}$. Since $\Delta = Q^2 - PR = 0$ at p, $Q \neq 0$ implies $P \cdot R \neq 0$. Now we note that the null space N_p

is given by

$$(4.1) \qquad p\,dx + Q\,dy = 0$$

since we have the identity

$$(4.2) \qquad P(ds)^2 = (P\,dx + Q\,dy)^2 - \Delta(dy)^2$$

By the assumption (III) the equation (4.1) should be equivalent to $\Delta_x\,dx + \Delta_y\,dy = 0$, i.e. the determinant $p\Delta_y - Q\Delta_x$ is divisible by Δ. There is thus a holomorphic function α near p such that

$$(4.3) \qquad P\,\Delta y - Q\Delta\,x = \alpha\Delta\ .$$

Now we want to pass to a sufficiently small neighborhood of p in the double covering $\tilde{X}$, where we can take $\theta = \sqrt{\Delta}$ and y as local coordinates. In fact, since $\Delta_x \neq 0$, we can locally express x as a holomorphic function of y and θ (Holomorphic functions and forms near p in X can be regarded as functions and forms near p in $\tilde{X}$ by projection $\tilde{X} \longrightarrow X$). By using (4.2), (4.3) and $\theta^2 = \Delta$ we can see that the induced conformal structure on $\tilde{X} - D$ is locally given by the tensor

$$(\alpha\theta^2 dx + 2\theta Q d\theta)^2 - (\Delta y)^2\theta^2(dy)^2$$

$$= \theta^2\{(\alpha\theta dx + 2Qd\theta)^2 - (\Delta y)^2(dy)^2\}\ .$$ Thus the induced structure can be defined by the tensor

$$(\alpha\theta dx + 2Qd\theta)^2 - (\Delta y)^2(dy)^2\ .$$

This extends to a non-degenerate quadratic form near $p \in \widetilde{X}$ since we have $\theta = 0$, $Q \neq 0$, $\Delta_y \neq 0$ at p.

Q.E.D.

5. Generalized conformal structures on $P_2(\mathbb{C})$

In this section we introduce a reasonable class of singular conformal structures on $P_2(\mathbb{C}) : (x_1, x_2, x_3)$. Let Ω_α be an affine part of $P_2(\mathbb{C})$ define by $x_\alpha \neq 0$ and $\pi : \mathbb{C}^3 - \{0\} \longrightarrow P_2(\mathbb{C})$ the natural projection. Let $\{\tau_\alpha\}$, $\tau_\alpha \in \Gamma(\Omega_\alpha, S^2T^*(P_2(\mathbb{C})))$, be a generalized holomorphic conformal structure on $P_2(\mathbb{C})$ defined by a line bundle L. Then, if $[D] = \mathcal{O}(2m)$, we have that $L = \mathcal{O}(m+3)$, since $[D] = 2(L+K)$ and $K = \mathcal{O}(-3)$. Note that $\mathcal{O}(1) = \{\frac{x_\beta}{x_\alpha}\}$. By multiplying suitable non-vanishing holomorphic functions to τ_α's we may assume that

$$\frac{\pi^*(\tau_\alpha)}{\pi^*(\tau_\beta)} = \left(\frac{x_\beta}{x_\alpha}\right)^{m+3} \quad \text{in} \quad \mathbb{C}^3 - \{0\}.$$

So, $\pi^*(\tau_\alpha)x_\alpha^{m+3} = \pi^*(\tau_\beta)x_\beta^{m+3}$ on the open set of $\mathbb{C}^3 - \{0\}$ defined by $x_\alpha \neq 0$, $x_\beta \neq 0$. This means that

$$ds^2 = \pi^*(\tau_\alpha)x_\alpha^{m+3}$$

is the restriction of a globally defined holomorphic covariant symmetric tensor to $\{x_\alpha \neq 0\}$. So, if we write down ds^2 as

$$ds^2 = \sum_{i,j=1}^{3} h_{ij}(x)\, dx_i dx_j \qquad (h_{ij} = h_{ji}) ,$$

$h_{ij}(x)$ is a <u>homogeneous polynomial of degree $m+1$</u>.

Note that $P_2(\mathbb{C})$ is the quotient of $\mathbb{C}^3-\{0\}$ by the $\mathbb{C}^*$-action generated by the Euler vector field $\xi = \sum_{i=1}^{3} x_i \frac{\partial}{\partial x_i}$. Thus, in order that ds^2 induces a generalized holomorphic conformal structure on $P_2(\mathbb{C})$, it is necessary and sufficient that ξ belongs to the null bundle of the tensor ds^2 ; $\xi \,\lrcorner\, ds^2 = 0$. This condition is expressed by the following identities

$$(5.1) \qquad \sum_{j=1}^{3} h_{ij}(x) x_j = 0 \qquad i = 1,2,3 .$$

In fact, if (5.1) is satisfied, we have for example

$$(5.2) \qquad \tau_\alpha = \frac{ds^2}{x_\alpha^{m+3}} = \sum_{i,j\neq\alpha} \frac{h_{ij}(x)}{x_\alpha^{m+1}}\, d\left(\frac{x_i}{x_\alpha}\right) d\left(\frac{x_j}{x_\alpha}\right) \quad \text{on } \Omega_\alpha,$$

where $\frac{h_{ij}(x)}{x_\alpha^{m+1}}$ can be viewed as a polynomial of degree $m+1$ in $\left(\frac{x_i}{x_\alpha}\right)_{i\neq\alpha}$. It follows from (5.1) that the cofactor matrix of (h_{ij}) is proportional to the matrix (x_{ij}) where $x_{ij} = x_i x_j$. This means in particular that $\frac{\det(h_{ij})_{i\cdot j\neq\alpha}}{x_\alpha^2} = \mathcal{D}$ is independent of α. $\mathcal{D}$ is a homogeneous polynomial in (x_1,x_2,x_3) of degree $2m$. It follows from

(5.2) that $\mathcal{D} = 0$ gives the discrimiant divisor D. Of course we must assume that $\mathcal{D}$ is not identically zero. We moreover assume the condition (II) of section four, i.e.,

(II)' the rank of ds^2 is equal to one along the regular locus of $\mathcal{D} = 0$.

By a suitable linear change of x_1, x_2, x_3 if necessary, we may assume that

(i) no prime factor of $\mathcal{D}$ divides one and the same column or row of H ,

(ii) $\mathcal{D}$ is not divisible by any coordinate x_i .

To formulate (III) of section four, i.e., $N_p = T_p(D)$ along the regular locus of D , we introduce homogeneous polynomials of degree 3m

$$\alpha_i(x) = h_{jj}\mathcal{D}_k - h_{jk}\mathcal{D}_j \qquad \text{(cf. (4.3))},$$

for every even permutation (i,j,k) of $(1,2,3)$. By using condition (II)' and (i), (ii) above, one sees easily that all α_i's are divisible by $\mathcal{D}$. Now (III) of section four is equivalent to the following:

(III)' $\quad \mathcal{D} \mid \alpha_i \qquad i = 1,2,3$

The condition (I) in section four is equivalent to the following:

(I)' no multiple factor occurs in the prime factorization of D .

If the condition (I)' as well as (II)' (III)' and (i)-(ii) are fulfilled, we can use the double covering trick (Theorem 2) to obtain an orbifold-holomorphic conformal structure on $(\tilde{X}-D) \cup D^*$, where $X = P_2(\mathbb{C})$. In particular if D is so "nice" that $\tilde{X}$ has at worst simple singularities and Hilbert modular cusps and that the minimal resolution Y of $\tilde{X}$ over Hilbert modular cusps has the adjoint bundle $K_Y + E$ which is orbifold-ample outside of E ,where E is the exceptional set over Hilbert modular cusps, and if the orbifold-holomorphic conformal structure on $(\tilde{X} - D) \cup D^* = X$ -{cusps} behaves logarithmically near E as a generalized one on Y, then we can use Theorem 1' to conclude that $\tilde{X}$ - {cusps} are uniformized by the bidisc. On the other hand, if a given generalized holomorphic conformal structure behaves logarithmically along Reg (D) then we do not need to consider the double covering along D because such D must come from the desingularization of Hilbert modular cusps and the blowing down of some (-1)-curves. So, if we are given a generalized holomorphic conformal structure behaving logarithmically along Reg (D) such that

(a) there exists a partial blowing up X of $\mathrm{Sing}\,(D)$ so that the proper transform $\tilde{D}$ of D is of simple normal crossings and the induced conformal structure behaves logarithmically along $\tilde{D}$ and is non-degenerate outside of $\tilde{D}$,

(b) $\tilde{D}$ consists of the exceptional sets for the minimal resolution of Hilbert modular cusps,

(c) $K_X + [\tilde{D}]$ is ample outside of $\tilde{D}$,

then $X - \tilde{D}$ is uniformized by the bidisc. In this case the holomorphic covariant symmetric tensor ds^2 on $\mathbb{C}^3$ induced by the projection $\mathbb{C}^3 - \{0\} \longrightarrow P_2(\mathbb{C})$ splits into the symmetric product of two differential 1-forms, because the two null directions are not confused by the discrete group representing $X - \tilde{D}$ as bidisc-quotient. Of course, since the homogeneous degree of ds^2 is $m+3$ where $\deg D = 2m$, the sum of homogeneous degrees of the forms is equal to $m+3$.

This last argument is for Example 6.4 in the following section. As this shows, the conditions (I), (I') in Sections 4 - 5 might sometimes be too restrictive. In some situations we have to allow double factors to occur in the discriminant locus D. Obviously they do not affect the double covering formation; $X(\sqrt{D}) = X(\sqrt{D'})$ where D' is one simple part of D. It is also an elementary calculation to prove that, if ds^2 is tangential to an irreducible component of D with multiplicity 2, then ds^2 behaves logarithmically near it. This remark suggests the naturality of dealing with mixed situations where only simple and double factors occur in the discriminant locus. For the precise formulation of it something should still be done.

6. Examples

In the previous section we have seen that the homogeneous symmetric tensor $ds^2 = \sum_{i \cdot j=1}^{3} h_{ij}(x)\, dx_i dx_j$ defines a generalized conformal structure on $P_2(\mathbb{C})$ provided that some additional conditions are satisfied. To show the richness of this category of conformal structures. We will now mention some examples:

[6.1] We set here for $d \geq 5$

$$h_{ii} = 2x_i^{d-2}\, x_j x_k$$

$$h_{ij} = x_k(x_k^{d-1} - x_i^{d-1} - x_j^{d-1})$$

where we should have always $\{i,j,k\} = \{1,2,3\}$. The double covering $\tilde{X}$ branched over the discriminant curve D is an oribifold of general type with $3(d-1)$ A_{d-2} - singular points. Since $\tilde{X}$ has a unique Einstein-Kähler orbifold metric with negative Ricci curvature, it follows that $\widetilde{X}$ is uniformizable by the bidisx $D \times D$. In fact K. Ivinskis [I] showed that X is uniformizable by the product of two isomorphic curves of genus $(d-2)(d-3)/2$. This is thus a so-called reducible quotient of the bidisc.

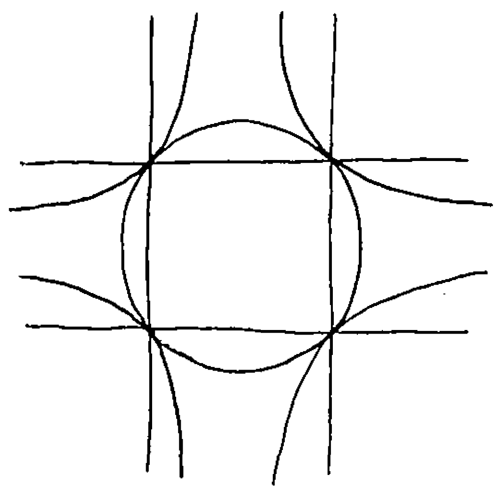
To give good examples for irreducible quotients of $D \times D$, we have to cite for the moment the earlier works [H1],[H2] of Hirzebruch in which he described $P_2(\mathbb{C})$ as some (quotients of) Hilbert modular surfaces. For such a description the generalized conformal structure induced on $P_2(\mathbb{C})$ should also be captured as a homogeneous symmetric tensor ds^2. It is naturally of particular importance to calculate the tensor explicitly since it should provide interesting identities between the modular forms describing the coordinates of $P_2(\mathbb{C})$ and their derivatives. We begin with the simplest one:

[6.2] Let $K = \mathbb{Q}(\sqrt{2})$ and $\mathcal{O}$ the ring of integers in K. We denote by $\Gamma(2)$ the principal congruence subgroup of $SL(2,\mathcal{O})$ associated with the ideal (2) of $\mathcal{O}$:

$$\left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathcal{O}) ; \alpha \equiv \delta \equiv 1, \beta \equiv \gamma \equiv 0 \mod. (2) \right\} .$$

Let further $\Gamma(2) \subseteq \Gamma \subseteq SL(2,\mathcal{O})$ be the group such that $\Gamma/\Gamma(2)$ is the center of $SL(2,\mathcal{O})/\Gamma(2)$. $([\Gamma:\Gamma(2)] = 2,\ SL(2,\mathcal{O})/\Gamma \simeq S_4)$. As usual $SL(2,\mathcal{O})$ acts on $H \times H : (z_1, z_2)$ in the non-trivial manner and we denote the compactified quotients $\overline{H \times H/\Gamma}$,$\overline{H \times H/\Gamma(2)}$

by $\check{Y}_1, \check{Y}_2$. ($\check{Y}_i$ has six cusps.) There is a natural covering map $\check{Y}_2 \longrightarrow \check{Y}_1$ of degree 2. The coordinate interchange $(z_1, z_2) \longrightarrow (z_2, z_1)$ induces an involution on $\check{Y}_1$ and we denote it by τ. Hirzebruch showed [H2] that $\check{Y}_1/\tau$ is isomorphic to $P_2(\mathbb{C})$ and that the ramification locus of the covering $\check{Y}_2 \longrightarrow P_2(\mathbb{C})$ is the curve of degree ten consisting of the following lines and conics : $x = \pm 1$, $y = \pm 1$, $xy = \pm 1$, $x^2 + y^2 = 2$, provided that the affine cocordinates x, y are suitably chosen.



(Note that this configuration has exactly six singular points, including the two at infinity. They correspond to the cusps on $\check{Y}_i$.) The mapping $\check{Y}_2 \longrightarrow P_2(\mathbb{C})$ is naturally factorized by the double covering of $P_2(\mathbb{C})$ ramified over this curve of degree 10. Thus there must be a unique tangential conformal structure on $P_2(\mathbb{C})$ with the curve as its discriminant. The explicit calculation shows that the structure is given by the tensor

$$(y^2 - 1)(2 - y^2 - x^2y^2)(dx)^2 + 2xy(x^2 - 1)(y^2 - 1)dxdy$$
$$+ (x^2 - 1)(2 - x^2 - x^2y^2)(dy)^2$$

in the affine patch (x,y), which obviously extends conformally to the whole plane $P_2(\mathbb{C})$.

[6.3] Next we observe the Hilbert modular surface $Y = \overline{H\times H/\Gamma(\sqrt{5})}$ completed with the six cusps of another type where $\Gamma(\sqrt{5})$ denotes the principal congruence subgroup of $SL(2,\mathcal{O})$ associated with the prime ideal $(\sqrt{5})$ in the ring $\mathcal{O}$ of integers in $K = Q(\sqrt{5})$. Each cusp is resolved by a 2-cycle of rational curves with self-intersection number -3. As before the transposition $(z_1,z_2) \longrightarrow (z_2,z_1)$ induces an involution, denoted also by τ, on $\check{Y}$ and it is shown in [H2] that the quotient $\check{Y}/\tau$ is A_5-isomorphic to $P_2(\mathbb{C}):(A_0,A_1,A_2)$ and that the branch locus of $Y \longrightarrow P_2(\mathbb{C})$ is the famous Klein curve $C = 0$ of degree 10:

$$C = 320\ A_0^6A_1^2A_2^2 - 160\ A_0^4A_1^3A_2^3 + 20\ A_0^2A_1^4A_2^4$$
$$+ 6\ A_1^5A_2^5 - 4\ A(A_1^5 + A_2^5)(32\ A_0^4 - 20\ A_0^2A_1A_2 + 5\ A_1^2A_2^2)$$
$$+ A_1^{10} + A_2^{10}$$

where (A_0, A_1, A_2) are the homogeneous coordinates used in [K]. Note that the operation of $SL(2,F_5) \cong SL(2,\mathcal{O})/\Gamma(\sqrt{5})$ on $\check{Y}$ induces the action of $A_5 \cong PSL(2,F_5)$ on $P_2(\mathbb{C})$ and that (A_0,A_1,A_2) are so chosen that the six points $(1,0,0)$, $(2^{-1},\varepsilon^{\nu},\varepsilon^{-\nu})$ $(\nu = 0,1,2,3,4; \varepsilon = \exp(2\pi i/5))$ form the unique minimal orbit of A_5. They are exactly the double cusps of the curve and are the images of the cusps of $\check{Y}$. Now the tangential conformal structure with the discriminant $C = 0$ is unique and given in the affine patch $A_0 \neq 0$ by the symmetric tensor $P(dx)^2 + 2Q\,dx\,dy + R\,(dy)^2$ with

$$P(x,y) = 2(8\,y^2 - 6\,xy^3 + x^2y^4 - x^4y + 4\,x^3)$$
$$Q(x,y) = -24\,xy + 10\,x^2y^2 - 2\,x^3y^3 + x^5 + y^5$$
$$R(x,y) = P(y,x).$$

[6.4] We operate again in the real quadratic field $\mathbb{Q}(\sqrt{5})$ and its ring $\mathcal{O}$ of integers. Since the ideal $p = (2)$ remains prime, the residue class field $\mathcal{O}/p$ is isomorphic to the Galois field F_4. This implies in particular that the icosahedral group $A_5 \cong PSL(2,F_4)$ operates on the completed quotient $\check{Y} = \overline{H\times H/\Gamma(2)}$ which has the five cusps, each of which is resolved by a triangle of rational curves with self-intersection number -3. Hirzebruch [H1] showed further that the image of the diagonal $\{(z,z)\}$ is lifted to an exceptional curve of the 1 st kind on the minimal resolution Y of $\check{Y}$

that is not A_5-invariant but there are exactly ten of its A_5-transforms which are disjoint from each other and that the separate blowing down of them is the classical diagonal surface X of Clebsch and Klein $(X:\Sigma_i y_i = \Sigma_i y_i^3 = 0$ in $P_4(\mathbb{C}):(y_0,y_1,y_2,y_3,y_4))$. The images of the ten (-1)-curves are the Eckardt points of X and the images of the fifteen rational curves forming the five exceptional sets of the resolution Y are fifteen among the twentyseven lines on X. The remaining twelve lines form two sixtriplets of disjoint lines which are permuted among themselves by A_5. If one blows down one system of disjoint lines of this double six, one arrives again at $P_2(\mathbb{C})$ on which A_5 acts and the six points forming the minimal orbit are given. The fifteen lines on X are mapped onto the fifteen lines on $P_2(\mathbb{C}):(A_0,A_1,A_2)$ each of which passes exactly two of the six points. They are thus given by the equations [K]:

$$(*)\qquad \begin{cases} (1\pm\sqrt{5})A_0 + \varepsilon^{\nu}A_1 + \varepsilon^{-\nu}A_2 = 0 \\ \\ \varepsilon^{\nu}A_1 - \varepsilon^{-\nu}A_2 = 0 \end{cases} \qquad \nu = 0,1,2,3,4$$

As we have seen in Section 3, the conformal structure on $P_2(\mathbb{C})$ induced by this manner is regular outside the fifteen lines and it behaves logarithmically near them. It follows that the structure is also written in the form of homogeneous symmetric tensor $ds^2 = \Sigma h_{ij}\, dx_i\, dx_j$ which is tangential to any of the fifteen lines and for which the polynomial $\mathcal{D}$ coincides (up to a non-zero constant factor) with the square of the product of the left hand sides of (*). Moreover, when lefted to X, the singularity of the structure along the (-1)-curves of the blowing up should be removed by factoring out suitable powers of local defining functions of the curves. One sees also the degree of h_{ij} should be equal to $15 + 1 = 16$. Together with A_5-invariance, these requirements determine uniquely the tensor ds^2. Since the discriminant $\mathcal{D}$ is a square, ds^2 should be decomposed into the product of two homogeneous 1-forms. In fact they are the following differential forms:

$$\begin{aligned} f_1 = {} & 10A_0(A_1^5 - A_2^5)\, dA_0 + (16A_0^5A_2 - 40A_0^3A_1A_2^2 - 10 \\ & A_0^2A_1^4 + 10A_0A_1^2A_2^3 + A_1^5A_2 + A_2^6)dA_1 + (-16A_0^5A_1 \\ & + 40A_0^3A_1^2A_2 + 10A_0^2A_2^4 - 10A_0A_1^3A_2^2 - A_1^6 - A_1A_2^5)dA_2 \end{aligned}$$

$$\begin{aligned} f_2 = {} & (64A_0^5A_1^5 - 64A_0^5A_2^5 - 40A_0^3A_1^6A_2 + 40A_0^3A_1A_2^6 \\ & + 10A_0A_1^7A_2^2 - 10A_0A_1^2A_2^7 - A_1^{10} + A_2^{10})DA_0 + (64 \\ & A_0^7A_1A_2^2 - 64A_0^6A_1^4 - 144A_0^5A_1^2A_2^3 + 80A_0^4A_1^5A_2 \\ & + 40A_0^4A_2^6 + 40A_0^3A_1^3A_2^4 - 26A_0^2A_1^6A_2^2 - 16A_0^2A_1 \\ & A_2^7 + A_0A_1^9 + 40A_1^4\ A_2^5 + A_1^7A_2^3 + A_1^2A_2^8)dA_1 + (- \\ & 64A_0^7A_1^2A_2 + 64A_0^6A_2^4 + 144A_0^5A_1^3A_2^2 - 40A_0^4A_1^6 \\ & -80A_0^4A_1^5 - 40A_0^3A_1^4A_2^3 + 16A_0^2A_1^7A_2 + 26A_0^2 \\ & A_1^2A_2^6 - A_0A_1^5A_2^4 - A_0A_2^9 - A_1^8A_2^2 - A_1^3A_2^7)dA_2 \end{aligned}$$

The strange impression for the fact that f_1 and f_2 have different degrees will disappear if one notes that X admits the action of S_5 extending that of A_5 and that odd elements of S_5 induce not projective but Cremona transformations on $P_2(\mathbb{C})$ under which the forms f_1, f_2 are interchanged.

In example [6.2-4], the double plane on a blown up plane satisfies the sufficient conditions for the existence of complete Einstein-Kähler metric with negative Ricci-curvature. It thus may be interesting to look at these examples as such (compatifiable) surfaces whose universal covering spaces are determined by looking at the geometry of these (using the uniformization theorem (Theorem 1')).

To close this section we will now give two double planes for which the modified proportionality is fulfilled but which can not be uniformized by the bidisc. The first one is rigid:

[6.5] We take a non-degenerate conic C on $P_2(\mathbb{C})$ and a point p_1 outside it. From p_1 we draw then the two tangent lines ℓ_1', ℓ_2'' to C and we form the polar line ℓ_1''' for p_1 by connecting the contact points of ℓ_1', ℓ_1'' with C. We further choose a point $p_2 \in \ell_1'''$ and form the two tangents ℓ_2', ℓ_2'' and the polar ℓ_2''' for p_2 in the same way. The point p_1 is then on the line ℓ_2'''.

The octic curve D consisting of C , $\ell_i', \ell_i'', \ell_i'''$ $(i = 1,2)$ is unique up to projective transformations. The double covering, denoted by X , of $P_2(\mathbb{C})$ branched over D has now only rational double points; they are five A_1, two P_4 and four D_5. One easily checks $\bar{c}_1^2(X) = 2\bar{c}_2(X)$. But the direct computation says that there is no tangential conformal structure on $P_2(\mathbb{C})$ with the discriminant D . One can even show that there is not any conformal structure with the discriminant D for which the six lines $\ell_i', \ell_i'', \ell_i'''$ $(i = 1,2)$ are integral curves, of $ds^2 = 0$. This implies in particular that X is not a quotient of the bidisc.

In the following final example we will observe a 1-parameter family of arrangements of five conics, each having generically sixteen tacnodes (ordinary contact points). It contains singular members with more tacnodes. (The existence of such arrangements is announced in Naruki [N].) If one considers the corresponding family of double planes branched over such arrangements, one has then the modified proportionality for generic members. But the existence of singular members as above allows us to apply the differential geometric consideration of Section 1 to this family, concluding that almost all members do not come from the bidisc.

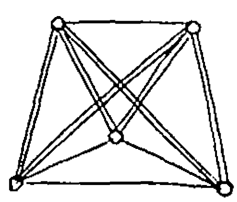
[6.6] The five conics of the family are given by the equations:

$$c_i : z^2 = x^2/a_i^2 + xy + a_j a_k y^2 = (x - a_i a_j y)\cdot(x - a_i a_k y)$$

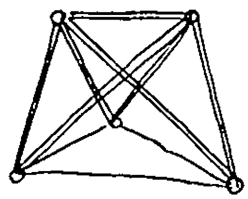
$$\{i,j,k\} = \{1,2,3\}$$

$$c_\pm : z^2 = (xy + py^2) \pm 2xy$$

here (x,y,z) are the homogeneous coordinates of $P_2(\mathbb{C})$, a_1, a_2, a_3 are the parameters of the family bound by the relation $a_1 + a_2 + a_3 = 0$ and we write p for $a_2 a_3 + a_1 a_3 + a_1 a_2$. This family is considered to depend essentially on only one parameter, since the corresponding arrangements are isomorphic for (a_1, a_2, a_3) and (ta_1, ta_2, ta_3) $t \neq 0$. One sees immediately that the point $z = x - a_j a_k y = 0$ is the contact point of c_j and c_k, and that $z = y = 0$ is the contact point of $c_\pm$. We have thus four tacnodes now. The simple subtraction of the equations shows also that each of $c_\pm$ is tangent to each of c_i at two points (on the line $x_i/a_i \mp a_i y = 0$). If $p = 0$, then $c_\pm$ are tangent also at point $z = x = 0$. Pictorially any generic configuration looks like



where o represents a non-singular conic in $P_2(\mathbb{C})$, o——o means two conics intersect transversely at two points and contact at one point, and o══o means two conics contact at two points. For any generic configuration D, the modified proportionality $\bar{c}_1^2 = 2\bar{c}_2$ are fulfilled for $X(\sqrt{D})$, which has eight A_1's and 16 A_3's. There is an complex analytic family D_t such that D_t for $t \neq 0$ is generic and D_0 is



i.e., two A_1 singularities collapse into one A_3 singularity. We claim that any sequence $\{t_n\}$ with $t_n \longrightarrow 0$ contains infinitely many t_n's, say $\{t_{n_k}\}$, with $t_{n_k} \longrightarrow 0$ and $X(\sqrt{D_{t_{n_k}}})$ is not uniformized by the bidisc. Suppose for some $\{t_n\}$ with $t_n \longrightarrow 0$ that $X(\sqrt{D_{t_n}})$ is uniformized by the bidisc. Then, by using Kazhdan-Margulis theorem (see [B-G-S] and [Bu]) as well as the general Schwarz Lemma for volume forms due to Yau [Y2], one sees that the canonical Einstein-Kähler metric on $X(\sqrt{D_{t_n}})$ converges to the canonical Einstein-Kähler metric on $X(\sqrt{D_0})$, which will satisfy $|W_+| \equiv |W_-|$. On the other hand, for $X(\sqrt{D_0})$,

$$-\frac{3}{4} = \bar{c}_2 - \frac{1}{2}\bar{c}_1^2 = c_2 - \frac{1}{2}c_1^2 - \sum_{p\in \mathrm{Sing}(X(\sqrt{D_0}))} \left(e(Ep) - \frac{1}{|G(p)|}\right)$$

$$= \frac{1}{8\pi^2}\int_{X(\sqrt{D_0})} (|W_-|^2 - |W_+|^2) * 1 = 0,$$

which is a contradiction.

o

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