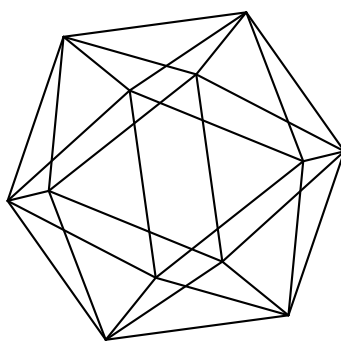


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Automorphism groups of compact complex surfaces

by

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AUTOMORPHISM GROUPS OF COMPACT COMPLEX SURFACES

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ABSTRACT. We study automorphism groups and birational automorphism groups of compact complex surfaces. We show that the automorphism group of such surface X is always Jordan, and the birational automorphism group is Jordan unless X is birational to a product of an elliptic and a rational curve.

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1. INTRODUCTION

It often happens that some infinite subgroups exhibit a nice and simple behavior on the level of their finite subgroups. An amazing example of such situation is given by the following result due to C. Jordan (see [CR62, Theorem 36.13]).

Theorem 1.1. *There is a constant $J = J(n)$ such that for every finite subgroup $G \subset \text{GL}_n(\mathbb{C})$ there exists a normal abelian subgroup $A \subset G$ of index at most J .*

This motivates the following definition.

Definition 1.2 (see [Pop11, Definition 2.1]). A group Γ is called *Jordan* (alternatively, we say that Γ *has Jordan property*) if there is a constant J such that for every finite subgroup $G \subset \Gamma$ there exists a normal abelian subgroup $A \subset G$ of index at most J .

In other words, Theorem 1.1 says that the group $\text{GL}_n(\mathbb{C})$ is Jordan. The same applies to any linear algebraic group, since it can be realized as a subgroup of a general linear group.

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It was noticed by J.-P. Serre that Jordan property sometimes holds for groups of birational automorphisms.

Theorem 1.3 ([Ser09, Theorem 5.3], [Ser10, Théorème 3.1]). *The group of birational automorphisms of \mathbb{P}^2 over the field \mathbb{C} (or any other field of characteristic 0) is Jordan.*

Yu. Zarhin pointed out in [Zar14] that there are projective complex surfaces whose birational automorphism groups are not Jordan; they are birational to products $E \times \mathbb{P}^1$, where E is an elliptic curve. The following result of V. Popov classifies projective surfaces with non-Jordan birational automorphism groups.

Theorem 1.4 ([Pop11, Theorem 2.32]). *Let X be a projective surface over \mathbb{C} . Then the group of birational automorphisms of X is not Jordan if and only if X is birational to $E \times \mathbb{P}^1$, where E is an elliptic curve.*

Automorphism groups having Jordan property were studied recently in many different contexts. Yu. Prokhorov and C. Shramov in [PS16b, Theorem 1.8] and [PS14, Theorem 1.8] proved that this property holds for groups of birational selfmaps of rationally connected algebraic varieties, and some other algebraic varieties of arbitrary dimension. Actually, their results were initially obtained modulo a conjectural boundedness of terminal Fano varieties (see e.g. [PS16b, Conjecture 1.7]), which was recently proved by C. Birkar in [Bir16, Theorem 1.1]. Also Yu. Prokhorov and C. Shramov classified Jordan birational automorphism groups of algebraic threefolds in [PS16d]. Some results about birational automorphisms of conic bundles were obtained by T. Bandman and Yu. Zarhin in [BZ17a]. For other results on Jordan birational automorphism groups see [PS16a], [PS16c], and [Yas16].

S. Meng and D.-Q. Zhang proved in [MZ15b] that an automorphism group of any projective variety is Jordan. T. Bandman and Yu. Zarhin proved a similar result for automorphism groups of quasi-projective surfaces in [BZ15], and also in some particular cases in arbitrary dimension in [BZ17b]. For a survey of some other relevant results see [Pop14].

É. Ghys asked (following a more particular question posed earlier by W. Feit) whether a diffeomorphism group of a smooth compact manifold is always Jordan. Recently B. Csikós, L. Pyber, and E. Szabó in [CPS14] provided a counterexample following the method of [Zar14]; see also [Mun17b] for a further development of this method, and [Pop16, Corollary 2] for a non-compact counterexample. However, Jordan property holds for diffeomorphism groups in many cases; see [Mun16a], [Mun14], [MT15], [Mun13], [GZ13], [Zim12], [Zim14a], [Zim14b], [MZ15a], and references therein. Also there are results for groups of symplectomorphisms, see [Mun17a] and [Mun16b].

The goal of this paper is to generalize Theorem 1.4, and to some extent the result of [MZ15b], to a different setting, namely, to the case of compact complex surfaces (see §3 below for basic definitions and background). We prove the following.

Theorem 1.5. *Let X be a connected compact complex surface. Then the automorphism group of X is Jordan.*

One can also show (see [Mun13, Theorem 1.3] or Theorem 2.9 below) that the number of generators of any finite subgroup of an automorphism group of a compact complex surface X , and actually of a diffeomorphism group of an arbitrary compact manifold, is bounded by a constant that depends only on X .

The main result of this paper is as follows.

Theorem 1.6. *Let X be a connected compact complex surface. Then the group of birational automorphisms of X is not Jordan if and only if X is birational to $E \times \mathbb{P}^1$, where E is an elliptic curve. Moreover, there always exists a constant $R = R(X)$ such that every finite subgroup of the birational automorphism group of X is generated by at most R elements.*

The plan of the paper is as follows. In §2 we collect some elementary facts about Jordan property, and other boundedness properties for subgroups. In §3 we recall the basic facts from the theory of compact complex surfaces, most importantly their Enriques–Kodaira classification. In §4 we recall some important general facts concerning automorphisms of complex spaces. In §5 we study automorphism groups of non-projective surfaces with non-zero topological Euler characteristic; an important subclass of such surfaces is formed by minimal surfaces of class VII with non-zero second Betti number (which are still not completely classified). In §6 and §7 we study automorphism groups of Hopf and Inoue surfaces, respectively; these are all possible minimal surfaces of class VII with trivial second Betti number. In §8 we study automorphism groups of Kodaira surfaces. In §9 we study automorphism groups of other minimal surfaces of non-negative Kodaira dimension, and prove Theorems 1.5 and 1.6. Finally, in Appendix A we collect some auxiliary group-theoretic results about infinite discrete groups and their automorphisms that we use in §7 and §8.

Our general strategy is to consider the compact complex surfaces according to Enriques–Kodaira classification. One feature of our proof that we find interesting to mention is that Inoue and Kodaira surfaces are treated by literally the same method which is based on the fact that they are (diffeomorphic to) solvmanifolds (cf. [Has05, Theorem 1]), and for which we never met a proper analog in the projective situation. It is possible that one can generalize this approach to higher dimensional solvmanifolds. Note also that some of our theorems follow from more general results of I. Mundet i Riera, cf. Theorems 5.1 and 5.12 (and also the discussion in the end of §5).

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2. JORDAN PROPERTY

In this section we recall some group-theoretic properties related to the Jordan property, and prove a couple of auxiliary results about them.

Definition 2.1. We say that a group Γ *has bounded finite subgroups* if there exists a constant $B = B(\Gamma)$ such that for any finite subgroup $G \subset \Gamma$ one has $|G| \leq B$.

The following result is due to H. Minkowski (see e.g. [Ser07, Theorem 5] and [Ser07, §4.3]).

Theorem 2.2. *For every n the group $GL_n(\mathbb{Q})$ has bounded finite subgroups.*

Definition 2.3. We say that a group Γ is *strongly Jordan* if it is Jordan, and there exists a constant $R = R(\Gamma)$ such that every finite subgroup in Γ is generated by at most R elements.

Note that Definition 2.3 is equivalent to a similar definition in [BZ15]. An example of a strongly Jordan group is given by $\mathrm{GL}_n(\mathbb{C})$. This follows from the fact that every abelian subgroup of $\mathrm{GL}_n(\mathbb{C})$ is conjugate to a group that consists of diagonal matrices. Note however that even the group \mathbb{C}^* contains *infinite* abelian subgroups of arbitrarily large rank.

The following elementary result will be useful to study Jordan property.

Lemma 2.4. *Let*

$$1 \longrightarrow \Gamma' \longrightarrow \Gamma \longrightarrow \Gamma''$$

be an exact sequence of groups. Then the following assertions hold.

- (i) *If Γ' is Jordan (respectively, strongly Jordan) and Γ'' has bounded finite subgroups, then Γ is Jordan (respectively, strongly Jordan).*
- (ii) *If Γ' has bounded finite subgroups and Γ'' is strongly Jordan, then Γ is strongly Jordan.*

Proof. Assertion (i) is obvious. For assertion (ii) see [PS14, Lemma 2.8] or [BZ15, Lemma 2.2]. \square

It is easy to see that if Γ_1 is a subgroup of finite index in Γ_2 , then Γ_2 is Jordan (respectively, strongly Jordan) if and only so is Γ_1 . At the same time Jordan property, as well as strong Jordan property, does not behave well with respect to quotients by infinite groups. Namely, a quotient of a strongly Jordan group by its subgroup may fail to be Jordan or to have all of its finite subgroups generated by a bounded number of elements. In spite of this we will be able to control the properties of some quotients by infinite groups that will be important for us.

Lemma 2.5. *Let A be an abelian group whose torsion subgroup A_t is isomorphic to $(\mathbb{Q}/\mathbb{Z})^n$, and let $\Lambda \subset A$ be a subgroup isomorphic to \mathbb{Z}^m . Then the quotient group $\Gamma = A/\Lambda$ is strongly Jordan.*

Proof. The group Γ is abelian and thus Jordan. Let $V \subset \Gamma$ be a finite subgroup and let $\tilde{V} \subset A$ be its preimage. Clearly, \tilde{V} is finitely generated and can be decomposed into a direct product $\tilde{V} = \tilde{V}_t \times \tilde{V}_f$ of its torsion and torsion free parts. In particular, \tilde{V}_f is a free abelian group. Since $\tilde{V}_f/(\tilde{V}_f \cap \Lambda)$ is a finite group, one has

$$\mathrm{rk} \tilde{V}_f = \mathrm{rk}(\tilde{V}_f \cap \Lambda) \leq \mathrm{rk} \Lambda = m.$$

The group \tilde{V}_t is contained in $A_t \cong (\mathbb{Q}/\mathbb{Z})^n$ and so it can be generated by n elements. Thus \tilde{V} can be generated by $n + m$ elements, and the images of these elements in Γ generate the subgroup V . \square

Lemma 2.6. *Let*

$$(2.7) \quad 1 \longrightarrow \Gamma' \longrightarrow \Gamma \longrightarrow \Gamma''$$

be an exact sequence of groups. Suppose that Γ' is central in Γ (so that in particular Γ' is abelian) and there exists a constant R such that every finite subgroup of Γ' is generated by at most R elements. Suppose also that there exists a constant J such that for every finite subgroup $G \subset \Gamma''$ there is a cyclic subgroup $C \subset G$ of index at most J (so that in particular Γ'' is strongly Jordan). Then the group Γ is strongly Jordan.

Proof. Let $G \subset \Gamma$ be a finite subgroup. The exact sequence (2.7) induces an exact sequence of groups

$$1 \longrightarrow G' \longrightarrow G \longrightarrow G'',$$

where G' is a subgroup of Γ' (in particular, G' is abelian), while G'' is a subgroup of Γ'' . There is a subgroup $\bar{G} \subset G$ of index at most J such that \bar{G} contains G' , and the quotient \bar{G}/G' is a cyclic group. To prove that the group Γ is Jordan it is enough to check that \bar{G} is an abelian group. The latter follows from the fact that G' is a central subgroup of \bar{G} .

The assertion about the bounded number of generators is obvious. \square

Lemma 2.8. *Let Λ be a finitely generated central subgroup of $\mathrm{GL}_2(\mathbb{C})$. Then the quotient group $\Gamma = \mathrm{GL}_2(\mathbb{C})/\Lambda$ is strongly Jordan.*

Proof. We have an exact sequence of groups

$$1 \longrightarrow \mathbb{C}^*/\Lambda \longrightarrow \Gamma \longrightarrow \mathrm{PGL}_2(\mathbb{C}) \longrightarrow 1.$$

The group \mathbb{C}^*/Λ is a central subgroup of Γ . Also, the group \mathbb{C}^*/Λ is strongly Jordan by Lemma 2.5.

On the other hand, we know from the classification of finite subgroups of $\mathrm{PGL}_2(\mathbb{C})$ that every finite subgroup therein contains a cyclic subgroup of bounded index. Therefore, the assertion follows from Lemma 2.6. \square

Most of the groups we will be working with in the remaining part of the paper will be strongly Jordan. However, we will only need to check Jordan property for them due to the following result.

Theorem 2.9 ([Mun13, Theorem 1.3]). *For any compact manifold X there is a constant R such that every finite group acting effectively by diffeomorphisms of X can be generated by at most R elements.*

3. MINIMAL SURFACES

In this section we recall the basic properties of compact complex surfaces. Everything here (as well as in §4 below) is well known to experts, but in some important cases we provide proofs for the reader's convenience. Starting from this point we will always assume that our complex surfaces are connected.

Throughout this paper \mathcal{K}_X denotes the canonical line bundle of a complex manifold X . One has $c_1(\mathcal{K}_X) = -c_1(X)$. Given a divisor D on X , we will denote by $[D]$ the corresponding class in $H^2(X, \mathbb{Z})$.

Definition 3.1. Let X and Y be compact complex surfaces. A proper holomorphic map $f: X \rightarrow Y$ is said to be a *proper modification* if there are analytic subsets $Z_1 \subsetneq X$ and $Z_2 \subsetneq Y$ such that the restriction $f_{X \setminus Z_1}: X \setminus Z_1 \rightarrow Y \setminus Z_2$ is biholomorphic. A *birational* (or *bimeromorphic*) *map* $X \dashrightarrow Y$ is an equivalence class of diagrams

$$\begin{array}{ccc} & Z & \\ f \swarrow & & \searrow g \\ X & \dashrightarrow & Y \end{array}$$

where f and g are proper modifications, modulo natural equivalence relation.

Birational maps from a given compact complex surface X to itself form a group, which we will denote by $\text{Bir}(X)$. As usual, we say that two complex surfaces are birationally equivalent, or just birational, if there exists a birational map between them.

Remark 3.2. If X and Y are birationally equivalent compact complex surfaces, then the fields of meromorphic functions on X and Y are isomorphic. The converse is not true if the algebraic dimension of X (and thus also of Y) is less than 2.

A (-1) -curve on a compact complex surface is a smooth rational curve with self-intersection equal to -1 . A compact complex surface is *minimal* if it does not contain (-1) -curves.

Lemma 3.3 (see [BHPVdV04, §IV.6]). *Let X be a compact complex surface. Suppose that there is a line bundle \mathcal{L} on X such that $\mathcal{L}^2 > 0$. Then X is projective.*

Proposition 3.4. *Let X be a minimal surface. Suppose that X is neither rational nor ruled. Then every birational map from an arbitrary compact complex surface X' to X is a proper modification. In particular, X is the unique minimal model in its class of birational equivalence, and $\text{Bir}(X) = \text{Aut}(X)$.*

Proof. We may assume that X is not projective, since otherwise the assertion is well known. Suppose that

$$\begin{array}{ccc} & Z & \\ g \swarrow & & \searrow f \\ X' & \text{-----} & X \end{array}$$

is a birational map that is not a proper modification. Then there exists a (-1) -curve C contracted by g but not contracted by f . Thus C meets a fiber $f^{-1}(x)$ for some point $x \in X$, since otherwise X would contain a (-1) -curve. Contracting (-1) -curves in $f^{-1}(x)$ consecutively, we get a surface S with a proper modification $h: Z \rightarrow S$, and a proper modification $t: S \rightarrow X$ such that $C_1 = h(C)$ is a (-1) -curve and there exists another (-1) -curve C_2 meeting C_1 and contracted by t . If $C_1 \cdot C_2 > 1$, then $(C_1 + C_2)^2 > 0$ and the surface S is projective by Lemma 3.3. Assume that $C_1 \cdot C_2 = 1$. Then for $n \gg 0$ we have

$$c_1(\mathcal{K}_S \otimes \mathcal{O}_S(-nC_1 - nC_2))^2 = c_1(S)^2 + 4n > 0,$$

so that the surface S is again projective by Lemma 3.3. The obtained contradiction completes the proof. \square

Given a compact complex surface X , we can consider its *pluricanonical map*, which is the rational map given by a linear system $|\mathcal{K}_X^{\otimes m}|$ for $m \gg 0$. The dimension of its image is called the *Kodaira dimension* of X and is denoted by $\kappa(X)$; if the linear system $|\mathcal{K}_X^{\otimes m}|$ is empty for all $m > 0$, we put $\kappa(X) = -\infty$. By $b_i(X)$ we denote the i -th Betti number of X . By $a(X)$ we denote the algebraic dimension of X , i.e. the transcendence degree of the field of meromorphic functions on X .

Theorem 3.5 (see [BHPVdV04, Corollary IV.6.5]). *A compact complex surface X is projective if and only if $a(X) = 2$.*

The following is the famous Enriques–Kodaira classification of compact complex surfaces, see e.g. [BHPVdV04, Chapter VI].

Theorem 3.6. *Let X be a minimal compact complex surface. Then X is of one of the following types.*

$\kappa(X)$	type	$a(X)$	$b_1(X)$	$\chi_{\text{top}}(X)$
$-\infty$	rational surfaces	2	0	3, 4
	ruled surfaces of genus $g > 0$	2	$2g$	$4(1 - g)$
	surfaces of class VII	0, 1	1	≥ 0
0	complex tori	0, 1, 2	4	0
	K3 surfaces	0, 1, 2	0	24
	Enriques surfaces	2	0	12
	bielliptic surfaces	2	2	0
	primary Kodaira surfaces	1	3	0
	secondary Kodaira surfaces	1	1	0
1	properly elliptic surfaces	1, 2		≥ 0
2	surfaces of general type	2	$\equiv 0 \pmod{2}$	> 0

4. AUTOMORPHISMS

In this section we recall some important general facts about automorphisms of complex spaces.

Let U be a reduced complex space, see e.g. [Ser56] or [Mal68] for a definition and basic properties. Recall that a complex space is called *irreducible* if it cannot be represented as a union of two proper closed analytic subsets. We denote by $T_{P,U}$ the Zariski tangent space (see [Mal68, §2]) to U at a point $P \in U$.

Proposition 4.1 (cf. [BB73, Lemma 2.4], [Pop14, Lemma 4]). *Let U be an irreducible Hausdorff reduced complex space, and $\Gamma \subset \text{Aut}(U)$ be a finite group. Suppose that Γ has a fixed point P on U . Then the natural representation*

$$\Gamma \longrightarrow \text{GL}(T_{P,U})$$

is faithful.

Proof. Assume the contrary. Let $\mathfrak{m} = \mathfrak{m}_{P,U}$ be the maximal ideal of the local ring $\mathcal{O}_{P,U}$. We claim that the exact sequence

$$0 \longrightarrow \mathfrak{m}^2 \xrightarrow{\nu} \mathfrak{m} \xrightarrow{\varsigma} \mathfrak{m}/\mathfrak{m}^2 \longrightarrow 0$$

of Γ -modules splits. Indeed, take elements $f_1, \dots, f_n \in \mathfrak{m}$ such that their images $\varsigma(f_i)$ generate $\mathfrak{m}/\mathfrak{m}^2$ and consider the vector space $W \subset \mathfrak{m}$ generated by all $g \cdot f_i$, $g \in \Gamma$. This space is finite-dimensional and Γ -invariant. Hence $\mathfrak{m}^2 \cap W$ is a direct summand, i.e.

$$W = V \oplus (\mathfrak{m}^2 \cap W)$$

as a Γ -module for some V . Thus the restriction $\varsigma|_V: V \rightarrow \mathfrak{m}/\mathfrak{m}^2$ is an isomorphism. Therefore, one has

$$(4.2) \quad \mathfrak{m} = V \oplus \mathfrak{m}^2.$$

It is clear that $T_{P,U} \cong V^\vee$, and so the action of Γ on V is not faithful. Let $\Gamma_0 \subset \Gamma$ be the kernel of this action, and let $V^d \subset \mathfrak{m}$ be the subspace generated by all products of at most d elements of V . We claim that

$$(4.3) \quad \mathfrak{m}^d = V^d + \mathfrak{m}^{d+1}.$$

We prove this claim by induction on d . For $d = 1$ it coincides with (4.2). Assume that this claim holds for some d . Take any element $f \in \mathfrak{m}^{d+1}$. It can be written in the form

$$f = \sum f_i w_i, \quad f_i \in \mathfrak{m}^d, \quad w_i \in \mathfrak{m}.$$

According to (4.2) and (4.3) we have

$$\begin{aligned} f_i &= s_i + h_i, & s_i &\in V^d, & h_i &\in \mathfrak{m}^{d+1}, \\ w_i &= u_i + v_i, & u_i &\in V, & v_i &\in \mathfrak{m}^2. \end{aligned}$$

Therefore,

$$f = \sum (s_i + h_i)(u_i + v_i) = \sum s_i u_i + \sum (s_i v_i + h_i u_i + h_i v_i) \in V^{d+1} + \mathfrak{m}^{d+2}.$$

This proves (4.3) for $d + 1$.

Therefore the restriction to V^d of the natural map

$$\mathfrak{m}^d \longrightarrow \mathfrak{m}^d / \mathfrak{m}^{d+1}$$

is surjective. Hence Γ_0 acts trivially on $\mathfrak{m}^d / \mathfrak{m}^{d+1}$ for any d .

Take any element $f \in \mathfrak{m}$. By the above we have

$$f - g \cdot f \in \mathfrak{m}^{d+1}$$

for every $g \in \Gamma_0$ and every $d > 0$. On the other hand, one has $\cap \mathfrak{m}^d = 0$ (see e.g. [AM69, Corollary 10.18]). This implies that $f = g \cdot f$, i.e. f is Γ_0 -invariant. Thus Γ_0 acts trivially on \mathfrak{m} and also on $\mathcal{O}_{P,U} \cong \mathbb{C} \oplus \mathfrak{m}$. This means that the action of Γ_0 on the germs of holomorphic functions at P is trivial.

Let U' be a sufficiently small Γ_0 -invariant irreducible neighborhood of P . By definition of a reduced complex space, U' is isomorphic to a subset in \mathbb{C}^N , and thus its points are separated by holomorphic functions. We claim that the action of Γ_0 on U' is trivial. Indeed, choose a non-trivial element $g \in \Gamma_0$, and suppose that there is a point $P_1 \in U'$ such that $P_2 = g(P_1)$ is different from P_1 . Let f be a holomorphic function on U' such that $f(P_1) \neq f(P_2)$. Then $g \cdot f \neq f$. However, the germs of f and $g \cdot f$ at P should be the same. Since U' is irreducible, this gives a contradiction.

Now let U_0 be the maximal open subset of U such that U_0 contains P , and the action of Γ_0 on U_0 is trivial; the above argument guarantees that U_0 is not empty. By assumption one has $U_0 \neq U$. Since U is irreducible, this implies that there is a point Q that is contained in the closure of U_0 , but not in U_0 itself. If Q is Γ_0 -invariant, one can choose a Γ_0 -invariant irreducible neighborhood U'_0 of Q that is isomorphic to a subset of \mathbb{C}^N . This neighborhood contains an open subset of Γ_0 -invariant points, means that the action of Γ_0 on the whole U'_0 is trivial. The latter is impossible by construction of U_0 . Thus for some element $g \in \Gamma_0$ one has $g(Q) \neq Q$. This is impossible because U is Hausdorff. The obtained contradiction completes the proof. \square

Remark 4.4. One cannot drop the assumption that U is irreducible in Proposition 4.1. Indeed, the assertion fails for the variety given by equation $xy = 0$ in \mathbb{A}^2 with coordinates x and y , the point P with coordinates $x = 1$ and $y = 0$, and the group $\Gamma \cong \mathbb{Z}/2\mathbb{Z}$ whose

generator acts by $(x, y) \mapsto (x, -y)$. Similarly, the assertion fails for the simplest example of a non-Hausdorff reduced complex space, namely, for two copies of \mathbb{A}^1 glued along the common open subset $\mathbb{A}^1 \setminus \{0\}$, and the natural involution acting on this space.

Remark 4.5. The following observation was pointed out to us by M. Brion. A crucial step in the proof of Proposition 4.1 is the fact that the Γ -orbit of a function from $\mathfrak{m}_{P,U}$ generates a finite-dimensional subspace in $\mathfrak{m}_{P,U}$. If U is an algebraic variety, this holds under a weaker assumption that Γ is a reductive group. However, in the holomorphic setting this is not true any more. Indeed, let $U = \mathbb{A}^1$, and let the group $\Gamma \cong \mathbb{C}^*$ act on U by scaling, so that the point $P = 0$ is fixed by Γ . Let f be a holomorphic function. Then the subspace of $\mathcal{O}_{P,U}$ generated by the Γ -orbit of f is finite dimensional if and only if f is a polynomial. We do not know if the assertion of Proposition 4.1 can be generalized to the case of reductive groups.

Proposition 4.1 easily implies the following result.

Corollary 4.6. *Let U be an irreducible Hausdorff reduced complex space, and $\Delta \subset \text{Aut}(U)$ be a subgroup. Suppose that Δ has a fixed point P on U , and let*

$$\varsigma: \Delta \longrightarrow \text{GL}(T_{P,U})$$

be the natural representation. Suppose that there is a subgroup $\Gamma \subset \Delta$ of finite index such that the restriction $\varsigma|_{\Gamma}$ is an embedding. Then ς is an embedding as well.

Proof. Let $\Delta_0 \subset \Delta$ be the kernel of ς . Since $[\Delta : \Gamma] < \infty$, we see that Δ_0 is finite. Thus Δ_0 is trivial by Proposition 4.1. \square

Another application of Proposition 4.1 is as follows.

Lemma 4.7. *Let X be a compact complex surface. Suppose that there is a finite non-empty $\text{Aut}(X)$ -invariant set \mathcal{S} of curves on X such that \mathcal{S} does not contain smooth elliptic curves. Then the group $\text{Aut}(X)$ is Jordan.*

Proof. Let C be one of the curves from \mathcal{S} . Then the group $\text{Aut}_C(X)$ of automorphisms of X that preserve the curve C has finite index in $\text{Aut}(X)$. Since C is not a smooth elliptic curve, there is a constant $B = B(C)$ such that every finite subgroup of $\text{Aut}_C(X)$ contains a subgroup of index at most B that fixes some point on C . Indeed, if C is singular, this is obvious; if C is a smooth rational curve, this follows from the classification of finite subgroups of $\text{Aut}(C) \cong \text{PGL}_2(\mathbb{C})$; if C is a smooth curve of genus at least 2, this follows from the Hurwitz bound. Now Proposition 4.1 implies that every finite subgroup of $\text{Aut}_C(X)$ contains a subgroup of index at most B that is embedded into $\text{GL}_2(\mathbb{C})$. Therefore, the group $\text{Aut}_C(X)$ is Jordan by Theorem 1.1, and hence the group $\text{Aut}(X)$ is Jordan as well. \square

5. NON-PROJECTIVE SURFACES WITH $\chi_{\text{top}}(X) \neq 0$

In this section we will (mostly) work with non-projective compact complex surfaces X with $\chi_{\text{top}}(X) \neq 0$. In this case, by the Enriques–Kodaira classification (see Theorem 3.6) one has $\chi_{\text{top}}(X) > 0$. The main purpose of this section is to prove the following result.

Theorem 5.1. *Let X be a non-projective compact complex surface with $\chi_{\text{top}}(X) \neq 0$. Then the group $\text{Aut}(X)$ is Jordan.*

Recall that an *algebraic reduction* of a compact complex surface X with $a(X) = 1$ is the morphism $\pi: X \rightarrow B$ given by the Stein factorization of the map $X \rightarrow \mathbb{P}^1$ defined by a non-constant meromorphic function. One can check that π is an elliptic fibration, see [BHPVdV04, Proposition VI.5.1].

Lemma 5.2. *Let X be a non-projective compact complex surface. If X contains an irreducible curve C which is not a smooth elliptic curve, then the group $\text{Aut}(X)$ is Jordan.*

Proof. We claim that the surface X contains at most a finite number of such curves. Indeed, if $a(X) = 0$, then X contains at most a finite number of curves at all [BHPVdV04, Theorem IV.8.2]. If $a(X) = 1$, then all curves on X are contained in the fibers of the algebraic reduction by Lemma 3.3. The latter fibration is elliptic, so every non-elliptic curve is contained in one of its degenerate fibers. Now the assertion follows from Lemma 4.7. \square

Lemma 5.3. *Let X be a compact complex surface with $\chi_{\text{top}}(X) \neq 0$. If $a(X) = 1$, then the group $\text{Aut}(X)$ is Jordan.*

Proof. Let $\pi: X \rightarrow B$ the algebraic reduction, so that B is a smooth curve and π is an elliptic fibration. Since $\chi_{\text{top}}(X) \neq 0$, the fibration π has at least one fiber X_b such that $F = (X_b)_{\text{red}}$ is not a smooth elliptic curve. So the group $\text{Aut}(X)$ is Jordan by Lemma 5.2. \square

For every compact complex surface X , we denote by $\overline{\text{Aut}}(X)$ the subgroup of $\text{Aut}(X)$ that consists of all elements acting trivially on $H^*(X, \mathbb{Z})$. This is a normal subgroup of $\text{Aut}(X)$, and the quotient group $\text{Aut}(X)/\overline{\text{Aut}}(X)$ has bounded finite subgroups by Theorem 2.2. Thus Lemma 2.4(i) implies that the group $\text{Aut}(X)$ is Jordan if and only if $\overline{\text{Aut}}(X)$ is Jordan.

Lemma 5.4. *Let X be a compact complex surface. Suppose that every irreducible curve contained in X is a smooth elliptic curve. Let $g \in \overline{\text{Aut}}(X)$ be an element of finite order, and $\Xi_0(g)$ be the set of all isolated fixed points of g . Then*

$$|\Xi_0(g)| = \chi_{\text{top}}(X).$$

Proof. The fixed locus $\Xi(g)$ of g is a disjoint union $\Xi_0(g) \sqcup \Xi_1(g)$, where $\Xi_1(g)$ is of pure dimension 1. Proposition 4.1 implies that every irreducible component of $\Xi_1(g)$ is its connected component. Thus every connected component of $\Xi_1(g)$ is a smooth elliptic curve, so that $\chi_{\text{top}}(\Xi_1(g)) = 0$. On the other hand, one has

$$\chi_{\text{top}}(\Xi(g)) = \chi_{\text{top}}(X)$$

by the topological Lefschetz fixed point formula, and the assertion follows. \square

Lemma 5.5. *Let X be a compact complex surface with $\chi_{\text{top}}(X) \neq 0$. Suppose that every irreducible curve contained in X is a smooth elliptic curve. Let $G \subset \overline{\text{Aut}}(X)$ be a finite subgroup. If G contains a non-trivial cyclic normal subgroup, then G contains an abelian subgroup of index at most $12\chi_{\text{top}}(X)$.*

Proof. Let $N \subset G$ be a non-trivial cyclic normal subgroup. By Lemma 5.4 the group N has exactly $\chi_{\text{top}}(X) > 0$ isolated fixed points on X (and maybe also several curves that consist of fixed points). Since N is normal in G , the group G permutes these points. Thus there exists a subgroup of index at most $\chi_{\text{top}}(X)$ in G acting on X with a fixed point.

Now the assertion follows from Proposition 4.1 and the classification of finite subgroups of $\mathrm{GL}_2(\mathbb{C})$ (cf. [PS16a, Corollary 2.2.2]). \square

Lemma 5.6. *Let X be a compact complex surface with $a(X) = 0$ and $\chi_{\mathrm{top}}(X) \neq 0$. If X contains at least one curve, then $\mathrm{Aut}(X)$ is Jordan.*

Proof. It is enough to prove that the group $\overline{\mathrm{Aut}}(X)$ is Jordan. The surface X contains at most a finite number of curves by [BHPVdV04, Theorem IV.8.2]. By Lemma 5.2 we may assume that all these curves are smooth and elliptic. Let C_1, \dots, C_m be all curves on X , and let $\mathrm{Aut}^\sharp(X) \subset \overline{\mathrm{Aut}}(X)$ be the stabilizer of C_1 . Clearly, the subgroup $\mathrm{Aut}^\sharp(X)$ has index at most m in $\overline{\mathrm{Aut}}(X)$, so it is sufficient to prove that $\mathrm{Aut}^\sharp(X)$ is Jordan. For any finite subgroup $G \subset \mathrm{Aut}^\sharp(X)$ we have an exact sequence

$$1 \longrightarrow \Gamma \longrightarrow G \longrightarrow \mathrm{Aut}(C_1),$$

where Γ is a finite cyclic group. If $\Gamma = \{1\}$, then G is contained in $\mathrm{Aut}(C_1)$. Since C_1 is an elliptic curve, the group G has an abelian subgroup of index at most 6. If $\Gamma \neq \{1\}$, then G has an abelian subgroup of index at most $12\chi_{\mathrm{top}}(X)$ by Lemma 5.5. Therefore, in both cases G also has a normal abelian subgroup of bounded index. \square

In the following lemmas we will deal with compact complex surfaces X that contain no curves. In particular, this implies that $a(X) = 0$, and the action of any finite subgroup of $\mathrm{Aut}(X)$ is free in codimension one.

Lemma 5.7. *Let X be a compact complex surface with $\chi_{\mathrm{top}}(X) \neq 0$. Suppose that X contains no curves. Then the group $\overline{\mathrm{Aut}}(X)$ has no elements of even order.*

Proof. Let $g \in \overline{\mathrm{Aut}}(X)$ be an element of order 2 (such elements always exist provided that there are elements of even order).

First assume that $\varkappa(X) = -\infty$. We have $b_1(X) = 1$ and $b_2(X) = \chi_{\mathrm{top}}(X) > 0$ (see Theorem 3.6). Moreover, we know that $h^{2,0}(X) = 0$ because $\varkappa(X) = -\infty$. Hodge relations (see e.g. [BHPVdV04, § IV.2]) give us

$$h^{0,1}(X) = 1, \quad h^{1,0}(X) = 0, \quad \text{and} \quad h^{2,0}(X) = h^{0,2}(X) = 0.$$

Therefore, one has $\chi(\mathcal{O}_X) = 0$. Since g acts trivially on $H^*(X, \mathbb{Z})$, the holomorphic Lefschetz fixed point formula shows that g has no fixed points. This contradicts Lemma 5.4.

Now assume that $\varkappa(X) \geq 0$. Since $a(X) = 0$, this implies that $\varkappa(X) = 0$ and X is a K3 surface (see Theorem 3.6). Therefore, one has $\chi_{\mathrm{top}}(X) = 24$ and $\chi(\mathcal{O}_X) = 2$. As above the holomorphic Lefschetz fixed point formula shows that g has exactly 8 fixed points. This contradicts Lemma 5.4. \square

Lemma 5.8. *Let X be a compact complex surface with $\chi_{\mathrm{top}}(X) \neq 0$. Suppose that X contains no curves. Let $G \subset \overline{\mathrm{Aut}}(X)$ be a finite subgroup. Suppose that G has a fixed point. Then G is cyclic.*

Proof. Let $x \in X$ be a fixed point of G . By Proposition 4.1 we have an embedding

$$G \subset \mathrm{GL}(T_{x,X}) \cong \mathrm{GL}_2(\mathbb{C}).$$

Since the order of G is odd by Lemma 5.7, it must be abelian. Since the action is free in codimension one, the group G is cyclic. \square

Lemma 5.9. *Let X be a compact complex surface with $\chi_{\text{top}}(X) \neq 0$. Suppose that X contains no curves. Let $G \subset \overline{\text{Aut}}(X)$ be a finite cyclic subgroup, and $g \in G$ be a non-trivial element. Then g has the same set of fixed points as G .*

Proof. For an arbitrary element $h \in G$ denote by $\text{Fix}(h)$ the fixed locus of h . By Lemma 5.4 one has

$$|\text{Fix}(h)| = \chi_{\text{top}}(X)$$

for every non-trivial element h .

Let f be a generator of G . Then for some positive integer n one has $f^n = g$, so that

$$\text{Fix}(f) \subset \text{Fix}(g).$$

Therefore, one has $\text{Fix}(f) = \text{Fix}(g)$, which means that every non-trivial element of G has one and the same set of fixed points. \square

Lemma 5.10. *Let X be a compact complex surface with $\chi_{\text{top}}(X) \neq 0$. Suppose that X contains no curves. Then every finite subgroup $G \subset \overline{\text{Aut}}(X)$ is a union $G = \bigcup_{i=1}^m G_i$ of cyclic subgroups such that $G_i \cap G_j = \{1\}$ for $i \neq j$.*

Proof. Choose some representation of G as a union $G = \bigcup_{i=1}^m G_i$, where G_i are cyclic groups that possibly have non-trivial intersections. Let G_1 and G_2 be subgroups such that $G_1 \cap G_2 \neq \{1\}$. Let $g \in G_1 \cap G_2$ be a non-trivial element. By Lemma 5.4 it has a fixed point, say x . By Lemma 5.8 the stabilizer G_x is a cyclic group. By Lemma 5.9 the groups G_1 and G_2 fix the point x , so that $G_1, G_2 \subset G_x$. Replacing G_1 and G_2 by G_x , we proceed to construct the required system of subgroups by induction. \square

Lemma 5.11. *Let X be a compact complex surface with $\chi_{\text{top}}(X) \neq 0$. Suppose that X contains no curves. Then the group $\text{Aut}(X)$ is Jordan.*

Proof. It is enough to prove that the group $\overline{\text{Aut}}(X)$ is Jordan. Let $G \subset \overline{\text{Aut}}(X)$ be a finite subgroup. Let $\Xi \subset X$ be the set of points with non-trivial stabilizers in G .

By Lemma 5.10 the group G is a union $G = \bigcup_{i=1}^m G_i$ of cyclic subgroups such that $G_i \cap G_j = \{1\}$ for $i \neq j$. We claim that the stabilizer of any point $x \in \Xi$ is one of the groups G_1, \dots, G_m . Indeed, choose a point $x \in \Xi$, and let G_x be its stabilizer. Let g_x be a generator of G_x , and let $1 \leq r \leq m$ be the index such that the group G_r contains g_x . Then $G_x \subset G_r$. Now Lemma 5.9 implies that $G_x = G_r$.

By Lemma 5.4 every element of G has exactly $b_2(X)$ fixed points. The set Ξ is a disjoint union of orbits of the group G . Therefore, for some positive integers k_i one has

$$|\Xi| = m b_2(X) = \sum_{i=1}^m k_i [G : G_i].$$

Hence, for some i we have $[G : G_i] \leq b_2(X)$, i.e. G contains a cyclic subgroup G_i of index at most $b_2(X)$. This implies that G contains a normal abelian subgroup of bounded index. \square

Now we are ready to prove Theorem 5.1.

Proof of Theorem 5.1. If $a(X) = 1$, then the assertion follows from Lemma 5.3. If $a(X) = 0$ and X contains at least one curve, then the assertion follows from Lemma 5.6. Finally, if X contains no curves, then the assertion follows from Lemma 5.11. \square

An alternative way to prove Theorem 5.1 is provided by the following more general result due to I. Mundet i Riera. Our proof of Theorem 5.1 is a simplified version of the proof of this result given in [Mun16a].

Theorem 5.12 ([Mun16a, Theorem 1.1]). *Let X be a compact, orientable, connected four-dimensional smooth manifold with $\chi_{\text{top}}(X) \neq 0$. Then the group of diffeomorphisms of X is Jordan. In particular, if X is a compact complex surface with non-vanishing topological Euler characteristic, then the group $\text{Aut}(X)$ is Jordan.*

Note however that our proof of Theorem 5.1 implies that for a compact complex surface X with $\chi_{\text{top}}(X) \neq 0$ and containing no curves, there exists a constant J such that for every finite subgroup $G \subset \text{Aut}(X)$ there exists a normal *cyclic* subgroup of index at most J , while the results of [Mun16a] provide only a normal abelian subgroup of bounded index generated by at most 2 elements.

6. HOPF SURFACES

In this section we study automorphism groups of Hopf surfaces.

Recall that a *Hopf surface* X is compact complex surface whose universal cover is (analytically) isomorphic to $\mathbb{C}^2 \setminus \{0\}$. Thus $X \cong (\mathbb{C}^2 \setminus \{0\}) / \Gamma$, where $\Gamma \cong \pi_1(X)$ is a group acting freely on $\mathbb{C}^2 \setminus \{0\}$. A Hopf surface X is said to be *primary* if $\pi_1(X) \cong \mathbb{Z}$. One can show that a primary Hopf surface is isomorphic to a quotient

$$X(\alpha, \beta, \lambda, n) = (\mathbb{C}^2 \setminus \{0\}) / \Lambda,$$

where $\Lambda \cong \mathbb{Z}$ is a group generated by the transformation

$$(6.1) \quad (x, y) \mapsto (\alpha x + \lambda y^n, \beta y).$$

Here n is a positive integer, and α and β are complex numbers satisfying

$$0 < |\alpha| \leq |\beta| < 1;$$

moreover, one has $\lambda = 0$, or $\alpha = \beta^n$ [Kod66, §10]. A *secondary* Hopf surface is a quotient of a primary Hopf surface by a free action of a finite group [Kod66, §10]. Every Hopf surface contains a curve, see [Kod66, Theorem 32]. Automorphisms of Hopf surfaces were studied in details in [Kat75], [Kat89], [Nam74], and [MN00]. Our approach does not use these results.

We will need the following easy observation.

Lemma 6.2. *Let*

$$M = \begin{pmatrix} \alpha & \lambda \\ 0 & \beta \end{pmatrix} \in \text{GL}_2(\mathbb{C})$$

be an upper triangular matrix, and $Z \subset \text{GL}_2(\mathbb{C})$ be the centralizer of M . The following assertions hold.

- (i) *If $\alpha = \beta$ and $\lambda = 0$, then $Z = \text{GL}_2(\mathbb{C})$.*
- (ii) *If $\alpha \neq \beta$ and $\lambda = 0$, then $Z \cong (\mathbb{C}^*)^2$.*
- (iii) *If $\alpha = \beta$ and $\lambda \neq 0$, then $Z \cong \mathbb{C}^* \times \mathbb{C}^+$.*

Proof. Simple linear algebra. □

Lemma 6.3. *Let X be a Hopf surface. Then the group $\text{Aut}(X)$ is Jordan.*

Proof. The non-compact surface $\mathbb{C}^2 \setminus \{0\}$ is the universal cover of X ; moreover, X is obtained from $\mathbb{C}^2 \setminus \{0\}$ as a quotient by a free action of some group Γ that contains a normal subgroup $\Lambda \cong \mathbb{Z}$ of finite index. We may shrink Λ if necessary and suppose that Λ is a characteristic subgroup of Γ ; in fact, it is enough to replace Λ by its subgroup of index $|\Gamma/\Lambda|$. A generator of Λ is given by formula (6.1). There is an exact sequence of groups

$$1 \longrightarrow \Gamma \longrightarrow \widetilde{\text{Aut}}(X) \longrightarrow \text{Aut}(X) \longrightarrow 1,$$

where $\widetilde{\text{Aut}}(X)$ acts by automorphisms of $\mathbb{C}^2 \setminus \{0\}$. By Hartogs theorem the action of $\widetilde{\text{Aut}}(X)$ extends to \mathbb{C}^2 so that $\widetilde{\text{Aut}}(X)$ fixes the origin $0 \in \mathbb{C}^2$. The image of the generator of Λ is mapped by the natural homomorphism

$$\varsigma: \widetilde{\text{Aut}}(X) \longrightarrow \text{GL}(T_{0,\mathbb{C}^2}) \cong \text{GL}_2(\mathbb{C})$$

to the matrix

$$M = \begin{pmatrix} \alpha & \lambda \delta_1^n \\ 0 & \beta \end{pmatrix}$$

where δ is the Kroneker symbol.

Let $G \subset \text{Aut}(X)$ be a finite subgroup, and \tilde{G} be its preimage in $\widetilde{\text{Aut}}(X)$. Thus, one has $G \cong \tilde{G}/\Gamma$. By Corollary 4.6 we know that $\varsigma|_{\tilde{G}}$ is an embedding. Let Ω be the normalizer of $\varsigma(\Lambda)$ in $\text{GL}_2(\mathbb{C})$. By construction $\varsigma(\tilde{G})$ is contained in the normalizer of $\varsigma(\Gamma)$ in $\text{GL}_2(\mathbb{C})$, which in turn is contained in Ω because Λ is a characteristic subgroup of Γ . We see that every finite subgroup of $\text{Aut}(X)$ is contained in the group $\Omega/\varsigma(\Lambda)$. Since $\varsigma(\Lambda) \cong \mathbb{Z}$, the group Ω has a (normal) subgroup Ω' of index at most 2 that coincides with the centralizer of the matrix M .

It remains to check that the group $\Omega'/\varsigma(\Lambda)$ is Jordan. If $\lambda = 0$ and $\alpha = \beta$, then this follows from Lemmas 6.2(i) and 2.8. If either $\lambda = 0$ and $\alpha \neq \beta$, or $\lambda \neq 0$ and $n \geq 2$, then this follows from Lemmas 6.2(ii) and 2.5. If $\lambda \neq 0$ and $n = 1$, then this follows from Lemmas 6.2(iii) and 2.5. \square

Remark 6.4. Suppose that for a primary Hopf surface $X \cong X(\alpha, \beta, \lambda, n)$ one has $\lambda = 0$ and $\alpha^k = \beta^l$ for some $k, l \in \mathbb{Z}$. Then there is an elliptic fibration

$$X \cong (\mathbb{C}^2 \setminus \{0\}) / \Lambda \rightarrow \mathbb{P}^1 \cong (\mathbb{C}^2 \setminus \{0\}) / \mathbb{C}^*,$$

and one has an exact sequence of groups

$$1 \longrightarrow E \longrightarrow \text{Aut}(X) \longrightarrow \text{PGL}_2(\mathbb{C}),$$

where E is the group of points of the elliptic curve \mathbb{C}^*/\mathbb{Z} .

7. INOUE SURFACES

In this section we study automorphism groups of Inoue surfaces, and make some general conclusions about automorphisms of surfaces of class VII.

Inoue surfaces are quotients of $\mathbb{C} \times \mathbb{H}$, where \mathbb{H} is the upper half-plane, by certain infinite discrete groups. They were introduced by M. Inoue [Ino74]. These surfaces contain no curves and their invariants are as follows:

$$a(X) = 0, \quad b_1(X) = 1, \quad b_2(X) = 0, \quad h^{1,0}(X) = 0, \quad h^{0,1}(X) = 1.$$

The following result shows the significance of Hopf and Inoue surfaces from the point of view of Enriques–Kodaira classification.

Theorem 7.1 (see [Bog77] and [Tel94]). *Every minimal surface of class VII with vanishing second Betti number is either a Hopf surface or an Inoue surface.*

Lemma 7.2. *Let X be an Inoue surface, and $G \subset \text{Aut}(X)$ be a finite subgroup. Then the action of G on X is free, and the quotient $\hat{X} = X/G$ is again an Inoue surface.*

Proof. Assume that the action of G on X is not free. To get a contradiction we may assume that G is a cyclic group of prime order. Let δ be its generator. Since X contains no curves, the fixed point locus of G consists of a finite number of points P_1, \dots, P_n . On the other hand, by the topological Lefschetz fixed point formula, one has

$$n = \text{Lef}(\delta) = 2 - 2 \text{tr}_{H^1(X, \mathbb{R})} \delta^* > 0.$$

Hence the action of δ^* on $H^1(X, \mathbb{R}) \cong \mathbb{R}$ is not trivial. This is possible only if δ is of order 2 and $n = 4$. Then \hat{X} has exactly 4 singular points which are Du Val of type A_1 . Let $Y \rightarrow \hat{X}$ be the minimal resolution of singularities. Then

$$c_1(Y)^2 = c_1(\mathcal{K}_{\hat{X}})^2 = \frac{1}{2}c_1(X)^2 = 0,$$

and $\chi_{\text{top}}(Y) = 4 + \chi_{\text{top}}(\hat{X}) = 6$. This contradicts the Noether's formula, see e.g. [BHPVdV04, §I.5].

Thus $X \rightarrow \hat{X}$ is an unramified finite cover. This implies that $\chi_{\text{top}}(\hat{X}) = 0$. Furthermore, one has

$$b_2(\hat{X}) = \text{rk } H^2(X, \mathbb{Z})^G = 0,$$

and so $b_1(\hat{X}) = 1$. Therefore, \hat{X} is a minimal surface of class VII (see Theorem 3.6). Clearly, \hat{X} contains no curves. Thus by Theorem 7.1 the surface \hat{X} is either a Hopf surface or an Inoue surface. Since Hopf surfaces contain curves, we conclude that \hat{X} is an Inoue surface. \square

There are three types of Inoue surfaces: S_M , $S^{(+)}$ and $S^{(-)}$. They are distinguished by the type of their fundamental group $\Gamma = \pi_1(X)$, see [Ino74]:

type	generators	relations
S_M	$\delta_1, \delta_2, \delta_3, \gamma$	$[\delta_i, \delta_j] = 1, \gamma \delta_i \gamma^{-1} = \delta_1^{m_{1,i}} \delta_2^{m_{2,i}} \delta_3^{m_{3,i}}, (m_{j,i}) \in \text{SL}_3(\mathbb{Z})$
$S^{(\pm)}$	$\delta_1, \delta_2, \delta_3, \gamma$	$[\delta_i, \delta_3] = 1, [\delta_1, \delta_2] = \delta_3^r, \gamma \delta_i \gamma^{-1} = \delta_1^{m_{1,i}} \delta_2^{m_{2,i}} \delta_3^{p_i}$ for $i = 1, 2,$ $\gamma \delta_3 \gamma^{-1} = \delta_3^{\pm 1}, (m_{j,i}) \in \text{GL}_2(\mathbb{Z}), \det(m_{j,i}) = \pm 1$

In the notation of Appendix A one has $\Gamma \cong \Gamma_0 \rtimes \Gamma_1$, where $\Gamma_1 \cong \mathbb{Z}$, while $\Gamma_0 \cong \mathbb{Z}^3$ for Inoue surfaces of type S_M , and $\Gamma_0 \cong \mathcal{H}(r)$ for Inoue surfaces of types $S^{(\pm)}$ (see §A.5 and §A.14 for more details). In the former case the matrix $M \in \text{SL}_3(\mathbb{Z})$ that defines the semi-direct product has eigenvalues α, β , and $\bar{\beta}$, where $\alpha \in \mathbb{R}, \alpha > 1$, and $\beta \notin \mathbb{R}$. In the latter case the matrix $M \in \text{GL}_2(\mathbb{Z})$ that defines the action of \mathbb{Z} on $\mathcal{H}(r)/\mathbb{Z}(\mathcal{H}(r)) \cong \mathbb{Z}^2$ has real eigenvalues α and β , where $\alpha\beta = \pm 1$ (depending on whether Γ is of type $S^{(+)}$ or $S^{(-)}$), and both α and β are different from 1, see [Ino74, §§2–4].

Lemma 7.3. *Let Γ be a group of one of the types $S_M, S^{(+)}$, or $S^{(-)}$. Then*

- (i) Γ is of type S_M if and only if Γ contains a characteristic subgroup isomorphic to \mathbb{Z}^3 ;

- (ii) Γ is of type $S^{(+)}$ if and only if Γ contains no subgroups isomorphic to \mathbb{Z}^3 and $z(\Gamma) \neq \{1\}$;
- (iii) Γ is of type $S^{(-)}$ if and only if Γ contains no subgroups isomorphic to \mathbb{Z}^3 and $z(\Gamma) = \{1\}$.

Proof. This follows from Lemmas A.6(ii),(iii) and A.15(ii) and Remark A.17. \square

Corollary 7.4. *Let X be an Inoue surface, and $G \subset \text{Aut}(X)$ be a finite subgroup. Then the action of G on X is free, and the following assertions hold.*

- (i) *If X is of type S_M , then so is X/G ;*
- (ii) *If X is of type $S^{(-)}$, then so is X/G ;*
- (iii) *If X is of type $S^{(+)}$, then X/G is of type $S^{(+)}$ or $S^{(-)}$.*

Proof. Put $\hat{X} = X/G$. Then the action of G on X is free, and X is an Inoue surface by Lemma 7.2. Put $\hat{\Gamma} = \pi_1(\hat{X})$. Then $\hat{\Gamma}$ is a group of one of the types S_M , $S^{(+)}$, or $S^{(-)}$, and $\Gamma \subset \hat{\Gamma}$ is a normal subgroup of finite index. Now everything follows from Lemma 7.3. \square

Lemma 7.5. *Let X be an Inoue surface of type S_M . Then the group $\text{Aut}(X)$ is Jordan.*

Proof. Let $G \subset \text{Aut}(X)$ be a finite subgroup, and put $\hat{X} = X/G$. By Corollary 7.4 the action of G on X is free, and \hat{X} is also an Inoue surface of type S_M . Put $\Gamma = \pi_1(X)$ and $\hat{\Gamma} = \pi_1(\hat{X})$. Then Γ is a normal subgroup of $\hat{\Gamma}$, and $\hat{\Gamma}/\Gamma \cong G$; moreover, both Γ and $\hat{\Gamma}$ are semi-direct products as in §A.5. Now it follows from Lemma A.8 that there is a constant ν that depends only on Γ (that is, only on X), such that G has a normal abelian subgroup of index at most ν . \square

Lemma 7.6. *Let X be an Inoue surface of type $S^{(+)}$ or $S^{(-)}$. Then the group $\text{Aut}(X)$ is Jordan.*

Proof. Let $G \subset \text{Aut}(X)$ be a finite subgroup, and put $\hat{X} = X/G$. By Corollary 7.4 the action of G on X is free, and \hat{X} is also an Inoue surface of type $S^{(+)}$ or $S^{(-)}$. Put $\Gamma = \pi_1(X)$ and $\hat{\Gamma} = \pi_1(\hat{X})$. Then Γ is a normal subgroup of $\hat{\Gamma}$, and $\hat{\Gamma}/\Gamma \cong G$; moreover, both Γ and $\hat{\Gamma}$ are semi-direct products as in §A.14. Now it follows from Lemma A.18 that there is a constant ν that depends only on Γ (that is, only on X), such that G has a normal abelian subgroup of index at most ν . \square

We summarize the results of Lemmas 7.5 and 7.6 as follows.

Corollary 7.7. *Let X be an Inoue surface. Then the group $\text{Aut}(X)$ is Jordan.*

Finally, we put together the information about automorphisms of surfaces of class VII.

Corollary 7.8. *Let X be a minimal surface of class VII. Then the group $\text{Aut}(X)$ is Jordan.*

Proof. If the second Betti number $b_2(X)$ vanishes, then X is either a Hopf surface or an Inoue surface by Theorem 7.1. Thus the assertion follows from Lemma 6.3 and Corollary 7.7 in this case. If $b_2(X)$ does not vanish, then the assertion follows from Theorem 5.1. \square

Remark 7.9. Except for Hopf surfaces, there are some other classes of minimal compact complex surfaces of class VII whose automorphism groups are studied in details. For

instance, this is the case for so called hyperbolic and parabolic Inoue surfaces, see [Pin84] and [Fuj09], respectively. Note that surfaces of both of these types have positive second Betti numbers (and thus they are not to be confused with Inoue surfaces studied in this section).

8. KODAIRA SURFACES

In this section we study automorphism groups of Kodaira surfaces. Our approach here is similar to what happens in §7.

Recall (see e.g. [BHPVdV04, § V.5]) that a Kodaira surface is a compact complex surface of Kodaira dimension 0 with odd first Betti number. There are two types of Kodaira surfaces: primary and secondary ones. A primary Kodaira surface is a compact complex surface with the following invariants [Kod64, §6]:

$$\mathcal{H}_X \sim 0, \quad a(X) = 1, \quad b_1(X) = 3, \quad b_2(X) = 4, \quad \chi_{\text{top}}(X) = 0, \quad h^1(X, \mathcal{O}_X) = 2, \quad h^2(X, \mathcal{O}_X) = 1.$$

Let X be a primary Kodaira surface and let $\phi: X \rightarrow B$ be its algebraic reduction. Then B is an elliptic curve and ϕ is a principal elliptic fibration [Kod64, §6], [BHPVdV04, § V.5]. The universal cover of X is isomorphic to \mathbb{C}^2 , and the fundamental group $\Gamma = \pi_1(X)$ has the following presentation:

$$(8.1) \quad \Gamma = \langle \delta_1, \delta_2, \delta_3, \gamma \mid [\delta_1, \delta_2] = \delta_3^r, [\delta_i, \delta_3] = [\delta_i, \gamma] = 1 \rangle,$$

where r is a positive integer [Kod64, §6]. In the notation of Appendix A one has $\Gamma \cong \mathcal{H}(r) \times \mathbb{Z}$.

A secondary Kodaira surface is a quotient of a primary Kodaira surface by a free action of a finite cyclic group. The invariants of a secondary Kodaira surface are [Kod66, §9]:

$$a(X) = 1, \quad b_1(X) = 1, \quad b_2(X) = 0, \quad \chi_{\text{top}}(X) = 0, \quad h^1(X, \mathcal{O}_X) = 1, \quad h^2(X, \mathcal{O}_X) = 0.$$

For both types of Kodaira surfaces the algebraic reduction $\phi: X \rightarrow B$ is an $\text{Aut}(X)$ -equivariant elliptic fibration, so that in particular the group $\text{Aut}(X)$ acts on the curve B . Denote by $\overline{\text{Aut}}(X) \subset \text{Aut}(X)$ the subgroup that consists of all elements acting trivially on $H^*(X, \mathbb{Z})$ and $H^*(B, \mathbb{Z})$.

Lemma 8.2 (cf. Lemma 7.2). *Let X be a primary Kodaira surface, and $G \subset \overline{\text{Aut}}(X)$ be a finite subgroup. Then the action of G on X is free, and the quotient $\hat{X} = X/G$ is again a primary Kodaira surface.*

Proof. The curve B is elliptic; thus the group G acts on B without fixed points. This means that there are no fibers of ϕ that consist of points fixed by G . On the other hand, every curve on X is a fiber of $\phi: X \rightarrow B$ by Lemma 3.3. Hence there are no curves that consist of points fixed by G on X at all. Now the topological Lefschetz fixed point formula implies that G has no fixed points on B and on X . Therefore, \hat{X} is a smooth surface, and the quotient morphism $X \rightarrow \hat{X}$ is unramified. Hence $\kappa(\hat{X}) = \kappa(X) = 0$. Moreover, we have

$$c_1(\hat{X})^2 = c_1(X)^2 = 0.$$

This means that the surface \hat{X} is minimal. Since $G \subset \overline{\text{Aut}}(X)$, we have

$$b_1(\hat{X}) = b_1(X) = 3.$$

Therefore, \hat{X} is a primary Kodaira surface by Theorem 3.6. □

Lemma 8.3. *Let X be a primary Kodaira surface. Then the group $\text{Aut}(X)$ is Jordan.*

Proof. Let $G \subset \text{Aut}(X)$ be a finite subgroup. By Theorem 2.2 we can assume that $G \subset \overline{\text{Aut}}(X)$. Put $\hat{X} = X/G$. It follows from Lemma 8.2 that G acts freely on X , and \hat{X} is a primary Kodaira surface. Put $\Gamma = \pi_1(X)$ and $\hat{\Gamma} = \pi_1(\hat{X})$. Then Γ is a normal subgroup of $\hat{\Gamma}$, and $\hat{\Gamma}/\Gamma \cong G$; moreover, both Γ and $\hat{\Gamma}$ are as in §A.19. Now it follows from Lemma A.21 that there is a constant r that depends only on Γ (that is, only on X), such that G has a normal abelian subgroup of index at most r . \square

Lemma 8.4. *Let X be a secondary Kodaira surface. Then the group $\text{Aut}(X)$ is Jordan.*

Proof. Since $a(X) = 1$, the algebraic reduction is an $\text{Aut}(X)$ -equivariant elliptic fibration $\pi: X \rightarrow B$. Thus there is an exact sequence of groups

$$1 \longrightarrow \text{Aut}(X)_\pi \longrightarrow \text{Aut}(X) \longrightarrow \Gamma \longrightarrow 1,$$

where the action of $\text{Aut}(X)_\pi$ is fiberwise with respect to π , and Γ is a subgroup of $\text{Aut}(B)$.

We claim that the group $\text{Aut}(X)_\pi$ is Jordan. Indeed, if H is a finite subgroup of $\text{Aut}(X)_\pi$, then H acts faithfully on a typical fiber of π , which is a smooth elliptic curve. This implies that H has a normal abelian subgroup of index at most 6.

Since

$$h^0(X, \Omega_X) = b_1(X) - h^1(X, \mathcal{O}_X) = 0,$$

the base curve B is rational. Furthermore, one has

$$\chi(\mathcal{O}_X) = h^0(X, \mathcal{O}_X) - h^1(X, \mathcal{O}_X) + h^2(X, \mathcal{O}_X) = 1 - 1 + 0 = 0.$$

By the canonical bundle formula (see e.g. [BHPVdV04, Theorem V.12.1]) we have

$$\mathcal{K}_X \sim \pi^*(\mathcal{K}_B \otimes \mathcal{L}) \otimes \mathcal{O}_X \left(\sum (m_i - 1)F_i \right),$$

where F_i are all (reduced) multiple fibers of π , the fiber F_i has multiplicity m_i , and \mathcal{L} is a line bundle of degree $\chi(\mathcal{O}_X) = 0$. Since X has Kodaira dimension 0, we see that

$$\sum (1 - 1/m_i) = 2.$$

In particular, the number of multiple fibers equals either 3 or 4. This means that Γ has a finite non-empty invariant subset in B that consists of 3 or 4 points. Hence Γ is finite, so that the assertion follows by Lemma 2.4(i). \square

We summarize the results of Lemmas 8.3 and 8.4 as follows.

Corollary 8.5. *Let X be a Kodaira surface. Then the group $\text{Aut}(X)$ is Jordan.*

An alternative way to prove the Jordan property for the automorphism group of a secondary Kodaira surface is to use the fact that its canonical cover is a primary Kodaira surface together with Lemma 2.4(ii) and Theorem 2.9.

9. NON-NEGATIVE KODAIRA DIMENSION

In this section we study automorphism groups of surfaces of non-negative Kodaira dimension, and prove Theorems 1.5 and 1.6.

The case of Kodaira dimension 2 is well known.

Theorem 9.1. *Let X be a (minimal) surface of general type. Then the group $\text{Aut}(X)$ is finite.*

Proof. The surface X is projective, see Theorem 3.6. Thus the group $\text{Aut}(X)$ is finite, see for instance [HMX13] where a much more general result is established for varieties of general type of arbitrary dimension. \square

Now we consider the case of Kodaira dimension 1.

Lemma 9.2 (cf. [PS16d, Lemma 3.3]). *Let X be a minimal surface of Kodaira dimension 1. Then the group $\text{Aut}(X)$ is Jordan.*

Proof. Let $\phi: X \rightarrow B$ be the pluricanonical fibration, where B is some (smooth) curve. It is equivariant with respect to the action of $\text{Aut}(X)$. Thus there is an exact sequence of groups

$$1 \longrightarrow \text{Aut}(X)_\phi \longrightarrow \text{Aut}(X) \longrightarrow \Gamma \longrightarrow 1,$$

where the action of $\text{Aut}(X)_\phi$ is fiberwise with respect to ϕ , and Γ is a subgroup of $\text{Aut}(B)$. As in the proof of Lemma 8.4, we see that the group $\text{Aut}(X)_\phi$ is Jordan. Hence by Lemma 2.4(i) it is enough to check that Γ has bounded finite subgroups. In particular, this holds if the genus of B is at least 2, since the group $\text{Aut}(B)$ is finite in this case. Thus we will assume that the genus of B is at most 1.

Suppose that ϕ has a fiber F such that F_{red} is not a smooth elliptic curve. Then every irreducible component of F is a rational curve, see e.g. [BHPVdV04, § V.7]. Hence Lemma 4.7 applied to the set of irreducible components of fibers of the morphism ϕ shows that the group $\text{Aut}(X)$ is Jordan.

Therefore, we will assume that all (set-theoretic) fibers of ϕ are smooth elliptic curves. Then the topological Euler characteristic $\chi_{\text{top}}(X)$ equals 0. By the Noether's formula one has

$$\chi(\mathcal{O}_X) = \frac{1}{12} (c_1(X)^2 + \chi_{\text{top}}(X)) = 0.$$

By the canonical bundle formula we have

$$\mathcal{K}_X \sim \phi^*(\mathcal{K}_B \otimes \mathcal{L}) \otimes \mathcal{O}_X \left(\sum (m_i - 1)F_i \right),$$

where F_i are all (reduced) multiple fibers of ϕ , the fiber F_i has multiplicity m_i , and \mathcal{L} is a line bundle of degree $\chi(\mathcal{O}_X) = 0$. Since X has Kodaira dimension 1, we see that

$$(9.3) \quad 2g(B) - 2 + \sum (1 - 1/m_i) = \deg(\mathcal{K}_B \otimes \mathcal{L}) + \sum (1 - 1/m_i) > 0.$$

Suppose that B is an elliptic curve, so that $g(B) = 1$. Then (9.3) implies that ϕ has at least one multiple fiber. This means that Γ has a finite non-empty invariant subset in B , so that Γ is finite.

Now suppose that B is a rational curve, so that $g(B) = 0$. Then (9.3) implies that ϕ has at least three multiple fibers, cf. the proof of Lemma 8.4. This means that Γ has a finite non-empty invariant subset in B that consists of at least three points. Therefore, Γ is finite in this case as well. \square

Finally, we consider the case of Kodaira dimension 0. The following result is well known.

Theorem 9.4. *Let $X = \mathbb{C}^n/\Lambda$ be a complex torus. Then*

$$(9.5) \quad \text{Aut}(X) \cong (\mathbb{C}^n/\Lambda) \rtimes \Gamma,$$

where Γ is isomorphic to the stabilizer of the lattice Λ in $\text{GL}_n(\mathbb{C})$.

Proof. The proof is standard, but we include it for the readers convenience. Let Γ be the stabilizer of the point $0 \in X$. Then the decomposition (9.5) holds, and it remains to prove that Γ is isomorphic to the stabilizer of the lattice Λ in $\mathrm{GL}_n(\mathbb{C})$.

Since \mathbb{C}^n is the universal cover of X , there is an embedding $\Gamma \hookrightarrow \mathrm{Aut}(\mathbb{C}^n)$, and there is a point in Λ that is invariant with respect to Γ . We may assume that this is the origin in \mathbb{C}^n .

Let g be an element of Γ . One has $g(\Lambda) = \Lambda$. We claim that $g \in \mathrm{GL}_n(\mathbb{C})$. Indeed, let λ be an arbitrary element of the lattice Λ . Consider a holomorphic map

$$f_\lambda: \mathbb{C}^n \rightarrow \mathbb{C}^n, \quad f_\lambda(z) = g(z + \lambda) - g(z).$$

One has $f_\lambda(z) \in \Lambda$ for every $z \in \mathbb{C}^n$. This means that $f_\lambda(z)$ is constant, so that all partial derivatives of f_λ vanish. Hence the partial derivatives of $g(z)$ are periodic with respect to the lattice Λ . This in turn means that these partial derivatives are bounded and thus constant, so that $g(z)$ is a linear function in z . \square

Remark 9.6. A complete classification of finite groups that can act by automorphisms of a two-dimensional complex torus (preserving a point therein) was obtained in [Fuj88].

Theorem 9.4 immediately implies the following result.

Corollary 9.7. *Let X be a complex torus. Then the group $\mathrm{Aut}(X)$ is Jordan.*

Proof. By Lemma 2.4(i) it is enough to check that in the notation of Theorem 9.4 the group Γ has bounded finite subgroups. Since Γ is a subgroup in the automorphism group of Λ , the latter follows from Theorem 2.2. \square

Lemma 9.8. *Let X be either a K3 surface, or an Enriques surface. Then the group $\mathrm{Aut}(X)$ has bounded finite subgroups.*

Proof. Suppose that X is a K3 surface. If X is projective, then the assertion follows from [PS14, Theorem 1.8(i)]. If X is non-projective, then the assertion follows from Theorem 5.1, or from a stronger result of [Ogu08, Theorem 1.5].

Now suppose that X is an Enriques surface. Then it is projective (see Theorem 3.6), so that the assertion again follows from [PS14, Theorem 1.8(i)]. \square

Note that in the assumptions of Lemma 9.8, the (weaker) assertion that the group $\mathrm{Aut}(X)$ is Jordan follows directly from Theorem 5.12.

Lemma 9.9. *Let X be a bielliptic surface. Then the group $\mathrm{Aut}(X)$ is Jordan.*

Proof. The surface X is projective (see Theorem 3.6). Thus the assertion follows from Theorem 1.4 (or [BZ15], or [MZ15b], or [PS14, Theorem 1.8(ii)]). \square

Corollary 9.10. *Let X be a minimal surface of Kodaira dimension 0. Then the group $\mathrm{Aut}(X)$ is Jordan.*

Proof. We know from Theorem 3.6 that X is either a complex torus, or a K3 surface, or an Enriques surface, or a bielliptic surface, or a Kodaira surface.

If X is a complex torus, then the assertion holds by Corollary 9.7. If X is a K3 surface or an Enriques surface, then the assertion is implied by Lemma 9.8. If X is a bielliptic surface, then the assertion holds by Lemma 9.9. If X is a Kodaira surface, then the assertion holds by Corollary 8.5. \square

Proposition 9.11. *Let X be a minimal surface. Then the group $\mathrm{Aut}(X)$ is Jordan.*

Proof. We check the possibilities for the birational type of X listed in Theorem 3.6 case by case. If X is rational or ruled, then X is projective (see Theorem 3.6), and thus the group $\text{Aut}(X)$ is Jordan by [Zar15, Corollary 1.6] or [MZ15b]. If X is a surface of class VII, then the group $\text{Aut}(X)$ is Jordan by Corollary 7.8. If the Kodaira dimension of X is 0, then the group $\text{Aut}(X)$ is Jordan by Corollary 9.10. If the Kodaira dimension of X is 1, then the group $\text{Aut}(X)$ is Jordan by Lemma 9.2. Finally, if the Kodaira dimension of X is 2, then the group $\text{Aut}(X)$ is finite by Theorem 9.1. \square

Now we are ready to prove Theorem 1.5.

Proof of Theorem 1.5. If X is rational or ruled, then X is projective (see Theorem 3.6), and thus the group $\text{Aut}(X)$ is Jordan by [BZ15] or [MZ15b]. Otherwise Proposition 3.4 implies that there is a unique minimal surface X' birational to X , and

$$\text{Aut}(X) \subset \text{Bir}(X) \cong \text{Bir}(X') = \text{Aut}(X').$$

Now the assertion follows from Proposition 9.11. \square

Finally, we are going to prove Theorem 1.6.

Proof of Theorem 1.6. There always exists a minimal surface birational to a given one, so we may assume that X is a minimal surface itself.

If X is rational, then the group $\text{Bir}(X)$ is Jordan by Theorem 1.3. Also, by [PS16b, Theorem 4.2] and Proposition 4.1 every finite subgroup of $\text{Bir}(X)$ contains a subgroup of bounded index that can be embedded into $\text{GL}_2(\mathbb{C})$. Hence every finite subgroup of $\text{Bir}(X)$ can be generated by a bounded number of elements.

If X is ruled and non-rational, let $\phi: X \rightarrow B$ be the \mathbb{P}^1 -fibration over a (smooth) curve. Since X is projective (see Theorem 3.6), the group $\text{Bir}(X)$ is Jordan if and only if B is not an elliptic curve by Theorem 1.4. Moreover, we always have an exact sequence of groups

$$1 \rightarrow \text{Bir}(X)_\phi \rightarrow \text{Bir}(X) \rightarrow \text{Aut}(B),$$

where the action of the subgroup $\text{Bir}(X)_\phi$ is fiberwise with respect to ϕ . In particular, the group $\text{Bir}(X)_\phi$ acts faithfully on the schematic general fiber of ϕ , which is a conic over the field $\mathbb{C}(B)$. This implies that finite subgroups of $\text{Bir}(X)_\phi$ are generated by a bounded number of elements. Also, finite subgroups of $\text{Aut}(B)$ are generated by a bounded number of elements. Therefore, the same holds for finite subgroups of $\text{Bir}(X)$ as well.

In the remaining cases we have $\text{Bir}(X) = \text{Aut}(X)$ by Proposition 3.4, so the assertion follows from Proposition 9.11 and Theorem 2.9. \square

APPENDIX A. DISCRETE GROUPS

In this section we prove some auxiliary results about discrete infinite groups and their finite quotient groups. Our main goal will be so called Wang groups (see [Has05]) which include in particular fundamental groups of Inoue and Kodaira surfaces. For every group Γ we denote by $z(\Gamma)$ the center of Γ , and for a subgroup $\Gamma' \subset \Gamma$ we denote by $z(\Gamma', \Gamma)$ the centralizer of Γ' in Γ .

A.1. Integer matrices. The following facts from number theory are well known to experts; we include them for the reader's convenience.

Lemma A.2. *The following assertions hold.*

- (i) Let α be an algebraic integer such that for every Galois conjugate α' of α one has $|\alpha'| = 1$. Then α is a root of unity.
- (ii) Let n be a positive integer. Then there exists a constant $\varepsilon = \varepsilon(n)$ with the following property: if an algebraic integer α of degree n is such that for every Galois conjugate α' of α one has $1 - \varepsilon < |\alpha'| < 1 + \varepsilon$, then α is a root of unity.

Proof. To prove assertion (i), fix an embedding $\mathbb{Q}(\alpha) \subset \mathbb{C}$. Then $\bar{\alpha}$ is a root of the minimal polynomial of α over \mathbb{Q} . Hence $\bar{\alpha}$ is an algebraic integer. On the other hand, one has $\bar{\alpha} = \alpha^{-1}$. Since both α and α^{-1} are algebraic integers, we conclude that all non-archimedean valuations of α equal 1. At the same time all archimedean valuations of α equal 1 by assumption. Therefore, assertion (i) follows from [Cas67, Lemma II.18.2].

Now consider an algebraic integer α of degree n such that all its Galois conjugates have absolute values less than, say, 2. The absolute values of the coefficients of its minimal polynomial are bounded by some constant $C = C(n)$. Consider the set of polynomials with integer coefficients

$$\mathcal{Q} = \{Q = x^n + a_{n-1}x^{n-1} + \dots + a_0 \mid |a_i| \leq C, \text{ and } Q \text{ is irreducible}\}.$$

The set \mathcal{Q} is finite. Put

$$\Pi = \{(x_1, \dots, x_n) \mid x_1, \dots, x_n \text{ are different roots of some polynomial } Q \in \mathcal{Q}\} \subset \mathbb{C}^n.$$

Then Π is a finite subset of \mathbb{C}^n ; furthermore, all algebraic integers of degree n such that all their Galois conjugates have absolute values at most 2 appear as coordinates of the points of Π . There is a number $\mu = \mu(n)$ such that for every point $P = (\alpha_1, \dots, \alpha_n) \in \Pi$ the inequalities $1 - \mu < |\alpha_i| < 1 + \mu$ for all i imply that $|\alpha_i| = 1$ for all i . In the latter case α_i are roots of unity by assertion (i). Thus it remains to put $\varepsilon = \min(\mu, 1)$ to prove assertion (ii). \square

Lemma A.3. Let $M \in \text{GL}_n(\mathbb{Z})$ be a matrix. Suppose that for every C there is an integer $k > C$ such that there exists a matrix $R_k \in \text{GL}_n(\mathbb{Z})$ with $R_k^k = M$. Then all eigen-values of M are roots of unity.

Proof. Let λ_k be an eigen-value of R_k . Then λ_k is an algebraic integer of degree at most n , because it is a root of the characteristic polynomial of the matrix R_k . Moreover, λ_k^k is an eigen-value of M . This means that

$$(A.4) \quad \sqrt[k]{l_{\min}} \leq |\lambda_k| \leq \sqrt[k]{l_{\max}},$$

where l_{\min} and l_{\max} are the minimal and the maximal absolute values of the eigen-values of the matrix M , respectively. Both of the above bounds converge to 1 when k goes to infinity. All Galois conjugates of λ_k are also eigen-values of R_k , hence the inequality (A.4) holds for them as well. Therefore, for k large enough all eigen-values of R_k are roots of unity by Lemma A.2(ii), and thus so are the eigen-values of M . \square

A.5. Lattices and semi-direct products. Consider the groups $\Gamma_0 \cong \mathbb{Z}^3$ and $\Gamma_1 \cong \mathbb{Z}$. Let γ be a generator of Γ_1 . Fix a basis $\{\delta_1, \delta_2, \delta_3\}$ in Γ_0 . Then $\text{End}(\Gamma_0)$ can be identified with $\text{Mat}_{3 \times 3}(\mathbb{Z})$, and so for any integral 3×3 -matrix $M = (m_{j,i})$ one can define its action on Γ_0 via

$$M(\delta_i) = \delta_1^{m_{1,i}} \delta_2^{m_{2,i}} \delta_3^{m_{3,i}}.$$

If $M \in \text{GL}_3(\mathbb{Z})$, this defines a semi-direct product $\Gamma = \Gamma_0 \rtimes \Gamma_1$.

The following facts are easy exercises in group theory.

Lemma A.6. *Suppose that the matrix M does not have eigen-values equal to 1. Then the following assertions hold:*

- (i) $[\Gamma, \Gamma] = \text{Im}(M - \text{Id}) \subset \Gamma_0$ is a free abelian subgroup of rank 3;
- (ii) one has $\Gamma_0 = \text{z}([\Gamma, \Gamma], \Gamma)$; in particular, Γ_0 is a characteristic subgroup of Γ ;
- (iii) the center $\text{z}(\Gamma)$ is trivial.

It also appears that one can easily describe all normal subgroups of finite index in Γ (and actually do it in a slightly more general setting).

Lemma A.7. *Let Δ_0 be an arbitrary group (and $\Gamma_1 \cong \mathbb{Z}$ as before be a cyclic group generated by an element γ). Consider a semi-direct product $\Delta = \Delta_0 \rtimes \Gamma_1$. Let $\Delta' \subset \Delta$ be a normal subgroup of finite index. Then*

- (i) $\Delta' = \Delta'_0 \rtimes \Gamma'_1$, where $\Delta'_0 = \Delta' \cap \Delta_0$, and $\Gamma'_1 \cong \mathbb{Z}$ is generated by $\gamma^k \delta'$ for some positive integer k and $\delta' \in \Delta_0$;
- (ii) Δ/Δ' has a normal subgroup of index k isomorphic to Δ_0/Δ'_0 .

Lemma A.8. *Let $\Gamma = \Gamma_0 \rtimes \Gamma_1$ be a semi-direct product defined by a matrix M as above. Suppose that the matrix M does not have eigen-values equal to 1, and at least one of its eigen-values is not a root of unity. Then there exists a constant $\nu = \nu(\Gamma)$ with the following property. Let $\hat{\Gamma} = \hat{\Gamma}_0 \rtimes \hat{\Gamma}_1$, where $\hat{\Gamma}_0 \cong \mathbb{Z}^3$ and $\hat{\Gamma}_1 \cong \mathbb{Z}$. Suppose that $\hat{\Gamma}$ contains Γ as a normal subgroup. Then the group $G = \hat{\Gamma}/\Gamma$ is finite and has a normal abelian subgroup of index at most ν .*

Proof. The group G is finite for obvious reasons. By Lemma A.6(ii) we have $\Gamma_0 = \Gamma \cap \hat{\Gamma}_0$. Thus by Lemma A.7 there is a positive integer k with the following properties: the subgroup $\Gamma_1 \subset \Gamma$ is generated by an element $\hat{\gamma}^k \hat{\delta}$, where $\hat{\gamma}$ is a generator of $\hat{\Gamma}_1$, and $\hat{\delta}$ is an element of $\hat{\Gamma}_0$; and the group G contains a normal abelian subgroup of index k . Note that the subgroup Γ_0 is normal in $\hat{\Gamma}$, because Γ is normal in $\hat{\Gamma}$, while Γ_0 is a characteristic subgroup of Γ by Lemma A.6(ii). Let $R \in \text{GL}_3(\mathbb{Z})$ be the matrix that defines the semi-direct product $\hat{\Gamma} = \hat{\Gamma}_0 \rtimes \hat{\Gamma}_1$. Considering the action of the element $\hat{\gamma}$ on the lattice $\hat{\Gamma}_0 \cong \mathbb{Z}^3$ and its sublattice $\Gamma_0 \cong \mathbb{Z}^3$, we see that R^k is conjugate to M . Thus k is bounded by some constant ν that depends only on M (that is, only on Γ) by Lemma A.3. \square

A.9. Heisenberg groups. Let r be a positive integer. Consider a group

$$(A.10) \quad \mathcal{H}(r) = \langle \delta_1, \delta_2, \delta_3 \mid [\delta_i, \delta_3] = 1, [\delta_1, \delta_2] = \delta_3^r \rangle.$$

One can think about $\mathcal{H}(r)$ as the group of all matrices

$$\begin{pmatrix} 1 & a & \frac{c}{r} \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \in \text{GL}_3(\mathbb{Q}),$$

where a , b , and c are integers. One can choose the generators so that the element δ_1 corresponds to $a = 1$, $b = c = 0$, the element δ_2 corresponds to $a = 0$, $b = 1$, $c = 0$, and the element δ_3 corresponds to $a = b = 0$, $c = 1$. The group $\mathcal{H}(1)$ is known as the discrete Heisenberg group. The center $\text{z}(\mathcal{H}(r)) \cong \mathbb{Z}$ is generated by δ_3 , while the commutator $[\mathcal{H}(r), \mathcal{H}(r)]$ is generated by δ_3^r . For the quotient group $\mathcal{H}(r) = \mathcal{H}(r)/\text{z}(\mathcal{H}(r))$ one has $\mathcal{H}(r) \cong \mathbb{Z}^2$.

The following properties are easy to establish.

Lemma A.11. *Every subgroup of finite index in $\mathcal{H}(r)$ is isomorphic to $\mathcal{H}(r')$ for some positive integer r' . Every subgroup of infinite index in $\mathcal{H}(r)$ is abelian.*

Note that a subgroup in $\mathcal{H}(r)$ generated by δ_1^a , δ_2 , and δ_3^c is isomorphic to $\mathcal{H}\left(\frac{ar}{c}\right)$. Hence any group $\mathcal{H}(r')$ is realized as a subgroup of $\mathcal{H}(r)$. We will be interested in properties of normal subgroups of $\mathcal{H}(r)$.

Lemma A.12. *Let $\Gamma_0 \subset \mathcal{H}(r)$ be a normal subgroup of finite index, and put $G_0 = \mathcal{H}(r)/\Gamma_0$. Then there are integers $a_1, a_2, a_3, b_1, b_2, b_3$ with $a_1b_2 - a_2b_1 \neq 0$, and $c > 0$ such that*

$$\Gamma_0 = \langle \delta_1^{a_1} \delta_2^{a_2} \delta_3^{a_3}, \delta_1^{b_1} \delta_2^{b_2} \delta_3^{b_3}, \delta_3^c \rangle.$$

Moreover, c divides $r \gcd(a_1, a_2, b_1, b_2)$, one has

$$\Gamma_0 \cong \mathcal{H}\left(\frac{r|a_1b_2 - a_2b_1|}{c}\right),$$

and the group G_0 contains a normal abelian subgroup of index at most $\gcd(a_1, b_1)$.

Proof. Since G_0 is finite, the image $\bar{\Gamma}_0$ of Γ_0 in $\bar{\mathcal{H}}(r)$ is isomorphic to \mathbb{Z}^2 . Choose the vectors (a_1, a_2) and (b_1, b_2) in $\mathcal{H}(r) \cong \mathbb{Z}^2$ generating $\bar{\Gamma}_0$. The group Γ_0 contains the elements $\zeta = \delta_1^{a_1} \delta_2^{a_2} \delta_3^{a_3}$ and $\xi = \delta_1^{b_1} \delta_2^{b_2} \delta_3^{b_3}$ for some integers a_3 and b_3 . The subgroup of Γ_0 generated by ζ and ξ maps surjectively to $\bar{\Gamma}_0$. Hence Γ_0 is generated by ζ , ξ , and the intersection $\Gamma_0 \cap z(\mathcal{H}(r))$. The latter is a subgroup of $z(\mathcal{H}(r)) \cong \mathbb{Z}$, and thus is generated by some element of the form δ_3^c .

Since the subgroup Γ_0 is normal, we have $\delta_3^{rb_1} = [\delta_1, \zeta] \in \Gamma_0$, so that c divides ra_2 . Similarly, we see that c divides ra_1 , rb_1 , and rb_2 , and thus also divides $r \gcd(a_1, a_2, b_1, b_2)$.

It is easy to compute that $[\zeta, \xi] = \delta_3^{r(a_1b_2 - a_2b_1)}$. Therefore, one has

$$\Gamma_0 \cong \mathcal{H}\left(\frac{r|a_1b_2 - a_2b_1|}{c}\right).$$

Let F be a subgroup of $\mathcal{H}(r)$ generated by the elements δ_2 and δ_3 , and \check{F} be its image in G_0 . The subgroup F is a normal abelian subgroup of $\mathcal{H}(r)$, hence \check{F} is a normal abelian subgroup of G_0 . Let $f: \mathcal{H}(r) \rightarrow G_0/\check{F}$ be the natural projection. Then the group G_0/\check{F} is generated by $f(\delta_1)$, and one has

$$f(\delta_1^{a_1}) = f(\zeta) = 1 = f(\xi) = f(\delta_1^{b_1}).$$

Hence $[G_0 : \check{F}] = |G_0/\check{F}|$ is bounded from above by (and actually equals) the number $\gcd(a_1, b_1)$. \square

An immediate consequence of Lemma A.12 is the following boundedness result.

Corollary A.13. *Let $\Gamma_0 \subset \hat{\Gamma}_0$ be a normal subgroup, where $\Gamma_0 \cong \mathcal{H}(r_1)$ and $\hat{\Gamma}_0 \cong \mathcal{H}(r_2)$. Then the quotient group $G_0 = \hat{\Gamma}_0/\Gamma_0$ is finite, and G_0 contains a normal abelian subgroup of index at most r_1 .*

Proof. The group G_0 is finite for obvious reasons. By Lemma A.12 there are integers a_1, a_2, b_1, b_2 with $a_1b_2 - a_2b_1 \neq 0$, and $c > 0$ such that c divides $r \gcd(a_1, a_2, b_1, b_2)$, one has

$$r_1 = \frac{r_2|a_1b_2 - a_2b_1|}{c},$$

and G_0 contains a normal abelian subgroup of index at most $\gcd(a_1, b_1)$. On the other hand, one has

$$\frac{r_2|a_1b_2 - a_2b_1|}{c} \geq \frac{r_2 \gcd(a_1, b_1) \gcd(a_2, b_2)}{c} \geq \frac{r_2 \gcd(a_1, a_2, b_1, b_2)}{c} \gcd(a_1, b_1) \geq \gcd(a_1, b_1).$$

□

A.14. Heisenberg groups and semi-direct products. Consider the groups $\Gamma_0 \cong \mathcal{H}(r)$ and $\Gamma_1 \cong \mathbb{Z}$. Let γ be a generator of Γ_1 . Consider a semi-direct product $\Gamma = \Gamma_0 \rtimes \Gamma_1$. The action of γ on Γ_0 gives rise to its action on

$$\bar{\Gamma}_0 = \Gamma_0 / z(\Gamma_0) \cong \mathbb{Z}^2,$$

which is given by a matrix $M \in \text{GL}_2(\mathbb{Z})$ if we fix a basis in $\bar{\Gamma}_0$ (cf. [Osi15] for a detailed description of the automorphism group of the discrete Heisenberg group).

Lemma A.15. *The following assertions hold.*

- (i) *One has $\gamma \delta_3 \gamma^{-1} = \delta_3^{\det M}$.*
- (ii) *The center $z(\Gamma)$ is trivial if and only if $\det M = -1$.*

Proof. For $i = 1, 2$ one has

$$\gamma \delta_i \gamma^{-1} = \delta_1^{m_{1,i}} \delta_2^{m_{2,i}} \delta_1^{p_i},$$

where $M = (m_{i,j})$, and p_i are some integers. Obviously, we have $\gamma \delta_3 \gamma^{-1} = \delta_3^t$ for some integer t . Therefore

$$\begin{aligned} \delta_3^{rt} &= \gamma \delta_3^r \gamma^{-1} = \gamma \delta_1 \delta_2 \delta_1^{-1} \delta_2^{-1} \gamma^{-1} = \delta_1^{m_{1,1}} \delta_2^{m_{2,1}} \delta_1^{m_{1,2}} \delta_2^{m_{2,2}} \delta_2^{-m_{2,1}} \delta_1^{-m_{1,1}} \delta_2^{-m_{2,2}} \delta_1^{-m_{1,2}} = \\ &= \delta_3^{r(m_{1,1}m_{2,2} - m_{1,2}m_{2,1})} = \delta_3^{r \det M}, \end{aligned}$$

which implies assertion (i). Assertion (ii) easily follows from assertion (i). □

We will need the following notation. Let Υ be a group, and Δ be its subset. Denote

$$\text{rad}(\Delta, \Upsilon) = \{g \in \Upsilon \mid g^k \in \Delta \text{ for some positive integer } k\}.$$

If Δ is invariant with respect to some automorphism of Υ , then $\text{rad}(\Delta, \Upsilon)$ is invariant with respect to this automorphism as well. If a group Υ has no torsion and $\Delta \subset \Delta'$ is a pair of subgroups in Υ such that the index $[\Delta' : \Delta]$ is finite, then $\Delta' \subset \text{rad}(\Delta, \Upsilon)$.

Using Lemma A.11, one can easily check the following.

Lemma A.16. *Suppose that the matrix M does not have eigen-values equal to 1. Then the following assertions hold:*

- (i) $[\Gamma, \Gamma] \subset \Gamma_0$, and

$$[\Gamma, \Gamma] / z([\Gamma, \Gamma]) = \text{Im}(M - \text{Id}) \subset \Gamma_0$$

is a free abelian group of rank 2;

- (ii) $[\Gamma, \Gamma] \cong \mathcal{H}(r')$ for some r' , and $[\Gamma, \Gamma]$ is a subgroup of finite index in Γ_0 ;
- (iii) $\Gamma_0 = \text{rad}([\Gamma, \Gamma], \Gamma)$; in particular, Γ_0 is a characteristic subgroup of Γ .

Remark A.17. If the matrix M does not have eigen-values equal to ± 1 , then the group Γ does not contain subgroups isomorphic to \mathbb{Z}^3 .

Lemma A.18. *Suppose that the eigen-values of the matrix M are not roots of unity. Then there exists a constant $\nu = \nu(\Gamma)$ with the following property. Let $\hat{\Gamma} = \hat{\Gamma}_0 \rtimes \hat{\Gamma}_1$, where $\hat{\Gamma}_0 \cong \mathcal{H}(\hat{r})$ and $\hat{\Gamma}_1 \cong \mathbb{Z}$, and suppose that $\Gamma \subset \hat{\Gamma}$ is a normal subgroup. Then the group $G = \hat{\Gamma}/\Gamma$ is finite and has a normal abelian subgroup of index at most ν .*

Proof. The group G is finite for obvious reasons (cf. Lemma A.11). By Lemma A.16(iii) we have $\Gamma_0 = \Gamma \cap \hat{\Gamma}_0$. Thus by Lemma A.7 there is a positive integer k with the following properties: the subgroup $\Gamma_1 \subset \Gamma$ is generated by an element $\hat{\gamma}^k \hat{\delta}$, where $\hat{\gamma}$ is a generator of $\hat{\Gamma}_1$, and $\hat{\delta}$ is an element of $\hat{\Gamma}_0$; and the group G contains a normal subgroup $G_0 \cong \hat{\Gamma}_0/\Gamma_0$ of index k . Note that the subgroup Γ_0 is normal in $\hat{\Gamma}$, because Γ is normal in $\hat{\Gamma}$, while Γ_0 is a characteristic subgroup of Γ by Lemma A.16(iii). Let $R \in \text{GL}_2(\mathbb{Z})$ be the matrix that defines the semi-direct product $\hat{\Gamma} = \hat{\Gamma}_0 \rtimes \hat{\Gamma}_1$. Considering the action of the element $\hat{\gamma}$ on the lattice $\hat{\Gamma}_0/z(\hat{\Gamma}_0) \cong \mathbb{Z}^2$ and its sublattice $\Gamma_0/z(\Gamma_0) \cong \mathbb{Z}^2$, we see that R^k is conjugate to M . Thus k is bounded by some constant that depends only on M (that is, only on Γ) by Lemma A.3. On the other hand, the group G_0 contains a normal abelian subgroup of index at most r by Corollary A.13, and the assertion easily follows. \square

A.19. Heisenberg groups and direct products. Consider the groups $\Gamma_0 \cong \mathcal{H}(r)$ and $\Gamma_1 \cong \mathbb{Z}$, and put $\Gamma = \Gamma_0 \times \Gamma_1$. One has $z(\Gamma) = \langle \delta_3, \gamma \rangle \cong \mathbb{Z}^2$, and $\bar{\Gamma} = \Gamma/z(\Gamma) \cong \mathbb{Z}^2$.

Unlike the situation in §A.5 and §A.14, the subgroup Γ_0 is not characteristic in Γ . Indeed, let δ_1, δ_2 , and δ_3 be the generators of Γ_0 as in (A.10), and γ be a generator of Γ_1 . Define an automorphism ψ of Γ by

$$\psi(\delta_1) = \delta_1 \gamma, \quad \psi(\delta_2) = \delta_2, \quad \psi(\delta_3) = \delta_3, \quad \psi(\gamma) = \gamma,$$

cf. [Osi15]. Then ψ does not preserve Γ_0 . However, the following weaker uniqueness result holds.

Lemma A.20. *Let $\Gamma'_0 \subset \Gamma$ be a normal subgroup isomorphic to $\mathcal{H}(r')$ for some positive integer r' . Suppose that $\varsigma: \Gamma/\Gamma'_0 \cong \mathbb{Z}$. Then the natural projection $\Gamma'_0 \rightarrow \Gamma_0$ is an isomorphism. In particular, one has $r = r'$.*

Proof. Put $\Upsilon_1 = \Gamma_1/\Gamma'_0 \cap \Gamma_1$ and $\Upsilon_0 = \Gamma_0/\varsigma(\Gamma'_0)$. Then $\Gamma/\Gamma'_0 \cong \Upsilon_0 \times \Upsilon_1$. Therefore, either Υ_0 is trivial and $\Upsilon_1 \cong \mathbb{Z}$, or $\Upsilon_0 \cong \mathbb{Z}$ and Υ_1 is trivial. In the former case ς provides an isomorphism from Γ'_0 to Γ_0 . In the latter case $\Gamma_1 \subset \Gamma'_0$ and the group $\varsigma(\Gamma'_0) \cong \Gamma'_0/\Gamma_1$ is abelian by Lemma A.11. Thus the group Γ'_0 is abelian as well, which is a contradiction. \square

Lemma A.21. *Suppose that Γ is a normal subgroup in a group $\hat{\Gamma} = \hat{\Gamma}_0 \times \hat{\Gamma}_1$, where $\hat{\Gamma}_0 \cong \mathcal{H}(\hat{r})$ and $\hat{\Gamma}_1 \cong \mathbb{Z}$. Then the group $G = \hat{\Gamma}/\Gamma$ is finite and has a normal abelian subgroup of index at most r .*

Proof. The group G is finite for obvious reasons. Put $\Gamma'_0 = \Gamma \cap \hat{\Gamma}_0$ and $G_0 = \hat{\Gamma}_0/\Gamma'_0$. By Lemma A.7 one has $\Gamma = \Gamma'_0 \rtimes \Gamma'_1$, where $\Gamma'_1 \cong \mathbb{Z}$ is generated by $\hat{\gamma}^k \hat{\delta}$ for some positive integer k , a generator $\hat{\gamma}$ of $\hat{\Gamma}_1$, and an element $\hat{\delta} \in \hat{\Gamma}_0$. Thus $\Gamma/\Gamma'_0 \cong \mathbb{Z}$. Since Γ'_0 is a subgroup of finite index in $\hat{\Gamma}_0$, by Lemma A.11 one has $\Gamma'_0 \cong \mathcal{H}(r')$ for some r' . By Lemma A.20 we know that $r' = r$. Thus G_0 contains a normal abelian subgroup N of index at most r by Corollary A.13. On the other hand, G is generated by G_0 and the image $\bar{\gamma}$ of $\hat{\gamma}$. Since $\bar{\gamma}$ is a central element in G , the group generated by N and $\bar{\gamma}$ is a normal abelian subgroup of index at most r in G . \square

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