ON THE CANCELLATION OF HYPERBOLIC FORMS OVER ORDERS IN SEMISIMPLE ALGEBRAS

by

I. Hambleton and M. Kreck

Max–Planck–Institut
für Mathematik
Gottfried–Claren–Str. 26
5300 Bonn 3
Federal Republic of Germany

MPI/89 – 91

On the Cancellation of Hyperbolic Forms over Orders in Semisimple Algebras

I. Hambleton[1] and M. Kreck

Let R be a Dedekind domain and K its field of quotients. The purpose of this note is to obtain an improvement in the stable range for cancellation of lattices over R–orders in separable K–algebras, assuming some local information about the lattices. Recall that a *lattice* over an R–order A is an A–module which is projective as an R–module. Our results are based on the work of H. Bass [2],[3], A. Bak [1] and L. N. Vaserstein [11].

The general stable range condition for cancellation of lattices over orders is free rank ≥ 2 for the linear case [2;(3.5), p.184], and free hyperbolic rank ≥ 2 for the unitary case [3;(3.6),p.238]. To state our condition, let A and B be orders in separable algebras over K [4;71.1,75.1], and suppose that there is a surjective ring homomorphism ϵ: A $\longrightarrow$ B. We say that a finitely generated A–module L has (A,B)-*free rank* ≥ 1 *at a prime* $\mathfrak{p} \in R$, if there exists an integer r such that $(B^r \oplus L)_{\mathfrak{p}}$ has free rank ≥ 1 over $A_{\mathfrak{p}}$. Here $A_{\mathfrak{p}}$ denotes the localized order $A \otimes R_{(\mathfrak{p})}$.

Theorem 1

Let L be an A-lattice and put $M = L \oplus A$. Suppose that there exists a surjection of orders ϵ: A $\longrightarrow$ B such that L has (A,B)-free rank ≥ 1 at all but finitely many primes. If $GL_2(A)$ acts transitively on unimodular elements in $B \oplus B$, then for any A–lattice N which is locally a direct summand of M^n for some integer n, $M \oplus N \cong M' \oplus N$ implies $M \cong M'$.

[1] Partially supported by NSERC grant A4000

For the corresponding result in the unitary case there is a similar condition involving hyperbolic rank ≥ 1 and a locally (A,B)–free submodule. A quadratic module V has (A,B)-*hyperbolic rank* ≥ 1 *at a prime* $\mathfrak{p} \in R$ if there exists an integer r such that $(H(B^r) \oplus V)_{\mathfrak{p}}$ has free hyperbolic rank ≥ 1 over $A_{\mathfrak{p}}$. The other terms used in the statement are defined precisely in §2 or in [3;pp. 80, 87]. Note in particular that a *unitary module* is a (λ,Λ)–quadratic form on a finitely-generated *projective* A-module. A totally isotropic submodule is one on which the quadratic form is identically zero.

Theorem 2

Let V be a (λ,Λ)–quadratic module over a unitary (R,λ) algebra (A,Λ) and put $(M,[h]) = V \perp H(A)$. Suppose that there exists a surjection of orders $\epsilon\colon A \longrightarrow B$ such that V has (A,B)-hyperbolic rank ≥ 1 at all but finitely many primes. If $U_2(A)$ acts transitively on the set of unimodular elements in $H(B \oplus B)$ of fixed length, then for any unitary module N, $M \perp N \cong M' \perp N$ implies $M \cong M'$.

In our work [5], [6] on the topological classification of 4–manifolds and algebraic surfaces we encounter locally $(\mathbb{Z}\pi,\mathbb{Z})$–free modules, where $\mathbb{Z}\pi$ is the integral group ring of a finite group. We check that for $B = \mathbb{Z}$, the conditions on "transitive action" in Theorems 1 and 2 are satisfied (see (1.4) and (2.10)), hence can be omitted from the statements.

For example consider the lattices arising from exact sequences

$$(0.1) \qquad 0 \longrightarrow L \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

with C_i finitely generated projective $\mathbb{Z}\pi$ modules. Such lattices with minimal $\mathbb{Z}$–rank need not contain any free direct summands over $\mathbb{Z}\pi$, but rationally contain all the representations of π except the trivial one. The simplest case occurs for π cyclic and $L = \ker\{\epsilon\colon \mathbb{Z}\pi \longrightarrow \mathbb{Z}\}$ the augmentation ideal.

More generally, if M is a $\mathbb{Z}\pi$-lattice such that $M \otimes \mathbb{Q}$ is a free module over $\mathbb{Q}[\rho]$ for some $\rho \triangleleft \pi$, then M is $(\mathbb{Z}\pi, \mathbb{Z}[\pi/\rho])$-locally free at all but finitely many primes.

In § 3 we discuss metabolic forms over group rings $\mathbb{Z}\pi$, leading to examples of (A,B)-hyperbolic rank ≥ 1 forms which contain no hyperbolic summand.

The purely algebraic results of this paper have consequences in several different geometric situations. These will be described elsewhere.

§ 1. The Linear Case

By an "A–module" we will mean a finitely generated right A–module. As above we suppose that ϵ: $A \longrightarrow B$ is a surjective ring homomorphism of R–orders in (possibly different) separable K–algebras. If M is an A–lattice and $N = M \otimes_A B := \epsilon_*(M)$, we get an induced homomorphism

$$\epsilon_* : GL(M) \longrightarrow GL(N).$$

If $M = M_1 \oplus M_2$ is a direct sum splitting of an A–module then $E(M_1, M_2)$ denotes the subgroup of GL(M) generated by the elementary automorphisms ([2;p.182]). Recall that for an element $x \in M$, $O_M(x)$ is the left ideal in A generated by

$$\{ f(x) \mid f \in \mathrm{Hom}_A(M, A) \}.$$

If $O_M(x) = A$ we say that x is unimodular.

The following result of Bass is an essential ingredient in the proofs of the cancellation theorems.

(1.1) Theorem [2;(3.1), p.178]: Let Q be a projective A-module and $P \cong A \oplus A$. For any unimodular element $x = (p,q) \in P \oplus Q$, there exists an A-homomorphism f: $Q \longrightarrow P$ such that $O_P(p + f(q)) = A$.

We also need two other facts.

(1.2) Lemma: Let M be a finitely generated right A–module, projective over R, and $A' = A/A\mathfrak{t}$ for an ideal $\mathfrak{t} \in R$ such that the localized order $A_{\mathfrak{t}}$ is maximal. Then the induced map

$$\mathrm{Hom}_A(M, A) \longrightarrow \mathrm{Hom}_{A'}(M', A')$$

is surjective, where $M' = M/M\mathfrak{t}$.

Proof: First note that $M_{\mathfrak{t}}$ is projective over $A_{\mathfrak{t}}$. Since $A' = A_{\mathfrak{t}}/A_{\mathfrak{t}}\mathfrak{t}$ we can lift any map $f'\colon M' \longrightarrow A'$ to $f\colon M_{\mathfrak{t}} \longrightarrow A_{\mathfrak{t}}$. After restricting to $M \subseteq M_{\mathfrak{t}}$ and multiplying by an element $r \in R$ prime to $\mathfrak{t}$, we obtain a lifting of $r'f'$. But r' (the image of r in R') is a unit in A'.

(1.3) Lemma [3;(2.5.2),p.225]: If C is a semisimple algebra, then for each $a, b \in C$ there exists $r \in C$ such that $C(a + rb) = Ca + Cb$.

We now come to the main result of the section.

(1.4) Theorem: Let A be an R–order in a separable K–algebra. Suppose that $M = L \oplus P$ is an A–lattice, where $P = p_0A \oplus p_1A$ is free of rank 2 and L is (A,B)–free of rank ≥ 1 at all but finitely many primes. Let $G_0 \subseteq GL(P)$ be a subgroup such that $\epsilon_*(G_0)$ acts transitively on the unimodular elements in $\epsilon_*(P)$. Then the group

$$G = \langle G_0, E(p_0A, L \oplus p_1A), E(p_1A, L \oplus p_0A)\rangle \subseteq GL(M)$$

acts transitively on the unimodular elements in M.

(1.5) Remark: In some cases there may be no subgroup G_0 with the required property. For example, if $B = \mathbb{Z}\pi$ is the integral group ring of a finite group π, then $GL_2(B)$ acts transitively on unimodular elements in $B \oplus B$ if and only if the relation $I \oplus B \cong B \oplus B$ for a projective ideal I implies $I \cong B$. In [8;Thm.3] Swan shows that this is not true for a certain ideal in $\mathbb{Z}\pi$ where π is the generalized quaternion group of order 32. Later in [10], extending the work of Jacobinski [7], Swan shows that cancellation in this sense fails for $\mathbb{Z}\pi$ if and only if π has a binary polyhedral quotient group in an explicitly given list.

Proof: We divide the proof into several parts. Let $x = v + p \in M$ be a unimodular element, where $p = p_0a + p_1b \in P$ and $v \in L$. We move x first into P to control the projection $\epsilon_*(x)$, and then use the stability assumption on L to move x so that its component in $p_0A \oplus L$ is unimodular. Finally we move x to p_0.

(i) Since M has free rank ≥ 2 we may now perform the first step, to get $v = 0$, so that x starts out in P. To see this note that $O(p) + O(v) = A$, so there exists $c \in O(v)$ such that $O(p) + c$ contains 1. Apply (1.2) to $A \oplus P$ and the element (c, p) to find $z \in P$ with $O(p + zc) = A$. There exists $g\colon L \longrightarrow P$ with $g(v) = zc$, and $f\colon P \longrightarrow L$ with $f(p + zc) = v$. Extend by zero on the complements. Then

$$\tau(x) = (1 - f)\,(1 + g)\,(x) \;\in P,$$

and $\tau \in E(P, L) \subseteq \; < E(p_0A, L \oplus p_1A), E(p_1A, L \oplus p_0A) > \; \subseteq G$.

(ii) Since G_0 acts transitively on unimodular elements in $\epsilon_*(P) = B \oplus B$, we may assume that $\epsilon_*(x) = \epsilon_*(p_0)$.

(iii) Write $x = p_0a + p_1b$, so that $O(x) = Aa + Ab$. Consider the quotient ring $\bar{A} = A/\mathfrak{g}A$ where $\mathfrak{g}$ is the ideal in R generated by all the primes $\mathfrak{p} \in R$ at which A is not maximal, or L does not have (A,B)–free rank ≥ 1. Then we claim that, after changing x by an element from G if necessary,

$$\text{(1.6)} \qquad O(\bar{x}) = \bar{A}\bar{a} = \bar{A}, \text{ and } \epsilon_*(x) = \epsilon_*(p_0)$$

or a projects to a unit in $\bar{A}$ without disturbing step (ii). To see this note that the quotient ring $\bar{A} = \bar{C} \times \bar{C}'$, where $\bar{C}$ is the smallest direct factor mapping onto $\bar{B}$ by $\bar{\epsilon}$. But by lifting idempotents, $\bar{\epsilon}$ induces an isomorphism $\bar{B}/\text{Rad}\,\bar{B} \cong \bar{C}/\text{Rad}\,\bar{C}$. Therefore the $\bar{C}$

component of a is already a unit since a projects to 1 in the semisimple quotient. Over the other factor we can apply [2;(2.8),p.87]: there exists $u \in A$, such that the element $a + ub$ projects to a unit in $\bar{C}'$ and to 1 in $\bar{B}$. Let $g: p_1A \longrightarrow p_0A \subseteq M$ such that $g(p_1) = p_0u$. Extend g to a map from M to M by zero on the complement. Then $\tau = 1 + g$ is an element of G and $\tau(x)$ has the desired properties (1.6).

(iv) From step (iii) we have $Aa + \mathfrak{g}A = A$ and so $(Ab)_\mathfrak{p} = A_\mathfrak{p}$ for all primes $\mathfrak{p}$ dividing $\mathfrak{g}$. Therefore if $\mathfrak{t} \subseteq R$ denotes the largest ideal such that $A\mathfrak{t} \subseteq Aa$, we see that $\mathfrak{p}$ does not divide $\mathfrak{t}$ for all primes $\mathfrak{p}$ dividing $\mathfrak{g}$ and in particular $\mathfrak{t} \neq 0$.

(v) Now we project to the semilocal ring $A' = A/A\mathfrak{t}$, which is the quotient of a maximal order $A_\mathfrak{t}$ and so the projection $\epsilon': A' \longrightarrow B'$ splits and $A' = B' \times C'$. Since over the B' factor a projects to 1, we have $(Aa)' = A'$. Over the complementary factor C' we use a suitable $\tau \in E(p_1'C', L')$, so that after applying τ we achieve the condition

$$A'a' + O(v') = A' \tag{1.7}$$

over both factors of A'. This is an application of (1.2) to the component of x in $L' \oplus p_1'C'$ using the fact that $C' \subseteq L'$. The necessary homomorphism $g \in \mathrm{Hom}_{A'}(P_1', L')$, which is the identity over B', can be lifted to $\mathrm{Hom}_A(P_1, L)$ since P_1 is projective and extended to M by zero on $L \oplus p_0A$.

(vi) We now lift the relation (1.7) to A using (1.2) and obtain

$$Aa + O(v) + A\mathfrak{t} = A.$$

But $A\mathfrak{t} \subseteq Aa$ so we can assume that $v + p_0a$ is unimodular.

(vii) The argument of [2; pp 183–184] now shows that there is an element $\tau \in E(p_1A, p_0A \oplus L)$ such that $\tau(x) = p_0$. In our situation start with the unimodular element $z = v + p_0a \in L \oplus P_0$. Write $L \oplus P_0 = zA \oplus N$ and let $g_2(z) = p_1(1-a-b)$, with $g_2(N) = 0$. Let $g_3(p_1) = p_0$, $g_4(p_0) = p_1(a-1)$, $g_5(p_0) = p_1$, $g_6(p_1) = -v$, where the homomorphisms are extended to the obvious complements by zero. If $\tau_i = 1 + g_i$, then

$$\tau_6\tau_5\tau_4\tau_3\tau_2(x) = p_0.$$

This completes the proof.

Proof of Theorem 1: By Swan's Cancellation Theorem ([9; 9.7] and the discussion on [9;p.169]), $M \oplus A \cong M' \oplus A$ since $M \oplus A$ is the direct sum of two faithful modules. We apply (1.4) following [2;IV,3.5] to cancel the free modules.

Remark: The method does not seem to prove either Swan's or Jacobinski's cancellation theorems independently.

§ 2. The Unitary Case

We adopt the notation and conventions of Bass in [3; pp.61–90,233] for (λ,Λ)-quadratic modules over a unitary (R,λ)-algebra (A,Λ). A *unitary* module is a non-singular (λ,Λ)-quadratic form on a finitely generated projective A module. Since R is a Dedekind domain, $X = \max(R_0)$ has dimension $d = 1$, where $R_0 \subseteq R$ is the subring generated by all norms $t\bar{t}$ $(t \in R)$. Note that $\lambda\bar{\lambda} = 1$. The form parameter Λ is *ample* at $\mathfrak{m} \in X$ if given a, b $\in A[\mathfrak{m}]$, the semisimple quotient of $A_{\mathfrak{m}}$, there exists $r \in \Lambda[\mathfrak{m}]$ such that

(2.1) $$A[\mathfrak{m}](a + rb) = A[\mathfrak{m}]a + A[\mathfrak{m}]b.$$

In [3;§2,p.218ff] there is a discussion of this condition. If $R = \mathbb{Z}$ and $\Lambda = \{ a-\lambda\bar{a} \mid a \in A \}$, the minimal form parameter, then Λ is not ample at any prime when $\lambda = 1$ and Λ is not ample at 2 if $\lambda = -1$. Let $\mathfrak{A}_\Lambda \subseteq R_0$ be the ideal such that Λ is ample at all $\mathfrak{m} \notin V(\mathfrak{A}_\Lambda) = \{ \mathfrak{p} \in X \mid \mathfrak{A}_\Lambda \subseteq \mathfrak{p} \}$, and d_Λ the dimension of the closed set $V(\mathfrak{A}_\Lambda)$ in X. Note that $d_\Lambda \leq 1$ for all Λ, and $d_\Lambda \leq 0$ when Λ is ample at all but finitely many primes.

If $(M,[h])$ is any (λ,Λ)-quadratic module over A [3;p.80], then a *transvection* [3;p.91] is a unitary automorphism $\sigma = \sigma_{u,a,v}\colon M \longrightarrow M$ given by

(2.2) $$\sigma(x) = x + u<v,x> - v\bar{\lambda}<u,x> - u\bar{\lambda}a<u,x>,$$

where u, v ∈ M and a ∈ A satisfy the conditions

(2.3) $$h(u,u) \in \Lambda,\ \ <u,v> = 0,\ \ h(v,v) \equiv a \pmod{\Lambda}.$$

Note that $<x,y> = h(x,y) + \lambda\overline{h(y,x)}$ is the associated hermitian form. For any submodule $L \subsetneq M$,

$$L^{\perp} = \{ x \in M \mid <x,y> = 0 \text{ for all } y \in L \}.$$

If $M = M' \perp M''$ is an orthogonal direct sum, with $L' \subsetneq M'$ a *totally isotropic* submodule (i.e. $h(x,y) = 0 \pmod{\Lambda}$ for all x, y ∈ L′), then we define

(2.4) $$EU(M', L'; M'') = < \sigma_{u,a,v} \mid u \in L' \text{ and } v \in M'' >.$$

We will need the relation (see [3;p.92]):

(2.5) if $\alpha : (M,[h]) \longrightarrow (M',[h'])$ is an isometry, then

$$\alpha \circ \sigma_{u,a,v} \circ \alpha^{-1} = \sigma'_{\alpha u,a,\alpha v}$$

where $\sigma \in U(M,[h])$ and $\sigma' \in U(M',[h'])$.

The hyperbolic rank of a (λ,Λ)-quadratic module (M,[h]) is ≥ 1 if $(M,[h]) = \mathbb{H}(A) \perp (M',[h'])$, where $\mathbb{H}(P)$ denotes the hyperbolic form on $P \oplus \bar{P}$ [3;p.82] and elements denoted by pairs x = (p,q) with $p \in P$, $q \in \bar{P}$. Here we are using the notation $\bar{P}$ for the dual module P^* regarded as a right A–module in the usual way. Since we will always be working with P containing at least one A-free direct summand, we will often write $P = p_0A \oplus P_1$, $\bar{P} = q_0A \oplus \bar{P}_1$ and denote the element

$$(p,q) = (p_0a + p_1, q_0b + q_1).$$

The main result of this section is a unitary analogue of (1.4), so we use some of the notation (e.g. A, ϵ, B). Before stating it, we need two lemmas.

(2.6) Lemma: Let V be a (λ,Λ) quadratic module which has (A,B)–hyperbolic rank ≥ 1 at a prime $\mathfrak{p} \in R_0$, for which $A_{\mathfrak{p}}$ is maximal. Then

(i) V contains a totally isotropic submodule L which has (A,B)-locally free rank ≥ 1 at all but finitely many primes, and

(ii) if $x \in H(P) \subseteq V \perp H(P)$ with $P \cong A^r$ and $f: P \longrightarrow L$ is an A–homomorphism, then there are elements $q_i \in \bar{P}$, $v_i \in L$ $(1 \le i \le r)$ such that

$$\prod \sigma_{q_i,o,v_i}(x) = x + f(x).$$

Proof: (i) Since A_p is maximal, we can write $A_p = B' \times C'$ and work over the C' factor V' of V_p. Then V' has free hyperbolic rank ≥ 1 and for L_p we choose a maximal rank totally isotropic C'-free direct summand. Let $L = L_p \cap V$ and compare it to a direct sum of copies of the A-lattice $C := \ker\{\epsilon: A \longrightarrow B\}$. Since $C_p \cong C'$ we may choose a direct sum $N = C^r$ with the same R-rank as L and so $N_p \cong L_p$. Therefore N and L are full lattices on the same K–vector space (K is the quotient field of R), and hence agree at all but finitely many primes. If we further avoid all the primes where A is not maximal, then L has (A,B)-free rank ≥ 1 at the remaining primes.

(ii) Let $\{q_1,\dots,q_r\}$ be a basis for $\bar{P}$. Then there exist $v_1, \dots, v_r \in L$ such that $f(x) = \Sigma\, \bar{\lambda} v_i \langle q_i, x\rangle$ for all $x \in P$.

(2.7) Lemma [3;(3.11),p.241]: Suppose that (C,Λ) is a semisimple unitary algebra over (R,λ). Assume either that (i) P has free rank ≥ 2, or (ii) Λ is ample in C and $P = C$. Write $x \in \mathbf{H}(P)$ as $x = (p_0a + p_1, q_0b + q_1)$. Then there is an element $\sigma \in \mathbf{H}(E(P)) \cdot EU(\mathbf{H}(P)$ such that $\sigma(x) = (p_0a' + p_1', q_0b' + q_1')$ and $O(x) = Aa'$. In case (i), $\sigma \in EU(\mathbf{H}(P_0), Q; \mathbf{H}(P_1))$ where $Q = P_0$ or $\bar{P}_0$.

Definition: Let (M, [h]) be a (λ,Λ)-quadratic module. An element $x \in M$ is [h]-unimodular if there exists $y \in M$ such that $\langle x, y\rangle = 1$.

If (M,[h]) is non-singular then an element is [h]-unimodular if and only if it is unimodular.

The following is our main result in the quadratic case.

(2.8) Theorem: Let V be a (λ,Λ)–quadratic module which has (A,B)–hyperbolic rank ≥ 1 at all but finitely many primes, and put $(M, [h]) = V \perp \mathbb{H}(P)$ where P is A-free of rank 2. Suppose there exists a subgroup $G_1 \subsetneq U(\mathbb{H}(P))$ such that $\epsilon_*(G_1)$ acts transitively on the set of unimodular elements in $\mathbb{H}(\epsilon_*(P))$ of fixed length $[h(x,x)]$. Then

$$G = < G_1, EU(\mathbb{H}(P), Q; V), \mathbb{H}(E(P))\cdot EU(\mathbb{H}(P))>$$

where $Q = P$ or $\bar{P}$, acts transitively on the set of [h]-unimodular elements of a fixed length, and the set of hyperbolic pairs and hyperbolic planes in M.

Proof: The same reduction used in [3;(3.5),p.236] shows that it is enough to prove that G acts transitively on the set of [h]-unimodular elements of a fixed length in M. One can check that G contains all transvections $\sigma_{p_0,a,v}$ with $v \in (p_0)^{\perp} = V \oplus \mathbb{H}(P_1) \oplus p_0A$ (see [3;(3.11),p.143] and [3;(5.6),p.98]).

(i) Let $x = (v; p,q) \in V \perp \mathbb{H}(P)$ be an [h]-unimodular element. Since P is free of rank 2, it follows as in [3;p.181] that we may assume (p,q) is unimodular. More precisely, there exists some $y \in M$ such that $< x, y > = 1$ and so $<V, v> + O(p) + O(q) = A$. Choose $w \in V$ so that $<v, w> + O(p) + O(q)$ contains 1; put $c = <v, w>$. From (1.1) there is a $p_1 \in P$ such that $O(p + p_1c) + O(q) = A$. Now apply the transvection $\sigma_{p_1,a,w}$ to x. This isometry lies in $EU(\mathbb{H}(P), P;V)$.

(ii) Since $\epsilon_*(G_1)$ acts transitively on the set of unimodular elements of fixed length in $\mathbb{H}(\epsilon_*(P))$ we may assume that $\epsilon_*(x) = \epsilon_*(v; p_0, q_0b)$, where $\bar{b} \equiv h_P(x,x) \bmod \Lambda$.

(iii) We may now achieve "$O(x) = Aa$ over $A[\mathfrak{g}]$" using (2.7) and the fact that P is free of rank 2. Here $\mathfrak{g} = \Pi \{m|\ m \in S\}$ where S is a finite set in X containing all the primes at which A is not maximal or V does not have (A,B)–hyperbolic rank ≥ 1. Furthermore by

(2.6) we may assume that V contains a non-zero totally isotropic submodule L which has (A,B)-free rank ≥ 1 at all primes not in S. Note that after step (ii), nothing needs to be done over B and (p,q) is still unimodular. This step uses $\mathbf{H}(E(P))\cdot EU(\mathbf{H}(P))$.

(iv) Let $\mathfrak{t} \subseteq R_0$ be the largest ideal such that $A\mathfrak{t} \subseteq Aa$, and put $X' = V(\mathfrak{t})$, $X'_\Lambda = V(\mathfrak{t}+\mathfrak{A}_\Lambda)$, $d'_\Lambda = \dim X'_\Lambda$. Let $\pi\colon A \longrightarrow A' = A/A\mathfrak{t}$ be the natural projection and note that $\dim X' = 0$ and $\dim X'_\Lambda \leq 0$. As in [3;p.244] we see that $\mathfrak{m} \notin X'$ for all $\mathfrak{m} \in S$, hence $\mathfrak{t} \neq 0$ and A' is semilocal. We have $O(p_1,q_1 + q_0 b) + Aa = A$ and so $O(\pi p_1, \pi(q_1+q_0 b)) + \pi(Aa) = A'$. Over $B' = B/B\mathfrak{t}$ we do nothing. Over the complementary factor C' of A', apply (1.2) to find an element $u \in \pi P_1$ such that u projects to zero over B' and

$$O(\pi p_1 - ub) + O(\pi q_1) + \pi(Aa) = A'.$$

(Note that this already holds over B' by step (ii)). Choose $z \in P_1$ such that $\pi z = u$ and $\epsilon_*(z) = 0$. Since $\mathfrak{t}$ and $\mathfrak{g}$ are relatively prime, we can choose $z \in (P_1)\cdot\mathfrak{g}$.

Note that $\sigma_{p_0,0,z} \in EU(\mathbf{H}(P))$ by [3;(3.10.2),p.142]. Then

$$\begin{aligned}\sigma(x) &= x + p_0 <z,x> - z<p_0,x> \\ &= (v;\, p_1 - zb + p_0(a + <z, q_1>),\, q).\end{aligned}$$

Therefore

$$O(p_1 - zb) + O(q_1) + A(a + <z, q_1>) + A\mathfrak{t} = A.$$

But $A\mathfrak{t} \subseteq Aa \subseteq O(q_1) + A(a + <z,q_1>)$, so after these changes, we may assume that

(2.9) $$O(p_1) + O(q_1) + Aa = A.$$

(v) Since πV has hyperbolic rank ≥ 1 over C' we can choose an isometry $\alpha\colon \pi V \cong \mathbf{H}(C') \perp W'$ and extend it to an isometry of $\pi V \perp \mathbf{H}(\pi P)$ by the identity on $\mathbf{H}(\pi P)$. We now apply the first case of (2.7) to the element $\alpha(\pi(p_1, q_1)) \in \mathbf{H}(C') \perp \mathbf{H}(\pi P_1)$ over the semisimple ring C', where $A' = B' \times C'$. This uses an element $\sigma' \in EU(\mathbf{H}(\pi P_1), \pi Q; \mathbf{H}(C'))$ where $Q = P_1$ or $\bar{P}_1$. By (2.5), $\alpha^{-1}\circ\sigma'\circ\alpha \in EU(\mathbf{H}(\pi P_1), \pi Q; \pi V)$. Then there exists a lift σ of $\alpha^{-1}\circ\sigma'\circ\alpha$ to $U(M, [h])$ which lies in

$EU(\mathbb{H}(P_1), Q; V)$. After moving $x = (v; p,q)$ by σ we get

$$A = O(p_1) + At \subseteq O(p_1) + Aa = O(p).$$

Finally note that after this change p is unimodular and $\epsilon_*(x) = \epsilon_*(v; p_0, q_0 b)$, where $\bar{b} \equiv h_P(x,x) \bmod \Lambda$.

(vi) Since p is unimodular and $h(p,p) = 0$, $\mathbb{H}(P) = \mathbb{H}(pA) \perp \mathbb{H}(pA)^{\perp}$. If $\mathbb{H}(pA) = pA \oplus \bar{p}A$, where $\bar{p} \in \bar{P}$ then $\sigma_{\bar{p},d,\lambda v} \in EU(\mathbb{H}(P), \bar{P}; V)$ and

$$\begin{aligned} \sigma_{\bar{p},d,\lambda v}(x) &= x + \bar{p}<\lambda v,x> - \lambda v\bar{\lambda}<\bar{p},x> - \bar{p}\bar{\lambda}d<\bar{p},x> \\ &= (0; p, q'). \end{aligned}$$

(vii) We now have a hyperbolic element $x = (p, q) \in \mathbb{H}(P)$ with p unimodular. Recall that V contains a non–zero totally isotropic submodule L which has (A,B)-free rank ≥ 1 at all but finitely many primes. We claim that after applying a suitable transformation in G, we can assume that $x = (p_0,q)$, with a possibly different q. For this we need to refer to the proof of (1.4) to find which linear automorphisms of $L \oplus P$ are necessary to move p to p_0, and then show that they are induced by isometries in G.

Since our element p starts out in P steps (i) and (ii) of (1.3) are not necessary. Step (iii) requires an element of $E(p_1A, P_0)$ which is induced by a suitable element of $\mathbb{H}(E(P)) \subseteq G$. Step (v) uses an elementary transformation given by a homomorphism $f: P_1 \longrightarrow L$, and these are induced elements of G using (2.6). Finally, in step (vi) we first construct a homomorphism $g_2: P_0 \oplus L \longrightarrow P_1$ by splitting $P_0 \oplus L = zA \oplus N$ where z is unimodular and $h(z,z) = 0 \pmod{\Lambda}$. To realise τ_2 by an isometry, we find a unitary submodule $\mathbb{H}(zA) \subseteq V \perp \mathbb{H}(P_0)$ and then work inside $\mathbb{H}(zA) \perp \mathbb{H}(P_1)$. By [3;(3.10.4), p.143] $\mathbb{H}(\tau_2|_{zA \oplus P_1}) \subseteq EU(\mathbb{H}(zA), \bar{z}\bar{A}; \mathbb{H}(P_1)) \subseteq G$. The remaining automorphisms τ_3, τ_4, τ_5 are in $E(P_0, P_1)$ and τ_6 is defined using $g_6 \in Hom(P_1, L)$. These are induced by elements of $\mathbb{H}(E(P))$ or $EU(\mathbb{H}(P),\bar{P}; V)$ using (2.6).

Write $q = q_0 b - q_1 \in q_0A \oplus \bar{P}_1$. The transvection $\sigma_{q_1,0,q_0}$ belongs to $EU(\mathbb{H}(P))$ by

[3;(3.10.1), p.142], and

$$\sigma x = x - q_1 <q_0,x> - q_0\bar{\lambda}< q_1,x >.$$

Note that $<q_1,x> = 0$ since x has no component in $\bar{P}_1$ and $<q_0,x> = <q_0,p_0> = 1$, so $\sigma x = x + q_1 = (p_0, q_0b)$. We are now finished if x was unimodular. If x was a hyperbolic element, then $h(\sigma x,\sigma x) = h(p_0, q_0b) = \bar{b} \pmod{\Lambda}$, and so $\bar{b} \in \Lambda$, since x and $\sigma(x)$ are isotropic.

In the hyperbolic plane $p_0A + q_0A$, the element $X_+(-b) = \begin{bmatrix} I & -b \\ 0 & I \end{bmatrix} \in EU(\mathbf{H}(P_0))$ transforms $p_0 + q_0b$ into p_0 (the notation X_+ is from [3; p.130]).

Proof of Theorem 2: The argument is the same as for [3;(3.6), p.238] using our (2.8).

(2.10) Lemma: The group $\mathbf{H}(GL_2(\mathbb{Z}))\cdot EU(\mathbf{H}(\mathbb{Z} \oplus \mathbb{Z}))$ acts transitively on unimodular elements in $\mathbf{H}(\mathbb{Z} \oplus \mathbb{Z})$ of fixed length.

Proof: Let $P = p_0\mathbb{Z} \oplus p_1\mathbb{Z}$ (with dual basis q_0, q_1 for $\bar{P}$) and let $x = (p_0a, p_1b; q_0c, q_1d)$ be a unimodular element in $\mathbf{H}(P)$. We may assume that $d = 0$ after applying an element of $\mathbf{H}(E(P))$, so there exists an integer r such that $b + rc$ is a unit (mod a). Then

$$X_+\begin{pmatrix} 0 & -\lambda r \\ r & 0 \end{pmatrix}(x) = (p_0a, p_1(b+rc); q_0c, 0)$$

so that $O(p_0a) + O(p_1(b+rc)) = \mathbb{Z}$. We may therefore assume in the beginning that for $x = (p_0a, p_1b; q_0c, q_1d)$, a and b are relatively prime. Using a suitable element of $\mathbf{H}(E(P))$ we get $x = (p_0, 0; q_0c, q_1d)$ and after applying $X_-\begin{pmatrix} 0 & \lambda d \\ -d & 0 \end{pmatrix}$ the result is $(p_0, 0; q_0c, 0)$, where [c] is the length of x.

§ 3. Metabolic Forms

One way to obtain quadratic modules V with (A,B)-hyperbolic rank ≥ 1 at all but finitely many primes is to assume that V has a submodule **H**(L) where L has (A,B)-free rank ≥ 1. A generalization of this would be to assume that V contains a "metabolic form" on L. In this section we define a suitable notion of metabolic forms general enough for our applications elsewhere. The notation and conventions of § 2 will be used.

If N is an A-lattice and g: $\bar{N} \times \bar{N} \longrightarrow A$ is an R-bilinear form, let

$$[g] = \{g_\tau \mid g_\tau(\phi,\phi') = g(\phi,\phi') + <\phi, \tau(\phi')>, \ \tau \in \mathrm{Hom}_R(\bar{N}, N)\}.$$

Any $\theta \in \mathrm{Ext}^1_A(\bar{N}, N)$ defines an extension

(3.1) $$0 \longrightarrow N \xrightarrow{i} E \xrightarrow{j} \bar{N} \longrightarrow 0$$

of A-lattices which splits over R. We say that [g] is *θ–sesquilinear* if there is a cocycle $\gamma \in \mathrm{Hom}_R(\bar{N} \otimes_R A, N)$ representing θ such that for all $a \in A$:

(3.2)
$$g(\phi a, \phi') = \bar{a}g(\phi, \phi')$$
$$g(\phi, \phi' a) = g(\phi, \phi')a - <\phi, \gamma(\phi', a)>.$$

Note that any cocycle γ satisfies the relation:

$$\gamma(\phi, a_1a_2) = \gamma(\phi, a_1)a_2 + \gamma(\phi a_1, a_2)$$

and serves as a way to specify the A-module structure E on the R-module $N \oplus \bar{N}$ given by θ: for $(x,\phi) \in N \oplus \bar{N}$ define

(3.3) $$(x, \phi)\cdot a = (xa + \gamma(\phi, a), \phi a).$$

If we vary the choice of representative $g_\tau \in [g]$, then the new γ is $\gamma_\tau = \gamma + \delta\tau$, where

$$(\delta\tau)(\phi, a) = \tau(\phi)a - \tau(\phi a),$$

for some $\tau \in \mathrm{Hom}_R(\bar{N}, N)$, and all $a \in A$. Then $g_\tau(\phi,\phi') = g(\phi,\phi') + <\phi, \tau(\phi')>$ satisfies (3.2). Given an extension (N, θ) and a θ-sesquilinear form [g], we define the *metabolic* (λ,Λ)-quadratic form **Met**(N, θ, [g]) = (E, [q]) as follows: pick a compatible γ, g satisfying (3.2) and set

$$q((x,\phi),(x',\phi')) = <\phi, x'> + g(\phi,\phi').$$

It is easy to check that q is sesquilinear in the usual sense if [g] is θ-sesquilinear.

These metabolic forms are non-singular when they exist but an arbitrary extension need not admit any such form. Suppose that N is reflexive and let τ denote the involution on $\mathrm{Ext}^1_A(\bar{N}, N)$ given by dualizing exact sequences $(N,\theta) \mapsto (N,\theta)^*$. An extension (N, θ) is *λ-self-dual* if N is reflexive and there is a commutative diagram

(3.4)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{i} & E & \xrightarrow{j} & \bar{N} & \longrightarrow & 0 \\ & & \| & & \downarrow h & & \|\bar{\lambda} & & \\ 0 & \longrightarrow & N & \xrightarrow{j^*} & \bar{E} & \xrightarrow{i^*} & \bar{N} & \longrightarrow & 0 \end{array}$$

If $h^* = \lambda h$ then h is the adjoint of a metabolic hermitian form on E. We will define a homomorphism

$$\rho\colon \{ (N,\theta)^* = \lambda(N,\theta) \} \subseteq \mathrm{Ext}^1_A(\bar{N}, N) \longrightarrow H^1(\mathbb{Z}/2; \mathrm{Hom}_A(\bar{N}, N))$$

where $\mathrm{Hom}_A(\bar{N}, N)$ has the involution $\alpha \mapsto \bar{\lambda}\alpha^*$. We will show that $\rho(N,\theta)$ is the obstruction for finding a λ-self-dual map h. Choose an R-section s: $\bar{N} \longrightarrow E$ inducing a cocycle γ and identify $E = N \oplus \bar{N}$ as above. Then the lower sequence is split over R by s^* leading to an identification of $\bar{E} = N \oplus \bar{N}$. In these coordinates, for any A-map h making the diagram (3.4) commute,

$$h(x,\phi) = (x + s^* hs(\phi), \bar{\lambda}\phi)$$

and similarly

$$h^*(x,\phi) = (\lambda x + s^* h^* s(\phi), \phi).$$

Now $(h^*)^{-1} \circ \lambda h(x,\phi) = (x + \rho(h)(\phi), \phi)$ where $\rho(h) = s^* hs - \bar{\lambda} s^* h^* s$. Note that $\rho(h)$ is independent of the choice of the section s. Since $(h^*)^{-1} \circ \lambda h$ is an A-map, we can check using (3.3) that $\rho(h)$ is also an A-map. Similarly, by computing $h^* \circ (\bar{\lambda} h^{-1})$ and comparing with the formula for the dual, we see that $\rho(h)^* = -\lambda\rho(h)$. Moreover the cohomology class

$$[\rho(h)] \in H^1(\mathbb{Z}/2; \mathrm{Hom}_A(\bar{N}, N))$$

is independent of the choice of h. Define $\rho(N, \theta) = [\rho(h)]$ for any h making the diagram (3.4) commute.

(3.5) Proposition: If N is a reflexive A-module and (N, θ) is a self-dual extension, then (N, θ) admits a metabolic λ-hermitian form if and only if $\rho(N, \theta) = 0 \in H^1(\mathbb{Z}/2; \mathrm{Hom}_A(\bar{N}, N))$.

Note that a metabolic λ-hermitian form is unique up to isometry if it admits a quadratic refinement. We want to identify the obstruction to obtaining a quadratic refinement for a given metabolic λ-hermitian form h. Let

$$\eta\colon \ker \rho \longrightarrow \mathrm{coker}\,\{\,\hat{H}^0(\mathbb{Z}/2; \mathrm{Hom}_A(\bar{N}, N)) \longrightarrow \hat{H}^0(\mathbb{Z}/2; \mathrm{Hom}_R(\bar{N}, N))\,\}.$$

be the homomorphism defined by $\eta(h) = [s^* hs]$.

(3.6) Proposition: Suppose that (N,θ) admits a metabolic λ-hermitian form. Then (N,θ) admits a metabolic (λ,Λ)-quadratic form with respect to the minimal form parameter if and only if $\eta(N,\theta) = 0$.

Suppose now that $R = \mathbb{Z}$ and $A = \mathbb{Z}\pi$ where π is a finite group. Then each lattice L over A is reflexive. Let $N = \Omega^k\mathbb{Z}$, the kernel of a projective resolution F_* of $\mathbb{Z}$ of length k (see (0.1) for the case k = 3). We will show that every element of $\mathrm{Ext}^1_A(\bar{N}, N)$ is $(-1)^{k+1}$-self-dual.

(3.7) Lemma: Let $N = \Omega^k\mathbb{Z}$. The involution τ given by dualizing exact sequences induces multiplication by $(-1)^{k+1}$ on $\mathrm{Ext}^1_A(\bar{N}, N)$.

Proof: Let $\bar{X}$ be a projective resolution of $\bar{N}$ and X the dual co-resolution of N. We have two isomorphisms $\alpha, \beta : \mathrm{Ext}^1(\bar{N}, N) \cong H^1(\mathrm{Hom}_A(\bar{X}, X))$ comparing an extension with $\bar{X}$ or

X respectively. Note that over $A = \mathbb{Z}\pi$ we can use X instead of an injective co-resolution for computing $\mathrm{Ext}^i(\bar{N}, N)$. It is not difficult to see that $\alpha = -\beta$. Let t be the involution on $H^1(\mathrm{Hom}_A(\bar{X}, X))$ induced by dualization. By construction, $\alpha\tau = t\beta$ implying $\alpha\tau\alpha^{-1} = -t$.

Note that $\mathrm{Hom}_A(\bar{X}, X) \cong \mathrm{Hom}_{\mathbb{Z}}(\bar{X}, X) \otimes_A \mathbb{Z}$, and that $\mathrm{Hom}_{\mathbb{Z}}(\bar{X}, X)$ is a co-resolution of $\mathrm{Hom}_{\mathbb{Z}}(\bar{N}, N)$. Thus

$$H^i(\mathrm{Hom}_A(\bar{X}, X)) = H^i(\mathrm{Hom}_{\mathbb{Z}}(\bar{X}, X) \otimes_A \mathbb{Z}) = H^i(\pi; N \otimes_{\mathbb{Z}} N)$$

and under these identifications t corresponds to the involution induced by the flip map s: $x \otimes y \mapsto y \otimes x$ on $N \otimes N$.

Now we follow an argument suggested by R. Swan. Extend the projective resolution F defining N to a projective resolution $\hat{F}$ of $\mathbb{Z}$. Let f be the chain map on $\hat{F} \otimes_{\mathbb{Z}} \hat{F}$ mapping $x \otimes y \mapsto (-1)^{\deg(x)\deg(y)} y \otimes x$. Since f induces the identity on $\mathbb{Z}$ it induces the identity on all the derived functors. We have the similar chain map on $F \otimes_{\mathbb{Z}} F$ which on $F_{2k} = N \otimes N$ is $(-1)^k s$. Now we consider $F \otimes_{\mathbb{Z}} F$ as part of a co-resolution of $N \otimes N$ ending in $\mathbb{Z}$. Similarly we consider $\hat{F} \otimes_{\mathbb{Z}} \hat{F}$ a part of a complete co-resolution of $\mathbb{Z}$. Then

$$H^1(\pi; N \otimes_{\mathbb{Z}} N) = H^1(\mathrm{Hom}_A(\mathbb{Z}, F \otimes_{\mathbb{Z}} F)) \cong H^1(\mathrm{Hom}_A(\mathbb{Z}, \hat{F} \otimes_{\mathbb{Z}} \hat{F})).$$

where the last isomorphism is induced by the obvious chain map $\hat{F} \longrightarrow F$. Thus

$$\alpha\tau\alpha^{-1} = -s = (-1)^{k+1} f^* = (-1)^{k+1}.$$

(3.8) Example: Now we restrict to groups π of odd order. Since $\mathrm{Ext}^1_{\mathbb{Z}\pi}(\bar{N}, N)$ then has odd order $\rho(N,\theta)$ and $\eta(N,\theta)$ vanish for each λ-self-dual extension. In particular for $N = \Omega^k\mathbb{Z}$, each extension (N,θ) admits a metabolic (λ,Λ)-quadratic form whose λ-symmetrization is unique up to isometry.

References

[1] Bak, A., "The stable structure of quadratic modules", Thesis, Columbia University, 1969.

[2] Bass, H., Algebraic K–Theory, W.A. Benjamin, New York 1968

[3] Bass, H., "Unitary algebraic K–theory", Algebraic K–Theory III: Hermitian K–theory and Geometric Applications. Lect. Notes in Math. 343, 57–209, Springer Verlag, 1973.

[4] C.W.Curtis and I.Reiner. Representation Theory of Finite Groups and Associative Algebras. John Wiley & Sons. New York, 1962.

[5] Hambleton, I., Kreck, M., "On the classification of topological 4–manifolds with finite fundamental group", Math. Ann. 280 (1988), 85–104.

[6] Hambleton, I., Kreck, M., "Smooth structures on algebraic surfaces with cyclic fundamental group", Invent. Math.91 (1988), 53–59.

[7] Jacobinski, H., "Genera and decompositions of lattices over orders", Acta. Math. 121 (1968), 1–29.

[8] Swan, R.G., "Projective modules over group rings and maximal orders", Annals of Mathematics 76 (1962), 55–61.

[9] Swan, R. G., "K–theory of finite groups and orders", Notes by E.G.Evens. Lect. Notes in Math. 149, Springer–Verlag, Berlin 1970.

[10] Swan, R.G., "Projective modules over binary polyhedral groups", J. reine u. angew. Math. 342 (1983), 66–172.

[11] Vaserstein, L. N., "Stability of unitary and orthogonal groups over rings with involution", Math. Sbornik 81 (1970), 328–351.

McMaster University,
Hamilton, Ontario, Canada

Max-Planck Institut für Mathematik
Bonn, West Germany

December 15, 1989

ON THE CANCELLATION OF HYPERBOLIC FORMS OVER ORDERS IN SEMISIMPLE ALGEBRAS

by

I. Hambleton and M. Kreck

Max–Planck–Institut
für Mathematik
Gottfried–Claren–Str. 26
5300 Bonn 3
Federal Republic of Germany

MPI/89 – 91

References

[1] Bak, A., "The stable structure of quadratic modules", Thesis, Columbia University, 1969.

[2] Bass, H., Algebraic K-Theory, W.A. Benjamin, New York 1968

[3] Bass, H., "Unitary algebraic K-theory", Algebraic K-Theory III: Hermitian K-theory and Geometric Applications. Lect. Notes in Math. 343, 57–209, Springer Verlag, 1973.

[4] C.W.Curtis and I.Reiner. Representation Theory of Finite Groups and Associative Algebras. John Wiley & Sons. New York, 1962.

[5] Hambleton, I., Kreck, M., "On the classification of topological 4-manifolds with finite fundamental group", Math. Ann. 280 (1988), 85–104.

[6] Hambleton, I., Kreck, M., "Smooth structures on algebraic surfaces with cyclic fundamental group", Invent. Math.91 (1988), 53–59.

[7] Jacobinski, H., "Genera and decompositions of lattices over orders", Acta. Math. 121 (1968), 1–29.

[8] Swan, R.G., "Projective modules over group rings and maximal orders", Annals of Mathematics 76 (1962), 55–61.

[9] Swan, R. G., "K-theory of finite groups and orders", Notes by E.G.Evens. Lect. Notes in Math. 149, Springer-Verlag, Berlin 1970.

[10] Swan, R.G., "Projective modules over binary polyhedral groups", J. reine u. angew. Math. 342 (1983), 66–172.

[11] Vaserstein, L. N., "Stability of unitary and orthogonal groups over rings with involution", Math. Sbornik 81 (1970), 328–351.

McMaster University,
Hamilton, Ontario, Canada

Max-Planck Institut für Mathematik
Bonn, West Germany

December 15, 1989

On the Cancellation of Hyperbolic Forms over Orders in Semisimple Algebras

I. Hambleton[1] and M. Kreck

Let R be a Dedekind domain and K its field of quotients. The purpose of this note is to obtain an improvement in the stable range for cancellation of lattices over R–orders in separable K–algebras, assuming some local information about the lattices. Recall that a *lattice* over an R–order A is an A–module which is projective as an R–module. Our results are based on the work of H. Bass [2],[3], A. Bak [1] and L. N. Vaserstein [11].

The general stable range condition for cancellation of lattices over orders is free rank ≥ 2 for the linear case [2;(3.5), p.184], and free hyperbolic rank ≥ 2 for the unitary case [3;(3.6),p.238]. To state our condition, let A and B be orders in separable algebras over K [4;71.1,75.1], and suppose that there is a surjective ring homomorphism $\epsilon\colon A \longrightarrow B$. We say that a finitely generated A–module L has (A,B)-*free rank* ≥ 1 *at a prime* $\mathfrak{p} \in R$, if there exists an integer r such that $(B^r \oplus L)_{\mathfrak{p}}$ has free rank ≥ 1 over $A_{\mathfrak{p}}$. Here $A_{\mathfrak{p}}$ denotes the localized order $A \otimes R_{(\mathfrak{p})}$.

Theorem 1

Let L be an A-lattice and put $M = L \oplus A$. Suppose that there exists a surjection of orders $\epsilon\colon A \longrightarrow B$ such that L has (A,B)-free rank ≥ 1 at all but finitely many primes. If $GL_2(A)$ acts transitively on unimodular elements in $B \oplus B$, then for any A–lattice N which is locally a direct summand of M^n for some integer n, $M \oplus N \cong M' \oplus N$ implies $M \cong M'$.

[1] Partially supported by NSERC grant A4000

For the corresponding result in the unitary case there is a similar condition involving hyperbolic rank ≥ 1 and a locally (A,B)–free submodule. A quadratic module V has (A,B)-*hyperbolic rank* ≥ 1 *at a prime* $\mathfrak{p} \in R$ if there exists an integer r such that $(H(B^r) \oplus V)_{\mathfrak{p}}$ has free hyperbolic rank ≥ 1 over $A_{\mathfrak{p}}$. The other terms used in the statement are defined precisely in §2 or in [3;pp. 80, 87]. Note in particular that a *unitary module* is a (λ,Λ)–quadratic form on a finitely-generated *projective* A-module. A totally isotropic submodule is one on which the quadratic form is identically zero.

Theorem 2

Let V be a (λ,Λ)–quadratic module over a unitary (R,λ) algebra (A,Λ) and put $(M,[h]) = V \perp H(A)$. Suppose that there exists a surjection of orders ϵ: A ⟶ B such that V has (A,B)-hyperbolic rank ≥ 1 at all but finitely many primes. If $U_2(A)$ acts transitively on the set of unimodular elements in $H(B \oplus B)$ of fixed length, then for any unitary module N, $M \perp N \cong M' \perp N$ implies $M \cong M'$.

In our work [5], [6] on the topological classification of 4–manifolds and algebraic surfaces we encounter locally $(\mathbb{Z}\pi,\mathbb{Z})$–free modules, where $\mathbb{Z}\pi$ is the integral group ring of a finite group. We check that for $B = \mathbb{Z}$, the conditions on "transitive action" in Theorems 1 and 2 are satisfied (see (1.4) and (2.10)), hence can be omitted from the statements.

For example consider the lattices arising from exact sequences

(0.1) $$0 \longrightarrow L \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

with C_i finitely generated projective $\mathbb{Z}\pi$ modules. Such lattices with minimal $\mathbb{Z}$–rank need not contain any free direct summands over $\mathbb{Z}\pi$, but rationally contain all the representations of π except the trivial one. The simplest case occurs for π cyclic and $L = \ker \{\epsilon: \mathbb{Z}\pi \longrightarrow \mathbb{Z}\}$ the augmentation ideal.

More generally, if M is a $\mathbb{Z}\pi$-lattice such that $M \otimes \mathbb{Q}$ is a free module over $\mathbb{Q}[\rho]$ for some $\rho \triangleleft \pi$, then M is $(\mathbb{Z}\pi, \mathbb{Z}[\pi/\rho])$-locally free at all but finitely many primes.

In § 3 we discuss metabolic forms over group rings $\mathbb{Z}\pi$, leading to examples of (A,B)-hyperbolic rank ≥ 1 forms which contain no hyperbolic summand.

The purely algebraic results of this paper have consequences in several different geometric situations. These will be described elsewhere.

§ 1. The Linear Case

By an "A–module" we will mean a finitely generated right A–module. As above we suppose that ϵ: A ⟶ B is a surjective ring homomorphism of R–orders in (possibly different) separable K–algebras. If M is an A–lattice and $N = M \otimes_A B := \epsilon_*(M)$, we get an induced homomorphism

$$\epsilon_* : GL(M) \longrightarrow GL(N).$$

If $M = M_1 \oplus M_2$ is a direct sum splitting of an A–module then $E(M_1, M_2)$ denotes the subgroup of GL(M) generated by the elementary automorphisms ([2;p.182]). Recall that for an element $x \in M$, $O_M(x)$ is the left ideal in A generated by

$$\{ f(x) \mid f \in \mathrm{Hom}_A(M, A) \}.$$

If $O_M(x) = A$ we say that x is unimodular.

The following result of Bass is an essential ingredient in the proofs of the cancellation theorems.

(1.1) Theorem [2;(3.1), p.178]: Let Q be a projective A-module and $P \cong A \oplus A$. For any unimodular element $x = (p,q) \in P \oplus Q$, there exists an A-homomorphism f: Q ⟶ P such that $O_P(p + f(q)) = A$.

We also need two other facts.

(1.2) Lemma: Let M be a finitely generated right A–module, projective over R, and $A' = A/At$ for an ideal $t \in R$ such that the localized order A_t is maximal. Then the induced map

$$\mathrm{Hom}_A(M, A) \longrightarrow \mathrm{Hom}_{A'}(M', A')$$

is surjective, where $M' = M/Mt$.

Proof: First note that M_t is projective over A_t. Since $A' = A_t/A_t t$ we can lift any map $f'\colon M' \longrightarrow A'$ to $f\colon M_t \longrightarrow A_t$. After restricting to $M \subseteq M_t$ and multiplying by an element $r \in R$ prime to t, we obtain a lifting of $r'f'$. But r' (the image of r in R') is a unit in A'.

(1.3) Lemma [3;(2.5.2),p.225]: If C is a semisimple algebra, then for each $a, b \in C$ there exists $r \in C$ such that $C(a + rb) = Ca + Cb$.

We now come to the main result of the section.

(1.4) Theorem: Let A be an R–order in a separable K–algebra. Suppose that $M = L \oplus P$ is an A–lattice, where $P = p_0A \oplus p_1A$ is free of rank 2 and L is (A,B)–free of rank ≥ 1 at all but finitely many primes. Let $G_0 \subseteq GL(P)$ be a subgroup such that $\epsilon_*(G_0)$ acts transitively on the unimodular elements in $\epsilon_*(P)$. Then the group

$$G = \langle G_0, E(p_0A, L \oplus p_1A), E(p_1A, L \oplus p_0A) \rangle \subseteq GL(M)$$

acts transitively on the unimodular elements in M.

(1.5) Remark: In some cases there may be no subgroup G_0 with the required property. For example, if $B = \mathbb{Z}\pi$ is the integral group ring of a finite group π, then $GL_2(B)$ acts transitively on unimodular elements in $B \oplus B$ if and only if the relation $I \oplus B \cong B \oplus B$ for a projective ideal I implies $I \cong B$. In [8;Thm.3] Swan shows that this is not true for a certain ideal in $\mathbb{Z}\pi$ where π is the generalized quaternion group of order 32. Later in [10], extending the work of Jacobinski [7], Swan shows that cancellation in this sense fails for $\mathbb{Z}\pi$ if and only if π has a binary polyhedral quotient group in an explicitly given list.

Proof: We divide the proof into several parts. Let $x = v + p \in M$ be a unimodular element, where $p = p_0 a + p_1 b \in P$ and $v \in L$. We move x first into P to control the projection $\epsilon_*(x)$, and then use the stability assumption on L to move x so that its component in $p_0 A \oplus L$ is unimodular. Finally we move x to p_0.

(i) Since M has free rank ≥ 2 we may now perform the first step, to get $v = 0$, so that x starts out in P. To see this note that $O(p) + O(v) = A$, so there exists $c \in O(v)$ such that $O(p) + c$ contains 1. Apply (1.2) to $A \oplus P$ and the element (c, p) to find $z \in P$ with $O(p + zc) = A$. There exists $g\colon L \longrightarrow P$ with $g(v) = zc$, and $f\colon P \longrightarrow L$ with $f(p + zc) = v$. Extend by zero on the complements. Then

$$\tau(x) = (1 - f)\,(1 + g)\,(x) \in P,$$

and $\tau \in E(P, L) \subseteq \langle E(p_0 A, L \oplus p_1 A), E(p_1 A, L \oplus p_0 A) \rangle \subseteq G$.

(ii) Since G_0 acts transitively on unimodular elements in $\epsilon_*(P) = B \oplus B$, we may assume that $\epsilon_*(x) = \epsilon_*(p_0)$.

(iii) Write $x = p_0 a + p_1 b$, so that $O(x) = Aa + Ab$. Consider the quotient ring $\bar{A} = A/\mathfrak{g}A$ where $\mathfrak{g}$ is the ideal in R generated by all the primes $\mathfrak{p} \in R$ at which A is not maximal, or L does not have (A,B)–free rank ≥ 1. Then we claim that, after changing x by an element from G if necessary,

$$O(\bar{x}) = \bar{A}\bar{a} = \bar{A}, \text{ and } \epsilon_*(x) = \epsilon_*(p_0) \tag{1.6}$$

or a projects to a unit in $\bar{A}$ without disturbing step (ii). To see this note that the quotient ring $\bar{A} = \bar{C} \times \bar{C}'$, where $\bar{C}$ is the smallest direct factor mapping onto $\bar{B}$ by $\bar{\epsilon}$. But by lifting idempotents, $\bar{\epsilon}$ induces an isomorphism $\bar{B}/\mathrm{Rad}\,\bar{B} \cong \bar{C}/\mathrm{Rad}\,\bar{C}$. Therefore the $\bar{C}$

component of a is already a unit since a projects to 1 in the semisimple quotient. Over the other factor we can apply [2;(2.8),p.87]: there exists $u \in A$, such that the element $a + ub$ projects to a unit in $\bar{C}'$ and to 1 in $\bar{B}$. Let $g: p_1A \longrightarrow p_0A \subseteq M$ such that $g(p_1) = p_0u$. Extend g to a map from M to M by zero on the complement. Then $\tau = 1 + g$ is an element of G and $\tau(x)$ has the desired properties (1.6).

(iv) From step (iii) we have $Aa + \mathfrak{g}A = A$ and so $(Ab)_{\mathfrak{p}} = A_{\mathfrak{p}}$ for all primes $\mathfrak{p}$ dividing $\mathfrak{g}$. Therefore if $\mathfrak{t} \subseteq R$ denotes the largest ideal such that $A\mathfrak{t} \subseteq Aa$, we see that $\mathfrak{p}$ does not divide $\mathfrak{t}$ for all primes $\mathfrak{p}$ dividing $\mathfrak{g}$ and in particular $\mathfrak{t} \neq 0$.

(v) Now we project to the semilocal ring $A' = A/A\mathfrak{t}$, which is the quotient of a maximal order $A_{\mathfrak{t}}$ and so the projection $\epsilon': A' \longrightarrow B'$ splits and $A' = B' \times C'$. Since over the B' factor a projects to 1, we have $(Aa)' = A'$. Over the complementary factor C' we use a suitable $\tau \in E(p_1'C', L')$, so that after applying τ we achieve the condition

$$A'a' + O(v') = A' \tag{1.7}$$

over both factors of A'. This is an application of (1.2) to the component of x in $L' \oplus p_1'C'$ using the fact that $C' \subseteq L'$. The necessary homomorphism $g \in \mathrm{Hom}_{A'}(P_1', L')$, which is the identity over B', can be lifted to $\mathrm{Hom}_A(P_1, L)$ since P_1 is projective and extended to M by zero on $L \oplus p_0A$.

(vi) We now lift the relation (1.7) to A using (1.2) and obtain

$$Aa + O(v) + A\mathfrak{t} = A.$$

But $A\mathfrak{t} \subseteq Aa$ so we can assume that $v + p_0a$ is unimodular.

(vii) The argument of [2; pp 183–184] now shows that there is an element $\tau \in E(p_1A, p_0A \oplus L)$ such that $\tau(x) = p_0$. In our situation start with the unimodular element $z = v + p_0a \in L \oplus P_0$. Write $L \oplus P_0 = zA \oplus N$ and let $g_2(z) = p_1(1-a-b)$, with $g_2(N) = 0$. Let $g_3(p_1) = p_0$, $g_4(p_0) = p_1(a-1)$, $g_5(p_0) = p_1$, $g_6(p_1) = -v$, where the homomorphisms are extended to the obvious complements by zero. If $\tau_i = 1 + g_i$, then

$$\tau_6\tau_5\tau_4\tau_3\tau_2(x) = p_0.$$

This completes the proof.

Proof of Theorem 1: By Swan's Cancellation Theorem ([9; 9.7] and the discussion on [9;p.169]), $M \oplus A \cong M' \oplus A$ since $M \oplus A$ is the direct sum of two faithful modules. We apply (1.4) following [2;IV,3.5] to cancel the free modules.

Remark: The method does not seem to prove either Swan's or Jacobinski's cancellation theorems independently.

§ 2. The Unitary Case

We adopt the notation and conventions of Bass in [3; pp.61–90,233] for (λ,Λ)-quadratic modules over a unitary (R,λ)-algebra (A,Λ). A *unitary* module is a non-singular (λ,Λ)-quadratic form on a finitely generated projective A module. Since R is a Dedekind domain, $X = \max(R_0)$ has dimension $d = 1$, where $R_0 \subseteq R$ is the subring generated by all norms $t\bar{t}$ $(t \in R)$. Note that $\lambda\bar{\lambda} = 1$. The form parameter Λ is *ample* at $m \in X$ if given a, b $\in A[m]$, the semisimple quotient of A_m, there exists $r \in \Lambda[m]$ such that

(2.1) $$A[m](a + rb) = A[m]a + A[m]b.$$

In [3;§2,p.218ff] there is a discussion of this condition. If $R = \mathbb{Z}$ and $\Lambda = \{ a-\lambda\bar{a} \mid a \in A \}$, the minimal form parameter, then Λ is not ample at any prime when $\lambda = 1$ and Λ is not ample at 2 if $\lambda = -1$. Let $\mathfrak{A}_\Lambda \subseteq R_0$ be the ideal such that Λ is ample at all m $\notin V(\mathfrak{A}_\Lambda) = \{ p \in X \mid \mathfrak{A}_\Lambda \subseteq p \}$, and d_Λ the dimension of the closed set $V(\mathfrak{A}_\Lambda)$ in X. Note that $d_\Lambda \leq 1$ for all Λ, and $d_\Lambda \leq 0$ when Λ is ample at all but finitely many primes.

If $(M,[h])$ is any (λ,Λ)-quadratic module over A [3;p.80], then a *transvection* [3;p.91] is a unitary automorphism $\sigma = \sigma_{u,a,v}\colon M \longrightarrow M$ given by

(2.2) $$\sigma(x) = x + u\langle v,x\rangle - v\bar{\lambda}\langle u,x\rangle - u\bar{\lambda}a\langle u,x\rangle,$$

where u, v $\in$ M and a $\in$ A satisfy the conditions

(2.3) $$h(u,u) \in \Lambda, \ \ \langle u,v\rangle = 0, \ \ h(v,v) \equiv a \ (\text{mod } \Lambda).$$

Note that $\langle x,y\rangle = h(x,y) + \lambda\overline{h(y,x)}$ is the associated hermitian form. For any submodule $L \subseteq M$,

$$L^{\perp} = \{ x \in M \mid \langle x,y\rangle = 0 \text{ for all } y \in L \}.$$

If $M = M' \perp M''$ is an orthogonal direct sum, with $L' \subseteq M'$ a *totally isotropic* submodule (i.e. $h(x,y) = 0 \ (\text{mod } \Lambda)$ for all x, y $\in L'$), then we define

(2.4) $$EU(M', L'; M'') = \langle \sigma_{u,a,v} \mid u \in L' \text{ and } v \in M'' \rangle.$$

We will need the relation (see [3;p.92]):

(2.5) if $\alpha : (M,[h]) \longrightarrow (M',[h'])$ is an isometry, then

$$\alpha\circ\sigma_{u,a,v}\circ\alpha^{-1} = \sigma'_{\alpha u,a,\alpha v}$$

where $\sigma \in U(M,[h])$ and $\sigma' \in U(M',[h'])$.

The hyperbolic rank of a (λ,Λ)-quadratic module (M,[h]) is ≥ 1 if $(M,[h]) = \mathbb{H}(A) \perp (M',[h'])$, where $\mathbb{H}(P)$ denotes the hyperbolic form on $P \oplus \bar{P}$ [3;p.82] and elements denoted by pairs $x = (p,q)$ with $p \in P$, $q \in \bar{P}$. Here we are using the notation $\bar{P}$ for the dual module P^* regarded as a right A–module in the usual way. Since we will always be working with P containing at least one A-free direct summand, we will often write $P = p_0A \oplus P_1$, $\bar{P} = q_0A \oplus \bar{P}_1$ and denote the element

$$(p,q) = (p_0a + p_1, q_0b + q_1).$$

The main result of this section is a unitary analogue of (1.4), so we use some of the notation (e.g. A, ϵ, B). Before stating it, we need two lemmas.

(2.6) Lemma: Let V be a (λ,Λ) quadratic module which has (A,B)–hyperbolic rank ≥ 1 at a prime $p \in R_0$, for which A_p is maximal. Then

(i) V contains a totally isotropic submodule L which has (A,B)-locally free rank ≥ 1 at all but finitely many primes, and

(ii) if $x \in H(P) \subseteq V \perp H(P)$ with $P \cong A^r$ and $f: P \longrightarrow L$ is an A–homomorphism, then there are elements $q_i \in \bar{P}$, $v_i \in L$ $(1 \leq i \leq r)$ such that

$$\prod \sigma_{q_i,0,v_i}(x) = x + f(x).$$

Proof: (i) Since A_p is maximal, we can write $A_p = B' \times C'$ and work over the C' factor V' of V_p. Then V' has free hyperbolic rank ≥ 1 and for L_p we choose a maximal rank totally isotropic C'-free direct summand. Let $L = L_p \cap V$ and compare it to a direct sum of copies of the A-lattice $C := \ker\{\epsilon: A \longrightarrow B\}$. Since $C_p \cong C'$ we may choose a direct sum $N = C^r$ with the same R-rank as L and so $N_p \cong L_p$. Therefore N and L are full lattices on the same K–vector space (K is the quotient field of R), and hence agree at all but finitely many primes. If we further avoid all the primes where A is not maximal, then L has (A,B)-free rank ≥ 1 at the remaining primes.

(ii) Let $\{q_1,\ldots,q_r\}$ be a basis for $\bar{P}$. Then there exist $v_1, \ldots, v_r \in L$ such that $f(x) = \Sigma\, \bar{\lambda} v_i \langle q_i, x\rangle$ for all $x \in P$.

(2.7) Lemma [3;(3.11),p.241]: Suppose that (C,Λ) is a semisimple unitary algebra over (R,λ). Assume either that (i) P has free rank ≥ 2, or (ii) Λ is ample in C and $P = C$. Write $x \in H(P)$ as $x = (p_0 a + p_1, q_0 b + q_1)$. Then there is an element $\sigma \in H(E(P)) \cdot EU(H(P)$ such that $\sigma(x) = (p_0 a' + p_1', q_0 b' + q_1')$ and $O(x) = Aa'$. In case (i), $\sigma \in EU(H(P_0), Q; H(P_1))$ where $Q = P_0$ or $\bar{P}_0$.

Definition: Let (M, [h]) be a (λ,Λ)-quadratic module. An element $x \in M$ is [h]-unimodular if there exists $y \in M$ such that $\langle x, y\rangle = 1$.

If (M,[h]) is non-singular then an element is [h]-unimodular if and only if it is unimodular.

The following is our main result in the quadratic case.

(2.8) Theorem: Let V be a (λ,Λ)–quadratic module which has (A,B)–hyperbolic rank ≥ 1 at all but finitely many primes, and put $(M, [h]) = V \perp \mathbb{H}(P)$ where P is A-free of rank 2. Suppose there exists a subgroup $G_1 \subseteq U(\mathbb{H}(P))$ such that $\epsilon_*(G_1)$ acts transitively on the set of unimodular elements in $\mathbb{H}(\epsilon_*(P))$ of fixed length $[h(x,x)]$. Then

$$G = < G_1, EU(\mathbb{H}(P), Q; V), \mathbb{H}(E(P))\cdot EU(\mathbb{H}(P))>$$

where $Q = P$ or $\bar{P}$, acts transitively on the set of [h]-unimodular elements of a fixed length, and the set of hyperbolic pairs and hyperbolic planes in M.

Proof: The same reduction used in [3;(3.5),p.236] shows that it is enough to prove that G acts transitively on the set of [h]-unimodular elements of a fixed length in M. One can check that G contains all transvections $\sigma_{p_0,a,v}$ with $v \in (p_0)^{\perp} = V \oplus \mathbb{H}(P_1) \oplus p_0A$ (see [3;(3.11),p.143] and [3;(5.6),p.98]).

(i) Let $x = (v; p,q) \in V \perp \mathbb{H}(P)$ be an [h]-unimodular element. Since P is free of rank 2, it follows as in [3;p.181] that we may assume (p,q) is unimodular. More precisely, there exists some $y \in M$ such that $< x, y > = 1$ and so $<V, v> + O(p) + O(q) = A$. Choose $w \in V$ so that $<v, w> + O(p) + O(q)$ contains 1; put $c = <v, w>$. From (1.1) there is a $p_1 \in P$ such that $O(p + p_1c) + O(q) = A$. Now apply the transvection $\sigma_{p_1,a,w}$ to x. This isometry lies in $EU(\mathbb{H}(P), P;V)$.

(ii) Since $\epsilon_*(G_1)$ acts transitively on the set of unimodular elements of fixed length in $\mathbb{H}(\epsilon_*(P))$ we may assume that $\epsilon_*(x) = \epsilon_*(v; p_0, q_0b)$, where $\bar{b} \equiv h_P(x,x) \bmod \Lambda$.

(iii) We may now achieve "$O(x) = Aa$ over $A[g]$" using (2.7) and the fact that P is free of rank 2. Here $g = \Pi \{m | m \in S\}$ where S is a finite set in X containing all the primes at which A is not maximal or V does not have (A,B)–hyperbolic rank ≥ 1. Furthermore by

(2.6) we may assume that V contains a non-zero totally isotropic submodule L which has (A,B)-free rank ≥ 1 at all primes not in S. Note that after step (ii), nothing needs to be done over B and (p,q) is still unimodular. This step uses $\mathbb{H}(E(P))\cdot EU(\mathbb{H}(P))$.

(iv) Let $\mathfrak{t} \subseteq R_0$ be the largest ideal such that $A\mathfrak{t} \subseteq Aa$, and put $X' = V(\mathfrak{t})$, $X'_\Lambda = V(\mathfrak{t}+\mathfrak{A}_\Lambda)$, $d'_\Lambda = \dim X'_\Lambda$. Let π: $A \longrightarrow A' = A/A\mathfrak{t}$ be the natural projection and note that $\dim X' = 0$ and $\dim X'_\Lambda \leq 0$. As in [3;p.244] we see that $\mathfrak{m} \notin X'$ for all $\mathfrak{m} \in S$, hence $\mathfrak{t} \neq 0$ and A' is semilocal. We have $O(p_1,q_1 + q_0b) + Aa = A$ and so $O(\pi p_1, \pi(q_1+q_0b)) + \pi(Aa) = A'$. Over $B' = B/B\mathfrak{t}$ we do nothing. Over the complementary factor C' of A', apply (1.2) to find an element $u \in \pi P_1$ such that u projects to zero over B' and

$$O(\pi p_1 - ub) + O(\pi q_1) + \pi(Aa) = A'.$$

(Note that this already holds over B' by step (ii)). Choose $z \in P_1$ such that $\pi z = u$ and $\epsilon_*(z) = 0$. Since $\mathfrak{t}$ and $\mathfrak{g}$ are relatively prime, we can choose $z \in (P_1)\cdot\mathfrak{g}$.

Note that $\sigma_{p_0,0,z} \in EU(\mathbb{H}(P))$ by [3;(3.10.2),p.142]. Then

$$\begin{aligned} \sigma(x) &= x + p_0 <z,x> - z<p_0,x> \\ &= (v;\ p_1 - zb + p_0(a + <z, q_1>),\ q). \end{aligned}$$

Therefore

$$O(p_1 - zb) + O(q_1) + A(a + <z, q_1>) + A\mathfrak{t} = A.$$

But $A\mathfrak{t} \subseteq Aa \subseteq O(q_1) + A(a + <z,q_1>)$, so after these changes, we may assume that

(2.9) $$O(p_1) + O(q_1) + Aa = A.$$

(v) Since πV has hyperbolic rank ≥ 1 over C' we can choose an isometry α: $\pi V \cong \mathbb{H}(C') \perp W'$ and extend it to an isometry of $\pi V \perp \mathbb{H}(\pi P)$ by the identity on $\mathbb{H}(\pi P)$. We now apply the first case of (2.7) to the element $\alpha(\pi(p_1, q_1)) \in \mathbb{H}(C') \perp \mathbb{H}(\pi P_1)$ over the semisimple ring C', where $A' = B' \times C'$. This uses an element $\sigma' \in EU(\mathbb{H}(\pi P_1), \pi Q; \mathbb{H}(C'))$ where $Q = P_1$ or $\bar{P}_1$. By (2.5), $\alpha^{-1}\circ\sigma'\circ\alpha \in EU(\mathbb{H}(\pi P_1), \pi Q; \pi V)$. Then there exists a lift σ of $\alpha^{-1}\circ\sigma'\circ\alpha$ to $U(M, [h])$ which lies in

$EU(\mathbb{H}(P_1), Q; V)$. After moving $x = (v; p,q)$ by σ we get

$$A = O(p_1) + At \subseteq O(p_1) + Aa = O(p).$$

Finally note that after this change p is unimodular and $\epsilon_*(x) = \epsilon_*(v; p_0, q_0 b)$, where $\bar{b} \equiv h_P(x,x) \bmod \Lambda$.

(vi) Since p is unimodular and $h(p,p) = 0$, $\mathbb{H}(P) = \mathbb{H}(pA) \perp \mathbb{H}(pA)^{\perp}$. If $\mathbb{H}(pA) = pA \oplus \bar{p}A$, where $\bar{p} \in \bar{P}$ then $\sigma_{\bar{p},d,\lambda v} \in EU(\mathbb{H}(P), \bar{P}; V)$ and

$$\begin{aligned}\sigma_{\bar{p},d,\lambda v}(x) &= x + \bar{p}<\lambda v,x> - \lambda v\bar{\lambda}<\bar{p},x> - \bar{p}\bar{\lambda}d<\bar{p},x> \\ &= (0; p, q').\end{aligned}$$

(vii) We now have a hyperbolic element $x = (p, q) \in \mathbb{H}(P)$ with p unimodular. Recall that V contains a non–zero totally isotropic submodule L which has (A,B)-free rank ≥ 1 at all but finitely many primes. We claim that after applying a suitable transformation in G, we can assume that $x = (p_0,q)$, with a possibly different q. For this we need to refer to the proof of (1.4) to find which linear automorphisms of $L \oplus P$ are necessary to move p to p_0, and then show that they are induced by isometries in G.

Since our element p starts out in P steps (i) and (ii) of (1.3) are not necessary. Step (iii) requires an element of $E(p_1A, P_0)$ which is induced by a suitable element of $\mathbb{H}(E(P)) \subseteq G$. Step (v) uses an elementary transformation given by a homomorphism $f: P_1 \longrightarrow L$, and these are induced elements of G using (2.6). Finally, in step (vi) we first construct a homomorphism $g_2: P_0 \oplus L \longrightarrow P_1$ by splitting $P_0 \oplus L = zA \oplus N$ where z is unimodular and $h(z,z) = 0 \pmod \Lambda$. To realise τ_2 by an isometry, we find a unitary submodule $\mathbb{H}(zA) \subseteq V \perp \mathbb{H}(P_0)$ and then work inside $\mathbb{H}(zA) \perp \mathbb{H}(P_1)$. By [3;(3.10.4), p.143] $\mathbb{H}(\tau_2|_{zA\oplus P_1}) \subseteq EU(\mathbb{H}(zA), \bar{z}\bar{A}; \mathbb{H}(P_1)) \subseteq G$. The remaining automorphisms τ_3, τ_4, τ_5 are in $E(P_0, P_1)$ and τ_6 is defined using $g_6 \in Hom(P_1, L)$. These are induced by elements of $\mathbb{H}(E(P))$ or $EU(\mathbb{H}(P),\bar{P}; V)$ using (2.6).

Write $q = q_0 b - q_1 \in q_0 A \oplus \bar{P}_1$. The transvection $\sigma_{q_1,0,q_0}$ belongs to $EU(\mathbb{H}(P))$ by

[3;(3.10.1), p.142], and

$$\sigma x = x - q_1 \langle q_0,x\rangle - q_0\bar{\lambda}\langle q_1,x\rangle.$$

Note that $\langle q_1,x\rangle = 0$ since x has no component in $\bar{P}_1$ and $\langle q_0,x\rangle = \langle q_0,p_o\rangle = 1$, so $\sigma x = x + q_1 = (p_0, q_0 b)$. We are now finished if x was unimodular. If x was a hyperbolic element, then $h(\sigma x,\sigma x) = h(p_0, q_0 b) = \bar{b} \pmod{\Lambda}$, and so $\bar{b} \in \Lambda$, since x and $\sigma(x)$ are isotropic.

In the hyperbolic plane $p_0A + q_0A$, the element $X_+(-b) = \begin{bmatrix} I & -b \\ 0 & I \end{bmatrix} \in EU(\mathbb{H}(P_0))$ transforms $p_0 + q_0 b$ into p_0 (the notation X_+ is from [3; p.130]).

Proof of Theorem 2: The argument is the same as for [3;(3.6), p.238] using our (2.8).

(2.10) Lemma: The group $\mathbb{H}(GL_2(\mathbb{Z}))\cdot EU(\mathbb{H}(\mathbb{Z} \oplus \mathbb{Z}))$ acts transitively on unimodular elements in $\mathbb{H}(\mathbb{Z} \oplus \mathbb{Z})$ of fixed length.

Proof: Let $P = p_0\mathbb{Z} \oplus p_1\mathbb{Z}$ (with dual basis q_0, q_1 for $\bar{P}$) and let $x = (p_0 a, p_1 b; q_0 c, q_1 d)$ be a unimodular element in $\mathbb{H}(P)$. We may assume that $d = 0$ after applying an element of $\mathbb{H}(E(P))$, so there exists an integer r such that $b + rc$ is a unit (mod a). Then

$$X_+\begin{pmatrix} 0 & -\lambda r \\ r & 0 \end{pmatrix}(x) = (p_0 a, p_1(b+rc); q_0 c, 0)$$

so that $O(p_0 a) + O(p_1(b+rc)) = \mathbb{Z}$. We may therefore assume in the beginning that for $x = (p_0 a, p_1 b; q_0 c, q_1 d)$, a and b are relatively prime. Using a suitable element of $\mathbb{H}(E(P))$ we get $x = (p_0, 0; q_0 c, q_1 d)$ and after applying $X_-\begin{pmatrix} 0 & \lambda d \\ -d & 0 \end{pmatrix}$ the result is $(p_0, 0; q_0 c, 0)$, where [c] is the length of x.

§ 3. Metabolic Forms

One way to obtain quadratic modules V with (A,B)-hyperbolic rank ≥ 1 at all but finitely many primes is to assume that V has a submodule H(L) where L has (A,B)-free rank ≥ 1. A generalization of this would be to assume that V contains a "metabolic form" on L. In this section we define a suitable notion of metabolic forms general enough for our applications elsewhere. The notation and conventions of § 2 will be used.

If N is an A-lattice and $g\colon \bar{N} \times \bar{N} \longrightarrow A$ is an R-bilinear form, let

$$[g] = \{g_\tau \mid g_\tau(\phi,\phi') = g(\phi,\phi') + \langle \phi, \tau(\phi')\rangle,\ \tau \in \mathrm{Hom}_R(\bar{N}, N)\}.$$

Any $\theta \in \mathrm{Ext}^1_A(\bar{N}, N)$ defines an extension

$$0 \longrightarrow N \xrightarrow{i} E \xrightarrow{j} \bar{N} \longrightarrow 0 \tag{3.1}$$

of A-lattices which splits over R. We say that [g] is *θ–sesquilinear* if there is a cocycle $\gamma \in \mathrm{Hom}_R(\bar{N} \otimes_R A, N)$ representing θ such that for all $a \in A$:

$$g(\phi a, \phi') = \bar{a} g(\phi, \phi')$$

(3.2)

$$g(\phi, \phi' a) = g(\phi, \phi')a - \langle \phi, \gamma(\phi', a) \rangle.$$

Note that any cocycle γ satisfies the relation:

$$\gamma(\phi, a_1 a_2) = \gamma(\phi, a_1)a_2 + \gamma(\phi a_1, a_2)$$

and serves as a way to specify the A-module structure E on the R-module $N \oplus \bar{N}$ given by θ: for $(x,\phi) \in N \oplus \bar{N}$ define

$$(x, \phi)\cdot a = (xa + \gamma(\phi, a), \phi a). \tag{3.3}$$

If we vary the choice of representative $g_\tau \in [g]$, then the new γ is $\gamma_\tau = \gamma + \delta\tau$, where

$$(\delta\tau)(\phi, a) = \tau(\phi)a - \tau(\phi a),$$

for some $\tau \in \mathrm{Hom}_R(\bar{N}, N)$, and all $a \in A$. Then $g_\tau(\phi,\phi') = g(\phi,\phi') + \langle \phi, \tau(\phi')\rangle$ satisfies (3.2). Given an extension (N, θ) and a θ-sesquilinear form [g], we define the *metabolic* (λ,Λ)-quadratic form $\mathbf{Met}(N, \theta, [g]) = (E, [q])$ as follows: pick a compatible γ, g satisfying (3.2) and set

$$q((x,\phi), (x',\phi')) = <\phi, x'> + g(\phi, \phi').$$

It is easy to check that q is sesquilinear in the usual sense if [g] is θ-sesquilinear.

These metabolic forms are non-singular when they exist but an arbitrary extension need not admit any such form. Suppose that N is reflexive and let τ denote the involution on $\mathrm{Ext}^1_A(\bar{N}, N)$ given by dualizing exact sequences $(N,\theta) \mapsto (N,\theta)^*$. An extension (N, θ) is *λ-self-dual* if N is reflexive and there is a commutative diagram

(3.4)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{i} & E & \xrightarrow{j} & \bar{N} & \longrightarrow & 0 \\ & & \| & & \downarrow h & & \| \bar{\lambda} & & \\ 0 & \longrightarrow & N & \xrightarrow{j^*} & \bar{E} & \xrightarrow{i^*} & \bar{N} & \longrightarrow & 0 \end{array}$$

If $h^* = \lambda h$ then h is the adjoint of a metabolic hermitian form on E. We will define a homomorphism

$$\rho\colon \{ (N,\theta)^* = \lambda(N,\theta) \} \subsetneq \mathrm{Ext}^1_A(\bar{N}, N) \longrightarrow H^1(\mathbb{Z}/2; \mathrm{Hom}_A(\bar{N}, N))$$

where $\mathrm{Hom}_A(\bar{N}, N)$ has the involution $\alpha \mapsto \bar{\lambda}\alpha^*$. We will show that $\rho(N,\theta)$ is the obstruction for finding a λ-self-dual map h. Choose an R-section s: $\bar{N} \longrightarrow E$ inducing a cocycle γ and identify $E = N \oplus \bar{N}$ as above. Then the lower sequence is split over R by s^* leading to an identification of $\bar{E} = N \oplus \bar{N}$. In these coordinates, for any A-map h making the diagram (3.4) commute,

$$h(x,\phi) = (x + s^*hs(\phi), \bar{\lambda}\phi)$$

and similarly

$$h^*(x,\phi) = (\lambda x + s^*h^*s(\phi), \phi).$$

Now $(h^*)^{-1} \circ \lambda h(x,\phi) = (x + \rho(h)(\phi), \phi)$ where $\rho(h) = s^*hs - \bar{\lambda}s^*h^*s$. Note that $\rho(h)$ is independent of the choice of the section s. Since $(h^*)^{-1} \circ \lambda h$ is an A-map, we can check using (3.3) that $\rho(h)$ is also an A-map. Similarly, by computing $h^* \circ (\bar{\lambda}h^{-1})$ and comparing with the formula for the dual, we see that $\rho(h)^* = -\lambda\rho(h)$. Moreover the cohomology class

$$[\rho(h)] \in H^1(\mathbb{Z}/2; \mathrm{Hom}_A(\bar{N}, N))$$

is independent of the choice of h. Define $\rho(N, \theta) = [\rho(h)]$ for any h making the diagram (3.4) commute.

(3.5) Proposition: If N is a reflexive A-module and (N, θ) is a self-dual extension, then (N, θ) admits a metabolic λ-hermitian form if and only if $\rho(N, \theta) = 0 \in H^1(\mathbb{Z}/2; \mathrm{Hom}_A(\bar{N}, N))$.

Note that a metabolic λ-hermitian form is unique up to isometry if it admits a quadratic refinement. We want to identify the obstruction to obtaining a quadratic refinement for a given metabolic λ-hermitian form h. Let

$$\eta\colon \ker \rho \longrightarrow \mathrm{coker}\,\{\, \hat{H}^0(\mathbb{Z}/2; \mathrm{Hom}_A(\bar{N}, N)) \longrightarrow \hat{H}^0(\mathbb{Z}/2; \mathrm{Hom}_R(\bar{N}, N)) \,\}.$$

be the homomorphism defined by $\eta(h) = [s^* hs]$.

(3.6) Proposition: Suppose that (N,θ) admits a metabolic λ-hermitian form. Then (N,θ) admits a metabolic (λ,Λ)-quadratic form with respect to the minimal form parameter if and only if $\eta(N,\theta) = 0$.

Suppose now that $R = \mathbb{Z}$ and $A = \mathbb{Z}\pi$ where π is a finite group. Then each lattice L over A is reflexive. Let $N = \Omega^k\mathbb{Z}$, the kernel of a projective resolution F_* of $\mathbb{Z}$ of length k (see (0.1) for the case k = 3). We will show that every element of $\mathrm{Ext}^1_A(\bar{N}, N)$ is $(-1)^{k+1}$-self-dual.

(3.7) Lemma: Let $N = \Omega^k\mathbb{Z}$. The involution τ given by dualizing exact sequences induces multiplication by $(-1)^{k+1}$ on $\mathrm{Ext}^1_A(\bar{N}, N)$.

Proof: Let $\bar{X}$ be a projective resolution of $\bar{N}$ and X the dual co-resolution of N. We have two isomorphisms $\alpha, \beta : \mathrm{Ext}^1(\bar{N}, N) \cong H^1(\mathrm{Hom}_A(\bar{X}, X))$ comparing an extension with $\bar{X}$ or

X respectively. Note that over $A = \mathbb{Z}\pi$ we can use X instead of an injective co-resolution for computing $\mathrm{Ext}^i(\bar{N}, N)$. It is not difficult to see that $\alpha = -\beta$. Let t be the involution on $H^1(\mathrm{Hom}_A(\bar{X}, X))$ induced by dualization. By construction, $\alpha\tau = t\beta$ implying $\alpha\tau\alpha^{-1} = -t$.

Note that $\mathrm{Hom}_A(\bar{X}, X) \cong \mathrm{Hom}_{\mathbb{Z}}(\bar{X}, X) \otimes_A \mathbb{Z}$, and that $\mathrm{Hom}_{\mathbb{Z}}(\bar{X}, X)$ is a co-resolution of $\mathrm{Hom}_{\mathbb{Z}}(\bar{N}, N)$. Thus

$$H^i(\mathrm{Hom}_A(\bar{X}, X)) = H^i(\mathrm{Hom}_{\mathbb{Z}}(\bar{X}, X) \otimes_A \mathbb{Z}) = H^i(\pi; N \otimes_{\mathbb{Z}} N)$$

and under these identifications t corresponds to the involution induced by the flip map s: $x \otimes y \mapsto y \otimes x$ on $N \otimes N$.

Now we follow an argument suggested by R. Swan. Extend the projective resolution F defining N to a projective resolution $\hat{F}$ of $\mathbb{Z}$. Let f be the chain map on $\hat{F} \otimes_{\mathbb{Z}} \hat{F}$ mapping $x \otimes y \mapsto (-1)^{\deg(x)\deg(y)} y \otimes x$. Since f induces the identity on $\mathbb{Z}$ it induces the identity on all the derived functors. We have the similar chain map on $F \otimes_{\mathbb{Z}} F$ which on $F_{2k} = N \otimes N$ is $(-1)^k s$. Now we consider $F \otimes_{\mathbb{Z}} F$ as part of a co-resolution of $N \otimes N$ ending in $\mathbb{Z}$. Similarly we consider $\hat{F} \otimes_{\mathbb{Z}} \hat{F}$ a part of a complete co-resolution of $\mathbb{Z}$. Then

$$H^1(\pi; N \otimes_{\mathbb{Z}} N) = H^1(\mathrm{Hom}_A(\mathbb{Z}, F \otimes_{\mathbb{Z}} F)) \cong H^1(\mathrm{Hom}_A(\mathbb{Z}, \hat{F} \otimes_{\mathbb{Z}} \hat{F})).$$

where the last isomorphism is induced by the obvious chain map $\hat{F} \longrightarrow F$. Thus

$$\alpha\tau\alpha^{-1} = -s = (-1)^{k+1} f^* = (-1)^{k+1}.$$

(3.8) Example: Now we restrict to groups π of odd order. Since $\mathrm{Ext}^1_{\mathbb{Z}\pi}(\bar{N}, N)$ then has odd order $\rho(N,\theta)$ and $\eta(N,\theta)$ vanish for each λ-self-dual extension. In particular for $N = \Omega^k\mathbb{Z}$, each extension (N,θ) admits a metabolic (λ,Λ)-quadratic form whose λ-symmetrization is unique up to isometry.

References

[1] Bak, A., "The stable structure of quadratic modules", Thesis, Columbia University, 1969.

[2] Bass, H., Algebraic K–Theory, W.A. Benjamin, New York 1968

[3] Bass, H., "Unitary algebraic K–theory", Algebraic K–Theory III: Hermitian K–theory and Geometric Applications. Lect. Notes in Math. 343, 57–209, Springer Verlag, 1973.

[4] C.W.Curtis and I.Reiner. Representation Theory of Finite Groups and Associative Algebras. John Wiley & Sons. New York, 1962.

[5] Hambleton, I., Kreck, M., "On the classification of topological 4–manifolds with finite fundamental group", Math. Ann. 280 (1988), 85–104.

[6] Hambleton, I., Kreck, M., "Smooth structures on algebraic surfaces with cyclic fundamental group", Invent. Math.91 (1988), 53–59.

[7] Jacobinski, H., "Genera and decompositions of lattices over orders", Acta. Math. 121 (1968), 1–29.

[8] Swan, R.G., "Projective modules over group rings and maximal orders", Annals of Mathematics 76 (1962), 55–61.

[9] Swan, R. G., "K–theory of finite groups and orders", Notes by E.G.Evens. Lect. Notes in Math. 149, Springer–Verlag, Berlin 1970.

[10] Swan, R.G., "Projective modules over binary polyhedral groups", J. reine u. angew. Math. 342 (1983), 66–172.

[11] Vaserstein, L. N., "Stability of unitary and orthogonal groups over rings with involution", Math. Sbornik 81 (1970), 328–351.

McMaster University,
Hamilton, Ontario, Canada

Max-Planck Institut für Mathematik
Bonn, West Germany

December 15, 1989